Summer Project Report: Godel's Incompleteness Theorems

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Chaitin's Incompleteness Theorem

We first define the notion of Kolmogorov complexity which was used by Chaitin in his proof. Kolmogorov complexity would formally be defined only with respect to some specified programming language or by considering a universal Turing Machine.

Definition 1 (Kolmogorov complexity)**.** *The Kolmogorov complexity $K(x)$ of an integer x is defined, with respect to some language P, to be the length (in bits) of the shortest computer program, in P, that outputs x (and then stops).*

For the discussion below we may consider P to be any fixed language.

Chaitin's incompleteness theorem states that for any consistent mathematical theory in which we can formulate certain statements about the complexity of strings (we shall henceforth call this condition as being *rich enough*), there exists a threshold complexity and we cannot prove strings to be more complex than this threshold within the theory. This threshold complexity is called Chaitin's constant.

Theorem 1 (Chaitin's Incompleteness Theorem)**.** *For any rich enough and consistent mathematical theory, there exists some (large enough) integer L (where L depends on the theory and on P) such that, for any integer x, the statement $K(x)\rangle L$ cannot be proved within the theory.*

Proof. We prove by contradiction. Let $\forall L$ there is some x such that there is a proof of the statement: $K(x)\rangle L$ within the theory and this theory be consistent. Let p be the first such proof (that proves $K(x)\rangle L$) where the set of all the proofs of this statement have been given some order (say lexicographic order). Let z be the integer x for which p proves $K(x)\rangle L$. We can easily have a program to output z, the program runs trough all possible proofs lexicographically and stops when it reaches the first proof that proves a statement of the form $K(x)\rangle L$. z is precisely this x in the statement that this first proof proves. But the length of the program we have specified is bounded by some costant plus $\log L$. Thus $K(z)\langle L$ for large enough L. But p proves $K(z)\rangle L$ and hence the theory is not consistent. □

Godel's First Incompleteness Theorem

Theorem 2 (Godel's First Incompleteness Theorem)**.** *For any rich enough and consistent mathematical theory, there exists a statement that cannot be proved or disproved within the theory.*

Proof. We give the proof for the case where we consider the Natural numbers and the usual associated mathematical theory. As the total number of L bit long computer programs is at maximum 2^{L+1}, hence for any L there is always some number x between 0 and 2^{L+1} such that $K(x)\rangle L$. Thus, for this x, the statement $K(x)\rangle x$ is a true statement (over N) that, by Chaitin's Incompleteness Theorem, has no proof. As the theory is consistent the negation of this statement has no proof. Thus this is an undecidable statement in the theory. □

References

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- The Handbook of Mathematical Logic. John Barwise.
- The Surprise Examination Paradox and the Second Incompleteness Theorem. Shira Kritchman and Ran Raz. Notices of the AMS. December 2010.