MATH-1187 - CALCULUS AND MATHEMATICAL ANALYSIS

COURSEWORK - B

NAME: ISHAN ARUN
STUDENT ID: 001423670

Q2)

(a) To show $T_5^{(\sin 2x)} = 2T_5^{\sin(x)\cos(x)}$.

- For $\sin 2x$;

$f(0) = \sin 0 = 0$

$f'(0) = 2\cos 2x = 2$ [k = 1]

$f''(0) = -4\sin 2x = 0$

$f'''(0) = -4 \cdot 2\cos 2x = -8\cos 2x = -8$ [k = 3]

$f''''(0) = +16\sin 2x = 0$

$f'''''(0) = 32\cos 2x = 32.$ [k = 5]

Formula for taylor series at $x = 0$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}a(0)}{k!} \cdot x^k$$

$$T_5^{(\sin 2x)} = \frac{2 \cdot x}{1!} - \frac{8 \cdot x^3}{3!} + \frac{32 \cdot x^5}{5!}$$

$$T_5^{(\sin 2x)} = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} \quad \text{— (i)}$$

• For taylor series for $f(x) = \sin x \cos x$, we can multiply the 2 taylor series of $\sin x$ & $\cos x$.

$$T_{(5)} \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad — \text{ (a)}$$

$$T_{(5)} \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad — \text{ (b)}$$

Multiplying (a) & (b) and ignoring terms above x^5 (5th degree)

$$T_{(5)}^{\sin x \cos x} = T_{(5)}^{\sin x} \times T_{(5)}^{\cos x}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$$

$$T_{(5)}^{\sin x \cos x} = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^3}{3!} + \frac{x^5}{3!\,2!} + \frac{x^5}{5!}$$ {ignoring terms where power/degree of $x > 5$}

$$T_{(5)}^{\sin x \cos x} = x - \frac{x^3}{2!} - \frac{x^3}{3!} + \frac{x^5}{4!} + \frac{x^5}{3!\,2!} + \frac{x^5}{5!}.$$

$$T_{(5)}^{\sin x \cos x} = x - \frac{4x^3}{6} + \frac{2}{15}x^{5}$$ (multiplying both sides by 2)

$$2 \cdot T_{(5)}^{\sin x \cos x} = 2x - \frac{4x^3}{3} + \frac{4}{15}x^5 = T_5^{(\sin 2x)}$$ {from (i)}

$$\therefore T_5^{(\sin 2x)} = 2 \times T_5^{\sin x \cos x}$$ ∴ Hence Proved

(b) $\lim\limits_{x \to 0} \dfrac{\tan^{-1} 3x - \sin 2x - x}{2x^3}$

Taylor series $(\tan^{-1} 3x)$
— (using T.S of $\tan^{-1} x$ & sub. $3x = x$) $= 3x - \dfrac{3^3 \cdot x^3}{3} + \dfrac{3^5 x^5}{5} - \dfrac{3^7 \cdot x^7}{7} + \dots$

Taylor series $(\sin 2x) = 2x - \dfrac{2^3 \cdot x^3}{3!} + \dfrac{2^5 \cdot x^5}{5!} - \dfrac{2^7 \cdot x^7}{7!} + \dots$

$$\therefore \lim_{x \to 0} \frac{\tan^{-1} 3x - \sin 2x - x}{2x^3}$$

$$= \lim_{x \to 0} \frac{\left(3x - \frac{27x^3}{3} + \frac{3^5 \cdot x^5}{5} - \frac{3^7 \cdot x^7}{7} + \dots\right) - \left(2x - \frac{8x^3}{3!} + \frac{2^5 \cdot x^5}{5!} \dots\right) - x}{2x^3}$$

$$\lim_{x \to 0} \frac{(3x - 2x - x) + \left(-\frac{27x^3}{3} + \frac{8x^3}{3!}\right) + \left(\frac{3^5 x^5}{5} - \frac{3^7}{7} \cdot x^7 + \dots\right) - \left(\frac{2^5 x^5}{5!} + \dots\right)}{2x^3}$$

$$\lim_{x \to 0} \frac{\left(-\frac{27}{3} x^3 + \frac{4}{3} x^3\right) + \left(\frac{3^5 \cdot x^5}{5} - \frac{3^7 x^7}{7} + \dots\right) - \left(\frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \dots\right)}{2x^3}$$

$$\lim_{x \to 0} -\frac{23}{6} + \underbrace{\left(\frac{3^5 x^2}{10} - \frac{3^7 x^4}{14} + \dots\right) - \left(\frac{2^4 \cdot x^2}{5!} - \frac{2^6 \cdot x^4}{7!} + \dots\right)}_{\text{As } x \to 0 \text{, both these terms equal to } 0.}$$

$$\therefore \lim_{x \to 0} -\frac{23}{6} = -\frac{23}{6}$$

$$\therefore \lim_{x \to 0} \frac{\tan^{-1} 3x - \sin 2x - x}{2x^3} = -\frac{23}{6}$$

(c)(i) $T_{(3)} e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$$T_{(3)} e^{(-1/3)} = 1 - \frac{1}{3} + \frac{(-1/3)^2}{2} + \frac{(-1/3)^3}{6}$$

$$T_{(3)} e^{(-1/3)} = 1 - \frac{1}{3} + \frac{1}{18} - \frac{1}{162}$$

$$T_{(3)} e^{(-1/3)} = 1 - \frac{46}{162} = 1 - 0.28395$$

$$T_{(3)} e^{(-1/3)} = 0.71605$$

$\therefore e^{(-1/3)} \approx 0.71605$, is the approximate value of $e^{-1/3}$ using a third order polynomial.

(ii) The approximate value of $e^{-1/3}$ using the 3^{rd} order polynomial;

$$e^{-1/3} \approx 0.71605$$

The remainder term $R_3(x) = e^{4}(c) \frac{(x-a)^4}{(4)!}$. in b/w x & 0.

$$R_3(x) = \frac{e^{c} x^4}{4!} \quad \text{where } c$$

Substituting $x = -\frac{1}{3}$

$$R_3 = \frac{e^c}{24}\left(-\frac{1}{3}\right)^4 = \frac{e^c}{24} \cdot \frac{1}{81}$$

$R_3 = \frac{e^c}{1944}$ where c is a value b/w $-\frac{1}{3}$ & 0.

Since e^x is decreasing function for negative x, the maximum value of e^x in $(-\frac{1}{3}, 0)$ is at $c = 0$.

$$\therefore e^c \leq e^0 = 1$$

$$\therefore R_3 = \frac{e^c}{1944} \leq \frac{1}{1944} \approx 0.0005144$$

$\therefore$ The value of $e^{-1/3}$ lies $(\pm R_3)$ of our initial approximation (i.e $e^{-1/3} \approx 0.71605$)

$$\therefore e^{-1/3} \in (0.71605 - 0.0005144,\ 0.71605 + 0.0005144)$$

$$e^{-1/3} \in (0.7155356,\ 0.7165644)$$

$\therefore$ This is the narrowest interval we can define for which the exact value of $(e^{-1/3})$ is contained in, using a 3rd order taylor polynomial.

- Difficulties associated to finding the limits of this interval:-

1. The remainder depends on e^c, where c is ~~any~~ a

unknown value $\in (-\frac{1}{3}, 0)$, thus we can estimate only the maximum possible error, that is by taking e^0 instead of e^c, we assume the maximum error, while in reality the error could be smaller, thus meaning our calculated interval is wider than necessary.

2. Also, the more the no. of terms in the taylor polynomial, the more accurate is the result, smaller remainder which would decrease the width of the interval. But we were only allowed to use up to a 3rd order taylor polynomial

Q3)

(a)

x	$g(x)$
0	3
1	5
2	7
3	5
4	9
5	15
6	12

L_6

U_6

Interval [0,6] with $\Delta x = 1$ gives us 6 intervals i.e [0,1][1,2][2,3][3,4][4,5][5,6].

Thus to calculate both the Upper and lower reimann sums, we need the values of the function $g(x)$ at points 0, 1, 2, 3, 4, 5 and 6.

The values of $g(x)$ at these discrete values of x [0 to 6] were given in the table. Hence since we have the function values at the necessary points, we can calculate both the reimann sums

Thus in conclusion, the information given is sufficient for us to correctly calculate the upper and lower Reimann Sums b/w 0 and 6 with 6 intervals for the function.

(b) STUDENT ID: 001423670 $\boxed{a=7}$

(i) $f(x) = x^{a+2} + 2a$

$\therefore \quad f(x) = x^9 + 14$ [continous & increasing]

Interval $= [0, 10-a] = [0, 3]$

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

- Here the right-end points (x_i) used for the upper reimann sum is equal to

$$x_i = \Delta x \times i = \frac{3i}{n} \quad \text{for } i = 1, 2, 3, 4, \ldots, n$$

- The Upper Reimann Sum (U_n):-

$$U_n = \sum_{i=1}^{n} M_i \Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$U_n = \sum_{i=1}^{n} \left((x_i)^9 + 14\right)\Delta x \qquad \left[x_i = \frac{3i}{n}\right]$$

$$U_n = \sum_{i=1}^{n} \left(\left(\frac{3i}{n}\right)^9 + 14\right)\frac{3}{n}$$

$$U_n = \sum_{i=1}^{n} \frac{3}{n}\cdot\left(\frac{3i}{n}\right)^9 + \sum_{i=1}^{n} \frac{3}{n} \times 14$$

$$U_n = \frac{3\cdot 3^9}{n\cdot n^9}\sum_{i=1}^{n} i^9 + \frac{3\times 14}{n}\sum_{i=1}^{n} 1$$

$$U_n = \frac{3^{10}}{n^{10}}\sum_{i=1}^{n} i^9 + \frac{42}{\cancel{n}}\times \cancel{n} \qquad \left[\sum_{i=1}^{n} 1 = n\right]$$

$$U_n = \frac{3^{10}}{n^{10}}\sum_{i=1}^{n} i^9 + 42 \quad \text{(or)} \quad \frac{59049}{n^{10}}\sum_{i=1}^{n} i^9 + 42$$

(This is the formula equation for the Upper Reimann Sum for $f(x) = x^9 + 14$ over closed interval $[0,3]$ w/ n-sub-intervals)

(ii) $n = 2$ Interval $= [0,3]$

$$\Delta x = \frac{3}{2} = 1.5$$

$$\text{sub-intervals} = [\underset{m_1}{0}, \underset{M_1}{1.5}] \;\&\; [\underset{m_2}{1.5}, \underset{M_2}{3}]$$

$$f(0) = (0)^9 + 14 = 14 = m_1$$

$$f(1.5) = (1.5)^9 + 14 = 38.44 + 14 = 52.44 = M_1 \;\&\; m_2$$

$$f(3) = (3)^9 + 14 = 19683 + 14 = 19697 = M_2$$

$\therefore$ The Upper Reimann Sum :- $\sum_{i=1}^{n} M_i \Delta x$ $[n = 2]$

$$U_n = (M_1 + M_2)\Delta x$$

$$U_n = (52.44 + 19697)\,1.5 = (19749.44)\,1.5$$

$$U_n = 29624.16$$

(iii) $$U_n = \frac{3^{10}}{n^{10}} \sum_{i=1}^{n} i^9 + 42$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty}\left(\frac{3^{10}}{n^{10}} \sum_{i=1}^{n} i^9 + 42\right)$$

$$= \lim_{n\to\infty} \frac{3^{10}}{n^{10}} \sum_{i=1}^{n} i^9 + \lim_{n\to\infty} 42$$

(given to assume $\sum_{i=1}^{n} i^k \approx \frac{n^{k+1}}{k+1}$, i.e $\sum_{i=1}^{n} i^9 \approx \frac{n^{10}}{10}$)

$$\therefore = \lim_{n\to\infty} \frac{3^{10}}{\cancel{n^{10}}} \times \frac{\cancel{n^{10}}}{10} + 42$$

$$= \left(\lim_{n\to\infty} \frac{3^{10}}{10}\right) + 42 = \frac{3^{10}}{10} + 42$$

$$= \frac{59049}{10} + 42$$

$$= 5904.9 + 42$$

$$\lim_{n\to\infty} U_n = 5946.9$$

(c) $a = 7$

$$\therefore \text{Reimann Sum} = \sum_{i=1}^{2000} \frac{1}{(3 + 0.001i)^7} (0.001)$$

of form $\sum_{i=1}^{n} f(x_i) \cdot \Delta x$

where $f(x_i) = \dfrac{1}{(x_i)^7}$ $\quad \Delta x = 0.001$

$x_i = 3 + 0.001i$ for i from 1 to 2000

$\therefore x_{min} = 3 + 0.001 \quad (i = 1)$
$x_{min} = 3.001 \rightarrow$ [LOWER BOUND]

$x_{max} = 3 + 0.001(2000) \quad (i = 2000)$
$x_{max} = 3 + 2$
$x_{max} = 5 \rightarrow$ [UPPER BOUND]

Here we can write the integral in form :-

$$\int_{x_{min}}^{x_{max}} f(x) \cdot dx$$

$$= \int_{3.001}^{5} \frac{1}{x^7} \cdot dx \qquad \left[f(x) = \frac{1}{x^7} \right]$$

This is the integral approximated by the given reimann sum.

Q1) ~~considering~~

(a) Considering a truth table for A, B, C.

$2^3 = 8$ possibilities

A	B	C	$A \to B$	$(A \to B) \wedge C$	$\sim((A \to B) \wedge C)$
T	T	T	T	T	F
T	T	F	T	F	T
T	F	T	F	F	T
F	T	T	T	T	F
F	F	T	T	T	F
F	T	F	T	F	T
T	F	F	F	F	T
F	F	F	T	F	T

Continued.

$A \to B$	$\sim(A \to B)$	$\sim B$	$(\sim(A \to B) \vee (\sim B))$
T	F	F	F
T	F	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	F	F	F
F	T	T	T
T	F	T	~~F~~ T

To ~~Bx~~ show if :-

$$\sim((A \to B) \wedge C) \equiv (\sim(A \to B) \vee (\sim B))$$

Since the values of column entries (truth values) of $\sim((A \to B) \wedge C)$ and $(\sim(A \to B) \vee (\sim B))$ are not identical. Hence we can conclude both of them are not logically equivalent.

Therefore the statement $\sim((A \to B) \wedge C) \equiv (\sim(A \to B) \vee (\sim B))$ is false.

(b) considering 4 consecutive integers,

$i-1, i, i+1, i+2$ for any $i \in \mathbb{Z}$ (set of integers)

Sum of the 4 numbers
$$= (i-1) + i + (i+1) + (i+2)$$
$$= i - 1 + i + i + 1 + i + 2$$
$$= 4i + 2 = \underbrace{2(2i+1)}_{\text{of form } 2 \times m}$$

Since the sum of the 4 numbers can be written/represented as a multiple of 2 [$2 \times m$' form] or in other words, it shows the sum is divisible by 2, we can conclude and say the sum of 4 consecutive ~~num~~ integers is always even.

Hence Proved.

(c) To ~~prove~~ prove ~~26a~~ $26a - 18b = 5$ is not possible for any integers a & b By contradiction.

Thus, we assume $26a - 18b = 5$ is true for any integers $a, b \in \mathbb{Z}$.

$$26a - 18b = 5$$
$$\underbrace{2(13a) - 2(9b)}_{} = 5$$

of form '2m', thus showing it is both multiples of 2 i.e. it is even

$\therefore$ $26a$ is even for any integer 'a'
$18b$ is even for any integer 'b'

The difference of 2 ~~its~~ even numbers is also even [in this case $2(13a - 9b)$].

$\therefore$ $26a - 18b$ = even.
But ~~this~~, the R.H.S = 5, which is an odd number. ~~So~~ Hence, we have reached a contradiction where the R.H.S is odd & L.H.S always gives an even number.

Therefore, the statement is false. In ~~conclusio~~ conclusion, we can say there does not exist integers a & b for which $26a - 18b = 5$

(HENCE PROVED)

(d) To prove if mn is even, m is even or n is even

$$A \Rightarrow B$$

$$A \Rightarrow B = (\sim B) \Rightarrow (\sim A)$$

Therefore if mn is even, m is even or n is even is same as saying if m and n is odd, mn will also ~~w~~ be odd.

For two integers $m, n \in \mathbb{Z}$, assume they're both odd.
Thus we can write them in form,

$m = 2k+1 \quad n = 2j+1$, where $k, j \in \mathbb{Z}$

$m \times n = (2k+1)(2j+1)$

$mn = 4kj + 2k + 2j + 1$

$mn = 2(2kj + k + j) + 1$

Since $2kj + k + j$ is an integer, mn is of form $2z+1$, $(z = 2kj+k+j)$, thus showing mn is odd

∴ Hence Proved.

In conclusion, since we have proved the contrapositive ie $(\sim B) \Rightarrow (\sim A)$ and since that is logically equivalent to $A \Rightarrow B$, we can conclude that the original statement is true.

Thus if mn is even, m or n is even for 2 integers m & n.

∴ Hence Proved by reconfirmation

e) Evaluating the proof provided by the LLM:-

To prove by mathematical induction that for all integers $n \geq 1$

$$\sum_{r=1}^{n} \left(5^{2(r-1)} + 13^{r-1}\right) = \frac{5(5^{2n}) + 12(11^{n}) - 17}{120}$$

(let this be the predicate $P(n)$, where $n \in \mathbb{N}$)

1. BASE STEP :- (To show $P(1)$ is True)

For $(n=1)$

$$L.H.S :- \sum_{r=1}^{n} \left(5^{2(r-1)} + 13^{r-1}\right)$$

$$= \sum_{r=1}^{1} 5^{2(r-1)} + 13^{r-1} = 5^{2(1-1)} + 13^{(1-1)}$$

$$= 5^{0} + 13^{0} = 1+1 = 2$$

$$R.H.S :- \frac{5(5^{2n}) + 12(11^{n}) - 17}{120} \quad (n=1)$$

$$\Rightarrow \frac{5(5^{2}) + 12(11) - 17}{120} = \frac{125 + 132 - 17}{120}$$

$$\Rightarrow \frac{257-17}{120} = \frac{240}{120} = 2$$

$\therefore$ L.H.S = R.H.S

Therefore the Base Step holds true, which matches with what was ~~p~~ shown by the LLM.

2. INDUCTIVE STEP :-

We assume that for some integer $n \geq 1$, $P(k)$ is true

$$\therefore \sum_{r=1}^{k} \left(5^{2(r-1)} + 13^{r-1}\right) = \frac{5(5^{2k}) + 12(11^{k}) - 17}{120}$$

To show the statement holds true for $(k+1)$

$P(k+1) =$

$$\sum_{r=1}^{k+1}\left(5^{2(r-1)}+13^{r-1}\right)=\frac{5\left(5^{2(k+1)}\right)+12\left(11^{k+1}\right)-17}{120}$$

L.H.S of this equation :-

$$\sum_{r=1}^{k+1} 5^{2(r-1)}+13^{r-1}=\sum_{r=1}^{k} 5^{2(r-1)}+13^{r-1}+\left(5^{2(k+1-1)}+13^{(k+1-1)}\right)$$

$$\Rightarrow \underbrace{\sum_{r=1}^{k} 5^{2(r-1)}+13^{r-1}}_{P(k)}+\left(5^{2k}+13^{k}\right)$$

$$\therefore \sum_{r=1}^{k+1} 5^{2(r-1)}+13^{r-1}=\frac{5(5^{2k})+12(11^{k})-17}{120}+5^{2k}+13^{k}$$

$$=\frac{5(5^{2k})+12(11^{k})-17+120\cdot 5^{2k}+120\cdot 13^{k}}{120}$$

$$\Rightarrow \frac{5(5^{2k})+120(5^{2k})+12(11^{k})+120(13^{k})-17}{120}$$

$$\Rightarrow \frac{5(5^{2k}+24\cdot 5^{2k})+12(11^{k}+10(13^{k}))-17}{120}$$

$$\Rightarrow \frac{5\cdot 5^{2}\cdot 5^{2k}+12(11^{k}+10(13^{k}))-17}{120}$$

$$\Rightarrow \frac{5(5^{2k+2})+12(11^{k}+10(13^{k}))-17}{120}$$

(Final simplification)

Here in this step, the next simplification done by the LLM during the 'simplify the numerator step' is :-

$5(5^{2k} + 24(5^{2k})) + \underline{12(11^k + 10(13^k))} - 17 =$

$\Rightarrow 5(5^{2k+2}) + \underline{12(11^{k+1})} - 17$

THIS STEP OF THE SIMPLIFICATION IS WRONG.

Here the term $12(11^k + 10(13^k))$ ~~becomes~~ becomes $12(11^{k+1})$ with no explained simplification. Basically the term $10(13^k)$ is simply removed by the LLM to further simplify this step

Therefore I disagree with the proof provided by the LLM as it ~~was unable to show~~ had incorrectly simplified the term $12(11^k + 10(13^k))$ into $12(11^{k+1})$. Thus it was unable to conclude the inductive step.

∴ The proof provided by the LMM is incorrect.