Computational geometry and topology assignment 2: Ishan Agarwal

Ishan Agarwal

February 2017

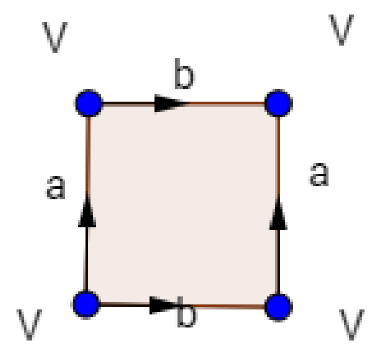
A Note on Notation

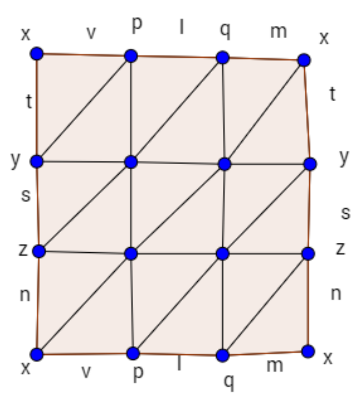
- All angles are in degrees
- b_i refers to the i^{th} betti number
- In all triangulations simplices left not named are distinct while those given the same name are identified as the same.
- By colouring we always mean a vertex colouring with neighbouring vertices having different colours

Problem 1(a)

Consider a tetrahedron T with the faces included. This is clearly a simplicial complex which is homotopy equivalent to the sphere. The triangulation given by the triangles formed by the edges of the tetrahedron clearly form a valid triangulation of the sphere. The Euler characteristic is $V-E+F = 4-6+4 = 2$ or we can calculate it in terms of betti numbers as $b_0-b_1+b_2 = 1-0+1 = 2$

We now give triangulation of a torus. If we do not insist that the three vertices of a triangle are distinct then the following works.
However here we are using a single vertex and as per the definition in class this fails to be a triangulation. We can however use the following





triangulation.

In fact we can see that the second triangulation gives us the Euler characteristic as $V - E + F = 9 - 27 + 18 = 0$. (The first complex is what is known as pseudo triangulation and gives the same answer as it is a CW complex though not a triangulation as per our definition, it can still be used to compute the Euler characteristic: $V - E + F = 1 - 2 + 1 = 0$)

Thus a sphere has Euler characteristic 2 but a torus has Euler characteristic 0. However two topologically equivalent spaces (whether homeomorphic or homotopy equivalent) must have same Euler characteristic and hence the torus and the sphere are not topologically equivalent.

Problem 1(b)

There are many such examples. Consider D^2 and D^3 which are the closed circle and closed sphere (including the inside regions in both cases). Both have $b_0 = 1$ and the rest betti numbers 0. This is easy to see by considering triangulations for each: for the first a triangle with the face included and for the second a tetrahedron with the inside tetrahedron included. However there are no examples for manifolds that are without boundary i.e open compact connected bounded manifolds as the n^{th} homology group of such a manifold is always Z.

Problem 2(a)

lemma 1:Pick a triangle t in a triangulation of a simple closed polygon and consider all the triangles that share an edge with it. Call this set S_1. Next define S_2 as S_1 union all the triangles sharing an edge with the boundary of S_1 and so on. By some S_n all the triangles in the triangulation would have been included.
Proof:
Suppose this were not so. If two triangles share an edge then their union is path connected. If a triangle w exists which does not belong in S_n for any n then say there was a path from some point in t to some point in this triangle. As the original polygon is bounded. If there exists such a path there must exist some such path of finite length, since the polygon is path connected. Further we can slightly change any such path to ensure it does not pass through any vertices. Thus the path can cross only finitely many edges, as it must be of finite length (again paths can be arbitarilly close to corners of triangles but we can deform them to ensure a uniform lower bound of the length of the path within each triangle.) But then look at the sequence of edges that this path crosses. Clearly the triangle w will lie in S_n as these triangles will get included in the S_i one by one (at worst). Contradiction. K is a triangulation. Pick any triangle as t_1. Include it in the shelling. Next pick any triangle which shares an edge with t_1. Include it as t_2. At the i^{th} step include any triangle t that shares an edge with the boundary set of the

shelling so far. Now suppose no triangles in K remain that are not already included in the shelling which share an edge with this boundary set, then we claim that all the triangles in the triangulation have been included. Say this were not so. Consider such a triangle t. This t has not already been included yet it shares no common edge with the boundary of the shelling up till now $(t_1, t_2, ...t_n)$. Say it shared a boundary with $(t_1, t_2, ...t_i)$ for some i. As it is not yet included, it will still share that boundary. Thus we assume that for no such i did this occur. But then t never had a common edge with any of the sub triangulations $(t_1, t_2, t_3...t_i)$. However t is part of the triangulation K so look at the triangles that have a common edge with t. None of these could have been included in the shelling as well. For if any of them were included, the sub - triangulation would have a boundary edge common with t at that point. But we can extend this argument for each of these triangles with some edge common to t as well. However note that in a triangulation, if we start at a triangle and consider all it's neighbouring triangles, then all their neighbours and so on, eventually we would have considered all triangles in the triangulation.(by lemma 1) Hence for such a t to exist the set $(t_1, t_2, t_3..., t_i)$ would have to be empty. But we can always include t_1. Contradiction.

It remains to show that for each i, $(t_1, t_2, ...t_i)$ would be the triangulation of some closed simple polygon. But if you have a closed simple polygon and we add to its triangulation a triangle with an edge common to one of it's boundary edges, it is trivial to check that this is still a closed simple polygon as we have just increased the number of sides by 1.

Problem 2(b)

Consider the triangulation K and a shelling S. Say $S = (t_1, t_2, t_3, ...t_n)$ Set some vertex of t_1 as 1 and another as 0. Set the third vertex as 0.5. Now we maintain the vertex with value 1 as max and vertex with value 0 as min. Whenever we add a triangle to the shelling it will have an edge in common with the boundary of the shelling. Call this common edge E with vertices v_1 and v_2. Set the value of the function at the vertex of the added triangle as the average of these values. Now this vertex has a neighbour with value lower than it as one with value higher than it. Thus it can never become a maxima or minima. The original maxima remains a maxima and so does the minima due to the nature of this construction and every new vertex value cannot either be a max or min. Hence this has exactly one max and one min. Furthermore to add each triangle takes only constant time, as wejust copute the average of the two values at the common vertices. Thus the algorithm takes time O(n) where there are n triangles in the triangulation.

Problem 4

Assumptions:

- The morse function is bounded
- The morse function has distinct values on different points.
- We can, using the triangle edge data structure, find neighbours of points, traverse the triangulation, as well as find the number of connected components of the link of a point, within some bounded constant time. The explicit means of doing this are not shown (i.e how to do this using enext and fnext operations) in the interests of brevity. This is in any case simple, for example to find the number of connected components of the link of v, take some edge e which originates at v. Look at v.fnext,v.fnext.fnext and so on till we reach v. At each step store whether the value of the vertex at the other end of $v.fnext^i$ is greater than or lower than v. Keep a count of how many times this flips from lower to greater or vice verca when we go from $v.fnext^i$ to $v.fnext^{i+1}$. This is clearly the number of connected components of the link.

We can search through the data structure and find this minima and saddle point in O(n) time just by traversal. This can be done in O(n) time. Now maintain a set of visited vertices V. Start with the minima in V. At each step do the following

- order V in increasing functional value order. Now in this order check each of the vertices as to whether it is s or not. If yes return that (m,s) belongs. If no then check if those points are critical points. If yes then return that (m,s) does not belong.
- For all v in V look at neighbours of v with higher functional value than v. Include them in V and remove the original v from V.

- If V becomes empty conclude that (m,s) does not belong.(this cannot happen as if there is a minima and we keep increasing the value of f being looked at then, for bounded functions, some critical point must be obtained.

- goto the first step.

Note that if (m,s) is on an arc of the reeb graph then s has index one and there cannot be any other critical point on the arc. Thus the above algorithm which checks from m onwards around m but looking at increasing functional values will be correct as it will claim (m,s) are on an arc if and only if s is the critical point after m in order of increasing functional values when we look starting at m and progressively follow the function along wherever the function value increases away from m.
Also this algorithm looks at each vertex only once. At any given vertex checking if it is a critical point can be assumed to be bounded by some b. Thus this algorithm takes O(bn) time where n is the number of vertices in the triangulation.Here for most reasonable data sets we can assume b to be a constant.

Problem 5

We give a three colouring and prove that it is indeed a valid three colouring. Consider a triangulation of an orientable 2 manifold T. We classify the vertices in the Barycentric subdivision into 3 types.

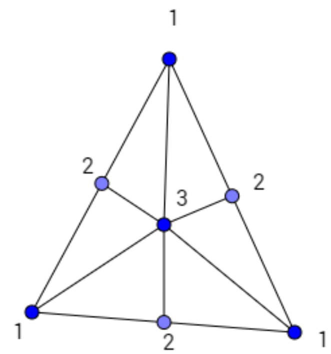
- 1: vertices originally present before the subdivision

- 2: new vertices introduced on the old edges

- 3: new vertices introduced within the old faces

We give these three kinds of vertices three different colours. All that remains to be proved is that in the Barycentric subdivision it is not possible that two vertices of the same class are neighbours.

Note that if two vertices of type 1 were not neighbours in the original triangulation then they cannot be neighbours now but if they were neighbours in the original triangulation, then they now have a vertex on the edge that



used to join them (of type 2) and so are not neighbours in the subdivision.

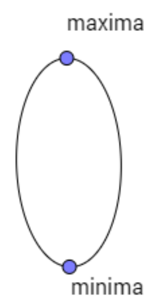
As for vertices of type 2, they have, by construction edges to vertices of type 1 and 3 only. Vertices of type 3 have edges to vertices of type 1 and 2 only. Thus this is a valid colouring.

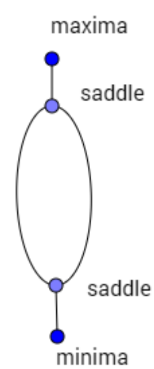
Problem 3(a)

Consider the function which is $g = f + \alpha x$ where x is the distance from the leftmost point on the torus. Then note that while f has, as has minima and maxima on two circles respectively, all the critical points of g are isolated and non-degenerate. Thus g is a Morse function. Now note that the total width of the Torus is bounded by some constant, call it B. We can set the value of $\alpha = \frac{\epsilon}{B}$. This ensures that f differs from g by at most ϵ. Furthermore, the height function fails to be a morse function due to it's maxima and minima being non-isolated and g does not have any such degeneracy.

Problem 3(b)

The first of the following sketches is for f and the second is for g.


maxima
minima


maxima
saddle
saddle
minima