Project Report

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Statement of Problem

We attempt to give a proof of the four colourability of Delauney triangulations of a point set in 2D Euclidean Space. This result trivially follows from the four colour theorem but we seek an elementary proof of the same in this restricted setting. Our attempt at the proof involves giving an efficient algorithm that outputs the four colouring of a given Delauney triangulation.

Past Results

A characterization of 3-colourable triangulations was given by Tsai and West(1), in terms of the 2-colourability of the dual graph of the triangulation. They also give an algorithm to 3-colour the triangulation if the dual graph is 2-colourable. However if the triangulation is not 3-colourable no efficient algorithm for giving a 4-colouring is known. We use an extension of the idea of this algorithm with suitable modifications to generate a 4-colouring of the Delauney triangulation from a 3-colouring of it's dual graph with certain assumptions on this 3-colouring.

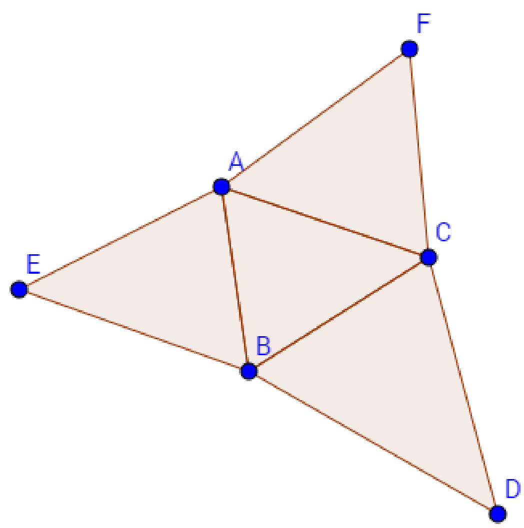
Results used regarding the dual graph of Delauney triangulations

Theorem 1 (Brooks 1941)**.** *Let G be a connected graph. If G is not complete or an odd cycle then G is δ colurable, where δ is the maximum degree in G.*

We note that the dual graph of any triangulation of a planar point set is connected and has maximum degree three. Also if the dual graph were a cycle, it is 3-colourable. Thus if the dual graph of a triangulation doesn't contain a K_4, by the above theorem it would be 3-colourable.

Lemma 1. *The dual graph of a triangulation of a planar point set doesn't contain K_4 as a subgraph.*

Proof. Let a triangulation T contain K_4 in it's dual graph. By definition, there exists a triangle ABC in T such that ABE, ACF and BCD are triangles in the triangulation whose duals form a K_4.



ACF must share edges with ABE and BCD. In particular FC must be identified with DC and AF with AE. Now $\angle FCA$ is less than 180 since it is a part of a triangle. Since $\angle FCA + \angle ACD = 360$, thus $\angle ACD$ is more than 180. Similarly $\angle DBA$ is also more than 180. Hence the sum of angles of the quadrilateral $ABDC$ will be more than 360, which is a contradiction. $\square$

A four colouring using a three colouring of the dual

Definition 1 (Unit Cycle)**.** *A cycle of a planar graph is a unit cycle if it's interior contains no vertices of the graph.*

Conjecture 1. *The dual graph of a Delaunay triangulation of a planar point set can be 3-coloured in a way in which the third colour is only used once per unit cycle. We shall call such a three colouring nice.*

In this section we use the numbers 1,2,3 and 4 for the colours of the vertices of the triangulation and the letters a, b and c for the colours of the dual graph.

We associate every triangle coloured with 1,2 and 3 an orientation α if these colours occur in clockwise order and an orientation β otherwise.

The following observation is easy to verify:

Lemma 2. *In a dual graph of a triangulation of a planar point set, two unit cycles can at most share one edge.*

Proof. This is trivial to check. □

Definition 2 (Complex Cycle)**.** *A complex cycle is either a unit cycle or consists of multiple unit cycles such that every unit cycle in it shares some edge with another unit cycle in it.*

Lemma 3. *The dual graph of a triangulation of a planar point set is a union of complex cycles and a set of vertices that do not contain any cycle.*

Proof. Suppose the dual graph has no unit cycles. By definition of unit cycle all the vertices form an acyclic graph. Hence we are done.

Consider, otherwise, some unit cycle and all unit cycles sharing some edge with it. Keep adding unit cycles that have some edge common to this set of unit cycles to this set. As the graph is finite this process must terminate. When it does note that the resulting subgraph is a complex cycle.

Now inductively either no more unit cycles remain, in which case we are done or they do. If they do repeat the above process to arrive at another complex cycle.

As the graph is finite eventually no more unit cycles shall remain and hence we would have a decomposition into paths and complex cycles as per the claim. □

Lemma 4. *Assuming Conjecture 1, there is a four colouring of any triangulation whose dual is a single complex cycle. Given the nice colouring of the dual graph from Conjecture 1, this can be computed in linear time.*

Proof. Given the nice colouring of the dual graph, we do the following:

1. Pick any triangle which is coloured a in the dual graph. Colour it's vertices using the colours 1,2 and 3 fixing the order α.

2. Perform a depth first graph traversal of the dual graph (starting from the triangle chosen before) traversing all the vertices which are not coloured c and colour the adjoining triangles (i.e. triangles which share an edge) in the following manner: when moving from a vertex coloured a to a vertex coloured b, we colour the new triangle with the colours 1,2 and 3 with orientation β while keeping the colouring consistent on the common edge of the old and the new triangle and vice-versa.

3. Now look at the triangles coloured c. If there is a miscolouring (i.e. a 1-1, 2-2 or a 3-3 edge), colour any one of the miscoloured vertices with 4.

This runs in time linear in the size of the graph.
We observe the following:

- The colouring is correct on the triangles coloured a and b at each iteration of step 2 of the algorithm.

  Proof. Let us assume that there is a miscolouring in a triangle PQR in some step of the traversal. We choose PQR such that it is the first such instance of a triangle in which a miscolouring occurs. Without loss of generality we assume that it has the colour a in the dual, P was coloured 1 at some previous traversal step and the vertex Q is now also coloured 1 in the latest step. We look at the traversal subgraph of the colouring algorithm till this step. Since this subgraph is connected, we get the shortest length path between the vertices P and Q through the triangles. It is easy to check that if the endpoints of such a path have the same vertex-colouring, then the number of triangles on this path will be even. If we add the triangle PQR to this path from P to Q which was of even length (note that PQR wasn't initially a part of this path as so far it hasn't been covered by the traversal algorithm), we get a cycle in the dual graph which is of odd length, and hence is not 2-colourable. Hence one of the triangles in this cycle is coloured with c, which is a contradiction as this cycle is a subgraph of the traversal subgraph, all whose vertices are triangles coloured a or b. □

- The only other miscolouring possible is when all the vertices of a triangle coloured c get the same colour. At least two of these three points will share a triangle which is coloured either a or b and by the previous observation, this is not possible.
- The algorithm colours all the vertices of triangles coloured a or b as the dual remains connected if we remove all the c coloured points by Conjecture 1.
- All the vertices of the triangles coloured c are shared with some triangle coloured either a or b.

  Hence our algorithm colours the whole triangulation.

□

Lemma 5. *Consider the decomposition given in lemma 3. Reduce each complex cycle to a point such that for any vertex v in the complex cycle if there was an edge between v and w, where w is some vertex not in that complex cycle, keep an edge between the reduced complex cycle and w or the reduced complex cycle to which w belonged. The resulting transformed dual graph will be acyclic.*

Proof. Let us assume that C is a cycle in the reduced dual graph. Since a complex cycle is path connected and each edge of C has a precursor in the dual graph, we have a cycle C' in the dual graph, which after the reduction (on the whole graph) forms C. But since C' lies entirely in some complex cycle (which follows from the fact that triangulation of a planar point set cannot have holes apart from triangles), it would have been reduced to a point in the reduced graph, which contradicts the existence of C. □

We now give our algorithm to four colour the triangulation. Look at the reduced dual graph of the triangulation. Consider that the dual graph has been given a nice colouring by Conjecture 1 with the special colour c appearing only once per unit cycle. We perform the reduction mentioned in Lemma 4 to get a tree whose nodes are reduced complex cycles. Consider a root complex cycle of this tree. Now look at all children of this complex cycle. There will be paths from the root to these children. As the maximum degree in the dual graph is three for any triangulation, these paths never intersect (they cannot even have the same starting point). After colouring the root complex cycle the colouring can be trivially extended along the paths (if the starting point in the dual is a triangle coloured c we do not have a unique choice to extend the colouring but we can pick either of the two valid choices). Eventually the path may reach another complex cycle. Consider the triangle in the triangulation which is common to the path and a new complex cycle. Suppose this triangle is not coloured c. Then by colouring it we have performed step 1 in the algorithm in Lemma 4 for this complex cycle and so we can proceed to colour it as defined there. (Clearly lemma 4 works even if you start with a b coloured triangle with orientation β).

The only special case we need to consider is if this triangle is coloured c. Call this triangle T. In this case simply begin the algorithm as defined in lemma 4 with some other triangle in that complex cycle and proceed. This algorithm will give some two colours to the vertices of T that lie in the complex cycle. However these have been given possibly some other colours. There can be the following cases: (note that paths and cycles must share an edge or not intersect. They cannot share just a point as the maximum degree in the dual graph is 3.

- to match on path, colours 1 and 2. In the colouring of the new complex cycle colours 1 and 2. In this case both colourings match on the common edge so we are done.

- to match on path, colours 1 and 2. In the colouring of the new complex cycle colours 2 and 3. In this case apply the permutation $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$, $4 \mapsto 4$ to the colouring of the new complex cycle given by the algorithm in lemma 4.

- to match on path, colours 1 and 2. In the colouring of the new complex cycle colours 3 and 4. In this case apply the permutation $1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto 1$, $4 \mapsto 2$ to the colouring of the new complex cycle given by the algorithm in lemma 4.

We can proceed inductively to colour all children of the complex cycle tree. The above algorithm's correctness follows from lemma 4 on each complex cycle and the above permuting ensures that interactions between complex cycles through paths are appropriately handled. It is also clear that the above algorithm terminates in time linear in the size of the triangulation.

Conclusion

Modulo Conjecture 1 we have shown that since any triangulation has a three colourable dual, any triangulation is four colourable. It is the belief of the authors that it should be possible to prove this conjecture at least for some classes of triangulations (in our case we would like to prove it for Delauney triangulations) and thus get the originally desired result. Our result can also be viewed as a tool to reducing a proof of four colourability of any given class of triangulations to proving the conjecture for that class. We also give, an algorithm to four colour a given triangulation given a *nice* three colouring of it's dual.

References

1. A new proof of 3-colorability of Eulerian triangulations, Mu-Tsun Tsai and Douglas B. West.

2. Brook's theorem, Diestel Graph Theory; A Graduate Text in Mathematics. Springer 2006.