Question **1**

Correct

Marked out of 42

(*Numerical Differentiation*). All numerical answers should be rounded to 8-digit floating-point numbers.

Consider the function

$$f(x) = \sqrt[3]{1 + \sqrt[6]{1 + \sqrt[7]{1 + x}}}.$$

In what follows, you will be asked to obtain a number of approximations of

$$f_0' = f(x_0) \text{ and } f_0'' = f''(x_0)$$

for the point $x_0 = 3$ using the grid centered at x_0 with step $h = 0.001$ and the approximation formulas we have studied during the lectures.

(i) Evaluate the function $f(x)$ at the grid points

$$x_{-1}, x_0, x_1, x_2, x_3, x_4.$$

Show your work by filling in the following standard table:

k	x_k	f_k
-1	2.999 ✓	1.2890737 ✓
0	3 ✓	1.2890744 ✓
1	3.001 ✓	1.2890752 ✓
2	3.002 ✓	1.2890759 ✓
3	3.003 ✓	1.2890767 ✓
4	3.004 ✓	1.2890774 ✓

Users of sci calculators are advised:

● to keep the values of the grid points in the variable X and to keep the grid size h in the variable D (as we did in class);

● to keep the values

$$f_{-1}, f_0, f_1, f_2, f_3, f_4$$

in the variables

A, B, C, E, F, M,

respectively;

● to store any approximation of either f_0', or f_0'' below in the variable Y, as, e.g., in

(A - 2× B + C)÷ D^2 → Y

(ii) Execute the command

```
N[ ReplaceAll[ D[ (1 + (1 + (1+x)^(1/7))^(1/6))^(1/3) ], {x -> 3} ], 15 ]
```

and the command

```
N[ ReplaceAll[ D[D[ (1 + (1 + (1+x)^(1/7))^(1/6))^(1/3) ]], {x -> 3} ], 15 ]
```

at the Wolfram Alpha website, thereby obtaining

● the 15-digit representation $\bar{z}$ of the value $z = f_0' = f'(x_0)$ and

● the 15-digit representation $\bar{t}$ of $t = f_0'' = f''(x_0)$.

● Recall that the command

D[g(x)]

returns the derivative of the function g(x) (this command, as is the case with the following two commands, is a simplified version of the corresponding command in the Wolfram Language);

● the command

ReplaceAll[h(x), {x -> c}]

is used to replace all occurrences of the variable x in a math expression h(x) with a *number* c;

● the command

N[..., d]

evaluates a *numerical* expression ..., and then rounds the result to a d-digit floating-point number.

(iii) Use (i) to find the following approximations z^* of $z = f_0'$ and the corresponding numbers $\mathrm{SD}(\bar{z} \approx z^*)$ of significant digits (recall that $\bar{z}$ is the 15-digit representation of $z = f_0'$ you should have found in (ii)).

(a) Apply the 2-point midpoint formula

$$z = f_0' \approx \frac{f_1 - f_{-1}}{2h} = z^*$$

to approximate $z = f_0'$. Then

$z^* \doteq$ 0.00074912693 ✔

and

$\mathrm{SD}(\bar{z} \approx z^*) =$ 8 ✔ .

Here and below, please *keep* the corresponding relative error(s) in your records for the future use.

To continue the instructions for the user of sci calculators: in this and in the following parts, if your approximation is saved in the variable `Y`, find the corresponding relative error $\mathrm{RE}(\bar{z} \approx z^*)$ by executing a command similar to

```
1 - Y/3.14159265358979
```

so that the number $\bar{z}$ is entered 'as is', with all its 15 digits present.

(b) Apply the 3-point endpoint formula

$$z = f_0' \approx \frac{-3f_0 + 4f_1 - f_2}{2h} = z^*$$

to approximate $z = f_0'$. Then

$z^* \doteq$ 0.00074912689 ✔

and

$\mathrm{SD}(\bar{z} \approx z^*) =$ 8 ✔ .

(c) Apply the 4-point endpoint formula

$$z = f_0' \approx \frac{-11f_0 + 18f_1 - 9f_2 + 2f_3}{6h} = z^*$$

to approximate $z = f_0'$. Then

$z^* \doteq$ 0.00074912691 ✔

and

$\mathrm{SD}(\bar{z} \approx z^*) =$ 10 ✔ .

(d) Apply the 5-point endpoint formula

$$z = f_0' \approx \frac{-25f_0 + 48f_1 - 36f_2 + 16f_3 - 3f_4}{12h} = z^*$$

to approximate $z = f_0'$. Then

$z^* \doteq$ 0.00074912691 ✔

and

$\mathrm{SD}(\bar{z} \approx z^*) =$ 10 ✔ .

(e) Now arrange the relative errors you have found in (a-d) in *increasing* order (this will show which approximation is the best, the second best, and so on). Describe then your arrangement by entering a sequence of the corresponding letters separated by commas, similar to

`a, d, b, c`.

Answer: c,d,a,b ✔ .

(iv) Use (i) to find the following approximations t^* of $t = f_0''$ and the corresponding numbers $\mathrm{SD}(\bar{t} \approx t^*)$ of significant digits (to repeat, $\bar{t}$ is the 15-digit representation of $t = f_0''$ you should have found in (ii)).

(a) Apply the 3-point midpoint formula

$$t = f''_0 \approx \frac{f_{-1} - 2f_0 + f_1}{h^2} = t^*$$

to approximate $t = f_0''$. Then

$t^* \doteq$ -0.00017364576 ✔

and

$\mathrm{SD}(\bar{t} \approx t^*) =$ 6 .

(b) Apply the 4-point endpoint formula

$$t = f_0'' \approx \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} = t^*$$

to approximate $t = f_0''$. Then

$t^* \doteq$ -0.00017364599

and

$\mathrm{SD}(\bar{t} \approx t^*) =$ 6 .

(c) As before in (iii), arrange the relative errors you have found in (a-b) in increasing order and enter a sequence of letters, separated by commas, describing your arrangement:

Answer: a,b .

Check

Question 2

Correct

Marked out of 16

(*Numerical Integration: Composite Rectangle Rule*). All numerical answers should be rounded to 8-digit floating-point numbers.

(i) (a) Use the Composite Rectangle Rule with $n = 20$ to find an approximation I^* of the integral

$$I = \int_0^{0.5} \ln(\,43.9 + \sin^2(x)\,)\,dx$$

(b) Show your work by filling in the following table of *selected* values of the grid points and the values of the integrand at these points:

k	x_k	y_k
0	0	3.7819143
5	0.125	3.7822683
10	0.25	3.7833076
15	0.375	3.7849656
19	0.475	3.7866674

(ii) By (i),

$I^* \doteq$ 1.8917942 .

In this and the following problems, it is a good idea to obtain a sufficiently accurate approximation of the integral in question by executing the commands similar to

```
evalf( integral(f(x), x = a .. b), 14 )
```

or to

```
N[ Integrate[ f[x], {x,a,b}], 14 ]
```

at the Wolfram Alpha website, or any other online resource/software, in order to compare with your answer.

Check

Question 3

Correct

Marked out of 18

(*Numerical Integration: Composite Trapezoidal Rule*). All numerical answers should be rounded to 8-digit floating-point numbers.

(i) (a) Use the Composite Trapezoidal Rule with $n = 10$ to find an approximation I^* of the integral

$$I = \int_0^{\pi/2} \sqrt{1.21\sin^2(t) + 14.44\cos^2(t)}\, dt.$$

(b) Show your work by filling in the following standard table

k	x_k	$(1/2 \mid 1) y_k$
0	0	1.9
1	0.15707963	3.7571583
2	0.31415927	3.6299652
3	0.4712389	3.4224552
4	0.62831853	3.1415199
5	0.78539816	2.7973201
6	0.9424778	2.4043404
7	1.0995574	1.9841372
8	1.2566371	1.5726896
9	1.4137167	1.2384511
10	1.5707963	0.55

(ii) By (i),

$I^* \doteq$ 4.146594 .

Check

Question **4**

Correct

Marked out of 24

(*Numerical Integration: Composite Simpson's Rule*). All numerical answers should be rounded to 8-digit floating-point numbers.

(i) (a) Find an approximation I^* of the integral

$$I = \int_0^{\pi/2} e^{2.08\sin(x)}\,dx$$

using the Composite Simpson's Rule with $n = 10$.

(b) Show your work by filling in the following standard table

k	x_k	$(1\,|\,2\,|\,4)y_k$
0	0	1
1	0.15707963	5.5382471
2	0.31415927	3.8034271
3	0.4712389	10.284055
4	0.62831853	6.7919664
5	0.78539816	17.410552
6	0.9424778	10.760721
7	1.0995574	25.523202
8	1.2566371	14.4594
9	1.4137167	31.208363
10	1.5707963	8.0044689

(ii) By (i),

$I^* \doteq$ 7.0572948 .

Check