Question **4**

Correct

Marked out of 20

(*Fixed-Point Iteration*). Unless otherwise required, all numerical answers should be rounded to 7-digit floating-point numbers. Given a real number z, the symbol $\tilde{z}$ denotes the result of rounding of z to a 7-digit floating point number.

Consider the polynomial

$$f(x) = 0.48x^3 + 0.52x^2 - 6.24x + 1.56.$$

In what follows, we will apply the Fixed-Point Iteration (FPI) method to approximate a unique root of the function $f(x)$ in $[0, 1]$.

(i) (a) Demonstrate that

$$0.48x^3 + 0.52x^2 - 6.24x + 1.56 = 0 \iff x = \frac{x^3}{13} + \frac{x^2}{12} + \frac{1}{4}$$

on $[0, 1]$, thereby obtaining a reduction to a fixed-point problem for the (iteration) function

$$g(x) = \frac{x^3}{13} + \frac{x^2}{12} + \frac{1}{4}$$

on $[0, 1]$.

(b) Clearly, the function $g(x)$ is strictly increasing ✔ on $[0, 1]$. Accordingly, for every $x \in [0, 1]$,

$$g(?_1) \leqslant g(x) \leqslant g(?_2),$$

where

$?_1 =$ 0 ✔ and $?_2 =$ 1 ✔

(please enter suitable points/numbers of the interval $[0, 1]$) and

$g(?_1) =$ 1/4 ✔ and $g(?_2) =$ 16/39 ✔

(please enter suitable *rational* numbers).

(c) Does the argument in (b) imply that the function $g(x)$ takes the interval $[0, 1]$ into itself?

Yes ✔

(d) Next, the derivative $g'(x)$ of the function $g(x)$ is also strictly increasing ✔ on $[0, 1]$. Consequently, for every $x \in [0, 1]$,

$$|g'(x)| \leqslant |g'(?_3)| = k$$

where

$?_3 =$ 1 ✔

(please enter a suitable point/number of the interval $[0, 1]$) and

$k = |g'(?_3)| =$ 31/78 ✔

(please enter a suitable *rational* number), and k is evidently less than 1 ✔ (please enter a *relevant* number).

(e) Now we see that both conditions from the main statement on convergence of the FPI in the lecture notes are ... (here and in the next part, please enter a suitable *word*)

true ✔

for the function $g(x)$ on $[0, 1]$, and hence the FPI for the function $g(x)$ on $[0, 1]$ ...

converges ✔ .

(ii) Use the Fixed-Point Iteration method to find an approximation p_N of the fixed-point p of $g(x)$ in $[0, 1]$, the root of the polynomial $f(x)$ in $[0, 1]$, satisfying

$$\mathrm{RE}(\tilde{p}_N \approx \tilde{p}_{N-1}) < 10^{-5}$$

by taking $p_0 = 1$ as the initial approximation.

(iii) Show your work by filling in the following table (if a particular row of the table is not necessary, please enter an asterisk $*$ in each input field in this row):

To repeat the advice given in Problem 2 once again, if you are going to use a scientific calculator, first create a similar table in an OpenOffice (or Excel) worksheet, and then copy-paste your answers.

n	p_{n-1}	p_n	$\mathrm{RE}(\tilde{p}_n \approx \tilde{p}_{n-1})$
1	1	0.4102564	1.4375
2	0.4102564	0.2693374	0.5232062
3	0.2693374	0.2575482	0.04577473
4	0.2575482	0.2568417	0.002750722
5	0.2568417	0.2568006	0.0001600464
6	0.2568006	0.2567982	9.34586e-06
7	*	*	*
8	*	*	*

(iii) Accordingly, by (i) and (ii),

$p_N \doteq$ 0.2567981 .

Check