Assignment 1: Shape representation

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Question 1)

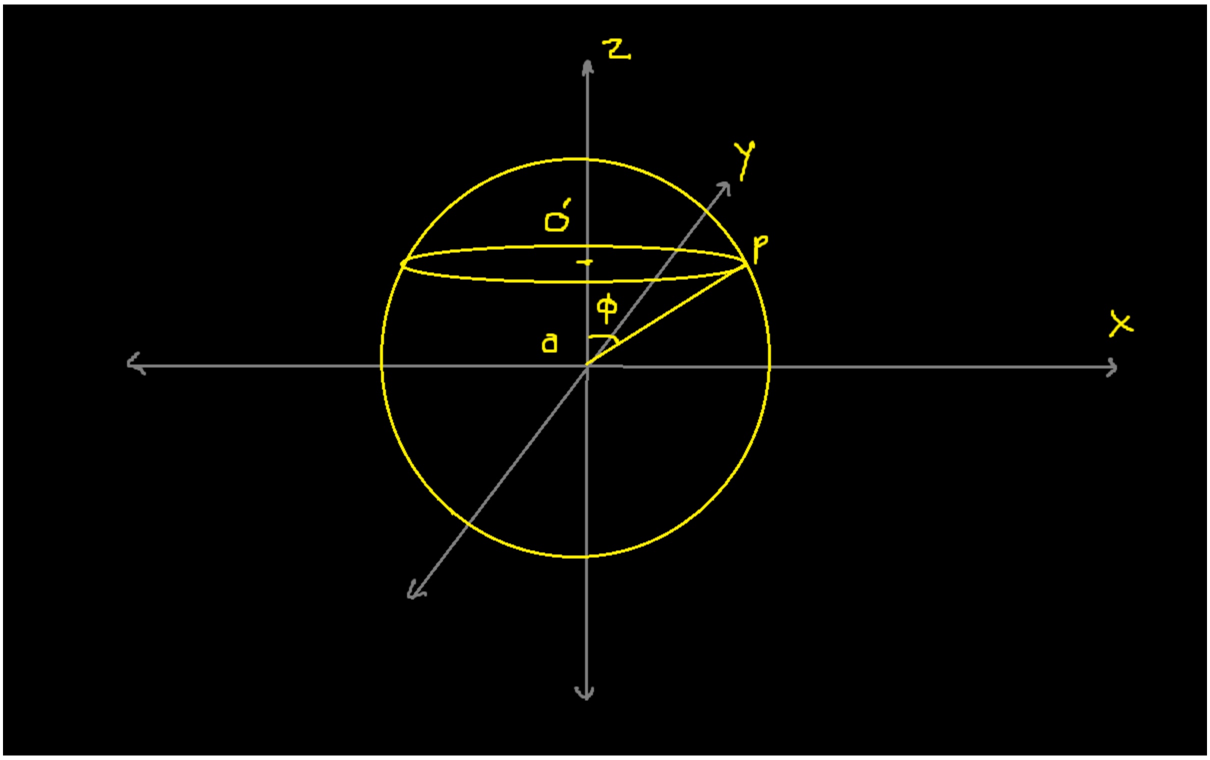
Parametric Equation of a sphere

Parametric Equation of a sphere of Radius R and cantered at (0,0,0) is –

$$C(\phi, \theta) = Rsin(\phi)\sin(\theta), Rsin(\phi)\cos(\theta), Rcos(\phi)$$

Where,

$\phi \in \{0, \pi\}$ and $\theta \in \{0{,}2\pi\}$



The parametric equation of a circle of unit radius having centre (0,0,0) is given by

$$C(\theta) = <\sin(\theta)\,,\cos(\theta)\,,0>$$

The surface of a sphere can be thought to be made up of many such circles of varying height and radii.

For Example,

Consider the circle centred at O′, having radius O′P, and lying the XY plane.

The radius of this circle $O'P = \mathrm{OP} * \ \sin(\phi)$. Let OP = R (i.e., Radius of the sphere) then,

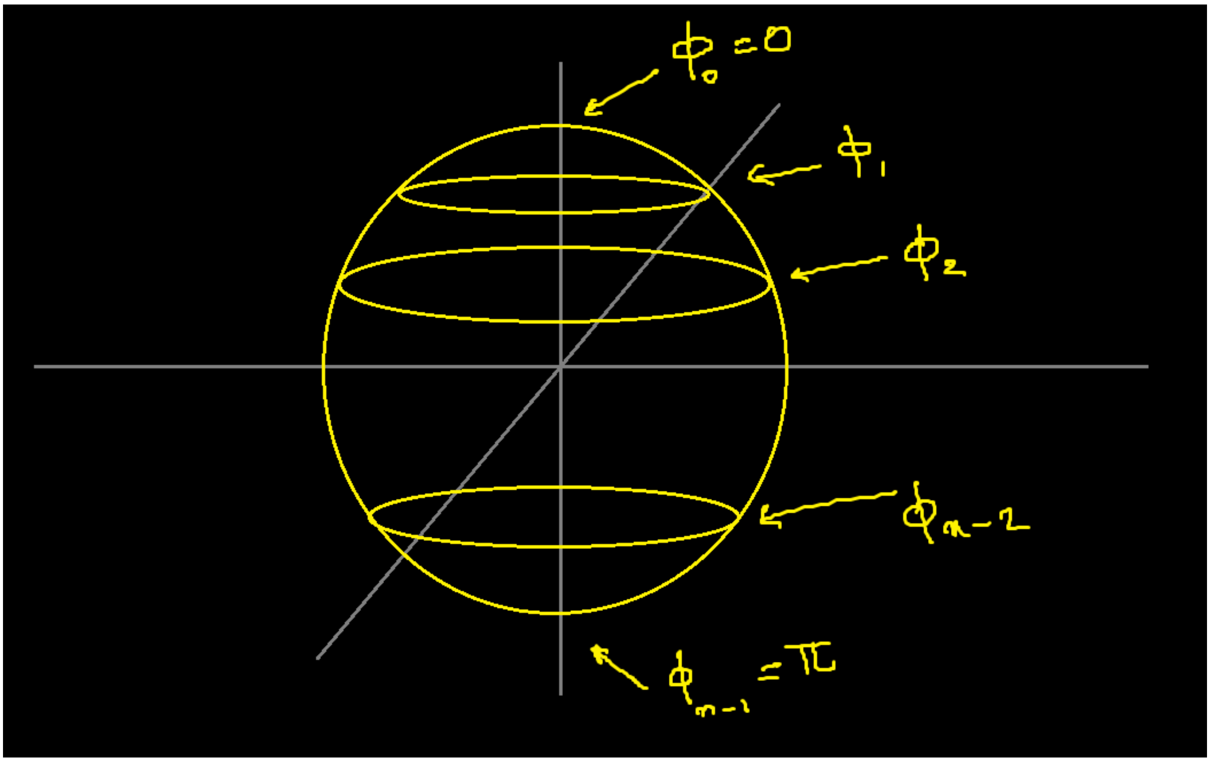
$$O'P = Rsin(\phi)$$

Now the above circle is also raised above XY plane by a value of $OO' = Rcos(\phi)$

If consider this shift in the Z coordinate of a circle. And, multiple the new radius, then we will get parametric equation of the entire sphere as,

$$C(\phi, \theta) = Rsin(\phi)\sin(\theta), Rsin(\phi)\cos(\theta), Rcos(\phi)$$

Covering the Sphere with triangles



Let's assume that ,

ϕ can take values from the set $\{ \phi_0, \phi_1, \dots \phi_{10}\}$

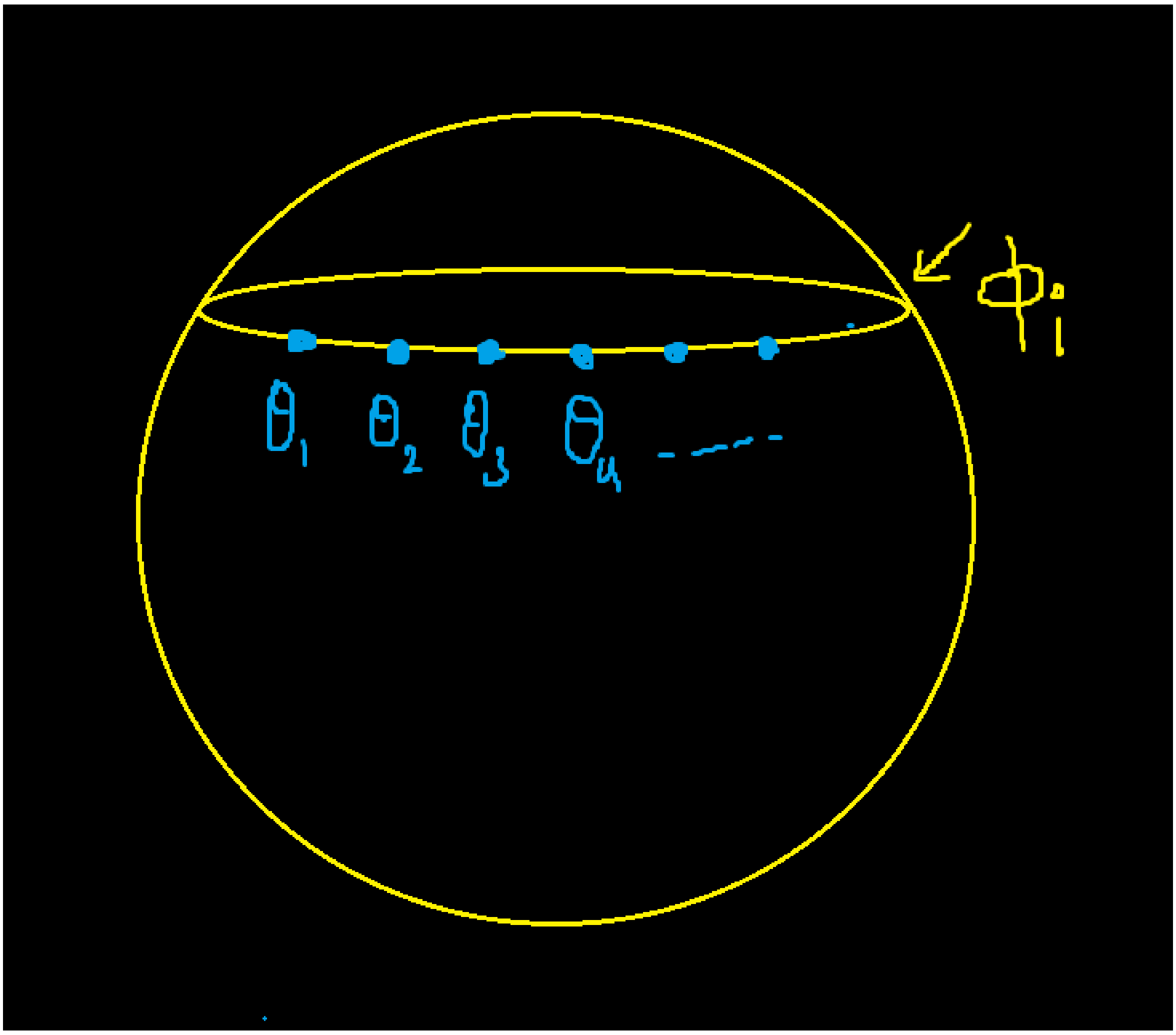
θ can take values from the set $\{ \theta_0, \theta_1, \dots \theta_{45}\}$

Where,

$\phi_i = (18 * i)^\circ \quad \forall\, i \in \{0{,}10\}$

$\theta_i = (8 * i)^\circ \quad \forall\, i \in \{0{,}45\}$

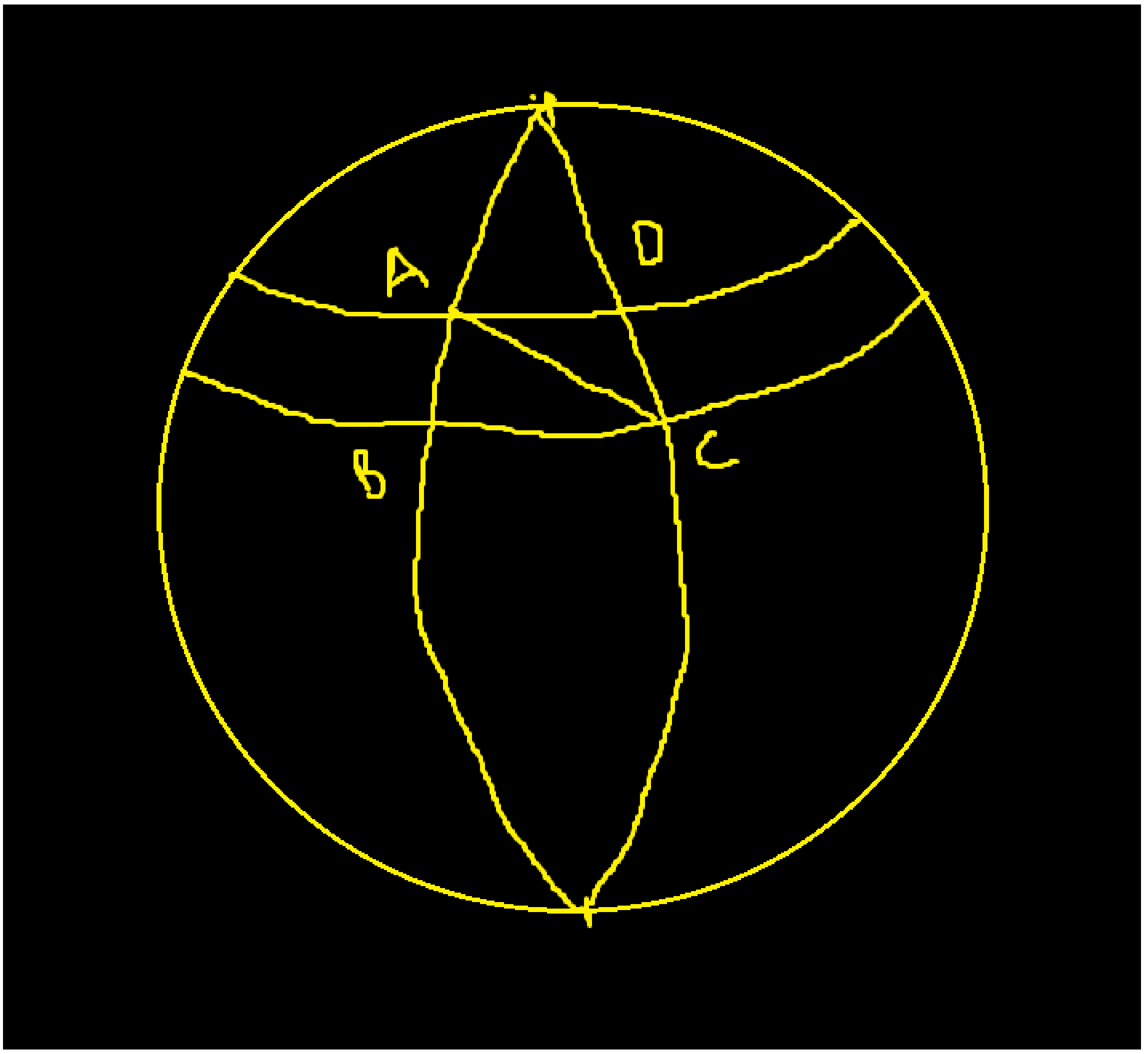
By varying the ϕ value in the parametric equation (example $\phi_0, \phi_1, \phi_2 \;\ldots\; \phi_{10}$) we can get multiple circles as shown above. These lines can be considered as latitudes.



If we keep $\phi = \phi_i$ fixed and vary the value of θ(example $\theta_0, \theta_1, \theta_2 \;\ldots$). Then we will get points on this same circle as shown above.

If we keep the value of θ fixed and vary, ϕ then we will get a longitude on the sphere.

If there are many latitudes and longitudes. Then the distance between each one of them will be small. And the area enclosed on the surface of the sphere between 2 adjacent latitudes and longitudes can be considered as a rectangle. This rectangle can be broken down into 2 triangles ACB and ADC, as shown below



Now let's say that,

Coordinates of A are given by $C(\phi_i, \theta_j)$.

Then we can say that,

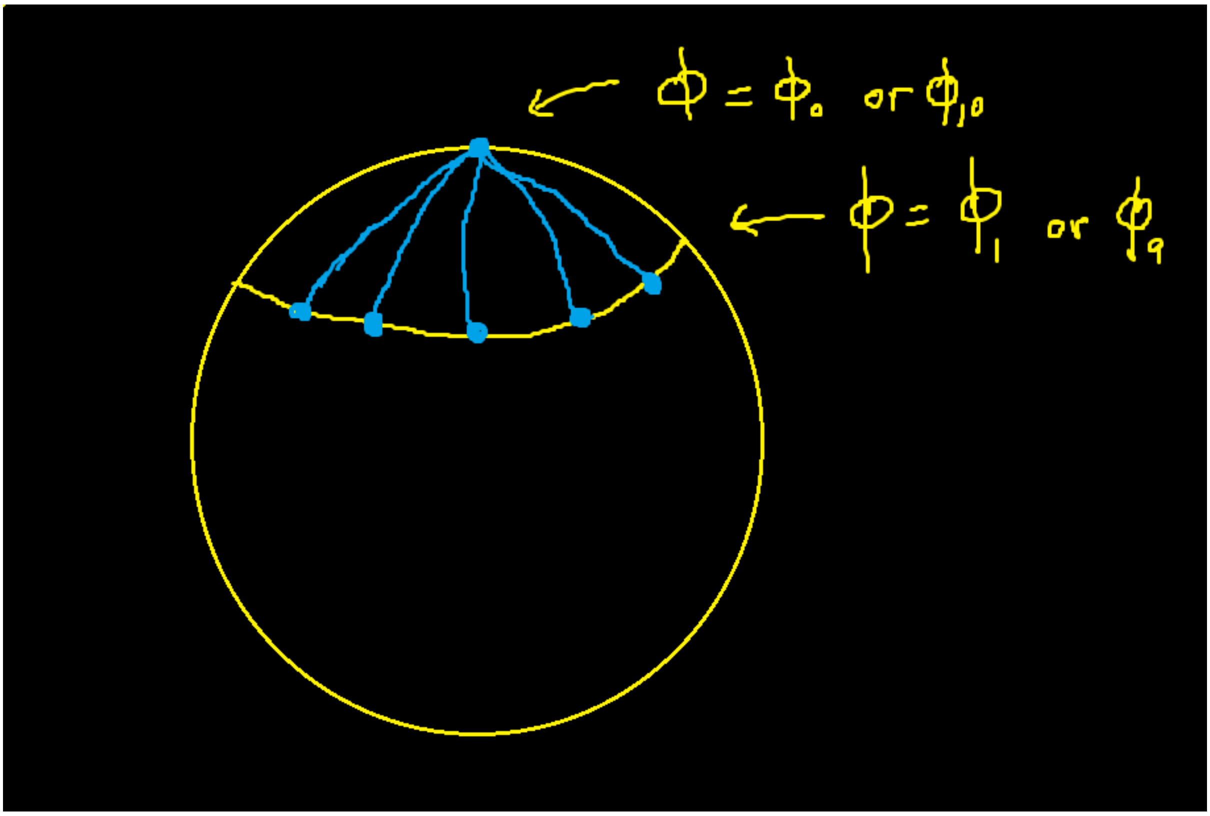
coordinates of B will be given by $C(\phi_{i+1}, \theta_j)$

coordinates of C will be given by $C(\phi_{i+1}, \theta_{j+1})$

coordinates of D will be given by $C(\phi_i, \theta_{j+1})$

Now we can easily find the coordinates of the triangles ACB and ADC.

Also, for the circles having $\phi = \phi_1 \ or \ \phi_9$, there are no circles above and below them respectively. The circle having $\phi = \phi_0 \ and \ \phi_{10}$ are just points. So, in this case we will not get a rectangle but a triangle only. As shown below.



Now we just to iterate over all $\phi\ and\ \theta$ and then we will get coordinates of vertices of all triangles. Then we will add these coordinates into the shape_vertices array and pass it to the rest of the program to plot these triangles in 3D.

Question – 2)

The parametric equation of Cubic Bezier Curve which goes through the points R_0, R_1, R_2, R is given by –

$$B(t) = (1-t)^3 R_0 + 3(1-t)^2 t R_1 + 3(1-t)t^2 R_2 + t^3 R_3$$

where $0 \leq t \leq 1$

C^0 continuity means that the two curves meet at one point.

C^1 continuity means C^0 continuity is satisfied, and the curves have the same first derivative at the point of intersection.

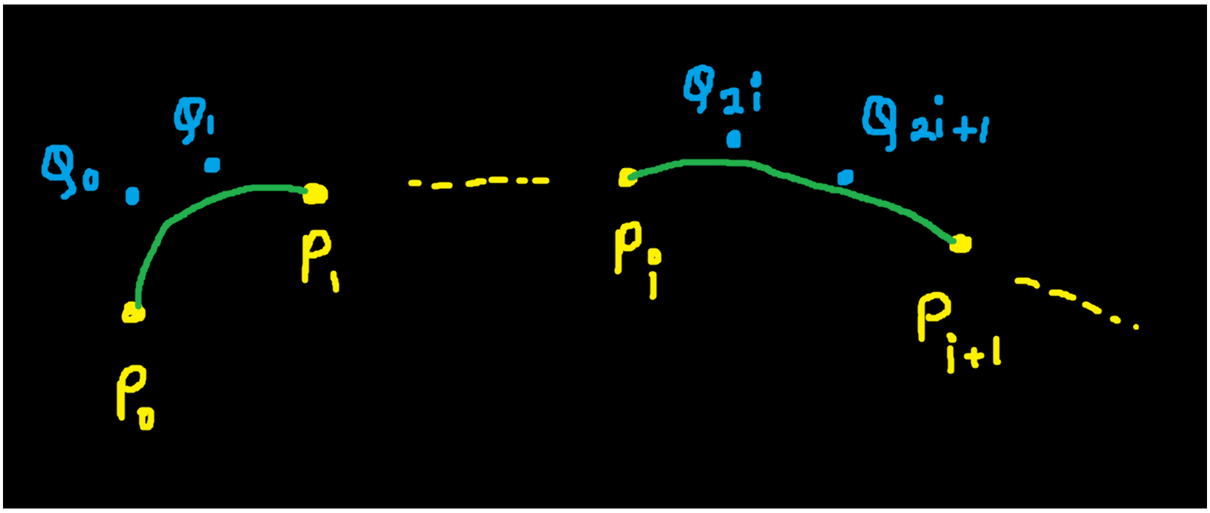
Let $P_0, P_1, P_2, \dots P_{n-1}$ be n points given to us.

Now we want to plot n – 1 cubic piecewise Bezier curves which start from P_i and end at p_{i+1}.

where $0 \leq i \leq n-2$.

A Bezier curve requires 4 points. So, this means we must add two more dummy points between the above-mentioned points

P_i and P_{i+1} (Yellow Coloured). Let these points be Q_{2i} and Q_{2i+1} (Blue Coloured). As shown below.



Checking for C^0 continuity –

There are n-1 cubic Bezier curves (lets number them from 0 to n-2). The i^{th} Bezier curve has the control points $P_i, Q_{2i}, Q_{2i+1}, P_{i+1}$.

Now for the entire interpolated curve to be C^0 continuous, Bezier curve for the control points $P_i, Q_{2i}, Q_{2i+1}, P_{i+1}$ and $P_{i+1}, Q_{2i+2}, Q_{2i+3}, P_{i+2}$ should intersect with each other. This should be true for all $0 \leq i \leq n-3$.

The equations of the two Bezier curves are –

$$B_1(t_1) = (1-t_1)^3 P_i + 3(1-t_1)^2 t_1 Q_{2i} + 3(1-t_1) {t_1}^2 Q_{2i+1} + t_1^3 P_{i+1}$$

And

$$B_2(t_2) = (1-t_2)^3 P_{i+1} + 3(1-t_2)^2 t_2 Q_{2i+2} + 3(1-t_2) t_2^2 Q_{2i+3} + t_2^3 P_{i+2}$$

Where $0 \leq t1 \leq 1 \ \ and \ \ 0 \leq t2 \leq 1$.

Now we want the ending point B_1 to be equal to the starting point of B_2.

Or, for C^0 we want $B_1(1) = B_2(0)$

Putting $t_1 = 1 \ and \ t_2 = 0$ in the above equations we get,

$$B_1(1) = (0)^3 P_i + 3(0)^2 Q_{2i} + 3(0) Q_{2i+1} + P_{i+1} = P_{i+1}$$

$$B_2(0) = P_{i+1} + 3(1)^2(0) Q_{2i+2} + 3(1)(0^2) Q_{2i+3} + (0) P_{i+2} = \ P_{i+1}$$

Now we can see that these two Bezier curves meet at the same point P_{i+1}. Now this is true for all the intersections as we have considered all cases where $0 \leq i \leq n-3$. This means the entire curve is C^0 continuous.

Checking for C^1 continuity –

We know from above that the curve is C^0.

The additional condition for C^1 is that the two curves meeting at every intersection should have the same first derivative at that intersection.

Just like above we can say that let the Bezier curves be numbered from 0 to n-2. Then for all $0 \leq i \leq n-3$, the first derivative of the i^{th} and the $(i+1)^{th}$ Beizer curve should be the same at the point of intersection (i.e., $t_1 = 1 \ and \ t_2 = 0$).

The equations of the first derivative of the two Bezier curves are –

$$B_1' \ (t1) =$$
$$3(1-t_1)^2 (Q_{2i} - P_i) + 6(1-t_1) t_1 (Q_{2i+1} - Q_{2i}) + 3t_1^2 (P_{i+1} \ - Q_{2i+1} \ )$$

And

$$B_2'(t2) =$$
$$3(1-t_2)^2(Q_{2i+2}-P_{i+1})+6(1-t_2)t_2(Q_{2i+3}-Q_{2i+2})+3t_2^2(P_{i+2}-Q_{2i+3})$$

Now for C^1 we want $B_1'(1) = B_2{}'(0)$. Putting $t_1 = 1\ and\ t_2 = 0$ in the above equations we get.

$$B_1'(1) =$$
$$3(0)^2(Q_{2i}-P_i)+6(0)(Q_{2i+1}-Q_{2i})+3(P_{i+1}-Q_{2i+1}) = 3(P_{i+1}-Q_{2i+1})$$

And

$$B_2'(0) =$$
$$3(Q_{2i+2}-P_{i+1})+6(0)(Q_{2i+3}-Q_{2i+2})+3(0)(P_{i+2}-Q_{2i+3}) = 3(Q_{2i+2}-P_{i+1})$$

Now we want

$(Q_{2i+2}-P_{i+1}) = (P_{i+1}-Q_{2i+1})$ or,

$p_{i+1} = (Q_{2i+1}+Q_{2i+2})/2$

This means that the points P_{i+1} should lie at the midpoint of the line joining $Q_{2i+1}\ and\ Q_{2i+2}$. This also means all these 3 points are collinear.

Pseudocode.

Given n points $P_0, P_1, \dots P_{n-1}$.

Let D be some distance which is smaller than the smallest distance among all $P_i\ and\ P_{i+1}$, $0 \le i \le n-2$.

Now we iterate from i=0 to n-3

All points p_{i+1} should be the midpoint of two dummy control points as proved above(blue coloured).

The distance between Q_{2i+2} and Q_{2i+1} $= 2d$. The direction of line connecting these two points can be anything.

After that 2 more dummy points are to be added (green coloured). One for the first Bezier curve and for the last one.

Let Q_0 be the midpoint of P_0 and Q_1

And Q_{2n-3} be the midpoint of Q_{2n-2} and P_{n-1}.

See figure Below.