Finance

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1. Abstract

The objective is to briefly introduce the basic concepts of basic probability theory, oriented to the application of quantitative analysis and qualitative analysis of financial processes. This includes various probaility distributions in finance, basic models of option pricing theory, portfolio optimization, credit risk, and financial dynamics such as stock market.

2. Introduction

Probability is one of the most basic concepts of mathematics. It plays a vital role in every field of human activity. These are quantitative tools widely used in the areas of finance. Knowledge of modern probability is essential for the development of finance theories and for the testing of their validity through robust analysis of real-world data. Our goal is to study the application of probability in the field of finance. There are many problems which can be generalized by using probability or extended by applying the methods of the theory.

Probability theory is useful in understanding and investigating the changes in prices, riskiness, and uncertainty about financial instruments. Probability theory is also useful to understand and investigate the changes in state variables, such as financial factors and macroeconomic fundamentals, that affect consumption and investment opportunities for investors. In the field of finance, the theory of stochastic orders is closely related to the expected utility theory which describes how choices under uncertainty are made.

3. Use of Various Probability Distributions in Finance

Variability is one of the most important concepts in finance. To model variability along with changes, we need the concept of a random variable. A random variable can denote :

- The return on an investment
- The price of an equity
- The sales volume of a store
- The turnover rate at your organization

The financial model can be divided into 2 categories :

- Discrete Time model
- Continuous Time model

3.1. Discrete Distributions

3.1.1 Bernoulli Distribution

It is one of the simplest distributions when it comes to Finance. A random variable X is said to follow a **Bernoulli distribution** with a parameter p if it takes values any two vales (say 1 and 0) with probability p and $1 - p$.

The use of this distribution in finance is pedogogical. In one period model it is convinient to modelize variations of logarithms of a stock price by Bernoulli distribution. If the day-0 price is S_0 then the day-1 price (S_1) can be uS_0 or dS_0.

$$ln(S_1) = ln(S_0) + X$$

where X is a Bernpoulli random variable taking values ln(u) and ln(d), u and d being up and down levels respectively.

3.1.2 Binomial Distribution

Stock returns in one period were represented by a Bernoulli distribution. Now considering a multi-period model and successive returns to be independent (Binomial) Bernoulli random variables. If we assume the parameters u and d to be constant over time (constant volitality) the log price variations are driven by binomial distributions. Binomial distribution was also the foundation of famous option valuaton model developed by Cox-Ross-Rubinstein (1979). The model states that :

$$S_{t+1} = S_t \times X_{t+1}$$

where S_t is day-t stock price s_{t+1} is the price of stock on $(t+1)^t h$ day and X_{t+1} is the up/down level of the stock (i.e. u or d) with probabilities p and 1-p. As said earlier that the random variables $X_1, X_2, ...$ are independent. From the above equation S_t can also be written as

$$S_t = S_0 \times \Pi_{s=1}^{t} X_t$$

this can be rewritten as

$$ln(\frac{S_t}{S_0}) = \sum_{s=1}^{t} X_t$$

The probabilty of stock being $k \cdot u$ price up after t days is :

$$ln((S_t) = ln(S_0) + k \cdot u) = \binom{n}{k} p^k (1-p)^{t-k}$$

The above models also gives us some moments of log returns :

$$E[ln(\frac{S_t}{S_0})] = t(pln(u) + (1-p)ln(d))$$

$$\sigma^2[ln(\frac{S_t}{S_0})] = tp(1-p)ln(\frac{u}{d})^2$$

3.2. Continuous Distribution

3.2.1 Guassian Distribution

It is one of the most common and most used probability distribution. It is the best distribution with respect to value of stocks. The pdf of a gaussian random variable with expectation μ and deviation σ is :

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2}$$

The curve of f_X of Gaussian Distribution is parallel to the line $x = \mu$. Many other mathematical and statistical processes have been derived using Gaussian distribution. The main motivation behind this distribution is the Central Limit Theorm (CLT) which states that for sufficiently large samples, the average calculated from independent, identically distributed random variables have approximately normal (gaussian) distributions, irrespective of the distribution from which the sampled variables were taken.

The assumption of a normal (gaussian) distribution is applied to asset prices as well as price action. Traders plot price points over time to fit recent price action into a normal distribution. The further price action moves from the mean, the more it is being over or undervalued. Traders can use the standard deviations to suggest potential trades. This type of trading is generally done on very short time frames as larger timescales make it much harder to pick entry and exit points.

3.2.2 Log-Normal Distribution

It is a derived distribution from Gaussian. If the day-0 price is S_0 and day t price is S_t then

$$r = ln(\frac{S_t}{S_0})$$

It can be rewritten as $S_t = S_0 \cdot e^r$. The Log-Normal Distribution for parameters (μ and σ) [1] is

[1] Here μ and σ are the mean and deviation of Gaussian random variable respectively

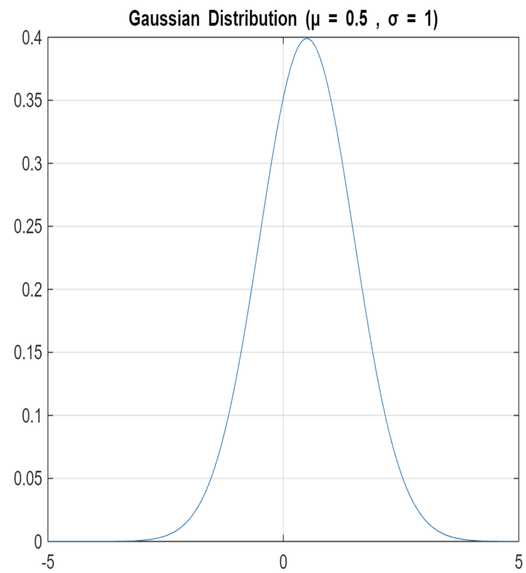


Figure 1. Gaussian(Normal) Distribution

given by :

$$f_X(x) = \begin{cases} \frac{1}{x\cdot\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}(\frac{ln(x)-\mu}{\sigma})^2} & \text{if } x > 0 \\ 0 & otherwise \end{cases}$$

It is also very useful while determining stock prices. It is base for a very famous option valuation model namely Black-Scholes.

$$E(X) = e^{(\mu+\frac{\sigma^2}{2})}$$
$$Var(X) = e^{(2\mu+\sigma^2)} \cdot (e^{\sigma^2} - 1)$$

4. Black Scholes

Everybody who wants to do finance wants to know about Black-Scholes formula or equation.

So here we are going to derive a partial differential equation which has to be satisfied by the something called an option price.

So option is one of the most important financial derivative which is nothing but a secondary financial instrument but its price depends on the primary instrument like stocks.

Suppose the buyer is buying a stock from the seller at price called the option price after time called the expiration date but the buyer is not obligated to buy it from the buyer only, it can also buy the stock from the market also depending on the price of the stock but this is not the free deal that means when the buyer makes the contract then he have to pay an upfront price (premium) which is called the option price.

Now, how you determine that option price is the issue of Black Scholes.

Hence,

let S_t be the stock buyer made a deal on with option price K and expiration date T.

then,

Value of the option $\rightarrow C(T,\, S(T)) = maxS_T - K,\, 0.$

Option price = $C(0,\, S(0))$ at $T = 0$.

and,

Price of the option seller's portfolio $\rightarrow X(t) = C(T,\, S(T))$ as the seller don't want his loss.

$\rightarrow$ The main characteristics of Black Scholes formula are:

(1) A financial market with the one stock + money market

(2) Market is complete (which works on the above condition of the value of the option)

(3) Free of Arbitrage $\rightarrow$ which is earning money without spending the single penny.

Then, in this setting the stock has the stock price satisfies the below differential equation called the drift.

$$dS(t) = \alpha\, S(t)\, dt + \sigma\, S(t)\, dW(t)$$

where σ captures the variation of the stock price and α is the constant.

So, the buyer at time T = 0 pays the Option price to the seller i.e. $X(0)$ which he invests in the market and the stocks and retrieving it so that at the end the Option price could be equal to the value of the option which is exactly the meaning of the complete market.

The instantaneous interest rate of this deposit would be e^{rt} and the money after the time t would be $S_t = S_0\, e^{rt}$ where B_0 is the money deposited at the initial and e^{-rt} is the discounted stock price which we are interested in.

In the pricing structure of Black and Scholes we also do the risk neutral pricing which shows us how we get the discounted stock price values and see their evaluation.

Hence,

we will say that at every time t, the value of the option will be $C(t,\, S(t))$ (**Delta-Hedging formula**) means if the buyer don't buy between the time 0 and T but buys at the time t where $0 < t < T$ then $C(t,\, S(t))$ would be the money buyer have to pay and that should be at every time should be $X(t)$ essentially

4.1. Black Scholes Formula

The Black-Scholes call option formula is calculated by multiplying the stock price by the cumulative standard normal probability distribution function.

In mathematical notation:

$$C = S_t\, N(d_1) - K\, e^{-rt}\, N(d_2)$$

where,

$$d_1 = \frac{ln\frac{S_t}{K} + (r + \frac{\sigma_v^2}{2})\, t}{\sigma_S\, \sqrt{t}}$$

and,

$$d_2 = d_1 - \sigma_S\, \sqrt{t}$$

where,
C = Call option price
S = Current stock (or other underlying) price
K = Strike price
r = Risk-free interest rate
t = Time to maturity
N = A normal distribution

Volitality The value of stocks (or any other financial assets) vary on a daily basis. Volitality is an indicator to quantify these changes.

Volitality is the measure of the dispersion of returns for a given security or market index. In most cases, the higher the volitality, the riskier the security (the more the security can give returns). Volatility is often measured as either the standard deviation or variance between returns from that same security or market index.

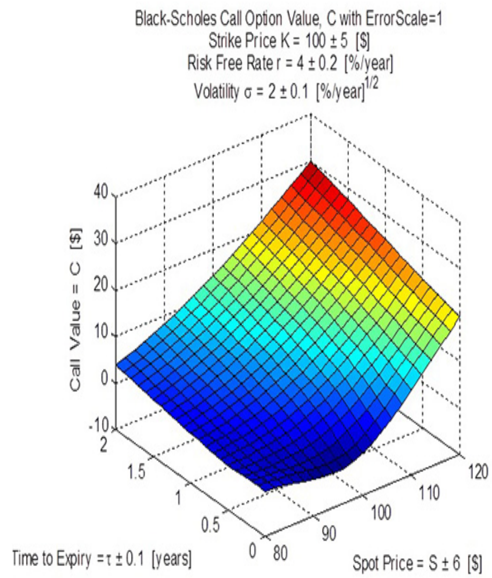


Figure 2. Black-Scholes-option-pricing

Annualized Volitality To have a look at the different volatilities over a period of a year, we multiply the volatility by a factor that accounts for the variability of the assets for one year.To do this we use the variance. Since in the given example we are using Black-Scholes model $r(t) = ln(\frac{S_t}{S_{t-1}})$. The daily variance is taken as $\sigma_d^2 = \frac{\sum_{i=2}^{n} r(t)^2}{n-1}$.

The annualized variance is $365 \cdot \sigma_d^2$

Volitality = $\sqrt{annualizedvariance} = \sqrt{\sigma_d^2}$

4.2. Simulation and Results

(Figure 3) has the data of any hypothetical european stock for 20 trading days. The price is in the range $95 - 100$ generated using RANDBETWEEN command of excel. Then as given above we calculate the annular volitality. The code is for Black-Scholes option pricing model for european stocks (or any other asset). Price denotes the

DAY	Price		r(t)	r(t)^2	
1	100				
2	95		-0.051293	0.002631	
3	97		0.020834	0.000434	
4	99		0.020409	0.000417	
5	95		-0.041243	0.001701	
6	95		0	0	
7	95		0	0	
8	99		0.041243	0.001701	
9	98		-0.010152	0.000103	
10	95		-0.031091	0.000967	
11	99		0.041243	0.001701	
12	99		0	0	
13	98		-0.010152	0.000103	
14	100		0.020203	0.000408	
15	95		-0.051293	0.002631	
16	99		0.041243	0.001701	
17	97		-0.020409	0.000417	
18	99		0.020409	0.000417	
19	95		-0.041243	0.001701	
20	100		0.051293	0.002631	
				0.019662	sum of entries
				0.001035	daily variance
				0.377726	annualarised variance
				0.614594	Annualarised volitality
				61.45943	percentage

Figure 3. stock data

```
price = 95;
sp = 105;
r = 0.1;
t = 0.25;
T = xlsread('Stock values.xlsx','F27:F27');
vol = T;
[Call,Put] = blsprice(price,sp,r,t,vol);
Call
Put
```

```
Call =

    8.7042


Put =

   16.1118
```

Figure 4. Call and Put options

current market price (cmp) of the stock (or other assets), sp denotes the strike price of the option, r denotes the compounded risk-free rate of return over the life of the option(annualized rate), t is the time period for option expiry in years, and vol is the volitality as calculated above. the blsprice command calculates the option pricing in the way shown in Section 5. It gives us the Call and Put option (Figure 4)

5. Geometric Brownian Motion

Brownian motion, or pedesis, is the random motion of particles suspended in a medium. This pattern of motion typically consists of random fluctuations in a particle's position inside a fluid sub-domain, followed by a relocation to another sub-domain. Each relocation is followed by more fluctuations within the new closed volume.

When we can represent this motion using random processes, we shall be able to apply this concept in different spheres of finance. One of them is prediction of price of equity.
Whenever we analyze the stock price of any publicly traded company, it is impossible that the graph is uniformly going upwards or downwards such as that of e^x. While the stock price generally increases, there are always minor variations, ups and downs, which can be represented with the help of the geometric brownian model. A stochastic process $B(t), t \geq 0$ is a Brownian motion (BM) if and only if it satisfies:
1. For any time points $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$, the increments $B(t_4) - B(t_3)$ and $B(t_2) - B(t_1)$ are independent.
2. Each increment is a zero-mean Gaussian random variable with variance equals the difference

in time: $B(t) - B(s) \sim N(0, t-s)$.
3. $B(0) = 0$.

We also notice that the Brownian Motion is a non-stationary Gaussian process but its increments $B_{t+h} - B_t$ is a stationary Gaussian process keeping t fixed and varying $h \geq 0$.
We now define a geometric brownian motion as a stochastic process $S(t), t \geq 0$ such that:

$$S(t) = S_0.e^{\frac{(\mu-\sigma^2)}{2}t+\sigma B(t)}, \forall t \geq 0$$

where $B(t), t \geq 0$ is a brownian motion.

The above formula has practical use as a stock-price model, where S_0 is the initial price of the stock, $S(t)$ is the price after t days, μ is the drift of the stock (the gradual movement of stock towards an equilibrium) and σ is the volatility of the stock.
Now, we can also write the logarithmic return of the stock price as follows:

$$R_{\Delta t}(t) = \ln \frac{S(t+\Delta t)}{S(t)}$$
$$= (\mu - \frac{\sigma^2}{2})\Delta t + \sigma[B(t+\Delta t) - B(t), t \geq 0$$

Also $[B(t + \Delta t) - B(t)$ can be represented as a normal distribution with mean 0 and variance δt, therefore, the distribution of the logarithmic return for the Geometric Brownian Motion can be shown as:

$$R_{\Delta t}(t) \sim N\bigg((\mu - \frac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t\bigg)$$

6. Risk Analysis

Risk Analysis is a mathematical approach involving the use of various tools to assess and rank risks and formulate methods for resolving them.
For any financial model, risk analysis is essential for handling potential threats and to develop procedures to handle them.
The crucial steps involved in risk analysis are:

- Identification of risks
- Assessment of risks
- Developing a response strategy
- Monitoring and control

In financial models and simulations, the probability of a variable represents the probability of a random phenomenon that affects the price or determines the level of investment risk.

6.1. Measures of Risk

- Mean and Variance

  Let X be a random variable.
  The mean of X is termed as the expected value of X. It is denoted by $\mu = E[X]$.
  The variance of X is defined as the average of squared difference from mean. It is denoted by $Var[X] = E[(X - \mu)^2]$
  It is often assumed that a factor having more variance has more risk, but it is not a sole factor determining risk.

  Standard deviation of a random variable is the square root of the variance. It is denoted by
  $\sigma_X = \sqrt{Var[X]}$

- Correlation

  It is defined as a measure of dependence between 2 random variables. Its value is between -1 and 1.

  Let us assume 2 stocks A and B. Let the two normal probability distributions corresponding to A and B be R_A and R_B, then correlation is measured as

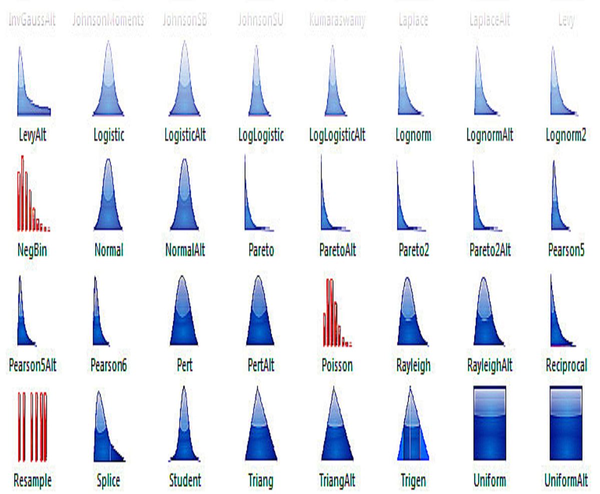


Figure 5. Distributions in Monte Carlo

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = \frac{E[(R_A - \mu_A)(R_B - \mu_B)]}{\sigma_A \sigma_B}$$

- Value at risk (VaR)

  It is termed as the worst case scenario for any business model. It is the maximum risk of an investment over a specific interval of time.

  The value of VaR depends on 4 factors:

  - Investment Value
  - Expected volatility
  - Confidence level
  - Time Horizon

6.2. Monte Carlo Simulation

Monte Carlo Simulation is a mathematical algorithm that predicts all the possible outcomes of our decisions, help us analyze the impact of risk and thus making the decision making easier for us under uncertainties.

This technology is used by professionals in a wide range of fields including finance, project management, energy, manufacturing, engineering, research and development, insurance, oil and gas, transportation, and the environment.

6.2.1 Working

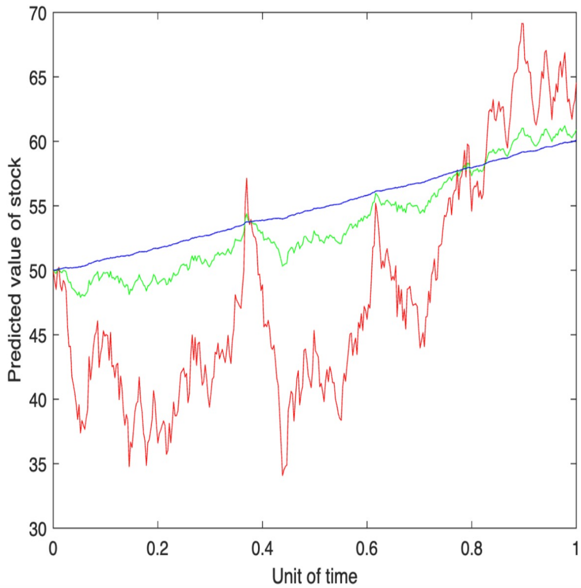
The Monte Carlo simulation performs a risk analysis by replacing a series of values (probability distribution) with any factor with inherent uncertainty to build a model of possible outcomes. Then he calculates the result over and over again, each time using a different set of random values of the probability function.

Depending on the amount of uncertainty and the assigned range, the Monte Carlo simulation can involve thousands or tens of thousands of new calculations before completion. The Monte Carlo simulation produces a distribution of possible result values.

6.2.2 Various aspects

- Explains probability of occurrence of each outcome
- Generates plots for various outcomes
- Analyzes every possible outcome
- Determines which input has the most impact on the overall revenue

7. Results and Simulations



https://github.com/lakshbalani/PnRP-project

Other results are in the reports and codes are in matlab.

8. Conclusion

The report analyses the various aspects of probability used in the field of finance ranging from stock market to risk analysis. We simulated a stock price prediction in MATLAB and determined call and put pricing options for a stock from an excel sheet of stock prices. We briefly studied about risk analysis, Monte Carlo Simulation, Geometric Brownian Motion Model, Black-Scholes option pricing. As we notice from the plot, as the value of sigma changes from 0.5 to 0.01, the deviations of the stock decreases as the volatility decreases and hence the fluctuations in the values of the stock decreases. We also notice on changing the mu values the amplitude of the stock increases, hence the maximum value of the stock throughout the trading period increases. As we deviate the Strike price, the call pricing decreases.

9. Contibutions

- Ishanya Sethi (2020102014): Probability Distributions and their use in finance, Black-Scholes model coding,Gaussian Coding
- Laksh Balani (2020102019) : Risk Management,report,Monte-carlo
- Anish Mathur (2020102044) : Geometric Brownian Motion, GBM coding,report
- Prasoon Garg (2020102049) : Black-Scholes Option Pricing and Report

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Lastly, we would like to thank the reader for taking out time to peruse our project. We hope you enjoyed going through the different applications of probability in finance as much as we did.

11. References

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