01. Algorithms. Time Complexity. Merge Sort.

Algorithms

Def. **Algorithm:** Basically a formalized way to solve a problem.

Input Data -> [Algorithm] -> Output Data

Eg. Calculate sum of numbers in an array.
Input: a[0..n-1]
Output: Σa[i]

```
S = 0
for i=0..n-1:
    s+=a[i]
print(s)
```

Time Complexity

We can't measure the running time of an algorithm in seconds as different computers will give different speeds, based on the computer's hardware.

Def. **Computational Model:** Mathematical abstraction that is used to calculate time and memory complexity.

Eg. **RAM-Model**
Basically simulates the regular processor in the computer.

Think of memory as one big array. Any element can be accessed in one operation in the RAM model.

```
S = 0 //1 operation
for i=0..n-1: //2 operations per loop. So 2n
    s+=a[i] //3 operations (access, add, assign) per loop. So 3n
print(s) //1 operation
```

T(n) = 2 + 5n
This is a function which explains how time grows when you increase the size of input.

We take T(n) and remove what is not important.
T(n) = 5n, since 2 is not important compared to 5n
T(n) = n, since 5 is not important as it depends on various things and is not a property of the algorithm.

In T(n) = a + bn, a and b might vary, but n will remain.
T(n) = O(n)
O(n) is the upper bound of the time

If `f(n) = O(g(n))`
∃ n0, c ∀ n>=n0 : f(n)<=c.g(n)

Proof. 2+5n = O(n)
f(n) = 2+5n
g(n) = n

Consider n0=2 and c=6
2+5n<=6n
2<=n
So the proof works.

Proof. n = O(n^2)
f(n) = n
g(n) = n^2

Consider n0=1 and c=1
n<=n^2
1<=n

If `f(n) = Ω(g(n))`
∃ n0, c ∀ n>=n0 : f(n)>=c.g(n)

Proof. 2+5n = Ω(n)
Consider n0=1 and c=1
2+5n >= n

So T(n) = O(n) and T(n) = Ω(n)
If upper and lower bounds are same, T(n) = Θ(n)

How to calculate

eg.

```
for i=0..n-1
    for j=0..n-1
```

O(n^2)

eg.

```
for i=0..n-1
    for j=0..i-1  //1+2+3..n-1
```

O(n^2)

eg.

```
i=0
while i*i<n
    i++;
```

O(sqrt(n))

eg.

```
i=1
while i<n
    i*=2
```

After k iterations, i=2^k
2^k >=n
k>=logn
O(logn)
Note:

$\log_a n = O(\log_b n)$
$a, b > 1$

Recursive Algorithms

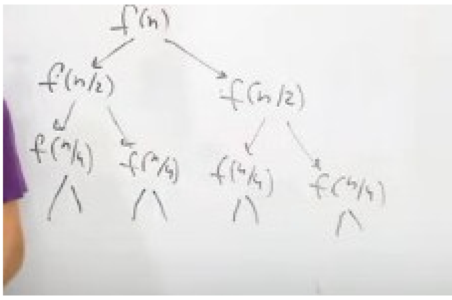
```
def f(n)
    if n=0
        return
    f(n-1)
```

To calculate time complexity, calculate no of times we call the recursive function, and for every call, calcuate no of operations.

f(n) -> f(n-1) -> f(n-2) -> ... -> f(0)
n recursive calls, each of O(1), so O(n)

```
def f(n)
    if n=0
        return
    f(n/2)
    f(n/2)
```

We have a tree of recursive calls



To calculate no. of recursive calls, we find the number of nodes.
Height of tree, H = log2(n)
Total elements = 2^H = 2^(log2(n)) = n

So O(n)

```
def f(n)
    if n=0
        return
    f(n/2)
    f(n/2)
    f(n/2)
```

Height of tree, H = log2(n)
Total elements = 3^H = 3^(log2(n)) = n^(log2(3))

Sorting Algorithms

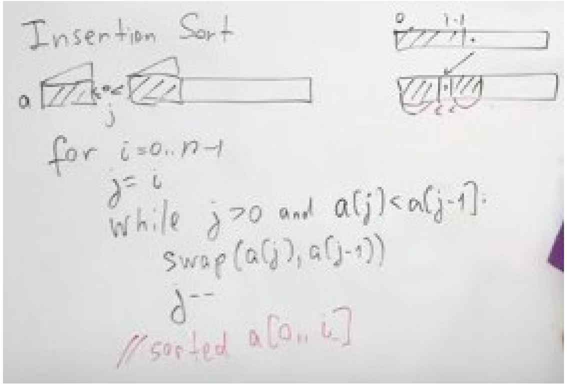
Input: a[0..n-1]
Output: b[0..n-1], same as a, but sorted

Insertion Sort

```
for i=0..n-1  //For every element in the array
    j=i //Take the element
```

```
    while j>0 and a[j]<a[j-1]: //While index is positive and element is smaller than its left neighbor (i.e. unsorted)
        swap(a[j], a[j-1]) //Swap element with the left neighor
        j-- //Decrease the index
    //At this point we have sorted a[0..i]
```

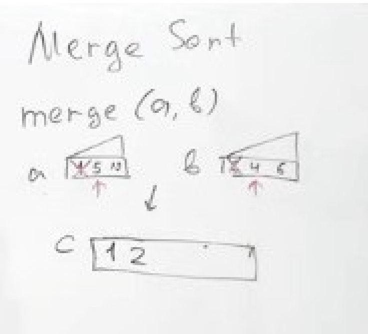
Note: Correctedness can be proved using the invariant that after every iteration we have a sorted prefix.



Time complexity:
Best case: Array is already sorted. Traverse all elements once. O(n)
Worst case: Array sorted in reverse. O(n^2)
∴ O(n^2)

Merge Sort

`merge(a,b)` merges two sorted arrays into one sorted array.

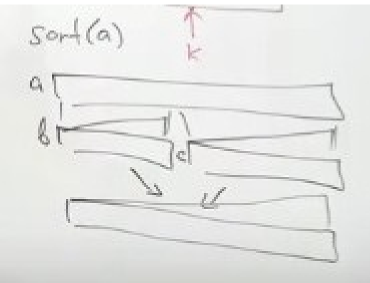



```
merge(a,b)
    n = len(a) //length of array a
    m = len(b) //length of array b
    i=0, j=0, k=0
    c = int[n+m] //array of size n+m
    while i<n or j<m: //while there are elements
        if j=m or (i<n and a[i]<b[j]) //If 2nd array is empty
        // or if element of a < element of b
            c[k++]=a[i++] //move element of a to array c
```

```
                           //and increase counters
         else
             c[k++]=b[j++] //move element of b to array c
                           //and increase counters
     return c //c is sorted array
```

Time complexity: O(n+m)

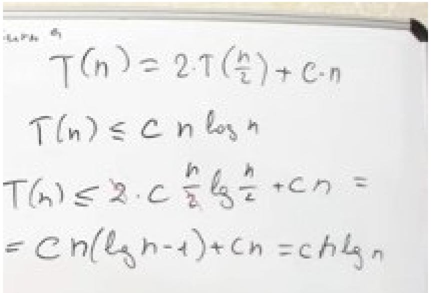
`sort(a)` sorts the array using Divide and Conquer




```
sort(a):
    if len(a)<2 //If array has less than 2 elements
        return a //Return it
    b=[0..(n/2)-1] //First half
    c=[n/2..n-1] //Second half
    b=sort(b) //Recursively sort first half
    c=sort(c) //Recursively sort second half
    return merge(b,c) //Merge both halves
```

Time complexity: T(n) = 2T(n/2) + c.n since two recursive calls of size n/2 + linear time merging

= O(nlogn) (either calculate by Recursion tree or by Master Theorem or by induction)
Proof. (by Induction) T(n) <= cnlogn
T(n) <= 2.c.(n/2)log(n/2) + c.n
T(n) <= c.n.log(n-1) + cn = c.n.logn



Master Theorem

Suppose a problem is split into 'b' parts and 'a' recursive calls are made, and some extra operations are made.
T(n) = aT(n/b) + g(n)

and

$g(n)$'s complexity is some $c.n^k$

Then

$T(n) = n^k$ if $a<b^k$

$T(n) = (n^k)\log n$ if $a=b^k$

$T(n) = n^{(\log_b(a))}$ if $a>b^k$