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Pricing Interest Rate Swaps in Traditional and Decentralised Finance

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Abstract

A self contained review of the pricing of interest rate swaps across both traditional and decentralised finance. The pricing formulas for the traditional case are examined and two popular methodologies for curve construction—explicit and implicit interpolation of the discount factors—are implemented in Python using market data. One specific example of crypto interest rate swaps, the BitMEX Bitcoin-USD Funding Rate Swap, is discussed and its pricing formula examined. Our project concludes with some opportunities for new interest rate swap-like products within crypto markets.

Keywords

pricing; interest rate swaps; cryptocurrencies; crypto interest rate swaps

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In loving memory of Archie, always in our hearts.

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Chapter 1

Introduction

Derivatives are financial weapons of mass destruction.

—Warren Buffett, Billionaire Investor

Bank accounts. Credit Cards. Loans. Mortgages. These products have one thing in common: they are all related to interest rates. Of course, that is not to say they are the same. A savings account offering 30% interest may be wishful thinking, but on a credit card, that is fairly standard. The important thing is that interest rates are everywhere.

As we transition from the world of retail to wholesale finance, interest rates become no less important—only, the sums become much larger. Such vast amounts of money introduce much greater levels of uncertainty and a keen desire to manage risk. Enter derivatives. Specifically interest rate derivatives. The $500 trillion market to help companies manage their interest rate risks (BIS, 2008).

One of the most popular products is interest rate swaps. Since their introduction in the 1980's they have become a key tool for those looking to convert variable, floating rate exposure into known fixed rate payments. Their theoretical pricing amounts to no more than straightforward time value of money calculations, using an appropriate discounting rate, and has been well known since their inception. Most financial textbooks with a section on interest rate products will include at least some discussion of interest rate swaps (Sadr, 2009) (Flavell, 2012) (Wilmott, 2013) (Veronesi, 2016).

In recent years, however, the previously illusive world of cryptocurrencies has exploded in popularity. Despite its volatility, in part due its sensitivity to an army of fearless (or perhaps reckless) retail traders and celebrity endorsements such as serial tweeter Elon Musk, it is clear that the cryptocurrency market is here to stay.

This dramatic rise in attention has been coupled with increased academic focus, yet current research tends to focus on the creation, mechanics, and sustainability of existing cryptocurrencies and their associated blockchain protocols, with a little reserved for the analysis and extension of traditional products and derivatives to the world of decentralised finance.

Companies at the forefront of new cryptocurrency products have begun to consider the world of interest rate derivatives—and specifically interest rate swaps—as an opportunity to extend the traditional case to the crypto market. Of those few products that exist though, there is little analysis or research specifically devoted to them, and even fewer which provide a coherent bridge between traditional interest rate swaps and their crypto counterparts.

As a result, we state the primary objective of this project: to bridge the gap between the established literature of interest rates and its derivatives and the emerging world of cryptocurrency derivatives. Rather than considering traditional and decentralised finance as two distinct and separate areas, we take a product focused approach by considering interest rate swaps as a whole, under which we group and review the two markets. This ensures the project is concise and self-contained. This will likely be beneficial for traditional and crypto market participants seeking to understand the role of interest rate swaps and its possible use with crypto, in addition to academics for whom this project may generate interest in further, deeper research into the mathematical and computational aspects of crypto interest rate swaps.

The project is structured as follows. Chapter 2 introduces the background and context to the topic, including some necessary mathematical preliminaries. In this chapter, we also cover the historical and regulatory context of interest rate derivatives and cryptocurrencies. Chapter 3 covers the world of traditional finance and the definition and terminology of various interest rate derivatives. We include the pricing of traditional interest rate swaps; a description of two curve construction methodologies; their Python implementation using market data; and a discussion of the practical considerations for pricing traditional interest rate swaps. Chapter 4 extends the coverage of the previous chapter to cryptocurrencies, and includes the definition and terminology of various cryptocurrency products; a discussion of a particular crypto interest rate model; the pricing of crypto interest rate swaps; and possible avenues for new interest rate swap-like products within the crypto market. Finally, Chapter 5 ends with some concluding remarks reflecting upon whether we have achieved our primary objective, and offering some thoughts and recommendations on future work.

Chapter 2

Background

It's a safe banking system, a sound banking system. Our regulators are on top of it. This is a very manageable situation.

—Henry Paulson, Former Secretary of the Treasury
Two months prior to Lehman Brothers' collapse.

2.1 Introduction

This chapter provides some wider context to the project. It begins with a broad overview of interest rate derivative and cryptocurrency markets in general, including their history, current landscape, and regulatory regimes. We then review the established literature relating to interest rates, specifically those pertaining to interest rate models, and some more recent papers regarding the interest rate environment and associated products within the crypto sector. Finally, we look at some of the fundamental mathematical definitions and results that will be used extensively throughout the project.

2.2 History & Context

2.2.1 A Brief History of Interest Rate Derivatives

It was 1981 when the World Bank entered into the first *currency swap*. The comparatively low level of interest rates in Germany and Switzerland meant they were facing increasing demand from borrowers seeking loans denominated in Deutsche marks and Swiss francs respectively. Unfortunately, the governments of both countries had imposed limits on the amount the World Bank could borrow and by August 1981 those limits had been reached. As it happened the U.S. company IBM was looking to increase its dollar holdings, having already garnered a sizable amount of bonds issued in marks and francs, and as such IBM and the World Bank agreed to swap currencies. Thus, the swap market was born.

It was not until 1985 that the World Bank entered into its first *interest rate swap*, initiated at par, for which an agreement was made to pay a floating amount of U.S. dollar interest, based upon the three-month World Bank Treasury bill rate, and receive a fixed rate of U.S. dollar interest.

From this point the market for swaps grew rapidly as both currency and interest rate swaps meant that parties could access new, alternative sources of funding in a financially practical way. In addition, it allowed parties to actively consider their risk management by means of converting their liabilities, either into a more appropriate currency or a more appropriate structure of interest payments (World Bank, 2018).

As the size grew, so too did the number of interest rate derivative products. *Basis rate swaps*, *interest rate swaptions*, and *caps* and *floors* were just a few of those introduced, while exotic features were added to established contracts as the market saw an explosion in demand, and opportunities. The swiftness of their introduction was thanks, in large part, to the fact that such products are traded *Over-the-Counter* (OTC)—that is, the contracts simply involved the two respective parties with no central exchange or clearing house acting as mediator. As a result, the level of transparency was low.

For over thirty years the market continued to flourish, quickly becoming one of the largest and most active in the world. The Bank for International Settlements reported that in the first half of 2008, the notional principle outstanding for interest rate derivatives stood at just under $503 *trillion* (BIS, 2008), dominating other OTC asset classes as shown in Figure 2.1. However, the financial crisis brought the market squarely into focus for regulators.

2.2.2 The Financial Crisis and the Current Landscape

The financial crisis of 2007–2009 brought about a worldwide overhaul of financial regulations. In the U.S. the Dodd-Frank Wall Street Reform and Consumer Protection Act ("Dodd-Frank Act") was passed, whilst the European Market Infrastructure Regulation ("EMIR") was implemented in Europe—both of which were specifically designed to secure OTC derivatives markets in an effort to prevent another global collapse (CFTC, 2013).

Despite these fundamental changes, the popularity of interest rate derivatives contracts did not waiver. In fact, the outstanding notional principle of the market peaked at $587 trillion in 2011, although it has since stabilised to somewhere in the region of $480–$520 trillion in the last few years.

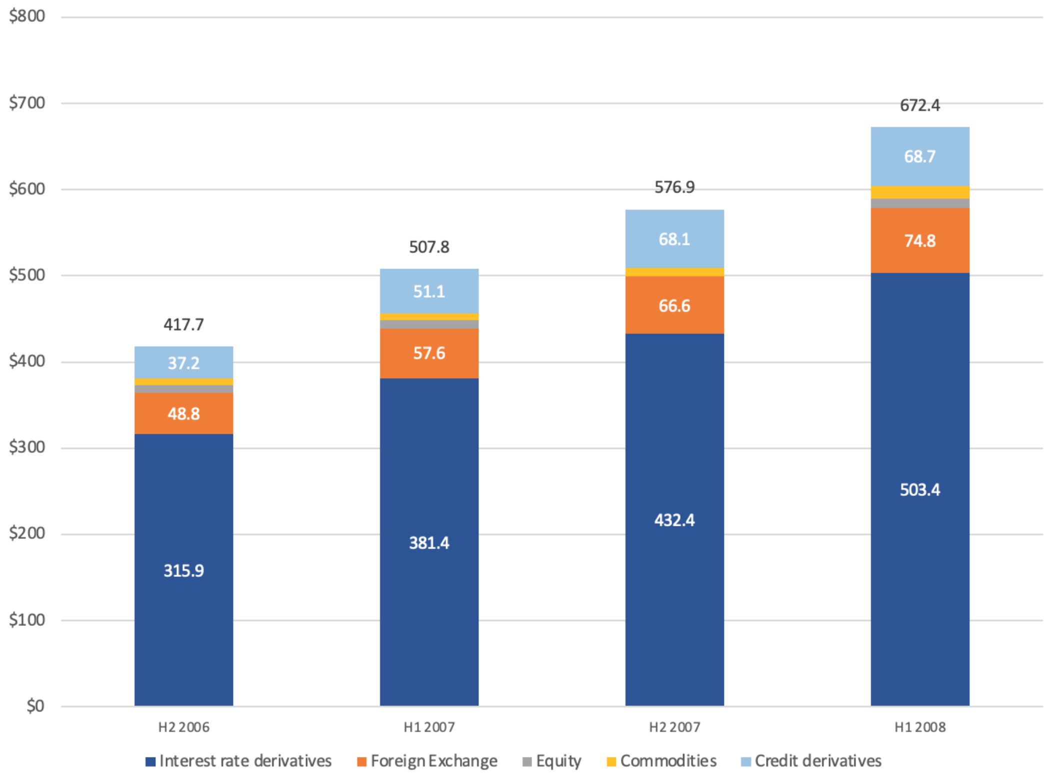

Figure 2.1: The notional amounts outstanding across OTC derivatives markets for each half year from H1 2006 to H1 2008 in USD (trillions). We can clearly see the continuous growth over the period, and the domination of interest rate derivatives which had an outstanding notional greater than the sum of the other categories over the range. Data from (BIS, 2008).

What did change, however, was the logistics of the transactions and obligations of each party entering into such contracts. The primary objectives of the regulations can be summarised as follows (European Commission, 2012):

- Introduction of central clearing for those OTC derivatives classed as standardised;
- Increased usage of an exchange or electronic platform for the trading of OTC derivatives classed as standardised;
- Increased capital requirements for organisations trading OTC derivatives contracts that are not centrally cleared, i.e. are not standardised;
- Increased minimum margin requirements for those OTC derivatives contracts that are not centrally cleared;
- Reporting of OTC derivatives trades to trade data repositories.

These factors are critical to well functioning and transparent OTC derivatives markets and there has been a great deal of investment by market participants into the underlying infrastructure. In addition, they will play an important role when we consider the extension of traditional interest rate derivatives to the world of cryptocurrencies so it is well worth examining the main points in more detail.

2.2.2.1 The Role of Central Clearing

The introduction of central clearing for those contracts classed as standardised—essentially those that were the most liquid—was aimed at reducing the counterparty credit exposure of previous bilateral agreements. This meant, for certain interest rate derivatives, contracts *must* involve a central counterparty ('CCP'). The CCP (normally a clearing house) would then act as 'a buyer to every seller and a seller to every buyer' thereby reducing overall risk through netting[1].

Initially, there were four contracts identified as standardised: *fixed-for-floating swaps*, *basis rate swaps*, *overnight index swaps* ('OIS'), and *Forward Rate Agreements* ('FRA'). In addition, only those contracts denominated in U.S. dollars, Pound Sterling, Euros, and Japanese Yen, were required to be cleared. This requirement has since been expanded to include a wider range of interest rate derivatives.

Of course, the opportunity to reduce the level of risk involved in trades is certainly an attractive prospect. As such, in the first half of 2020, as much as 90.1% of traded notional principle for interest rate derivatives was centrally cleared. The International Swaps and Derivatives Association ('ISDA') confirmed that market participants cleared a greater dollar amount than mandated under the current legislation (ISDA, 2021a).

2.2.2.2 Increased Capital Requirements

For some contracts, however, the use of a CCP is either infeasible due to the bespoke nature of the specific contract or unwarranted due to the additional cost, after all, the services provided by the CCP are not free. In this scenario, the credit risk associated with a default by either party remains.

To combat this, the Basel Committee on Banking Standards ('BCBS') proposed a new BASEL III framework. This far reaching regulatory initiative included a section proposing new rules specifically relating to OTC derivatives by introducing a standardised methodology for calculating counterparty credit risk and credit valuation adjustments. This standardisation of approach resulted in market participants being obliged to hold increased capital reserves. BCBS reported that the amount of capital held by 'Group 1'[2] banks more than doubled from €2.3 trillion in 2011 to €4.9 trillion in 2019 (BCBS, 2020).

In a wider context, the BASEL III framework also had an impact on banks financial

[1] The general term used for the aggregation and subsequent simplification of multiple contractual obligations (JSCC, 2012).

[2] BCBS defines *Group 1* banks as those whose Tier 1 capital is greater than €3 billion, in addition to those classified as 'global systemically important' (BCBS, 2020)

ratios. The *net stable funding ratio*, *liquidity coverage ratio*, and *leverage ratio* all now have additional rules regarding the treatment of derivatives. For a detailed discussion of the impacts of BASEL III on financial institutions post-financial crisis see Blundell-Wignall and Atkinson (2010), Cosimano and Hakura (2011), Slovik and Cournède (2011).

2.2.2.3 Increased Margin Requirements

In a similar manner, BCBS and the International Organisation of Securities Commissions ('IOSCO') produced a joint framework covering increased margin requirements for those contracts not involving a CCP (IOSCO, 2020). This meant that both the *initial margin* and *variable margin* obligations were higher, again in an effort to reduce the credit exposure should one party experience default.

Like central clearing, posting greater margin protects both parties and is therefore mutually beneficial. As a result, an ISDA survey of those participants which were subject to the requirements under the first phase of implementation found that $83 billion of initial margin not subject to the regulations was received. Furthermore, total initial margin levels for Phase 1 firms had more than doubled from $130 billion in 2017 to $286 billion in 2021 (ISDA, 2021b).

2.2.2.4 Transaction Reporting

Since OTC derivatives pre-financial crisis were simply bilateral agreements, by definition, there was very little transparency of the underlying contracts. Regulators could investigate the transactions and portfolios of individual participants but could not easily obtain an aggregated view of the market.

The requirement for derivatives transactions to be reported to Trade Repositories[3], which would be registered and supervised by national competent authorities, meant that regulators now had improved visibility, and therefore sufficient oversight, of OTC markets.

The local regulatory regimes (EMIR for Europe, Dodd-Frank for the U.S.) set forth the mandatory data that must be provided to Trade Repositories. The interested reader is referred to the websites of regulatory authorities for more information (see the Financial Conduct Authority post-EU Withdrawal (FCA, 2022d); the European Securities & Markets Authority (ESMA, 2022); and the Securities & Exchange Commission (SEC, 2022)).

2.2.3 Market Participants

The history and current landscape of interest rate derivatives provides important context, however, it does nothing to answer the question: who are the major market participants?

[3]In the U.S. interest rate derivatives transactions are reported to *Swap Data Repositories* (SDR)—a diminutive of the more general term, Trade Repository.

In general, derivatives markets can be broadly considered to have four distinct, but invariably interlinked, participants: Hedgers, Speculators, Systematic Traders, and Arbitrageurs. For more detail, see Chernenko and Faulkender (2011), Chui (2012), Stankovska (2017).

Hedgers

Participants classed as *hedgers* are those who wish to reduce their risk. They do so by entering into appropriate derivatives contracts that reduce the uncertainty of future prices and protect against any future market volatility.

In general, for companies entering derivatives markets as hedgers, such activities do not form part of their core business. Whilst their main objective may be to reduce uncertainty, they still fundamentally require the assets to perform business functions and as such are likely to have an asymmetrical approach to losses. Consider the aviation industry. Taking a loss on a delivery of jet fuel is clearly less critical than the necessity of having the jet fuel available to fly planes.

Speculators

Hedging can be considered a common sense strategy for those companies that require derivatives products and their underlying securities in order for their business to function. Yet it should be obvious that it is not only those companies which partake in such transactions. Generally speaking, those that are simply making predictions about the future, and seeking to profit should their view be correct, are called *speculators*. In contrast to hedgers, this forms the core business of speculators.

The key point under these circumstances is that the motivation to enter the derivatives market is driven by profit seeking opportunities rather than as a risk mitigation strategy. A company may hold a strong belief, but does not truly know—that is, they are *speculating*—how the price of an underlying asset will develop over the time horizon. If the market moves in the opposite direction, significant losses can quickly occur.

Systematic Traders

Systematic trading relies on a combination of technology, algorithms, and market signals. By setting clear, well defined strategies based on empirical data analysis and backtesting, systematic traders aim to remove any behavioral characteristics associated with the humans involved. The algorithmic implementation has the additional benefits of executing orders, taking profits, cutting losses, and managing risks in real time without the need for constant oversight.

In practice this requires significant initial investments in technology and data in order to perform the necessary analyses to identify appropriate signals and markets—as well as employees with knowledge of mathematics, computer science, economics, and finance—which can act as a barrier to entry. Therefore systematic trading lends itself to hedge funds, and large global financial institutions.

Arbitrageurs

Financial markets exist across the world, with a security sometimes being traded in multiple locations. For instance, a share may be traded on both the London and New York Stock Exchanges. Should those markets make a mistake and display different prices for the same underlying share (accounting for exchange rates), it would be possible for a party to make riskless profit by buying on the cheaper exchange and instantly selling on the more expensive exchange until the mispricing was corrected. Those participants that look for such opportunities are called *arbitrageurs* and play a vital role in the efficient functioning of global financial markets.

In practice, if and when such mispricings occur they may only exist for milliseconds. To take advantage, companies pursuing arbitrage strategies usually invest heavily into technology that can handle such low latencies, with companies even relocating offices closer to exchanges in an effort to receive information marginally faster than competitors. These barriers to entry tend to limit participation to specialist firms.

2.2.4 The LIBOR Transition

Naturally, interest rate derivatives rely on an interest (or reference) rate and for much of their history this was the *London Interbank Offer Rate* ('LIBOR'). LIBOR was essentially the rate at which large, global banks would lend to each other, and was calculated on a daily basis for five currencies and seven maturities. This benchmark was then used extensively throughout financial markets involving interest rates.

By virtue of its definition, LIBOR itself was not impacted by the financial crisis and changing regulations—although that is not to say the *level* of LIBOR was not impacted, it was, in fact, much higher during this period as one would expect. However, LIBOR is not without its own historical issues. In 2008, critics suggested that banks could be understating their submissions, thereby giving the impression that other banks could borrow at a rate cheaper than they could in reality. This led to Mervyn King, former Governor of the Bank of England, to say that LIBOR was "*the rate at which banks didn't lend to each other*" (TSC, 2022). Furthermore, in 2012 it was discovered that companies had been attempting to manipulate LIBOR through their submissions in what later became known as the *LIBOR scandal*. Barclays, for instance, was fined a total of $435 million for its role (CFR, 2022).

More recently, however, a new regulatory regime was established that sought to transition away from LIBOR and towards alternative *risk free rates* instead. The aim of this transition was to produce a rate that was based on historical data, rather than being forward-looking, which would make the benchmark more robust. As a result, the new benchmarks are now locally administered and overseen, that is, each jurisdiction will govern their own rate.

At the end of 2021, LIBOR ceased to be applicable and was replaced by the *Sterling Overnight Index Average* ('SONIA') in the U.K.; the *Secured Overnight Financing Rate* ('SOFR') in the U.S.; and the European Short Term Rate ('ESTER') in Europe—with the exception of certain U.S. dollar denominated maturities which will continue for slightly longer (FCA, 2022c).

2.2.5 A Brief History of Cryptocurrencies

Despite its prolific rise over the past decade, the origins of cryptocurrencies can be traced back to the 1980's when an American computer scientist by the name of David Chaum created an electronic, cryptographic currency called electronic cash (*e-cash*) (Chaum, 1983). In fact, a few years before its creation, a dissertation by the same author provided what is considered to be the first implementation of the blockchain concept (Chaum, 1979).

E-cash did not survive, however, but the idea had been firmly established. In the late 1990's, *B-Money* (Dai, 1998) and *Bit Gold* (Szabo, 1998) were attempts to develop digital currencies utilising encrypted ledger technology but never fully materialised. Then, in 2008, Satoshi Nakamoto (a presumed pseudonym) published a paper entitled *Bitcoin: A peer-to-peer electronic cash system* (Nakamoto, 2008). The following year Bitcoin was made open-source, and still remains the flagship cryptocurrency to this day.

Bitcoin itself has a rich history from those early days to the present, however, this project is focused more generally on the concept of cryptocurrencies and derivatives thereof. As such, we will not go into any further detail. The interested reader is referred to Böhme et al. (2015) and Zohar (2015) for more information regarding the Bitcoin concept, or Urquhart (2016) for some of the challenges it faces.

2.2.6 The Current Landscape, Products & Regulatory Oversight

With increasing reliance on digital technologies across all aspects of our lives, it seems impossible to ignore the world of cryptocurrencies. After all, Bloomberg (2021) reported that the total market cap at the end of 2021 was over \$3 trillion. Since Bitcoin's successful introduction, the number of available tokens has exploded and CNBC (2022) reported there are approximately 19,000 cryptocurrencies in existence. This is in part due to the ease with which one can create new tokens by simply launching on the desired existing framework. Naturally, there are drastic differences in both the value, maturity, and legitimacy of these digital currencies, and the vast majority of newly issued coins (through a so called Initial Coin Offering, or 'ICO') fail rapidly (Blaseg, 2018). However, these facts do characterise the recent popularity and increased attention being paid to the sector. Of course, as popularity increases so too does the demand for products.

Despite their decentralised nature, cryptocurrencies are still conceptually similar to currencies, and the world of derivatives can easily be extended from their traditional counterparts. As a result, several crypto exchanges have appeared offering two major benefits: firstly, reducing the barriers to entry for those investors interested in trading cryptocurrencies but who lack the technical expertise (or resources) to participate in the underlying computational activities; and secondly, extending the product offering for major cryptocurrencies through the likes of derivatives. In fact, the introduction of derivatives caused a jump in value across the sector and was seen as a major statement regarding the seriousness of crypto markets (Gemeni, 2021).

As a result, crypto futures and options are now available. These contracts behave largely the same as their traditional counterparts, with features like initial and variable margin requirements, and both delivery and perpetual contracts. However, there are additional non-technical features such as *Leaderboards* and *Battle*—which pits two traders head-to-head to see who is the most profitable over a set time period. These are designed to enhance the experience of traders through the so called "*gamification of cryptocurrency trading*" (Binance, 2021).

These changes have not gone unnoticed by regulators and the sector has been placed under the spotlight in the last couple of years. However, cryptocurrencies present a challenging situation for regulators worldwide as they fall right on the border of the regulatory perimeter. The UK Financial Regulator has stated that "*cryptoassets are not underpinned by any currency or other asset and are not considered to be a currency or money*" (FCA, 2022a). This means that the only oversight provided to cryptocurrencies themselves (e.g. Bitcoin) is for money laundering purposes. On the other hand, they have also stated that "*cryptocurrency derivatives are capable of being financial instruments under MiFID II*" (FCA, 2022b), thereby requiring compliance with the authorisation and regulation in the same manner as for traditional financial derivatives—this includes futures and options contracts.

The onus, though, is on the firm themselves to seek out and establish whether their products and services fall under this regime. Consequently, the number of scams has been high, but even legitimate firm collapses have left consumers with significant losses. Of course, where the responsibility lies in such cases and the extent to which regulations should be enforced is left to the opinion of the reader. Nevertheless, it is expected that the sector will continue to receive increasing regulatory attention over the coming years.

2.3 Literature Review

Much of the literature regarding interest rates and their respective derivatives has been established for some time since the flurry of proposals between 1970 and 2000. For the most part the proposals could be classified as short rate models, but there were some alternative ideas as well. For an excellent and incredibly detailed treatment of the world of interest rates and their derivatives, the reader is referred to Brigo et al. (2001).

Short rate models are built on the instantaneous *spot rate* being the underlying variable for the stochastic process, thereby allowing interest rate derivatives to be priced under an appropriate risk-neutral measure. There are numerous famous models of this type including simpler one-factor models such as that of Vasicek (1977) who proposed a mean-reverting stochastic process (although, it was simply a specific case of the Ornstein-Uhlenbeck process from Physics). This model was criticised for allowing negative interest rates, however, the longstanding belief that interest rates could not turn negative has since been overturned.

To account for this, Cox, Ingersoll, and Ross proposed a modified stochastic process (Cox et al., 1985) which included the square root of the instantaneous interest rate in the diffusion term preventing non-zero rates and, provided the Feller Condition was satisfied, would not become absorbed at zero either. Since then, numerous other models have been proposed, each with slight modifications aimed at better capturing the term structure including that of Ho and Lee (1986), Hull and White (1993), Black and Karasinski (1991), and Black, Derman, and Toy (Black et al., 1990). There have also been two-factor models proposed, such as that of Longstaff and Schwartz (1992) which separates the behaviour of short term and long term interest rate movements and produces a linear combination of the two.

One of the alternatives to short rate models, were those developed under the Heath-Jarrow-Morton (HJM) framework (Heath et al., 1992). Such models are based on the instantaneous *forward rate* and are aimed at describing the full dynamics of the forward curve as opposed to simply a single point on the curve, as with short rate models. Similarly, another alternative was a new class called the LIBOR market model, or Brace-Gatarek-Museila (BGM) model (Brace et al., 1997), which proposed modelling a *set of forward rates*. This was a significant change when pricing interest rate derivatives as it now meant using an appropriate *forward* measure.

Of course, the natural question is: which interest rate term structure model should be used? In truth, the nature of the derivatives contract itself largely dictates the model with more complicated, exotic contracts restricting the choice. For instance, the LIBOR market model is well suited to Bermudan swaptions, while plain vanilla swaptions can be priced under the simpler Black (1976) model.

Interestingly, however, the focus of our project—pricing interest rate swaps—does not require the interest rate model. As a result, the literature relating to the topic is largely homogeneous, that is, ignoring notational differences the method to price interest rate swaps is standardised, clearly understood, and widely accepted in practice. Hence, we have used Brigo et al. (2001), Sadr (2009), Flavell (2012), Wilmott (2013), Veronesi (2016) for reference throughout this project.

Since crypto is still a relatively immature sector, there is little to no literature discussing crypto interest rate swaps. In fact, one of the more pressing, fundamental challenges is what do we mean by the concept of 'interest' in the crypto market. Several papers have been published over the last few years proposing new ideas, models, and mathematical frameworks aimed at addressing this current gap.

A paper put forth by Brody et al. (2020) proposed a model for the term structure of interest rates in the world of cryptocurrencies, in which the typical numeraire—the money market account—does not exist. This model builds on the extensive literature regarding the short rate, some of which we have just mentioned, and establishes a non-trivial 'no interest' environment, that is, the spot rate is zero for all time and yet the term structure is non-zero and we can construct yield curves. This will be examined in greater detail in Chapter 4.

The paper by Kaneko et al. (2019) suggests a model based on Unspent Transaction Output ("UTXO") which adds a second layer to an underlying blockchain technology. This second layer extends the transaction data captured on the existing blockchain to include another program capable of managing the concept of interest. To do so, their model proposes issuance of two, equal and offsetting transactions—one acting as the loan itself, which would subsequently be transferred to the borrower, whilst the other would play the dual role of maintaining the '*currency conservation law*' and a claim for the loan amount against the borrower.

The strength of the model lies in its ability to account for both fixed and floating rates as follows. For floating rates of interest, a further issuance of two equal and offsetting transactions in the amount of the interest, determined precisely at (or before) the point of issuance. This could then be repeated as many times as required, with each transaction being stored and completely verifiable by other participants on the *secondary layer of the blockchain*. Fixed rates of interest are managed once again by the issuance of two equal and offsetting transactions, however, this time they are all issued at initiation and rely on programs that implement a suspension period before becoming active.

Unfortunately, the model is not free of challenges. Mainly, the issuance and subsequent closure of any loans requires what they classify as a *Bank authority*. This introduces the idea of specific roles performing specific functions with respect to the loan and subsequent handling—for instance, they state that only such Bank authorities should have the ability to close the loan. This, in turn, greatly increases the complexity of the model both from a conceptual and implementation perspective.

2.4 Mathematical Preliminaries

We now introduce the mathematical notation used throughout this project and present some fundamental terminology along with their precise definition.

2.4.1 Discount Factors

The world of fixed income is synonymous with the concept of the *time value of money*. The idea that a specific amount of money received in the future must have some present value. Since the amount of money is arbitrary, we can consider the case of receiving \$1 at some time in the future. We will define the *discount factor* to be the present value of that \$1.

Let us formalise this concept, following the same notation as that of Wilmott (2013) and Veronesi (2016).

Definition 2.1 (Discount factor)**.** Let $t_0 = 0$ be today, and t_i denote some time in the future, that is, $0 = t_0 < t_i$. Then we will define the *discount factor* to be the present value of \$1 received at time t_i and denote this as $Z(0, t_i)$.

We can think of $Z(0, t_i)$ as the price today of a zero coupon bond. Assuming interest rates are non-negative (which is usually, but certainly not always, the case), we can construct a downward sloping curve of discount factors called the *discount curve*.

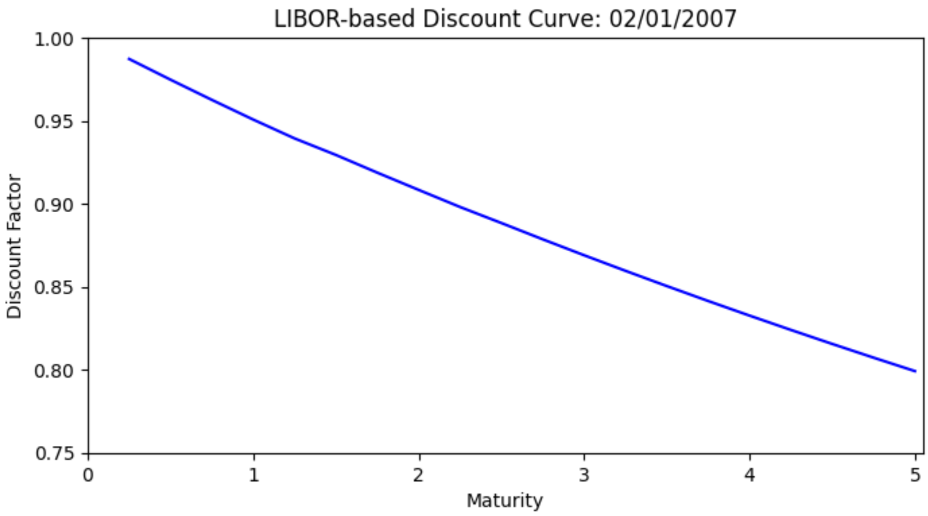

Figure 2.2: An example of a discount curve out to five years constructed from LIBOR-based instruments on 02/01/2007. Note the decreasing nature of the curve, indicating that receiving a hypothetical \$1 further into the future has a lower present value today. Reproduced from Veronesi (2016). Source data from Bloomberg.

2.4.2 Spot Rates

Our discount factors give us the present value of \$1 received at some time in the future. Suppose we now flip the problem and instead of thinking about the present value of \$1, we consider what \$1 will be worth at some future point. To answer this question, we can use the concept of *spot rates* (also called *zero rates*), although we must take care with the compounding frequency. In what follows we standardise the notation of Sadr (2009) and Brigo et al. (2001). Let us first introduce the simplest case: the money market account.

Definition 2.2 (Money market account)**.** Let $B(t)$ be the value of a money market account at some point, t, in the future. We will assume an initial deposit of B_0, that is $B(0) = B_0$. Further, assume a *constant, continuously compounded annual return*, r. Then the value of the money market account evolves according to the following ordinary differential equation,

$$\frac{dB(t)}{dt} = rB(t) \tag{2.1}$$

which has solution

$$B(t) = B_0 e^{rt}. \tag{2.2}$$

Our assumption of a single constant continuous rate, r, makes this calculation almost trivial. However, as we shall see in Chapter 4, when developing traditional interest rate models using the money market numeraire, we relax the constraint and assume the rate, r_s, is a positive function of time for $0 < s < t$. This leads to the more general expression,

$$B_s = \exp\left(\int_0^s r_u du\right). \tag{2.3}$$

Our money market account is, in fact, a direct relation between present and future values. In practice, rather than having one single continuous rate, we may have multiple rates (quoted on an annualised basis) which differ depending on the time period—consider fixed rate savings accounts which offer greater returns the longer the initial deposit is locked up.

Definition 2.3 (Continuous zero/spot rate)**.** Let $r_c(t_i)$ be the *continuously compounded annual return* of \$1 invested today until a future time, t_i (in years). Then we will refer to $r_c(t_i)$ as the *continuous spot rate* which relates present and future values via

$$FV = PVe^{r_c(t_i)t_i}. \tag{2.4}$$

Note that the spot rate, $r_c(t_i)$, depends on the time as discussed.

Assuming a present value of unit currency to simplify notation, we can relate discount factors and spot rates via

$$\frac{1}{Z(0,t_i)} = e^{r_c(t_i)t_i}, \tag{2.5}$$

which upon rearranging gives

$$r_c(t_i) = -\frac{1}{t_i} \ln Z(0, t_i). \tag{2.6}$$

The treatment of the discrete case is analogous.

Definition 2.4 (Discrete spot rate)**.** Let $r_d(t_i)$ be the *discretely compounded annualised return* of \$1 invested today until a future time, t_i (in years). In addition, suppose the return is compounded a discrete number, say k, times per year. Thus, kt_i represents the total number of compounded periods. Then the return $r_d(t_i)$ is called the *discrete spot rate* which relates present and future values via

$$FV = PV \left(1 + \frac{r_d(t_i)}{k}\right)^{kt_i}. \tag{2.7}$$

Once again, we can see the intrinsically linked nature of discount factors and spot rates via

$$\frac{1}{Z(0, t_i)} = \left(1 + \frac{r_d(t_i)}{k}\right)^{kt_i}. \tag{2.8}$$

Increasing the number of periods, k, per year and taking the limit we find

$$\frac{1}{Z(0, t_i)} = \lim_{k \to \infty} \left[\left(1 + \frac{r_d(t_i)}{k}\right)^{k}\right]^{t_i} \tag{2.9}$$

$$= \left[\lim_{k \to \infty} \left(1 + \frac{r_d(t_i)}{k}\right)^{k}\right]^{t_i}. \tag{2.10}$$

Noting that the inner limit is precisely the definition of the exponential, we arrive at the continuous result above.

This ability to transition between discrete and continuous spot rates is an important result and will be utilised when we consider curve construction in Chapter 3. To make the relation explicit we can equate our two methods to compute future values,

$$PVe^{r_c(t_i)t_i} = PV \left(1 + \frac{r_d(t_i)}{k}\right)^{kt_i} \tag{2.11}$$

$$e^{r_c(t_i)t_i} = \left(1 + \frac{r_d(t_i)}{k}\right)^{kt_i} \tag{2.12}$$

$$r_c(t_i) = k \log \left(1 + \frac{r_d(t_i)}{k}\right). \tag{2.13}$$

Alternatively, using the fact that the number of compounding periods per year is inversely related to the time in years, that is $t_i = \frac{1}{k}$, we obtain the equivalent expression (Banque du Canada, 2000)

$$r_c(t_i) = \frac{1}{t_i} \ln(1 + r_d(t_i)t_i). \tag{2.14}$$

Throughout this project we will use the subscripts, d and c, to indicate whether the interest rate is discretely or continuously compounded.

Given a series of spot rates we can construct the *spot curve*.

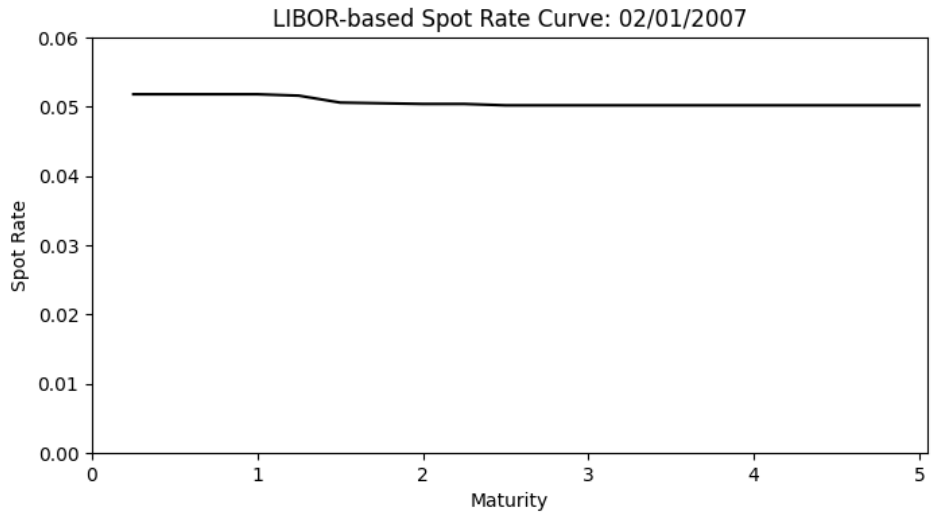

Figure 2.3: An example of a spot curve out to five years constructed from LIBOR-based instruments on 02/01/2007. The results in this graph can be constructed from the data given in Figure 2.2 and vice versa. Reproduced from Veronesi (2016). Source data from Bloomberg.

2.4.3 Forward Rates

The spot curve is a useful tool for examining rates today, however, it gives little information about rates in the future. For that, we must look at *forward rates*.

Forward rates establish a *fixed* rate of interest for a specified time period *starting at some point in the future*. That is, in order to understand a forward rate we need, not one, but two times: the start of the period and the end. We will follow the notation of Veronesi (2016) and denote this as $f(0, t_{i-1}, t_i)$. As with spot rates, there is also the compounding frequency to consider.

A natural question arises: how do we determine a rate of interest for a period of time in the future? After all, we can only observe today's spot rates. Fortunately, we can rely on the concept of 'no arbitrage' to establish the appropriate forward rate.

Consider two future times, $0 < t_1 < t_2$. From either the discrete or continuous definition of spot rates, we know that the amount received at time t_1 is $1/Z(0, t_1)$ and likewise at time t_2 is $1/Z(0, t_2)$.

Then by the principle of no arbitrage we must have that investing at the spot rate associated with time t_1 and then reinvesting for the period $[t_1, t_2]$ should earn no more or less than simply investing at the rate associated with time t_2. In mathematical terms, we have

$$\frac{1}{Z(0,t_1)} \times (1 + f(0,t_1,t_2)) = \frac{1}{Z(0,t_2)}. \tag{2.15}$$

Rearranging gives rise to the following definition.

Definition 2.5 (Discrete forward rate)**.** Let $f_d(0,t_{i-1},t_i)$ be the *annualised forward rate* between t_{i-1} and t_i and let $Z(0,t_{i-1})$, $Z(0,t_i)$ be the associated discount factors. Then we can compute the discretely compounded forward rates as

$$f_d(0,t_{i-1},t_i) = \frac{\frac{Z(0,t_{i-1})}{Z(0,t_i)} - 1}{t_i - t_{i-1}}. \tag{2.16}$$

Definition 2.6 (Continuous forward rate)**.** As with the spot rate, we can compute the continuously compounded forward rate.

$$f_c(0,t_{i-1},t_i) = \frac{\ln[Z(0,t_{i-1})/Z(0,t_i)]}{t_i - t_{i-1}}. \tag{2.17}$$

Note that by starting our forward period today in definitions 2.5 and 2.6 we simply arrive at the spot rate. Once again, we can construct the *forward curve* given knowledge of the appropriate forward rates.

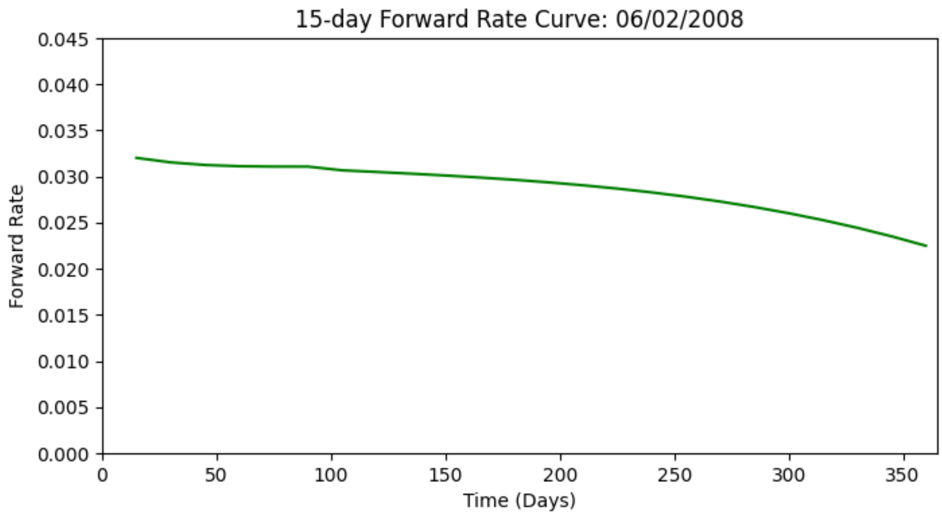

Figure 2.4: An example of a 15-day forward rate curve over a one year horizon on 06/02/2008. Constructed from the data provided in Flavell (2012).

Overall, we can see from Figure 2.5 the linked nature of the discount factors, spot rates, and forward rates. With knowledge of one, we can compute the others giving us a full picture of the current interest rate environment. We can consider discount factors as 'pure', by which we mean the choice of compounding does not affect them. For forward and spot rates, however, it is important to understand which compounding convention is being applied.

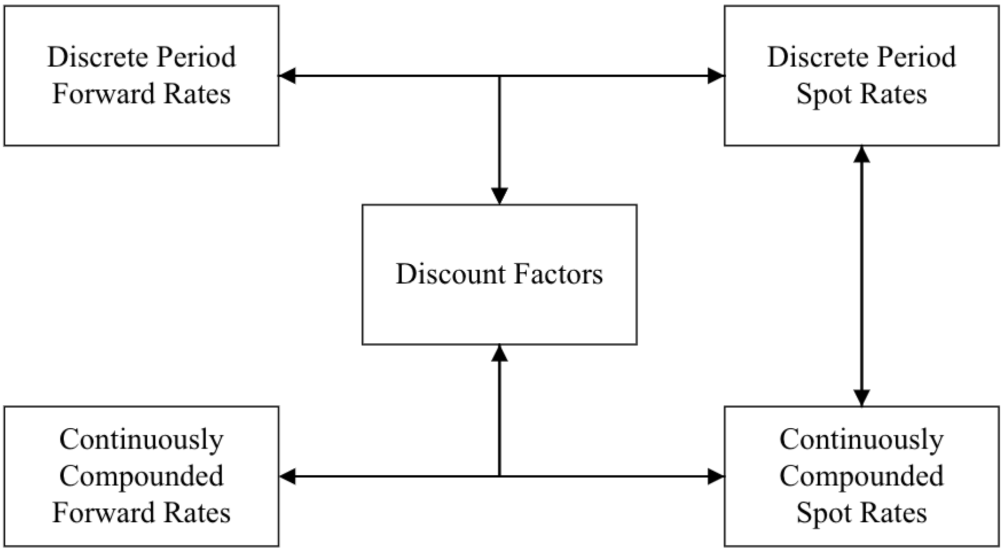

Figure 2.5: A graphical representation of the links between different rates.

The definitions and formulae presented in this section are critical to the understanding and subsequent pricing of interest rate derivatives. To fully appreciate their use, the reader is referred to Appendix A.1 and A.2 which contains detailed numerical examples.

Chapter 3

Interest Rate Swaps

The problem with interest rates is that you are not modeling a single number, you are modeling a whole term structure, so its a sort of different type of problem.

—John Hull, Professor of Risk Management & Derivatives

3.1 Introduction

In this chapter we explore interest rate derivatives—specifically swaps—in more detail, building on the mathematical foundations presented at the end of the previous chapter. We begin with a selective overview of some available products and the terminology used to describe them, before proceeding to discuss their pricing in more detail. We then consider the process of curve construction, describing two simple methodologies and produce a Python implementation based upon two historical datasets. Finally, we discuss some of the practical challenges faced and how our assumptions must be loosened before any real world application could be viable.

3.2 Terminology & Products

So far we have only spoken about interest rate derivatives in a general sense. Before considering the pricing of such derivatives, we must first understand the products available, their mechanics, and the associated terminology. The volume of products on offer is huge and as such we will not trouble ourselves by attempting to discuss and categorise them all, instead we will focus on those explored in this project. We summarise the detailed descriptions of the instruments presented in Sadr (2009).

3.2.1 Forward Rate Agreement

A forward rate agreement is an OTC contract in which two parties agree a rate of interest to be paid for some specific time period, starting at some point in the future. The fixed rate of interest, which is agreed upon at the inception of the contract, is called the *forward rate*. See definitions 2.5 and 2.6.

The future period for which the interest rate applies can be thought of as the borrowing and lending period. Forward rate agreements are quoted as $A \times B$, where A represents the point from which the fixed forward rate applies, and B represents the end of the contract.

3.2.2 General Fixed-for-Floating Swap

A general fixed-for-floating interest rate swap is an OTC agreement in which two parties agree to exchange a fixed rate of interest for a floating rate of interest. The floating rate of interest is tied to some benchmark reference rate and is set at the beginning of the interest period but paid at the end of the interest period.

The cash flows on the either side of a fixed-for-floating interest rate swap are called *legs*, and as such we will refer to the *fixed leg* and the *floating leg*. In addition, fixed-for-floating swaps are quoted from the perspective of the fixed leg. This means a swap for which an entity is paying the fixed rate has entered into a *payer swap*. Likewise, a swap which receives the fixed rate is called a *receiver swap*.

At initiation, the fixed-for-floating swap is issued at par such that it has zero value to either party. The fixed rate of this swap, which we will refer to as the *swap fixed rate*, is the price of the contract.

The length of the contract is referred to as the *term* or *maturity* of a swap, while the frequency of the payments is called the *tenor* of the swap. The fixed leg and the floating leg may have different tenors. The actual dollar amount of interest paid on each payment date is determined by the monies underlying the contract called the *notional principle*. This notional principle is not exchanged at initiation of the contract. Furthermore, the two monetary interest payments are not typically exchanged, instead the two interest rates are netted and a single payment is made to the beneficiary.

3.2.3 Overnight Indexed Swap

An *Overnight Indexed Swap* ('OIS') is a specific contract falling under the broader category of fixed-for-floating interest rate swaps described above. As the name suggests, the benchmark rate for an OIS is the overnight rate (of which there are multiple depending on the market) and the floating interest payments are a geometric average of the overnight rates during the interest period.

3.2.4 Basis Swap

Interest rate swaps need not necessarily have one fixed and one floating leg. A contract that has two floating legs, each tied to different benchmark rates, is called a *basis swap*.

The two floating rates of a basis swap can be the *same* benchmark rate but with *different* tenors (e.g. 3M LIBOR vs. 6M LIBOR) or two *different* benchmarks (e.g. 3M LIBOR vs. Commercial Paper). Often the two floating rates are denominated in the same currency, but benchmarks in two *different* currencies are also possible (e.g. 3M-USD LIBOR vs. 3M-GBP LIBOR).

3.2.5 General Interest Rate Swaption

We can extend the concept of the swaps to consider swap options, or *swaptions*. A swaption is the right, but not the obligation, to exercise the contract and enter into a fixed-for-floating interest rate swap. The terminology is analogous with that of the swap and a payer swaption is the right to enter the swap as the fixed rate payer, while a receiver swaption is the opposite. Clearly the optionality in the contract is valuable, hence, swaptions (as with options in general) require an upfront premium payment.

3.2.6 Plain Vanilla vs. Exotic

While each of the products above have different mechanics, they are all still considered standard by the market. That is, their features are simple, clearly understood, and used extensively by market participants. As such these contracts are classed as *plain vanilla*.

However, there may be times when a market participant has a very specific need and wishes to customise or tailor one of these contracts. By modifying the terms of the contract, it can no longer be classed as standardised, and is thus deemed to be *exotic*.

Exotic features tend to reduce the liquidity of the contract and as such it may be difficult to close out the contract early should the need arise. Whether an exotic contract is cheaper or more expensive will depend on the nature of features added, that is, are they favourable for the purchaser. In addition, the exotic features make pricing such contracts far more challenging.

3.3 Pricing a Forward Rate Agreement

The pricing of a forward rate agreement was innocuously covered in Chapter 2 when we discussed forward rates. This stems from the fact that the forward rate, calculated using discount factors and the concept of no arbitrage, is precisely the definition of a forward rate agreement. That is, the forward rate, $f(0, t_1, t_2)$, is an agreement to pay a fixed rate of interest starting at t_1 and ending at t_2. We repeat the prices here on a discretely compounded basis

$$f_d(0, t_{i-1}, t_i) = \frac{\frac{Z(0,t_{i-1})}{Z(0,t_i)} - 1}{t_i - t_{i-1}} \tag{3.1}$$

and on a continuously compounded basis

$$f_c(0, t_{i-1}, t_i) = \frac{\ln[Z(0, t_{i-1})/Z(0, t_i)]}{t_i - t_{i-1}}. \tag{3.2}$$

We will see in the next section that forward rate agreements can be used to price general fixed-for-floating interest rate swaps.

3.4 Pricing a General Fixed-for-Floating Swap

Pricing an interest rate swap amounts to no more than the discounting of future cash flows. From our general introduction to a plain vanilla fixed-for-floating interest rate swap, we know that there are two streams of cash flows to consider: the (known) fixed interest payments and the (mostly unknown) floating interest payments.

Our aim is to find the appropriate fixed rate of interest that sets the present value of the fixed leg equal to the present value of the floating leg, thus giving the contract zero initial value. We consider each leg in turn.

3.4.1 The Fixed Leg

We will examine the fixed leg first, but before proceeding let us introduce some new notation to make our analysis less cumbersome. Following the convention of Wilmott (2013) we denote the swap fixed rate, r_s. In addition, we define the period of time between interest payments to be τ.

For our purposes we will consider payments which occur at the same time on both sides of the swap and thus simply writing τ will suffice. In practice, the fixed and floating legs will likely have different payment frequencies, hence, there will be two separate definitions of the time between interest payments.

Recall the meaning of discount factors, $Z(0, t_i)$, from definition 2.1. To establish the present value of each of the fixed cash flows of the fixed leg we can use the corresponding discount factors. In doing so, we arrive at the following expression for the present value of the fixed leg.

Definition 3.1 (Present value of the fixed leg)**.** Let r_s be the annualised fixed rate of a plain vanilla fixed-for-floating interest rate swap. Let τ be the time period between interest payments. Then the present value of the fixed leg is given by,

$$PV_{\text{fixed}} = r_s \tau \sum_{i=1}^{N} Z(0, t_i). \tag{3.3}$$

Where N represents the total number of interest payments during life of the contract.

3.4.2 The Floating Leg

The floating leg presents more of challenge since we are working with unknown future interest rates. From the description of the swap, we know that the floating rate of interest is set at the beginning of the interest period and paid at the end. However, as Wilmott (2013) points out, the underlying benchmark used to establish the floating rate payments is itself investable. This means we can appeal to the very definition of the benchmark rate.

At the beginning of each interest period, we know the floating amount we will be required to pay. Now suppose we invest \$1 today at the prevailing floating rate. At the end of the interest period we will have \$1 plus the (time period scaled) return, which will exactly cover the floating interest payment. This means we can view each interest payment of the floating leg as a \$1 inflow at the start of the period and the outflow of \$1 at the end.

This characterisation in terms of \$1 allows us to establish a link between floating leg payments and discount factors. An inflow at time t_i has a present value of $+Z(0, t_i)$, while an outflow at time t_{i+1} has a present value of $-Z(0, t_{i+1})$. As we repeat this over the life of the contract, it becomes a telescoping sum. This leads to the following definition.

Definition 3.2 (Present value of the floating leg)**.** Let $Z(0, t_i)$ be the discount factor associated with time t_i. Then the present value of a stream of floating interest rate payments is given by,

$$\begin{aligned} PV_{\text{float}} &= \sum_{i=1}^{N} \left[Z(0, t_{i-1}) - Z(0, t_i) \right] && (3.4) \\ &= Z(0, t_0) - Z(0, t_N) && (3.5) \\ &= 1 - Z(0, t_N). && (3.6) \end{aligned}$$

3.4.3 The Price

Returning to the question at hand, we are now in a position to determine the appropriate swap fixed rate, r_s. Equating the two present values

$$\begin{aligned} PV_{\text{fixed}} &= PV_{\text{float}} && (3.7) \\ r_s \tau \sum_{i=1}^{N} Z(0, t_i) &= 1 - Z(0, t_N) && (3.8) \end{aligned}$$

we arrive at the formula for a plain vanilla fixed-for-floating interest rate swap (Wilmott, 2013)

$$r_s = \frac{1 - Z(0, t_N)}{\tau \sum_{i=1}^{N} Z(0, t_i)}. \tag{3.9}$$

This representation of the swap fixed rate, r_s, using the discount factors (and associated discount curve) is not the only one. Using the relation between discount factors and forward rates, Veronesi (2016) presents an equivalent formulation of the swap fixed rate,

$$r_s = \frac{\sum_{i=1}^{N} Z(0, t_i) f(0, t_{i-1}, t_i)}{\sum_{i=1}^{N} Z(0, t_i)}. \tag{3.10}$$

To understand how these formulas are applied, the reader is referred to Appendix A.3 which provides a detailed numerical example using both the discount factors and the forward rates.

3.4.4 A Note on Overnight Indexed Swaps

Much of the pricing of Overnight Indexed Swaps is analogous to that of general fixed-for-floating swaps—an OIS is, after all, simply a specific instance of the general form. However, whilst general fixed-for-floating swaps may use LIBOR as a reference, meaning the rate for the interest period is known in advance, OIS use a sequence of daily overnight rates, hence, the future interest payment is not known until it is due.

Using the notation of Veronesi (2016), we will denote $j = 1, \ldots, n$ to be the days in the interest period and r_j to be the overnight rate associated with day j. Then the interest payment for an arbitrary time t_i is given as,

$$CF(t_i) = \prod_{j=1}^{n} (1 + r_j \delta) - 1. \tag{3.11}$$

Where δ is the time fraction corresponding to one day.

The same argument still holds regarding the present values of the fixed and floating legs, only this time we rely on the discount curve associated with Overnight Indexed Swaps. We will refer to the OIS discount factors by the necessary superscript. We can compute the Overnight Indexed Swap fixed rate via

$$r_s^{\mathrm{OIS}} = \frac{1 - Z^{\mathrm{OIS}}(0, t_N)}{\tau \sum_{i=1}^{N} Z^{\mathrm{OIS}}(0, t_i)} \tag{3.12}$$

using only OIS discount factors, and via

$$r_s^{\mathrm{OIS}} = \frac{\sum_{i=1}^{N} Z^{\mathrm{OIS}}(0, t_i) f(0, t_{i-1}, t_i)}{\sum_{i=1}^{N} Z^{\mathrm{OIS}}(0, t_i)} \tag{3.13}$$

using forward rates, just as before.

3.5 Pricing a Basis Swap

The pricing of a Basis Swap follows a somewhat similar approach to a generic fixed-for-floating swap, only now we have two floating rates. As before, the swap is usually structured such that its value is zero at initiation, but what is its price? After all, there is no swap fixed rate to calculate. For basis swaps, the price is essentially the spread paid over one of the floating legs.

When working with a basis swap of two different tenors of the same benchmark rate it may seem counter-intuitive to have any kind of spread, since there would appear to be an arbitrage opportunity. However, this spread reflects some of the subtler features of the market, namely the relative supply and demand of different tenors of the benchmark rate, in addition to some compensation for credit risk exposure.

The inclusion of a spread when the two benchmarks are different is a natural result given they will inherently have differing exposures and characteristics. In practice, with LIBOR-based basis swaps, the calculated spread, or price, is always added to the non-LIBOR side.

3.5.1 The LIBOR Floating Leg

Once again, we are looking to determine present values. The pricing of the LIBOR floating leg follows exactly the same approach as that of the generic fixed-for-floating swap in definition 3.2.

3.5.2 The Alternative Floating Leg

Let us introduce some further notation. We define the alternative reference rate, $r^{alt}(t_i)$ and the spread, m, following the convention of Flavell (2012). Similarly to the fixed leg of a generic fixed-for-floating swap, we are discounting a series of cash flows. Unfortunately, this time they are a sequence of variable unknown future cash flows, so we cannot move $r^{\text{alt}}(t_i)$ outside of the sum. Instead we can simply write

$$PV_{\text{alt}} = \tau \sum_{i=1}^{N} \left(r^{\text{alt}}(t_i) + m \right) Z(0, t_i). \tag{3.14}$$

Where τ represents the time between interest payments, as before.

3.5.3 The Price

Equating the two floating rates, we arrive at the following equation

$$PV_{\text{float}} = PV_{\text{alt}} \tag{3.15}$$

$$1 - Z(0, t_N) = \tau \sum_{i=1}^{N} \left(r^{\text{alt}}(t_i) + m \right) Z(0, t_i). \tag{3.16}$$

It turns out that we cannot simply invert this equation to find the appropriate spread, m, and must use optimisation and numerical methods to do so. In practice, these spreads are already quoted in the market and thus our main task is inferring the appropriate forward curve for the alternative benchmark—using a technique called *bootstrapping* (discussed shortly).

3.6 Curve Construction

With the formulas presented above you could be forgiven for thinking that the pricing of those interest rate derivative contracts is straightforward. Given a starting point, say, a set of discount factors, one could compute fair prices using little more than a spreadsheet.

The real challenge lies in establishing the correct starting point. Recall, throughout our discussions of forward rates, spot rates, and discount factors we have not mentioned where they come from in the first place. Can they simply be observed in the market? Sadly, the answer is no.

It turns out that our presentation so far needs to be reversed. Instead of using the discount factors or other rates to find the prices of products like fixed-for-floating swaps, we must use the prices of those swaps to find the discount factors and rates.

We can take this approach in large part due to the size and liquidity of the interest rate derivatives market discussed in Chapter 2. The market is sufficiently large that products such as swaps are, in fact, priced by the principles of supply and demand, independent of any models (Wilmott, 2013). As such, we use the market quoted rates on various interest rate instruments to compute discount factors and rates, and *construct their associated curves.* This then allows us to price back any contracts, not just those observable in the market.

3.6.1 Methodology

Constructing *appropriate* curves is a challenging feat, partly due the limited number of available instruments in comparison to the number of points required to properly cover the length of the curve. We will discuss the specifics shortly, but in general the method proceeds as follows (Banque du Canada, 2000). For the purposes of our project, we will use a continuously compounded basis.

3.6.1.1 Choosing Appropriate Instruments

The first step is to decide on a set of interest rate instruments to be used as inputs. We have mentioned the liquidity of interest rate derivatives in general, but of course some will be more liquid than others. The concept of stale pricing—that is, financial instruments which do not trade actively and therefore their quoted price may not be representative of current market conditions—is of serious concern, since they are likely to produce incorrect curves.

To identify the best products, the curve can be split into three sections, comprising of the short, medium, and long term. For the short term, deposit instruments such as LIBOR rates could be used as these were previously available up to one year. For the medium term, either forward rate agreements or Eurodollar futures (adjusted for convexity) could be used, constituting somewhere between six months and two years—if overlapping instruments are used, the result is a *blended curve*. In the long term, from one or two years and beyond, swap rates are used.

3.6.1.2 Constructing the Short Section

At the short end of the curve, we are fortunate that the instruments chosen, say LIBOR deposit rates, are inherently zero coupon. This means we can extract both the spot rates and the discount factors directly using the results from Chapter 2.

Let $\ell(t_i)$ be the LIBOR rate associated with time t_i. Then we can compute the spot rate via

$$r_c(t_i) = \frac{1}{t_i} \ln(1 + \ell(t_i)t_i). \tag{3.17}$$

Which can then be used to compute the discount factor through

$$Z(0, t_i) = e^{-r_c(t_i)t_i}. \tag{3.18}$$

Alternatively, we can jump directly from the LIBOR rate to the discount factor using

$$Z(0, t_i) = \frac{1}{1 + \ell(t_i)t_i}. \tag{3.19}$$

3.6.1.3 Constructing the Middle Section

We have a choice of approaches for the middle section of the curve. If we use forward rate agreements then we can extend our discussion of forward rates in Chapter 2 and invert definition 2.6 to express discount factors in terms of forward rates

$$Z(0, t_i) = Z(0, t_{i-1})e^{-f_c(0,t_{i-1},t_i)(t_{i-1}-t_i)}. \tag{3.20}$$

We can proceed recursively, noting that $Z(0,0) = 1$ to find that

$$Z(0,t_i) = \prod_{j=1}^{i-1} e^{-f_c(0,t_{i-1},t_i)(t_{i-1}-t_i)} \tag{3.21}$$

$$= \exp\left(-\sum_{j=1}^{i-1} f_c(0,t_{i-1},t_i)(t_{i-1}-t_i)\right). \tag{3.22}$$

Hence, we can find the discount factors and obtain spot rates and forward rates through the usual means.

Since forward rate agreements are OTC instruments, they are not subject to the conditions of exchanges. An alternative to forward rate agreements is to use Eurodollar futures contracts. By definition, futures contracts are exchange traded and thus are marked-to-market at the end of each day, with gains and losses being reflected in the margin account of each counterparty. This daily fluctuation bring about additional opportunities as gains can be reinvested at higher rates of return, on the other hand losses can be financed at lower rates (Piterbarg and Renedo, 2004). The asymmetrical nature of this technical feature means that Eurodollar futures values contain convexity—that is, the value will rise by a greater amount than it would fall given equal and opposite movements—a second order effect.

In practice, this requires an adjustment to the value of the Eurodollar futures contracts. The specific adjustment applied depends on the choice of short term interest rate model, however, generally speaking the change is minimal for shorter maturities (in the region of a few basis points) and so its true implications are beyond the scope of this project. The interested reader is referred to Hull (2003) and Piterbarg and Renedo (2004) for a more detailed treatment.

We must convert the quoted rate to a continuous basis, noting they settle quarterly on specific fixed days of the month, and then apply the following recursive formula to generate the spot rate. We will denote $EF(t_i)$ be the continuous Eurodollar future rate,

$$r_c(t_i) = \frac{EF(t_i)(t_i - t_{i-1}) + r_c(t_{i-1})t_{i-1}}{t_i}. \tag{3.23}$$

Once again, we can obtain the discount factors and forward rates from their respective formulas.

3.6.1.4 Constructing the Long Section

To construct the long end of the curve we turn to fixed-for-floating swap quotes. Returning to our swap pricing formula and inverting, we obtain

$$Z(0,t_i) = \frac{1 - r_s(t_i)\tau \sum_{j=1}^{i-1} Z(0,t_j)}{1 + r_s(t_i)\tau}. \tag{3.24}$$

In words, this means we can iteratively compute the discount factor associated with time t_i using the market quoted swap rate, $r_s(t_i)$, the time between interest payments τ, and the discount factors associated with all interest payments prior to that time. This progressive method, which uses all previously obtained discount factors to compute the next one, is called *bootstrapping* (Wilmott, 2013) (Veronesi, 2016).

3.6.1.5 Interpolation Schemes

At first glance this curve construction technique seems intuitive. We simply begin by computing the discount factors and spot rates at the short end and proceed along the curve utilising what we have previously calculated.

Unfortunately, any implementation of such method will soon run into a problem. At the longer end of the curve, we require the discount factors associated with *every interest payment over the life of the swap* (excluding the final payment of course, which is what we are trying to work out). But it is highly unlikely that there are liquid quotes for all of these times.

Consider a 15-year swap with semi-annual fixed and floating interest payments. Suppose we find that the most liquid swap maturity prior to this is 12-years. We may have computed the discount factors for every six month period from initiation to 12-years, but cannot continue further due to the lack of data. We still need the discount factors for 12.5, 13, 13.5, 14, and 14.5, hence, we must *interpolate* between these points.

Which interpolation technique to use is not a decision to be taken lightly. This presents the biggest challenge during curve construction of which there is no universally agreed correct answer. Hagan and West (2006) conducted extensive analysis into the different interpolation methods used in curve construction, ranging from simple linear interpolation to more complicated monotone convex interpolation—which is used by the U.S. Department of the Treasury for their yield curve construction (US Treasury, 2021). The interested reader can find a concise comparison of the many different methods in Hagan and West (2008). As our project is focused on the pricing of interest rate swaps, rather than analysing and evaluating different curve construction techniques, there is insufficient benefit in considering the complex, optimisation based approaches and instead we will focus on two simple, but still popular, methods: linear interpolation and cubic spline interpolation (Flavell, 2012).

Given the interlinked nature of discount factors and spot rates we must also decide *which* to interpolate. We can interpolate the discount factors directly, or we can interpolate the spot rates and infer the appropriate discount factors from those.

3.6.2 Python Implementation

Having covered the general curve construction methodology, we can now move on and consider its computational implementation in more detail. We have used Python and describe our step-by-step curve construction in words, before presenting the pseudocode. We then discuss the results of a real-world example. Our full code can be found in Appendix C.1.

3.6.3 Step-by-Step Curve Construction

Method: Interpolation of the discount factors, or *explicit discount factor* method.

1. **Short Section:** Compute the discount factors and spot rates for the short section of the curve using the using the approach outlined in Section 3.6.1.2.

2. **Middle Section:** Compute the discount factors and spot rates for the middle section of the curve using either of the two approaches outlined in Section 3.6.1.3 depending on the intermediate instruments being used.

3. **Long Section:**

   (a) Identify the term of the swap being used for the long section of the curve, t_i.

   (b) Identify the frequency of the interest payments of the swap, say n.

   (c) Construct an array containing the interest payment times up to, but not including, the final interest payment, say, $\mathbf{p} = \left[\frac{1}{n}, \frac{2}{n}, \ldots, \frac{nt_i - 1}{n}\right]$. We will define p_j, where $j = 1, \ldots i-1$, as the $\mathrm{j^{th}}$ interest payment.

   (d) Construct an array of all the times associated with the discount factors computed so far, say x.

   (e) Construct an array of all the discount factors at those times, say y.

   (f) Interpolate (x, y) using the intended method (e.g. linear or cubic spline). We used the SciPy package available in Python.

   (g) Initialise a dummy variable to store the sum of the discount factors, $DF = 0$.

   (h) Perform a *for* loop over the interest payment times, p_j, evaluating the discount factor, $Z(0, p_j)$, on the interpolated curve of (x, y), and summing the result, $DF = \sum_{j=1}^{i-1} Z(0, p_j)$.

   (i) Compute the final bootstrapped discount factor associated with t_i using the formula in Section 3.6.1.4

   (j) Repeat from step 3(a) for the next swap maturity.

We can present the construction of the long end in a more concise manner using pseudocode.

Algorithm 1 Constructing the Long Section: Explicit Discount Factors

Require: t_i
Require: n ▷ Frequency of interest payments
Require: $r_s(t_i)$
$\mathbf{p} = [\frac{1}{n}, \frac{2}{n}, \ldots, \frac{nt_i-1}{n}]$ ▷ Interest payment dates
$\tau = p_j - p_{j-1}$ ▷ Assume constant for our purposes
$x \leftarrow$ 'times'
$y \leftarrow$ 'discount factors'
Interpolate (x, y)
$DF = 0$
for $j = 1, \ldots, i-1$ **do**
    Extract $Z(0, p_j)$ from curve
    $DF = DF + Z(0, p_j)$
end for
$Z(0, t_i) = \frac{1 - r_s(t_i)\tau DF}{1 + r_s(t_i)\tau}$
repeat
until Curve construction complete.

Alternatively, we can interpolate the spot rates. The construction of the short and middle sections remain the same, so we must only alter the method for the long section.

Method: Interpolation of the spot rates and inference of the discount factors, or *implicit discount factor* method.

3. **Long Section:**

    (a) Identify the term of the swap being used for the long section of the curve, t_i.

    (b) Identify the frequency of the interest payments of the swap, say n.

    (c) Construct an array containing the interest payment times up to, but not including, the final interest payment, say $\mathbf{p} = \left[\frac{1}{n}, \frac{2}{n}, \ldots, \frac{nt_i-1}{n}\right]$. We will define p_j, where $j = 1, \ldots i-1$, as the $\mathrm{j^{th}}$ interest payment.

    (d) Construct an array of all the times associated with the *spot rates* computed so far, say x.

    (e) Construct an array of all the *spot rates* at those times, say y.

    (f) Interpolate (x, y) using the intended method (e.g. linear or cubic spline). We used the SciPy package available in Python.

    (g) Initialise a dummy variable to store the sum of the discount factors, $DF = 0$.

    (h) Perform a *for* loop over the interest payment times, p_j, evaluating the *spot rate*, $r_c(p_j)$ on the interpolated curve of (x, y). *Convert that spot rate into a discount factor*, $Z(0, p_j)$, and sum the result, $DF = \sum_{j=1}^{i-1} Z(0, p_j)$.

    (i) Compute the final bootstrapped discount factor associated with t_i using the formula in Section 3.6.1.4

(j) Repeat from step 3(a) for the next swap maturity.

Once again, this step can be concisely presented as pseudocode.

Algorithm 2 Constructing the Long Section: Implicit Discount Factors

---

Require: t_i
Require: n ▷ Frequency of interest payments
Require: $r_s(t_i)$
$\mathbf{p} = [\frac{1}{n}, \frac{2}{n}, \ldots, \frac{nt_i - 1}{n}]$ ▷ Interest payment dates
$\tau = p_j - p_{j-1}$ ▷ Assume constant for our purposes
$x \leftarrow$ 'times'
$y \leftarrow$ 'spot rates'
Interpolate (x, y)
$DF = 0$
for $j = 1, \ldots, i-1$ **do**
    Extract $r_c(p_j)$ from curve
    Convert $r_c(p_j) \longrightarrow Z(0, p_j)$
    $DF = DF + Z(0, p_j)$
end for
$Z(0, t_i) = \frac{1 - r_s(t_i)\tau DF}{1 + r_s(t_i)\tau}$
repeat
until Curve construction complete.

3.6.4 Datasets

For our implementation, we have used two datasets as this will allow us to compare the differences in curve construction under two separate scenarios.

The first dataset is a combination of USD LIBOR rates, obtained from IBORate (2022), and mid-quotes on USD fixed-for-floating LIBOR-based swaps obtained from Flavell (2012), with the underlying data from ICAP, on the 15$^{\text{th}}$ February 2008. Note for this first dataset, our curve will be only be constructed from deposit rates and swaps. That is, we will omit any FRA or Eurodollar future quotes and simply interpolate between the short and long sections.

Our second dataset is a combination of USD LIBOR rates, a selection of Eurodollar futures quoted rates, and mid-quotes on USD fixed-for-floating LIBOR-based swaps, once again obtained from Flavell (2012), on the 6$^{\text{th}}$ February 2008. This time our curve will be constructed using short, medium, and long instruments. Tables containing both sets of data can be found in Appendix B.1.

3.6.5 Results

Figures 3.1, 3.2, 3.3, and 3.4 show the results of our curve constructions for the two datasets. Tables containing detailed numerical results can be found in Appendix B.2.

From Figure 3.1 we can see from both panels that the spot curves under all four construction techniques appear smooth with only marginal differences between the rates across all maturities. However, the same is not true for the forward curve in Figure 3.3.

We need only consider the forward curve out to 10-years to get a clear understanding of the impacts of the different construction techniques. We notice that the Explicit Discount Factor method (both linear and cubic spline) and the linear Implicit Discount Factors produce a smooth, appropriate-looking forward curve, however, we begin to see instability under the cubic spline Implicit Discount Factor method.

This wave-like curve structure stems from the fact that, at the short end, the discount factors used to compute forward rates are close together, hence, minor differences can dominant and noticeably alter the forward rate. As the time extends, the differential between discount factors plays a less important role and we can see convergence under all four techniques.

Turning our attention to Figure 3.2, both panels indicate the long end of spot curve is once again smooth, however, this time the shorter end is not. This follows from the choice of instruments used, as dataset 1 only contained LIBOR deposit rates and swaps, whereas dataset 2 included Eurodollar futures in the time period between 1- and 2-years. Whilst this provides additional information, since these are market quoted prices, it highlights the challenges faced when stitching the different sections of the curve together. In practice, the instruments may be chosen such that they overlap, creating a blended curve, to compensate for the different contracts.

This issue pales in comparison to the jagged nature of the forward rate curve. As we can see from Figure 3.4, all four construction methodologies give unstable and inappropriate forward curves. This highlights the sensitive nature of the forward curves as there is apparently little difference between the data used to produce this forward curve on 06/02/2008 and less than ten days later on 15/02/2008, yet the latter appears relatively smooth, while the former is particularly unstable.

The inappropriate appearance of this curve does not invalidate the fact that both curves produced can be considered no-arbitrage forward curves for the instruments used in their construction. That is, using the forward curve and its associated discount curve under any of the methodologies, we can price back those same instruments to the observed market rates.

Clearly, however, it will not produce a reasonable price for those instruments *not* used in its construction. And therein lies the challenge. We are faced with a trade-off between strictly following the steps to construct the curve, and facing whatever forward curve stems as a result, or try and control the appearance of the curve and bear the consequences that it will no longer be considered free of arbitrage opportunities. In practice, this is a fine balance, and one in which the use of optimisation plays a major role.

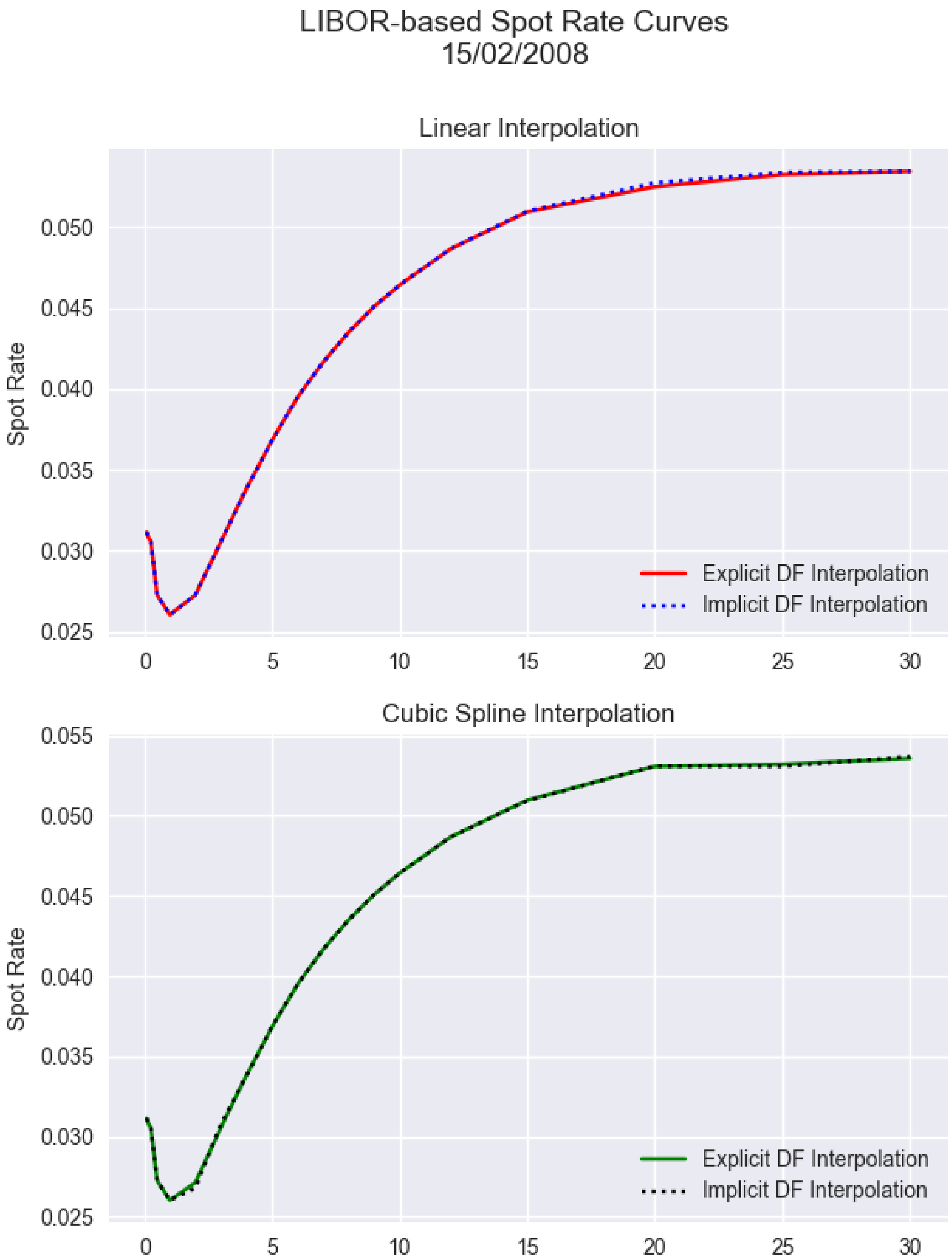

Figure 3.1: LIBOR-based Spot Rate Curves on 15/02/2008. **Panel 1**—The spot rate curves constructed using linear interpolation of the discount factors ('Explicit DF') and linear interpolation of the spot rates and inference of discount factors ('Implicit DF'). **Panel 2**—The spot rate curves constructed using cubic spline interpolation of the discount factors ('Explicit DF') and cubic spline interpolation of the spot rates and inference of discount factors ('Implicit DF').

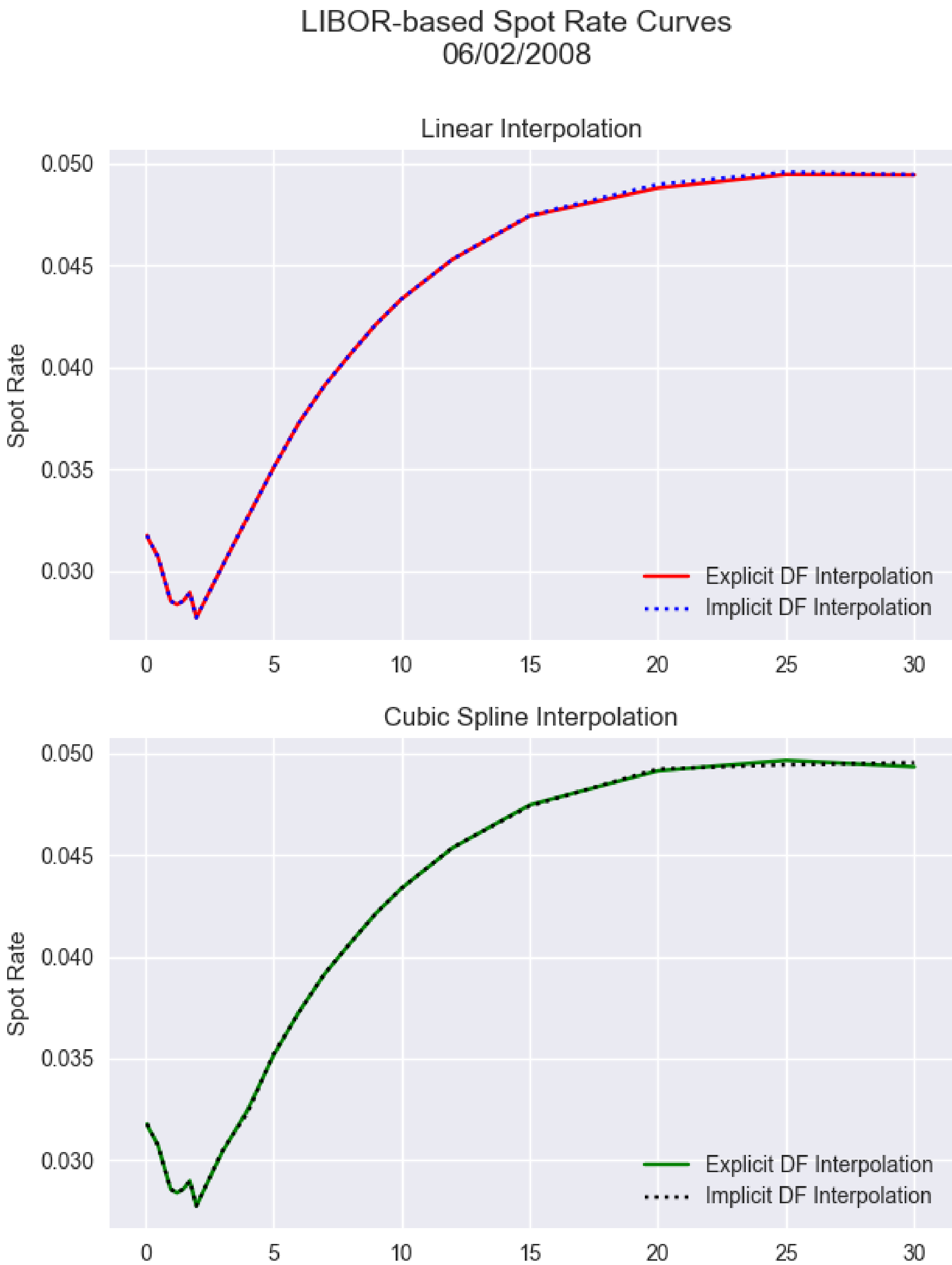

Figure 3.2: LIBOR-based Spot Rate Curves on 06/02/2008. **Panel 1**—The spot rate curves constructed using linear interpolation of the discount factors ('Explicit DF') and linear interpolation of the spot rates and inference of discount factors ('Implicit DF'). **Panel 2**—The spot rate curves constructed using cubic spline interpolation of the discount factors ('Explicit DF') and cubic spline interpolation of the spot rates and inference of discount factors ('Implicit DF').

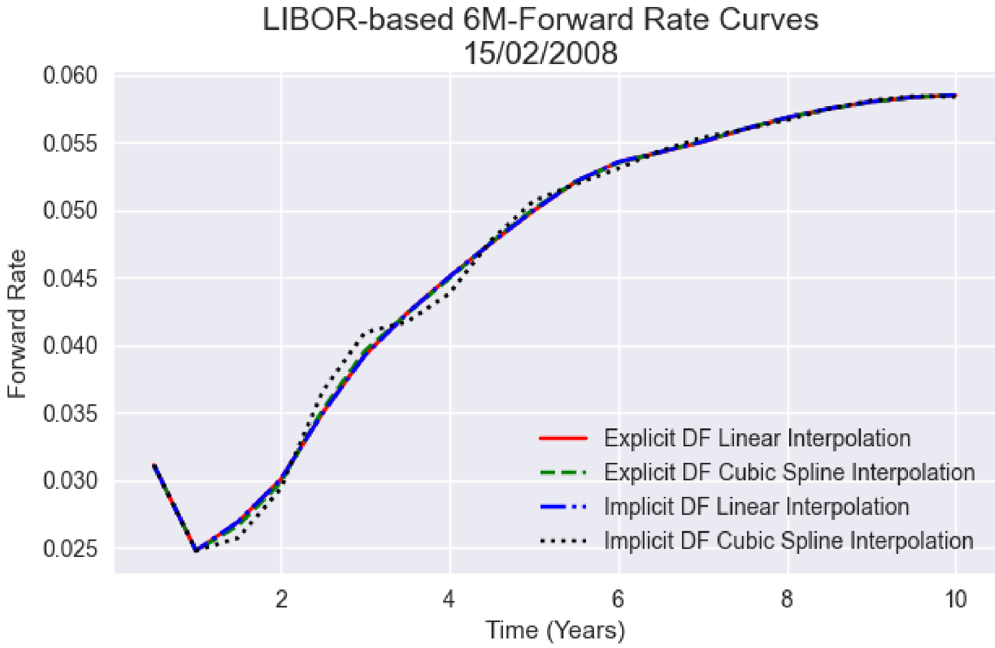

Figure 3.3: LIBOR-based 6M-Forward Rate Curves on 15/02/2008 associated with each of the four construction techniques.

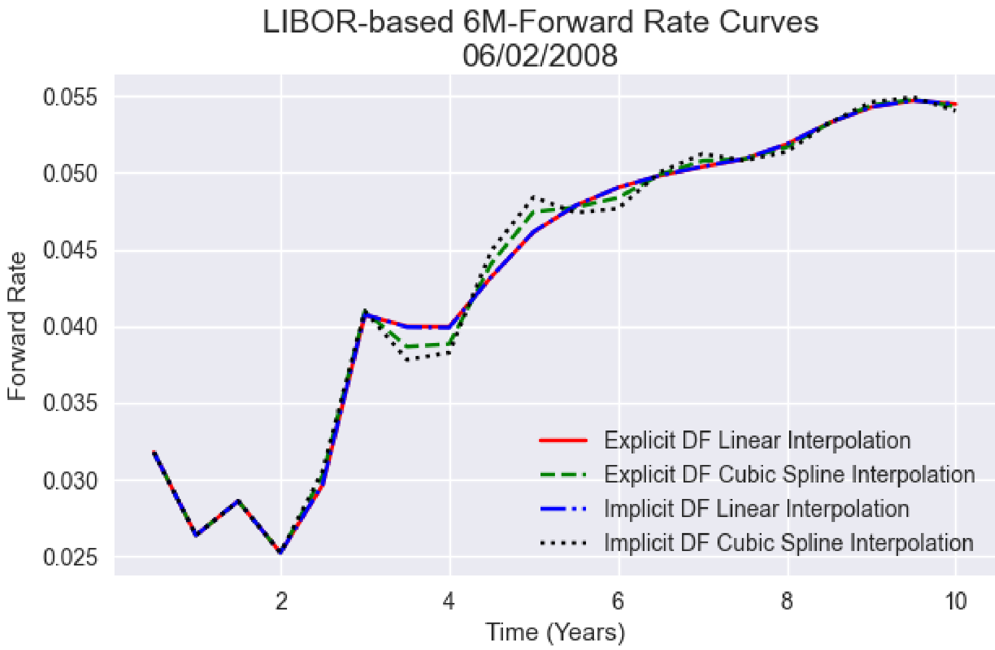

Figure 3.4: LIBOR-based 6M-Forward Rate Curves on 06/02/2008 associated with each of the four construction techniques.

3.7 Practical Considerations

Our Python implementation is a useful tool for illustrating the curves constructed from the different methodologies. However, our underlying assumptions mean it is a simplified version of the problem faced in the real-world. In this section we discuss some of the additional factors that must be considered and describe how those changes could be incorporated.

3.7.1 Day Count Convention

A major component of curve construction is the instruments used. If we dig a little deeper into those products, and consider their settlement characteristics in more detail, we find several complications relating to day count convention.

For our purposes we assumed that a year consisted of 360 days and each month was 30 days. This allowed us to consider months as simple fraction, for instance 3-months is simply 0.25. In practice, however, this may not always be the case. Different products use different conventions—actual/360 and actual/365 are just a couple of the possibilities, where actual refers to the actual number of days between two dates.

In addition, we must consider the concept of business days, and factor in things such as public holidays for the market in question. As a result, not only do we need to know the quoted swap rates to be used in our curve construction, we must also gather the relevant payment and settlement information and compute the true length of time where required. Whilst this adjustment is not conceptually difficult, its implementation requires a great deal of care.

3.7.2 Asymmetrical Interest Payments

When constructing the long section of the curve we made the simplifying assumption that interest payments on both the fixed and floating sides of the swaps were made semi-annually. This does not have to be (and usually is not) the case.

For instance, the fixed leg of a generic USD swap is paid semi-annually, whereas the floating rate leg is paid quarterly. This complicates our curve construction somewhat, as our initial interpolation between swap rates would be on an *semi-annual* basis, but we may be interested in the quarterly forward curve associated with the LIBOR fixing of the floating rate side. We would then need to make a further decision about the interpolation between the calculated discount factors.

As we saw in Figures 3.3 and 3.4, the forward curve is particularly sensitive and this interest payment complexity can cause the forward curve to become erratic. In fact, it is the stabilisation and appropriate construction of the forward curve that demands the most attention and, in general, a combination of interpolation techniques and optimisation methods may be used.

To summarise the challenges faced by day count conventions and interest payment frequencies, we display the following table based on the information given in Sadr (2009).

Currency	Fixed Leg Freq & Day Count	Float Leg Freq & Day Count	Float Leg Index	Spot Date
USD	S/A 30/360	Q Act/360	USD 3m LIBOR	2d LONDON
EUR	A 30/360	S/A Act/360	EUR 6m LIBOR	2d Target
EUR-1Y	Single	Q Act/360	EUR 3m LIBOR	2d Target
JPY	S/A Act/365	S/A Act/360	JPY 6m LIBOR	2d TOK
GBP	S/A Act/365	S/A Act/365	GBP 6m LIBOR	Same Day
GBP-1Y	Q Act/365	Q Act/365	GBP 3m LIBOR	Same Day
CHF	A 30/360	S/A Act/360	CHF 3m LIBOR	2d ZUR

Table 3.1: A summary of the different day count conventions and payment frequencies of general fixed-for-floating interest rate swaps in major markets.

3.7.3 Single vs. Multi-curve Pricing

The approach that we have been discussing so far is the so called *single curve framework*. That is, we are using observable market quotes for LIBOR based instruments and constructing the curve which then prices those same instruments under the assumption of no arbitrage. This approach was popular prior to the financial crisis when LIBOR rates were assumed to be representative of risk-free rates.

After the financial crisis, however, LIBOR rates were no longer deemed truly risk-free and as such the market moved towards the *multi-curve framework*. This involved using observed OIS quotes to construct the discount curve, computing the associated forward rates from a combination of the OIS discount curve and observed LIBOR-based swap quotes, before inferring the LIBOR-based discount factors and spot curve.

We can distinguish the two frameworks by their respective flow charts, summarised from Veronesi (2016).

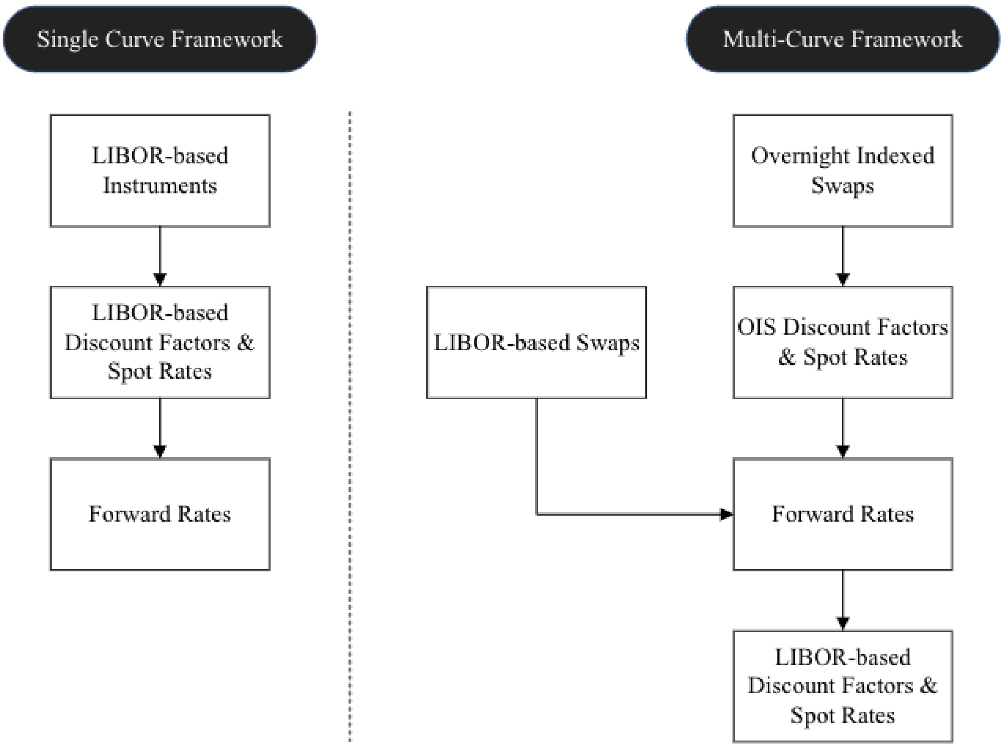

Figure 3.5: A flowchart depicting the difference between the single curve and multi-curve frameworks. The single curve was used extensively prior to the financial crisis, however, the multi-curve approach has been used since owing to LIBOR rates not being deemed truly risk-free.

Chapter 4

Crypto Interest Rate Swaps

If you don't believe it or don't get it, I don't have the time to try to convince you, sorry.

—Satoshi Nakamoto, Founder of Bitcoin

4.1 Introduction

In this chapter we extend our analysis of interest rate swaps to consider the case within cryptocurrency markets. We begin with a discussion of the terminology and existing products before reviewing traditional interest rate models, examining a proposed crypto interest rate model and constructing some indicative yield curves. We then consider a specific crypto interest rate swap: the Bitcoin-USD Funding Rate Swap, its mechanics and pricing formula. We finish by briefly proposing some new areas for interest rate swap-like products within crypto markets.

4.2 Terminology & Products

The world of cryptoassets is considerably different to that of traditional finance. It represents a new approach towards storing and transacting value, based on new technologies. This difference introduces new terminology that, once again, must be understood in order to fully appreciate the products on offer. Some elements have close ties with traditional counterparts, just with different names, whilst others are completely new.

Given the decentralised nature of cryptocurrencies, a mechanism is required to solve the so-called 'double spending' problem—that is, some way of ensuring one cannot spend the same money twice. In traditional finance there would simply be a central authority, such as a central bank, responsible for keeping track of such issues but in crypto there is no such thing. To solve this challenge, and give cryptocurrencies credibility and inherent value, two approaches have since been used: *Proof-of-Work* and *Proof-of-Stake*.

4.2.1 Proof-of-Work (PoW)

The proof-of-work technique relies on miners seeking to be the first to solve a cryptographic puzzle. The first miner to do so then adds the next block to the blockchain and is rewarded with a specified amount of cryptocurrency. The challenge arises from the complexity of the cryptographic puzzle itself, as solving such a problem requires enormous amounts of computational power (and has since been criticized for its energy inefficiency). Although given the sheer number of participating miners around the world, it only takes an average of ten minutes to solve (Balan, 2021). The technique can be considered self-regulating since each miner has an incentive to be the first to provide the correct answer and be the one to update the blockchain, thereby earning the reward. The open source nature of the protocol means an incorrect answer, submitted by a miner in a bid to earn the reward, would easily be identified by other participants and would be rejected (CoinDesk, 2022b).

4.2.2 Proof-of-Stake (PoS)

The proof-of-stake technique proposes an alternative to proof-of-work. The method relies on participants contributing, or *staking*, their cryptocurrency for the chance to become a validator, add the next block to the blockchain, and subsequently be rewarded with a specified amount of cryptocurrency. Unlike proof-of-work, this is not a race. The selection of the validators depends on how much cryptocurrency is staked and the length of time it has been staked for. Those who have been selected as validators confirm that the block is correct and other participants (those not initially selected) can review and attest to the correctness of that block. If there are no issues, then the selected validators earn a reward. With proof-of-stake, however, the self-regulation stems from consequences of an incorrect block. If the non-selected validators disagree, and prove the proposed block to be incorrect, then the initial validators will lose some of their staked cryptocurrency in a move known as *slashing* (CoinDesk, 2022a).

4.2.3 Staking

Whilst the proof-of-stake mechanism is conceptually simple, the actual investment required to become a potential validator can be substantial, not to mention requiring a large amount of computational resources and an in depth understanding of the fundamental technology. This often puts it out of reach for moderate investors.

To reduce the barriers to entry, crypto exchanges now offer staking as a product to their clients. This means that smaller investors can contribute their cryptocurrency into a pool which is then used participate in the proof-of-stake mechanism, and earn crypto 'interest' in return. This often requires some lock-up period, in which contributors to the pool will not have access to their assets, with any withdrawal attempts triggering significant penalties. The approach is very similar to, say, a traditional US Treasury bond fund—few investors could buy the bonds outright, but any investor can purchase a bond fund and earn their share of interest (coupon payments) through participation.

Given the high volatility in crypto markets, this product poses serious risks. If the price of the cryptocurrency staked drops over the lock-up period, then it is possible that the loss on the underlying will outweigh the interest earned over the period. This widespread problem is referred to as the *impermanent loss* and is felt by many participants, including market makers (CoinBase, 2022).

4.2.4 Crypto Saving & Lending

Staking should not be confused with crypto *savings products*. As with traditional finance, it is possible to deposit cryptocurrency into a savings account, offered by crypto exchanges, and earn interest in the form of crypto on that deposit. One has the option of locked or flexible savings accounts, with flexible providing instant access and locked only being available for withdrawal after a specified time period, as expected (Binance, 2022b).

A *crypto loan* is somewhat analogous to a traditional loan in that it refers to the borrowing of funds over a specific time period with an agreement to repay the loan with interest. The difference, however, is that in the majority of cases the funds being borrowed are cryptocurrencies, and the collateral is another cryptocurrency. It is worth noting that this is not always the case, as firms like Celsius (prior to filing for bankruptcy) offered USD loans, with collateral of cryptocurrency (Celsius, 2022).

To protect themselves, firms offering these products often require overcollateralisation, that is, the value of the cryptocurrency secured against the loan is significantly more than the loan itself. In addition, there are aggressive margin requirements that require further collateral deposits should the markets move negatively, with a threshold at which the company has the right to sell the secured cryptocurrency and close the loan. Once again, given the high volatility in crypto markets, the products pose serious risks (Binance, 2022a).

4.2.5 Crypto Swaps

Due to its early divergence from traditional finance, the term *crypto swap* already has meaning, hence, we must be explicit when discussing forthcoming *crypto interest rate swaps*. A crypto swap refers to the direct swap of one cryptocurrency for another and is analogous to an FX swap. In general, there are two methods for doing so. Firstly, one can use a centralised crypto exchange, however, this leads to certain unwanted characteristics, such as the power the exchange has over any transaction. They could limit the available cryptocurrencies or charge a fee for their services. The second method involves a trustless exchange using the atomic swap protocol which utilises a specific type of cryptographic holding period called Hashed Time Lock Contracts (Liu, 2020).

4.2.6 Liquidity Pooling

Of course, pursuing a crypto swap through an exchange requires there to be sufficient demand for each of the tokens in question, such that the exchange can offer the desired swaps. To establish a broader range of swap pairs, exchanges set up liquidity pools. These are essentially large collections of tokens that enable parties to swap their cryptocurrencies. To increase the size of the collection—since a larger pool equates to a more liquid swap pair—individuals can contribute their own cryptocurrencies to one, or both, sides of the pair (provided they own those assets prescribed by the swap pair). As a result, they are rewarded by earning a 'portion' of the returns on the transactions that occur.

The level of risk is primarily determined by the swap pair itself. For more stable tokens the lower volatility can lead to more predictable returns, albeit they are generally smaller. Alternatively, for pairs of which one, or both, are not stable tokens market volatility plays a much larger role and the returns are subject to greater variation, perhaps even resulting in losses.

4.3 Crypto Interest Rates

In the case of traditional markets, the concept of interest is simply one that rewards existing holders of assets. The idea that one can deposit their assets for a specific period of time after which they will receive slightly more than they previously held. This extends throughout the traditional finance system and is not limited to individuals; generic companies can deposit cash or other marketable instruments with financial institutions to earn interest, and likewise financial institutions themselves can make deposits to central banks. This has fueled the development of the numerous short rate models over the years, and is fundamental to derivatives pricing.

This is contradictory, however, to the world of cryptocurrencies. It is not the owners of cryptoassets that are rewarded, but the miners in a proof-of-work protocol and the validators in the proof-of-stake case. In addition, the decentralised nature of crypto means there is no central bank or authority by definition. This implies that the short rate is zero. How, then, can one construct a non-trival interest rate term structure model? Before examining the model itself, it will be useful to review the traditional case.

4.3.1 Traditional Interest Rate Models

The development of traditional short rate models begins with the assumption of the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\{\mathcal{F}\}_{s\geq 0}$ is the associated filtration. We then proceed to define the adapted short rate process $\{r_s\}_{s\geq 0}$.

Recall from definition 2.1 the idea of a discount factor, $Z(0, t_i)$, the present value of \$1 received at some point, t_i, in the future. We briefly mentioned the fact that this was equivalent to the price today of a zero coupon bond, with a principal of \$1, maturing at time t_i, before proceeding to demonstrate its links to spot rates and forward rates.

We can define this discount factor, or zero coupon bond, more generally using the short rate process, r_s instead. Let $\mathbb{Q}$ be an appropriate risk-neutral measure equivalent to our initial measure $\mathbb{P}$, and $0 \leq s < t_i$ be any time prior to maturity, then we can write

$$Z(s, t_i) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_s^{t_i} r_u du\right)\middle|\mathcal{F}_s\right]. \tag{4.1}$$

As it so happens, the representation given in 4.1 is not the only one possible. Constantinides (1992) introduced a new approach to interest rate modelling using *pricing kernels*. A pricing kernel is an $\mathcal{F}_s$-adapted càdlàg semimartingale $\{\pi_s\}_{s\geq 0}$ satisfying certain conditions. This leads to an alternative, but equivalent, representation of our zero coupon bond

$$Z(s, t_i) = \frac{1}{\pi_s}\mathbb{E}\left[\pi_{t_i}|\mathcal{F}_s\right]. \tag{4.2}$$

We can quickly recover the expression 4.1 as follows. Using the money market account as our numeraire, given by

$$B_s = \exp\left(\int_0^s r_u du\right) \tag{4.3}$$

we have that the combination of the inverse of the money market account (that is, discounting) and the Radon-Nikodym derivative forms our pricing kernel,

$$\pi_s = \exp\left(-\int_0^s r_u du\right)\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_s}. \tag{4.4}$$

As a result, we can use the typical change of measure technique on 4.2 to obtain our original expression. For more information regarding risk-neutrality, change of measure, and change of numeraire, the reader is referred to the excellent discussion in Chapter 2 of Brigo et al. (2001).

4.3.2 A Crypto Interest Rate Model

Regardless of the approach used, it is difficult to imagine the circumstances under which we can produce a non-trivial term structure given an identically zero short rate ($r_u = 0$). In addition, our previous comments on the concept of traditional versus decentralised 'interest' excludes any reliance on the money market account, since there will be no growth in cryptoassets held, that is, the process B_s is constant.

To produce a non-trivial term structure model, Brody et al. (2020) relies on choosing a pricing kernel that is a strict local martingale, which produces the desired effect of a zero short rate. The pricing kernel in question is the reciprocal of a Bessel(3) process (this is later extended to consider Bessel(n) processes, however, we will omit this generalisation from our work).

Assume the existence of a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, with adapted filtration $\{\mathcal{F}_s\}_{s\geq 0}$. Further, define three standard, independent Brownian motions $\{W_s^{(1)}, W_s^{(2)}, W_s^{(3)}\}_{s\geq 0}$ and let σ_s be a bounded, strictly positive deterministic function. Then

$$X_s = \int_0^s \sigma_u dW_u^{(1)} \tag{4.5}$$

$$Y_s = \int_0^s \sigma_u dW_u^{(2)} \tag{4.6}$$

$$Z_s = \int_0^s \sigma_u dW_u^{(3)} \tag{4.7}$$

and our pricing kernel, the reciprocal of a Bessel(3) process, is given by

$$\pi_s = \frac{1}{\sqrt{(X_s - a)^2 + (Y_s - b)^2 + (Z_s - c)^2}} \tag{4.8}$$

where $a, b, c \neq 0$. We also require the initial condition $\pi_0 = 1$, hence, we must have that $\sqrt{a^2 + b^2 + c^2} = 1$.

Now, using equation 4.2 we find,

$$Z(s, t_i) = \frac{1}{\pi_s}\mathbb{E}\left[\frac{1}{\sqrt{(X_{t_i} - a)^2 + (Y_{t_i} - b)^2 + (Z_{t_i} - c)^2}}\middle|\mathcal{F}_s\right]. \tag{4.9}$$

At this point we can rely on the definition of Brownian motion and observe that $X_{t_i} - a = (X_{t_i} - X_s) + (X_s - a)$, similarly for Y_{t_i} and Z_{t_i}. Hence, we find that

$$X_{t_i} \sim \mathcal{N}(X_s - a, \Sigma_{s,t_i}) \tag{4.10}$$

$$Y_{t_i} \sim \mathcal{N}(Y_s - b, \Sigma_{s,t_i}) \tag{4.11}$$

$$Z_{t_i} \sim \mathcal{N}(Z_s - c, \Sigma_{s,t_i}) \tag{4.12}$$

$$\Sigma_{s,t_i} = \int_s^{t_i} \sigma_u^2 du. \tag{4.13}$$

After a great deal of algebra, Brody et al. (2020) arrives at the expression for the zero coupon bond price. Given by,

$$Z(s, t_i) = \operatorname{erf}\left(\sqrt{\frac{(X_s - a)^2 + (Y_s - b)^2 + (Z_s - c)^2}{2\Sigma_{s,t_i}}}\right) \tag{4.14}$$

where erf(·) is the standard error function defined as,

$$\text{erf}(z) := \frac{1}{\sqrt{\pi}} \int_{-z}^{z} e^{-u^2} du. \tag{4.15}$$

Now that we have an explicit formula for the price of zero coupon bond, we can use the following relation to compute the yield curve (Wilmott, 2013)

$$Y(s, t_i) = -\frac{1}{(t_i - s)} \log Z(s, t_i). \tag{4.16}$$

Thus we obtain the explicit expression, for $s = 0$

$$Y(0, t_i) = -\frac{1}{t_i} \log \left[\text{erf} \left(\frac{1}{\sqrt{2\Sigma_{0,t_i}}} \right) \right]. \tag{4.17}$$

4.3.3 Crypto Yield Curves

From 4.17 we have a closed form expression for the yield curve provided we can appropriately identify Σ_{s,t_i}, or more specifically σ_u underlying it. Using a constant value for volatility, similarly to Brody et al. (2020), we can produce some illustrative yield curves displayed in Figure 4.1.

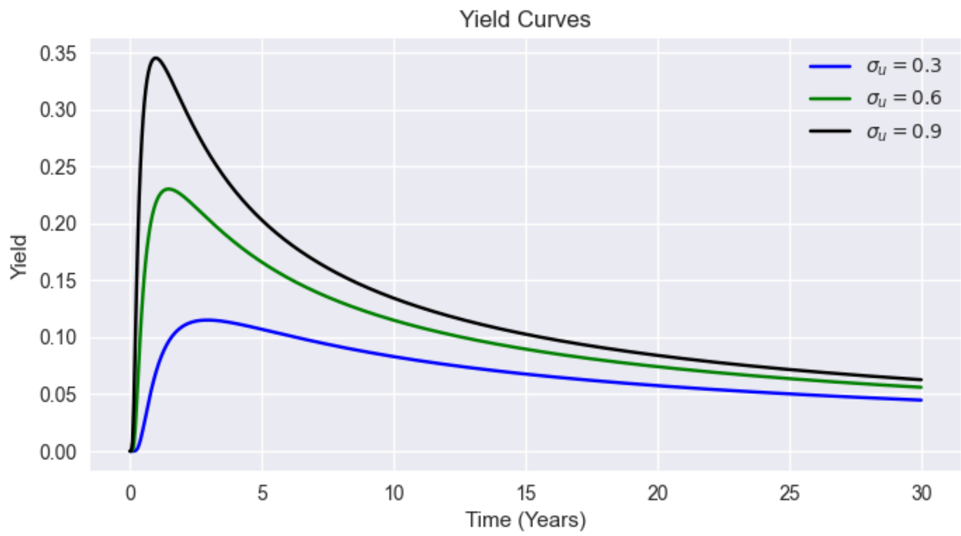

Figure 4.1: Examples of a crypto yield curve based on constant values of σ_u. Reproduced from Brody et al. (2020).

In practice, we would be more likely to work in the opposite direction. We would use the explicit formula for the yield curve to calibrate our volatility function σ_u to market data, just as one can compute implied volatility from the original Black and Scholes (1973) solution using an iterative technique.

4.4 Crypto Interest Rate Swaps

Cryptocurrency interest rate models are still in their infancy and there is yet to be a widely recognised and agreed upon approach. The intention, however, has led some crypto market participants to offer an early foray into the world of interest rate derivatives, and specifically interest rate swaps.

As discussed in Chapter 3, interest rate swaps are model independent, by which we mean that they sidestep the challenges faced by the lack of established crypto interest rate models. A general fixed-for-floating interest rate swap simply exchanges a series of fixed interest rate payments for a series of floating ones, but there are no restrictions on what the 'interest rate' is. This makes it a prime candidate to be extended to crypto markets provided there exists a product that, one way or another, involves variable payments.

The cryptocurrency exchange BitMEX offers one such product: *perpetual swaps* (recall, in this instance we are referring to swaps in the crypto sense—the exchange of one asset for another—similar to traditional currency exchange). Let us first understand one specific example of BitMEX's perpetual swaps, the Bitcoin-USD ('XBTUSD') perpetual swap, before considering its associated interest rate swap.

4.4.1 Bitmex XBTUSD Perpetual Swap

A perpetual swap behaves similarly to a traditional futures contract, however, as the name suggests, there is no contract settlement date. The XBTUSD contract specifically is an *inverse* contract, that is, the opening and closing prices are the reciprocal of the quoted XBTUSD conversion rate. In mathematical terms, this means the profit or loss to the long party is calculated as

$$\text{Profit/Loss} = \text{No. of Contracts} \times \$1 \times \left(\frac{1}{\text{XBTUSD}_{open}} - \frac{1}{\text{XBTUSD}_{close}} \right) \tag{4.18}$$

where the $1 reflects the multiplier for the contract.

As with traditional futures, the contracts are marked-to-market daily, although in crypto this process is called *fair price marking*, and initial and maintenance margin are required in tandem with traditional derivatives exchanges.

Whilst the XBTUSD contract is perpetual, holding the contract (either long or short) still requires a form of 'rolling over'. This occurs via a *funding rate*, which is either paid or received depending on the computed rate and the direction of the position held, calculated every 8 hours.

The actual calculation of the funding rate is non trivial and is beyond the scope of this project. Nevertheless, it is important to note some of its features.

- The amount paid is calculated upon the notional value of the contract, rather than the margin position, hence is independent of leverage;
- The XBTUSD funding rate consists of three components: the interest rate, the discount or premium, and a dampening element;
- The interest differential is determined using the prevailing XBT and USD lending rates, divided by the number of funding periods per day (in this case three);
- The funding rate is then calculated every minute over the period and a time-weighted average price is applied. This determines the funding rate for the next period, that is, the funding rate is always known eight hours in advance.

In general, the funding rate is positive and long parties pay the funding rate, while short parties receive.

4.4.2 Bitmex Funding Rate Swap

The variability of the funding rate is analogous to the floating rate exposure associated with traditional markets discussed in Chapter 3—consider a borrower with an outstanding loan tied to LIBOR rates. This means parties involved in XBTUSD perpetual swaps may naturally look towards derivatives contracts to hedge the uncertainty of future funding rates.

The BitMEX Funding Rate Swap, offered by the Singapore based Delta Exchange, precisely filled that gap. Functioning in a similar way to a general fixed-for-floating interest rate swap in traditional markets, the funding rate swap is an agreement to exchange a fixed rate payment for a series of floating rate payments tied to the funding rate of the BitMEX XBTUSD perpetual swap, thus swapping the floating rate for a fixed rate exposure.

It is worth noting that the funding rate swap is similar, but not identical, to traditional interest rate swaps since there is a lump sum payment associated with fixed rate side at the initiation of the contract, while the floating payments are paid in series.

Furthermore, under normal circumstances when the funding rate is positive and long parties compensate short parties, the long party entering into a funding rate swap is said to be *buying floating-for-fixed*. This means they are paying a fixed rate and receiving a floating rate (this would be deemed a *payer swap* in traditional terms). On the contrary, the short party entering into a funding rate swap is said to be *selling floating-for-fixed*, by which we mean they are receiving a fixed rate and paying the floating funding rate (a *receiver swap* in traditional terms).

4.5 Pricing a Bitmex Funding Rate Swap

4.5.1 The Pricing Formula

In traditional markets we observed that the 'price' of a general fixed-for-floating interest rate swap was simply the fixed rate quoted on an annualised basis. This is somewhat the case in decentralised markets, as the prices observed in the orderbook are the fixed rates associated with the contracts, again quoted on an annualised basis.

However, since the payment related to the fixed leg of the contract is paid immediately at the start of the contract, it holds closer ties to the concept of a *premium* in option pricing, and as such holds the same name. This premium represents a cash outflow for those buying floating-for-fixed, whilst the opposite is true for those selling.

In a similar manner to traditional fixed-for-floating interest rate swaps, we can consider the fixed rate of the BitMEX Funding Rate Swap as being determined by market forces—Delta Exchange's orderbook contains bids and offers, submitted by market participants, of annualised fixed rates. This means we are concerned with computing the premium of the contract.

Let r_s be the fixed rate of the funding rate swap. Given the increased frequency of funding periods, every eight hours, the premium is calculated using the time to maturity in seconds. To emphasise this point, we will denote the time to maturity as $\bar{\tau}$. Then the premium can be calculated as

$$\text{Premium} = \text{Notional} \times r_s \left(\frac{1}{\text{XBTUSD}}\right)\left(\frac{\bar{\tau}}{31536000}\right) \tag{4.19}$$

where the 31,536,000 is the number of seconds in a year. The notional principal merely acts as a scaling factor, although given the price of XBTUSD (around \$21,500 at time of writing) and the fact that the XBTUSD perpetual swap is an inverse contract, it will be important in the forthcoming numerical example to ensure the calculations remain at an appropriate scale.

4.5.2 Numerical Example

To firmly cement our understanding of the mechanics and calculations associated with the BitMEX Funding Rate Swap we will consider a numerical example. In an effort not to overcomplicate the scenario, we will make the following assumptions:

1. The price of XBTUSD remains constant throughout the life of the funding rate swap. This prevents us from needing the true spot rate at the time of each funding period in order to convert the USD denominated floating rate payments into XBT;

2. The funding rate remains constant throughout the life of the contract. Of course, the future funding rates would be unknown by definition, however, this allows us to illustrate the balancing of fixed and floating payments;

3. We will consider the example from the perspective of the long party under normal circumstances—that is, the funding rate is positive;

4. The fixed rate is determined by supply and demand.

Our input data is as follows.

BitMEX XBTUSD Funding Rate Swap	
CONTRACT DETAILS	
Notional	\$100,000
XBTUSD Spot	\$21,500
Fixed Rate (p.a.)	17%
Funding Rate (8h)	0.015%
Funding Rate (p.a.)	16.425%
Time to maturity (days)	90
Time to maturity (seconds)	7,776,000

Table 4.1: The observed market data used as inputs for the funding rate swap numerical example.

4.5.2.1 The Fixed Rate Leg (Upfront Premium)

The fixed rate leg requires upfront payment of the premium using 4.19

$$\text{Premium} = 100{,}000 \times 0.17 \left(\frac{1}{21{,}500}\right)\left(\frac{7,776,000}{31,536,000}\right) \tag{4.20}$$

$$= 0.19496655. \tag{4.21}$$

This value is denominated in XBT and must therefore be converted using the spot rate and quoted in USD in line with the terms of the contract. Therefore,

$$\text{Premium} = \$4191.78. \tag{4.22}$$

4.5.2.2 The Floating Rate Leg

In practice, we would not be able to compute the amount paid by the floating leg until the maturity of the contract. For our purposes, however, we have assumed a constant funding rate to illustrate the concept.

The floating leg is required to pay the funding rate every eight hours. At three funding periods per day for 90 days we have a total of 270 payment frequencies. In addition, we observed that the funding rate was 0.015%. Thus we have that,

$$\begin{aligned}
\text{Total Payment} &= \text{Notional} \times \text{No. of Periods} \times \text{Funding Rate} && (4.23)\\
&= 100,000 \times 270 \times 0.00015 && (4.24)\\
&= \$4050. && (4.25)
\end{aligned}$$

4.5.2.3 Discussion

As we can see the two payments are similar but not equal. The willingness by the long party to pay an annualised fixed rate of 17% suggests that they believe the funding rate will increase during the life of the contract. They could simply be speculating with the sole aim of earning a profit should their view be correct, or they could be seeking to manage their risk. This would likely be the case if they held an existing position in the underlying BitMEX XBTUSD Perpetual Swap, with a long term strategic view and wanted to reduce the uncertainty of future funding rates.

Of course, if one truly did believe that the funding rate would remain constant during the life of the contract, then the fair price would be that which has a value of zero at initiation, just as with traditional interest rate swaps. In our example, the fixed rate would be quoted as 16.425%—identical to the annualised funding rate—and the premium would be precisely \$4050. This observation puts forward a strong case for the further development of cryptocurrency interest rate models, applied directly to the funding rate of perpetual contracts.

4.6 Extensions to Alternative Cryptocurrencies

4.6.1 DAI Savings Rate Swap

As mentioned, the availability of interest rate swap-like products within the the crypto market is sparse. We focused on one of only two which have clear documentation, the XBTUSD Funding Rate Swap, with the other being the DAI Savings Rate Swap.

As the name suggests, the product is an agreement to swap the variable rates earned for depositing collateral into the DAI 'vault' for a fixed rate. The mechanics are almost identical to that of the XBTUSD Funding Rate Swap in that the premium associated with the fixed leg is paid at initiation of the contract, and the fixed rate is determined by market forces. We will omit the details, although it is worth noting there is no underlying inverse contract, hence, the premium is a simple multiplication.

4.6.2 Further Generalisations

Both the XBTUSD Funding Rate Swap and Dai Savings Rate Swap provide excellent blueprints for further generalisations to other crypto offerings. We now present some potential avenues within the crypto market which could offer interest rate swap-like products.

4.6.2.1 Other Perpetual Swap Contracts

The BitMEX XBTUSD Funding Rate Swap relies on the perpetual nature of the underlying and the fact it has an associated funding rate as the contract is ‘rolled over’. This practice, however, is not exclusive to this currency pair. In fact, there are plenty of other perpetual swap contracts citing different currencies, some of which continue to act as a bridge between traditional fiat currencies and cryptocurrencies, while others are pure crypto pairs. Regardless, each contract has its own associated funding rate and therefore the same principal can be applied.

Consider, for instance, the ETHUSD Perpetual Swap. Other than the name of the token, there is virtually no mechanical difference between an XBTUSD funding rate swap and an ETHUSD funding rate swap. Excluding external factors such as the demand and economic cost of offering such a product, there is clearly a theoretical basis for the extension of the specific funding rate swap previously discussed to cover more general currency pairs.

4.6.2.2 Staking

Staking has become a popular option for crypto investors to earn a yield on their assets by contributing them to a proof-of-stake mechanism and being rewarded for doing so. The lockup period can range from 10–120 days at which point the yield earned on the stake (quoted on an annualised basis) may have changed.

It is conceivable then that those looking to continuously stake their assets for an extended period of time may be willing to exchange the uncertain future staking yields for a stream of known fixed interest payments. Consider the following example.

The token ‘UCL’ functions using a proof-of-stake mechanism. An investor holding UCL tokens is bullish about its future prospects and believes its value will rise one year from now. He or she is therefore unwilling to sell their position, but wishes to maximise returns over this period.

The investor decides to contribute their UCL tokens to a staking pool, which has a lockup period of 30 days, after which they will receive their staked position and the associated interest payment, before recommitting the same position to the staking pool. At each rollover date, there is uncertainty surrounding the staking yield offered.

As a result, the investor enters into a contract—for which we will coin the term *Staking Yield Swap*—in which they agree to pay the variable rate of interest earned on their staked investment every 30 days and receive a fixed rate. By definition such an arrangement could be considered an interest rate swap-like product.

4.6.2.3 Crypto Saving & Lending

The concept of the Dai Savings Rate Swap could easily be applied to any crypto savings products that pay a variable rate of interest. In principal this is very similar to the Staking Yield Swap defined above since an investor is essentially depositing their investment and receiving interest, however, savings products may allow for more flexibility in terms of redemption. Once again, there are currently no other products besides the Dai Savings Rate Swap but there is sound theoretical basis for extension to general *Savings Rate Swaps*.

The increasing prevalence of crypto lending offers an opportunity for interest rate swaps to be used analogously to traditional markets. That is, a company offering floating rate crypto loans can hedge their risk by entering into what we might deem a plain vanilla pay floating receive fixed *crypto interest rate swap*. At present no such products exist. Of course, this precludes the challenges crypto loan platforms have faced with recent market volatility and wider bankruptcies.

4.6.2.4 Liquidity Pooling

Our final consideration is liquidity pooling in which depositing one, or both, cryptocurrencies into a pool allows investors to earn a variable yield. Given the previous extensions it is clear that we can once more imagine a scenario in which an investor wishes to swap the variable floating yield received from the pool for a fixed rate and, hence, we coin the term *Liquidity Yield Swap*. As before, no such products currently exist.

Chapter 5

Conclusion

> *In each revolution, we create a brand new way of trading, transacting and storing value––but we don't get rid of the old ones.*
>
> —Chris Skinner, Financial Markets Commentator

5.1 Project Summary

In this project we set out to examine the pricing of traditional interest rate swaps and their extension to the world of cryptocurrencies. We discussed the simplicity of the pricing formulas for the traditional case and identified that the calculation merely consists of a time value of money problem for the associated cash flows. Further, we found that we could use spot rates, forward rates, or discount factors to price a plain vanilla fixed-for-floating interest rate swap—and that knowledge of one allowed us to compute each of the others through their interlinked nature.

We then considered this premise from the opposite perspective, noting that it is market forces which determine the price of such swaps, hence, the primary challenge is constructing appropriate curves. We looked at two straightforward approaches to curve construction: explicit and implicit interpolation of the discount factors, and presented a simplified Python implementation using observed market data. Our results indicated the difficulty of such construction, highlighting the instability of the forward curve and the fact that the condition of no arbitrage, based only on a finite number of instruments, is not sufficient to produce smooth curves.

We concluded our study of the traditional case by discussing some of the practical considerations omitted from our analysis. This included the idea of day count convention, which is different depending on the geographical market for the interest rate swap itself. In addition, we noted the structural changes that have occurred since the financial crisis with the switch to multi-curve pricing—that is, using a combination of overnight indexed swaps and LIBOR-based swaps to produce forward curves—and more recently, financial markets' transition away from the LIBOR benchmark rate to alternative risk free rates.

The project then proceeded to explore the crypto market. We first analysed the concept of interest, stating that the owners of cryptocurrencies are not those that are rewarded, in contradiction to traditional markets. We reviewed traditional interest rate models and found that the short rate would be identically zero in the crypto case, questioning the existence of a non-trivial interest rate model. However, we discussed a recent paper which proposed use of a Bessel(3) process which could satisfy the aforementioned condition whilst remaining non-trivial, before producing some indicative yield curves.

We covered the pricing of one specific crypto interest rate swap: the BitMEX XBTUSD Funding Rate Swap. We found it had parallels to the traditional case, with a simplistic pricing formula—albeit based on a smaller timescale—but a slightly more complicated underlying in the funding rate. Our analysis was limited, however, by the lack of available historical data. Finally, we concluded by briefly mentioning the only other product on the market: the DAI Savings Rate Swap, and proposed some possible avenues which could theoretically host interest rate swap-like products.

5.2 Outlook and Future Work

Overall, this project achieved its objective to provide an initial bridge between the extensive literature of traditional interest rate derivatives, specifically interest rate swaps, and the newer world of cryptocurrency derivatives. As such, it is likely of interest to both traditional and decentralised market participants seeking a concise introduction to the topic. Alternatively, this project could spark the interest of academics currently in the field of cryptocurrencies and digital finance who may wish to explore the topic in more detail.

As a result, there are plenty of potential avenues for further research across several levels. From an academic perspective, a deeper analysis and evaluation of different curve construction techniques, with computational implementation, would make an excellent undergraduate or postgraduate project for those programmes which have a finance and computer science component. We would recommend the student begin with a review of the context of curve construction, by which we mean the history and an understanding of the financial products to which it applies. This could be followed by researching the work of Hagan and West (2006), who performed something similar, summarising the mathematical principles and implementing a variety of the methods in the preferred coding language.

Continuing at the undergraduate or postgraduate level, an analysis traditional interest rate swaps in a *post-LIBOR* world could be another potential project. We recommend a thorough analysis of the alternative risk free rates across the different jurisdictions, their calculation methodologies, and an exploration of the changes to pricing interest rate swaps (and potentially other products related to benchmark rates).

As we saw in Chapter 4, there is little existing literature regarding cryptocurrency interest rates and specifically crypto interest rate swaps. This represents a clear gap for further research, this time at a more advanced academic level, into a mathematical model of crypto interest rates. Furthermore, the theoretical products mentioned in our project could be mathematically formalised and economically evaluated to understand the circumstances under which they could viable—which is clearly of benefit to industry partners.

The final, and perhaps simplest, option could be to repeat and extend the analysis of this project with improved data. This could be in the form of more recent traditional market data for the pricing of interest rate swaps, and especially for curve construction, or using obtainable market data for cryptocurrencies.

Bibliography

Balan, Nicoleta. 2021. How long does it take to mine 1 Bitcoin depending on your device? *SMU Data Science Review* **5**(2) 5.

Banque du Canada, . 2000. A Practical Guide to Swap Curve Construction. URL `https://www.banqueducanada.ca/wp-content/uploads/2010/01/wp00-17.pdf`. Accessed: 16/07/2022.

BCBS. 2020. BASEL III Monitoring Report. URL `https://www.bis.org/bcbs/publ/d500.pdf`. Accessed: 20/06/2022.

Binance, . 2022a. Binance: Loan Products. URL `https://www.binance.com/en/loan`. Accessed: 07/08/2022.

Binance, . 2022b. Binance: Savings Accounts. URL `https://www.binance.com/en/savings`. Accessed: 07/08/2022.

Binance. 2021. Binance: An Overview of Binance Futures Products & Features. URL `https://www.binance.com/en/support/faq/00bd4e5720bf4ffebfadb4b15e465a73`. Accessed: 03/08/2022.

BIS. 2008. OTC Derivatives Market Activity in the First Half of 2008. URL `https://stats.bis.org/statx/srs/table/d7?p=20081&c=`. Accessed: 16/06/2022.

Black, Fischer. 1976. The Pricing of Commodity Contracts. *Journal of financial economics* **3**(1-2) 167–179. URL `https://doi.org/10.1016/0304-405X(76)90024-6`.

Black, Fischer, Emanuel Derman, William Toy. 1990. A One-factor Model of Interest Rates and its Application to Treasury Bond Options. *Financial analysts journal* **46**(1) 33–39. URL `https://doi.org/10.2469/faj.v46.n1.33`.

Black, Fischer, Piotr Karasinski. 1991. Bond and Option Pricing when Short Rates are Lognormal. *Financial Analysts Journal* **47**(4) 52–59. URL `https://doi.org/10.2469/faj.v47.n4.52`.

Black, Fischer, Myron Scholes. 1973. The Pricing of Options and Corporate Liabilities. *World Scientific Reference on Contingent Claims Analysis in Corporate Finance:*

Volume 1: Foundations of CCA and Equity Valuation. World Scientific, 3–21. URL https://doi.org/10.1142/9789814759588_0001.

Blaseg, Daniel. 2018. Dynamics of Voluntary Disclosure in the Unregulated Market for Initial Coin Offerings. *Available at SSRN 3207641* URL http://dx.doi.org/10.2139/ssrn.3207641.

Bloomberg. 2021. Crypto world hits 3 trillion market cap. URL https://www.bloomberg.com/news/articles/2021-11-08/crypto-world-hits-3-trillion-market-cap-as-ether-bitcoin-gain. Accessed: 03/08/2022.

Blundell-Wignall, Adrian, Paul Atkinson. 2010. Thinking Beyond Basel III: Necessary Solutions for Capital and Liquidity. *OECD Journal: Financial Market Trends* **2010**(1) 9–33. URL https://doi.org/10.1787/19952872.

Böhme, Rainer, Nicolas Christin, Benjamin Edelman, Tyler Moore. 2015. Bitcoin: Economics, Technology, and Governance. *Journal of economic Perspectives* **29**(2) 213–38. doi:10.1257/jep.29.2.213.

Brace, Alan, Dariusz G¸ atarek, Marek Musiela. 1997. The Market Model of Interest Rate Dynamics. *Mathematical finance* **7**(2) 127–155. URL https://doi.org/10.1111/1467-9965.00028.

Brigo, Damiano, Fabio Mercurio, et al. 2001. *Interest Rate Models: Theory and Practice*, vol. 2. Springer.

Brody, Dorje, Lane Hughston, Bernhard Meister. 2020. Theory of Cryptocurrency Interest Rates. *SIAM Journal on Financial Mathematics* **11**(1) 148–168. URL https://doi.org/10.1137/19M1263042.

Celsius, . 2022. Celsius: USD Loans backed by Cryptocurrency. URL https://celsius.network/crypto-loans. Accessed: 07/08/2022.

CFR. 2022. Understanding the LIBOR Scandal; Council forForeign Relations. URL https://www.cfr.org/backgrounder/understanding-libor-scandal. Accessed: 21/06/2022.

CFTC. 2013. Cross-Border Regulation of Swaps/Derivatives. URL https://www.cftc.gov/sites/default/files/idc/groups/public/@newsroom/documents/file/jointdiscussionscftc_europeanu.pdf. Accessed: 15/06/2022.

Chaum, David. 1979. *Computer Systems Established, Maintained and Trusted by Mutually Suspicious Groups*. Electronics Research Laboratory, University of California. MSc Thesis.

Chaum, David. 1983. Blind Signatures for Untraceable Payments. *Advances in cryptology*. Springer, 199–203. doi:10.1007/978-1-4757-0602-4_18.

Chernenko, Sergey, Michael Faulkender. 2011. The Two Sides of Derivatives Usage: Hedging and Speculating with Interest Rate Swaps. *Journal of Financial and Quantitative Analysis* **46**(6) 1727–1754. URL https://doi.org/10.1017/S0022109011000391.

Chui, Michael. 2012. Derivatives Markets, Products and Participants: An Overview. *Ifc Bullefin* **35** 3–12.

CNBC. 2022. Crypto firms say thousands of digital currencies will collapse. URL https://www.cnbc.com/2022/06/03/crypto-firms-say-thousands-of-digital-currencies-will-collapse.html. Accessed: 03/08/2022.

CoinBase, . 2022. What is Staking? URL https://www.coinbase.com/learn/crypto-basics/what-is-staking. Accessed: 07/08/2022.

CoinDesk, . 2022a. What is Proof-of-Stake? URL https://www.coindesk.com/learn/2020/12/30/what-is-proof-of-stake/. Accessed: 07/08/2022.

CoinDesk, . 2022b. What is Proof-of-Work? URL https://www.coindesk.com/learn/2020/12/16/what-is-proof-of-work/. Accessed: 07/08/2022.

Constantinides, George M. 1992. A Theory of the Nominal Term Structure of Interest Rates. *The Review of Financial Studies* **5**(4) 531–552. URL https://doi.org/10.1093/rfs/5.4.531.

Cosimano, Thomas F, Dalia Hakura. 2011. Bank Behavior in Response to Basel III: A Cross-country Analysis. *IMF Working Papers* **2011**(11/119). URL http://dx.doi.org/10.2139/ssrn.1860182.

Cox, John C, Jonathan E Ingersoll Jr, Stephen A Ross. 1985. A Theory of the Term Structure of Interest Rates. *Theory of valuation*. World Scientific, 129–164. URL https://doi.org/10.1142/9789812701022_0005.

Dai, Wei. 1998. B-Money, An Anonymous, Distributed Electronic Cash System. URL http://www.weidai.com/bmoney.txt. Accessed: 03/08/2022.

ESMA. 2022. EMIR Reporting: Regulatory & Implementing Technical Standards. URL https://www.esma.europa.eu/policy-rules/post-trading/trade-reporting. Accessed: 20/06/2022.

European Commission, . 2012. Derivatives / EMIR. URL https://ec.europa.eu/info/business-economy-euro/banking-and-finance/financial-markets/post-trade-services/derivatives-emir_en. Accessed: 15/06/2022.

FCA. 2022a. Cryptoassets: Advice for Consumers. URL https://www.fca.org.uk/consumers/cryptoassets. Accessed: 03/08/2022.

FCA. 2022b. Cryptocurrency Derivatives. URL https://www.fca.org.uk/news/statements/cryptocurrency-derivatives. Accessed: 03/08/2022.

FCA. 2022c. LIBOR Transition: FCA Guidance. URL https://www.fca.org.uk/markets/libor-transition. Accessed: 21/06/2022.

FCA. 2022d. UK EMIR Reporting (Post EU Withdrawal). URL https://www.fca.org.uk/markets/uk-emir. Accessed: 20/06/2022.

Flavell, Richard R. 2012. *Swaps and Other Derivatives*. John Wiley & Sons.

Gemeni. 2021. Blockchain and Financial Derivatives. URL https://www.gemini.com/cryptopedia/cryptocurrency-derivatives-trading-synthetic-assets. Accessed: 21/08/2022.

Hagan, Patrick S, Graeme West. 2006. Interpolation Methods for Curve Construction. *Applied Mathematical Finance* **13**(2) 89–129. URL https://doi.org/10.1080/13504860500396032.

Hagan, Patrick S, Graeme West. 2008. Methods for Constructing a Yield Curve. *Wilmott Magazine, May* 70–81.

Heath, David, Robert Jarrow, Andrew Morton. 1992. Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica: Journal of the Econometric Society* 77–105URL https://doi.org/10.2307/2951677.

Ho, Thomas SY, Sang-Bin Lee. 1986. Term Structure Movements and Pricing Interest Rate Contingent Claims. *the Journal of Finance* **41**(5) 1011–1029. URL https://doi.org/10.1111/j.1540-6261.1986.tb02528.x.

Hull, John, Alan White. 1993. One-factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities. *Journal of financial and quantitative analysis* **28**(2) 235–254. URL https://doi.org/10.2307/2331288.

Hull, John C. 2003. *Options, Futures and Other Derivatives*. Pearson Education India.

IBORate, . 2022. Historical USD-LIBOR Rates. URL http://iborate.com/usd-libor/. Accessed: 03/08/2022.

IOSCO. 2020. Margin Requirements for Non-centrally Cleared Derivatives. URL https://www.iosco.org/library/pubdocs/pdf/IOSCOPD651.pdf. Accessed: 20/06/2022.

ISDA. 2021a. Evolution of OTC Derivatives Markets since the Financial Crisis. URL https://www.isda.org/a/8jjTE/Evolution-of-OTC-Derivatives-Markets-Since-the-Financial-Crisis.pdf. Accessed: 15/06/2022.

ISDA. 2021b. Margin Requirements for Non-centrally Cleared Derivatives. URL https://www.isda.org/a/TwVgE/ISDA-Margin-Survey-Year-End-2021.pdf. Accessed: 20/06/2022.

JSCC, . 2012. Definition of Netting. URL https://www.jpx.co.jp/jscc/en/cash/cash/assumption-obligation/netting.html. Accessed: 22/08/2022.

Kaneko, Yusuke, Shigeyuki Osada, Shigeyuki Azuchi, Hitoshi Okada, Shigeichiro Yamasaki. 2019. A Management Method of Interest Rate in UTXO Model. *2019 IEEE Social Implications of Technology (SIT) and Information Management (SITIM)*. IEEE, 1–6. doi:10.1109/SITIM.2019.8910267.

Liu, James A. 2020. Atomic Swaptions: Cryptocurrency Derivatives. *arXiv preprint arXiv:1807.08644* URL https://doi.org/10.48550/arXiv.1807.08644.

Longstaff, Francis A, Eduardo S Schwartz. 1992. Interest Rate Volatility and the Term Structure: A Two-factor General Equilibrium Model. *The Journal of Finance* **47**(4) 1259–1282. URL https://doi.org/10.1111/j.1540-6261.1992.tb04657.x.

Nakamoto, Satoshi. 2008. Bitcoin: A Peer-to-Peer Electronic Cash System. *Decentralized Business Review* 21260.

Piterbarg, Vladimir, Marco Renedo. 2004. Eurodollar Futures Convexity Adjustments in Stochastic Volatility Models. *Available at SSRN 610223* URL http://dx.doi.org/10.2139/ssrn.610223.

Sadr, Amir. 2009. *Interest Rate Swaps and their Derivatives: A Practitioner's Guide*. John Wiley & Sons.

SEC. 2022. Dodd-Frank Security-based Swap Data Repositories. URL https://www.sec.gov/divisions/marketreg/security-based-swap-data-repositories. Accessed: 20/06/2022.

Slovik, Patrick, Boris Cournède. 2011. Macroeconomic Impact of Basel III. *OECD Economics Department Working Papers* **2011**(844). URL http://dx.doi.org/10.2139/ssrn.2650033.

Smith, Donald J. 2011. *Bond Math: The Theory Behind the Formulas*. John Wiley & Sons.

Stankovska, Aleksandra. 2017. Global Derivatives Markets. *Seeu Review* **12**(1) 81–93. doi:10.1515/seeur-2017-0006.

Szabo, Nick. 1998. Bit Gold. URL https://nakamotoinstitute.org/bit-gold/. Accessed: 03/08/2022.

TSC. 2022. Treasury Select Committee Minutes, 25/11/2008. URL https://publications.parliament.uk/pa/cm200708/cmselect/cmtreasy/1210/8112503.htm. Accessed: 21/06/2022.

Urquhart, Andrew. 2016. The Inefficiency of Bitcoin. *Economics Letters* **148** 80–82. URL https://doi.org/10.1016/j.econlet.2016.09.019.

US Treasury, . 2021. Yield Curve Methodology Change Information Sheet. URL https://home.treasury.gov/policy-issues/financing-the-government/yield-curve-methodology-change-information-sheet. Accessed: 03/08/2022.

Vasicek, Oldrich. 1977. An Equilibrium Characterization of the Term Structure. *Journal of financial economics* **5**(2) 177–188. URL https://doi.org/10.1016/0304-405X(77)90016-2.

Veronesi, Pietro. 2016. *Handbook of Fixed Income Securities*. John Wiley & Sons.

Wilmott, Paul. 2013. *Paul Wilmott on Quantitative Finance*. John Wiley & Sons.

World Bank, . 2018. 70 Years Connecting Capital Markets, Chapter 4: Pioneering Swaps. URL https://thedocs.worldbank.org/en/doc/981111541019927135-0340022018/original/70yearschapterfour.pdf. Accessed: 15/06/2022.

Zohar, Aviv. 2015. Bitcoin: Under the Hood. *Communications of the ACM* **58**(9) 104–113. doi:10.1145/2701411.

Appendix A

Numerical Examples

A.1 Discrete Compounding

To illustrate the use of the formulas presented in Chapter 2, we present some numerical examples based on the data provided in Smith (2011). For simplicity, we will ignore the additional challenges posed by day count conventions and assume a 30/360 calendar.

Consider the following table for reference.

Forward Period	Forward Rates	Spot Period	Spot Rates	Discount Factors
0×3	0.500%	0×3	0.5000%	0.998751561
3×6	1.582%	0×6	1.0406%	0.994817233
6×9	2.669%	0×9	1.5827%	0.988223069
9×12	3.765%	0×12	2.1272%	0.979007831
12×15	3.747%	0×15	2.4506%	0.969922559
15×18	4.405%	0×18	2.7757%	0.959357293
18×21	5.069%	0×21	3.1025%	0.947352265
21×24	5.743%	0×24	3.4316%	0.933943555

Table A.1: A table of forward rates, spot rates, and discount factors given on a quarterly compounded (30/360) basis.

A.1.1 Discrete spot rates from discrete forward rates

Given a series of forward rates, we can compute the discrete spot rates.

Example:

$$\left(1 + \frac{r_d(3M)]}{4}\right)\left(1 + \frac{f_d(0, 3M, 6M)}{4}\right) = \left(1 + \frac{r_d(6M)}{4}\right)^2 \tag{A.1}$$

$$\left(1 + \frac{0.00500}{4}\right)\left(1 + \frac{0.01582}{4}\right) = \left(1 + \frac{r_d(6M)}{4}\right)^2 \tag{A.2}$$

$$r_d(6M) = 0.010406 \tag{A.3}$$

$$= 1.0406\%. \tag{A.4}$$

A.1.2 Discount factors from discrete spot rates

We can compute discount factors given a set of discrete spot rates.

Example:

$$Z(0, 15M) = \frac{1}{\left(1 + \frac{r_d(15M)}{4}\right)^5} \tag{A.5}$$

$$Z(0, 15M) = \frac{1}{\left(1 + \frac{0.024506}{4}\right)^5} \tag{A.6}$$

$$Z(0, 15M) = 0.969922559. \tag{A.7}$$

A.1.3 Discrete forward rates from discount factors

We can compute the discrete forward rates given a set of discount factors.

Example:

$$f_d(0, 18M, 21M) = \frac{\left(\frac{Z(0,18M)}{Z(0,21M)} - 1\right)}{21M - 18M} \tag{A.8}$$

$$f_d(0, 18M, 21M) = \frac{\left(\frac{0.959357293}{0.947352265} - 1\right)}{0.25} \tag{A.9}$$

$$f_d(0, 18M, 21M) = 0.05069 \tag{A.10}$$

$$= 5.069\%. \tag{A.11}$$

A.2 Continuous Compounding

A.2.1 Continuous spot rates

Suppose we have the discrete spot rates given in Table A.1. Then we can convert them to continuously compounded spot rates.

Example:

$$r_c(9M) = \frac{1}{9M} \ln\left(1 + (9M) r_d(9M)\right) \tag{A.12}$$

$$r_c(9M) = \frac{1}{0.75} \ln\left(1 + (0.75)(0.015827)\right) \tag{A.13}$$

$$r_c(9M) = 0.015796 \tag{A.14}$$

$$= 1.5796\%. \tag{A.15}$$

Suppose we have the following table, with all data now presented on a continuous basis.

Forward Period	Forward Rates	Spot Period	Spot Rates	Discount Factors
0 × 3	0.4997%	0 × 3	0.4997%	0.998751561
3 × 6	1.5788%	0 × 6	1.0392%	0.994817233
6 × 9	2.6602%	0 × 9	1.5796%	0.988223069
9 × 12	3.7475%	0 × 12	2.1216%	0.979007831
12 × 15	3.7294%	0 × 15	2.4431%	0.969922559
15 × 18	4.3811%	0 × 18	2.7661%	0.959357293
18 × 21	5.0370%	0 × 21	3.0905%	0.947352265
21 × 24	5.7020%	0 × 24	3.4170%	0.933943555

Table A.2: A table of spot rates, forward rates, and discount rates given on a continuously compounded basis. Derived from the data given in Smith (2011).

A.2.2 Discount factors from continuous spot rates

We can compute discount factors given a set of continuous spot rates.

Example:

$$Z(0, 18M) = \exp(-r_c(18M)(18M)) \tag{A.16}$$

$$Z(0, 18M) = e^{-(0.027761)(1.5)} \tag{A.17}$$

$$Z(0, 18M) = 0.959357293. \tag{A.18}$$

A.2.3 Continuous forward rates from discount factors

We can compute the continuous forward rates given a set of discount factors.

Example:

$$f_c(0, 9M, 12M) = \frac{\ln\left(\frac{Z(0,9M)}{Z(0,12M)}\right)}{12M - 9M} \tag{A.19}$$

$$f_c(0, 9M, 12M) = \frac{\left(\frac{0.988223069}{0.979007831}\right)}{0.25} \tag{A.20}$$

$$f_c(0, 9M, 12M) = 0.037475 \tag{A.21}$$

$$= 3.7475\%. \tag{A.22}$$

A.3 Computing a Swap Price

To illustrate the formulas used to compute the swap fixed rate, which is equivalent to the price of a swap, we present a numerical example based on that given in Smith (2011).

A.3.1 Using discount factors only

Suppose we have the discretely compounded spot rates and discount factors given in table A.1. Selectively repeated here for the comfort of the reader.

Spot Period	Spot Rates	Discount Factors
0×3	0.5000%	0.998751561
0×6	1.0406%	0.994817233
0×9	1.5827%	0.988223069
0×12	2.1272%	0.979007831
0×15	2.4506%	0.969922559
0×18	2.7757%	0.959357293
0×21	3.1025%	0.947352265
0×24	3.4316%	0.933943555

Table A.3: The repeated table of spot rates and discount rates given on a quarterly compounded (30/360) basis.

Example: We want to compute the swap fixed rate, r_s, of a two year, quarterly paid plain vanilla fixed-for-floating swap.

$$r_s = \frac{1 - Z(0, 24M)}{\tau \sum_{i=3M}^{24M} Z(0, i)} \tag{A.23}$$

$$= \frac{0.066056445}{1.942843842} \tag{A.24}$$

$$= 0.033999873 \tag{A.25}$$

$$= 3.40\%. \tag{A.26}$$

A.3.2 Using forward rates and discount factors

Suppose we have the discretely compounded forward rates and discount factors given in table A.1. Selectively repeated here for the comfort of the reader.

Forward Period	Forward Rates	Spot Period	Discount Factors
0 × 3	0.500%	0 × 3	0.998751561
3 × 6	1.582%	0 × 6	0.994817233
6 × 9	2.669%	0 × 9	0.988223069
9 × 12	3.765%	0 × 12	0.979007831
12 × 15	3.747%	0 × 15	0.969922559
15 × 18	4.405%	0 × 18	0.959357293
18 × 21	5.069%	0 × 21	0.947352265
21 × 24	5.743%	0 × 24	0.933943555

Table A.4: The repeated table of forward rates and discount rates given on a quarterly compounded (30/360) basis.

Example: Let's compute the same swap fixed rate (SFR) for a two year, quarterly paid plain vanilla fixed-for-floating swap.

$$r_s = \frac{\sum_{i=3M}^{24M} Z(0, i) f_d(0, i-1, i)]}{\sum_{i=3M}^{24M} Z(0, i)} \tag{A.27}$$

$$= \frac{0.264227437}{7.771375367} \tag{A.28}$$

$$= 0.034000087 \tag{A.29}$$

$$= 3.40\%. \tag{A.30}$$

Appendix B

Curve Construction

B.1 Datasets

Dataset 1—15/02/2008

Product	Time	Cumulative Year Fraction	Observed Market Rate
Deposit	1M	0.0833	3.11875%
Deposit	2M	0.1667	3.08625%
Deposit	3M	0.2500	3.07000%
Deposit	6M	0.5000	2.74438%
Deposit	1Y	1.0000	2.63800%
Swap	2Y	2.0000	2.74300%
Swap	3Y	3.0000	3.06600%
Swap	4Y	4.0000	3.38600%
Swap	5Y	5.0000	3.67100%
Swap	6Y	6.0000	3.92100%
Swap	7Y	7.0000	4.12100%
Swap	8Y	8.0000	4.28800%
Swap	9Y	9.0000	4.42900%
Swap	10Y	10.000	4.54600%
Swap	12Y	12.000	4.73800%
Swap	15Y	15.000	4.93200%
Swap	20Y	20.000	5.08800%
Swap	25Y	25.000	5.14300%
Swap	30Y	30.000	5.16300%

Table B.1: The first dataset used in the Python curve construction implementation. A combination of USD LIBOR deposit rates and mid-quotes for USD fixed-for-floating LIBOR-based swaps on 15/02/2008.

Dataset 2—06/02/2008

Product	Time	Cumulative Year Fraction	Observed Market Rate
Deposit	1M	0.0833	3.11875%
Deposit	2M	0.1667	3.08625%
Deposit	3M	0.2500	3.07000%
Deposit	6M	0.5000	2.74438%
Deposit	1Y	1.0000	2.63800%
Eurodollar	15M	1.2500	2.78000%
Eurodollar	18M	1.5000	2.96000%
Eurodollar	21M	1.7500	3.15000%
Swap	2Y	2.0000	2.79500%
Swap	3Y	3.0000	3.03500%
Swap	4Y	4.0000	3.27500%
Swap	5Y	5.0000	3.50500%
Swap	6Y	6.0000	3.71500%
Swap	7Y	7.0000	3.88500%
Swap	8Y	8.0000	4.02500%
Swap	9Y	9.0000	4.15500%
Swap	10Y	10.000	4.26500%
Swap	12Y	12.000	4.43500%
Swap	15Y	15.000	4.61500%
Swap	20Y	20.000	4.75500%
Swap	25Y	25.000	4.80500%
Swap	30Y	30.000	4.81500%

Table B.2: The second dataset used in the Python curve construction implementation. A combination of USD LIBOR deposit rates, Eurodollar futures quoted rates, and mid-quotes for USD fixed-for-floating LIBOR-based swaps on 06/02/2008.

B.2 Results

Results: 15/02/2008

			Explicit DF Linear		Explicit DF Cubic spline		Implicit DF Linear		Implicit DF Cubic spline	
Product	Time	Observed Rate	Spot Rate	Discount Factor	Spot Rate	Discount Factor	Spot Rate	Discount Factor	Spot Rate	Discount Factor
Deposit	1M	3.11875%	3.1147%	0.997408	3.1147%	0.997408	3.1147%	0.997408	3.1147%	0.997408
Deposit	2M	3.08625%	3.0783%	0.994883	3.0783%	0.994883	3.0783%	0.994883	3.0783%	0.994883
Deposit	3M	3.07000%	3.0583%	0.992383	3.0583%	0.992383	3.0583%	0.992383	3.0583%	0.992383
Deposit	6M	2.74438%	2.7257%	0.986464	2.7257%	0.986464	2.7257%	0.986464	2.7257%	0.986464
Deposit	1Y	2.63800%	2.6038%	0.974298	2.6038%	0.974298	2.6038%	0.974298	2.6038%	0.974298
Swap	2Y	2.74300%	2.7267%	0.946925	2.7145%	0.947158	2.7277%	0.946908	2.6804%	0.947804
Swap	3Y	3.06600%	3.0557%	0.912405	3.0568%	0.912375	3.0555%	0.912412	3.0793%	0.91176
Swap	4Y	3.38600%	3.3845%	0.873382	3.3832%	0.873428	3.3841%	0.873398	3.3787%	0.873587
Swap	5Y	3.67100%	3.6827%	0.831825	3.6838%	0.831778	3.6826%	0.831828	3.6875%	0.831623
Swap	6Y	3.92100%	3.9492%	0.789028	3.9499%	0.788998	3.9495%	0.789015	3.9479%	0.789092
Swap	7Y	4.12100%	4.1657%	0.747069	4.1667%	0.747015	4.1662%	0.747044	4.1677%	0.746966
Swap	8Y	4.28800%	4.3499%	0.706104	4.351%	0.706040	4.3505%	0.706069	4.3505%	0.706069
Swap	9Y	4.42900%	4.5082%	0.666486	4.509%	0.666437	4.5089%	0.666446	4.5092%	0.666427
Swap	10Y	4.54600%	4.6415%	0.628672	4.6423%	0.62862	4.6421%	0.628629	4.6421%	0.628631
Swap	12Y	4.73800%	4.8652%	0.557762	4.8681%	0.557569	4.8664%	0.55768	4.8687%	0.557528
Swap	15Y	4.93200%	5.0936%	0.465779	5.0979%	0.465478	5.0980%	0.46547	5.0946%	0.46571
Swap	20Y	5.08800%	5.2495%	0.349969	5.3091%	0.345828	5.2731%	0.348322	5.3107%	0.345714
Swap	25Y	5.14300%	5.3240%	0.264216	5.322%	0.264348	5.3371%	0.263352	5.3095%	0.26517
Swap	30Y	5.16300%	5.3441%	0.201243	5.3583%	0.200389	5.3445%	0.201219	5.3682%	0.199793

Table B.3: The spot rates and discount factors from the four curve construction techniques for the 15/02/2008 dataset.

Forward Results: 15/02/2008

	Explicit DF Linear	Explicit DF Cubic spline	Implicit DF Linear	Implicit DF Cubic spline
Forward Period	Forward Rate			
$f(0,0,0.5)$	3.11470%	3.11470%	3.11470%	3.11470%
$f(0,0.5,1)$	2.48188%	2.48188%	2.48188%	2.48188%
$f(0,1,1.5)$	2.69428%	2.66629%	2.69641%	2.57571%
$f(0,1.5,2)$	3.00509%	2.98389%	3.00674%	2.93828%
$f(0,2,2.5)$	3.50001%	3.52393%	3.49814%	3.65984%
$f(0,2.5,3)$	3.92720%	3.95902%	3.92386%	4.09428%
$f(0,3,3.5)$	4.23648%	4.23335%	4.23455%	4.17354%
$f(0,3.5,4)$	4.50569%	4.49170%	4.50550%	4.38015%
$f(0,4,4.5)$	4.75596%	4.76336%	4.75711%	4.77867%
$f(0,4.5,5)$	4.99423%	5.00887%	4.99606%	5.06707%
$f(0,4,5.5)$	5.21010%	5.21115%	5.21199%	5.19276%
$f(0,5.5,6)$	5.35408%	5.34920%	5.35590%	5.30641%
$f(0,6,6.5)$	5.42631%	5.42789%	5.42808%	5.43724%
$f(0,6.5,7)$	5.50246%	5.50777%	5.50422%	5.53552%
$f(0,7,7.5)$	5.59709%	5.60025%	5.59882%	5.59624%
$f(0,7.5,8)$	5.68191%	5.68230%	5.68348%	5.66498%
$f(0,8,8.5)$	5.74881%	5.74737%	5.75009%	5.74709%
$f(0,8.5,9)$	5.79969%	5.79781%	5.80071%	5.80945%
$f(0,9,9.5)$	5.83359%	5.83294%	5.83443%	5.84060%
$f(0,9.5,10)$	5.84845%	5.85086%	5.84916%	5.83672%

Table B.4: The continuously compounded forward rates derived from the discount factors on 15/02/2008.

Results: 06/02/2008

			Explicit DF Linear		Explicit DF Cubic spline		Implicit DF Linear		Implicit DF Cubic spline	
Product	Time	Observed Rate	Spot Rate	Discount Factor	Spot Rate	Discount Factor	Spot Rate	Discount Factor	Spot Rate	Discount Factor
Deposit	1M	3.1813%	3.177%	0.997356	3.177%	0.997356	3.177%	0.997356	3.177%	0.997356
Deposit	2M	3.1575%	3.1492%	0.994765	3.1492%	0.994765	3.1492%	0.994765	3.1492%	0.994765
Deposit	3M	3.145%	3.1327%	0.992199	3.1327%	0.992199	3.1327%	0.992199	3.1327%	0.992199
Deposit	6M	3.0975%	3.0738%	0.984749	3.0738%	0.984749	3.0738%	0.984749	3.0738%	0.984749
Deposit	1Y	2.8962%	2.8551%	0.971853	2.8551%	0.971853	2.8551%	0.971853	2.8551%	0.971853
Eurodollar	15M	2.78%	2.8382%	0.965145	2.8382%	0.965145	2.8382%	0.965145	2.8382%	0.965145
Eurodollar	18M	2.96%	2.8566%	0.958055	2.8566%	0.958055	2.8566%	0.958055	2.8566%	0.958055
Eurodollar	21M	3.15%	2.8968%	0.95057	2.8968%	0.95057	2.8968%	0.95057	2.8968%	0.95057
Swap	2Y	2.795%	2.7732%	0.946047	2.7732%	0.946047	2.7732%	0.946047	2.7732%	0.946047
Swap	3Y	3.035%	3.0218%	0.913335	3.0359%	0.912946	3.0227%	0.913308	3.0415%	0.912795
Swap	4Y	3.275%	3.2657%	0.877544	3.2459%	0.878239	3.2653%	0.877558	3.2323%	0.878717
Swap	5Y	3.505%	3.5062%	0.839198	3.5122%	0.838946	3.5061%	0.839202	3.5194%	0.838645
Swap	6Y	3.715%	3.7294%	0.799505	3.728%	0.79957	3.7295%	0.799501	3.7251%	0.799708
Swap	7Y	3.885%	3.9125%	0.760427	3.9145%	0.760318	3.9127%	0.760414	3.9166%	0.760209
Swap	8Y	4.025%	4.0657%	0.72234	4.0664%	0.722303	4.0661%	0.722319	4.0656%	0.722347
Swap	9Y	4.155%	4.2114%	0.68453	4.2125%	0.684462	4.2118%	0.684504	4.2131%	0.684421
Swap	10Y	4.265%	4.3362%	0.648159	4.3368%	0.648123	4.3366%	0.648131	4.3366%	0.64813
Swap	12Y	4.435%	4.5316%	0.580546	4.536%	0.580234	4.5323%	0.580492	4.538%	0.580094
Swap	15Y	4.615%	4.7436%	0.490887	4.7472%	0.490619	4.7469%	0.490647	4.7417%	0.49103
Swap	20Y	4.755%	4.881%	0.376737	4.9153%	0.374162	4.899%	0.375386	4.9242%	0.373502
Swap	25Y	4.805%	4.9477%	0.290276	4.9665%	0.288915	4.9588%	0.289468	4.9455%	0.290434
Swap	30Y	4.815%	4.9455%	0.22681	4.9348%	0.22754	4.9467%	0.226726	4.9551%	0.226159

Table B.5: The spot rates and discount factors from the four curve construction techniques for the 06/02/2008 dataset.

FORWARD RESULTS: 06/02/2008

	EXPLICIT DF LINEAR	EXPLICIT DF CUBIC SPLINE	IMPLICIT DF LINEAR	IMPLICIT DF CUBIC SPLINE
FORWARD PERIOD	Forward Rate			
$f(0,0,0.5)$	3.17704%	3.17704%	3.17704%	3.17704%
$f(0,0.5,1)$	2.63644%	2.63644%	2.63644%	2.63644%
$f(0,1,1.5)$	2.85974%	2.85974%	2.85974%	2.85974%
$f(0,1.5,2)$	2.52273%	2.52273%	2.52273%	2.52273%
$f(0,2,2.5)$	2.96496%	3.02633%	2.9683%	3.05642%
$f(0,2.5,3)$	4.07288%	4.09662%	4.07544%	4.09965%
$f(0,3,3.5)$	3.99862%	3.86715%	3.99467%	3.78183%
$f(0,3.5,4)$	3.99659%	3.88447%	3.99135%	3.82799%
$f(0,4,4.5)$	4.32211%	4.40913%	4.32192%	4.49325%
$f(0,4.5,5)$	4.61381%	4.74553%	4.61633%	4.84171%
$f(0,4,5.5)$	4.78697%	4.77612%	4.78837%	4.74079%
$f(0,5.5,6)$	4.9039%	4.83817%	4.90453%	4.76745%
$f(0,6,6.5)$	4.98271%	4.9896%	4.98375%	5.0068%
$f(0,6.5,7)$	5.03978%	5.07783%	5.04113%	5.1238%
$f(0,7,7.5)$	5.0906%	5.09062%	5.09188%	5.08093%
$f(0,7.5,8)$	5.18621%	5.16775%	5.18733%	5.13668%
$f(0,8,8.5)$	5.32462%	5.32328%	5.32557%	5.32448%
$f(0,8.5,9)$	5.42815%	5.43924%	5.42893%	5.46194%
$f(0,9,9.5)$	5.47153%	5.47845%	5.47213%	5.49255%
$f(0,9.5,10)$	5.44781%	5.43205%	5.44822%	5.40399%

Table B.6: The continuously compounded forward rates derived from the discount factors on 06/02/2008.

Appendix C

Code Listing

C.1 Curve Construction: Python Code

Our curve construction code consists of four separate files for each of different methodologies. In order to reduce duplication, we present one full script for the method: *Linear Interpolation of Explicit Discount Factors*, and then only a subsection of the remaining three scripts. The construction of the forward curve, using cubic spline interpolation, was identical for all four methodologies, hence, we only include the full script once. Python scripts that plot the curves have been omitted, but are available upon request.

C.1.1 Linear Interpolation of Explicit Discount Factors

```python
# Function to generate discount factors and zero/spot rates using:
#   ~~~~Explicit Discount Factors, Linear Interpolation~~~~
def explicitDF_lin(df):
    for i in range(len(df)):
        # LIBOR Deposit calculations #
        if df['Product'].iloc[i] == 'Deposit':
            # Continuously compounded zero/spot rate
             df.loc[df.index[i], 'ContinuousRate'] = (1 / df['CumYearFraction'].iloc[i]) * np.log(1 +
                                                    df['CumYearFraction'].iloc[i] * df['MktRate'].iloc[i])
            # Discount factor
             df.loc[df.index[i], 'DiscountFactor'] = np.exp(-1 * df['CumYearFraction'].iloc[i] *
                                                    df['ContinuousRate'].iloc[i])

        # Eurodollar calculations #
        elif df['Product'].iloc[i] == 'Eurodollar':
            # Eurodollar future rate converted to continuous basis
            euro_cont = 4 * np.log(1 + df['MktRate'].iloc[i] * 0.25)
            # Continuously compounded zero/spot rate
            df.loc[df.index[i], 'ContinuousRate'] = \
                (euro_cont * (df['CumYearFraction'].iloc[i] - df['CumYearFraction'].iloc[i-1]) +
                 df['ContinuousRate'].iloc[i-1] * df['CumYearFraction'].iloc[i-1]) / df['CumYearFraction'].iloc[i]
            # Discount factor
            df.loc[df.index[i], 'DiscountFactor'] = np.exp(-1 * df['CumYearFraction'].iloc[i] *
                                                    df['ContinuousRate'].iloc[i])

        elif df['Product'].iloc[i] == 'Swap':
            term = int(df['CumYearFraction'].iloc[i])               # Maturity of the swap
            freq = 2                                                # Semi-annual payments
            paym = np.arange(1/freq, term, 1/freq)                  # Interest payment times

            # EXPLICIT DISCOUNT FACTOR STEP #
            x = pd.Series(df['CumYearFraction'][:i])                # Times existing so far
            y = pd.Series(df['DiscountFactor'][:i])                 # Discount factors existing so far
            df_tck = interpolate.splrep(x,y, k=1)                   # Interpolate a curve, [k=1 for linear]

            DF_sum = 0.0                                            # Initialise sum of discount factors
            for payment in paym:
                disc_fact = 0.0
                disc_fact = interpolate.splev(payment, df_tck)      # Evaluate discount factor on interpolated curve
                DF_sum = DF_sum + disc_fact                         # Sum the discount factors
            # Discount factor
            df.loc[df.index[i], 'DiscountFactor'] = (1 - (1/freq) * df['MktRate'].iloc[i] * DF_sum) / (1 + (1/freq) *
                                                                                    df['MktRate'].iloc[i])
            # Continuously compounded zero/spot rate
            df.loc[df.index[i], 'ContinuousRate'] = (-1 / df['CumYearFraction'].iloc[i]) * \
                                                    np.log(df['DiscountFactor'].iloc[i])

    return df
```

C.1.2 Cubic Spline Interpolation of Explicit Discount Factors

```
    elif df['Product'].iloc[i] == 'Swap':
        term = int(df['CumYearFraction'].iloc[i])              # Maturity of the swap
        freq = 2                                               # Semi-annual payments
        paym = np.arange(1/freq, term, 1/freq)                 # Interest payment times

        # EXPLICIT DISCOUNT FACTOR STEP #
        x = pd.Series(df['CumYearFraction'][:i])               # Times existing so far
        y = pd.Series(df['DiscountFactor'][:i])                # Discount factors existing so far
        df_tck = interpolate.splrep(x,y, k=3)                  # Interpolate a curve, [k=3 for cubic-spline]

        DF_sum = 0.0                                           # Initialise sum of discount factors
        for payment in paym:
            disc_fact = 0.0
            disc_fact = interpolate.splev(payment, df_tck)     # Evaluate discount factor on interpolated curve
            DF_sum = DF_sum + disc_fact                        # Sum the discount factors
        # Discount factor
        df.loc[df.index[i], 'DiscountFactor'] = (1 - (1/freq) * df['MktRate'].iloc[i] * DF_sum) / (1 + (1/freq) *
                                                                                   df['MktRate'].iloc[i])
        # Continuously compounded zero/spot rate
        df.loc[df.index[i], 'ContinuousRate'] = (-1 / df['CumYearFraction'].iloc[i]) * \
                                                np.log(df['DiscountFactor'].iloc[i])

return df
```

C.1.3 Linear Spline Interpolation of Implicit Discount Factors

```
    elif df['Product'].iloc[i] == 'Swap':
        term = int(df['CumYearFraction'].iloc[i])              # Maturity of the swap
        freq = 2                                               # Semi-annual payments
        paym = np.arange(1/freq, term, 1/freq)                 # Interest payment times

        # IMPLICIT DISCOUNT FACTOR STEP #
        x = pd.Series(df['CumYearFraction'][:i])               # Times existing so far
        y = pd.Series(df['ContinuousRate'][:i])                # Zero/spot rates existing so far
        df_tck = interpolate.splrep(x,y, k=1)                  # Interpolate a curve, [k=1 for linear]

        DF_sum = 0.0                                           # Initialise sum of discount factors
        for payment in paym:
            disc_fact = 0.0
            spot_rate = interpolate.splev(payment, df_tck)     # Evaluate zero/spot rate on interpolated curve
            disc_fact = np.exp(-1 * payment * spot_rate)       # Convert zero/spot rate to discount factor
            DF_sum = DF_sum + disc_fact                        # Sum the discount factors
        # Discount factor
        df.loc[df.index[i], 'DiscountFactor'] = (1 - (1/freq) * df['MktRate'].iloc[i] * DF_sum) / (1 + (1/freq) *
                                                                                   df['MktRate'].iloc[i])
        # Continuously compounded zero/spot rate
        df.loc[df.index[i], 'ContinuousRate'] = (-1 / df['CumYearFraction'].iloc[i]) * \
                                                np.log(df['DiscountFactor'].iloc[i])

return df
```

C.1.4 Cubic Spline Interpolation of Implicit Discount Factors

```
        elif df['Product'].iloc[i] == 'Swap':
            term = int(df['CumYearFraction'].iloc[i])              # Maturity of the swap
            freq = 2                                                # Semi-annual payments
            paym = np.arange(1/freq, term, 1/freq)                  # Interest payment times

            # IMPLICIT DISCOUNT FACTOR STEP #
            x = pd.Series(df['CumYearFraction'][:i])                # Times existing so far
            y = pd.Series(df['ContinuousRate'][:i])                 # Zero/spot rates existing so far
            df_tck = interpolate.splrep(x,y, k=3)                   # Interpolate a curve, [k=3 for cubic-spline]

            DF_sum = 0.0                                            # Initialise sum of discount factors
            for payment in paym:
                disc_fact = 0.0
                spot_rate = interpolate.splev(payment, df_tck)      # Evaluate zero/spot rate on interpolated curve
                disc_fact = np.exp(-1 * payment * spot_rate)        # Convert zero/spot rate to discount factor
                DF_sum = DF_sum + disc_fact                         # Sum the discount factors
            # Discount factor
            df.loc[df.index[i], 'DiscountFactor'] = (1 - (1/freq) * df['MktRate'].iloc[i] * DF_sum) / (1 + (1/freq) *
                                                                                    df['MktRate'].iloc[i])
            # Continuously compounded zero/spot rate
            df.loc[df.index[i], 'ContinuousRate'] = (-1 / df['CumYearFraction'].iloc[i]) * \
                                                    np.log(df['DiscountFactor'].iloc[i])

    return df
```

C.1.5 Forward Curve

```
# Function to generate 6M forward rates out to 10-years:
def forward_rates(df):
    term = 10                                                   # The length of the forward curve
    freq = 2                                                    # Semi-annual payments, i.e. 6m forward curve
    period = np.arange(1/freq, term+1/freq, 1/freq)             # Array of forward times
    j = df[df['Time'] == '10Y'].index[0]                        # Identify the row number of 10Y swap

    x = pd.Series(df['CumYearFraction'][:j+1])                  # Times up to and including 10Y
    y = pd.Series(df['DiscountFactor'][:j+1])                   # Discount factors up to and including 10Y
    fwd_tck = interpolate.splrep(x,y, k=3)                      # Interpolate a curve, [k=1 for linear]

    fwd_rates = np.array([])                                    # Initialise an array for storing fwd rates
    fwd_rates = np.append(fwd_rates, df['ContinuousRate'].iloc[0])  # First fwd rate == first zero/spot rate
    for k in range(1, len(period)):                             # Iterate through forward times
        DF_k_1 = interpolate.splev(period[k-1], fwd_tck)        # Evaluate discount factor at the start
        DF_k = interpolate.splev(period[k], fwd_tck)            # Evaluate discount factor at the end
        fwd_rates = np.append(fwd_rates, freq*np.log(DF_k_1/DF_k))  # Compute fwd rates

    return fwd_rates
```