Question **1**

Correct

Marked out of 42

(*Numerical Differentiation*). All numerical answers should be rounded to 8-digit floating-point numbers.

Consider the function

$$f(x) = \sqrt[3]{1 + \sqrt[4]{1 + \sqrt[8]{1+x}}}.$$

In what follows, you will be asked to obtain a number of approximations of

$$f_0' = f(x_0) \text{ and } f_0'' = f''(x_0)$$

for the point $x_0 = 3$ using the grid centered at x_0 with step $h = 0.001$ and the approximation formulas we have studied during the lectures.

(i) Evaluate the function $f(x)$ at the grid points

$$x_{-1}, x_0, x_1, x_2, x_3, x_4.$$

Show your work by filling in the following standard table:

k	x_k	f_k
-1	2.999 ✔	1.3038116 ✔
0	3.0 ✔	1.3038126 ✔
1	3.001 ✔	1.3038137 ✔
2	3.002 ✔	1.3038147 ✔
3	3.003 ✔	1.3038157 ✔
4	3.004 ✔	1.3038167 ✔

Users of sci calculators are advised:

● to keep the values of the grid points in the variable `X` and to keep the grid size h in the variable `D` (as we did in class);

● to keep the values

$$f_{-1}, f_0, f_1, f_2, f_3, f_4$$

in the variables

`A, B, C, E, F, M,`

respectively;

● to store any approximation of either f_0', or f_0'' below in the variable `Y`, as, e.g., in

`(A - 2× B + C)÷ D^2 → Y`

(ii) Execute the command

```
N[ ReplaceAll[ D[ (1 + (1 + (1+x)^(1/8))^(1/4))^(1/3) ], {x -> 3} ], 15 ]
```

and the command

```
N[ ReplaceAll[ D[D[ (1 + (1 + (1+x)^(1/8))^(1/4))^(1/3) ]], {x -> 3} ], 15 ]
```

at the Wolfram Alpha website, thereby obtaining

● the 15-digit representation $\bar{z}$ of the value $z = f_0' = f'(x_0)$ and

● the 15-digit representation $\bar{t}$ of $t = f_0'' = f''(x_0)$.

● Recall that the command

`D[ g(x) ]`

returns the derivative of the function `g(x)` (this command, as is the case with the following two commands, is a simplified version of the corresponding command in the Wolfram Language);

● the command

`ReplaceAll[ h(x), {x -> c} ]`

is used to replace all occurrences of the variable x in a math expression `h(x)` with a *number* c;

● the command

`N[ ..., d]`

evaluates a *numerical* expression `...`, and then rounds the result to a `d`-digit floating-point number.

(iii) Use (i) to find the following approximations z^* of $z = f_0'$ and the corresponding numbers $\mathrm{SD}(\bar{z} \approx z^*)$ of significant digits (recall that $\bar{z}$ is the 15-digit representation of $z = f_0'$ you should have found in (ii)).

(a) Apply the 2-point midpoint formula

$$z = f_0' \approx \frac{f_1 - f_{-1}}{2h} = z^*$$

to approximate $z = f_0'$. Then

$z^* \doteq$ 0.0010122336

and

$\mathrm{SD}(\bar{z} \approx z^*) =$ 8 .

Here and below, please *keep* the corresponding relative error(s) in your records for the future use.

To continue the instructions for the user of sci calculators: in this and in the following parts, if your approximation is saved in the variable `Y`, find the corresponding relative error $\mathrm{RE}(\bar{z} \approx z^*)$ by executing a command similar to

```
1 - Y/3.14159265358979
```

so that the number $\bar{z}$ is entered 'as is', with all its 15 digits present.

(b) Apply the 3-point endpoint formula

$$z = f_0' \approx \frac{-3f_0 + 4f_1 - f_2}{2h} = z^*$$

to approximate $z = f_0'$. Then

$z^* \doteq$ 0.0010122336

and

$\mathrm{SD}(\bar{z} \approx z^*) =$ 8 .

(c) Apply the 4-point endpoint formula

$$z = f_0' \approx \frac{-11f_0 + 18f_1 - 9f_2 + 2f_3}{6h} = z^*$$

to approximate $z = f_0'$. Then

$z^* \doteq$ 0.0010122336

and

$\mathrm{SD}(\bar{z} \approx z^*) =$ 10 .

(d) Apply the 5-point endpoint formula

$$z = f_0' \approx \frac{-25f_0 + 48f_1 - 36f_2 + 16f_3 - 3f_4}{12h} = z^*$$

to approximate $z = f_0'$. Then

$z^* \doteq$ 0.0010122336

and

$\mathrm{SD}(\bar{z} \approx z^*) =$ 11 .

(e) Now arrange the relative errors you have found in (a-d) in *increasing* order (this will show which approximation is the best, the second best, and so on). Describe then your arrangement by entering a sequence of the corresponding letters separated by commas, similar to

`a, d, b, c`.

Answer: d,c,a,b .

(iv) Use (i) to find the following approximations t^* of $t = f_0''$ and the corresponding numbers $\mathrm{SD}(\bar{t} \approx t^*)$ of significant digits (to repeat, $\bar{t}$ is the 15-digit representation of $t = f_0''$ you should have found in (ii)).

(a) Apply the 3-point midpoint formula

$$t = f''_0 \approx \frac{f_{-1} - 2f_0 + f_1}{h^2} = t^*$$

to approximate $t = f_0''$. Then

$t^* \doteq$ -0.00023588531

and

$\mathrm{SD}(\bar{t} \approx t^*) =$ 6 .

(b) Apply the 4-point endpoint formula

$$t = f_0'' \approx \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} = t^*$$

to approximate $t = f_0''$. Then

$t^* \doteq$ -0.00023588465

and

$\mathrm{SD}(\bar{t} \approx t^*) =$ 6 .

(c) As before in (iii), arrange the relative errors you have found in (a-b) in increasing order and enter a sequence of letters, separated by commas, describing your arrangement:

Answer: a,b .

Check

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Question **2**

Correct

Marked out of 16

(*Numerical Integration: Composite Rectangle Rule*). All numerical answers should be rounded to 8-digit floating-point numbers.

(i) (a) Use the Composite Rectangle Rule with $n = 20$ to find an approximation I^* of the integral

$$I = \int_0^{0.5} \ln(\,40.1 + \sin^2(x)\,)\,dx$$

(b) Show your work by filling in the following table of *selected* values of the grid points and the values of the integrand at these points:

k	x_k	y_k
0	0 ✔	3.6913763 ✔
5	0.125 ✔	3.6917639 ✔
10	0.25 ✔	3.6929016 ✔
15	0.375 ✔	3.6947163 ✔
19	0.475 ✔	3.6965787 ✔

(ii) By (i),

$I^* \doteq$ 1.8466044 ✔ .

In this and the following problems, it is a good idea to obtain a sufficiently accurate approximation of the integral in question by executing the commands similar to

```
evalf( integral(f(x), x = a .. b), 14 )
```

or to

```
N[ Integrate[ f[x], {x,a,b}], 14 ]
```

at the Wolfram Alpha website, or any other online resource/software, in order to compare with your answer.

Check

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Question **3**

Correct

Marked out of 18

(*Numerical Integration: Composite Trapezoidal Rule*). All numerical answers should be rounded to 8-digit floating-point numbers.

(i) (a) Use the Composite Trapezoidal Rule with $n = 10$ to find an approximation I^* of the integral

$$I = \int_0^{\pi/2} \sqrt{2.89 \sin^2(t) + 2.25 \cos^2(t)}\, dt.$$

(b) Show your work by filling in the following standard table

k	x_k	$(1/2 \mid 1) y_k$
0	0	0.75
1	0.15707963	1.5052116
2	0.31415927	1.520235
3	0.4712389	1.5433434
4	0.62831853	1.5719779
5	0.78539816	1.603122
6	0.9424778	1.6336724
7	1.0995574	1.6607502
8	1.2566371	1.6819291
9	1.4137167	1.6953873
10	1.5707963	0.85

(ii) By (i),

$I^* \doteq$ 2.5157291 .

Check

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Question **4**

Correct

Marked out of 24

(*Numerical Integration: Composite Simpson's Rule*). All numerical answers should be rounded to 8-digit floating-point numbers.

(i) (a) Find an approximation I^* of the integral

$$I = \int_0^{\pi/2} e^{1.64\sin(x)}\,dx$$

using the Composite Simpson's Rule with $n = 10$.

(b) Show your work by filling in the following standard table

k	x_k	$(1\,\vert\,2\,\vert\,4)y_k$
0	0	1.0
1	0.15707963	5.1698666
2	0.31415927	3.3199013
3	0.4712389	8.421928
4	0.62831853	5.2441596
5	0.78539816	12.755333
6	0.9424778	7.5378353
7	1.0995574	17.245393
8	1.2566371	9.5150986
9	1.4137167	20.208499
10	1.5707963	5.1551695

(ii) By (i),

$I^* \doteq$ 5.0042002 .

Check

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