Double Pendulum Dynamics

Q-1: SI Units of Stiffness and Damping Coefficients

The SI units for the stiffness coefficient S and damping coefficient D are:

- Stiffness coefficient (S): Nm
- Damping coefficient (D): N·sm

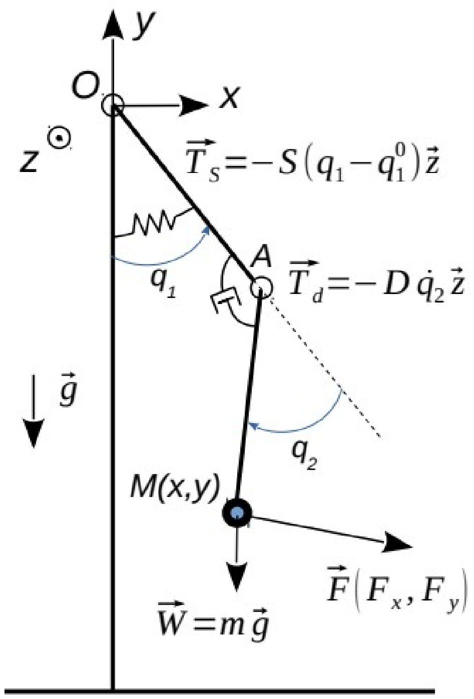

Figure 1: Double pendulum

Q-2: Geometric Model

The coordinates x and y as functions of the generalized coordinates q_1 and q_2 are:

$$x = L_1 \sin(q_1) + L_2 \sin(q_1 + q_2) \tag{1}$$

$$y = -L_1 \cos(q_1) - L_2 \cos(q_1 + q_2) \tag{2}$$

Q-3: Jacobian Matrix

The Jacobian matrix J for the double pendulum system is:

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{bmatrix} && (3) \\ &= \begin{bmatrix} L_1 \cos(q_1) + L_2 \cos(q_1 + q_2) & L_2 \cos(q_1 + q_2) \\ L_1 \sin(q_1) + L_2 \sin(q_1 + q_2) & L_2 \sin(q_1 + q_2) \end{bmatrix} && (4) \end{aligned}$$

Q-4: Generalized Forces

The generalized forces with respect to q_1 and q_2 are:

$$\delta W(W) = Q_{q_1}(W)\delta q_1 + Q_{q_2}(W)\delta q_2 \tag{5}$$
$$Q_{q_1}(W) = -mg(L_1 \sin(q_1) + L_2 \sin(q_1 + q_2)) \tag{6}$$
$$Q_{q_2}(W) = -mgL_2 \sin(q_1 + q_2) \tag{7}$$

Q-5: Spring Force

The generalized forces due to the spring force are:

$$Q_{q_1}(S) = -S(q_1 - q_1^0) \tag{8}$$
$$Q_{q_2}(S) = 0 \tag{9}$$

Q-6: Damping Force

The generalized forces due to the damping force are:

$$Q_{q_1}(D) = 0 \tag{10}$$
$$Q_{q_2}(D) = -D\dot{q}_2 \tag{11}$$

Q-7: Force Resolution

The force is defined as:

$$\begin{aligned} \vec{F} &= F_x \delta x + F_y \delta y && (12) \\ \delta W(F) &= F_x \left(L_1 \cos(q_1)\delta q_1 + L_2 \cos(q_1 + q_2)(\delta q_1 + \delta q_2)\right) && (13) \\ &\quad + F_y \left(L_1 \sin(q_1)\delta q_1 + L_2 \sin(q_1 + q_2)(\delta q_1 + \delta q_2)\right) && (14) \\ Q_{q_1}(F) &= F_x \left(L_1 \cos(q_1) + L_2 \cos(q_1 + q_2)\right) && (15) \\ &\quad + F_y \left(L_1 \sin(q_1) + L_2 \sin(q_1 + q_2)\right) && (16) \\ Q_{q_2}(F) &= F_x L_2 \cos(q_1 + q_2) && (17) \\ &\quad + F_y L_2 \sin(q_1 + q_2) && (18) \end{aligned}$$

Q-8: Potential Energy

The potential energy U due to gravity and the spring force is:

$$U = -mg(L_1\cos(q_1) + L_2\cos(q_1+q_2)) + c_1 \tag{19}$$

$$\text{At boundary condition: } c_1 = 0 \tag{20}$$

$$U = -mg(L_1\cos(q_1) + L_2\cos(q_1+q_2)) \tag{21}$$

$$E = \frac{1}{2}Sq_1^2 + c_2 \tag{22}$$

$$\text{At } q_1 = \frac{\pi}{2}, \text{ we get } c_2 = -\frac{\pi^2}{8}S \tag{23}$$

$$U = \frac{1}{2}Sq_1^2 - \frac{\pi^2}{8}S - mg(L_1\cos(q_1) + L_2\cos(q_1+q_2)) \tag{24}$$

Q-9: Translational Kinetic Energy

The translational kinetic energy T_t is given by:

$$\dot{x}_2 = L_1\cos(q_1)\dot{q}_1 + L_2\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \tag{25}$$

$$\dot{y}_2 = L_1\sin(q_1)\dot{q}_1 + L_2\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2) \tag{26}$$

$$T_t = \frac{1}{2}m\left[(L_1\dot{q}_1)^2 + L_2^2(\dot{q}_1+\dot{q}_2)^2\right. \tag{27}$$

$$\left. +2L_1L_2(\dot{q}_1+\dot{q}_2)\dot{q}_1\left(\cos(q_1)\cos(q_1+q_2) + \sin(q_1)\sin(q_1+q_2)\right)\right] \tag{28}$$

Q-10: Rotational Kinetic Energy

The rotational kinetic energy T_r is:

$$T_r = \frac{1}{2}I_g(\dot{q}_1+\dot{q}_2)^2 \tag{29}$$

$$I_g = \frac{2}{5}mR^2 \tag{30}$$

$$T_r = \frac{1}{5}mR^2(\dot{q}_1+\dot{q}_2)^2 \tag{31}$$

Q-11: Lagrangian

The Lagrangian L for the double pendulum is:

$$L = T - U \tag{32}$$

$$= T_t + T_r - U \tag{33}$$

$$L = \frac{1}{2}m\left[(L_1\dot{q}_1)^2 + L_2^2(\dot{q}_1+\dot{q}_2)^2\right. \tag{34}$$

$$\left. +2L_1L_2(\dot{q}_1+\dot{q}_2)\dot{q}_1\cos(q_2)\right] \tag{35}$$

$$+\frac{1}{5}mR^2(\dot{q}_1+\dot{q}_2)^2 \tag{36}$$

$$-\left(\frac{1}{2}Sq_1^2 - \frac{\pi^2}{8}S - mg(L_1\cos(q_1) + L_2\cos(q_1+q_2))\right) \tag{37}$$

Summary of Terms

The following terms are defined as:

$$A = \frac{1}{2}mL_1^2 + \frac{1}{2}mL_2 + mL_1L_2\cos(q_2) + \frac{2}{5}mR^2 \tag{38}$$

$$B = mL_2 + \frac{1}{5}mR^2 \tag{39}$$

$$C = mL_1L_2\cos(q_2) + \frac{2}{5}mR^2 \tag{40}$$

Q-12: Equations of Motion Using Lagrange's Equations

Using Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = Q_{q_1} \tag{41}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = Q_{q_2} \tag{42}$$

$$\frac{\partial L}{\partial \dot{q}_1} = 2A\dot{q}_1 + C\dot{q}_2 \tag{43}$$

$$\frac{\partial L}{\partial \dot{q}_2} = 2B\dot{q}_2 + C\dot{q}_1 \tag{44}$$

$$\frac{\partial L}{\partial q_1} = -Sq_1 - mg(L_1\sin(q_1) + L_2\sin(q_1+q_2)) \tag{45}$$

$$\frac{\partial L}{\partial q_2} = -mL_1L_2\sin(q_1+q_2)\dot{q}_1^2 - mgL_2\sin(q_1+q_2)\dot{q}_1\dot{q}_2 \tag{46}$$

Q-13: Equations of Motion

The equations of motion are derived from:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = Q_{q_1} \tag{47}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = Q_{q_2} \tag{48}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = 2A\ddot{q}_1 + C\ddot{q}_2 \tag{49}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = 2B\ddot{q}_2 + C\ddot{q}_1 \tag{50}$$

$$2A\ddot{q}_1 + C\ddot{q}_2 + Sq_1 + mg(L_1\sin(q_1) + L_2\sin(q_1+q_2)) = F_x(L_1\cos(q_1) + L_2\cos(q_1+q_2)) \tag{51}$$

$$+ F_y(L_1\sin(q_1) + L_2\sin(q_1+q_2)) \tag{52}$$

$$2B\ddot{q}_2 + C\ddot{q}_1 + mL_1L_2\sin(q_1+q_2)\dot{q}_1^2 + mgL_2\sin(q_1+q_2)\dot{q}_1\dot{q}_2 = F_xL_2\cos(q_1+q_2) \tag{53}$$

$$+ F_yL_2\sin(q_1+q_2) - D\dot{q}_2 \tag{54}$$

Q-14: Matrix Form

The matrix form of the equations of motion is:

$$\begin{bmatrix} 2A & C \\ C & 2B \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} F_x(L_1\cos(q_1) + L_2\cos(q_1+q_2)) \\ +F_y(L_1\sin(q_1) + L_2\sin(q_1+q_2)) \\ F_x L_2\cos(q_1+q_2) \\ +F_y L_2\sin(q_1+q_2) - D\dot{q}_2 \end{bmatrix} \tag{55}$$