GODEL'S INCOMPLETENESS THEOREM

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SUMMARY

The incompleteness theorems of Gödel are two mathematical logic theorems that deal with the provability limits of formal axiomatic systems. These are crucial in both mathematical logic and mathematics philosophy. These are usually regarded as proving Hilbert's programme of discovering a full and consistent set of axioms for all mathematics is unattainable.The incompleteness theorems apply to **complete, consistent, and effectively axiomatized formal systems** of sufficient complexity to express the basic arithmetic of the natural numbers.The incompleteness theorems show that systems which contain a sufficient amount of arithmetic cannot possess all three of these properties.

1 First incompleteness theorem:

"Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F."
The unprovable statement GF referred to by the theorem is often referred to as **"the Gödel sentence"** for the system F.The Gödel sentence is designed to refer, indirectly, to itself. The sentence states that, when a particular sequence of steps is used to construct another sentence, that constructed sentence will not be provable in F. However, the sequence of steps is such that the constructed sentence turns out to be GF itself. In this way, the Gödel sentence GF indirectly states its own unprovability within F.

2 Second incompleteness theorem:

"Assume F is a consistent formalized system which contains elementary arithmetic. Then $F \forall Cons(F)$.
For each formal system F containing basic arithmetic, it is possible to canonically define a formula **Cons(F)** expressing the consistency of F. This formula expresses the property that "there does not exist a natural number coding a formal derivation within the system F whose conclusion is a syntactic contradiction." The syntactic contradiction is often taken to be "0=1", in which case Cons(F) states "there is no natural number that codes a derivation of '0=1' from the axioms of F." It shows that, under general assumptions, the canonical consistency statement Cons(F) will not be provable in F.

This theorem is stronger than the first incompleteness theorem because the statement constructed in the first incompleteness theorem does not directly express the consistency of the system. The proof of the second incompleteness theorem is obtained by formalizing the proof of the first incompleteness theorem within the system F itself.

The incompleteness results affect the philosophy of mathematics, particularly versions of formalism, which use a single system of formal logic to define their principles.There are various philosophies of different authors like **Hilary Putnam** suggested that while Gödel's theorems cannot be applied to humans, since they make mistakes and are therefore inconsistent, it may be applied to the human faculty of science or mathematics in general, **Avi Wigderson** proposed that when knowability is interpreted by modern standards, namely via computational complexity, the Gödel phenomena are very much with us,Douglas Hofstadter refers Gödel's theorems as an example of what he calls a strange loop, a hierarchical, self-referential structure existing within an axiomatic formal system.
Although Gödel's theorems are usually studied in the context of classical logic, they also have a role in the study of paraconsistent logic and of inherently contradictory statements (dialetheia). Appeals and analogies are sometimes made to the incompleteness theorems in support of arguments that go beyond mathematics and logic. Several authors have commented negatively on such extensions and interpretations, including **Torkel Franzén ; Panu Raatikainen; Alan Sokal and Jean Bricmont; and Ophelia Benson and Jeremy Stangroom.**