Mathematics I

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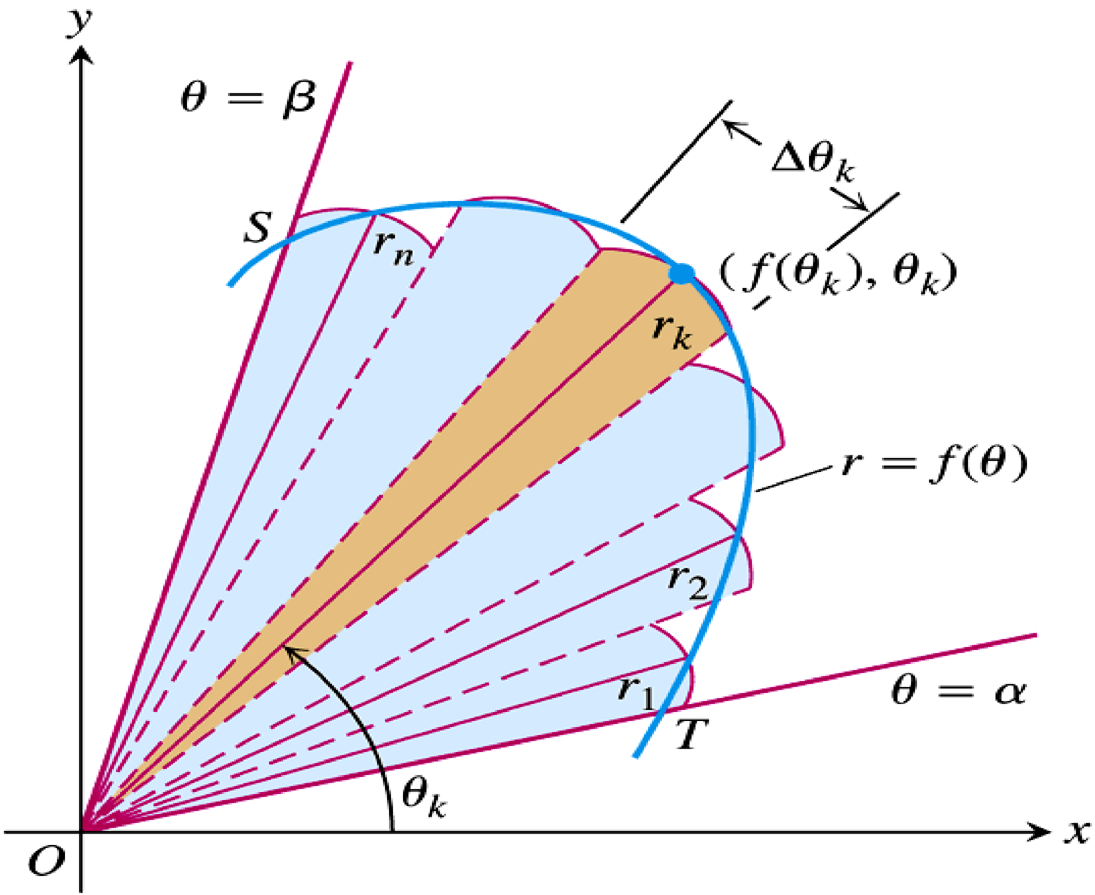
August 24, 2016

Section 11.5

Areas and Lengths in Polar Coordinates


y
θ = β
S
r_n
Δ$θ_k$
$(f(θ_k), θ_k)$
r_k
r = f(θ)
r_2
r_1
T
θ = α
$θ_k$
O
x

To derive a formula for the area A of region OTS, we approximate the region with n nonoverlapping fan shaped circular sectors based on a partition P of angle TOS. The typical sector has radius r_k and central angle $\Delta\theta_k$.

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Therefore the circular sector (with an angle $\Delta\theta_k$) is a part of circle of radius r_k. Thus the area of this sector is

$$A_k = \frac{\Delta\theta_k}{2\pi}(\pi r_k^2) = \frac{1}{2} r_k^2 \Delta\theta_k.$$

Then the area of region OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} r_k^2 \Delta\theta_k.$$

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$$A = \lim_{n\to\infty} \sum_{k=1}^{n} A_k = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta.$$

Then the area A of fan shaped region OTS between the origin and the curve $r = f(\theta)$ and also between the rays $\theta = \alpha$ and $\theta = \beta$ is given by (assuming $\alpha \leqslant \beta$)

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

Steps to Evaluate Area

- Plot the polar graphs correctly and clearly.

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- Label the relevant curves by their equations.
- Shade the required region.
- Use symmetries whenever required.

Remark

Whenever we find the area always make sure that the area is covered exactly once. Otherwise area obtained will be wrong. See the next example.

Example

Find the area of the region inside the circle $r = 2\cos\theta$.

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Sol. If we do not follow the above steps then see what happens?

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The area A is given by

$$A = \frac{1}{2}\int_0^{2\pi} 4\cos^2\theta\, d\theta = \int_0^{2\pi}(1+\cos 2\theta)\, d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 2\pi.$$

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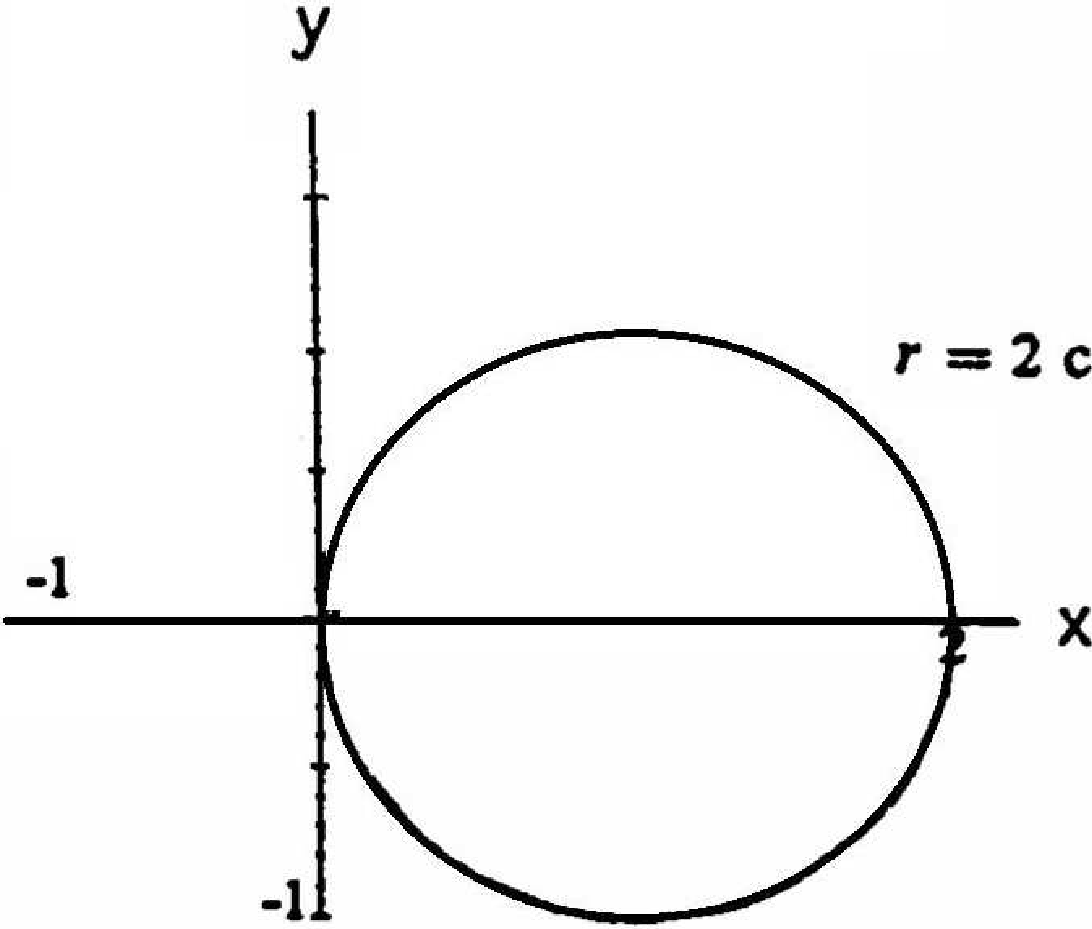
Note that $r = 2\cos\theta$ is a circle with center at $(1,0)$ and radius 1 so its actual area is π.


y
$r = 2\cos\theta$
-1
x
2
-1

The flaw happens because we have not traced the curve. When we trace the curve we find that the whole circle traces between 0 to π only. From π to 2π it traces twice. Thats why we got the area twice as the actual area.

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If we follow the above steps then required area A would be

$$\begin{aligned} A &= 2 \times \text{Area in first quadrant} \\ &= 2 \times \frac{1}{2} \int_0^{\pi/2} 4\cos^2\theta \, d\theta \\ &= 2 \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta = \pi. \end{aligned}$$

Example

Find the area of the region inside the lemniscate $r^2 = 2a^2 \cos 2\theta,\ a > 0.$

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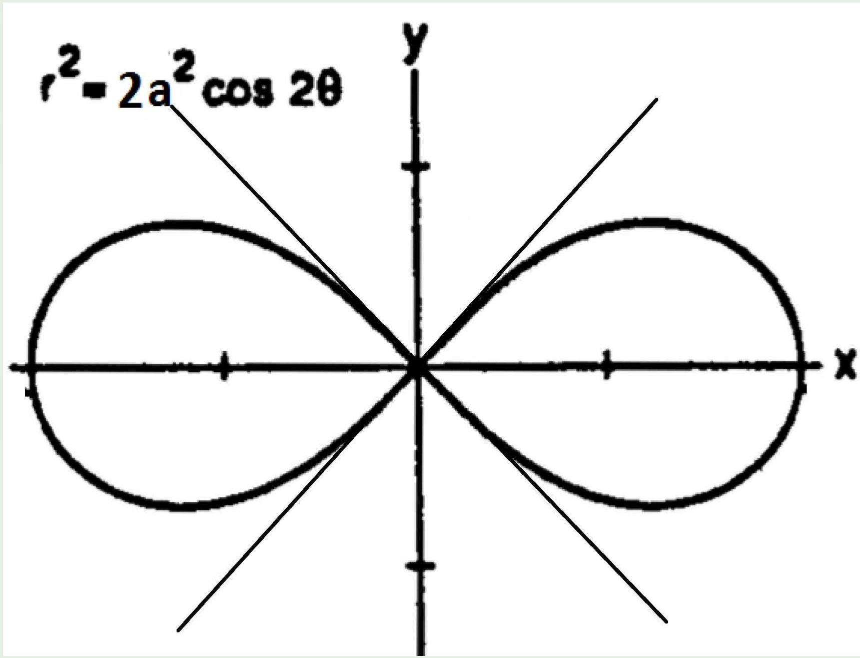

Using symmetries, the required area is given by

$$\begin{aligned} A &= 4 \times \text{Area in first quadrant} \\ &= 4 \times \frac{1}{2} \int_0^{\frac{\pi}{4}} r^2 \, d\theta \\ &= 2 \int_0^{\frac{\pi}{4}} 2a^2 \cos 2\theta \, d\theta \\ &= 2a^2. \end{aligned}$$

Remark

Again notice that in first quadrant do not take limits from 0 to $\pi/2$ as the curve becomes zero at $\pi/4$ and $\theta = \pi/4$ is tangent to the curve at pole. There is no graph between $\pi/4$ to $\pi/2$. Otherwise what happens see in next slide.

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$$
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&= 4 \times \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 \, d\theta \\
&= 2 \int_0^{\frac{\pi}{2}} 2a^2 \cos 2\theta \, d\theta \\
&= 0,
\end{aligned}
$$

which is not correct (as area cannot be zero).

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Find the area of the region inside the lemniscate $r^2 = \sin 2\theta$.

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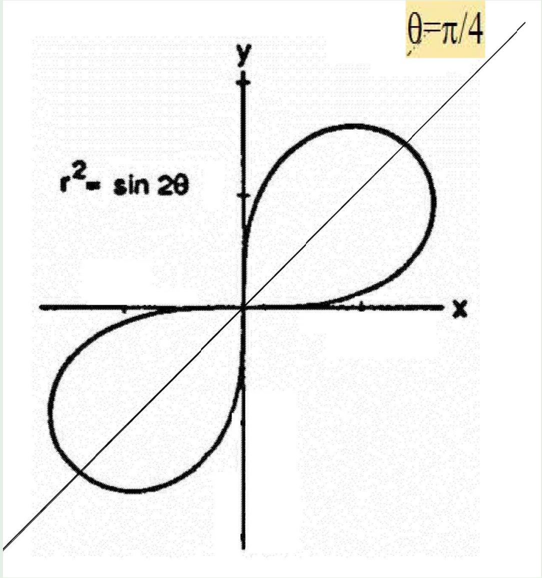
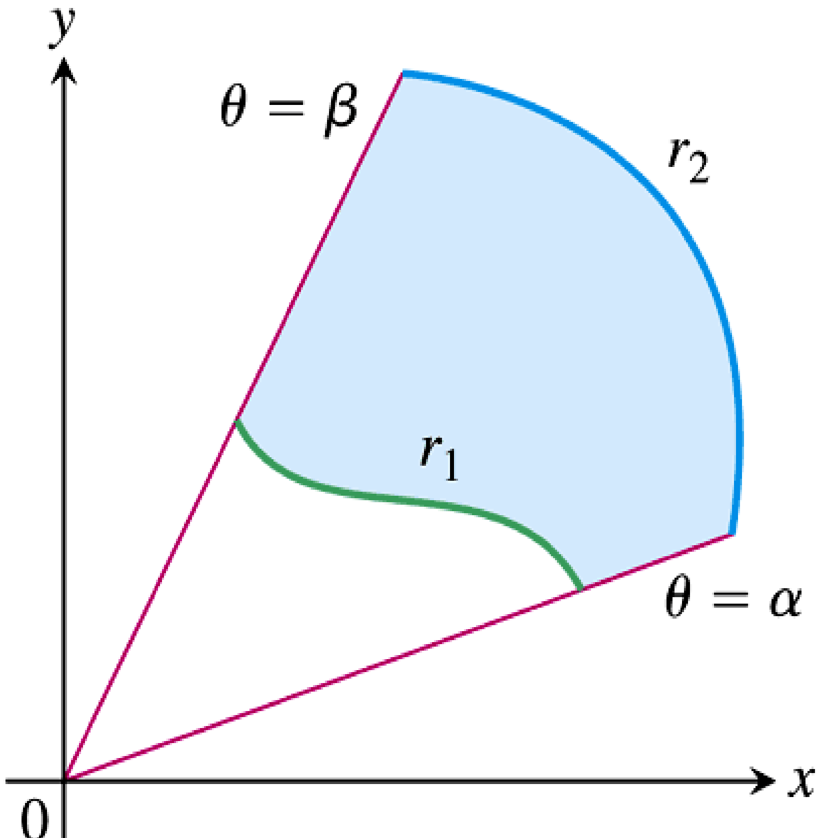


Using symmetries, the required area is given by

$$\begin{aligned} A &= 4 \times \text{Area lying between } 0 \text{ and } \pi/4 \\ &= 4 \times \frac{1}{2} \int_0^{\frac{\pi}{4}} r^2 \, d\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \sin 2\theta \; d\theta \\ &= 1. \end{aligned}$$

Area Shared by Two Curves



Area Shared by Two Curves

Area of the region $0 \leqslant r_1(\theta) \leqslant r \leqslant r_2(\theta),\ \alpha \leqslant \theta \leqslant \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_2^2 - r_1^2\right) d\theta.$$

Q:10. Find the area of the region shared by the circles $r = 1$ and $r = 2\sin\theta$.

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Sol.

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- Both the curves are symmetrical about y-axis.

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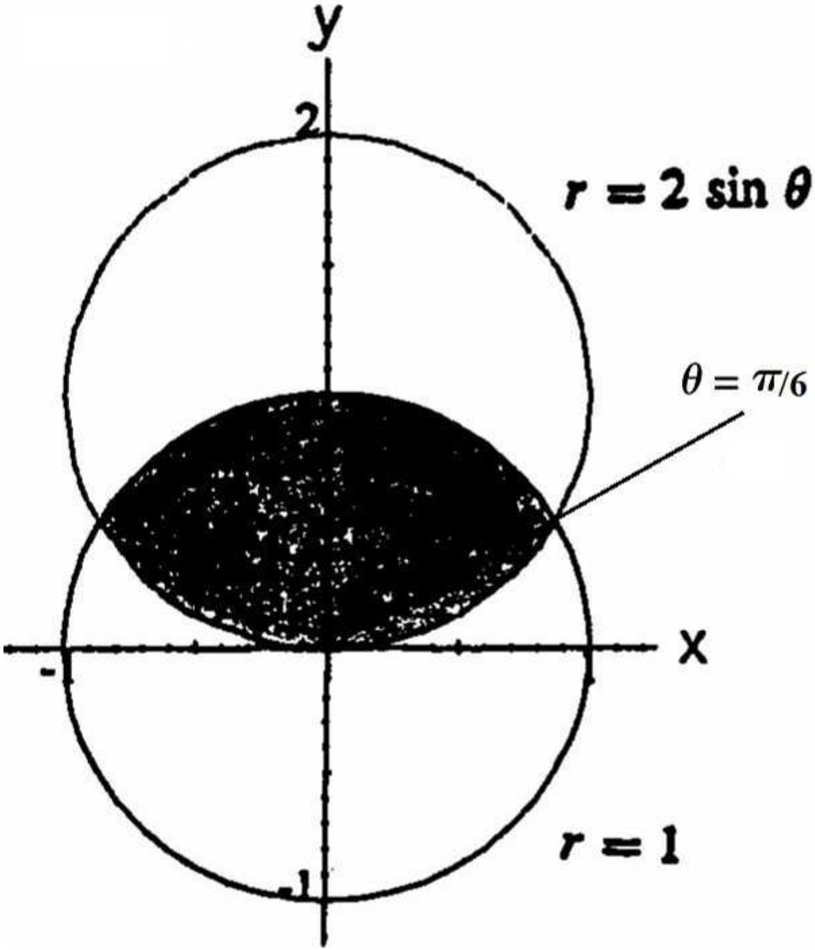
- Required region is the shaded region.


y
2
$r = 2 \sin \theta$
$\theta = \pi/6$
x
-1
$r = 1$
-1

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{6}} (2\sin\theta)^2 \, d\theta = \frac{\pi}{6} - \frac{\sqrt{3}}{4}.$$

$$A_2 = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1)^2 \, d\theta = \frac{\pi}{6}.$$

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The area in first quadrant $= A_1 + A_2 = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$.

Thus the required area $A = 2(A_1 + A_2) = 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$.

Q:14. Find the area of the region inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1+\cos\theta)$, $a > 0$.

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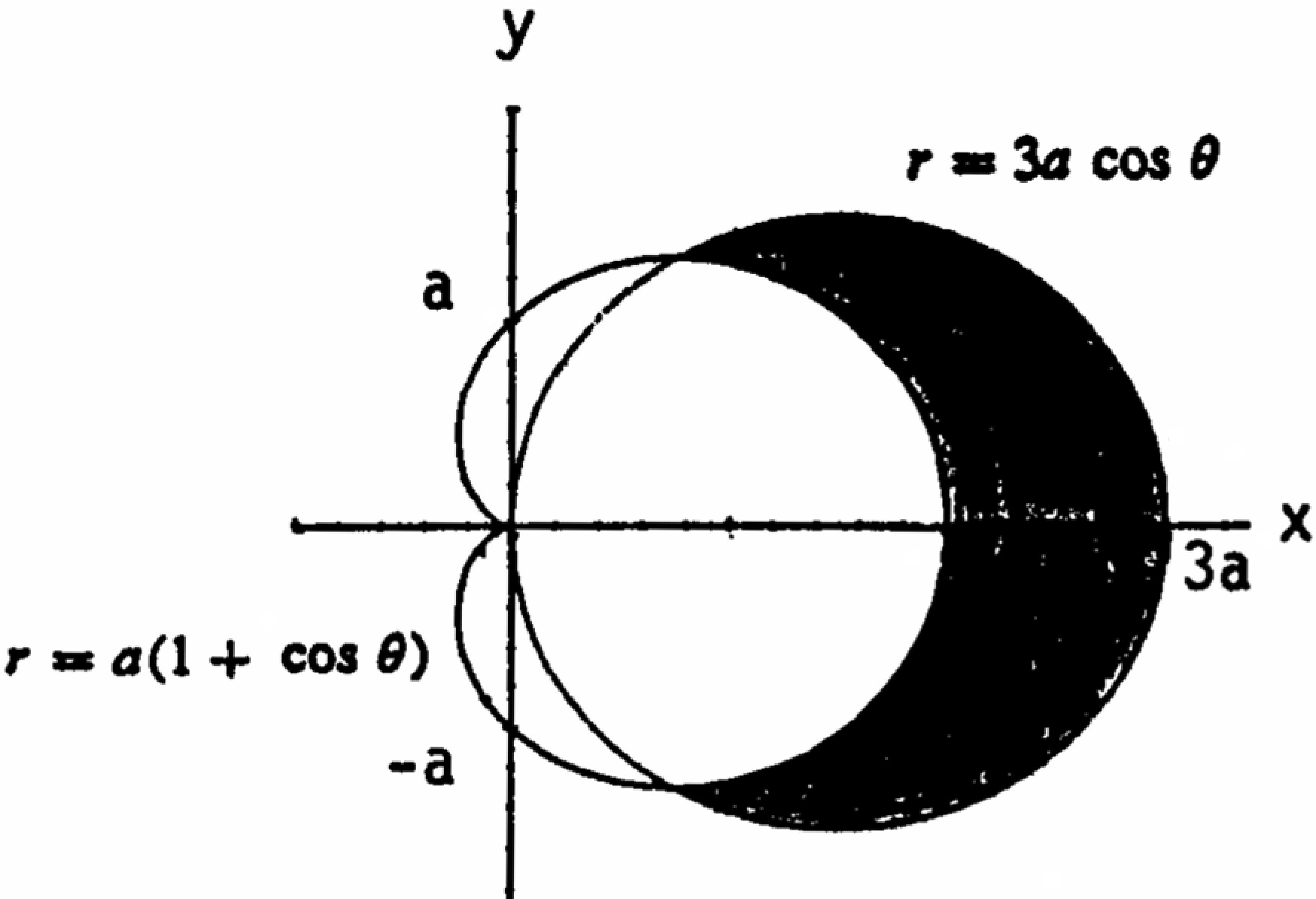
y
$r = 3a\cos\theta$
a
x
3a
$r = a(1 + \cos\theta)$
-a

$$
\begin{aligned}
A &= 2 \times \text{Area in first quadrant} \\
&= 2 \times \frac{1}{2} \int_0^{\frac{\pi}{3}} [(3a\cos\theta)^2 - a^2(1+\cos\theta)^2]\, d\theta \\
&= \pi a^2.
\end{aligned}
$$

Q:20. Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = \cos\theta$.

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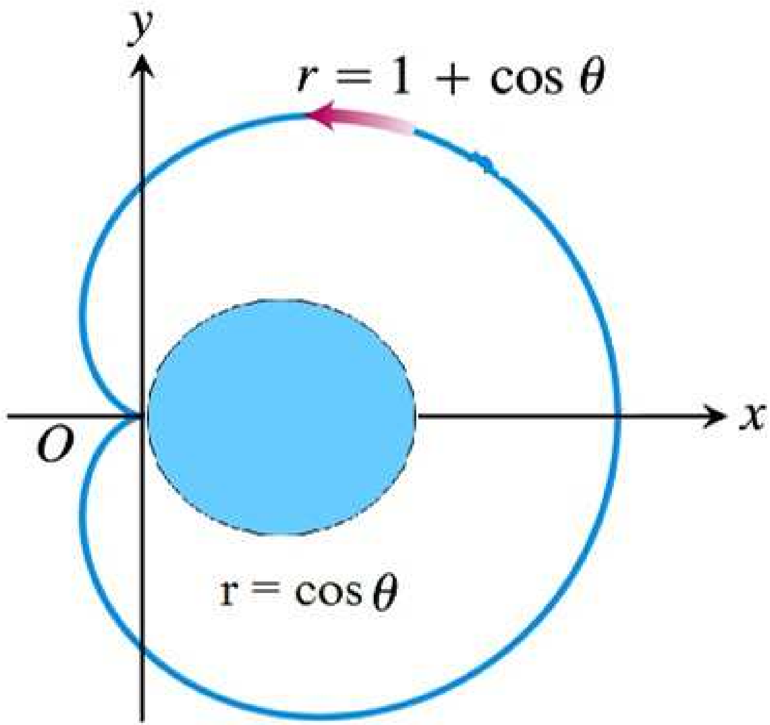
Sol.



Let A_1 and A_2 are the areas in first and second quadrant respectively. Then

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$$A_1 = \frac{1}{2}\int_0^{\pi/2} [(1+\cos\theta)^2 - (\cos\theta)^2]\, d\theta$$
$$= \frac{1}{2}\left(\frac{\pi}{2} + 2\right).$$

$$A_2 = \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2\, d\theta$$
$$= \frac{1}{2}\left(\frac{3\pi}{4} - 2\right).$$

Thus the required area

$$
\begin{aligned}
A &= 2(A_1 + A_2) \\
&= \left(\frac{\pi}{2} + 2\right) + \left(\frac{3\pi}{4} - 2\right) \\
&= \frac{5\pi}{4}.
\end{aligned}
$$

Length of a Polar Curve

Let $r = f(\theta)$ has a continuous first derivative for $\alpha \leqslant \theta \leqslant \beta$ and the point $P(r,\theta)$ traces the curve exactly once as θ varies from α to β. Then the length of the curve from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Q:22. Find the length of the spiral $r = \frac{e^{\theta}}{\sqrt{2}},\ 0 \leqslant \theta \leqslant \pi.$

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Sol. The required length is

$$L = \int_0^{\pi} \sqrt{\left(\frac{e^{\theta}}{\sqrt{2}}\right)^2 + \left(\frac{e^{\theta}}{\sqrt{2}}\right)^2}\, d\theta$$

$$= \int_0^{\pi} e^{\theta}\, d\theta$$

$$= [e^{\theta}]_0^{\pi}$$

$$= e^{\pi} - 1.$$

Q:26. Find the length of the parabolic segment

$$r = \frac{2}{1-\cos\theta}, \ \pi/2 \leqslant \theta \leqslant \pi.$$

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Sol. Better to convert $r = \frac{2}{1-\cos\theta} = \csc^2\frac{\theta}{2}$. Thus the required length is

Q:26. Find the length of the parabolic segment

$$r = \frac{2}{1-\cos\theta}, \quad \pi/2 \leqslant \theta \leqslant \pi.$$

Sol. Better to convert $r = \frac{2}{1-\cos\theta} = \csc^2\frac{\theta}{2}$. Thus the required length is

$$L = \int_{\frac{\pi}{2}}^{\pi} \sqrt{\left(\csc^2\frac{\theta}{2}\right)^2 + \left(-\csc^2\frac{\theta}{2}\cot\frac{\theta}{2}\right)^2}\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \csc^2\frac{\theta}{2}\sqrt{1+\cot^2\frac{\theta}{2}}\, d\theta$$

Put $\cot\frac{\theta}{2} = t$ so that $\csc^2\frac{\theta}{2}\,d\theta = -2dt$ and so

$$\begin{aligned} L &= 2\int_0^1 \sqrt{1+t^2}\,dt \\ &= 2\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\right]_0^1 \\ &= \sqrt{2} + \ln(1+\sqrt{2}). \end{aligned}$$

Example

Find the length of the part of cardioid $r = 1 + \cos\theta$ that lies outside the circle $r = \sin\theta$.

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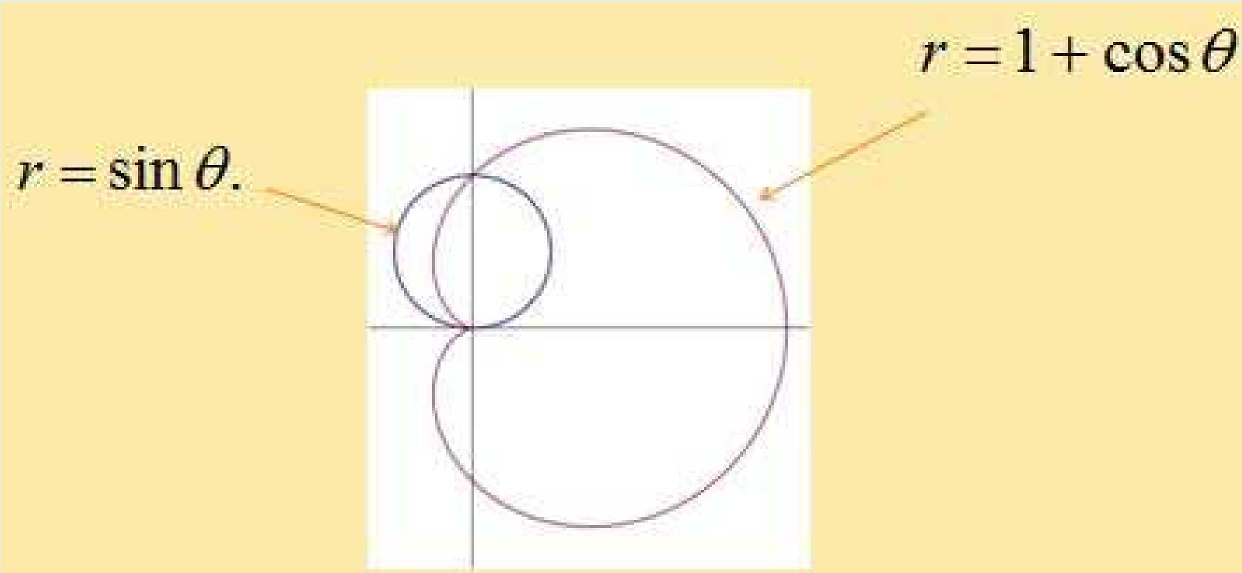
Find the length of the part of cardioid $r = 1 + \cos\theta$ that lies outside the circle $r = \sin\theta$.

Sol. The points of intersection are $(0,0)$ and $(1, \frac{\pi}{2})$.



The required length is the length of the cardioid from $-\pi$ to $\frac{\pi}{2}$. Thus

The required length is the length of the cardioid from $-\pi$ to $\frac{\pi}{2}$. Thus

$$\begin{aligned}
L &= \int_{-\pi}^{\frac{\pi}{2}} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \\
&= \sqrt{2} \int_{-\pi}^{\frac{\pi}{2}} \sqrt{1+\cos\theta}\, d\theta \\
&= 2 \int_{-\pi}^{\frac{\pi}{2}} \left|\cos\frac{\theta}{2}\right| d\theta \\
&= 2 \int_{-\pi}^{\frac{\pi}{2}} \cos\frac{\theta}{2}\, d\theta \\
&= 2\sqrt{2}(\sqrt{2}+1).
\end{aligned}$$