Oscillations and Waves

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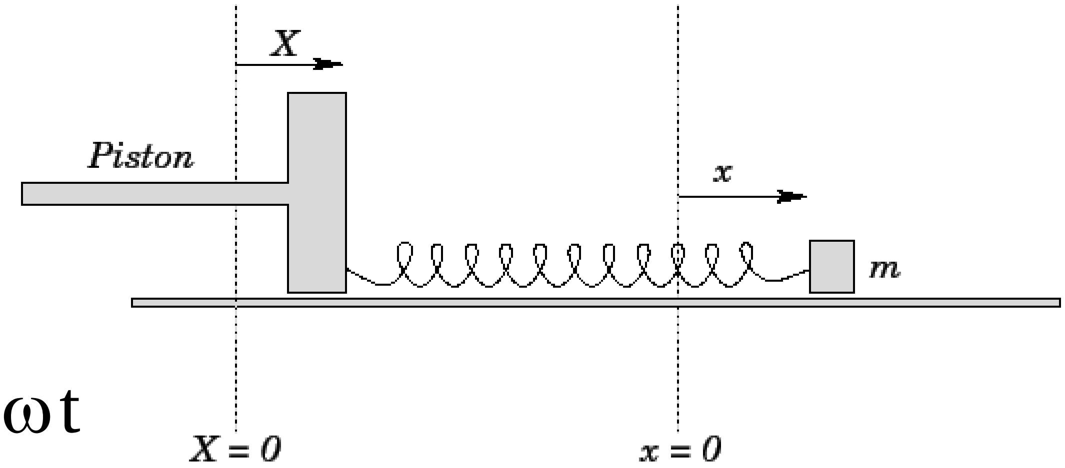
Forced/Driven Oscillations

An additional externally applied harmonic force acts on the oscillator

(i) Without Damping:

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos\omega t$$



Or, $$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

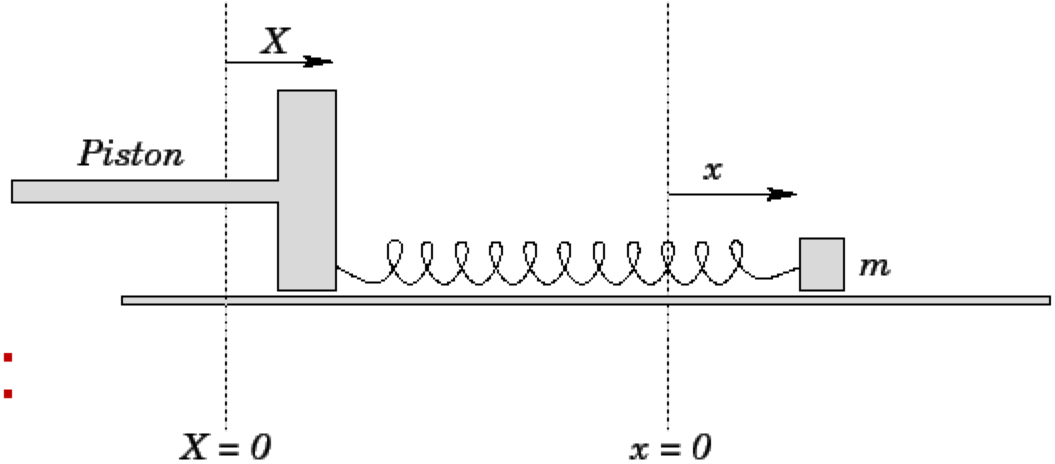
ω_0 : Natural angular frequency

ω : Angular frequency of driving force

Example 1. Spring-mass system with oscillating 'fixed' point

$$X = A\cos\omega t$$

$$F = -k(x - X)$$



Equation of motion :

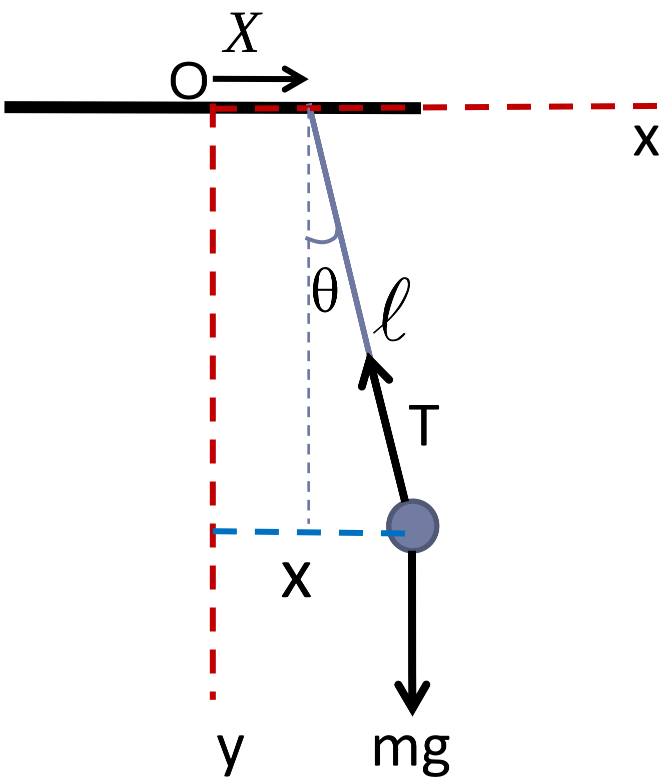
$$\frac{d^2x}{dt^2} = -\omega_0^2(x - A\cos\omega t)$$

Or, $$\frac{d^2x}{dt^2} + \omega_0^2 x = \omega_0^2 A\cos\omega t$$

Or, $$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \qquad \left(F_0 = m\omega_0^2 A\right)$$

Example 2 Pendulum With Oscillating Point of Suspension

$$X = A\cos\omega t$$



Balancing forces along y,

$$mg = T\cos\theta \approx T$$

The restoring force is :

$$F_{rest} = -T\sin\theta = -mg\frac{(x-X)}{\ell}$$

Equation of motion :

$$m\frac{d^2x}{dt^2} = -\frac{mg}{\ell}\left(x - A\cos\omega t\right)$$

Or,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \omega_0{}^2 A\cos\omega t$$

Or,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \qquad \left(F_0 = m\omega_0{}^2 A\right)$$

Solving the Equation of Motion

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

The above is a linear, inhomogeneous differential equation.

The homogeneous part of the equation is :

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

A particular solution of the equation of the driven oscillator :

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Complex form : $$\frac{d^2z}{dt^2} + \omega_0^2 z = \frac{F_0}{m} e^{j\omega t}$$

Obvious guess : $$z(t) = Ae^{j\omega t}$$

Putting this into the equation :

$$A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$$

Or,

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$\therefore \quad x_{p.s} = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = A(\omega) \cos \omega t$$

Most general solution for the driven oscillator :

$$x(t) = A\cos(\omega_0 t + \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

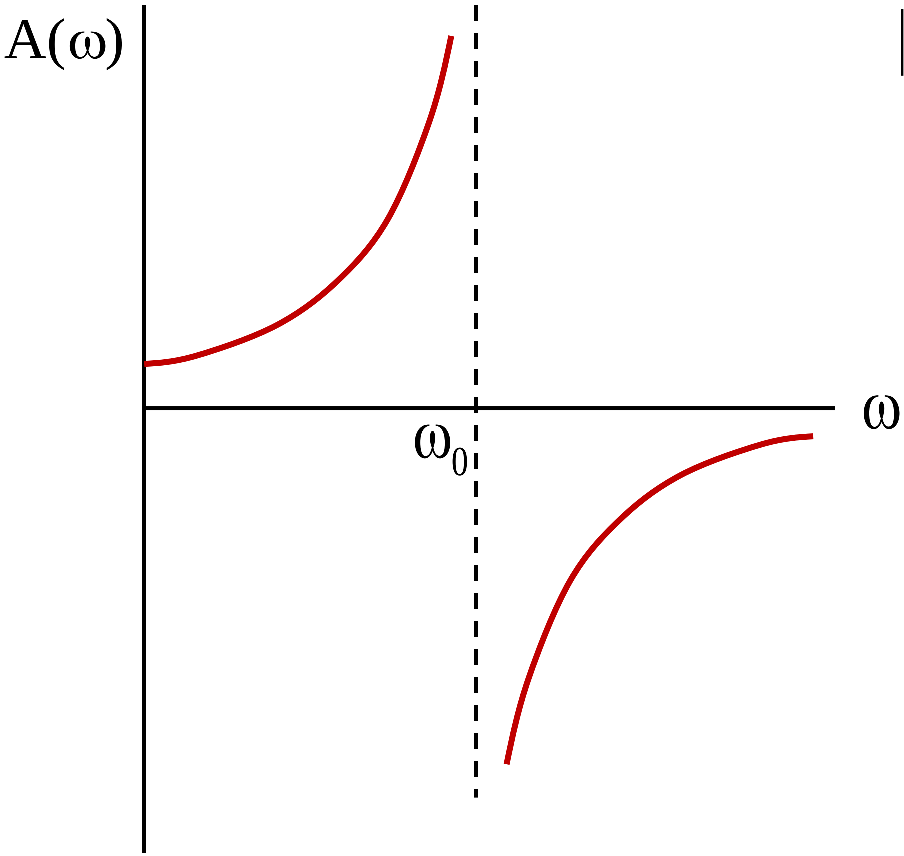
Resonance

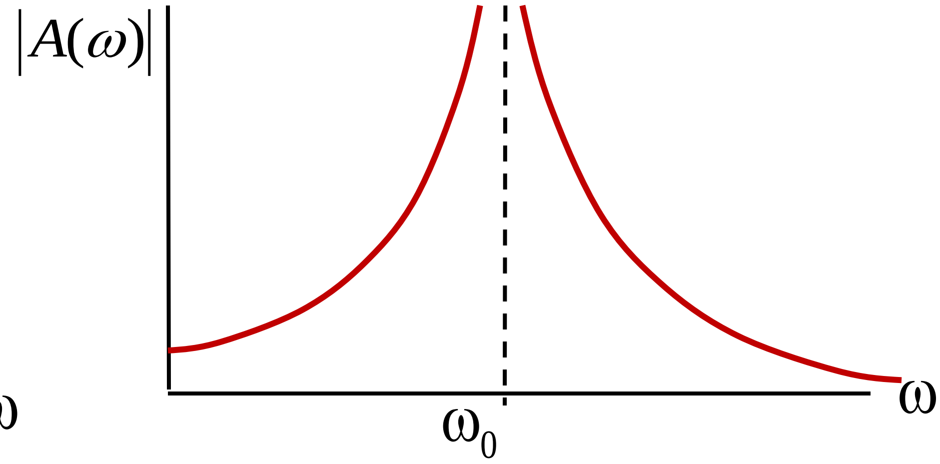
The amplitude of the oscillations :

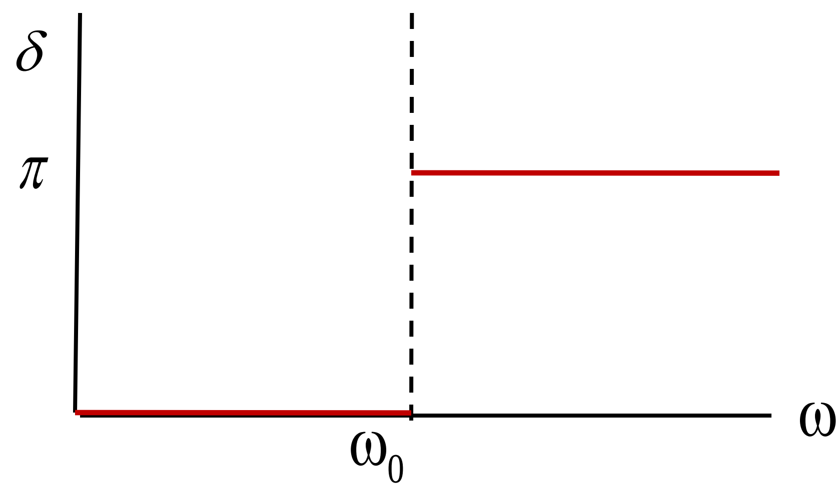
$$A(\omega) = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

varies with the driving frequency ω

There is a divergence of the amplitude at the matching of natural frequency with the forcing frequency.







δ is phase difference between F and x

(ii) Forced Oscillations with Damping

Equation of Motion :

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

To obtain the particular solution, take the complex form :

$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m}e^{j\omega t}$$

To obtain a particular solution, the obvious choice is :

$$z_{p.s}(t) = Ae^{j(\omega t - \delta)}$$

Substitution and subsequent solution for A gives :

$$A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

$$\delta = \tan^{-1}\left(\frac{\gamma\,\omega}{\omega_0^2 - \omega^2}\right)$$

$$z_{p.s} = \frac{F_0 \, e^{j(\omega t - \delta)}}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}$$

$$\therefore \quad x_{p.s} = \mathrm{Re}(z_{p.s})$$

$$\boxed{\mathrm{x}_{\mathrm{p.s.}} = \frac{F_0 \cos(\omega t - \delta)}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}}$$

$$\boxed{= \mathrm{A}(\omega)\cos(\omega \mathrm{t} - \delta)}$$

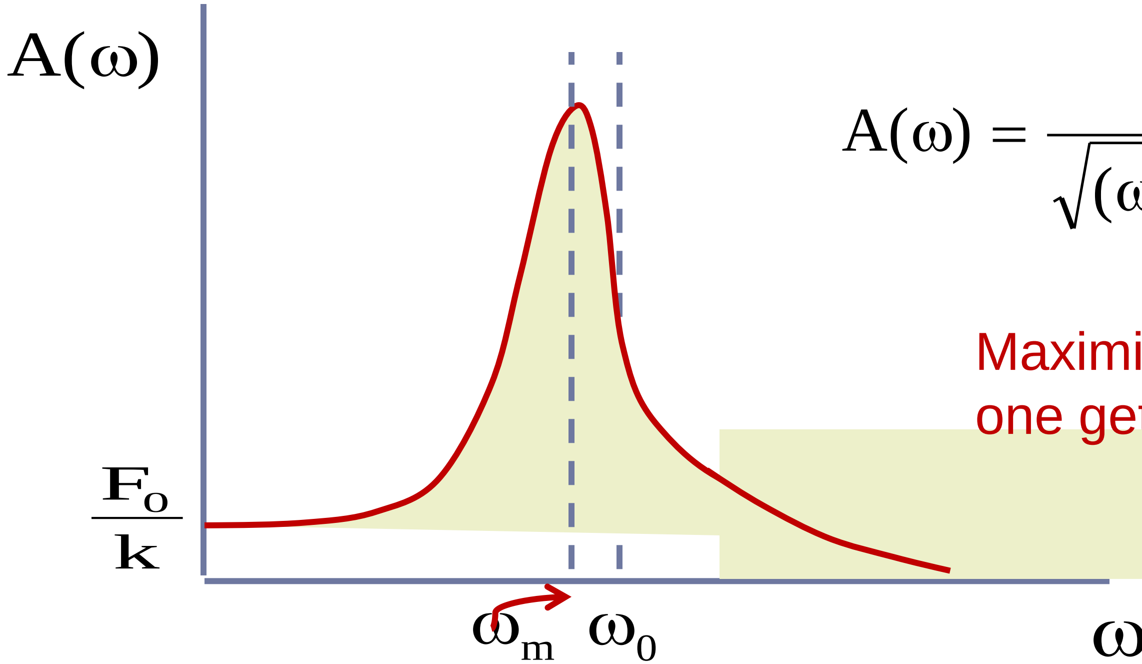
Most General Solution :

$$x(t) = B e^{-\frac{\gamma}{2}t} \cos(\omega' t + \phi) + A(\omega)\cos(\omega t - \delta)$$

Transient

Steady State

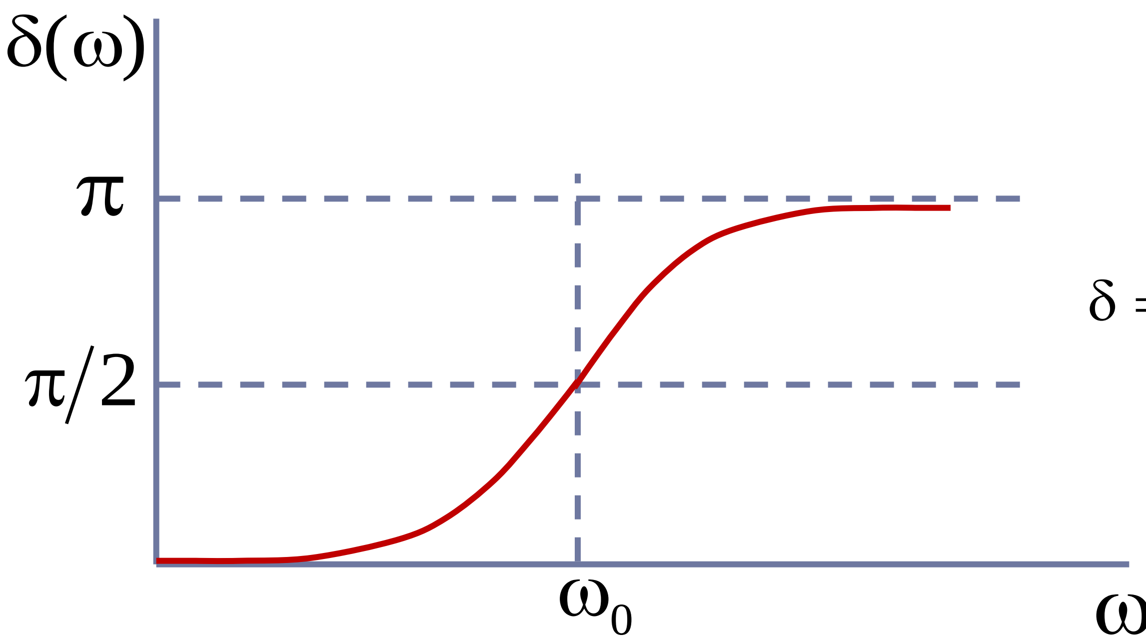
The transient part of the solution dies out after about Q oscillations, and after that the steady state oscillations go on unabated



$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

Maximizing A w. r. t. ω one gets

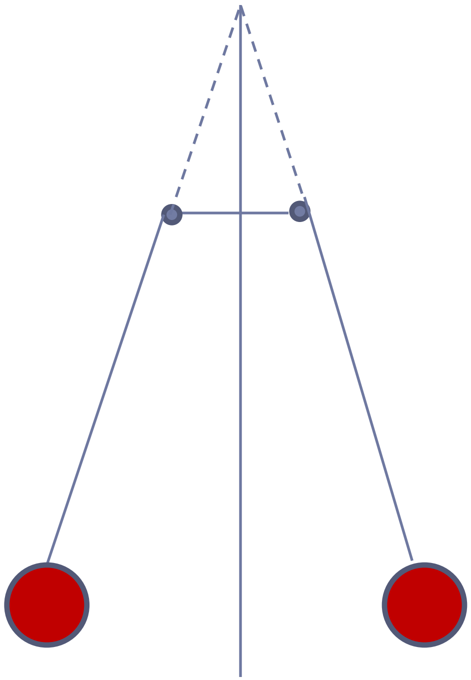
$$\omega_m = \left[\omega_0^{\,2} - \frac{\gamma^2}{2}\right]^{1/2}$$



$$\delta = \tan^{-1}\left(\frac{\gamma\,\omega}{\omega_0^2 - \omega^2}\right)$$

- In the presence of damping, the denominator does not vanish hence at resonance the maximum amplitude is finite and does not diverge.

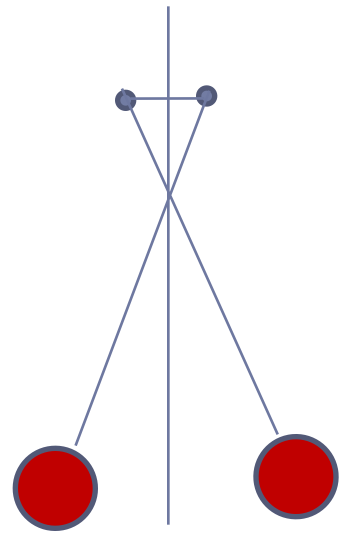
- The amplitude resonance does not occur when the natural frequency matches exactly with the driving frequency.



$\omega << \omega_0$

Point of suspension and bob in phase

Greater than l



$\omega >> \omega_0$

Point of suspension and bob out of phase

Shorter than l

Resonance in the presence of damping

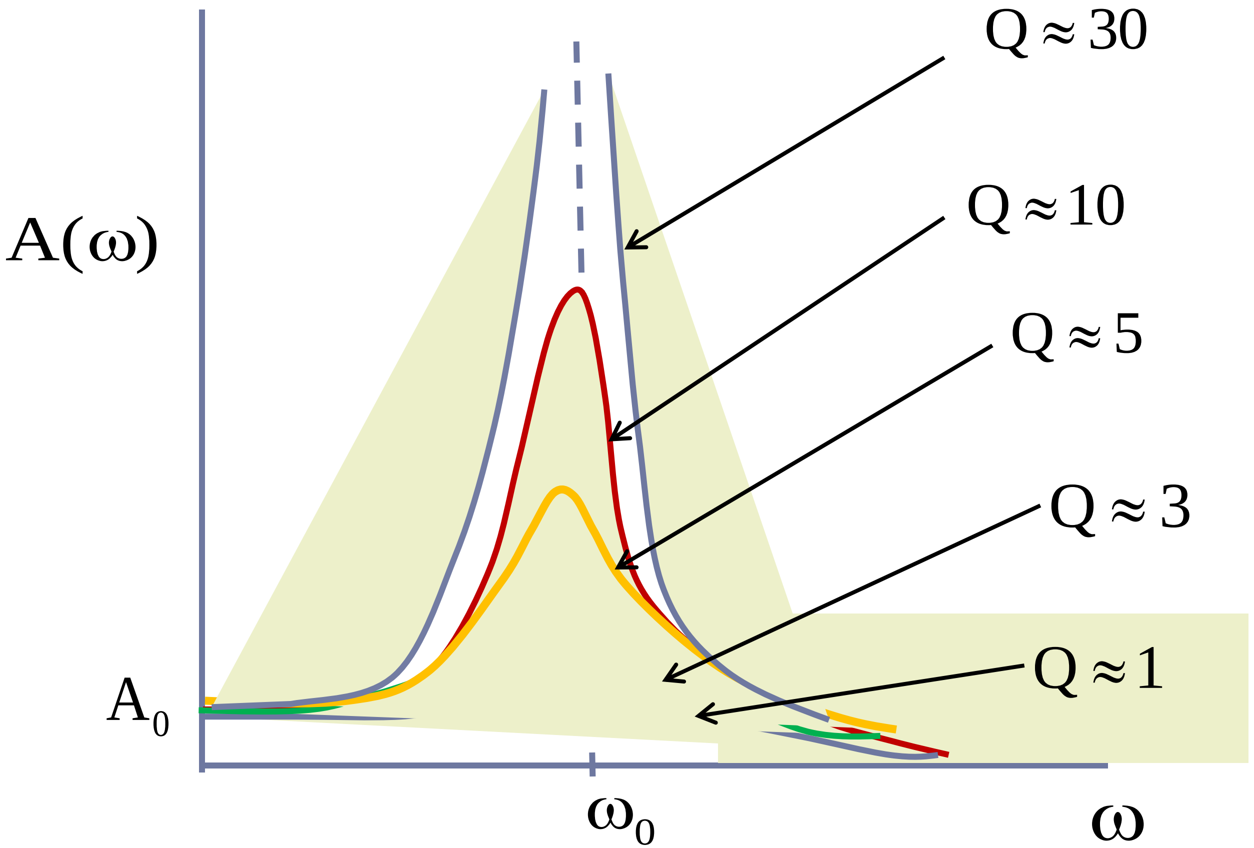
Assuming Q to be reasonably large :

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \frac{(\omega\omega_0)^2}{Q^2}}}$$

$$= \frac{A_0}{\frac{\omega}{\omega_0}\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}} \qquad (A_0 = A(0) = F_0/m\omega_0^2)$$

$$\omega_m = \omega_0\left[1 - \frac{1}{2Q^2}\right]^{1/2} \approx \omega_0\left(1 - \frac{1}{4Q^2}\right)$$

$$A_m = A(\omega_m) = \frac{A_0 Q}{\left(1 - \frac{1}{4Q^2}\right)^{1/2}}$$



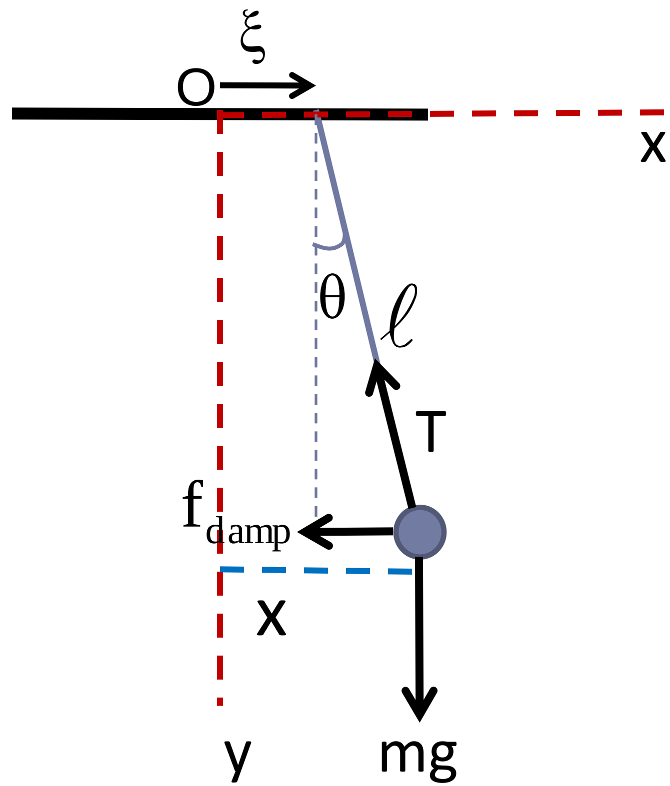
Amplitude for increasing quality

Prob. 4.5 **A simple pendulum has a length of 1 m. In free vibration the amplitude falls off by a factor e in 50 swings. The pendulum is set into forced vibration by moving its point of suspension horizontally in SHM with an amplitude of 1 mm.**

a) Show that if the horizontal displacement of the bob is $\mathbf{x}$ and the horizontal displacement of its point of suspension is ξ the equation of motion of the pendulum is :

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \frac{g}{\ell}x = \frac{g}{\ell}\xi$$

Answer :



In the x-y (inertial) frame, the Eq. of motion is :

$$m\frac{d^2x}{dt^2} = -m\frac{g}{\ell}(x - \xi) - b\frac{dx}{dt}$$

Or, $$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \frac{g}{\ell} x = \frac{g}{\ell} \xi$$

Or, $$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \omega_0^2 \xi_0 \cos \omega t$$

Or, $$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t \qquad F_0 = m \omega_0^2 \xi_0$$

b) **At exact resonance, what is the amplitude of motion of the bob of the pendulum**

$$A_m = A(\omega_m) = \frac{A_0 Q}{\left(1 - \frac{1}{4Q^2}\right)^{1/2}}$$

After n oscillations, the amplitude drops by a factor :

$$e^{-n\pi/Q}$$

$$\therefore \quad 50\pi/Q = 1 \Rightarrow Q = 50\pi$$

$$A_0 = \frac{F_0}{m\omega_0^2} = \xi_0 = 1\ \text{mm}$$

$$\Rightarrow A_m \approx A_0 Q = 50\pi\ mm = 15.7\ cm$$

c) At what angular frequency, is the amplitude half its resonance value?

$$A(\omega) = \frac{A_0}{\frac{\omega}{\omega_0}\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}} = \frac{A_0 Q}{2}$$

Putting $\frac{\omega}{\omega_0} = x,$ the equation to be solved :

$$\frac{4}{Q^2} = x^2\left[\left(\frac{1}{x} - x\right)^2 + \frac{1}{Q^2}\right] = (1 - x^2)^2 + \frac{x^2}{Q^2}$$

Since x is expected to be extremely close to 1, put : $x = 1 + \alpha$

$$\therefore\ 1 - x^2 \approx -2\alpha$$

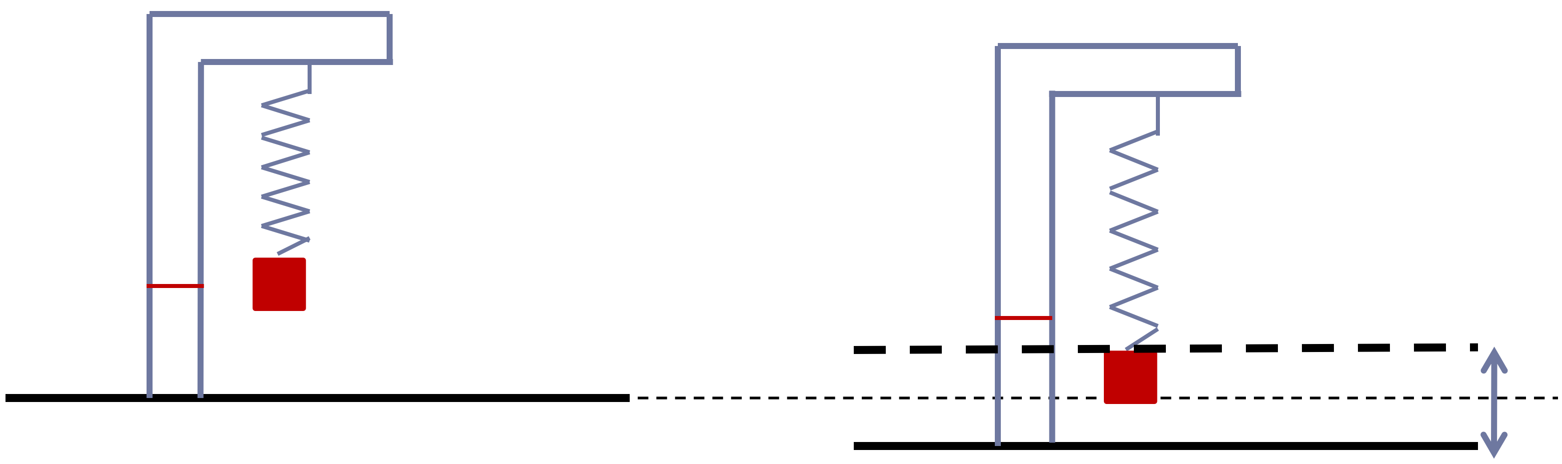
$$\therefore\ \frac{4}{Q^2} = 4\alpha^2 + \frac{1}{Q^2}$$

Or, $\alpha = \pm\frac{\sqrt{3}}{2Q} = \pm\frac{\sqrt{3}}{100\pi} = \pm 5.5 \times 10^{-3}$

$$\therefore\ \omega = \omega_0\left(1 \pm 0.0055\right) \quad ; \quad \omega_0 = \sqrt{10} = 3.16\ s^{-1}$$

Prob. 4.6. Simple Seismograph as in figure below.

It consists of a mass m hung from a spring on a rigid framework attached to the earth. The spring force and damping force depend on displacement and velocity relative to the earth's surface, but the dynamically significant acceleration is acceleration of m relative to the fixed stars.

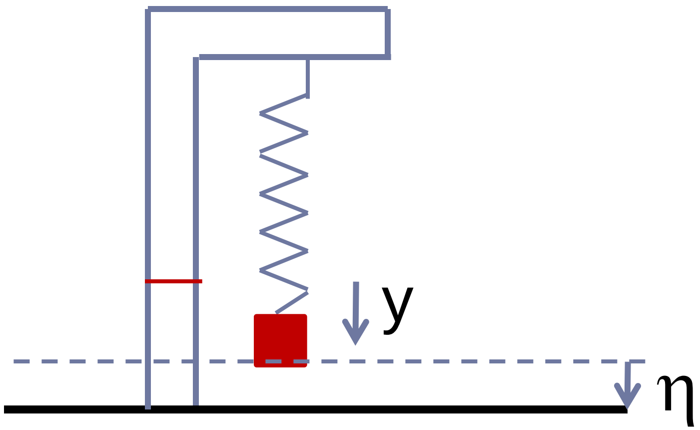


Equilibrium

The earth shaking under a tremor

a) Show that the equation of motion is :

$$\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + \omega_0^2 y = -\frac{d^2\eta}{dt^2}$$



y is displacement of m relative to earth and η is displacement of earth's surface itself.

Ans :

Since y is defined w.r.t. the earth's frame, which is non-inertial, the forces are as shown.

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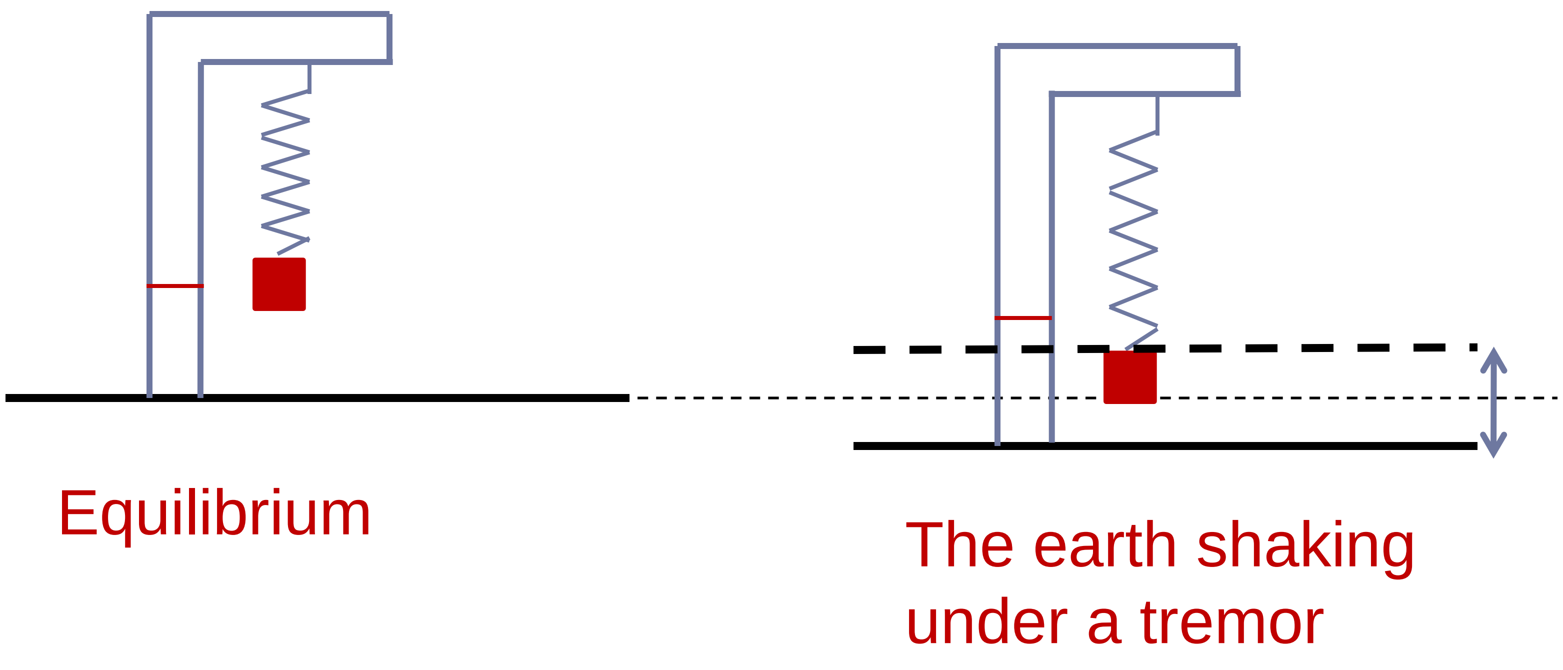

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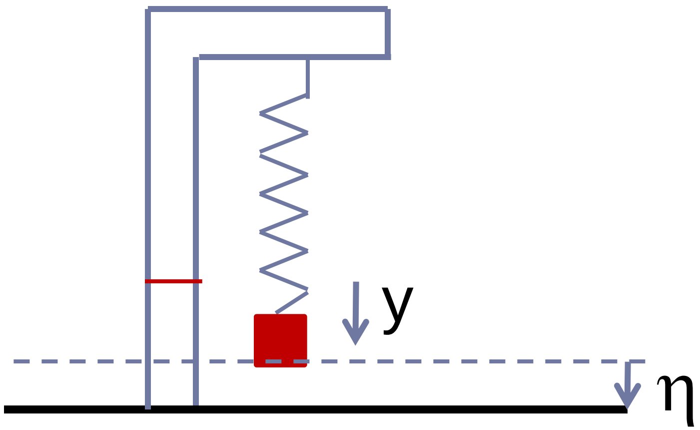

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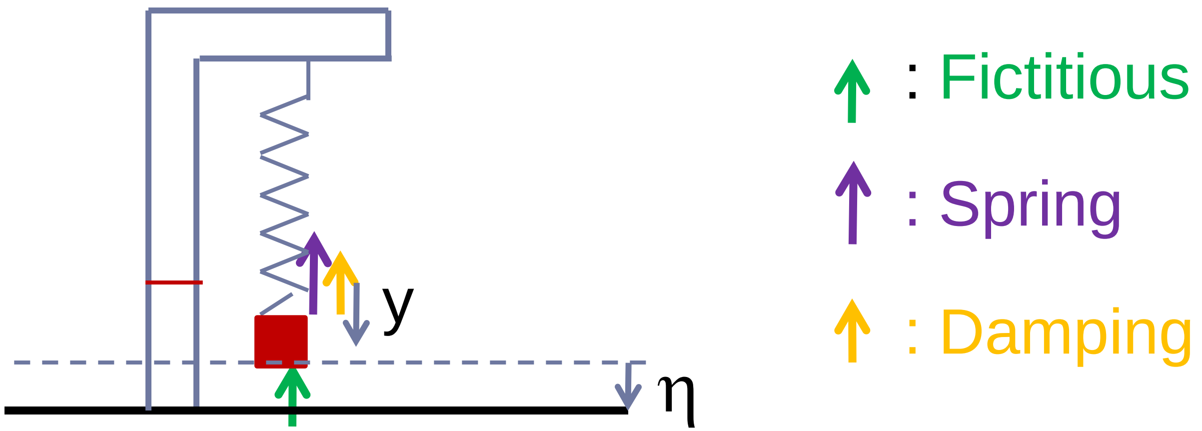


$$F_{fict} = -m\frac{d^2\eta}{dt^2} \quad ; \quad F_{spring} = -ky \quad ; \quad F_{damp} = -b\frac{dy}{dt}$$

Eq. of Motion :

$$m\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - ky - m\frac{d^2\eta}{dt^2}$$

b) Solve for y if $\eta = C\cos\omega t$

Dividing out by m and putting $\eta = C\cos\omega t$

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = C\omega^2\cos\omega t$$

$$\therefore \qquad F_0 = mC\omega^2 \qquad A_0 = \frac{C\omega^2}{\omega_0^2}$$

$$\boxed{y = \frac{F_0\cos(\omega t - \delta)}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}}$$

c) Plot a graph of amplitude versus driving frequency.

d) A typical long period seismometer has a period of about 30 sec. and quality of 2. As a result of earthquake the earth's surface may oscillate with a period of 20 min. and with an amplitude such that the maximum acceleration is about $10^{-9}\,\mathrm{m}-\mathrm{s}^{-2}$. How small a value of the displacement of the block must be observable, if the quake is to be detected.

$$\frac{\omega}{\omega_0} = \frac{30\text{ s}}{20\,\text{min}} = 0.025$$

$$a_{\text{max}} = C\,\omega^2$$

$$\therefore\ \ A_0 = \frac{a_{\text{max}}}{\omega_0^2} \approx 2.25 \times 10^{-8}\text{ m}$$

$$A(\omega) = \frac{A_0}{\frac{\omega}{\omega_0}\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}}$$

$$\boxed{A(\omega) \approx 22\ nm}$$

Transient Phenomena

In a driven oscillator, the motion in the beginning is not quite simple harmonic. This part of the motion is called the transients. Afterwards, the motion settles to a SHM of a frequency, that is equal to the driving frequency.

Complete motion :

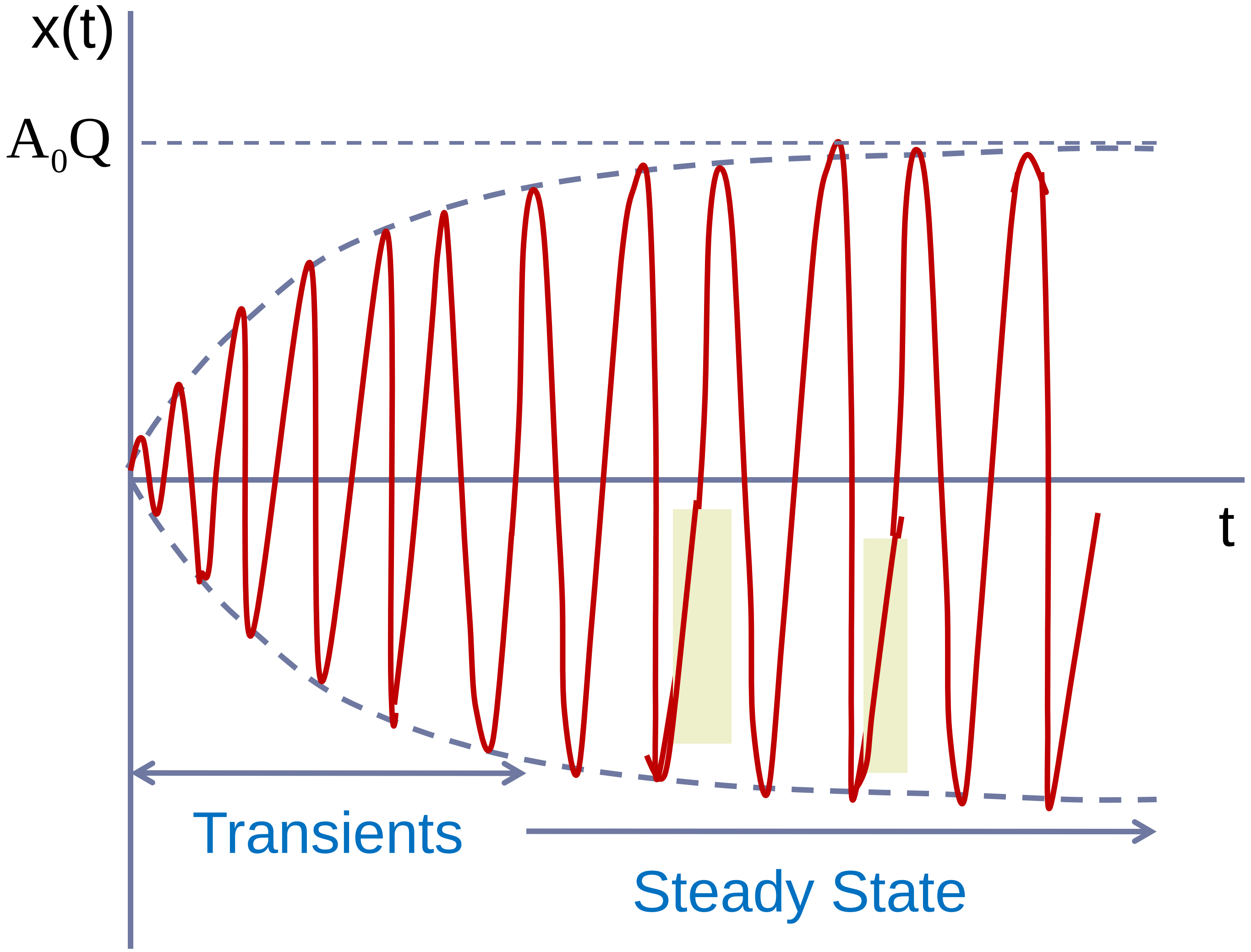
$$x(t) = Be^{-\frac{\gamma}{2}t}\cos(\omega' t + \phi) + A(\omega)\cos(\omega t - \delta)$$

At resonance, $\omega \approx \omega_0$, $\delta = \pi/2$, $A = A_0 Q$

$$\therefore\ x(t) = B e^{-\frac{\gamma}{2}t} \cos(\omega_0 t + \phi) + A_0 Q \sin \omega_0 t$$

With initial conditions : $x(0) = \dot{x}(0) = 0$

$$x(t) = A_0 Q \left(1 - e^{-\frac{\gamma}{2}t} \right) \sin \omega_0 t$$


x(t)
A_0Q
t
Transients
Steady State

Power Input to a Driven Oscillator in the Steady State

Instantaneous power input to the oscillator by the driving force :

$$P = Fv$$

$$F = F_0 \cos \omega t$$

$$v = \frac{dx}{dt} = -A(\omega)\,\omega \sin(\omega t - \delta)$$

$$= -v_0 \sin(\omega t - \delta)$$

$$v_0 = \omega A(\omega) = \frac{A_0\omega_0}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}}$$

Resonance for velocity amplitude occurs exactly at the natural frequency

$$P(t) = -F_0 v_0 \cos\omega t \sin(\omega t - \delta)$$

$$= -F_0 v_0 (\cos\delta \cos\omega t \sin\omega t - \sin\delta \cos^2\omega t)$$

$$\overline{P}(\omega) = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{2}F_0 v_0 \sin\delta$$

Power Input to a Driven Oscillator in the Steady State

Instantaneous power input to the oscillator by the driving force :

$$P = F v$$

$$F = F_0 \cos \omega t$$

$$v = \frac{dx}{dt} = -A(\omega)\,\omega \sin(\omega t - \delta)$$

$$= -v_0 \sin(\omega t - \delta)$$

$$v_0 = \omega A(\omega) = \frac{A_0\omega_0}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}}$$

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$$P(t) = -F_0 v_0 \cos\omega t \sin(\omega t - \delta)$$

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$$\overline{P}(\omega) = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{2} F_0 v_0 \sin\delta$$

Prob. 4.10 The power required to maintain forced vibration must be equal to the power loss due to damping.

a) Find the instantaneous rate of doing work against the damping force.

Ans

$$\frac{dW}{dt} = \;=\; -F_{damp}v$$

$$= \mathrm{b}\,\mathrm{v}^2$$

$$= \mathrm{b}\omega^2\mathrm{A}^2\sin^2(\omega\mathrm{t}-\delta)$$

b) Find the mean rate of doing work against damping

Ans :

$$\frac{\overline{d\overline{W}}}{dt} = b\omega^2 A^2 \frac{1}{T}\int_0^T \sin^2(\omega t - \delta)\,dt$$

$$= \frac{1}{2} b\omega^2 A^2$$

c) Substitute the value of A at any arbitrary frequency and hence obtain the expression for average P.

Ans: $$\overline{P}_{drive} = \frac{1}{2}F_0 v_0 \sin\delta = \frac{1}{2}F_0 \omega A \sin\delta$$

Since $$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

$$\sin\delta = \frac{b\omega}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

$$= \frac{b\omega A}{F_0}$$

$$\therefore\ \overline{P}_{drive} = \frac{1}{2}b\omega^2 A^2 \qquad = \frac{bF_0^2/(2m^2\omega_0^2)}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]}$$

Power Resonance Curve

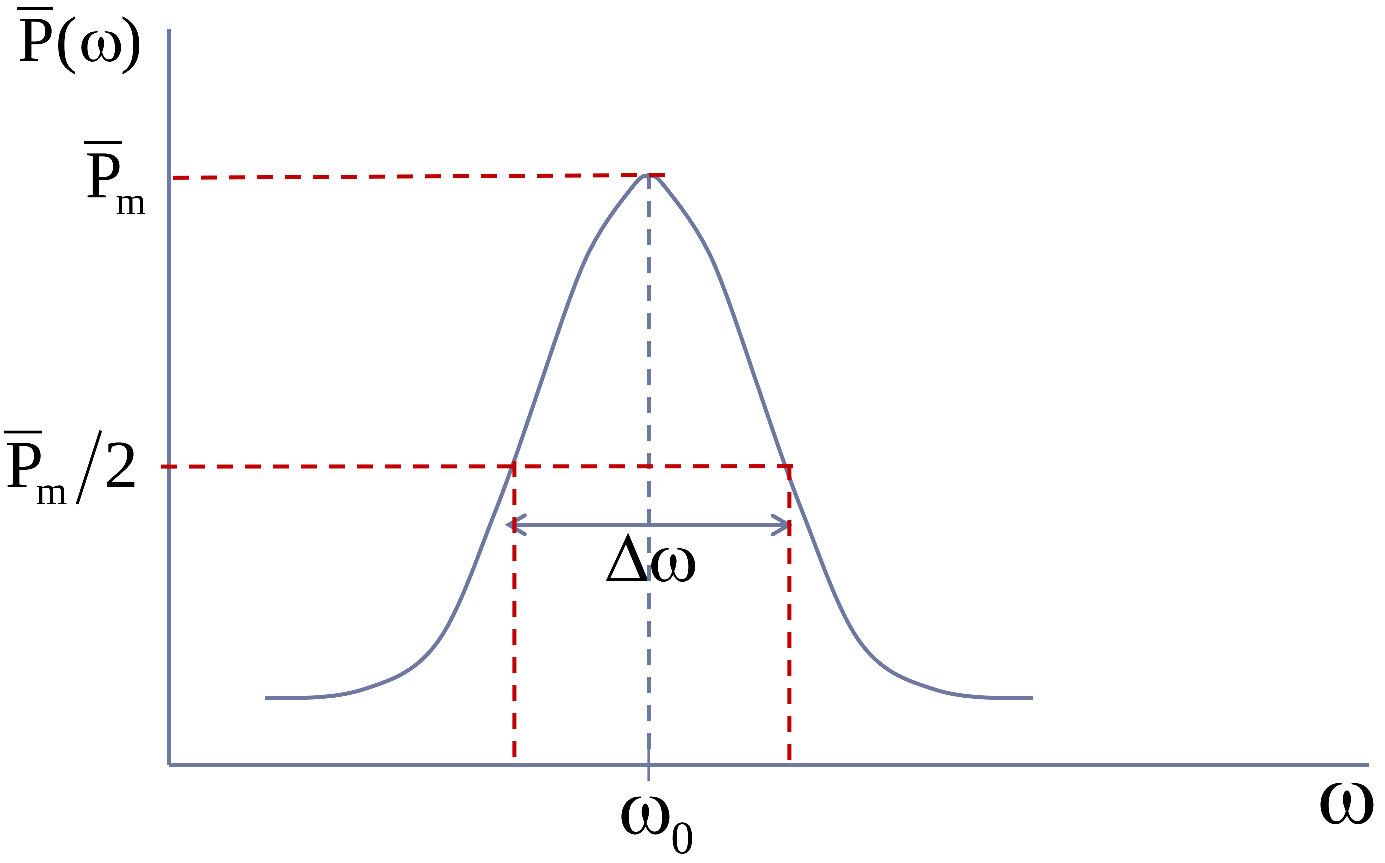
$$\overline{P}(\omega) = \frac{1}{2} b\,\omega^2 A^2 = \frac{bF_0^2 \big/ (2m^2\omega_0^2)}{\left[\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)^2 + \dfrac{1}{Q^2}\right]}$$

$\overline{P}$ is maximum at $\omega = \omega_0$

$$\boxed{\Rightarrow \bar{P}_m = \frac{bF_0^2\, Q^2}{2m^2\omega_0^{\,2}} = \frac{F_0^2}{2b}}$$

$$\therefore \overline{P}(\omega) = \frac{\overline{P}_m}{Q^2} \frac{1}{\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)^2 + \dfrac{1}{Q^2}}$$

Width of Power Resonance Curve (Full Width at Half Maximum (FWHM))



$\Delta\omega$: FWHM

Finding FWHM

Equating $\bar{P}(\omega)$ to $\dfrac{\bar{P}_m}{2}$

$$\frac{1}{2} = \frac{1}{Q^2} \frac{1}{\left(\dfrac{\omega_0}{\omega} - \dfrac{\omega}{\omega_0}\right)^2 + \dfrac{1}{Q^2}}$$

$$\Rightarrow \quad Q^2 \left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 = 1$$

$$\Rightarrow \quad \left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right) = \pm \frac{1}{Q}$$

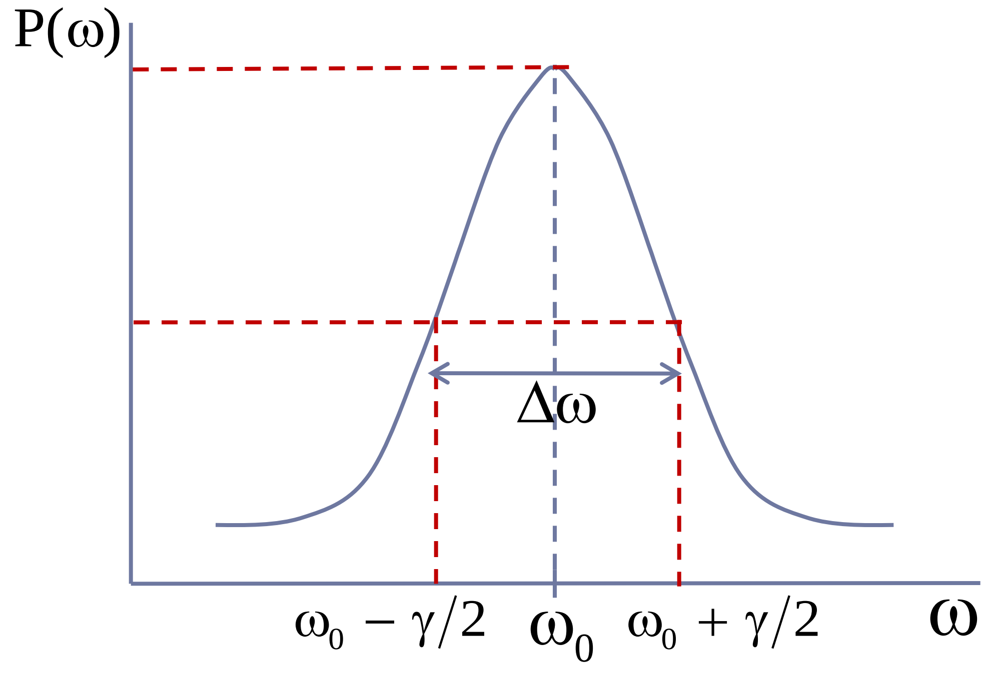
Putting $\dfrac{\omega}{\omega_0} = 1 + \alpha$, where $\alpha << 1$

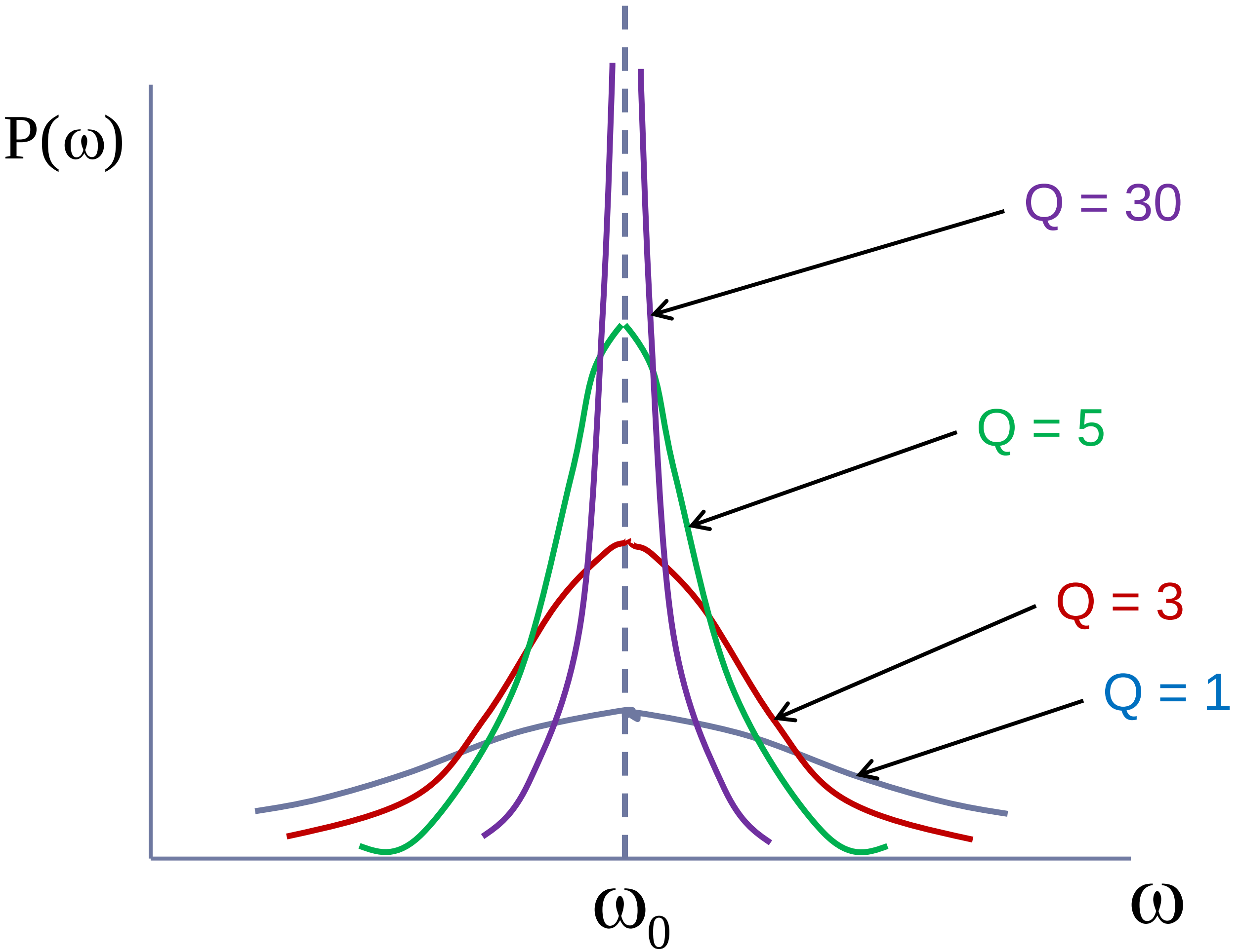
$$\frac{1}{1+\alpha} - (1+\alpha) = \pm\frac{1}{Q} \quad \Rightarrow \quad \alpha = \pm\frac{1}{2Q}$$

$$\frac{\omega}{\omega_0} = 1 \pm \frac{1}{2Q}$$

$$\therefore \quad \Delta\omega = \frac{\omega_0}{Q} = \gamma$$

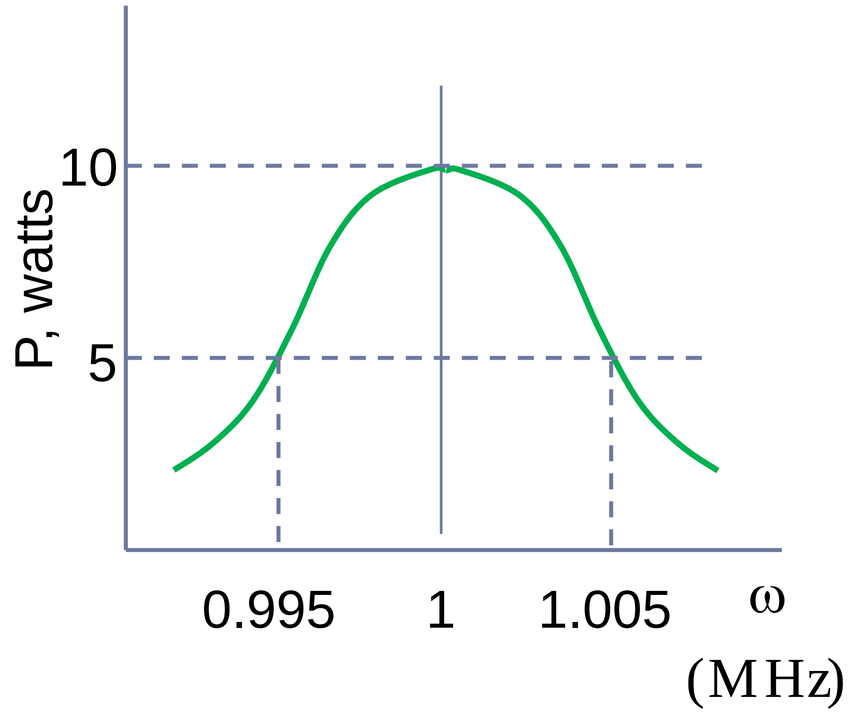
$$\boxed{\therefore \quad Q = \frac{\omega_0}{\Delta\omega}}$$





P(ω)
Q = 30
Q = 5
Q = 3
Q = 1
ω₀
ω

Prob. 4.17 The graph shows the mean power absorbed by an oscillator when driven by a force of constant magnitude but variable frequency.



a) At exact resonance, how much work per cycle is being done against the resistive force?

$$\omega_0 = 10^6\ \mathrm{s}^{-1} \quad \Rightarrow \; \mathrm{T}_0 = 2\pi \times 10^{-6}\,\mathrm{s}$$

$$\therefore \; Work/cycle = P_{\max} T_0 \; \boxed{= 2\pi \times 10^{-5}\ J}$$

b) At exact resonance, what is the total mechanical energy E_0 of the oscillator?

$$\overline{P}_m = \frac{1}{2} b\,\omega_0^2 A^2 = \gamma E_0 \qquad \text{where } E_0 = \frac{1}{2} m\,\omega_0^2 A^2$$

From the power resonance curve :

$$\therefore \quad \Delta\omega = \gamma = 0.01 \times 10^6 \ s^{-1} = 10^4\,\mathrm{s}^{-1}$$

$$\boxed{\therefore \quad E_0 = \frac{\overline{P}_m}{\gamma} = 1.0 \ \mathrm{mJ}}$$

c) If the driving force is turned off, how long does it take for the energy of the oscillator to drop to $E_0 e^{-1}$?

$$E(t) = E_0 e^{-\gamma t}$$

$$\boxed{\therefore \ t = 1/\gamma = 10^{-4} \text{ s}}$$

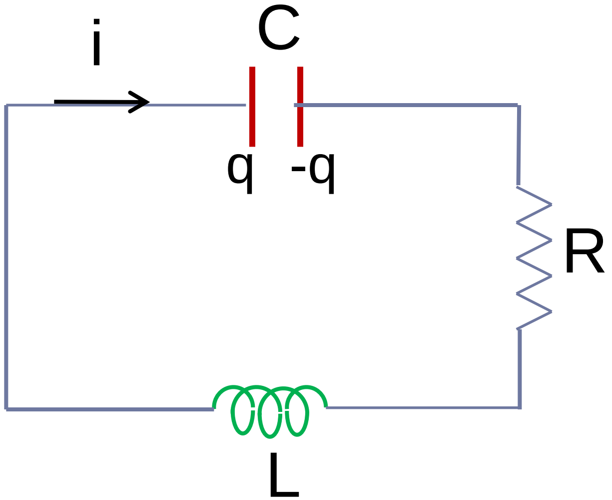
LCR Circuit : Angular frequency of free oscillations

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Free Oscillations LCR Circuit

$$V_L + V_R + V_C = 0$$

$$\therefore L\frac{di}{dt} + Ri + \frac{q}{C} = 0$$



Or, $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

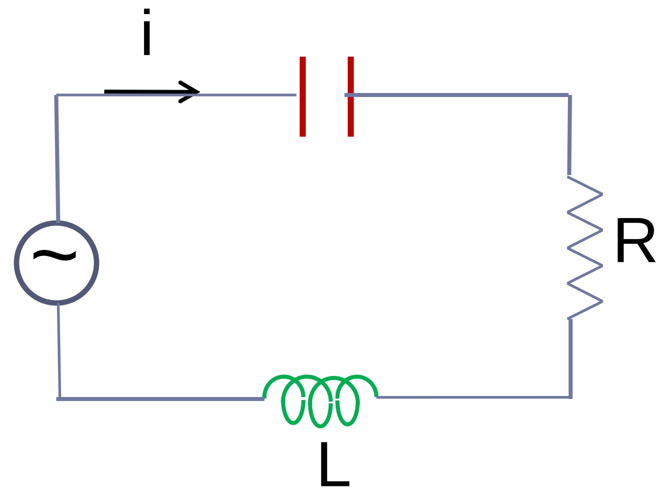
New correspondence :

$$\gamma \rightarrow \frac{\mathrm{R}}{\mathrm{L}}$$

$$q = q_0\, e^{-Rt/2L} \cos(\omega t + \phi)$$

Forced Oscillations of LCR Circuit

$$V = V_0 \cos \omega t$$



$$L \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \cos \omega t$$

Steady State Solution :

$$q(t) = q_0(\omega) \cos(\omega t - \delta)$$