Four Colouring a Delauney Triangulation

An attempt at an elementary proof

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- We want to give an efficient algorithm that outputs the four colouring of a given Delauney triangulation.

Importance

The existence of such a four colouring follows trivially from the Four Colour Theorem.

Theorem

Given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.

However we are looking at a much more restricted class and want an elementary proof.

Existing results and our result

- Efficient algorithms for checking if a Delauney triangulation is 3 colourable and outputing a three colouring if so are known.
- We attempt to give an algorithm that gives a four colouring of a Delauney triangulation of a planar point set.

Motivation for our algorithm

- A characterization of 3-colourable triangulations was given by Tsai and West(1), in terms of the 2-colourability of the dual graph of the triangulation. They also give an algorithm to 3-colour the triangulation if the dual graph is 2-colourable.
- We use an extension of the idea of this algorithm with suitable modifications to generate a 4-colouring of the Delauney triangulation from a 3-colouring of it's dual graph with certain assumptions on this 3-colouring.

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This result can be used to prove the 3-colourability of dual graphs of triangulations.

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Definitions and Background

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Definition (Complex Cycle)

A complex cycle is either a unit cycle or consists of multiple unit cycles such that every unit cycle in it shares some edge with another unit cycle in it.

The Conjecture

The dual graph of a triangulation of a planar point set can be 3-coloured in a way in which the third colour is only used once per unit cycle. We shall call such a three colouring *nice*.

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Lemma

In the above decomposition if we reduce complex cycles to points then the resulting reduced dual graph is a tree.

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- Show that this is correct on all triangles labelled a or b. Failures can be characterized as occuring when there are odd cycles in the dual and hence the colour *c* must occur on the dual in this cycle leading to a contradiction.

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- Errors thus occur only on triangles labelled *c* but these can be patched using colour 4. This is possible since such triangles are far apart.

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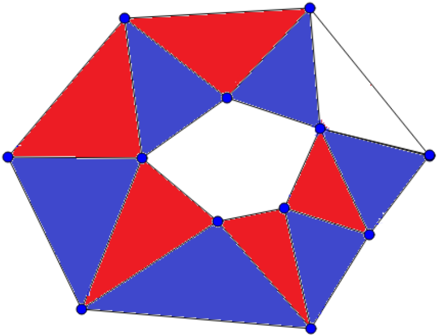

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- There can be an edge in common with the path. However we can permute the colouring appropriately to match on the common edge.

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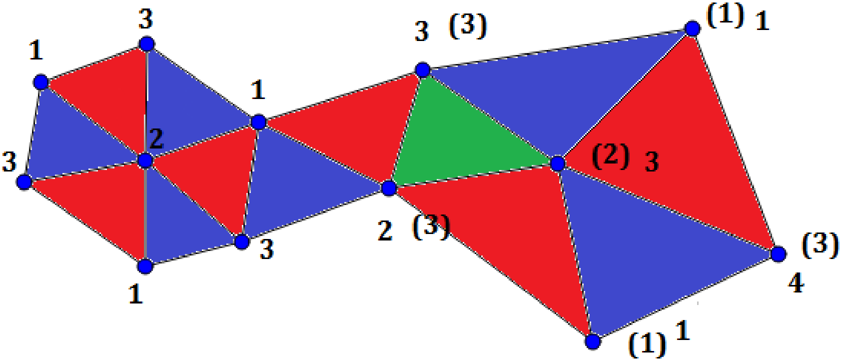

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- It is the belief of the authors that it should be possible to prove this conjecture at least for some classes of triangulations (in our case we would like to prove it for Delauney triangulations).
- Our result can also be viewed as a tool to reducing a proof of four colourability of any given class of triangulations to proving the conjecture for that class.
- We also give, an algorithm to four colour a given triangulation given a *nice* three colouring of it's dual.

References

1. A new proof of 3-colorability of Eulerian triangulations, Mu-Tsun Tsai and Douglas B. West.
2. Brook's theorem, Diestel Graph Theory; A Graduate Text in Mathematics. Springer 2006.

Thank you!