Project Report

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Introduction

This document serves as personal notes that should turn into a project report as things are formalized better and in a more organized fashion. Currently this is a log of all the results in the order in which we came upon them.

Problem statement

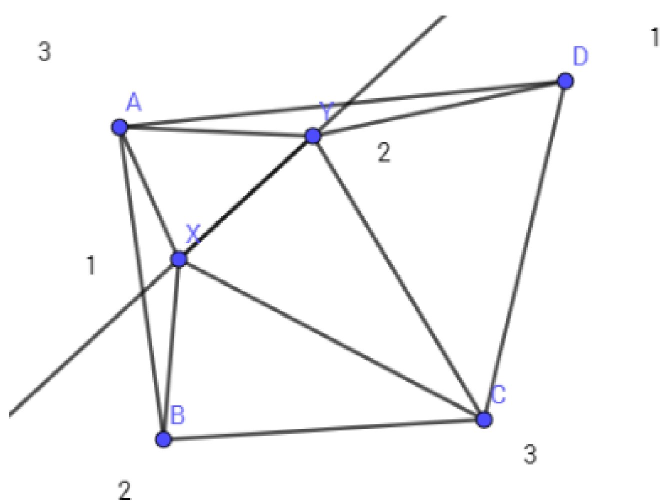
Consider S a set of n points in R^2. We wish to characterize point sets such that at least 1 possible triangulation of the point set is three colourable. Here we restrict our attention to triangulations which are connected and are such that if we include the triangles as faces we get a surface homeomorphic to R^2. We also require that the triangulation includes the convex hull of the point set in it's edge set. Furthermore non degeneracy of the point set is assumed throughout.

Results

A convex hull with one point within always has a three colourable triangulation as long as the convex hull has greater than 3 points. This is easy to see by showing it for the case of 4 points on the hull and inducting for larger hulls. Using a similar technique we show the following: Let P be any point set of n+2 points where n points lie on the convex hull. For $n \geq 4$ P has some 3 colourable triangulation.

Lemma 1. *Consider 4 vertices of a convex quadrilateral and 2 points within this quadrilateral. This pointset has a 3 colourable triangulation.*

Proof. Consider the quadrilateral named ABCD and points X and Y anywhere within it. Consider the line XY. Pick the side of XY in the plane which has lower number of points in it. Clearly there are either 1 or 2 points on this side. Without loss of generality call these A or A and B. Add the edges AX, AY, CX, CY as well as the edge XY. Note that these edges will not intersect except at end points as AX, AY and CX, CY are on opposite sides of XY. Furthermore it is easy to see that now adding one of the pairs BX, DY or BY, DX will not cause any intersections(this can be shown by easy but tedious angle arguments that we skip here). Note now that A,B,C,D,X,Y with the convex hull ABCD and edges AX,CX,AY,CY and either BX,DY or BY,DX does form a triangulation. Once again this is simple to check). We give a three colouring as shown below, which completes the proof of the lemma. □



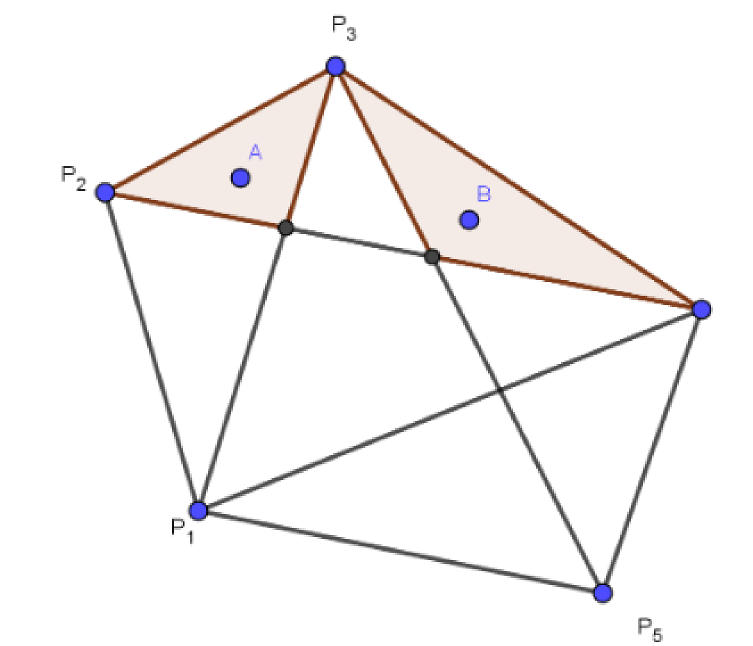
Theorem 1. *Let P be any point set of n+2 points where n points lie on the convex hull. For $n \geq 4$ P has some 3 colourable triangulation.*

Proof. We give a proof by induction. The base case ($n = 4$) is proved by

Lemma 1.

Now consider an n gon $\{P_1, ..., P_n\}$ and two points A and B such that A and B both lie in the triangle $P_{k-1}P_kP_{k+1}$ for some k. Consider Some j such that the triangles $P_{j-1}P_jP_{j+1}$ and $P_{k-1}P_kP_{k+1}$ do not have any common area. Then no points lie in $P_{j-1}P_jP_{j+1}$. Such a j clearly exists for all $n \geq 5$ as A and B lie in $P_{k-1}P_kP_{k+1}$ and so in particular $j = k + 2$ works.

We now claim that howsoever we select the position of A and B , for $n \geq 5$ such a j exists such that neither A nor B lie in $P_{j-1}P_jP_{j+1}$. Consider the triangles $X = P_1P_2P_3$, $Y = P_2P_3P_4$ and $Z = P_3P_4P_5$. Note that in any convex n gon, $X \cap Y \neq \emptyset$, $Y \cap Z \neq \emptyset$ and $X \cap Z = \emptyset$. Suppose no such j exists. Then at least 1 of A or B must lie in each of X, Y and Z or else we could set j as 2, 3 or 4 respectively. Clearly this means A and B must lie in $X \cap Y$ and $Y \cap Z$ respectively. But then both A and B lie in Y so consider $j = 5$ and the triangle $P_4P_5P_1$ and we have found such a j.



Now consider the point set $(\{P_i\} - \{P_j\}) \cup \{A, B\}$ where j is such that

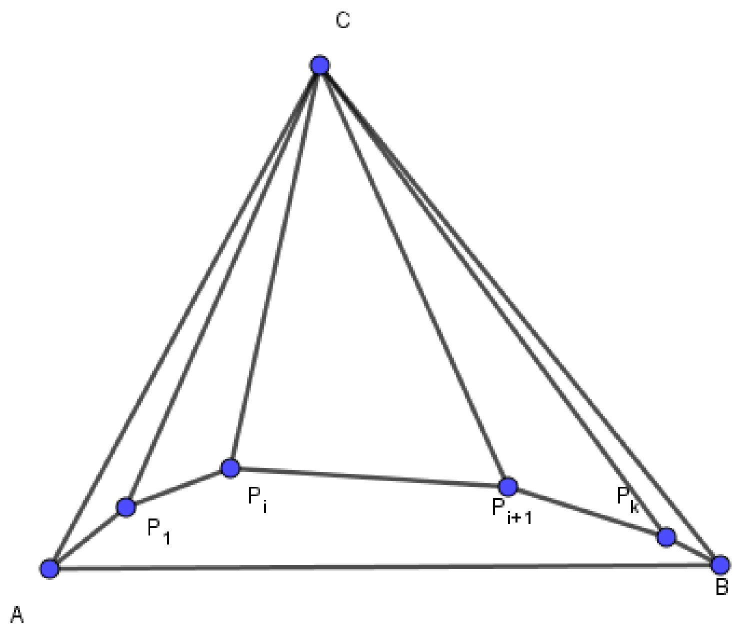
$P_{j-1}P_jP_{j+1}$ contains no points of the pointset. By induction hypothesis this point set has a three colourable triangulation T as it is of the form a convex n-1 gon and 2 points within it (since A and B do not lie in $P_{j-1}P_jP_{j+1}$). Now we extend this triangulation to the original point set by adding the point P_k and the triangle $P_{k-1}P_kP_{k+1}$. Note that this is clearly a triangulation as $\{P_i\}$ is a convex point set. We now note that, under the colouring of T, P_{k-1} and P_{k+1} have some distinct colours and under the extended triangulation P_k is a neighbour of only P_{k+1} and P_{k-1} so the colouring can also be extended to the extended triangulation which completes the proof. □

Consider now point sets P which have a triangle as convex hull. We characterize when such point sets have a three colourable triangulation. It is easy to see that if P has 3 points it always has a three colourable triangulation and if P has 4 or 5 points then there is no configurartion with any three colourable triangulation. First let us define an anomalous configuration.

Theorem 2. *Consider a point set P such that the convex hull of P is a triangle. Then we can always construct such a P with arbitrarily large cardinality n, at least 4, with no three colourable triangulation.*

Definition 1. *We name this configuration the anomalous configuration for n points.*

Proof. We construct such a point set. Consider P with 3 points on the convex hull. Now consider an edge AB of the triangle and any convex arc γ on AB as shown. Place k points on this arc γ. We claim that this point set with cardinality $k + 3$ with k $\geq$ 1 has no three colourable triangulation and we call it the anamolus configuration for $k + 3$ points.
First note that the edges of the form CP_i and P_iP_{i+1} are forced for all i. Once this is shown, we are done because consider the edge AB. Suppose there is some triangulation of the anomalous point set that is 3 colourable. We investigate what edges must be present in any triangulation of this point set. First note that if the edges of the form CP_i and P_iP_{i+1} are forced for all i. Once this is shown, we are done because consider the edge AB. It must be the face of at least one triangle which is not ABC, i.e we must have some P_j such that AP_jB is present in the triangulation. However consider the subgraph induced by the points A, B, C, P_j. It is a K_4 and hence the triangulation cannot be 3 colourable.



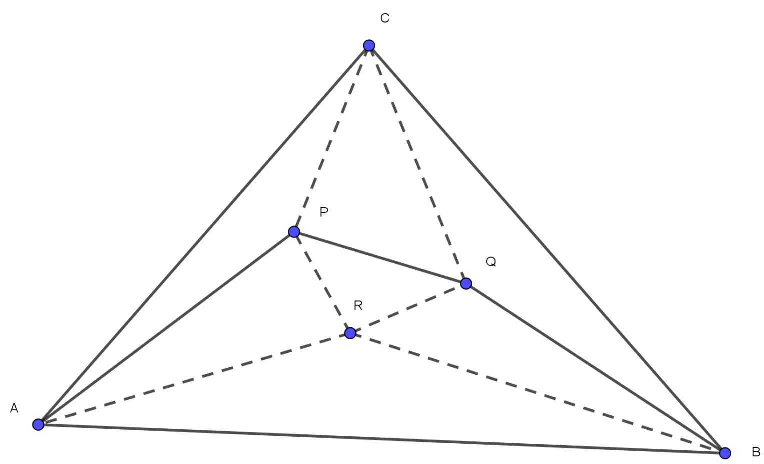
We now show why edges of the form CP_i and P_iP_{i+1} are forced for all i. Note first the following obvious fact: in a triangulation, consider any point which is not a vertex, in the region bounded by the convex hull. Such a point must lie, if it is not part of an edge, within some minimal triangle that contains no other vertices. as the faces of the triangulation are triangles. Consider some point lying in the area CP_iP_{i+1} This point two must have some such minimal triangle. The only possibilities are ABC or some CP_iP_j. However ABC contains other vertices. Also as the arc γ is convex, CP_iP_j contains all vertices P_k with k strictly between i and j. Hence the only possibility is to include the triangle CP_iP_{i+1}. This completes are proof as we have all edges of the form CP_i and P_iP_{i+1} as part of our triangulation.

□

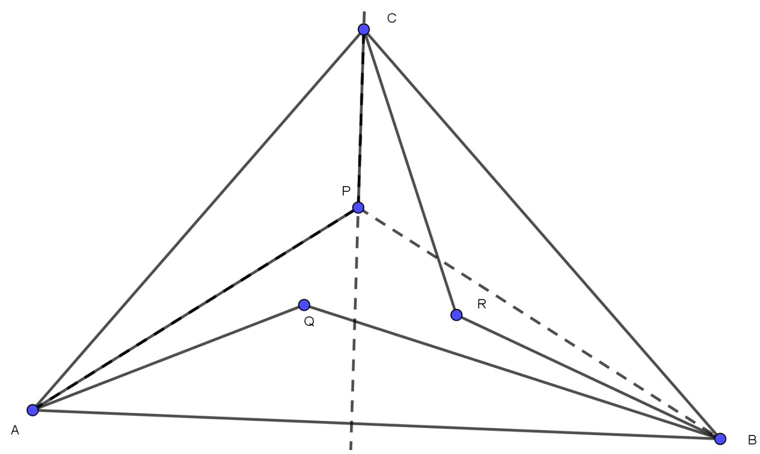
We now have the following results for pointsets with triangular convex hulls and a few points within. The case of 5 points, i.e a triangle with 2 interior points follows directly from the above result: these two points will always lie on a convex arc about some edge and so such a point set will have no three colourable triangulation.

Theorem 3. *If P has 6 points with 3 points on the convex hull and it is not anomalous then it has some three colourable triangulation.*

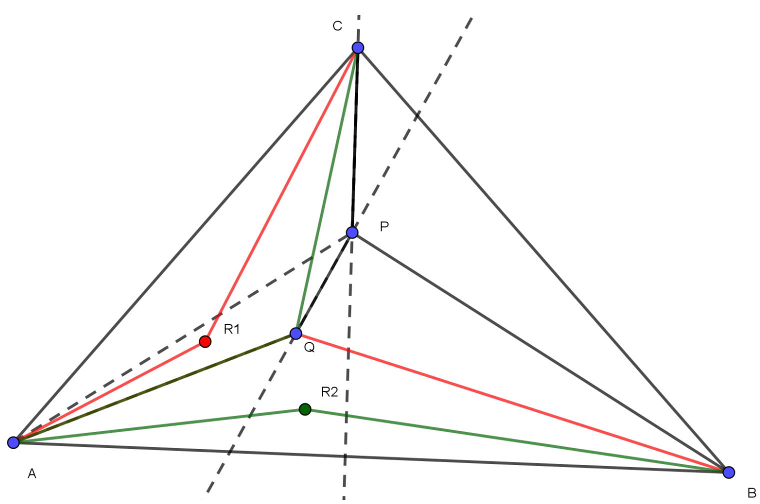
Proof. Consider any edge of the outer triangle ABC, say AB. Now of the interior points P,Q and R the angle CAX is minimized for some X out of P,Q and R. Similarly the angle CBY is also minimized for some Y out of P,Q and R. Let these minimizing points be P and Q respectively. Now clearly R must lie in the region APQB, as shown below. Say AP and BQ intersect at Z, then if R lies in PZQ then P,R,Q will form a convex arc on AB. If R lies outside both APQB and PZQ then it lies outside the triangle AZB, but this would contradict the way P and Q were chosen.



Thus we can use the triangulation shown below which is clearly 3 colourable. However it is possible that we do not get distinct angle minimizing X and Y. Say P was the interior point suct that angles CAX and CBY (where X and Y are some interior points) are both minimum for X=Y=P. Then extend CP to meet AB call this meeting point N. There can be two cases: either both points Q and R lie to one side of CN or they lie on either side. First let us suppose they lie on either side. Then by minimality properties of P we have that Q lies in PAN and R lies in PBN. Now we can see that we can get a three colourable triangulation by joining P,Q and R.



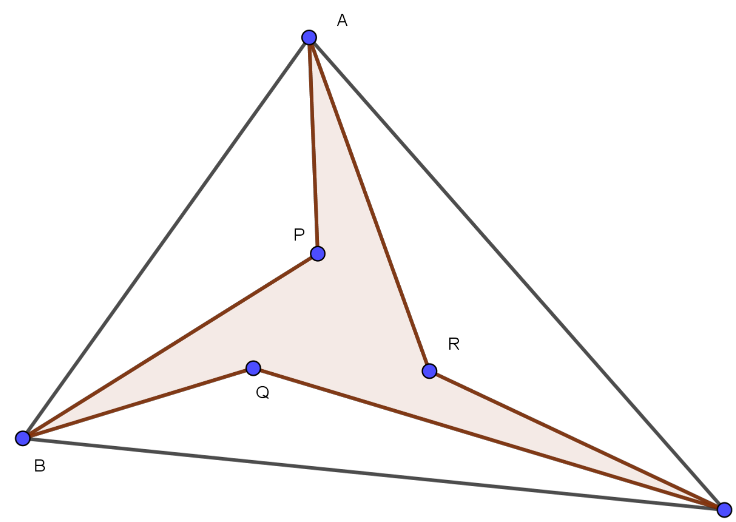
Finally we have the case where both Q and R lie on the same side of CN. This is shown below. Clearly, by the minimality property of P, both Q and R lie in PAN.



Now label as Q that interior point such that ACR is a smaller angle than ACQ. Join PQ and let this line intersect AB at M. Note that R cannot lie in MAQ as otherwise P, Q and R would lie on a convex arc about the side CA. Hence either R lies in the regions indicated by the two possible positions R_1 and R_2. (PAQ nad PMN respectively). In both these cases we can create a three colourable triangulation: In the case of R1 consider the faces ABQ, BCP nad CAR_1. In case of R_2 consider the faces ABR_2, BCP and ACQ. Finally include the triangle PQR. □

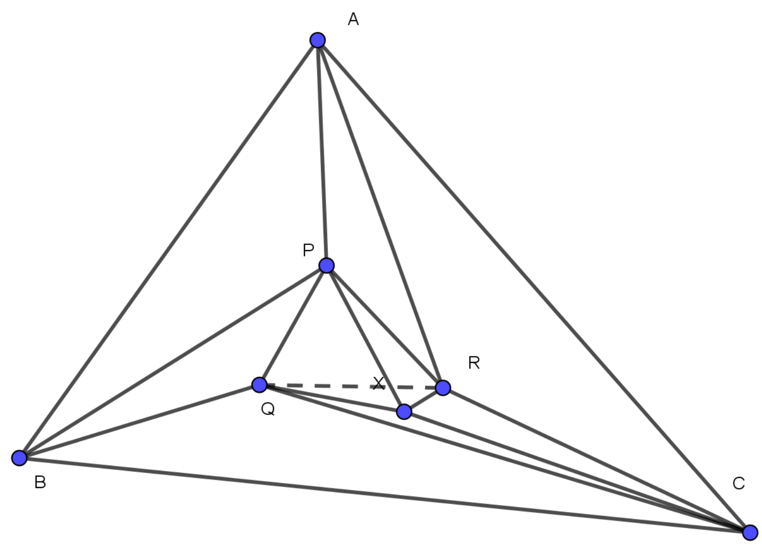
Theorem 4. *If P has 7 points with 3 points on the convex hull then it cannot have any three colourable triangulation.*

Proof. Suppose there is such a configuration. Let us call the outer triangle ABC. Now AB must be the side of some triangle other than ABC (this is easy to see from the obvious fact that any point in the triangle which is not a vertex and not on an edge must lie within some triangle that does not contain any vertex). The same is true of BC and CA. Let us call these interior points P,Q and R respectively. Note that n two of P, Q and R can be the same as this will cause the presence of a K_4, thus violating 3 colourability of the triangulation. Thus the following configuration is forced. The fourth interior point say X may lie anywhere in the shaded region.

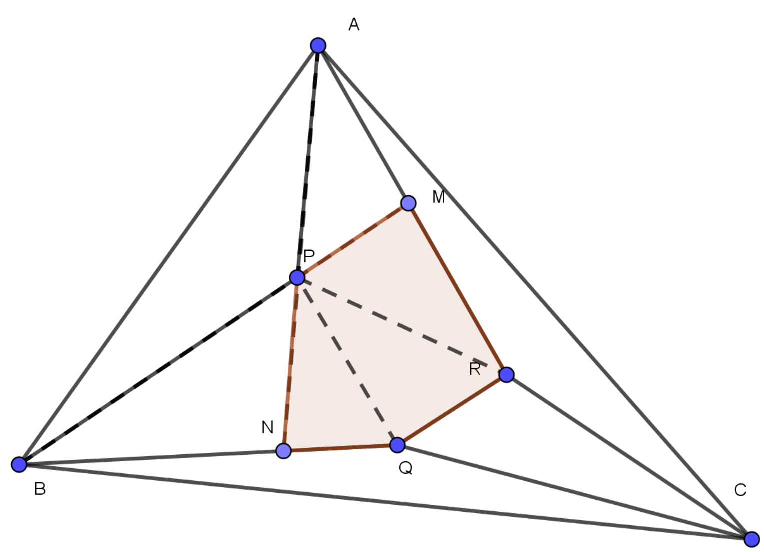


Note that X must have even degree as it is an interior point in a three colourable triangulation. The only possible degrees are thus 2,, 4 and 6. However 6 is impossible as then A,B,C,X will induce K_4 as a subgraph. Similarly we can easily verify that we cannot get a triangulation with degree of X as only 2. Hence degree must be exactly 4.

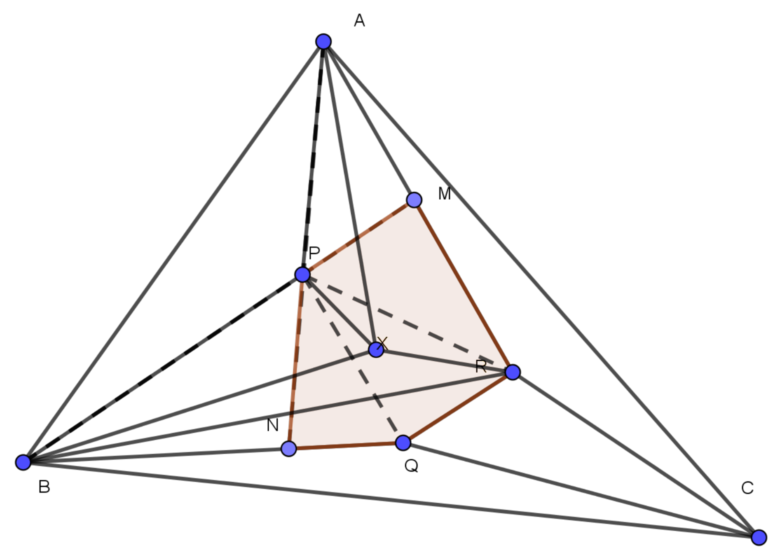
This implies that out of PQ, QR and RP exactly 1 or 2 must be missing. Because say all are missing then degree of X=4 implies that out of AQ, BR and CP exactly 2 should be present. However this is clearly impossible. If none are missing then degree of X is ≤ 3. Now consider that one are missing, say QR. Clearly X must lie within PQCR otherwise it will be a single vertex within a triangle which would lead to a K_4. However this means that to make the degree of X=4, the edges PX, QX, CX and RX would be forced and this triangulation(shown below) would result. However here degree of P would be 5 which is odd and hence this triangulation is not three colourable.



Finally we have to consider the case where two of PQ,QR and RP are missing, Let us say that only QR is present. Then we see that X must be in the region shaded below (PMRQN) because if not we cannot complete the triangulation without including at least one of PQ and PR.



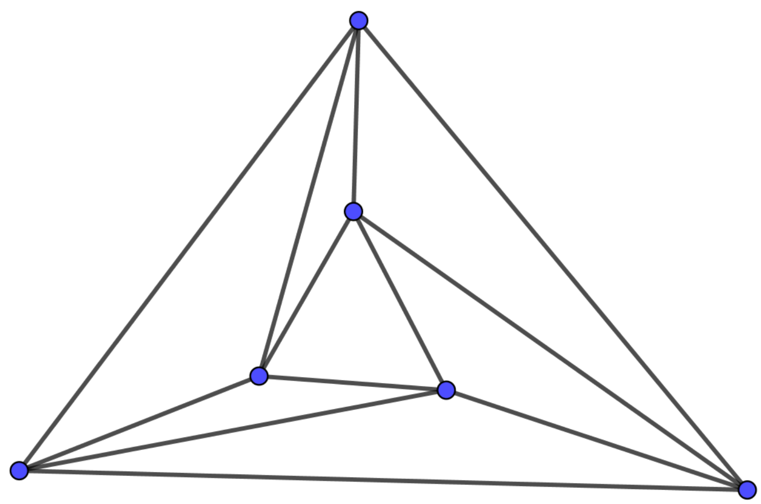
So we finally are limited to the case of only QR being in the triangulation, with PQ and PR absent. We can also assume X to lie in PMRQN. But as PR and PQ are absent this forces AX and BX to have a triangulation. Then we can see that XP is also forced. But degree of X is 4. So only one of XQ and XR can be chosen. However choosing either forces one ofPR and Q to have degree 5 as one of RB and QA get forced to complete the triangulation, Contradiction.



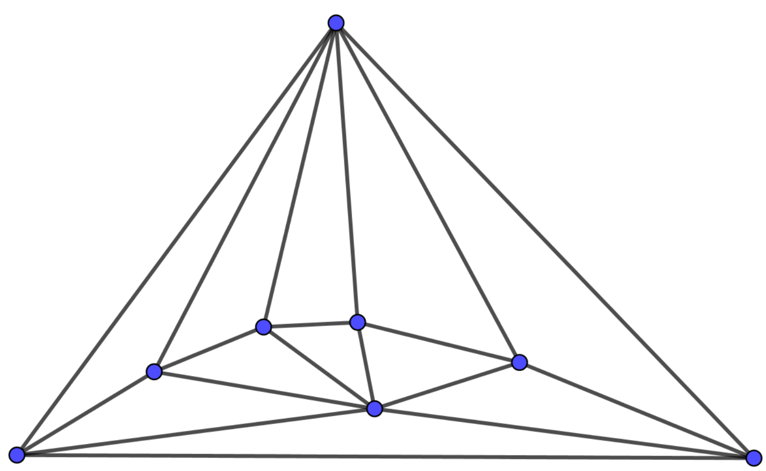
□

Theorem 5. *For $n \geq 8$ there is always some P with cardinality n with 3 points on the convex hull with at least 1 three colourable triangulation.*

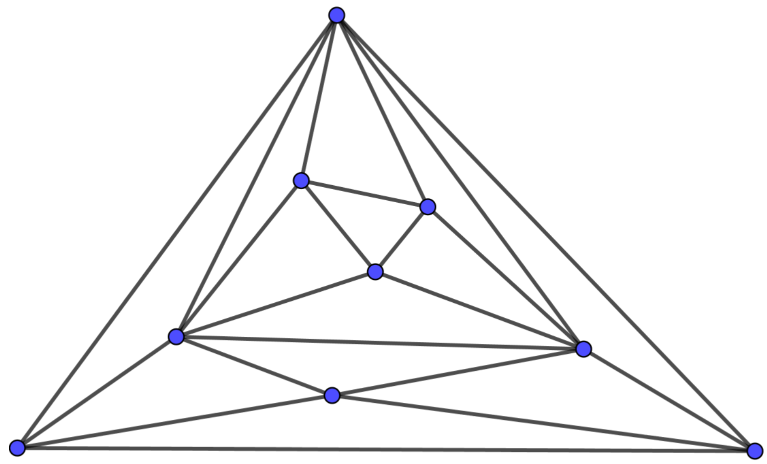
Proof. Note that we can reduce the problem of coming up with a point set with n points to one with n-3 points by the following construction. Create a triangle within the convex hull and triangulate as follows.



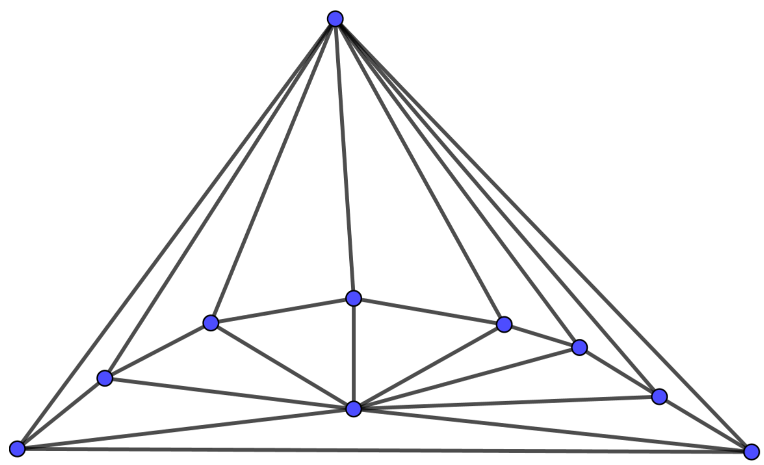
Then the problem clearly reduces to specifying the location of n-6 points with the 3 inner points as the triangular convex hull. Hence if we can show the existence of configurations with n= 8,9,10 points then we are done, by induction. We exhibit these below: For n=8:



For n=9:



For n=10:



□

We conjecture a much stronger result than Theorem 5.

Conjecture 1. *For all P with cardinality $n \geq 8$ and 3 points on the convex hull there is always at least one 3 colourable triangulation except if P is in anomalous configuration.*

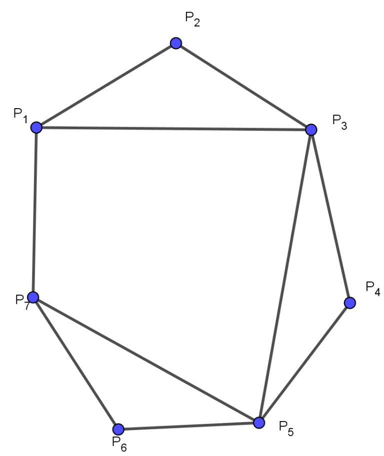
Proving the above completely solves our problem for point sets with triangular convex hulls. Also it would probably greatly facilitate a divide and conquer approach to solving the problem for general point sets which currently looks difficult given only Theorem 5 which merely gives an existence result. An idea is to use two lines of attack. For n-gons with small number of interior points show that there is always a three colourable triangulation, like we did for two points inside an n-gon. If the n-gon has a large number of points inside then intuitively we should be able to cut up the n-gon into triangles with atleast 5 points in them and then we would be done. This suggests three directions for future work

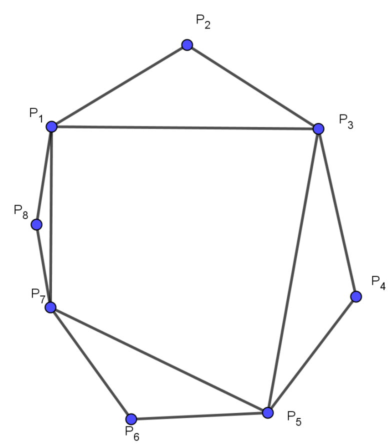
- show that an n -gon with enough interior points can be cut into triangles with atleast 5 points in each. In case this cannot be done for some point sets then that should intuitively require some special features that we should be able to expoit.
- show that an n gon with few points inside must always have a three colourable triangulation.
- most importantly and probably the most difficult, is to prove some form of Conjecture 1.

We now note that to solve the problem for the case of a convex hull with small number of interior points, we need only solve for the single case of $2k+1$ points on the boundary and k points on the interior. This follows from a simple reduction.
We first make some definitions:

Definition 2. *A **bordering** of a polygon $P_1P_2...P_n$ is the polygon with the lines P_1P_3, P_3P_5 and so on till either $P_{n-2}P_n$ or $P_{n-3}P_{n-1}$ depending upon the parity of n. Note that the bordering produces $\lceil \frac{n}{2} \rceil$ triangles which are disjoint in area and have the property that 'removing' any one triangle produces a $(n-1)$ gon.*

Here is a diagram illustrating the bordering for a 7 gon and 8 gon respectively.





Notation 1. *We use (r, k) to refer to a point set with r points on the convex hull and k interior points.*

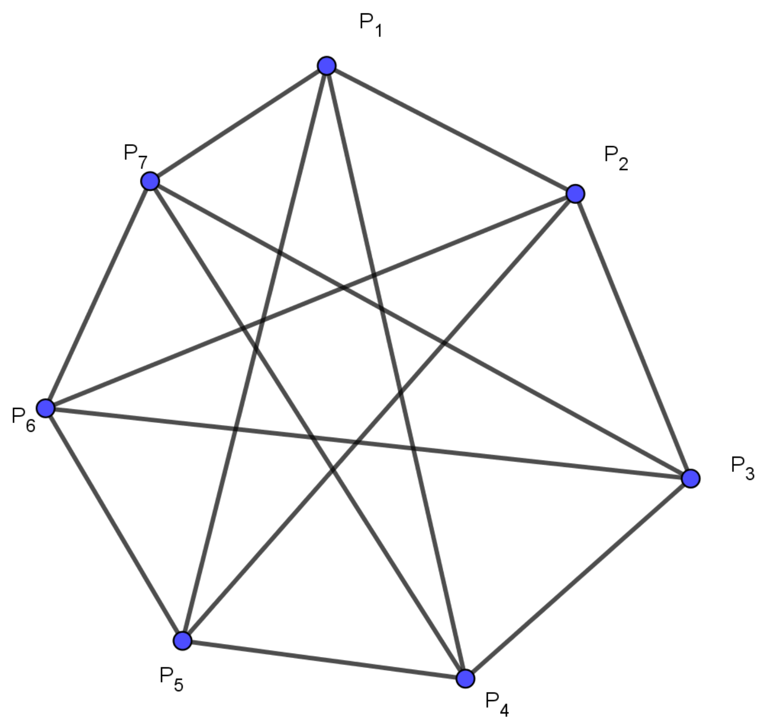
Theorem 6. *If any point set with ($2k+1$,k) has some three colourable triangulation then any point set with (n,k) and $n \geq 2k+1$ also has some three colourable triangulation.*

Proof. Consider the bordering of the n gon in the case of (n,k). This produces $\lceil \frac{n}{2} \rceil$ triangles which are disjoint in area and have the property that 'removing' any one triangle produces a $(n-1)$ gon. However if $n \geq 2k+2$ this means that more than k triangles are produced and so not all of them can contain an interior point. Hence we can find a triangle with no interior point and 'remove' it, reducing the problem to (n-1,k). This reduction can be performed until we hit $(2k + 1,k)$. Hence if we show the claim for $(2k + 1,k)$ we are done. □

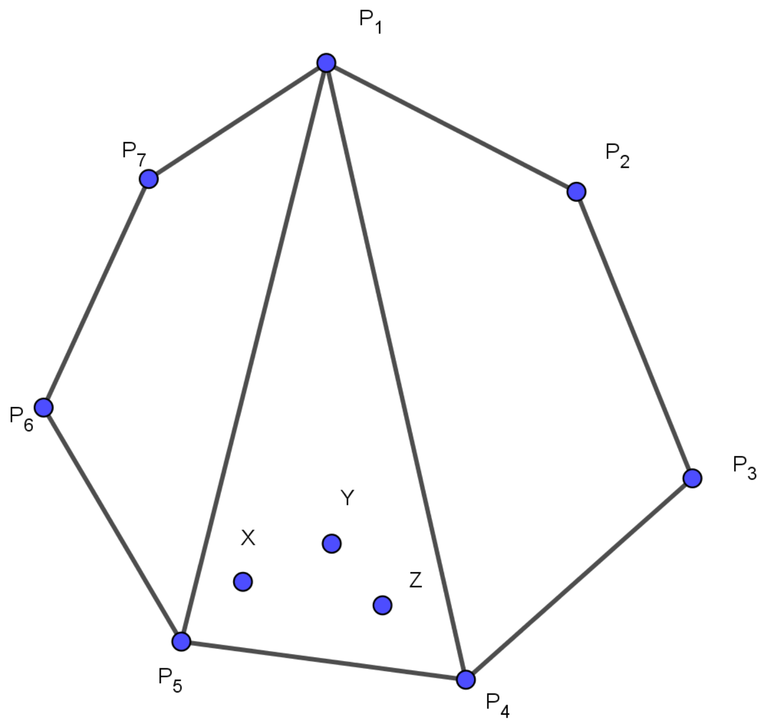
Using theorem 6 we will now show the claim for (n,3) for all $n \geq 7$.

Theorem 7. *Any point set with convex hull having 7 or more sides and 3 interior points has some three colourable triangulation.*

Proof. By theorem 6 we need only prove the case (7,3). Consider a *halving line* that splits the convex 7 gon into a 4 gon and a 5 gon. The case we need to consider is clearly only when all three interior points are on one side of the halving line, because other cases follow easily from our previous results. Now draw all possible halving lines as shown below. This splits the 7 gon into multiple regions. Note that all three of the interior points must lie in one region otherwise we have a halving line that separates them and we are done.



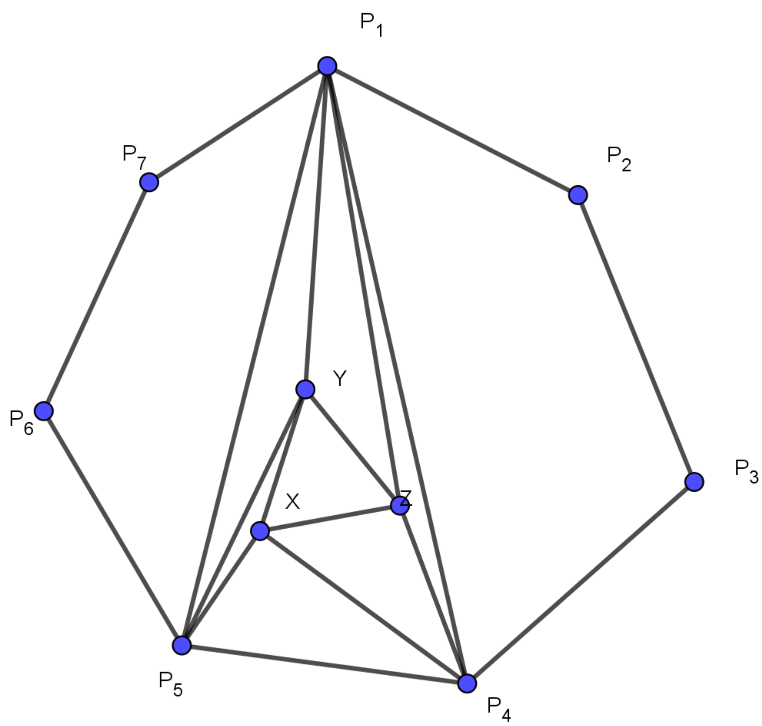
Now note that since the boundaries of these regions are all made up of halving lines, and all the regions are connected, infact each of these regions lies in some triangle which has one edge of the polygon as a side and has it's other two sides as halving lines which have a common end point. Thus we have that all three points lie in such a triangle as shown below.



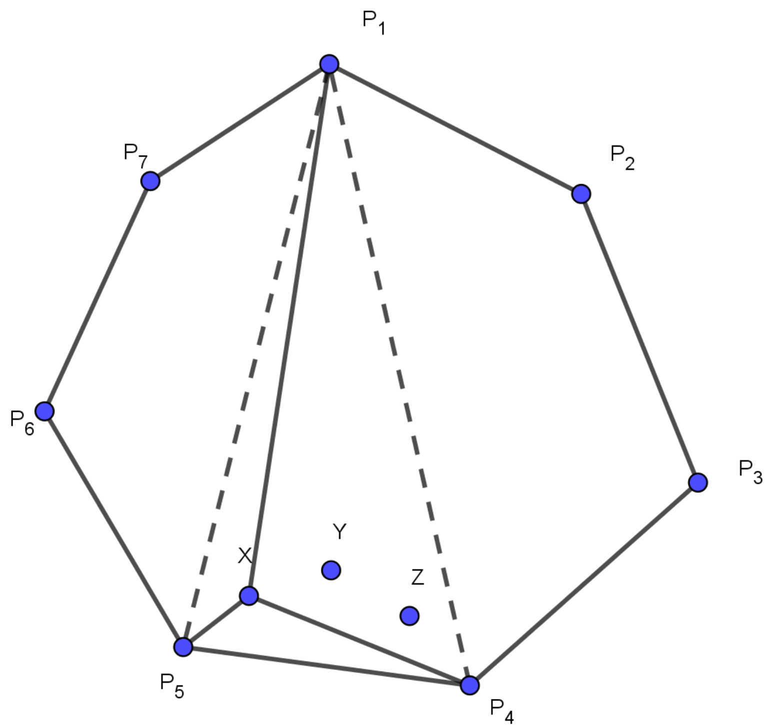
Now we consider three cases and in each one state how to construct a three colourable triangulation.

Case 1:

Suppose X,Y,Z in $P_1P_4P_5$ is not in the anomalous configuration, then there is exactly one way to triangulate three points in a triangle and we can use this. After this it is easy to see that we can complete the triangulation.

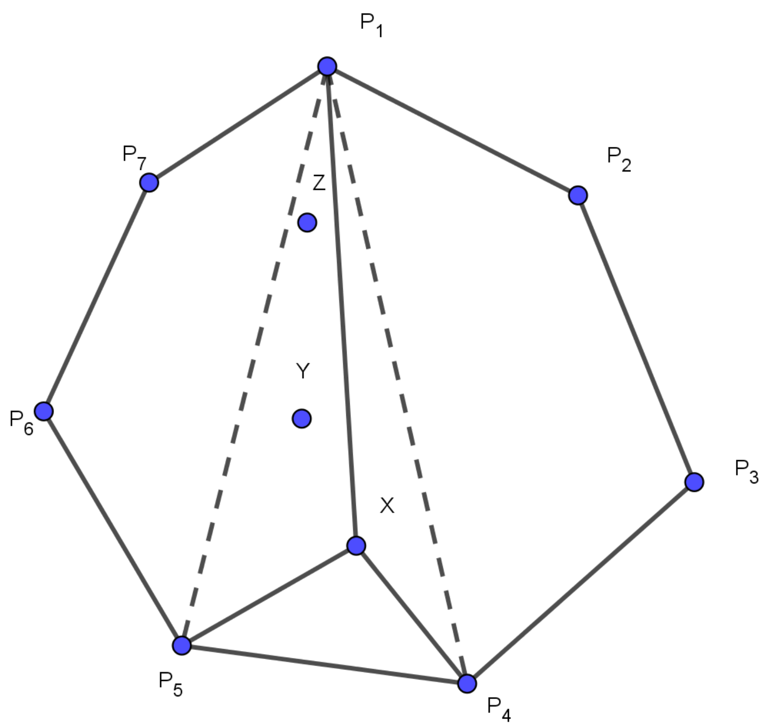


Case 2:
Suppose X,Y,Z are in anomalous configuration. Say they form a convex arc on P_4P_5. Then note that We can add the lines P_4X, XP_5 and P_1X. Now we have a pentagon with two points in it, which we can triangulate, however the point X may have even or odd degree. However note that there is an empty 5 gon one of whose vertices is X. But we can triangulate this as need be to add another odd or even number of edges to X, ensuring that all interior vertices have even degree.



Case 3:

Suppose X,Y,Z are in anomalous configuration. Say they form a convex arc on P_1P_5 (or P_1P_4 without loss of generality). Then note that we can do the same thing. We can add the lines P_4X, XP_5 and P_1X. Now we have a pentagon with two points in it, which we can triangulate, however the point X may have even or odd degree. However note that there is an empty 5 gon one of whose vertices is X. But we can triangulate this as need be to add another odd or even number of edges to X, ensuring that all interior vertices have even degree.



The fact that we can perform what we claim in cases 2 and 3 follows easily from the fact that the three points lie on a convex arc. □

The proof of theorem 7 is interesting because it suggests a way of generalizing the idea to show all (2k+1,k) which would prove the claim (using theorem 6) for all point sets (n,k) with $n \geq 2k + 1$. The idea is that a point in the interior, can be given any degree as long as we leave an n gon (with at least n=5) empty and having only one such point as a vertex, because this allows us control over the final degree of the point in the triangulation.

Recently, an interesting observation is that there may be some connection between the kinds of point sets where there is no three colourable triangulation and those with no triangulation that can be represented as a contact graph of rectangles. Note that there is clearly no direct exact correspondence between the classes of triangulations themselves, for example consider an odd wheel; it is not three colourable but can be represented as a contact graph. However there are two interesting questions:
Firstly is it true that any triangulation that is not three colourable, also not

representable as a contact graph of rectangles?
However more interesting given the nature of our original problem is the question of correspondence at the level of point sets and not triangulations; note for example the odd wheel is not three colourable but the underlying point set does have a contact graph representation except in the case of the K_4. However the underlying point set also has a three colourable triangulation as long as the odd wheel does not have only 4 points.
In fact consider the general anomalous configuration we described. It is trivial to see that this configuration is also not contact graph (of rectangles). Thus we have the interesting question regarding point sets: are those point sets which admit no three colourable triangulation also such that no triangulation admits a representation as a contact graph of rectangles. Clearly there are pointsets with three colourable triangulations that have no triangulation that can be represented as a contact graph of rectangles, for eg a triangle with 3 points in it (in non anamolus configuration). However it is still possible that we may have a necessary condition for a pointset to have some three colourable triangulation: that some triangulation of the pointset be representable as a contact graph of rectangles.
To address the above I plan to read the existing literature on the topic.

References

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