Assignment - 1

Problem 1:
DONE

Problem 2:
DONE

Problem 3:
Done

Problem 4:
(a) We observe that the program continuously halves the number beginning from 1 terminates with a 0. This happens because number 5e-324 **/** 2 was not representable in the IEEE double precision floating point system(smaller than ε_m) and thus it **underflowed** to 0. Last number printed before 0 is approximately the UFL.

(b) We observe that the code terminates with eps = 1.1102230246251565e-16 i.e adding 1 + eps = 1. This happened because **eps** is less than Machine precision (ε_m). Therefore adding a quantity less than will yield the same number.

(c)We observe that code terminates with inf. Just before inf it printed 8.98846567431158e+307. Numbers after 2*8.98846567431158e+307 are not printed and are merely represented as inf due to the fact because representable in the floating point system and thus it **overflowed** to 0. Last number printed before inf is approximately the OFL.

Problem 5:
Done

Problem 6:
Done

Problem 7:

a)

Euler's Constant at 100 iteration : 0.5822073316515288
Euler's Constant at 200 iteration : 0.5797135815734098
Euler's Constant at 300 iteration : 0.5788814056433012
Euler's Constant at 400 iteration : 0.5784651440685238
Euler's Constant at 500 iteration : 0.5782153315683285
Euler's Constant at 600 iteration : 0.5780487667534508
Euler's Constant at 700 iteration : 0.5779297805478292
Euler's Constant at 800 iteration : 0.5778405346932214
Euler's Constant at 900 iteration : 0.577771117576444
Euler's Constant at 1000 iteration : 0.5777155815682065
Euler's Constant at 1100 iteration : 0.5776701414855578
Euler's Constant at 1200 iteration : 0.5776322736978301
Euler's Constant at 1300 iteration : 0.5776002309764809
Euler's Constant at 1400 iteration : 0.5775727652416682
Euler's Constant at 1500 iteration : 0.5775489611978291
Euler's Constant at 1600 iteration : 0.5775281323494514
Euler's Constant at 1700 iteration : 0.5775097537135299
Euler's Constant at 1800 iteration : 0.5774934169591495
Euler's Constant at 1900 iteration : 0.5774787997122512
Euler's Constant at 2000 iteration : 0.5774656440682016
Euler's Constant at 2100 iteration : 0.577453741243179
Euler's Constant at 2200 iteration : 0.5774429204111771
Euler's Constant at 2300 iteration : 0.5774330404528918
Euler's Constant at 2400 iteration : 0.5774239837672805
Euler's Constant at 2500 iteration : 0.5774156515681996
Euler's Constant at 2600 iteration : 0.5774079602664202
Euler's Constant at 2700 iteration : 0.5774008386555254
Euler's Constant at 2800 iteration : 0.5773942257008384
Euler's Constant at 2900 iteration : 0.5773880687857824
Euler's Constant at 3000 iteration : 0.5773823223089227
Euler's Constant at 3100 iteration : 0.5773769465525795
Euler's Constant at 3200 iteration : 0.5773719067634975
Euler's Constant at 3300 iteration : 0.5773671724007521
Euler's Constant at 3400 iteration : 0.5773627165162711
Euler's Constant at 3500 iteration : 0.5773585152416505
Euler's Constant at 3600 iteration : 0.5773545473603523
Euler's Constant at 3700 iteration : 0.5773507939494777
Euler's Constant at 3800 iteration : 0.5773472380778735
Euler's Constant at 3900 iteration : 0.5773438645508655
Euler's Constant at 4000 iteration : 0.5773406596931707
Euler's Constant at 4100 iteration : 0.5773376111636459
Euler's Constant at 4200 iteration : 0.5773347077964406
Euler's Constant at 4300 iteration : 0.577331939464333
Euler's Constant at 4400 iteration : 0.5773292969607322
Euler's Constant at 4500 iteration : 0.5773267718973916
Euler's Constant at 4600 iteration : 0.5773243566154296
Euler's Constant at 4700 iteration : 0.5773220441077846
Euler's Constant at 4800 iteration : 0.5773198279512748
Euler's Constant at 4900 iteration : 0.5773177022470506
Euler's Constant at 5000 iteration : 0.577315661568166

b)

Euler's Constant at 100 iteration : 0.5772197901404903
Euler's Constant at 200 iteration : 0.5772167013748222
Euler's Constant at 300 iteration : 0.5772161263242399
Euler's Constant at 400 iteration : 0.5772159246680912
Euler's Constant at 500 iteration : 0.5772158312352449

Euler's Constant at 600 iteration : 0.577215780449559
Euler's Constant at 700 iteration : 0.5772157498141715
Euler's Constant at 800 iteration : 0.5772157299243794
Euler's Constant at 900 iteration : 0.5772157162847442
Euler's Constant at 1000 iteration : 0.5772157065265553
Euler's Constant at 1100 iteration : 0.5772156993055031
Euler's Constant at 1200 iteration : 0.5772156938126143
Euler's Constant at 1300 iteration : 0.577215689537403
Euler's Constant at 1400 iteration : 0.5772156861448545
Euler's Constant at 1500 iteration : 0.5772156834077089
Euler's Constant at 1600 iteration : 0.5772156811674058
Euler's Constant at 1700 iteration : 0.5772156793105871
Euler's Constant at 1800 iteration : 0.5772156777544755
Euler's Constant at 1900 iteration : 0.5772156764374801
Euler's Constant at 2000 iteration : 0.5772156753129938
Euler's Constant at 2100 iteration : 0.5772156743452568
Euler's Constant at 2200 iteration : 0.577215673506438
Euler's Constant at 2300 iteration : 0.5772156727746101
Euler's Constant at 2400 iteration : 0.5772156721323229
Euler's Constant at 2500 iteration : 0.5772156715655328
Euler's Constant at 2600 iteration : 0.5772156710628664
Euler's Constant at 2700 iteration : 0.5772156706149998
Euler's Constant at 2800 iteration : 0.5772156702142466
Euler's Constant at 2900 iteration : 0.5772156698542288
Euler's Constant at 3000 iteration : 0.5772156695296005
Euler's Constant at 3100 iteration : 0.5772156692358834
Euler's Constant at 3200 iteration : 0.5772156689692576
Euler's Constant at 3300 iteration : 0.5772156687264989
Euler's Constant at 3400 iteration : 0.5772156685048309
Euler's Constant at 3500 iteration : 0.5772156683019034
Euler's Constant at 3600 iteration : 0.5772156681156311
Euler's Constant at 3700 iteration : 0.577215667944273
Euler's Constant at 3800 iteration : 0.5772156677862554
Euler's Constant at 3900 iteration : 0.5772156676402354
Euler's Constant at 4000 iteration : 0.5772156675050191
Euler's Constant at 4100 iteration : 0.5772156673795781
Euler's Constant at 4200 iteration : 0.5772156672629993
Euler's Constant at 4300 iteration : 0.5772156671544533
Euler's Constant at 4400 iteration : 0.5772156670532187
Euler's Constant at 4500 iteration : 0.5772156669586632
Euler's Constant at 4600 iteration : 0.5772156668702006
Euler's Constant at 4700 iteration : 0.5772156667873283
Euler's Constant at 4800 iteration : 0.5772156667095789
Euler's Constant at 4900 iteration : 0.5772156666365351
Euler's Constant at 5000 iteration : 0.5772156665678327

We observe that value of euler's constant for 2 part is 0.577**2**156665678327 which is greater than the value of euler's constant obtained in 1 part which is 0.577**3**15661568166

Problem 8:

No Pivoting

Output:

n = 10
Solving Random Matrix

Condition number: (71.5453769293984436)
Error from un-pivoted solve is : 0.0000000000000035
Residual from un-pivoted solve is : 1.3248242049986465
Error from np.linalg.solve : 39.7652031456075790
Residual from np.linalg.solve : 0.0000000000000013

Solving Hilbert's Matrix

Condition number: (16024413512628.9003906250000000)
Error from un-pivoted solve is : 0.0000459395179676
Residual from un-pivoted solve is : 1.1333773853115643
Error from np.linalg.solve : 17638465.2618227750062943
Residual from np.linalg.solve : 0.0000000004454496

Solving Part 3

Condition number: (6.3137515146750447)
Error from un-pivoted solve is : 0.0000000000000000
Residual from un-pivoted solve is : 0.8519427513705972
Error from np.linalg.solve : 2.2360679774997898
Residual from np.linalg.solve : 0.0000000000000000

----
n = 20
Solving Random Matrix

Condition number: (55.0146776425699358)
Error from un-pivoted solve is : 0.0000000000000090
Residual from un-pivoted solve is : 1.9856453055939689
Error from np.linalg.solve : 10.0065665739301988
Residual from np.linalg.solve : 0.0000000000000012

Solving Hilbert's Matrix

Condition number: (1314422950044367104.0000000000000000)
Error from un-pivoted solve is : 3.1500166567989059
Residual from un-pivoted solve is : 1.4839954359892273

Error from np.linalg.solve : 4811595907.9607181549072266
Residual from np.linalg.solve : 0.0000001647100224

Solving Part 3

Condition number: (12.7062047361747101)
Error from un-pivoted solve is : 0.0000000000000000
Residual from un-pivoted solve is : 0.8598253374348430
Error from np.linalg.solve : 3.1622776601683791
Residual from np.linalg.solve : 0.0000000000000000

---

n = 30
Solving Random Matrix

Condition number: (1170.6696426025582696)
Error from un-pivoted solve is : 0.0000000000002354
Residual from un-pivoted solve is : 1.0923893984187669
Error from np.linalg.solve : 57.9834808179887560
Residual from np.linalg.solve : 0.0000000000000020

Solving Hilbert's Matrix

Condition number: (2895572130800129536.0000000000000000)
Error from un-pivoted solve is : 55.3011269933147815
Residual from un-pivoted solve is : 1.7275345448993327
Error from np.linalg.solve : 1725120419.5345187187194824
Residual from np.linalg.solve : 0.0000000698445846

Solving Part 3

Condition number: (19.0811366877282254)
Error from un-pivoted solve is : 0.0000000000000000
Residual from un-pivoted solve is : 0.8620709292256323
Error from np.linalg.solve : 3.8729833462074170
Residual from np.linalg.solve : 0.0000000000000000

---

n = 40
Solving Random Matrix

Condition number: (870.1900010803353780)
Error from un-pivoted solve is : 0.0000000000003612
Residual from un-pivoted solve is : 1.1386920370921423
Error from np.linalg.solve : 477.5000610680105524
Residual from np.linalg.solve : 0.0000000000000092

Solving Hilbert's Matrix

Condition number: (6507249795234633728.0000000000000000)
Error from un-pivoted solve is : 48.4290205961839320
Residual from un-pivoted solve is : 1.9212341785333358
Error from np.linalg.solve : 5262723604.2709436416625977
Residual from np.linalg.solve : 0.0000001273790155

Solving Part 3

Condition number: (25.4516995793570828)
Error from un-pivoted solve is : 0.0000000000000000
Residual from un-pivoted solve is : 0.8631255258376012
Error from np.linalg.solve : 4.4721359549995796
Residual from np.linalg.solve : 0.0000000000000000

Random Matrix -
Condition - The condition numbers are not very large
Error - Their exists some error but it is very low
Residual - Their residual is around 1 and 2 for all the cases
Necessity of pivoting: Condition number and error has reduced for most of the cases with pivoting.

Hilbert Matrix:
It has a very large condition number , and largest error and residual among all 3 matrices.

Necessity of pivoting: with pivoting error and residual number has increased

Type 3 Matrix:
It has the lowest condition number and 0 error from un-pivoted solve in all the cases.
It has low residuals compared to other matrices.
Necessity of pivoting:There is no effect of pivoting in this case.

Partially Pivoting

n = 10
Solving Random Matrix
Condition number: (36.9582414283928102)
Error from partially-pivoted solve is : 0.0000000000000015
Residual from partially-pivoted solve is : 1.4395239022829887
Error from np.linalg.solve : 6.6865471493018447
Residual from np.linalg.solve : 0.0000000000000006

Solving Hilbert's Matrix
Condition number: (16024413512628.9003906250000000)
Error from partially-pivoted solve is : 0.0001243116759689
Residual from partially-pivoted solve is : 1.2545354352529401
Error from np.linalg.solve : 84945023059.0608367919921875
Residual from np.linalg.solve : 0.0000036086213939

Solving Part 3
Condition number: (6.3137515146750447)
Error from partially-pivoted solve is : 0.0000000000000000
Residual from partially-pivoted solve is : 0.8519427513705972
Error from np.linalg.solve : 2.2360679774997898
Residual from np.linalg.solve : 0.0000000000000000

----
n = 20
Solving Random Matrix
Condition number: (299.6069320274039001)
Error from partially-pivoted solve is : 0.0000000000000062
Residual from partially-pivoted solve is : 2.2775819367800128

Error from np.linalg.solve : 23.7207488217389475
Residual from np.linalg.solve : 0.0000000000000021

Solving Hilbert's Matrix
Condition number: (1314422950044367104.0000000000000000)
Error from partially-pivoted solve is : 44.4964565223568655
Residual from partially-pivoted solve is : 1.6795455596500768
Error from np.linalg.solve : 5376804301778636.0000000000000000
Residual from np.linalg.solve : 0.1920812107341487

Solving Part 3
Condition number: (12.7062047361747101)
Error from partially-pivoted solve is : 0.0000000000000000
Residual from partially-pivoted solve is : 0.8598253374348430
Error from np.linalg.solve : 3.1622776601683791
Residual from np.linalg.solve : 0.0000000000000000

---

n = 30
Solving Random Matrix
Condition number: (765.4492065407964674)
Error from partially-pivoted solve is : 0.0000000000000278
Residual from partially-pivoted solve is : 2.9448493460223020
Error from np.linalg.solve : 31.6073030203569694
Residual from np.linalg.solve : 0.0000000000000047

Solving Hilbert's Matrix
Condition number: (2895572130800129536.0000000000000000)
Error from partially-pivoted solve is : 58.4654923838652536
Residual from partially-pivoted solve is : 1.9071677579528352
Error from np.linalg.solve : 6478050265679351.0000000000000000
Residual from np.linalg.solve : 0.2425982847681172

Solving Part 3
Condition number: (19.0811366877282254)
Error from partially-pivoted solve is : 0.0000000000000000
Residual from partially-pivoted solve is : 0.8620709292256323
Error from np.linalg.solve : 3.8729833462074170
Residual from np.linalg.solve : 0.0000000000000000

---

n = 40
Solving Random Matrix
Condition number: (2887.1689182390000497)
Error from partially-pivoted solve is : 0.0000000000000219
Residual from partially-pivoted solve is : 3.0203620978347114
Error from np.linalg.solve : 81.4337012606102206
Residual from np.linalg.solve : 0.0000000000000097

Solving Hilbert's Matrix
Condition number: (6507249795234633728.0000000000000000)
Error from partially-pivoted solve is : 20.2496214518533009
Residual from partially-pivoted solve is : 2.0156509111177816
Error from np.linalg.solve : 22377390919216360.0000000000000000
Residual from np.linalg.solve : 0.7075767821240854

Solving Part 3
Condition number: (25.4516995793570828)
Error from partially-pivoted solve is : 0.0000000000000000
Residual from partially-pivoted solve is : 0.8631255258376012
Error from np.linalg.solve : 4.4721359549995796
Residual from np.linalg.solve : 0.0000000000000000

HW-1

Ishit Bajpai

1. We have to find solution $x_0 = b/a$ to the linear equation $ax = b$, $a, x, b \in R$.

Absolute forward error $= |x - x_0|$

Relative forward error $= \dfrac{|x - x_0|}{|x|}$

Absolute backward error $= |b - ax| = |a(x - x_0)|$

Relative backward error $= \dfrac{|b - ax|}{|b|} = \dfrac{|a(x - x_0)|}{|ax|}$

- Absolute Condition Number $= \dfrac{\text{Absolute Forward Error}}{\text{Absolute Backwar Error}}$

$$= \frac{|x - x_0|}{|a(x - x_0)|} = \frac{1}{|a|}$$

- Relative Condition Number $= \dfrac{\text{Relative Forward Error}}{\text{Relative Backward Error}}$

$$= \frac{\dfrac{|x - x_0|}{|x|}}{\dfrac{|a(x - x_0)|}{|ax|}} = |1| = \underline{\underline{1}}$$

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2. $K_{rel}(x) = \left|\frac{x f'(x)}{f(x)}\right|$ proved in Lecture-2

$K_{abs}(x) = ?$, let $y = f(x)$ be
for a Δx change in x, $\hat{y} = f(x+\Delta x)$

Using taylor expansion, $f(x+\Delta x) = f(x) + \frac{\Delta x}{1!} f'(x)$

↑ Absolute forward Error (ignoring higher order terms)

$K_{abs}(x) = \frac{|f(x) - f(x+\Delta x)|}{|x - (x+\Delta x)|}$

Absolute Backward Error.

$= \frac{|f(x) - f(x) + \Delta x f'(x)|}{|x - x - \Delta x|} = |f'(x)|$

(a) $(x-1)^{\alpha}$

$K_{abs}(x) = |f'(x)| = |\alpha(x-1)^{\alpha-1}|$

There are 2 cases $\alpha > 1$ and $\alpha < 1$ (for $\alpha = 1$, $K_{abs}(x)$ is constant)

(i) $\alpha < 1$

$K_{abs}(x) = \frac{\alpha}{(x-1)^{1-\alpha}}$

as $x \to 1$, $K_{abs}(x) \to \infty$

$\therefore$ $K_{abs}(x)$ will be large for values near $x = 1$.

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(ii) $\alpha > 1$

$k_{abs}(x) = |\alpha (x-1)^{\alpha-1}|$

~~here as $x \to \infty$, $k_{abs}(x) \to \infty$~~ $n \in \mathbb{N} \cup \{0\}$

~~also as $x \to -\infty$, $k_{abs}(x) \to \infty$~~ if $\alpha = 2n+1$

~~OR $x \to -\infty$,~~ $n \in \mathbb{N}$

as $x \to \infty$, $(x-1)^{\alpha-1} \to \infty \Rightarrow k_{abs}(x) \to \infty$

also as

$x \to -\infty$, $(x-1)^{\alpha-1} \to \infty$ $(\alpha = 2n+1, n \in \mathbb{N} \cup \{0\})$

$\Rightarrow k_{abs}(x) \to \infty$

or

$x \to \infty$, $(x-1)^{\alpha-1} \to -\infty$ $(\alpha = 2n, n \in \mathbb{N}\}$

$\Rightarrow k_{abs}(x) \to \infty$

$\therefore$ for $x = \infty, -\infty$ $k_{abs}(x)$ will have large values. if $\alpha > 1$

Now for relative condition number

$k_{rel}(x) = \left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{x \cdot \alpha (x-1)^{\alpha-1}}{(x-1)^{\alpha}}\right|$

$= \left|\frac{x\alpha}{(x-1)}\right|$

(i) $k_{rel}(x)$ will have large values if $x \to 1$, as $\left|\frac{x\alpha}{x-1}\right| \to \infty$

$\therefore$ $k_{rel}(x)$ will have large values for value near 1.

(b) $x^{-1}e^{x}$

$$K_{abs}(x) = \cancel{x^{-1}e^{x}}\ |f'(x)| = \left|\frac{d}{dx}(x^{-1}e^{x})\right|$$

$$= \left|\frac{d}{dx}\left(\frac{e^{x}}{x}\right)\right| = \left|\frac{xe^{x}-e^{x}}{x^2}\right| = \left|\frac{e^{x}(x-1)}{x^2}\right|$$

as $x \to 0$, $\frac{e^{x}(x-1)}{x^2} \to \infty \Rightarrow K_{abs}(x) \to \infty$

$\therefore$ for values near $x=0$, $K_{abs}(x)$ will be large.

$$\cancel{K_{rel}(x) = \left|\frac{x f'(x)}{f(x)}\right| = \left|x \Big/ \frac{e^{x}(x-1)}{x^2} \cdot \frac{1}{\frac{e^{x}}{x}}\right|}$$

$\cancel{\neq \sqrt{(x-1)}},$

$$K_{abs}(x) = \left|e^{x}\left(\frac{1}{x} - \frac{1}{x^2}\right)\right|$$

as $x \to \infty$, $\left|e^{x}\left(\frac{1}{x} - \frac{1}{x^2}\right)\right| \to \infty$

(as e^{x} increases much faster than $\frac{1}{x}$ and $\frac{1}{x^2}$)

$\Rightarrow K_{abs}(x) \to \infty$

$\therefore$ for absolute ~~relative~~ condition number, condition number takes large values near $x=0$ and ~~$x \to \infty$~~ for x tending to ∞.

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Relative Condition number.

$$K_{rel}(x) = \left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{x \cdot e^x(x-1)}{x^2} \cdot \frac{1}{e^x x^{-1}}\right|$$

$$= |(x-1)|$$

as $x \to \infty$ $(x-1) \to \infty$
$\Rightarrow K_{rel}(x) \to \infty$

" $x \to -\infty$, $(x-1) \to -\infty$
$\Rightarrow K_{rel}(x) \to \infty$

therefore $K_{rel}(x)$ takes larges values as $x \to \infty$ or $x \to -\infty$.

(b) $\ln x$

$$K_{abs}(x) = |f'(x)| = \left|\frac{1}{x}\right|$$

as $x \to 0^+$, $\left|\frac{1}{x}\right| \to \infty$

$\therefore$ $K_{abs}(x)$ takes large values for values, $x = 0^+$

$$K_{rel}(x) = \left|\frac{x f'(x)}{f(x)}\right| = \left|x \cdot \frac{1}{x} \cdot \frac{1}{\ln x}\right|$$

$$= \left|\frac{1}{\ln x}\right|$$

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as $x \to 1$, $\left|\frac{1}{\ln x}\right| \to \infty \Rightarrow K_{rel}(x) \to \infty$

$\therefore K_{rel}(x)$ is large as $x \to \infty$.

(d) $\frac{1}{(1+x^{-1})}$

$K_{abs}(x) = |f'(x)| = \left|\frac{d}{dx}\left(\frac{x}{x+1}\right)\right|$

$= \frac{(x+1).1 - x}{(x+1)^2} = \frac{1}{(x+1)^2}$

as $x \to -1$, $\frac{1}{(x+1)^2} \to \infty$

$K_{abs}(x)$ is large for values near $x = -1$

$K_{rel}(x) = \left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{x \cdot \frac{1}{(x+1)^2}}{\frac{x}{(x+1)}}\right|$

$= \left|\frac{1}{(x+1)}\right|$

as $x \to -1$, $\frac{1}{|(x+1)|} \to \infty$

$\therefore K_{rel}(x)$ is large for values near $x = -1$.

3.

(a) Given : $f(x(\varepsilon)) + \varepsilon p(x(\varepsilon)) = 0$ with $x(0) = x^*$

Assumption : function exists and is differentiable

differentiating wrt ε

$$f'(x(\varepsilon)) \cdot x'(\varepsilon) \cdot 1 + \left(\varepsilon \cdot p'(x(\varepsilon)) \cdot x'(\varepsilon) \cdot 1 + 1 \cdot p(x(\varepsilon))\right) = 0$$

$$x'(\varepsilon)\left(f'(x(\varepsilon)) + \varepsilon \cdot p'(x(\varepsilon))\right) = -p(x(\varepsilon))$$

$$\frac{dx(\varepsilon)}{d\varepsilon} = \frac{-p(x(\varepsilon))}{(f'(x(\varepsilon)) + \varepsilon p'(x(\varepsilon))}$$

$$\Rightarrow \frac{dx}{d\varepsilon} = \frac{-p(x(\varepsilon))}{(f'(x(\varepsilon)) + \varepsilon p'(x(\varepsilon)))}$$

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = \frac{-p(x(0))}{(f'(x(0)) + 0)} = \frac{-p(x^*)}{f'(x^*)}$$

Hence proved $\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = \frac{-p(x^*)}{f'(x^*)}$

(b) $f(x) = (x-1)(x-2) \text{ ---- } (x-20)$

or

$f(x) = a_0 + a_1 x^1 - \text{ --- } - a_{20} x^{20}$

for abbrorpriat coefficient.

$p(x) = x^{19}$

$f'(x) = (+1)(x-2) \cdots (x-20) +$

$(x-1)(+1) \cdots (x-20) +$

$\vdots$

$(x-1)(x-2) \cdots (1)$

$\Rightarrow$ ~~$f'(x) =$~~

for any $x^* = j$, $1 \le j \le 20$

$f'(j) = 0 + 0 \cdots + \underbrace{\prod_{\substack{k=1 \\ j \neq k}}^{20} (j-k)}_{j^{th} \text{ term}} + 0 - + 0$

$f'(j) = \prod_{\substack{k=1 \\ j \neq k}}^{20} (j-k), \quad 1 \le k \le 20$ &

Now $p(j) = j^{19}$

Using previous result

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,\, x^*=j} = -\frac{p(j)}{f'(j)}$$

Plugging Values

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,\, x^*=j} = -\frac{j^{19}}{\prod\limits_{j\neq k}(j-k)} = -\prod_{\substack{k=1\\ j\neq k}}^{20}\left(\frac{j}{j-k}\right)$$

(c) $x^* = 1$

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,\, x^*=1} = -\left(\frac{1}{(1-2)(1-3)\cdots(1-20)}\right)$$

$$= +\frac{1}{19!}$$

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,x^*=20} = -\prod_{k\neq 20}\left(\frac{20}{(20-k)}\right); \; k=1,2 \dots 20$$

$$= -\left(\frac{(20)}{(19)} \times \frac{(20)}{(18)} \dots \times \frac{(20)}{(1)}\right)$$

$$= -\frac{\cancel{20^{20}}}{\cancel{19!}} \quad -\frac{20^{19}}{19!}$$

for $x^*=1$, $\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,x^*=1} = \frac{1}{19!}$ whereas for

$x^*=20$, $\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,x^*=20} = -\frac{20^{19}}{19!}$.

Since for small perturbation ~~is~~
~~x*~~ for $x^*=1$.

$x^*=1$, is more stable to perturbation

$$\left(\left|\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,x^*=1}\right| < \left|\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,x^*=20}\right|\right)$$

than $x^*=20$.

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Q6

Given: $\|\|$ is a vector norm in R^m
$A \in R^{m \times n}$
$\text{Rank}(A) = n$
$\Rightarrow$ A will have pivot in all columns.

To prove: $\|x\|_A = \|Ax\|$ is vector norm on R^n.

Therefore, we will have to verify three properties of norms

(i) $\|x\|_A \geq 0$ and $\|x\|_A = 0 \Leftrightarrow x = 0$ $\forall x \in R^n$

$\|x\|_A = \underbrace{\|Ax\|}_{R^m}$

Let $b \in$ ~~R^m~~ such that $\overset{m \times n}{A}\overset{n \times 1}{x} = b$, then there will be a unique solution x for the same as rank(A) = n.
Therefore it will have pivot in all columns, and using backward subsitution we will get a unique soln.

$\Rightarrow \|x\|_A = \|b\|$, $b \in R^m$

Since it's given $\|\cdot\|$ is a norm, therefore $\|b\| \geq 0$, $\forall b \in R^m$.

$\Rightarrow \|x\|_A = \|b\| \geq 0$ $\forall b \in R^m$

$\Rightarrow \|x\|_A \geq 0 \quad \forall\ Ax \in R^m$

Now, $\|x\|_A = 0 \Rightarrow \|Ax\| = 0$
$\Rightarrow \|b\| = 0$

~~Since~~ Since $\|\cdot\|$ is a norm on R^m
$\Rightarrow b = 0$
$\Rightarrow Ax = 0$
$\Rightarrow$ $\cancel{x=0}$

$\therefore \|x\|_A = 0 \Rightarrow Ax = 0$
$\Rightarrow x = 0$ (as Rank $A = n$)

$\sim$ $x = 0 \Rightarrow$ $Ax = 0 \Rightarrow b = 0 \Rightarrow \|x\|_A = 0$.

$\therefore Ax = 0 \Rightarrow \|x\|_A = 0$

$\therefore \|x\|_A = 0 \Leftrightarrow Ax = 0$
$\cancel{x=0}$

(ii) $\|\alpha x\|_A = |\alpha|\, \|x\|_A \quad , \forall\ \alpha \in R, x \in R^n$.

$\|\alpha x\|_A = \|A(\alpha x)\| = \|\alpha(Ax)\|$
$= |\alpha|\, \|Ax\|$
$= |\alpha|\, \|x\|_A$

(As $\|\cdot\|$ is a norm on R^m
$\|\alpha b\| = |\alpha| \|b\|$)

(iii) $\|x+y\|_A \le \|x\|_A + \|y\|_A \quad \forall$ ~~x, y~~ $Ax, Ay \in R^m$

~~Similar to argument line Q1, let~~

~~Proof~~

$$\|x+y\|_A = \|A(x+y)\|$$
$$= \|Ax + Ay\| \quad \text{(distributive property)}$$

(since $\|\ \|$ is a norm on R^m
$\|B_1 + B_2\| \le \|B_1\| + \|B_2\|, \ \forall B_1, B_2 \in R^m$)

~~Since~~ Using this property

$$\|x+y\|_A = \|Ax + Ay\| \le \|Ax\| + \|Ay\|$$

$$\forall \ Ax, Ay \in R^m$$

$$\Rightarrow \|x+y\|_A \le \|Ax\| + \|Ay\|$$
$$\le \|x\|_A + \|y\|_A$$

$$\therefore \|x+y\|_A \le \|x\|_A + \|y\|_A$$

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b) for $A \in R^{m \times n}$, verify norm properties on Frobenius norm.

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

(i) $\|A\|_F \geq 0$ and $\|A\|_F = 0 \iff A = 0$

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

$$\geq 0 \quad \forall\, a_{ij} \in R \;\; 1 \leq i \leq m, \; 1 \leq j \leq n.$$

if $\|A\|_F = 0 \Rightarrow \|A\|_F^2 = 0$

$$\Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 = 0$$

$$\Rightarrow a_{ij} = 0 \quad \text{for } \forall\; 1 \leq i \leq m, \; 1 \leq j \leq n.$$

$$\Rightarrow A = 0 \quad (\text{zero matrix})$$

& if $A = 0$, $\Rightarrow a_{ij} = 0 \quad \forall\; 1 \leq i \leq m, \; 1 \leq j \leq n$

$$\Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 = 0$$

$$\Rightarrow \|A\|_F = 0$$

$\therefore \; \|A\|_F = 0 \iff A = 0$

(ii) $\|\alpha A\|_F = |\alpha| \|A\|_F \quad \forall \alpha \in R$

$$\|\alpha A\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha a_{ij}|^2 \right)^{1/2}$$

$$= \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha|^2 |a_{ij}|^2 \right)^{1/2}$$

$$= \left(|\alpha|^2 \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

$$= |\alpha| \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

$$= |\alpha| (\|A\|_F) = |\alpha| \|A\|_F$$

$\therefore \; \|\alpha A\|_F = |\alpha| \|A\|_F \quad \forall \alpha \in R$

(iii) $\|A+B\|_F \le \|A\|_F + \|B\|_F$

$$\|A+B\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij} + b_{ij}|^2$$

$$\le \sum_{i=1}^{m} \sum_{j=1}^{n} \left(|a_{ij}| + |b_{ij}| \right)^2$$

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Since $|a_{ij}+b_{ij}|^2 = |a_{ij}^2+b_{ij}^2 + 2a_{ij}b_{ij}|$
and $(|a_{ij}|+|b_{ij}|)^2 = (a_{ij}^2 + b_{ij}^2 + 2|a_{ij}||b_{ij}|)$

Therefore $2|a_{ij}||b_{ij}| \geq 0$ and add to $a_{ij}^2 + b_{ij}^2$ make it more larger, however $2a_{ij}b_{ij}$ can be less than 0, which will decrease the value of $|a_{ij}+b_{ij}|^2$.

$$\Rightarrow \|A+B\|_F^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\left(|a_{ij}| + |b_{ij}|\right)^2$$

$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n}\left(a_{ij}^2 + b_{ij}^2 + 2|a_{ij}||b_{ij}|\right)$$

$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2 + \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}^2 + 2\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}||b_{ij}| \quad \text{— (1)}$$

Using Cauchy-Schwartz,

$(|a \cdot b| \leq \|a\|\|b\|$, here a, b are vectors and $\|a\| = \sqrt{a_1+a_2 \cdots a_n^2}$
$\|b\| = \sqrt{b_1^2+b_2^2 \cdots b_n^2}$

OR

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right)$$

Using this

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}||b_{ij}| \leq \sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|b_{ij}|^2\right)}$$

Plugging in (1)

$\|A+B\|_F$

$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n} |b_{ij}|^2 + 2\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2 \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}^2\right)^{1/2}$$

$$\leq \left(\left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} + \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |b_{ij}|^2\right)^{1/2}\right)^2$$

$$\leq \left(\|A\|_F + \|B\|_F\right)$$

Therefore all 3 properties of norm are proved.

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(c) Given: $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$

$$E = uv^T = \begin{bmatrix} u_1v_1 & \cdots & u_1v_n \\ \vdots & & \vdots \\ u_mv_1 & \cdots & u_mv_n \end{bmatrix}$$

To Prove: $\|E\|_F = \|E\|_2 = \|u\|_2 \|v\|_2$

$E \in \mathbb{R}^{m \times n}$

$$\|E\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |e_{ij}|^2 \right)^{1/2}$$

$$= \|E\|_2$$

$$\|E\|_2 = \max_{x \in \mathbb{R}^n} \|Ex\|_2 = \max_{x \in \mathbb{R}^n} \|uv^Tx\|_2$$

$$= \max_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \|u\|_2 |v^Tx| \qquad \underset{1\times n}{v^T} \underset{n\times 1}{x} \in \mathbb{R}$$

Using Cauchy-Schwarz ($|u \cdot v| \leq \|u\|\|v\|$)

$$|v^T \cdot x| \leq \|v^T\|_2 \|x\|_2$$

$$\Rightarrow |v^Tx| \leq \|v\|_2 \, 1$$

Subsituting in above equation

$$\|E\|_2 = \|U\|_2 \, \|V\|_2$$

$\therefore$ proved.

Q5

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Using the defination $k_{rel}(x) = \left|\frac{x f'(x)}{f(x)}\right|$

Condition number is calculated in the Code

$$k_{rel}(x) = \left|\frac{x}{\sin x \cos x}\right|$$

due to uncorrect representation of X,
$X = \frac{\pi}{4} + 2\pi \times 10^j \quad j = 0, \ldots 20.$

$$\tan\left(\frac{\pi}{4} + 2\pi \times 10^j\right) \neq 1$$

as $X\uparrow$, $k_{rel}(x)\uparrow$ and maximum

input error is $\underline{\underline{1.2018}}$ observed

for $j = 17$.

$$|x - \hat{x}| = \frac{|1 - \tan x|}{k(x)} \times x$$

~~Ans~~

- It is given that error is due to rounding of π to 16 decemal places.

As $j\uparrow$, π is multiplied by 2×10^j, thus magnifying the error in representation of x, which can be seen ~~evidently~~ from results.
(~~Ans~~ Roughly as $j\uparrow$, absolute input error$\uparrow$)