Oscillations and Waves

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<u>Text Book:</u>

Vibrations and Waves by A. P. French

Topics from Text Book II (A.P. French)

(4)	Simple harmonic motion (SHM)	The basic mass-spring system, solving SHM equation using complex exponentials, examples of SHM, the decay of free vibrations, effect of very large damping	Ch. 3 pp: 41-53, 62-70
(3)	Forced oscillator and resonance	Undamped oscillator with harmonic forcing, forced oscillator with damping	Ch. 4 pp. 78-95
(2)	Forced oscillator and resonance	Power absorbed by a driven oscillator, resonance	Ch.4 pp: 96-101
(5)	Coupled Oscillations	Normal modes, normal frequencies and forced oscillations of two coupled oscillators, normal modes and their properties for N coupled oscillators	Ch.5 pp: 121–127 129-151
(3)	Normal modes of continuous systems	The free oscillations of stretched strings, normal modes of a stretched string, forced oscillations of a stretched string	Ch.6 pp: 161-170
(3)	Progressive waves	Waves in one one direction, dispersion, Phase and group velocities. Energy in a mechanical wave, transport of energy by a wave.	Ch. 7 Pp 201-207, 230 237-241.

<u>Periodic motion</u>:- Any motion that repeats itself after some time.

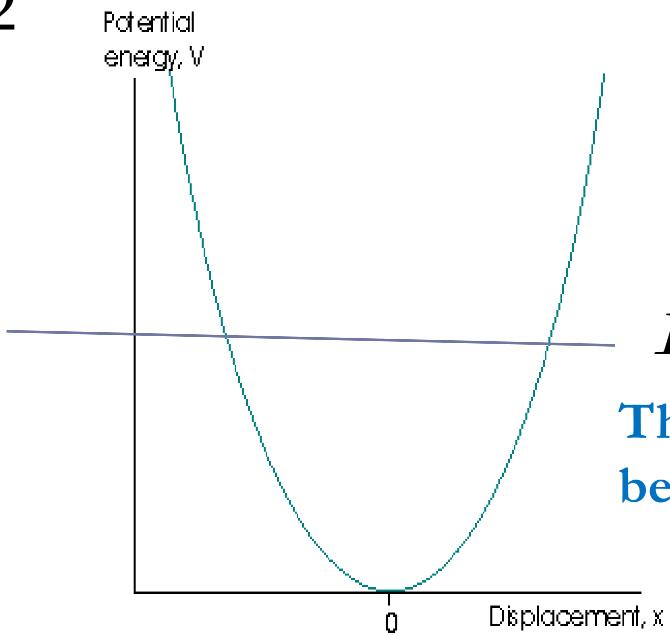
<u>Oscillatory motion</u>:- If a particle moves back and forth over the same path (we shall later give a more quantitative definition).

<u>Simple Harmonic motion</u>:- Oscillatory motions which can be expressed in terms of sine and cosine functions.

SHM is characterized by a potential energy which is a quadratic function of the coordinates

This implies that there is a restoring force proportional to the displacement of the coordinate from its equilibrium position

$$V(x) = \frac{1}{2}kx^2$$



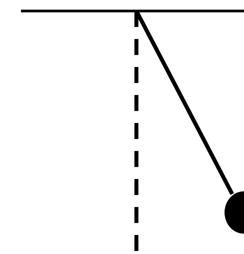
$$F = -\frac{dV(x)}{dx} = -kx$$

E

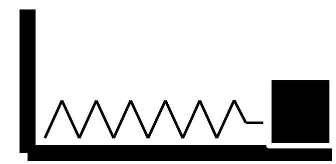
The energy E can not be negative because is a sum of two squares

Simple Harmonic Oscillators

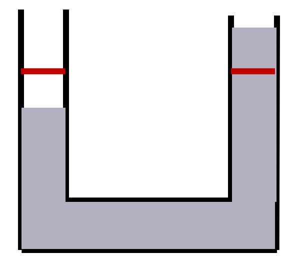
A simple pendulum



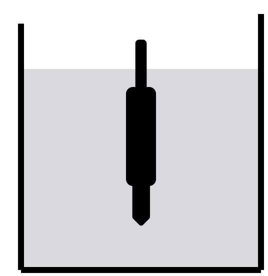
A mass fixed to a wall via a spring



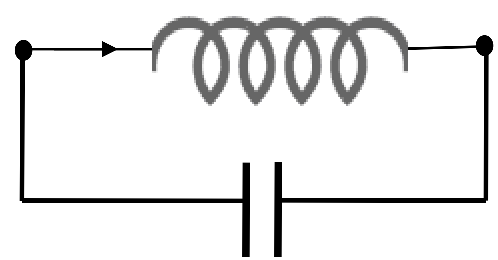
A frictionless U tube containing liquid



A hydrometer floating in a liquid

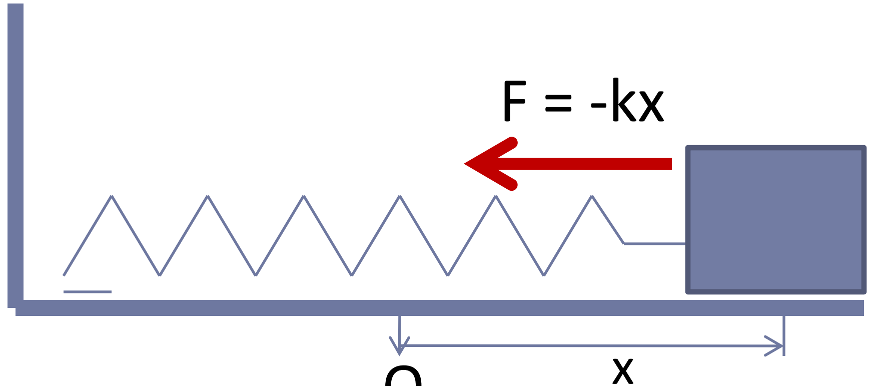


An inductor connected across
a capacitor carrying a charge q

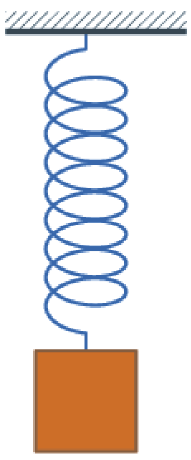


Simple Harmonic Motion

Example I : The idealized SHO is a spring-mass system



(The equilibrium position)



Equation of motion :

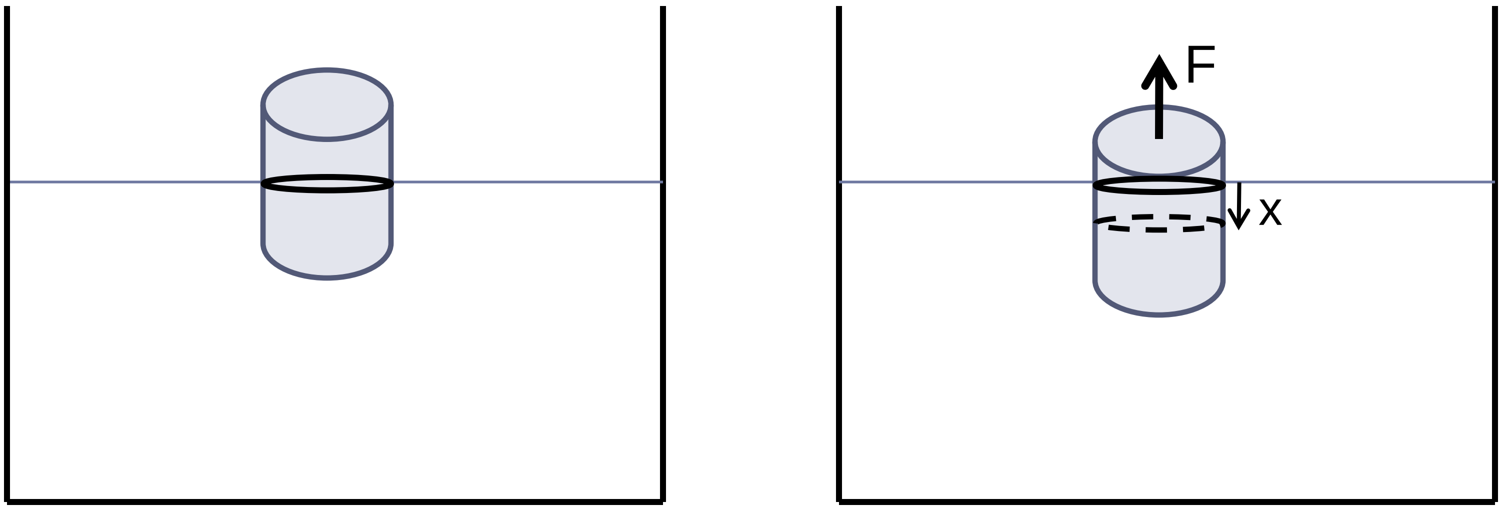
$$\boxed{m\frac{d^2x}{dt^2} = -kx}$$

Or,

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \left(\omega^2 = \frac{k}{m}\right)}$$

(The animation of the mass on a spring is from WikiMedia Commons. Thanks to Oleg Alexandrov for his creation.)

Example II : The up-down motion of a partially immersed solid



Equilibrium Position

Pushed down by x

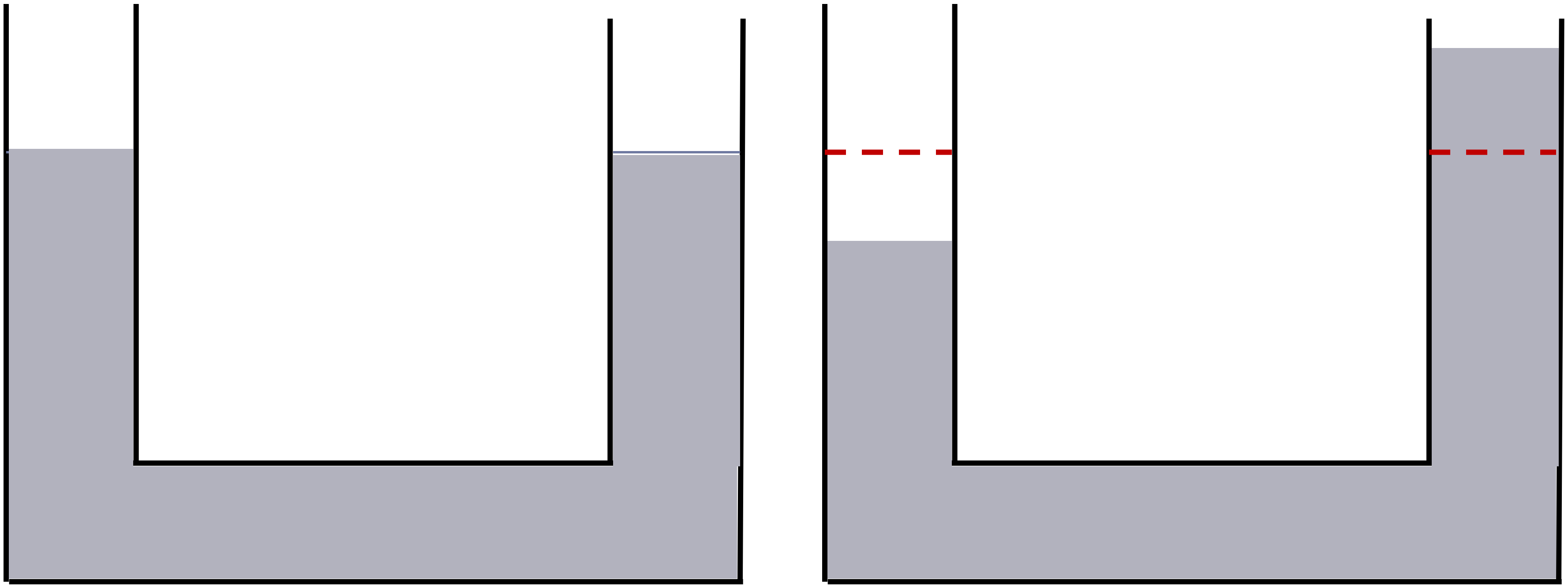
$$F = Additional\ \ Buoyancy\ \ Force = A\rho xg$$

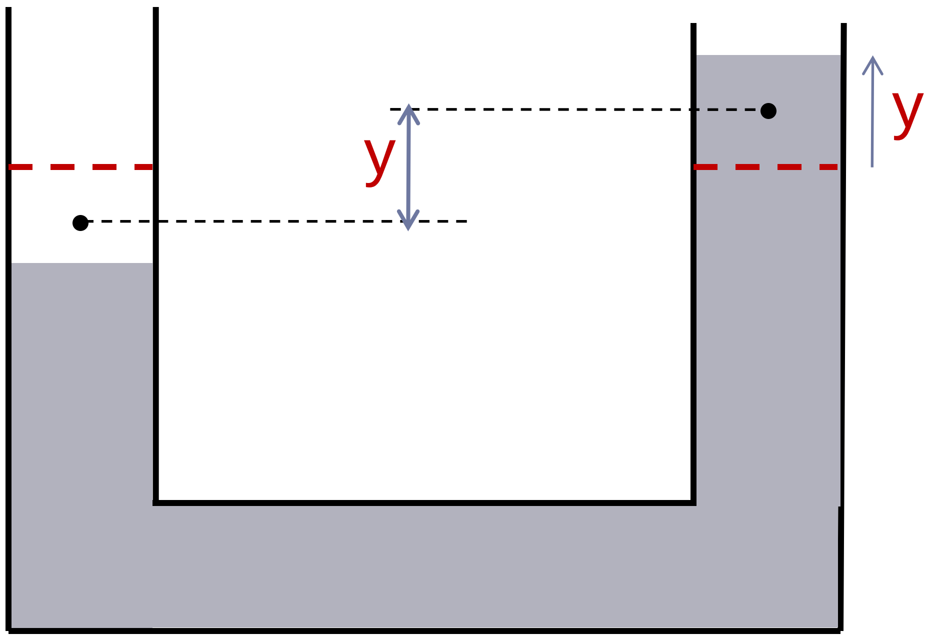
$\therefore$ **Equation of motion of the body is :**

$$m\frac{d^2x}{dt^2} = -\,A\rho g\,x$$

Simple Harmonic motion with $\omega = \sqrt{\frac{A\rho g}{m}}$

Example III : Oscillation of water column in a U-tube





M : Total mass of liquid

L : Total length of the water column

$$U(y) = \frac{M}{L} g y^2 \qquad KE = \frac{1}{2} M \dot{y}^2$$

$$E = \frac{1}{2} M \dot{y}^2 + \frac{M}{L} g y^2$$

Energy conservation :

$$\frac{dE}{dt} = 0$$

$$\Rightarrow M \dot{y} \ddot{y} + 2g \frac{M}{L} y \dot{y} = 0$$

$$\Rightarrow \ddot{y} + \frac{2g}{L} y = 0$$

SHM of angular frequency : $\omega = \sqrt{\dfrac{2g}{L}}$

Example IV

Prob. 6.17 (K & K):

A rod of length l and mass m, pivoted at one end, is held by a spring at its midpoint and a spring at its far end, both pulling in opposite directions. The springs have spring constant k, and at equilibrium their pull is perpendicular to the rod. Find the frequency of small oscillations about the equilibrium position.

Solution of SHM Equation

$$\ddot{x} + \omega^2 x = 0$$

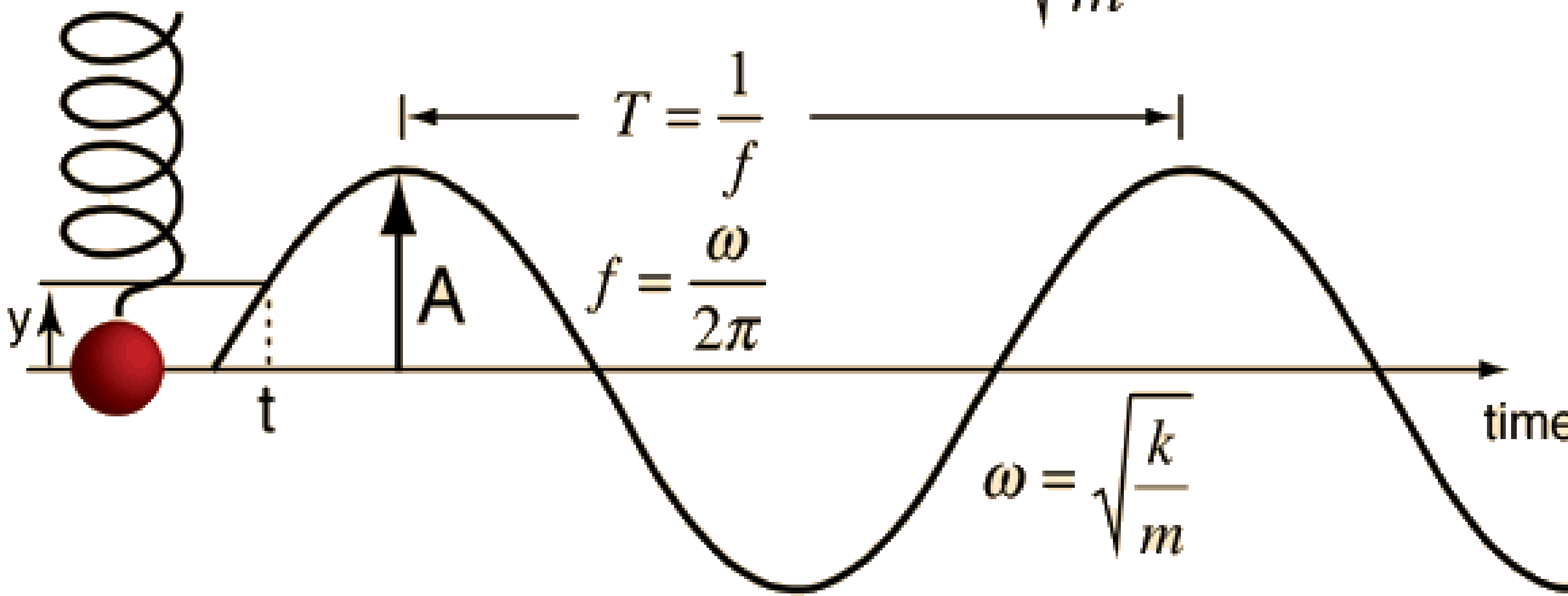
The two independent solutions are harmonic functions : $\cos \omega t$ & $\sin \omega t$

The most general solution of SHM equation is :

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Where, A & B are arbitrary constants to be fixed from initial conditions.

$$y = A\sin\omega t = A\sin\sqrt{\frac{k}{m}}\,t$$



Generally if the initial ($t = 0$) position and initial velocity are x_0 & v_0 respectively we have

$$x_0 = A \;\; ; \;\; v_0 = B\omega$$

$$\therefore \;\; x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$$

Alternative form for the solution

Given a pair of arbitrary constants A & B, one can express them in terms of another pair of arbitrary constants C & ϕ as

$$\mathrm{A} = C\cos\phi \; ; \; \mathrm{B} = -\mathrm{C}\sin\phi$$

This gives us the solution as

$$x(t) = C\cos\phi\cos\omega t - C\sin\phi\sin\omega t = C\cos(\omega t + \phi)$$

We may write $x(t) = C\cos(\omega t + \phi),$

C: Amplitude of oscillation ; ϕ : Phase Angle

Let us consider a trial solution of the form:

$$x(t) = C e^{pt}$$

Substituting this into the eq. $$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

We get $$p = \pm i\omega$$

So the most general solution is a linear combination

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

$C_1 \,\&\, C_2$ **must be complex constants to ensure that x(t) is real**

$$= |\mathrm{C}| e^{i(\omega t + \phi)} + |\mathrm{C}| e^{-i(\omega t + \phi)} = 2\mathrm{C}\ \mathrm{Cos}(\omega t + \phi)$$

$$= \mathrm{A}\ \mathrm{Cos}(\omega t + \phi)$$

Consider a complex function $z(t)$, that satisfies the SHO equation:

$$\frac{d^2 z}{dt^2} + \omega^2 z = 0$$

The real part of the complex solution, satisfies the original real form of the equation

It is often convenient to obtain the complex solution, and then take its real part as the actual solution.

$$z(t) = Ae^{i(\omega t)}$$

Where, A is a complex amplitude

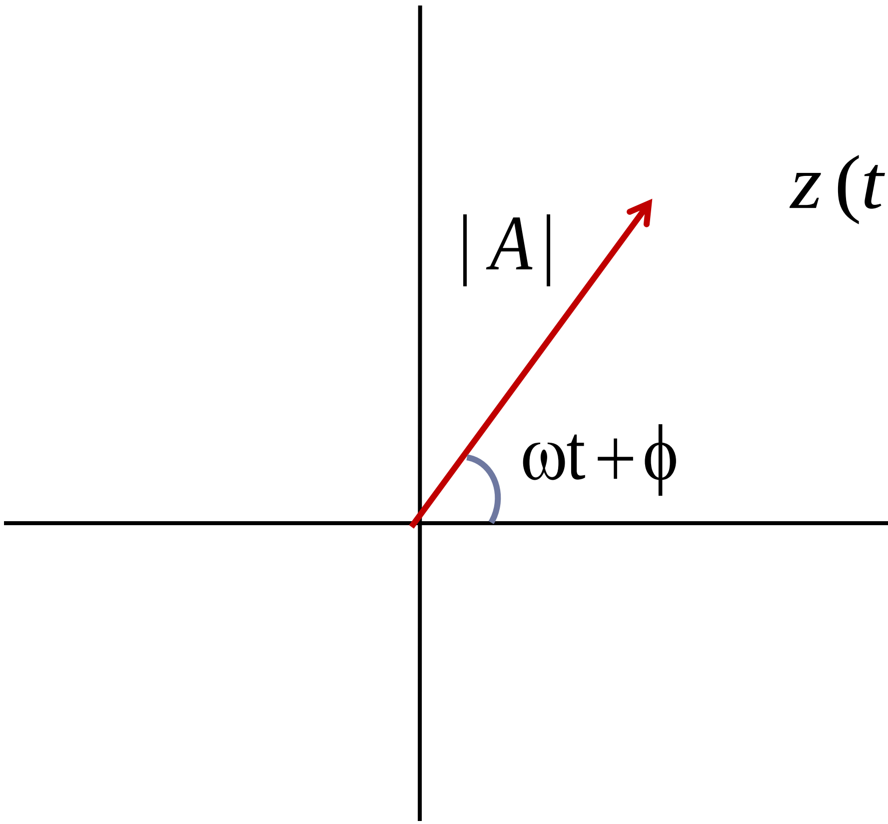
$$A = |A| e^{i\phi}$$

The real (or, imaginary) part of the above solution is the most general real solution of Simple harmonic motion.

$$x(t) = \mathrm{Re}[\mathrm{z(t)}] = \mathrm{Re}\left[|A| e^{i(\omega t + \phi)}\right]$$

$$= A\cos(\omega t + \phi)$$

The complex solution :

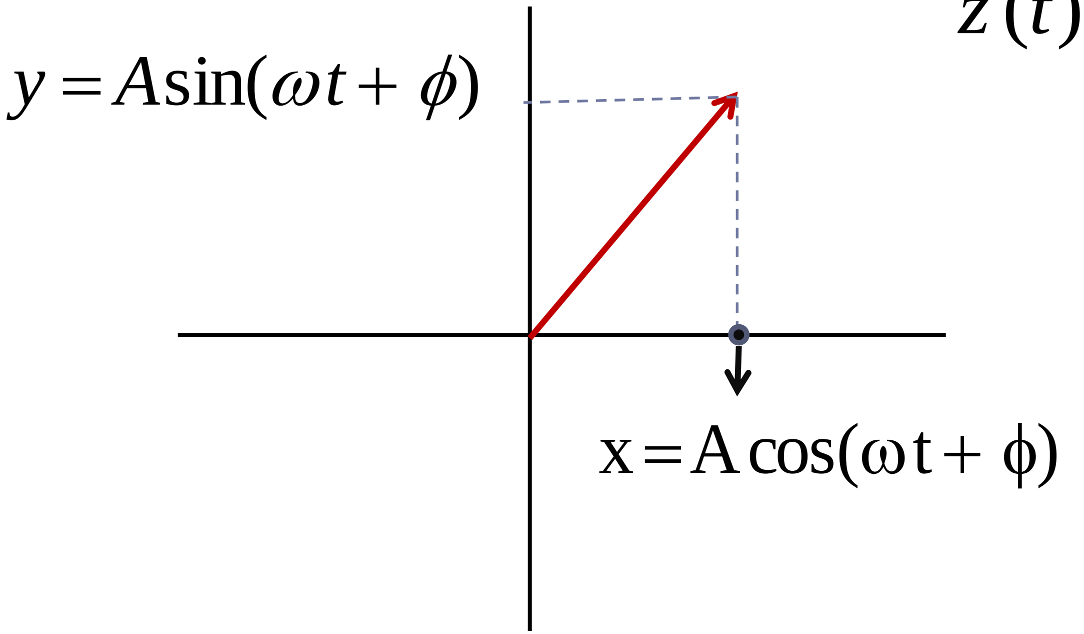


$$z(t) = Ae^{i(\omega t)}$$

is thus, a rotating vector of fixed length $|A|$ rotating counter-clockwise, with an angular velocity ω

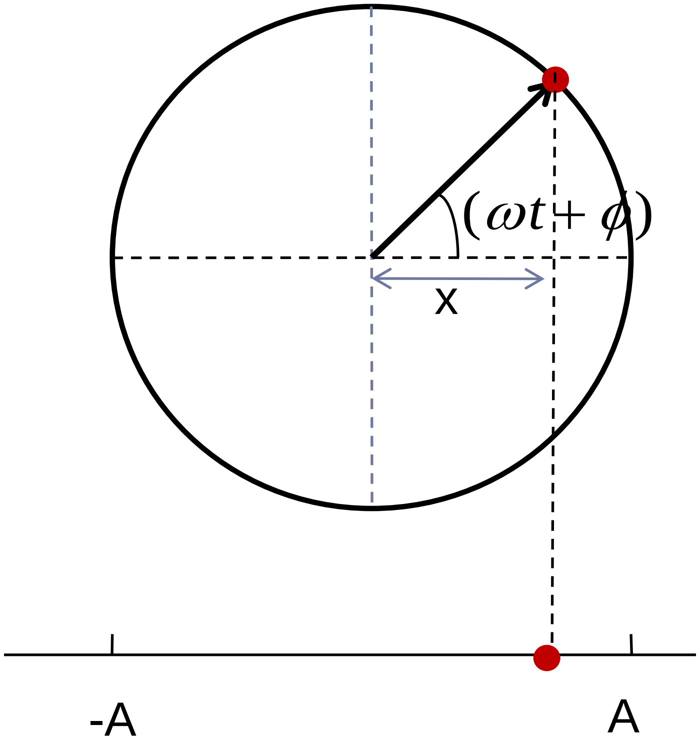
The SHM is the projection of the vector on the x-axis.

$$z(t) = Ae^{i(\omega t)}$$

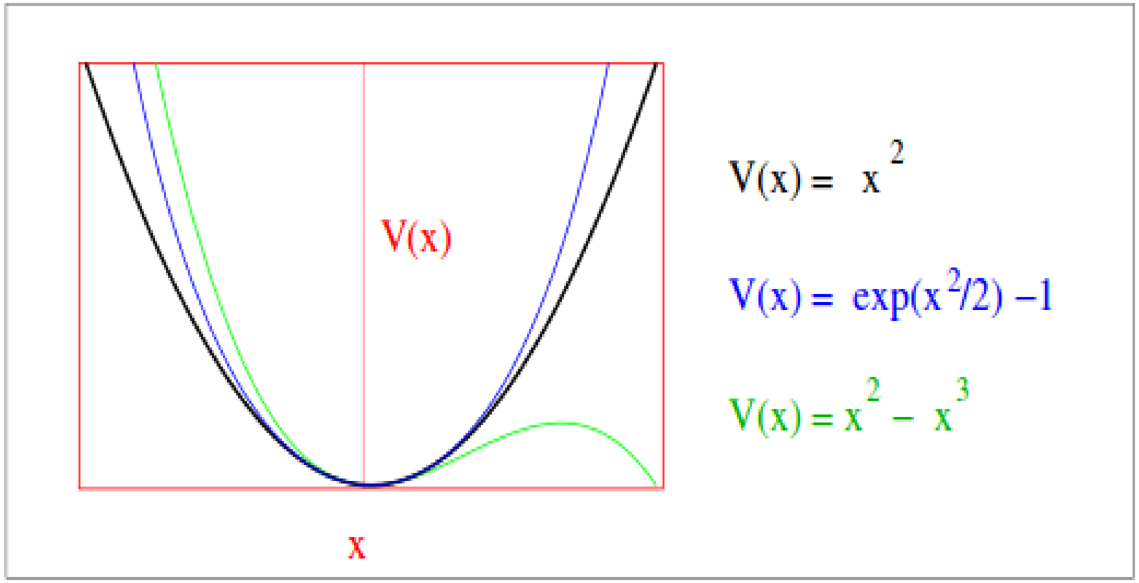


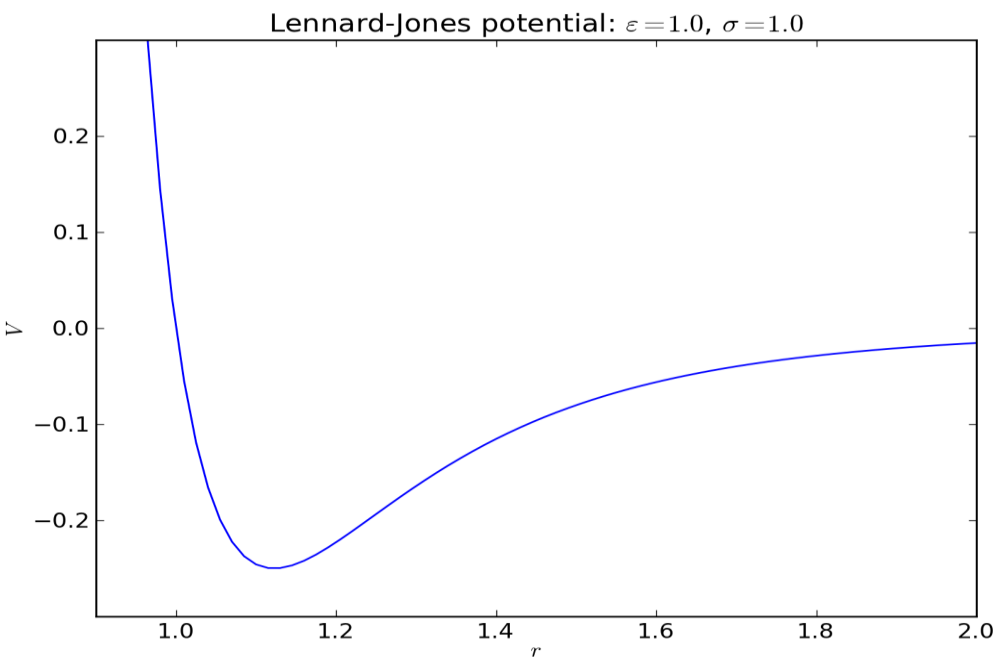
SHM as projection of uniform circular motion.

$$z = Ae^{i\,\omega t}$$



$$x = A\cos(\omega t + \phi)$$





What is the similarity between these potentials ?

Why study SHM ?

The reason is that any potential V(x) is well represented By a SHO in the neighborhood of points of stable equilibrium.

$$V(x) = V(x)\,|_{x=0} + \cancel{\frac{dV(x)}{d(x)}\Big|_{x=0} x} + \frac{d^2V(x)}{dx^2}\Big|_{x=0} x^2 + \dots\dots\dots$$

or

$$V(x) = V(x)\,|_{x=0} + \frac{1}{2} kx^2$$

where $$k = \frac{d^2V(x)}{dx^2}\Big|_{x=0}$$

Prob. 3.12

The total energy of an undamped oscillator, a constant, is given by :

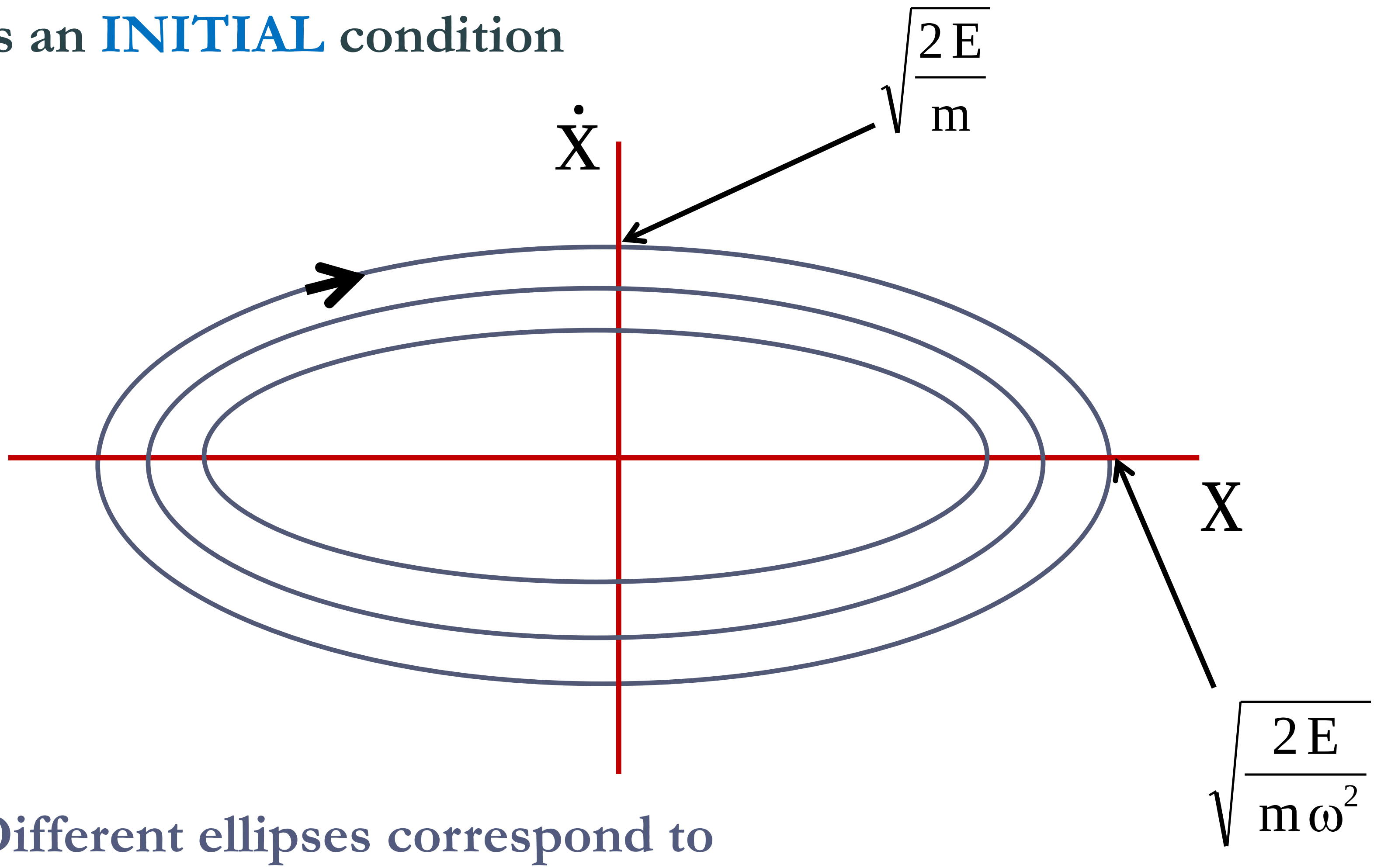
$$E = \frac{1}{2} m \omega_0{}^2 x^2 + \frac{1}{2} m \dot{x}^2$$

$$\Rightarrow \frac{x^2}{\left(\sqrt{\frac{2E}{m\omega_0{}^2}}\right)^2} + \frac{\dot{x}^2}{\left(\sqrt{\frac{2E}{m}}\right)^2} = 1$$

This eliminates time. The phase space description visualizes the motion
As this ellipse

Phase space of SHO

Every point on a ellipse
Is an INITIAL condition



Different ellipses correspond to
different values of the energy E.