Mathematics I

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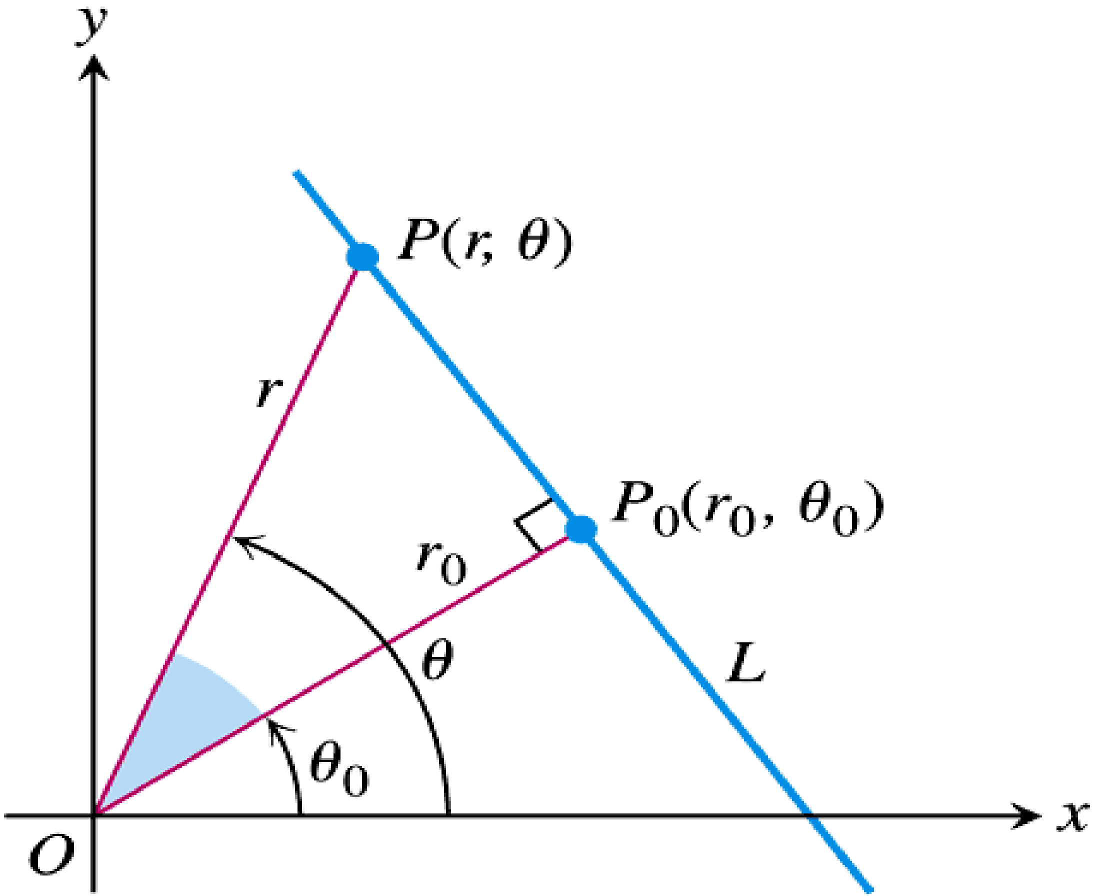
Section 11.7

Conics in Polar Coordinates

Polar Equation of a Straight Line

If the point $P(r_0, \theta_0)$ is the foot of the perpendicular from the pole to the line L and $r_0 \geqslant 0$, then the equation of L is

$$r\cos(\theta - \theta_0) = r_0.$$


y
$P(r, \theta)$
r
$P_0(r_0, \theta_0)$
r_0
θ
L
θ_0
x
O

Q:46. Sketch the line and find the cartesian equation for $r\cos\left(\theta + \frac{3\pi}{4}\right) = 1$.

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Sol.

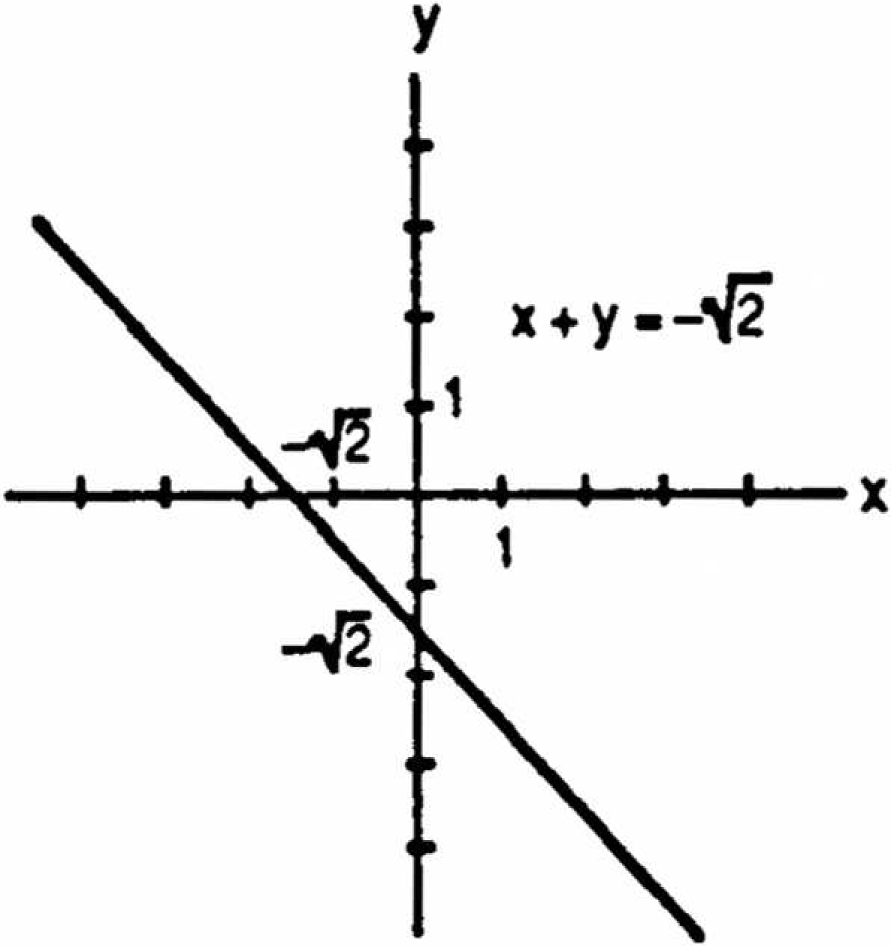
$$r\cos\left(\theta+\frac{3\pi}{4}\right)=1$$

$$r\left(\cos\theta\cos\frac{3\pi}{4}-\sin\theta\sin\frac{3\pi}{4}\right)=1$$

$$(r\cos\theta)\cos\frac{3\pi}{4}-(r\sin\theta)\sin\frac{3\pi}{4}=1$$

$$-\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y=1$$

$$x+y=-\sqrt{2}.$$


y
x + y = −√2
1
−√2
x
1
−√2

Q:50. Find a polar equation for $\sqrt{3}x - y = 1$ in the form $r\cos(\theta - \theta_0) = r_0$.

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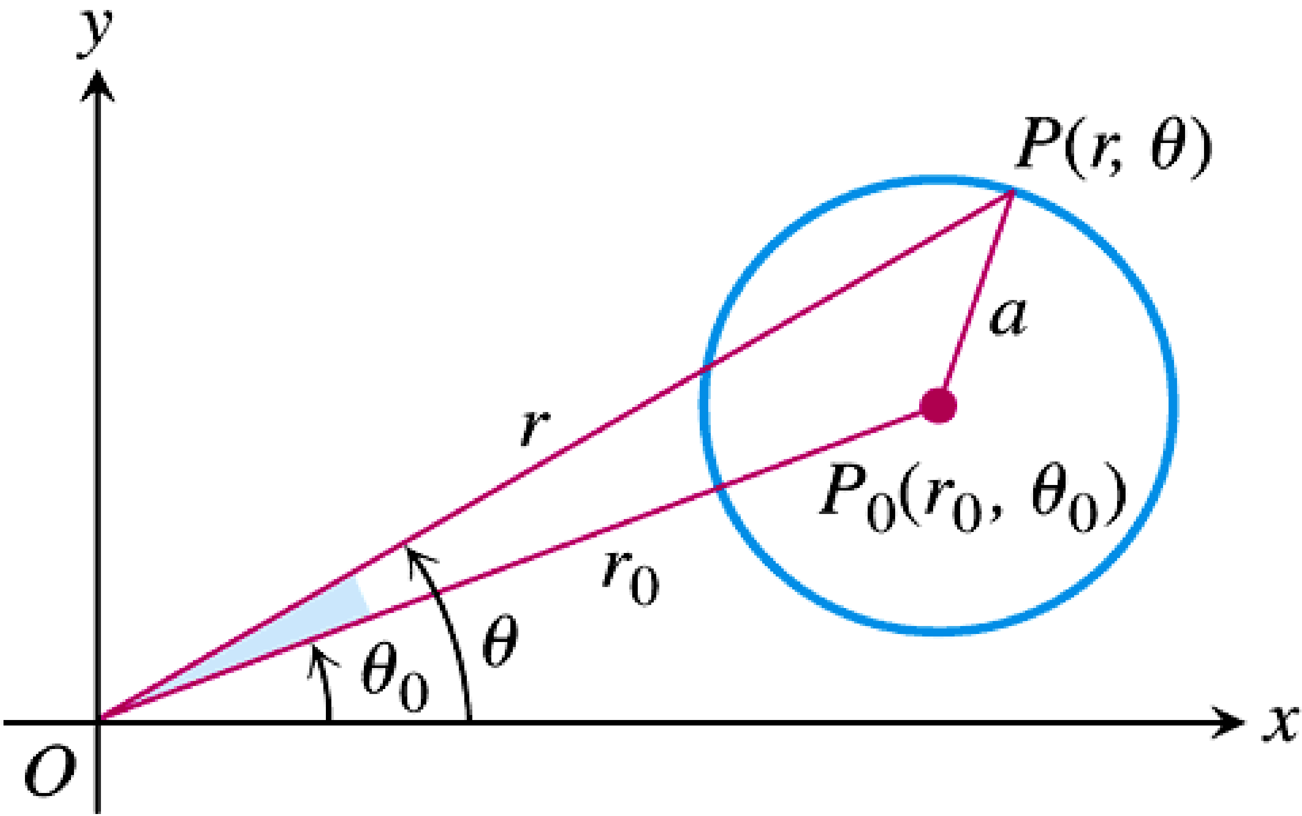
Sol.

$$\begin{aligned}
\sqrt{3}x - y &= 1\\
\sqrt{3}r\cos\theta - r\sin\theta &= 1\\
r\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) &= \frac{1}{2}\\
r\left(\cos\frac{\pi}{6}\cos\theta - \sin\frac{\pi}{6}\sin\theta\right) &= \frac{1}{2}\\
r\cos\left(\theta + \frac{\pi}{6}\right) &= \frac{1}{2}.
\end{aligned}$$

Polar Equation of a Circle



Polar Equation of a Circle

The polar equation of a circle of radius a and centered at (r_0, θ_0) is (using cosines law)

$$r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0) = a^2.$$

Circle passes through the pole

If the circle passes through the origin, then $r_0 = a$ and the equation simplifies to

$$r = 2a\cos(\theta - \theta_0).$$

Special Cases

1. **Equation of a circle centered at** $(a, 0)$ **and radius** a**.** If the center lies on positive x-axis then the equation becomes

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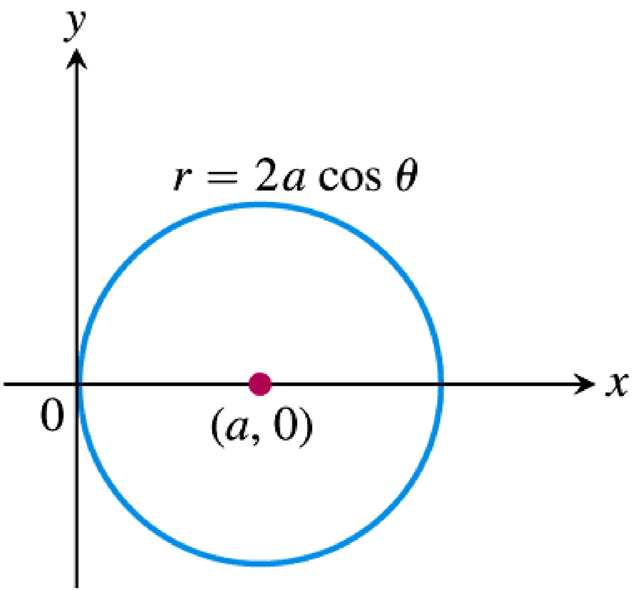
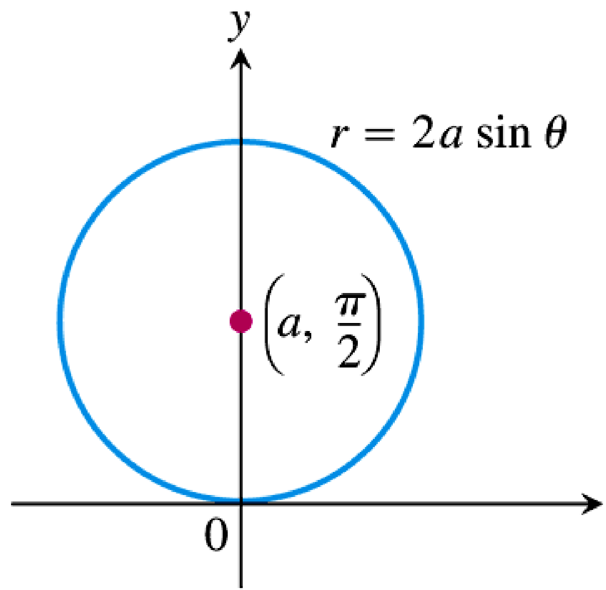
2. **Equation of a circle centered at** $\left(a,\frac{\pi}{2}\right)$ **and radius** a**.** If the center lies on positive y-axis then the equation becomes

$$r = 2a\sin\theta.$$



y
r = 2a cos θ
x
0
(a, 0)



y
r = 2a sin θ
(a, π/2)
0

3. **Equation of a circle centered at** $(-a, 0)$ **and radius** a**.** If the center lies on negative x-axis then the equation becomes

$$r = -2a\cos\theta.$$

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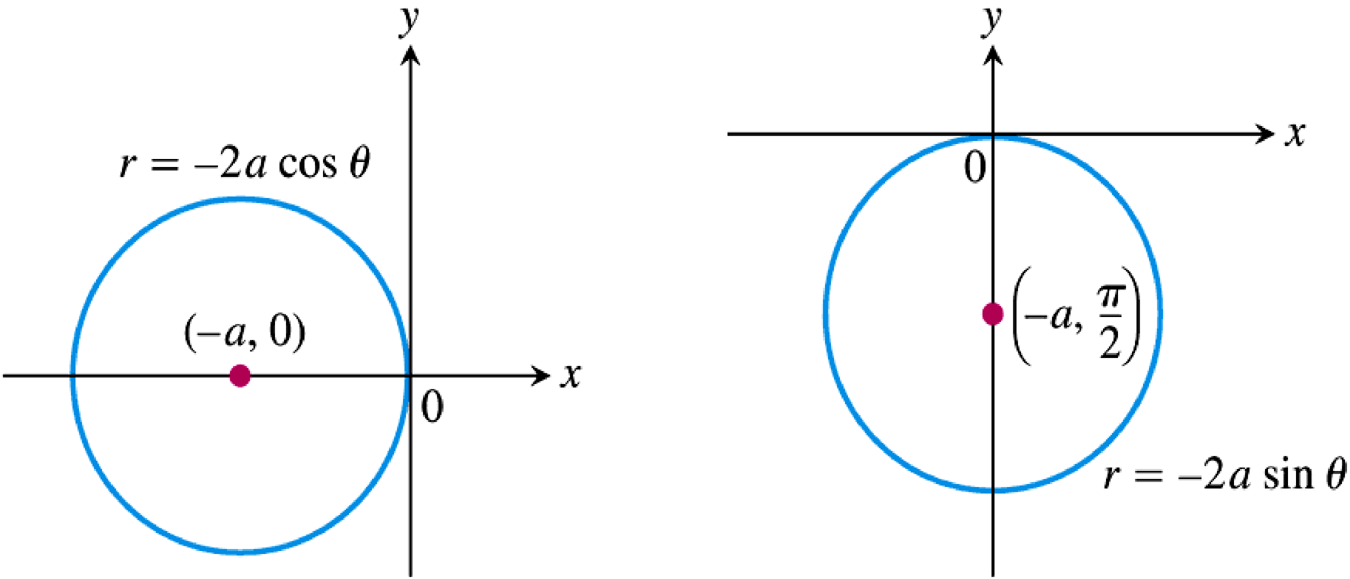
4. **Equation of a circle centered at** $\left(-a,\frac{\pi}{2}\right)$ **and radius** a**.** If the center lies on negative y-axis then the equation becomes

$$r = -2a\sin\theta.$$



y
x
0
$r = -2a\cos\theta$
$(-a, 0)$
y
x
0
$\left(-a, \frac{\pi}{2}\right)$
$r = -2a\sin\theta$

Q:56. Sketch the circle $r = -8\sin\theta$. Find polar coordinate of the center and identify the radius.

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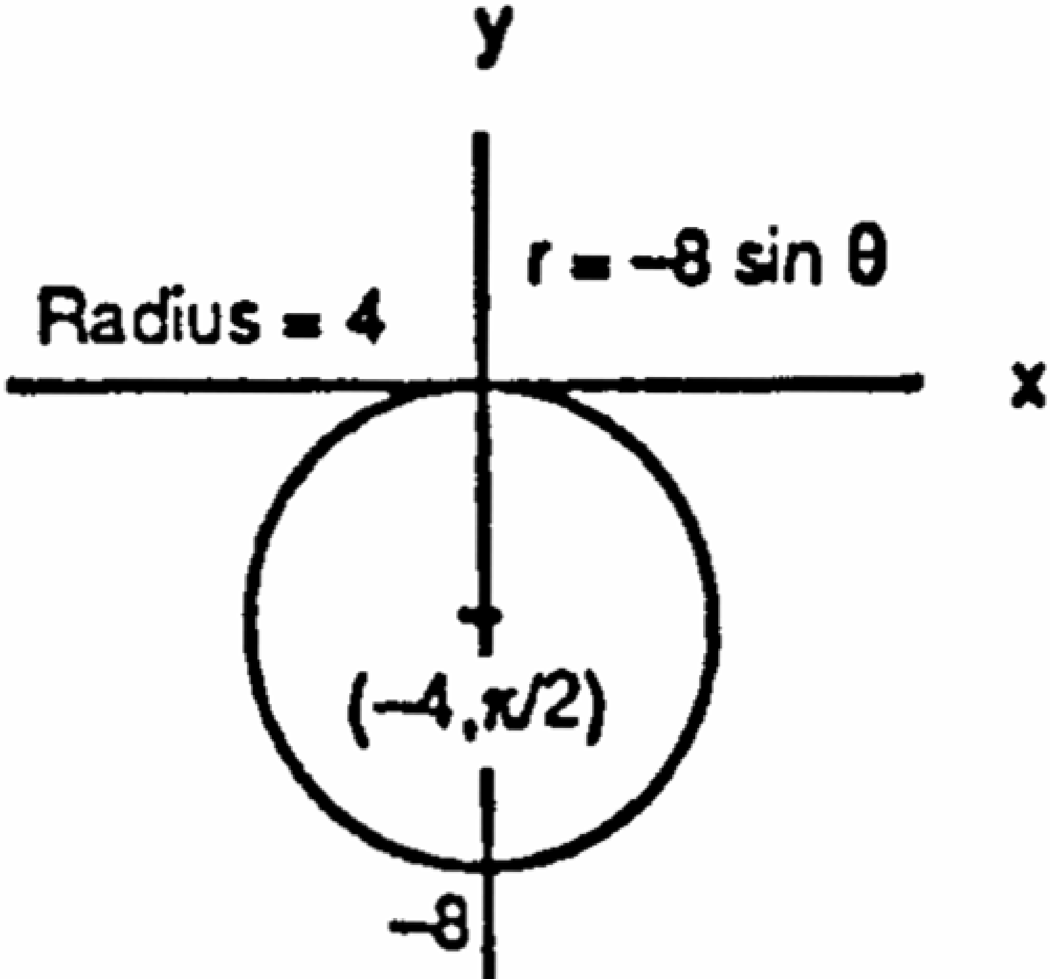
Sol. Compare with $r = -2a\sin\theta$, we get $a = 4$. Therefore the polar coordinate of the center is $\left(-4, \frac{\pi}{2}\right)$ and radius is 4.


y
r = −8 sin θ
Radius = 4
x
(−4, π/2)
−8

Q:64. Find polar equation for the circle $x^2 + y^2 - \frac{4}{3}y = 0$. Sketch the circle and label it with both its cartesian and polar equations.

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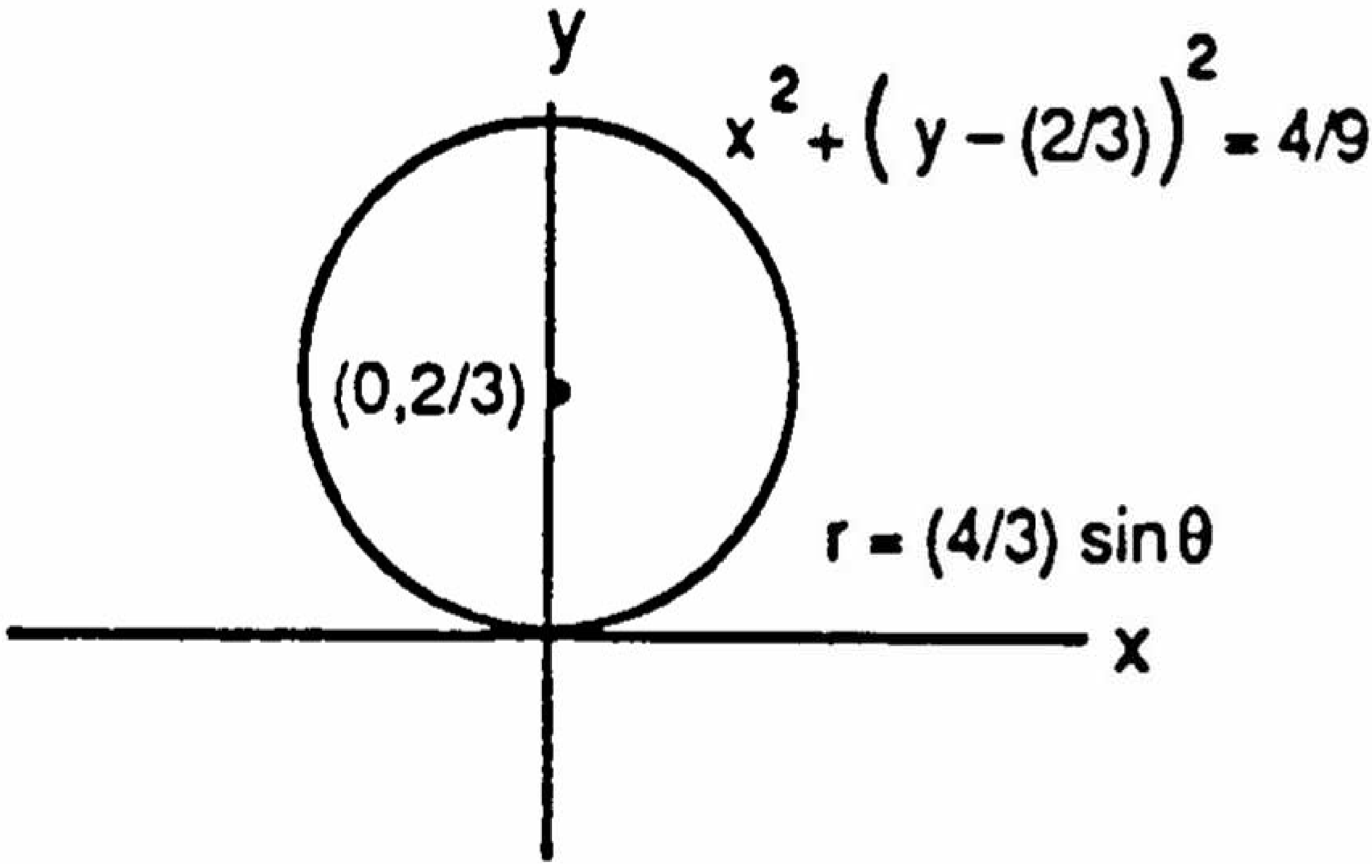
Sol. Rewrite the equation as $x^2 + \left(y - \frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^2$. Compare with $(x - x_0)^2 + (y - y_0)^2 = a^2$. The center is $\left(0, \frac{2}{3}\right)$ and radius is $a = \frac{2}{3}$. Therefore the polar equation is $r = \frac{4}{3}\sin\theta$ or one can find by changing into polar form as follows: $x^2 + y^2 - \frac{4}{3}y = 0$ gives $r^2 - \frac{4}{3}r\sin\theta = 0$ or $r = \frac{4}{3}\sin\theta$.


y
$x^2 + (y - (2/3))^2 = 4/9$
(0,2/3)
$r = (4/3)\sin\theta$
x

Conic Section

The curve obtained from the intersection of a double cone with a plane is called conic section.

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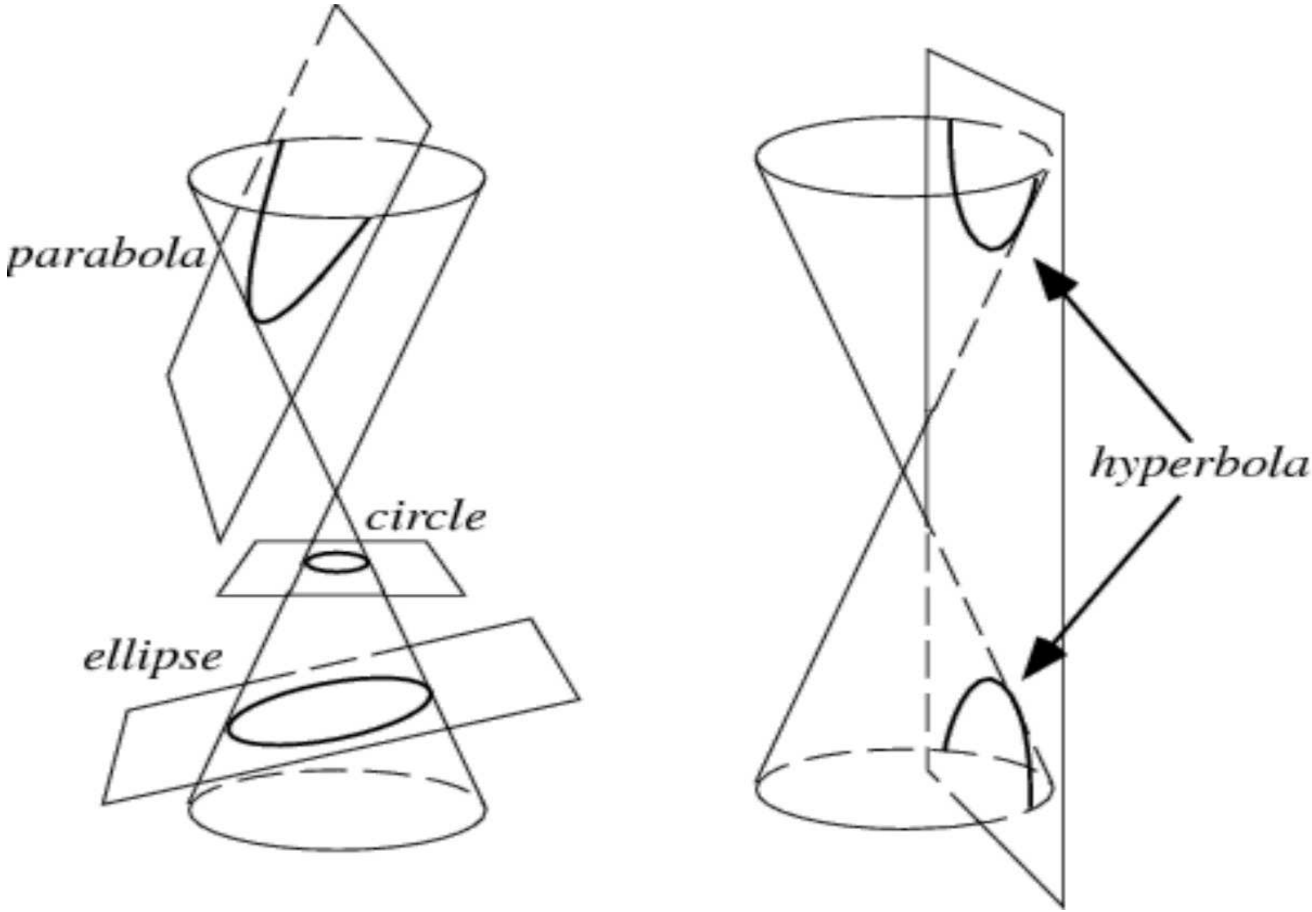
Each conic section can be defined in several ways.


parabola
circle
ellipse
hyperbola

Directrix

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A directrix of a conic section is a line which together with the focus (a fixed point) define a conic section as the locus of the points whose distance from the focus is proportional to the distance from the directrix.

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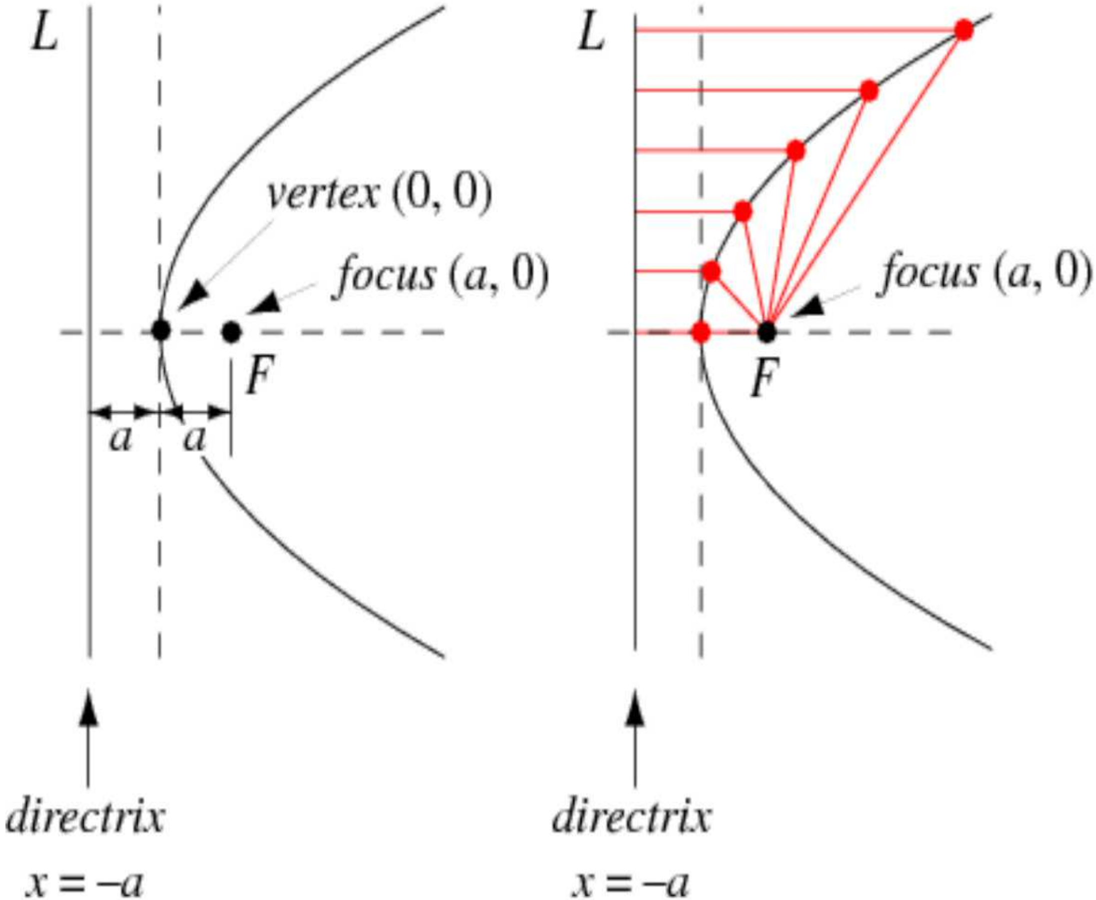
The constant of proportionality is called the eccentricity of the conic section.

Parabola

A parabola is a locus of points equidistant from a fixed point, called the focus of the parabola, and a line, called the directrix of the parabola.


L
vertex (0, 0)
focus (a, 0)
F
a
a
directrix
x = −a
L
focus (a, 0)
F
directrix
x = −a

Ellipse

For ellipse and hyperbola, there are two special points - their foci - in terms of which the definitions are set. Denote the foci F_1 and F_2.

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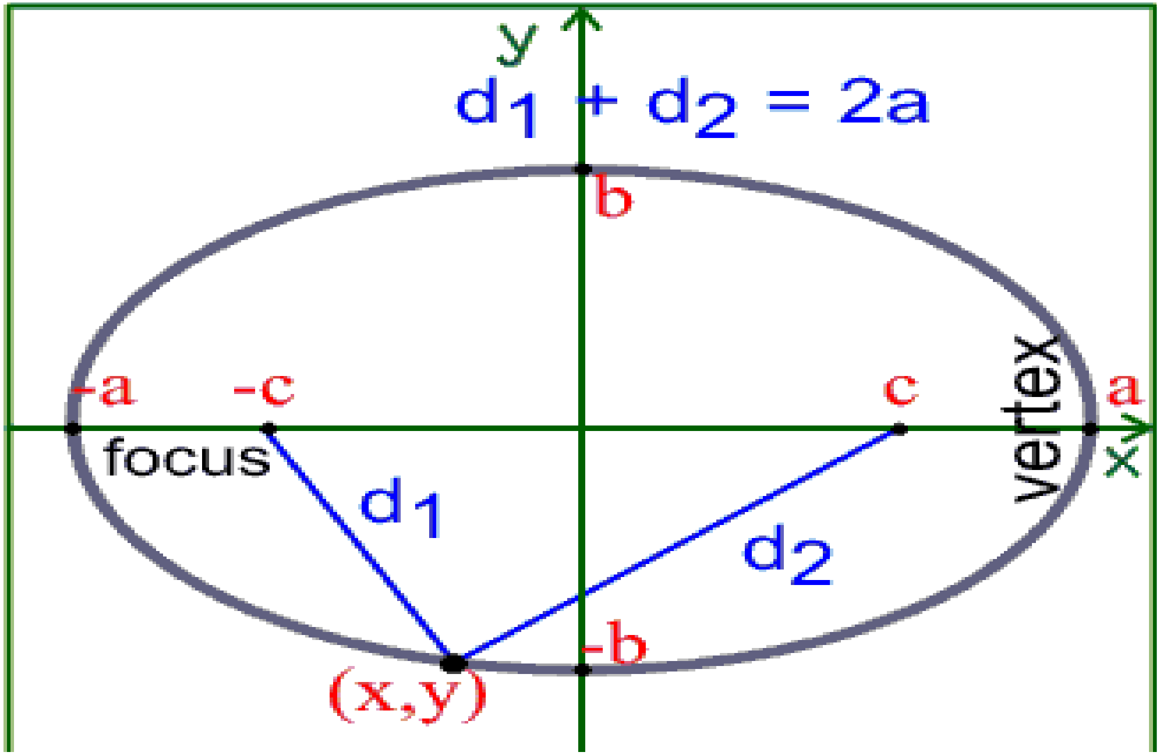
Ellipse

An ellipse is a locus of points the sum of whose distances to F_1 and F_2 is a constant.


y
$d_1 + d_2 = 2a$
b
-a
-c
c
vertex
a
x
focus
d_1
d_2
-b
(x,y)

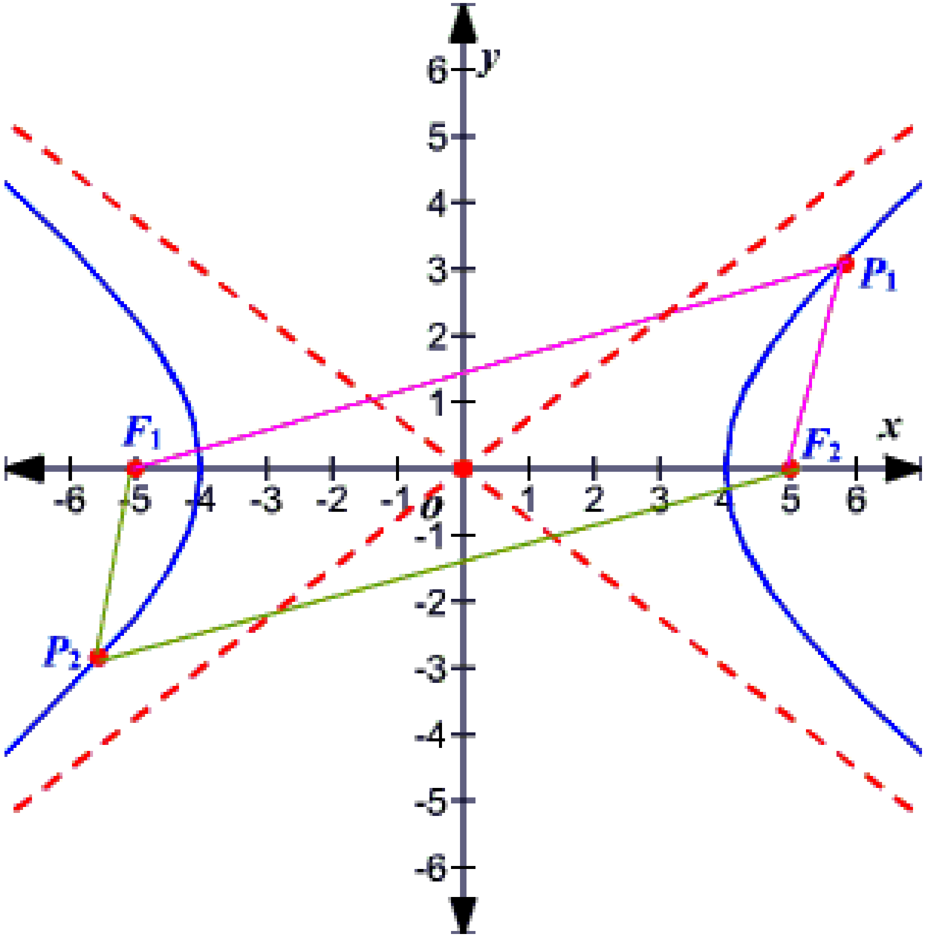
Hyperbola

Hyperbola is a locus of points for which the absolute value of the difference of the distances to F_1 and F_2 is a constant.

Hyperbola

Hyperbola is a locus of points for which the absolute value of the difference of the distances to F_1 and F_2 is a constant.

Thus in the graph given on next slide we have $|P_1F_1 - P_1F_2| = |P_2F_1 - P_2F_2|$.



y
x
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
F_1
F_2
P_1
P_2

Remark

For parabola there is one focus and one directrix while for ellipse and hyperbola there are two foci and two directrices.

Let $e > 0$ be the eccentricity. Then we have

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- $e < 1$: ellipse

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- $e < 1$: ellipse
- $e = 1$: parabola

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- $e < 1$: ellipse
- $e = 1$: parabola
- $e > 1$: hyperbola

Let $e > 0$ be the eccentricity. Then we have

- $e < 1$: ellipse
- $e = 1$: parabola
- $e > 1$: hyperbola

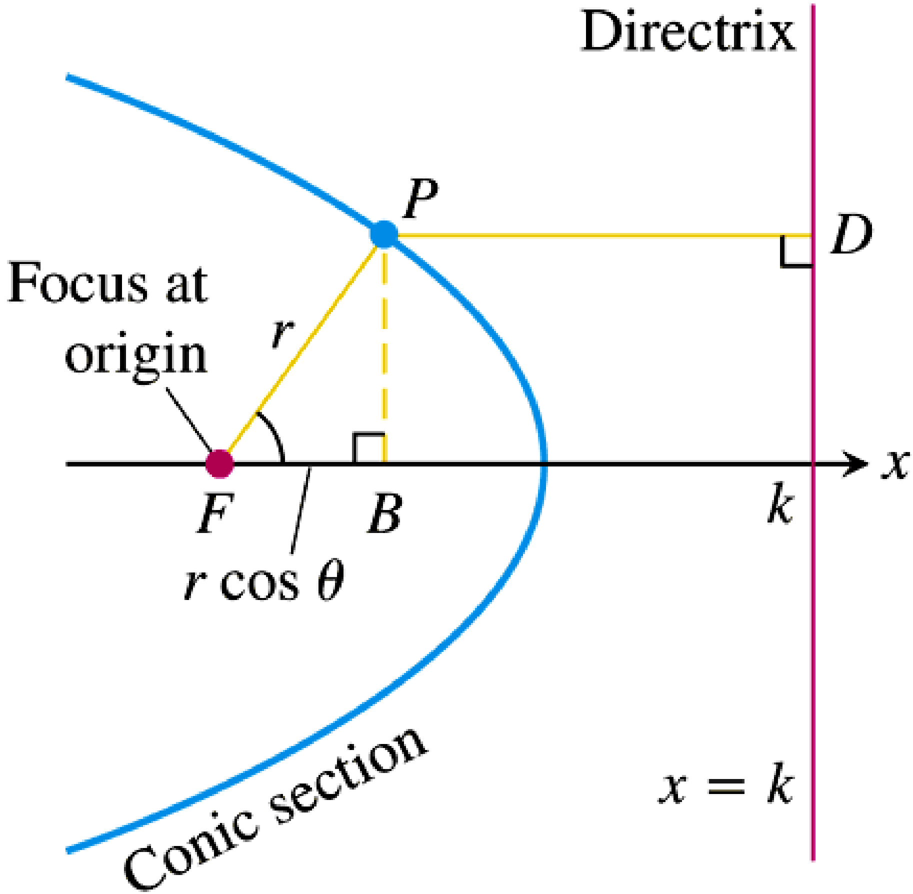
Remark

In all the problems, we consider one focus of the conic section at the origin.

Polar Equation of a Conic Section: Case I

If the directrix be $x = k$, $(k > 0)$ (vertical, to the right of the origin). Then the polar equation of the conic section is

$$r = \frac{ke}{1 + e\cos\theta}.$$


Directrix
P
D
Focus at origin
r
F
B
k
x
r cos θ
x = k
Conic section

Case II

If the directrix be $x = -k$, $(k > 0)$ (vertical, to the left of the origin). Then the polar equation of the conic section is

$$r = \frac{ke}{1 - e\cos\theta}.$$

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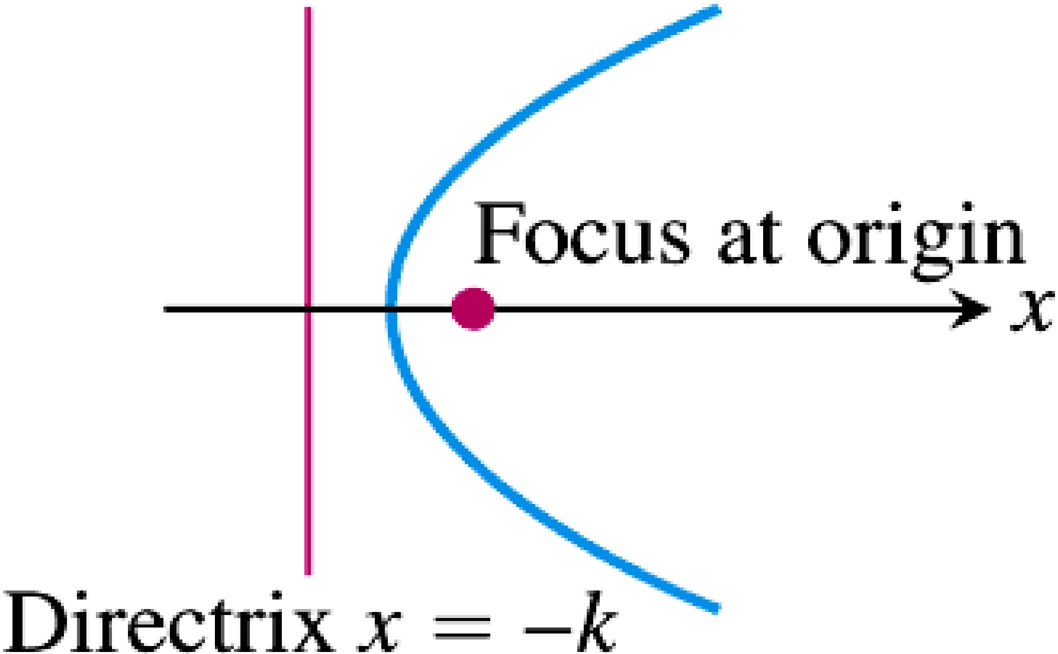
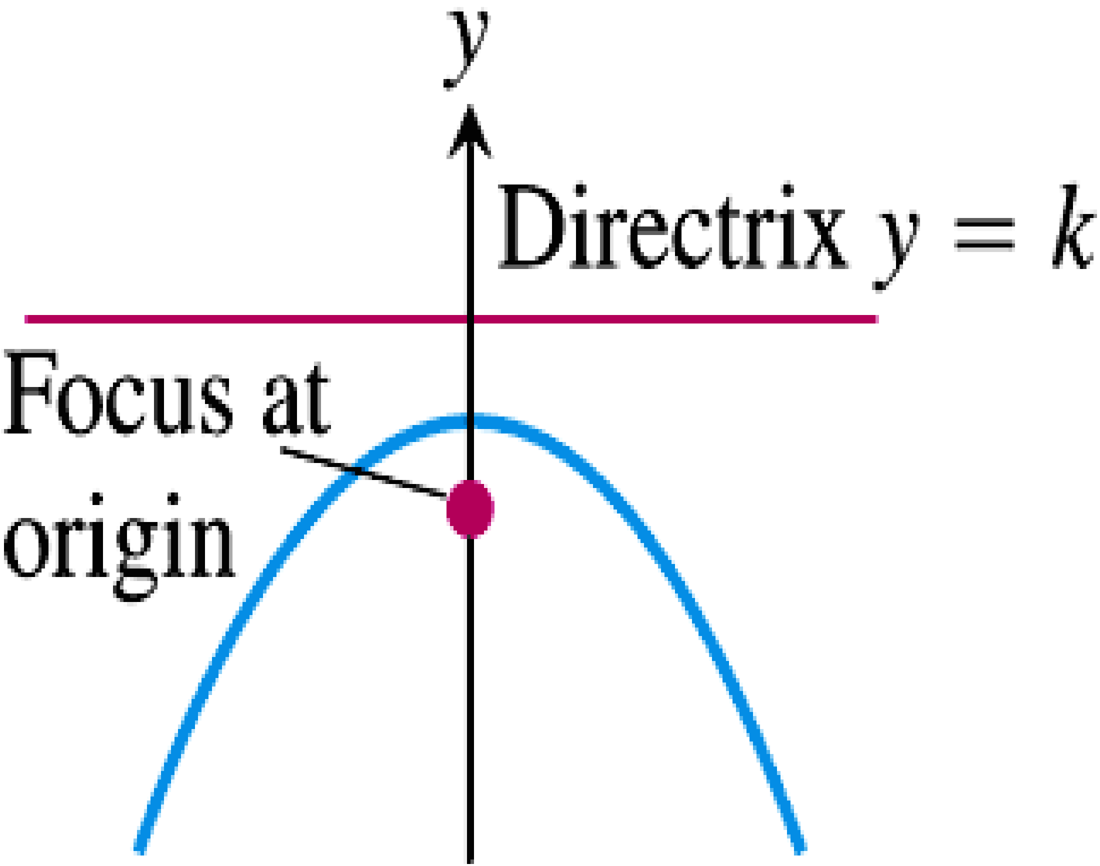


Case III

If the directrix be $y = k$, $(k > 0)$ (horizontal, above to the origin). Then the polar equation of the conic section is

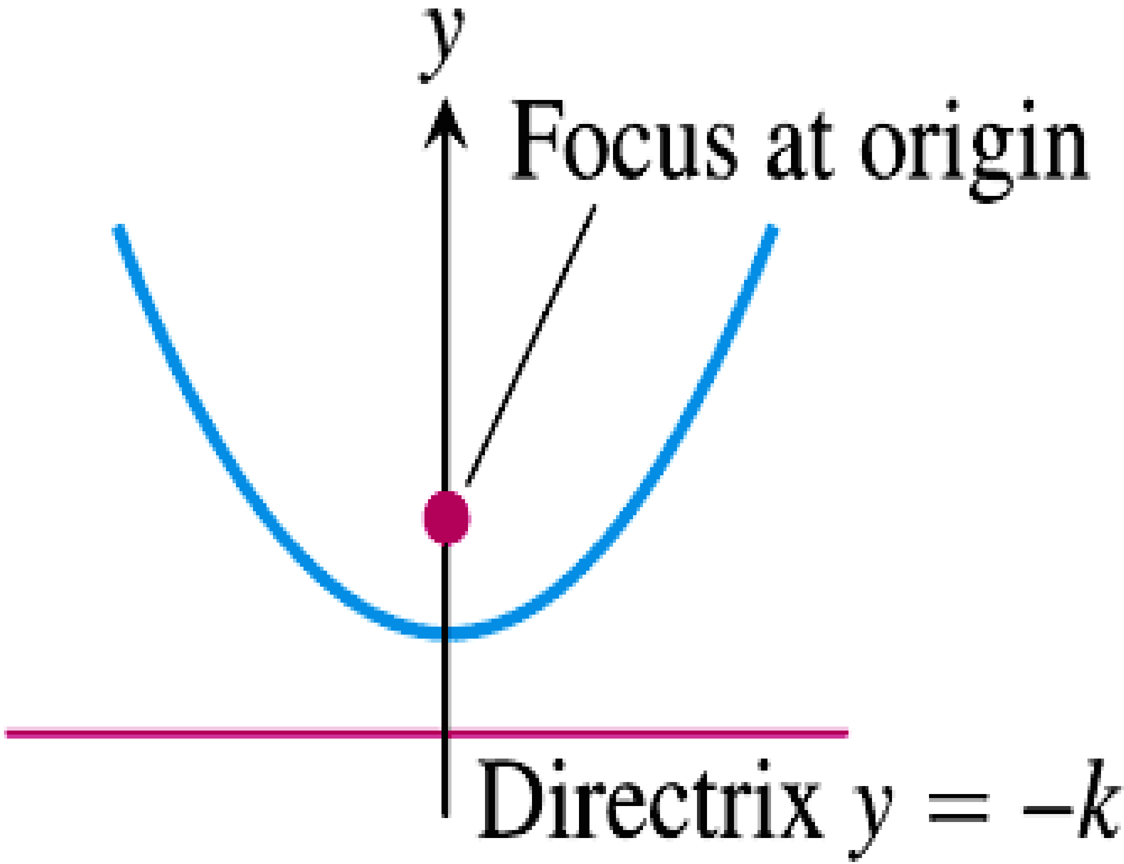
$$r = \frac{ke}{1 + e\sin\theta}.$$


y
Directrix $y = k$
Focus at origin

Case IV

If the directrix be $y = -k$, $(k > 0)$ (horizontal, below to the origin). Then the polar equation of the conic section is

$$r = \frac{ke}{1 - e\sin\theta}.$$


y
Focus at origin
Directrix $y = -k$

Polar equation of an ellipse

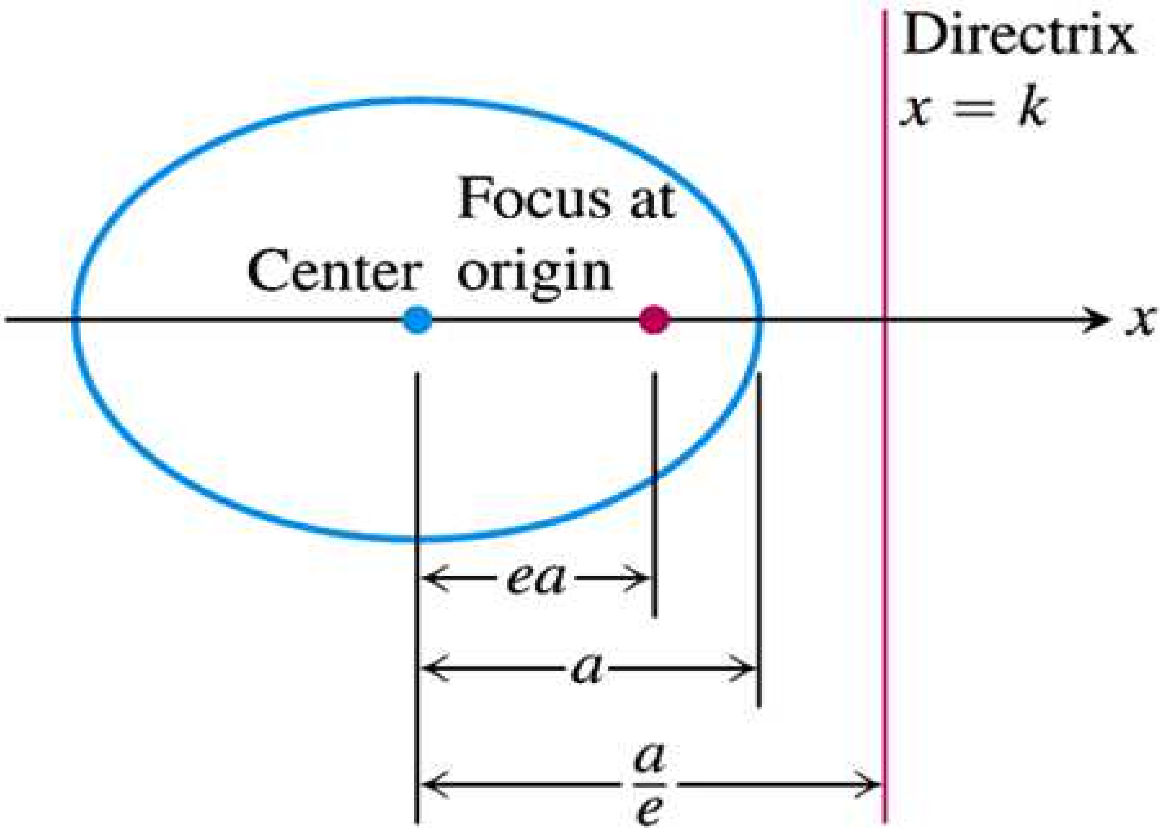
For an ellipse with semi-major axis a and eccentricity e (with focus at the origin), we have

$$k = \frac{a}{e} - ea$$

$$ke = a(1-e^2).$$

Hence from case I,

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$


Directrix
$x = k$
Focus at
Center origin
x
ea
a
$\frac{a}{e}$

Q:34. If $e = \frac{1}{4}$, and $x = -2$, then find polar equation of the conic section (Assume one focus at the origin).

Q:34. If $e = \frac{1}{4}$, and $x = -2$, then find polar equation of the conic section (Assume one focus at the origin).

Sol.

$$r = \frac{ke}{1 - e\cos\theta}$$

$$r = \frac{(2)\left(\frac{1}{4}\right)}{1 - \frac{1}{4}\cos\theta}$$

$$r = \frac{2}{4 - \cos\theta}.$$

Q:36. If $e = \frac{1}{3}$, and $y = 6$, then find polar equation of the conic section (Assume one focus at the origin).

Q:36. If $e = \frac{1}{3}$, and $y = 6$, then find polar equation of the conic section (Assume one focus at the origin).

Sol.

$$r = \frac{ke}{1 + e\sin\theta}$$

$$r = \frac{(6)\left(\frac{1}{3}\right)}{1 + \frac{1}{3}\sin\theta}$$

$$r = \frac{6}{3 + \sin\theta}.$$

Q:39. Sketch $r = \frac{25}{10-5\cos\theta}$. Include the directrix that corresponds to the focus at the origin. Level the vertices with appropriate polar coordinates. Label the center in the case of ellipse.

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Sol.

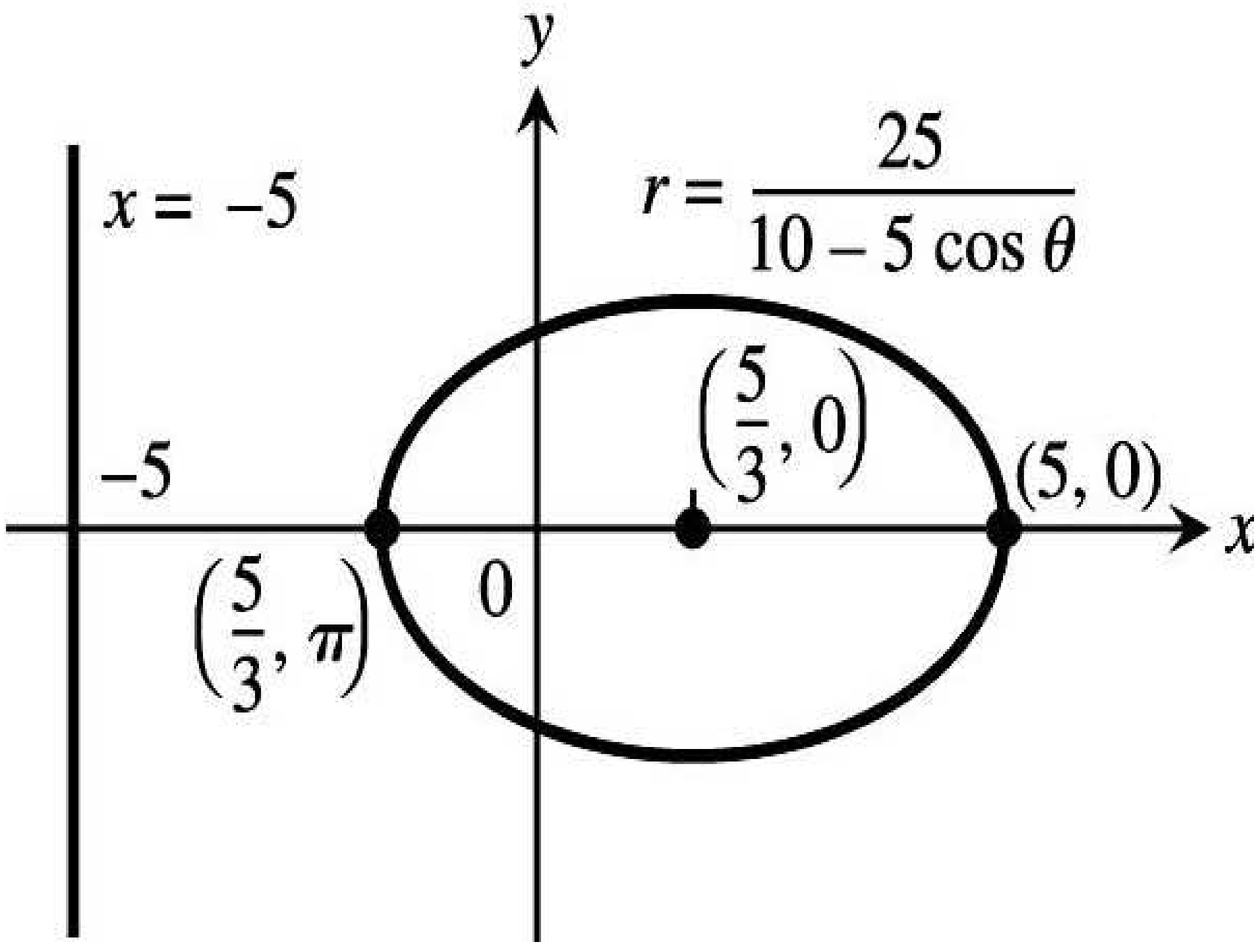
$$r = \frac{25}{10-5\cos\theta}$$
$$r = \frac{\frac{5}{2}}{1-\frac{1}{2}\cos\theta}.$$

So $e = \frac{1}{2}$ and $ke = \frac{5}{2} \Rightarrow k = 5$ and so $x = -5$ is the directrix. Since $e < 1$ so the curve is an ellipse.

Now for the ellipse

$$\begin{aligned} ke &= a(1-e^2) \\ \Rightarrow \frac{5}{2} &= a\left(1-\frac{1}{4}\right) \\ \Rightarrow a &= \frac{10}{3} \\ \Rightarrow ae &= \frac{5}{3}. \end{aligned}$$

Thus center $= \left(\frac{5}{3}, 0\right)$.



y
$r = \dfrac{25}{10 - 5\cos\theta}$
$x = -5$
-5
$\left(\frac{5}{3}, 0\right)$
$(5, 0)$
x
$\left(\frac{5}{3}, \pi\right)$
0

Q:41. Sketch $r = \frac{400}{16+8\sin\theta}$. Include the directrix that corresponds to the focus at the origin. Level the vertices with appropriate polar coordinates. Label the center in the case of ellipse.

Q:41. Sketch $r = \frac{400}{16+8\sin\theta}$. Include the directrix that corresponds to the focus at the origin. Level the vertices with appropriate polar coordinates. Label the center in the case of ellipse.

Sol.

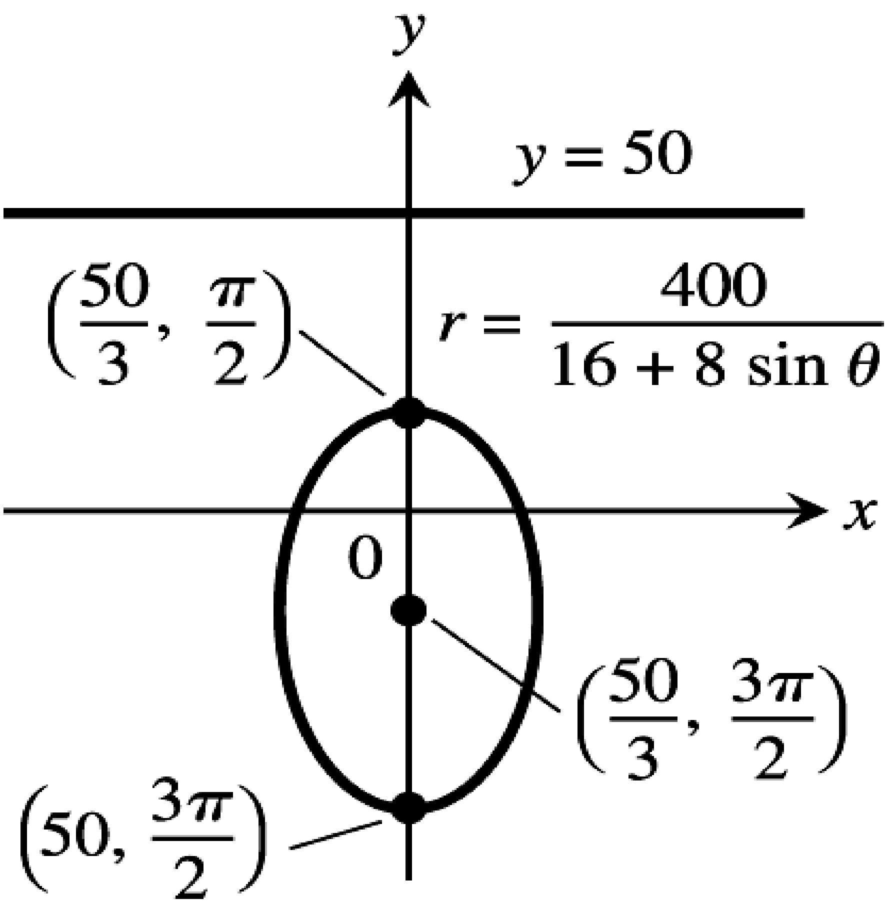
$$r = \frac{400}{16+8\sin\theta} = \frac{25}{1+\frac{1}{2}\sin\theta}.$$

So $e = \frac{1}{2}$ and $k = 50$. Thus $y = 50$ is the directrix. Since $e < 1$ so the curve is an ellipse.

Now for the ellipse

$$
\begin{aligned}
ke &= a(1-e^2) \\
\Rightarrow 25 &= a\left(1-\frac{1}{4}\right) \\
\Rightarrow a &= \frac{100}{3} \\
\Rightarrow ae &= \frac{50}{3}.
\end{aligned}
$$

Thus center $= \left(\frac{50}{3}, \frac{3\pi}{2}\right)$.



y
y = 50
$\left(\frac{50}{3}, \frac{\pi}{2}\right)$
$r = \frac{400}{16 + 8\sin\theta}$
x
0
$\left(\frac{50}{3}, \frac{3\pi}{2}\right)$
$\left(50, \frac{3\pi}{2}\right)$

Q:42. Sketch $r = \frac{12}{3+3\sin\theta}$. Include the directrix that corresponds to the focus at the origin. Level the vertices with appropriate polar coordinates. Label the center in the case of ellipse.

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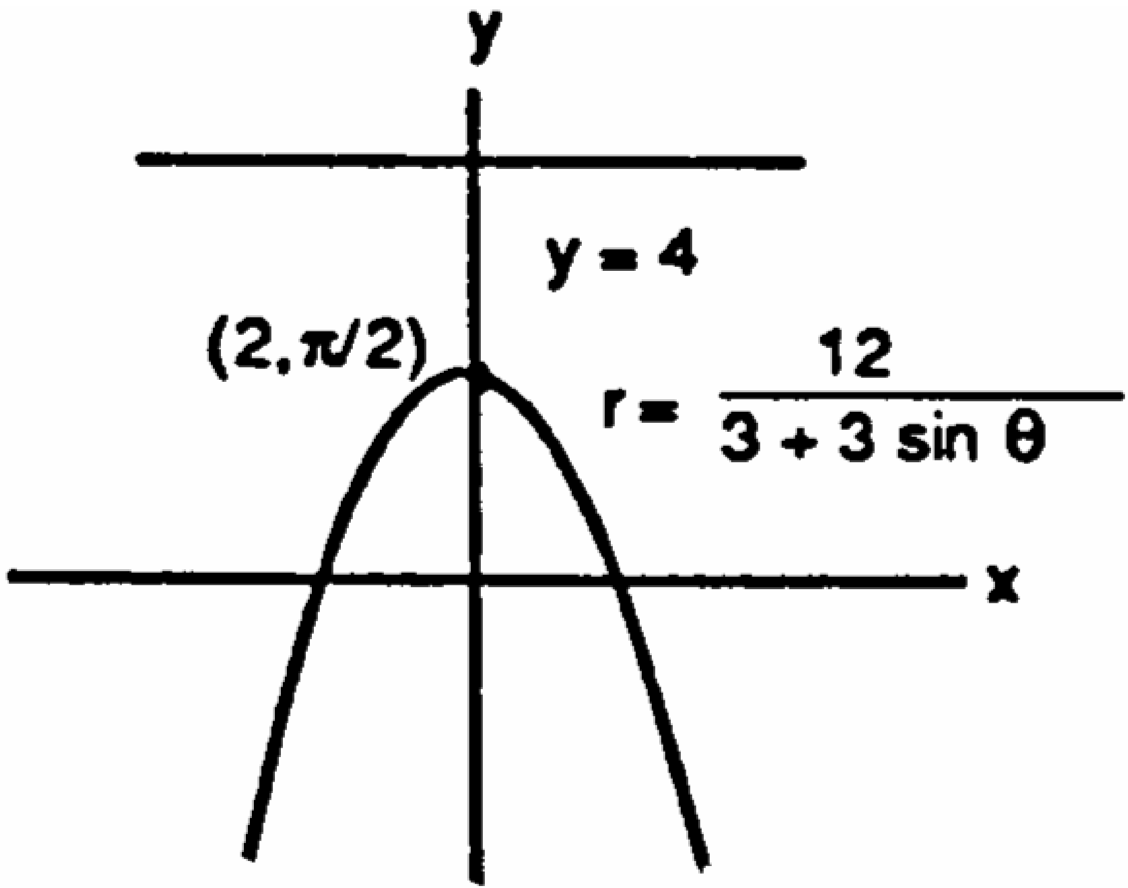
Sol.

$$r = \frac{12}{3+3\sin\theta}$$
$$r = \frac{4}{1+\sin\theta}.$$

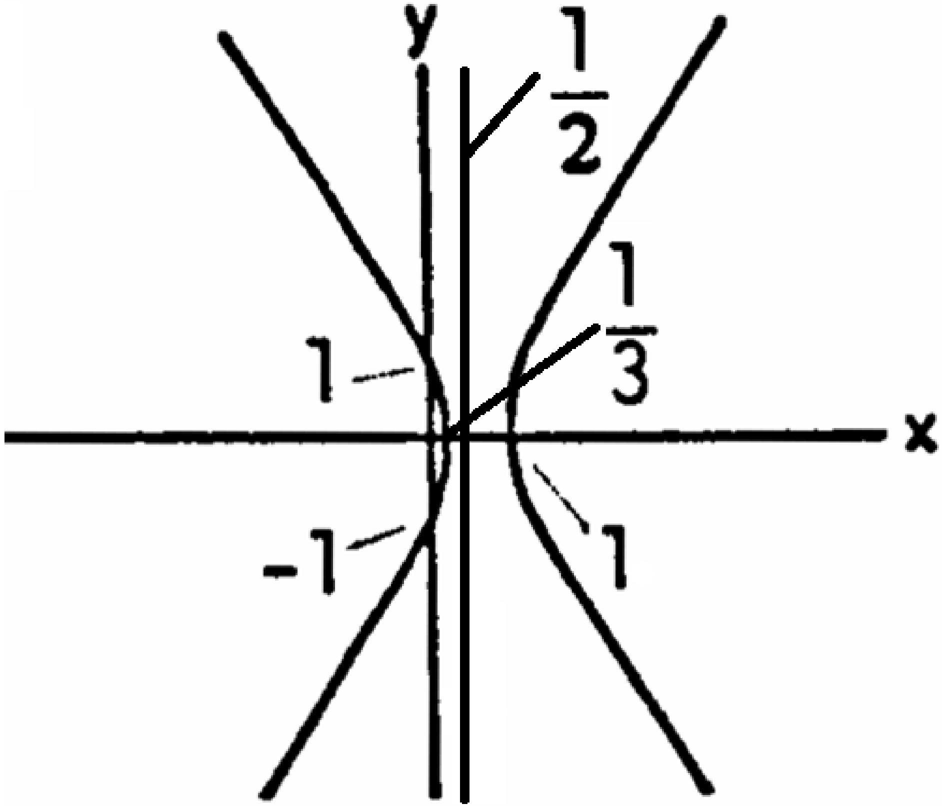
So $e = 1$ and $ke = 4 \Rightarrow k = 4$ and so $y = 4$ is the directrix. Since $e = 1$ so the curve is a parabola.


y
y = 4
(2, π/2)
r = 12 / (3 + 3 sin θ)
x

Q:74. Sketch $r = \frac{1}{1+2\cos\theta}$. Include the directrix that corresponds to the focus at the origin.

Q:74. Sketch $r = \frac{1}{1+2\cos\theta}$. Include the directrix that corresponds to the focus at the origin.

Sol. Given $r = \frac{1}{1+2\cos\theta}$, so $e = 2$ and $ke = 1 \Rightarrow k = 0.5$ and so $x = 0.5$ is the directrix that corresponds to the focus at the origin. Since $e > 1$ so the curve is a hyperbola.



$$r = 1/(1 + 2\cos\theta)$$