Oscillations and Waves

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Damped Simple Harmonic Motion

In addition to the restoring force, there is a damping force, always opposing the motion of the oscillator

We shall be considering a damping force is usually proportional to the velocity of the oscillator :

$$F_{damp} = -b\,v = -b\,\frac{dx}{dt}$$

Damped Simple Harmonic Motion

Equation of motion :

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

Or, $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

where $\gamma = \dfrac{b}{m}$ has dimension of frequency

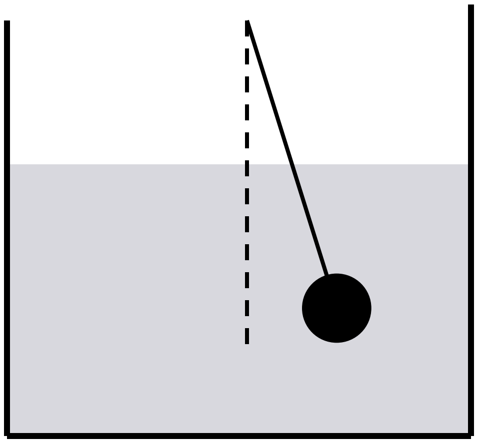
and $\omega_0 = \sqrt{\dfrac{k}{m}}$ is angular frequency when damping is absent

It is called undamped frequency or natural frequency

Case 1: Heavily Damped or Over Damped Motion

$$\frac{\gamma^2}{4} > \omega_0^2$$

Or damping force > restoring force



For example Pendulum inside thick syrup

Let us write $\sqrt{\dfrac{\gamma^2}{4} - \omega_0^2} = \beta$

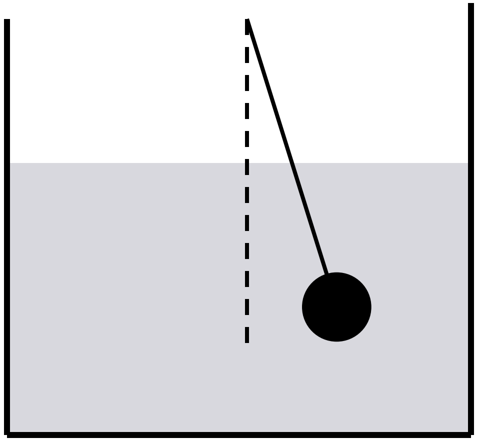
Most general solution is

$$x(t) = e^{-\frac{\gamma}{2}t}\left(C_1 e^{\beta t} + C_2 e^{-\beta t}\right)$$

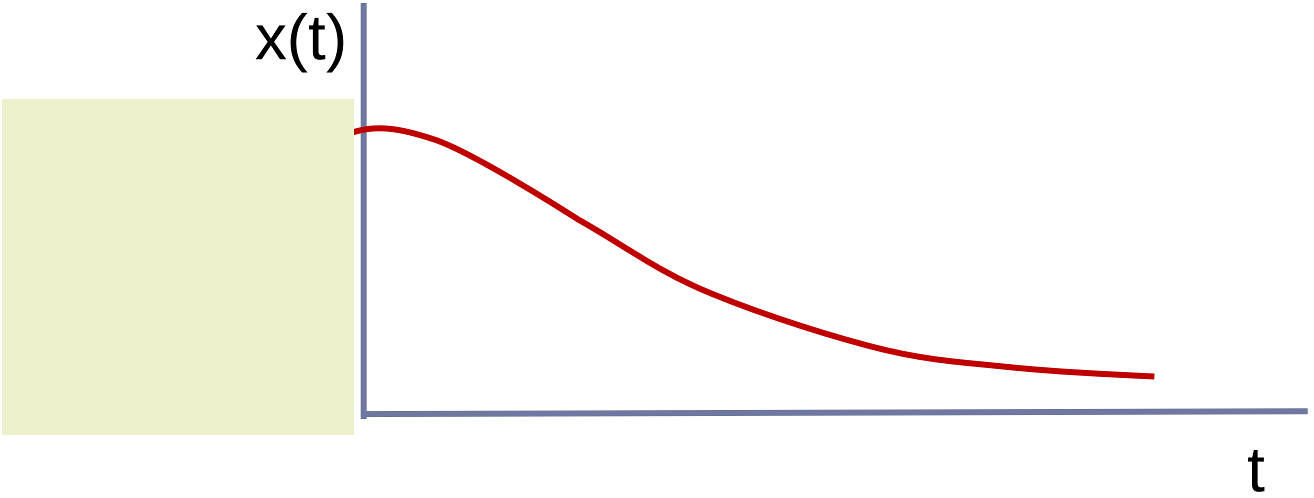
Non Oscillatory Motion

i) Initial conditions : Pendulum released from rest

i.e. $x(0) = x_0$; $\dot{x}(0) = 0$



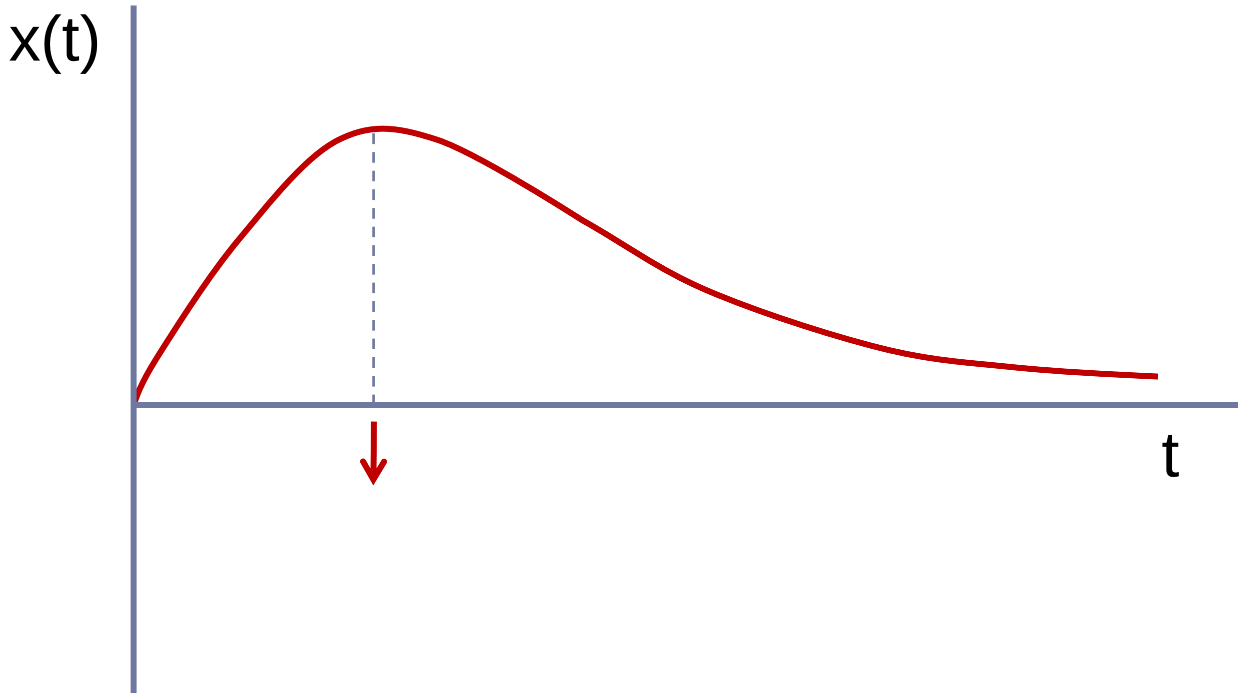
$$x(t) = \frac{x_0}{4q} e^{-\frac{\gamma}{2}t} \left[\left(\gamma + 2\beta\right) e^{\beta t} - \left(\gamma - 2\beta\right) e^{-\beta t} \right]$$



ii) With the initial conditions :

$$x(0) = 0 \; ; \; \dot{x}(0) = v_0$$

$$x(t) = \frac{v_0}{2} e^{-\frac{\gamma}{2}t} \left[e^{\beta t} - e^{-\beta t} \right]$$



Case 2: Critical Damping

square root term is zero i.e. q = 0

i.e. $$\frac{\gamma^2}{4} = \omega_0^2$$

This is the limiting case of behaviour of case 1

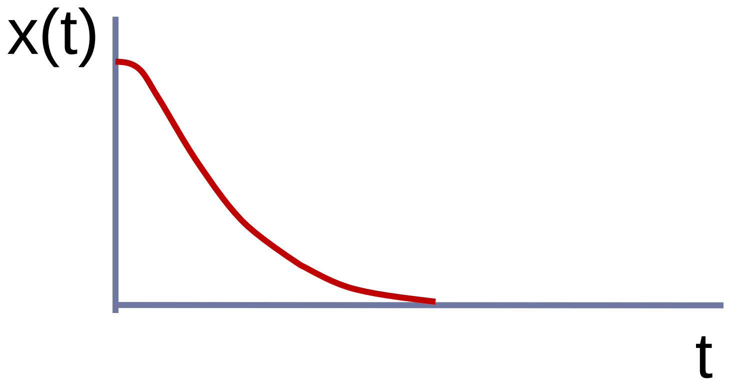
The most general solution in this case is

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$

i) With the initial conditions :

$$x(0) = x_0 \; ; \; \dot{x}(0) = 0$$

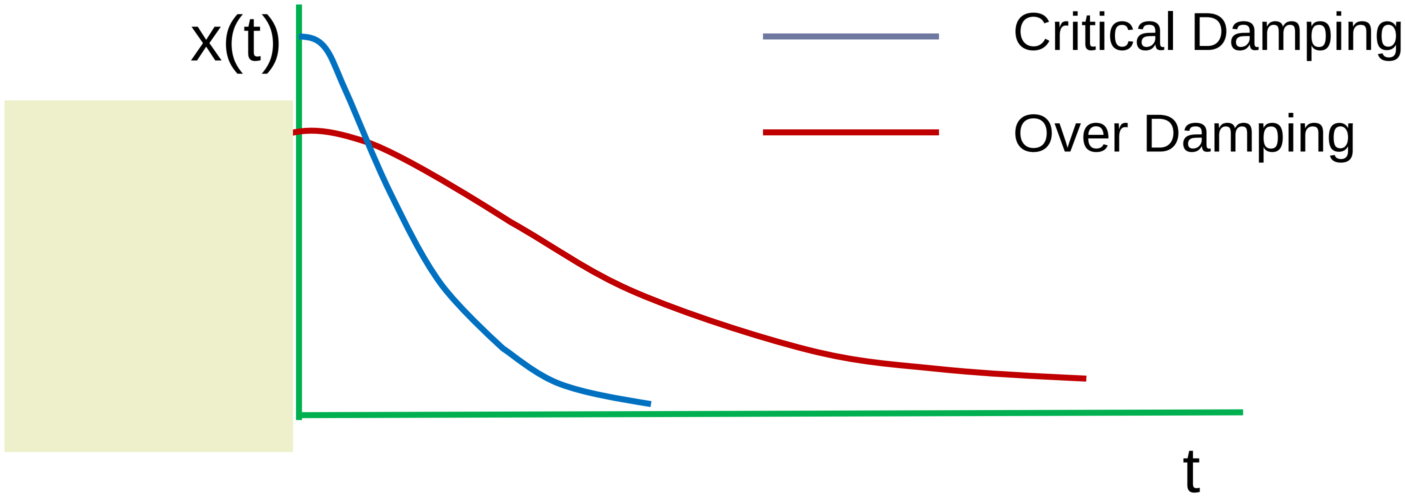
$$x(t) = x_0 \left(1 + \frac{\gamma}{2} t\right) e^{-\frac{\gamma}{2} t}$$



ii) With the initial conditions :

$$x(0) = 0\ ;\ \dot{x}(0) = v_0$$

$$x(t) = v_0\, t\, e^{-\frac{\gamma}{2}t}$$



$$\lim_{t\to\infty} \frac{x_{cd}(t)}{x_{od}(t)} = 0$$

Case 3: Under Damped SHM

$$\frac{\gamma^2}{4} < \omega_0^2$$

Or damping force < restoring force

We have $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

as the angular frequency

of under damped motion

The most general complex solution :

$$z(t) = e^{-\frac{\gamma}{2}t}\left(Ae^{i\omega t} + Be^{-i\omega t}\right)$$

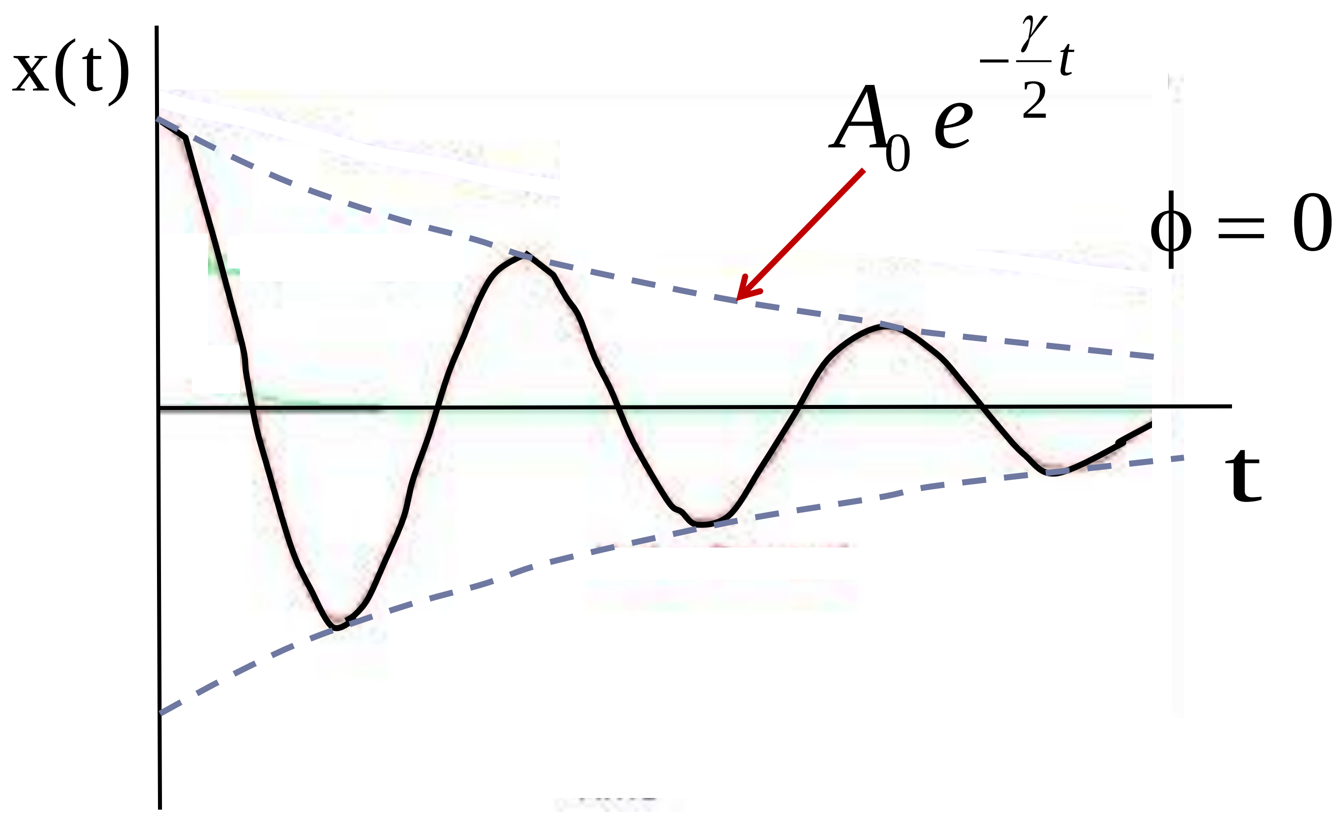
The most general real solution in this case is :

$$x(t) = A_0\, e^{-\frac{\gamma}{2}t}\cos(\omega t + \phi)$$

A_0 and ϕ are obtained from initial condition

The under-damped system is a SHM with decaying amplitude and altered frequency

Under damped Simple Harmonic motion



Amplitude of the oscillator decays with time as

$$A(t) = A_0\, e^{-\frac{\gamma}{2}t}$$

Energy of the oscillator also decays with time as

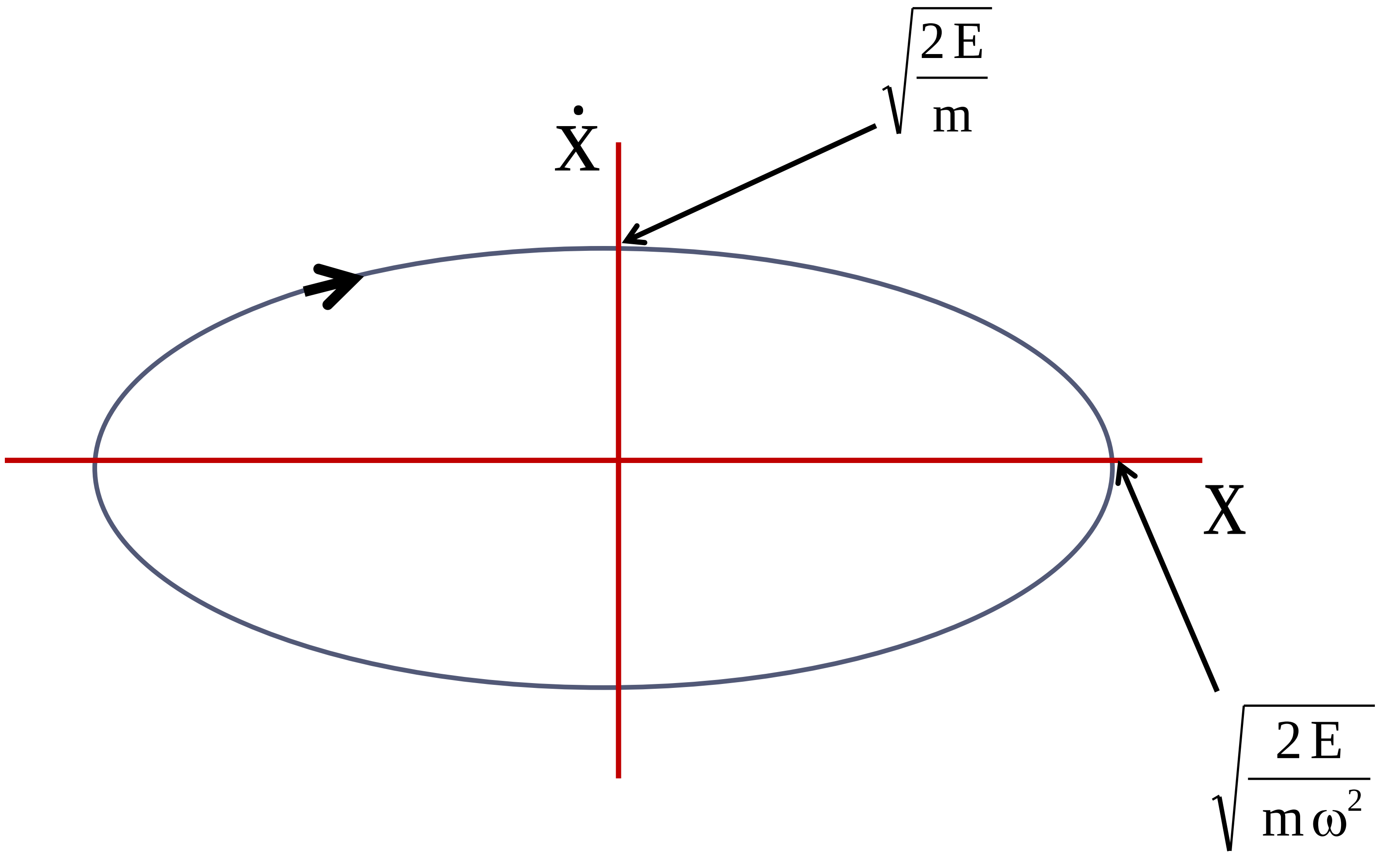
$$E = \frac{1}{2} k A^2(t) = \frac{1}{2} m \omega_0^{\;2} A_0^2 e^{-\gamma t}$$

$$= E_0\, e^{-\gamma t}$$

Frequency of damped oscillator is less than the undamped oscillator

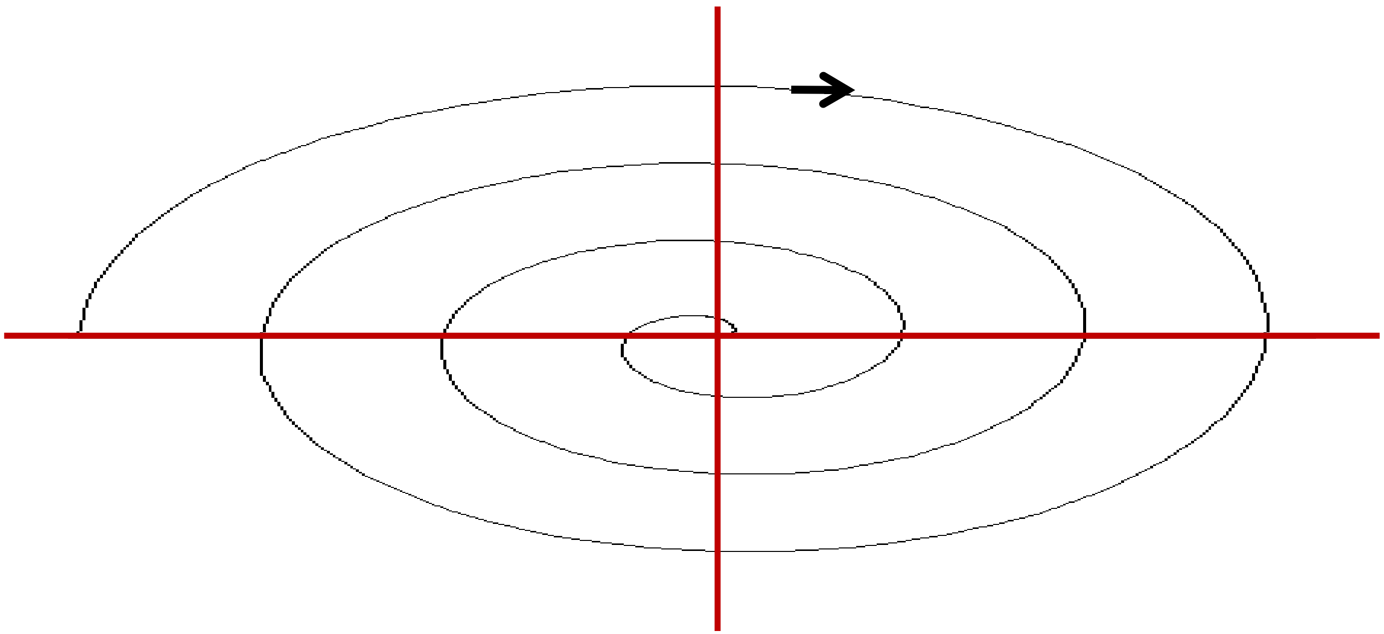
$$\omega\ (< \omega_0),\ \omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

Phase space Path is an ellipse for pure SHM



In the under-damped case one gets a curve spiraling into the origin

Total energy E of the damped oscillator decreases with time, so both the semi-major and semi-minor axes continuously decrease with time.



Quality Factor or Q – value

It describes the characteristics of Damped Harmonic Motion

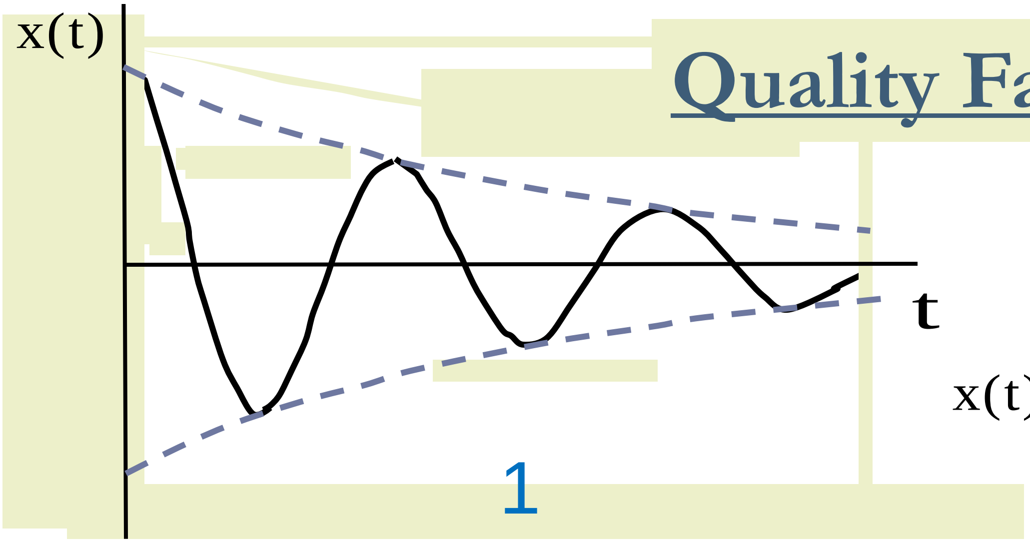
It is defined as the ratio of the time scale of damping to the time scale of oscillation

$$E = E_0 e^{-\gamma t} = E_0 e^{-1}$$

$$\Rightarrow \mathrm{t} = \frac{1}{\gamma}$$

$$\boxed{\mathrm{Q} = \frac{\omega}{\gamma}}$$

Quality Factor



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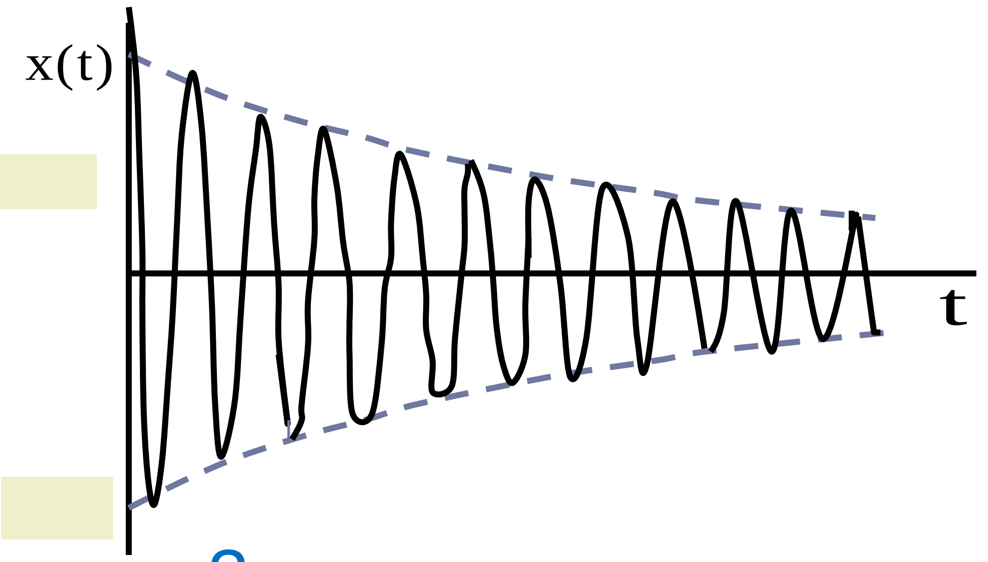

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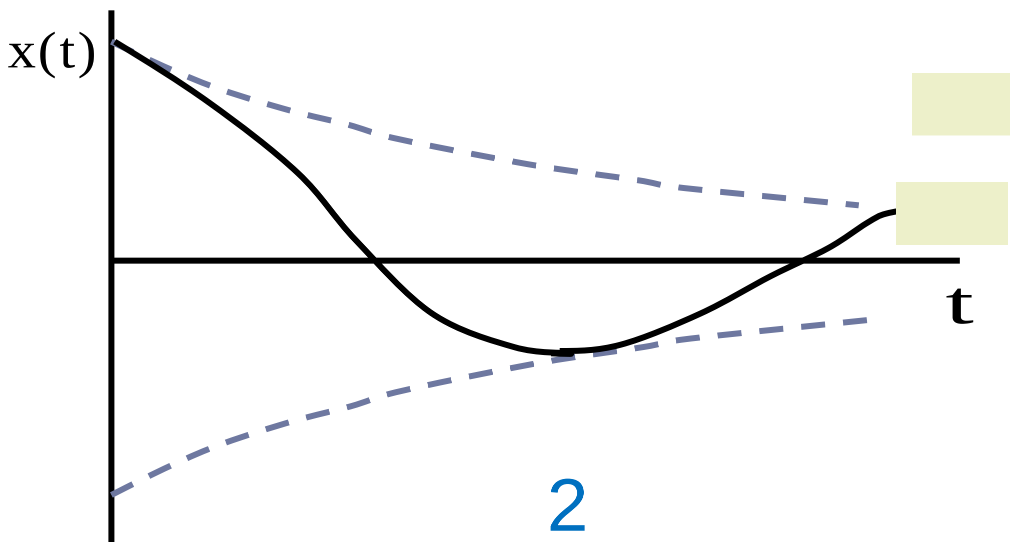

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The true measure of quality of a damped SHO is not determined by how long it lives (time in which the amplitude drops substantially), but rather, by how many cycles of oscillations it completes in this lifetime.

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \omega_0^2 - \frac{\omega^2}{4Q^2}$$

For large Q (Q > 5) or for small damping ,

$$\boxed{\therefore \quad Q \approx \frac{\omega_0}{\gamma} \quad (\text{Large Q})}$$

Amplitude after time t :

$$A(t) = A_0 e^{-\frac{\omega t}{2Q}}$$

Amplitude after n cycles :

$$A_n = A_0 e^{-\frac{n\pi}{Q}}$$

Energy after n cycles :

$$E_n = E_0 e^{-\frac{2n\pi}{Q}}$$