Itô Formulae for Stochastic Calculus

Introduction:

Throughout what follows, assume W_t is a standard Wiener Process (otherwise known as a Brownian Motion) with the following properties:

1. $W_0 = 0$;
2. The process W_t has independent increments. That is, for $t_0 < t_1 \leq t_2 < t_3$, we have $W_{t_3} - W_{t_2}$ and $W_{t_1} - W_{t_0}$ are independent random variables;
3. The process has Gaussian increments. That is, for $s < t$, we have $W_t - W_s \sim \mathcal{N}(0, |t-s|)$;
4. The process W_t has continuous paths.

1.1. Itô Formula I: Differential Form

$$dF(W_t) = \frac{dF}{dW_t} dW_t + \frac{1}{2} \frac{d^2F}{dW_t^2}. \tag{1}$$

Q: When do I use it?
A: When we have $F = F(W_t)$ which depends on W_t only.

Q: How do I derive this?
A: Write the Taylor expansion of $F(W_t + dW_t)$ up to second order and note that $dW_t^2 = dt$.

1.2. Itô Formula I: Integral Form

$$\int_0^t \frac{dF}{dW_s} dW_s = F(W_t) - F(W_0) - \frac{1}{2} \int_0^t \frac{d^2F}{dW_s^2} ds. \tag{2}$$

Q: When do I use it?
A: When we have $F = F(W_t)$ which depends on W_t only and we are required to work in integral form.

Q: How do I derive this?
A: Rearrange (1) and integrate between 0 and t, note the use of the dummy variable s.

2.1. Itô Formula II: Differential Form

$$dF(t, W_t) = \left(\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial W_t^2} \right) dt + \frac{\partial F}{\partial W_t} dW_t. \tag{3}$$

Q: When do I use it?
A: When we have $F = F(t, W_t)$ which now depends on both t and W_t.

Q: How do I derive this?
A: Write the Taylor expansion of $F(t + dt, W_t + dW_t)$ up to first order in t and second order in W_t, and note that $dW_t^2 = dt$.

2.2. Itô Formula II: Integral Form

$$\int_0^t \frac{\partial F}{\partial W_s} dW_s = F(t, W_t) - F(0, W_0) - \int_0^t \left(\frac{\partial F}{ds} + \frac{1}{2} \frac{\partial^2 F}{\partial W_s^2} \right) ds. \tag{4}$$

Q: When do I use it?
A: When we have $F = F(t, W_t)$ which now depends on both t and W_t and we are required to write it in integral form.

Q: How do I derive this?
A: Rearrange (3) and integrate between 0 and t, note the use of the dummy variable s.

3. Itô Formula III

Suppose we now have an additional SDE given by:

$$dS = \mu S dt + \sigma S dW_t.$$

And we wish to find $V = V(S)$, a function of S. Then Itô III is:

$$dV(S) = \left(\mu S \frac{dV}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 V}{dS^2} \right) dt + \sigma S \frac{dV}{dS} dW_t. \tag{5}$$

Q: When do I use it?
A: When we have $V = V(S)$ which depends on another SDE given by S.

Q: How do I derive this?
A: Perform a Taylor expansion of $V(S + dS)$ up to second order and then substitute in the value for dS where it appears, noting that higher orders of dt such as $dt^{\frac{3}{2}}$ and dt^2 get ignored.

4. Itô Formula IV

Suppose we have dS as before and we wish to find $V = V(t, S)$, a function of both t and S now. Then Itô IV is:

$$dV(t,S) = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW_t. \tag{6}$$

Q: When do I use it?
A: When we have $V = V(t, S)$ which depends on t and another SDE given by S.

Q: How do I derive this?
A: Perform a Taylor expansion of $V(t + dt, S + dS)$ up to first order in t and second order in S. Then substitute in the value for dS where it appears, noting that higher orders of dt such as $dt^{\frac{3}{2}}$ and dt^2 get ignored.

5. Itô Formula V: General Itô

Suppose we have some G_t that satisfies the SDE given by:

$$dG_t = A(t, G_t)dt + B(t, G_t)dW_t.$$

Now let $F = F(t, G_t)$ depend on both t and G_t. Then Itô V, otherwise known as the general Itô formula, is:

$$dF(t,G_t) = \left(\frac{\partial F}{\partial t} + A(t,G_t)\frac{\partial F}{\partial G_t} + \frac{1}{2}B^2(t,G_t)\frac{\partial^2 F}{\partial G_t^2}\right)dt + B(t,G_t)\frac{\partial F}{\partial G_t}dW_t. \tag{7}$$

Q: When do I use it?
A: When we have $F = F(t, G_t)$ which depends on t and another SDE given by G_t.

Q: How do I derive this?
A: Perform a Taylor expansion of $F(t + dt, G_t + dG_t)$ up to first order in t and second order in G_t. Then substitute in the value for dG_t where it appears, noting that higher orders of dt such as $dt^{\frac{3}{2}}$ and dt^2 get ignored. Also note, this is simply the more general form of Itô IV given above.

6. Itô Formula VI: Higher Dimensional Itô

Suppose we now have a multi-factor model, depending on two SDE's given by:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_t,$$

and

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_t.$$

Suppose the following correlation also exists:

$$\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt.$$

Then Itô VI is:

$$\begin{aligned} &dV(t, S_1, S_2) = \\ &\left(\frac{\partial V}{\partial t} + \mu_1 S_1 \frac{\partial V}{\partial S_1} + \mu_2 S_2 \frac{\partial V}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \right) dt + \\ &\qquad \sigma_1 S_1 \frac{\partial V}{\partial S_1} dW_t^{(1)} + \sigma_2 S_2 \frac{\partial V}{\partial S_2} dW_t^{(2)}. \end{aligned} \tag{8}$$

Q: When do I use it?
A: When we have $V = V(t, S_1, S_2)$ which depends on t and two more SDE's given by S_1 and S_2 to form a multi-factor model.

Q: How do I derive this?
A: Perform a Taylor expansion of $V(t + dt, S_1 + dS_1, S_2 + dS_2)$ up to first order in t and second order in S_1 and S_2, remembering to include the mixed partial term of the product of the two SDE's. Then substitute in the value for dS_1 and dS_2 where they appears, noting that higher orders of dt such as $dt^{\frac{3}{2}}$ and dt^2 get ignored. And of course, don't get screwed by the algebra...