[UNIT-1 - RELATIVISTIC MECHANICS]

* Event :- Occurrence of something at one point at an instant in space. Ex- shot of a bullet, swing of a pendulum when it comes in the mean position.

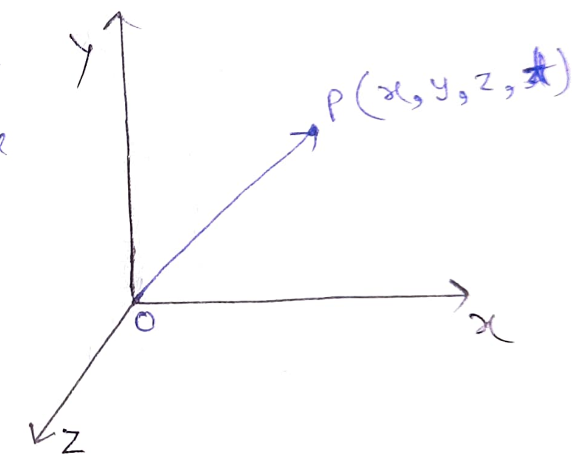
* Observer :- who observes the occurrence of the event in the space. (Rest or moving observer).

* Frame of refrence :- Geometrical frame-work (normal cartesian system) required to describe the occurance of event in the space.

$(x, y, z, t) \rightarrow$ Space-time

$\downarrow$

Spatial coordinate



The position vector of the moving object (at any instant) -

$$\boxed{\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}}$$

Velocity of the moving object -

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}}$$

acceleration -

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}}$$

→ Classification of frame of Reference:—

Frame of Reference

Inertial Frame

* The frame which follows the law of inertia (Newton's Ist law) (Gallilean's law). In Inertial frame

$$a = \frac{d^2 r}{dt^2} = 0 \quad , \text{ or}$$

$$\frac{d^2 x}{dt^2} = \frac{d^2 y}{dt^2} = \frac{d^2 z}{dt^2} = 0$$

* unaccelerated frame
* Theory of relativity which is app to inertial frame known as special theory of relativity.

Non-Inertial frame

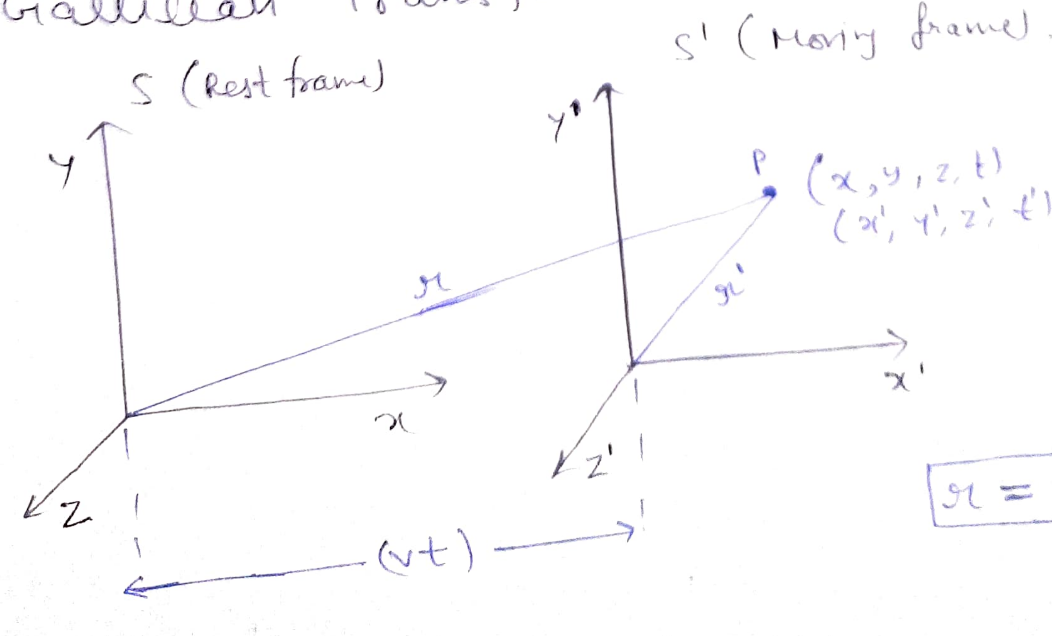
* Doesn't follow law of inertia. In Non-Inertial frame

$$a = \frac{d^2 r}{dt^2} \neq 0 \quad \text{or}$$

$$\frac{d^2 x}{dt^2} \neq 0, \frac{d^2 y}{dt^2} = 0, \frac{d^2 z}{dt^2} = 0$$

* accelerated frame
* Partical at rest can also experiance force, because of acc. of FOR known as Fictitious frame or Pshendo force.
* General theory of relativity.

1 * Gallilean Transformation—



Initially — when $t = t' = 0$ then origin coincides.

$$\boxed{r = r' + vt}$$

The position vector can be connected by the eqn –

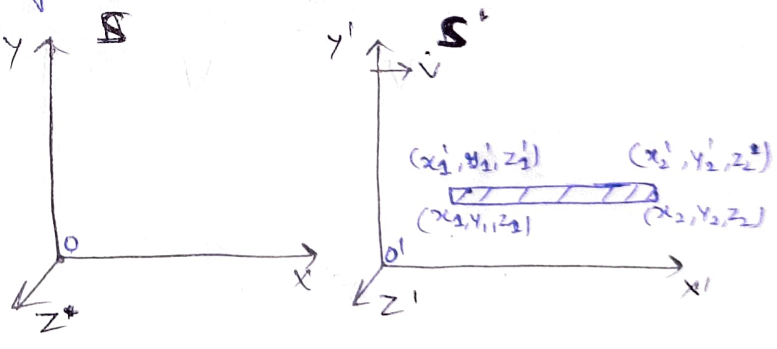
$$r = r' + vt$$
$$r' = r - vt \quad \text{—(1)}$$

In component form –

$$\boxed{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}} \longrightarrow \text{Gallilean Transformation.}$$

→ Consequence of Gallilean Transformation –

(1) length of an object is absolute.



length of the ~~observer~~ rod for s' - frame observer –

$$L' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}$$

Transforming the eqn using GT :–

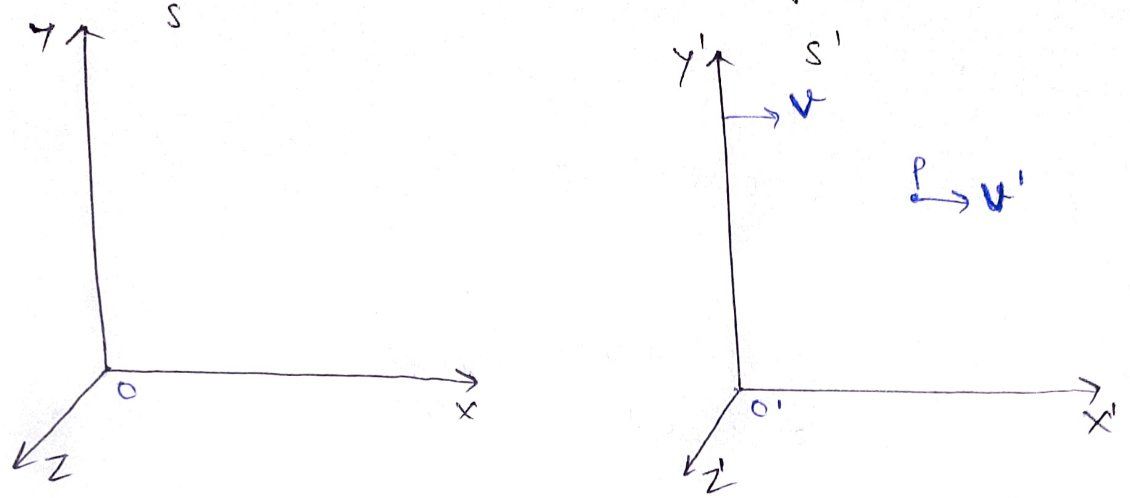
$$x_1' = x_1 - vt, \quad y_1' = y_1, \quad z_1' = z_1$$
$$x_2' = x_2 - vt, \quad y_2' = y_2, \quad z_2' = z_2$$

Now,

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \boxed{L = L'} \rightarrow \text{length is invarient under GT.}$$

it is for S- frame observer.

Velocity of moving object –



Using Gallilian Transformation –

$$r' = r - vt \quad , \quad t' = t$$

on diff. –

$$dr' = dr - v\,dt \quad , \quad dt' = dt$$

$$\frac{dr'}{dt} = \frac{dr}{dt} - v$$

$$\boxed{V' = V - v} \rightarrow \text{classical velocity addition}$$

V → velocity of object observed by observer S.
V' → velocity of object observed by observer S'.

→ acceleration –
similarly –

$$\boxed{a' = a}$$

Along with above facts, the law of conservation of momentum, energy will also have same form for both observer/frame of reference.

On summaryzing one can conclude that "All the laws of physics are identical for fundamental.

→ Michelson - Morley Experiment :-

* Historical Background -

Light wave should also require some material medium like other waves.

↓

Hypothetical medium throught the ~~medium~~ (universe).

↓

Ether.

* It is highly elastic.
* transparent
* negligible density
* The material bodies can move through it without disturbing it.

Also known as Ether hypothesis [concept of absolute frame].

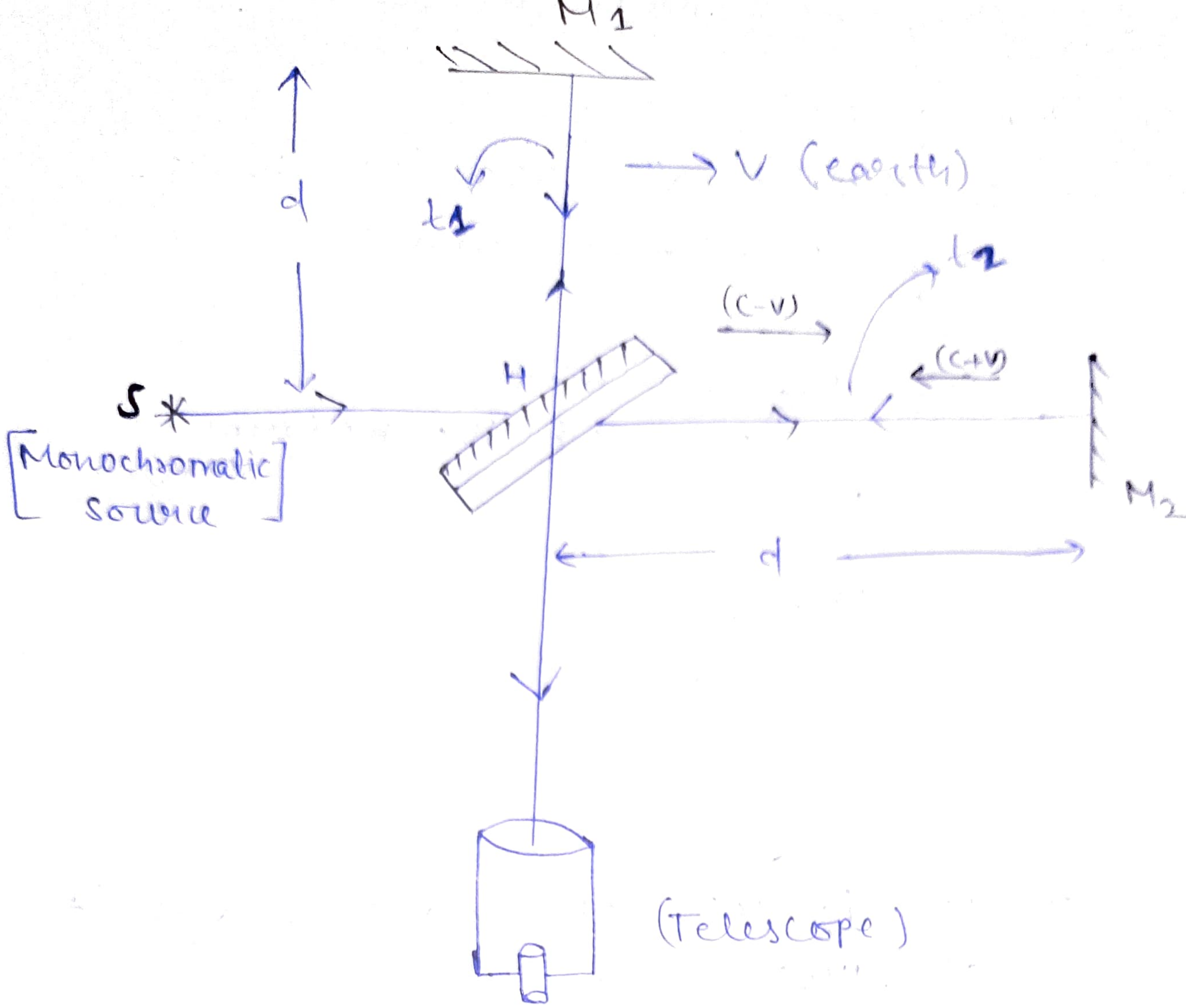
→ Objective :-

① Establish, experimentally the presence of Ether.

② to detect the motion of earth relative to ether.

③ to see wheather G.T are applicable for 'c' or not.

④ & to Justify the concept of 'Absolute frame of reference'.



→ Expectations :-

Suppose the earth is absolutely at rest.

$$t_1 = t_2 = \frac{2d}{c} \quad \text{---- (1)}$$

$$\Delta t = t_1 - t_2 = 0 \Rightarrow \Delta x = 0$$

No path difference and hence no fringe pattern will be visible.

But the reality is not so (earth is not at rest) Assuming the G.T valid for c -

$$t_2' = \frac{d}{c-v} + \frac{d}{c+v} = \frac{d(c+v) + d(c-v)}{c^2 - v^2}$$

$$t_2' = \frac{2dc}{c^2 - v^2} = 2dc\,[c^2 - v^2]^{-1} = \frac{2d}{c}\left[1 - \frac{v}{c}\right.$$

$$t_2' = \frac{2d}{c}\left[1 + \frac{v^2}{c^2} + \cdots\right] \quad \text{---- (2)}$$

$$\Rightarrow \quad t_1' = \frac{2d}{\sqrt{c^2 - u^2}}$$

similarly –

$$t_1' = \frac{2d}{c}\left[1 + \frac{u^2}{2c^2} + \cdots\right] \quad \cdots (3)$$

On comparing (2) + (3) –

$$t_2' > t_1'$$

$$\Delta t = \frac{2d}{c}\left[1 + \frac{u^2}{2c^2}\right]$$

$$\boxed{\Delta t = \frac{du^2}{c^3}}$$

{ when earth motion is taken in experiment then this was the expectation. }

Path difference –

$$\boxed{\Delta x = c\Delta t = \frac{du^2}{c^2}}$$

$$\frac{du^2}{c^2} = n\lambda$$

$$\boxed{n = \frac{du^2}{\lambda c^2}}$$

shift in fringe pattern will be observable.

...formation Equations :-

* Experimental Result :-

Actually (Practically) no fringe shift was observed in the experiment. Experiment was repeated many many times in different seasons, at different altitudes, every case was taken to remove any possible error but again and again no fringe shift was practically observed. This result was known as negative result or null result.

* Interpretation of Negative result :-

Earth is at rest in Ether.

1. "i.e. Therefore speed of light is a universal constant, identical for all observers of inertial frame independent upon the velocity velocity of source and observer."

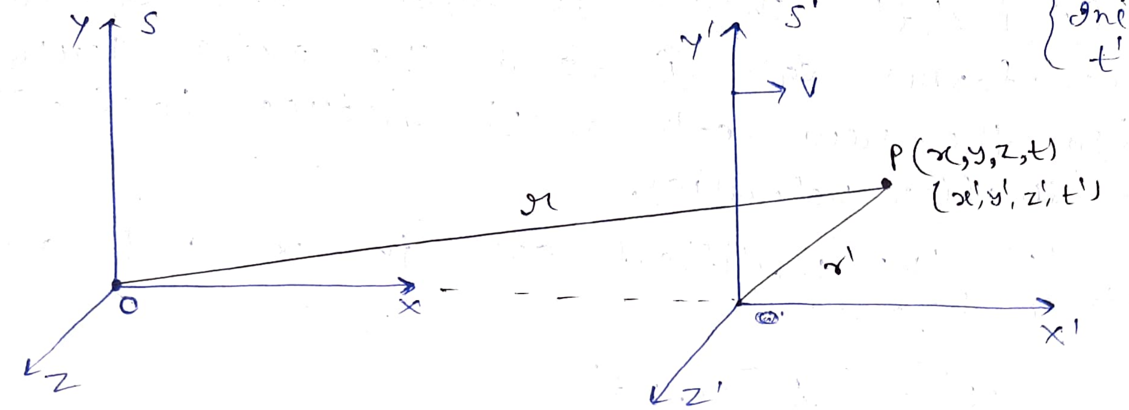
2. This was then supposed to be II basic postulate of theory of the relatively. Therefore, the II postulate of relativity is a natural consequence of Michelson - Morley experiment.

* Basic postulades of theory of relativity-

① All the laws of physics are identical for all the observers of the inertial frame that move with a constant velocity relative to one-another

② The ~~...~~ speed of light in vacuum is a universal constant for every inertial frame observer. This is also known as constancy of velocity of light

→ Lorentz Transformation Equations :-



{Initially $t' = t = 0$}

Time taken by light source from point O to P —

$$t = \frac{OP}{c} = \frac{\sqrt{x^2+y^2+z^2}}{c}$$

$$x^2+y^2+z^2 = c^2t^2 \quad \text{--- (1)}$$

Then, acc. to from S' –

$$t' = \frac{O'P}{c} = \frac{\sqrt{(x')^2+(y')^2+(z')^2}}{c}$$

$$(x')^2+(y')^2+(z')^2 = c^2(t')^2 \quad \text{--- (2)}$$

Using G.T –

$$(x-vt)^2 + (y)^2 + (z)^2 = c^2t^2 \qquad [\because t' = t]$$

$$x^2+u^2t^2-2xvt + y^2+z^2 = c^2t^2 \quad \text{--- (3)}$$

On comparing (3) & (1) we find that it is not in agreement with eq^n (1). Thus G.T fail if the constancy of speed of light is to be maintained. eq^n (3) contains an extra term $u^2t^2 - 2vxt$. This extra term indicates that there should be some modification in x and t.

→ Our transformation eqⁿs in (x, y, z, t) and (x', y', z', t') should be such that eqⁿ (3) transforms eqⁿ (1) and the extra term cancel. Also the new eqⁿ must reduce to G.T for smaller velocities.

Let the modified eqⁿ in x & t are given below :-

$$x' = \alpha\,[x - vt]$$

$$t' = \alpha'\,[t + fx]$$

using these in eqⁿ (2)

$$\alpha^2\,[x - vt]^2 + y^2 + z^2 = c^2 (\alpha')^2\,[t + fx]^2$$

$$x^2\left[\alpha^2 - f^2(\alpha')^2 c^2\right] - 2xt\left[\alpha^2 v + f c^2 (\alpha')^2\right] + y^2 + z^2 = \left((\alpha')^2 - \frac{\alpha^2 v^2}{c^2}\right) c^2 t^2 \quad \text{---} (4)$$

comparing (4) & (1)

$$\alpha^2 - f^2(\alpha')^2 c^2 = 1 \quad \text{-----} (5)$$

$$\alpha^2 v + f c^2 (\alpha')^2 = 0 \quad \text{-----} (6)$$

$$(\alpha')^2 - \frac{\alpha^2 v^2}{c^2} = 0 \quad \text{-----} (7)$$

Solving eqⁿ (5) (6) & (7)

$$\alpha = (\alpha') = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{-----} (8)$$

$$f = \frac{-v}{c^2} \quad \text{-----} (9)$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{---} (10), \qquad t' = \frac{t - xv/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{---} (11)$$

eqn (10) & (11) are Lorentz transformation eqn.

Can we use this for smaller velocity too — ans.

→ for smaller velocity —

$\frac{v}{c} \rightarrow , \frac{v^2}{c^2} \rightarrow 0$

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

which are well known G.T.

This indicates that the new idea of Lorentz is a most profound as it equally valid for the high velocities as well as for the smaller velocities.

→ Inverse L.T eqns :-

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} , \quad y = y' , \quad z = z'$$

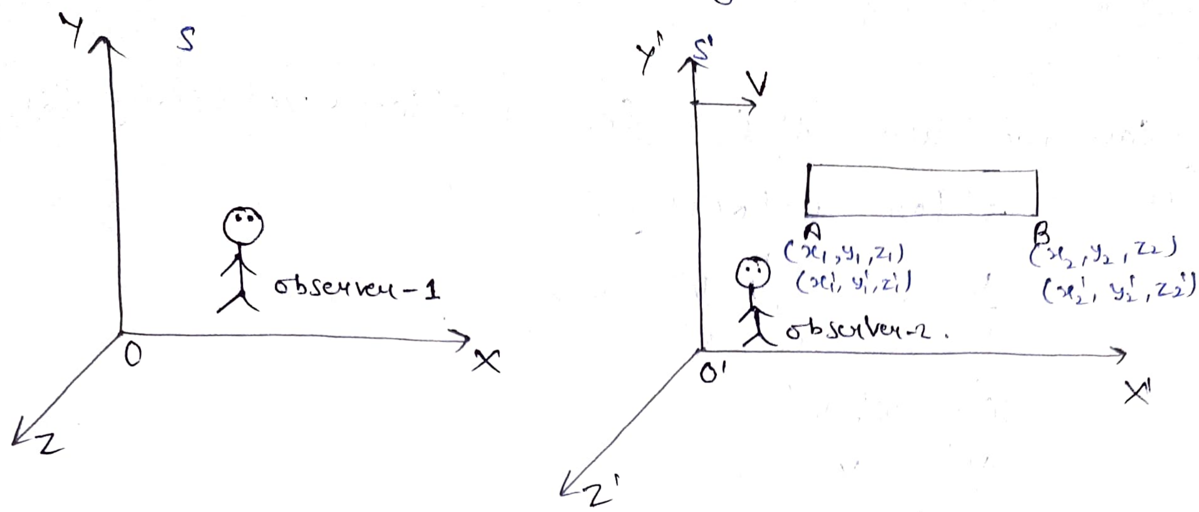
$$t = \frac{t' + \frac{x'v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

→ Consequences of Lorentz Transformation — consequences of LT :-

There are various consequences of LT :-

(i) Length contraction :— The rod (distance b/w two points in space) moving with a very high velocity gets contracted by a factor $\sqrt{1 - v^2/c^2}$ along the direction of motion. This is commonly known as 'length contraction'.

This is not noticed for smaller velocities.



Length of rod measured by S' frame observer is given by –

proper length $L_0 = x_2' - x_1'$ ------ (1)

The length measured by S frame observer –

observed length $L = x_2 - x_1$ --- (ii)

$\Rightarrow$ Using L.T we get

$$L_0 = \frac{x_2 - vt - x_1 + vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{(L_0)_{rest} = \frac{(L)_{motion}}{\sqrt{1 - v^2/c^2}}}$$

$$\boxed{(L)_{motion} < (L_0)_{rest}}$$

This is known as length contraction.

for smaller velocity $v^2/c^2 \to 0$

$$[L_{motion}] = [L_0]_{rest}$$

which is valid for smaller velocities and confirm G.T.

* Relativistic kinetic Energy :-

→ The relativistic K.E is given by-

$$K = c^2[m - m_0]$$

$$= c^2\left[\frac{m_0}{\sqrt{1-v^2/c^2}} - m_0\right]$$

$$= m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$K = m_0c^2\left[\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\cdots\right)-1\right]$$

for smaller velocities -

$v << c$,

, $v^4 <<< c$ so we can neglect -

$$K = \left[\frac{1}{2}\frac{v^2}{c^2}\right]\times m_0c^2$$

$$\boxed{K \approx \frac{1}{2}mv^2}$$

* Relation b/w Relativistic momentum and Energy –

The relativistic energy of the body of mass m is :

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$\because$ $p = mv \Rightarrow v = \frac{p}{m}$

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{p^2}{m^2c^2}}} = \frac{m_0 c^2}{\sqrt{1-\frac{p^2c^2}{m^2c^4}}}$$

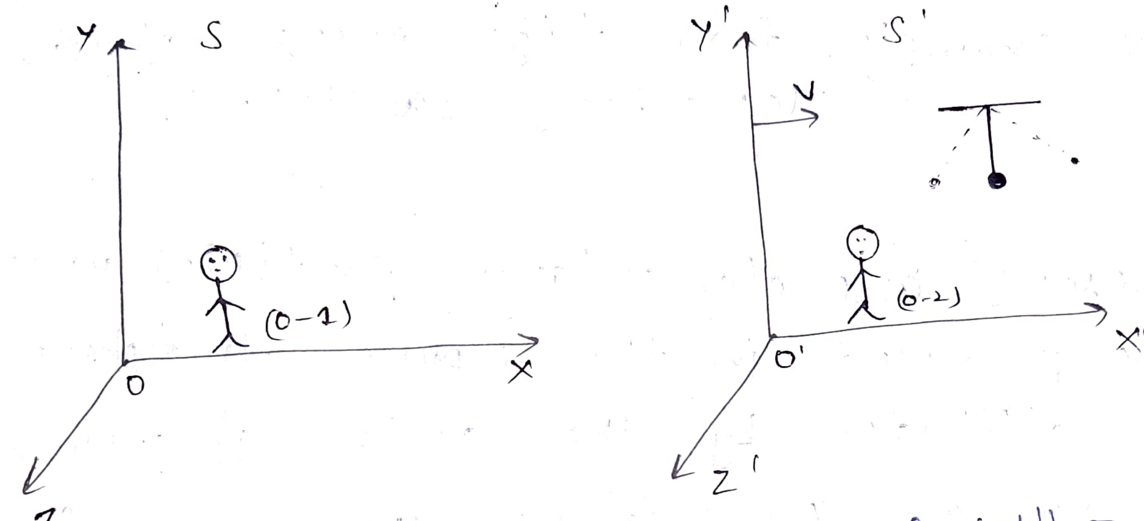
$$E = \frac{m_0 c^2}{\sqrt{1-\frac{p^2c^2}{E^2}}} \Rightarrow E^2 = \frac{m_0^2 c^4}{\left(1-\frac{p^2c^2}{E^2}\right)}$$

$$\boxed{E^2 - p^2c^2 = m_0^2 c^4}$$

Since m_0 & c both are constant, Therefore the quantity $E^2 - p^2c^2$ is unvarient under L.T, i.e its value remains unchanged in any inertial frame of reference.

(ii) Time dilation :-

Moving clock appears to go slow relative to stationary observers. This implies that time-interval b/w the two events measured by a clock (moving) is more than measured by a stationary clock.



The proper time interval $(\Delta t') = t_2' - t_1'$

The observer [S' frame] can easily measure it.

But the observer of S-frame cannot because the place of occurrence of event will also change for him when the bob again come to mean position.

~~The~~ Time interval by the S-frame observer—

$$\Delta t = t_2 - t_1$$

t_1 and t_2 cannot be measured by the clock, as the position of occurrence of event will change. So one has to use and replace t_1 & t_2 using Lorentz T.

$$\Delta t = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{(t_2' - t_1')}{\sqrt{1-v^2/c^2}}$$

$$[\Delta t]_{motion} = \frac{[\Delta t']_{rest}}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{[\Delta t]_{motion} > [\Delta t']_{rest}}$$

Time dilation is a real effect :-

* Example from high energy physics. (Elementary particles).

μ- mesons are elementary particles which are ~~the~~ produced in the upper atmosphere at high altitudes by the action of cosmic ray showers on π - measons. These are highly unstable and their life in own frame is 2.2×10^{-6} s and their velocity is very close to c. So the distance transversed by μ- mesons in this time-

$$d = 2.2 \times 10^{-6} \times 0.998c$$

$$= 660 \text{ metre}$$

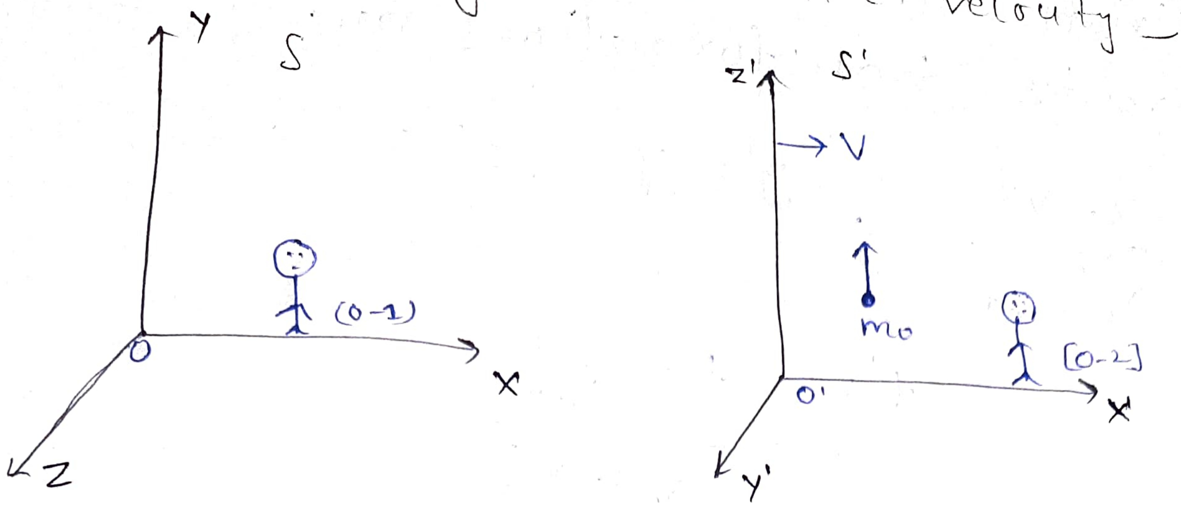
Therefore there is no expectation of finding these particles near to the surface of the earth. But experimentally detected near to the surface of the earth. This can be justified, considering time - dilation effect.

(For lab. frame observer) $\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} = \frac{2.2 \times 10^{-6}}{\sqrt{1 - (0.998)^2}} = 3.17 \times 10^{-5}$ Sec

Distance travelled $= 2.994 \times 10^{8} \times 3.17 \times 10^{-5}$
$= 9500$ m.

Therefore, μ mesons are able to reach the surface of earth.

* Variation of mass with velocity –



Momentum of particle for S' – from observer –

$$p'_y = v'_y \times m_0 = m_0 \times \frac{dy'}{dt'} \quad \text{--- (1)}$$

transforming dy' & dt' –

$$dy' = dy \quad , \quad dt = \frac{dt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

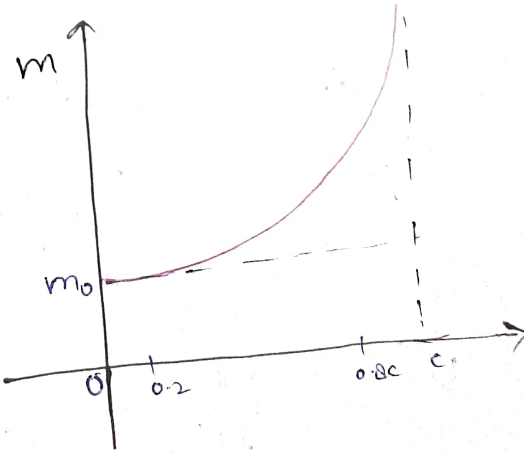
$$m_0 \frac{dy'}{dt} = \frac{m_0\, dy}{dt\sqrt{1-\frac{v^2}{c^2}}}$$

$$p_y = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \frac{dy}{dt} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \times u_y$$

Momentum for S- frame observer –

$$m \cdot u_y = \frac{m_0\, u_y}{\sqrt{1-v^2/c^2}}$$

$$\boxed{m = \frac{m_0}{\sqrt{1-v^2/c^2}}}$$

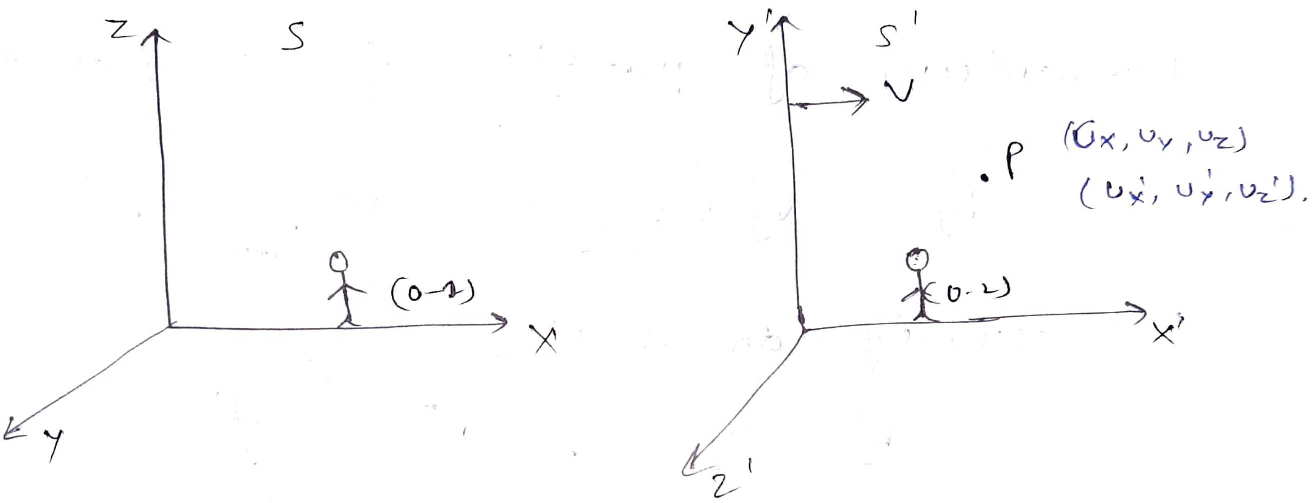


→ Relativistic Addition of velocities :-

Lorentz transformations are -

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-v^2/c^2}}$$



Velocity component -

$$u_x' = \frac{dx'}{dt'}, \quad u_y' = \frac{dy'}{dt'}, \quad u_z' = \frac{dz'}{dt'}$$

On differentiating LT eq^ns :-

$$dx' = \frac{dx - v\,dt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad dy' = dy, \quad dz' = dz$$

$$dt' = \frac{dt - \frac{v}{c^2}dx}{\sqrt{1-v^2/c^2}}$$

$$u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx}$$

$$u_x' = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

$$\boxed{u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}}$$

Similarly,

$$u_y' = \frac{dy'}{dt'} = \frac{dy'}{\dfrac{dt - \frac{v}{c^2}dx}{\sqrt{1-v^2/c^2}}}$$

$$\boxed{u_y' = \frac{u_y\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c^2}u_x}}$$

$$\boxed{u_z' = \frac{u_z\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c^2}u_x}}$$

lets the velocity of object as c -

$$u_x' = \frac{c-v}{1-\frac{v.c}{c^2}} = c$$

Confirms second basic postulate.

→ Einsteins Mass Energy Relation -

Consider a body moving with velocity v under the action of a force F, therefore,

$$F = \frac{dP}{dt} = \frac{d(m.v)}{dt} \quad \text{--- (1)}$$

In relativistic mechanics both the mass and velocity are variable -

$$F = m\frac{dv}{dt} + v\frac{dm}{dt} \quad \text{--- (2)}$$

Work done on the body –

$$dW = dK = F \cdot dx$$

$$dK = m\frac{dv}{dt} \cdot dx + v\frac{dm}{dt} \cdot dx$$

$$dK = mv\,dv + v^2\,dm \quad --- (3)$$

Mass variation can be expressed as –

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \Rightarrow m^2\left[1 - \frac{v^2}{c^2}\right] = m_0^2$$

$$m^2\left[c^2 - v^2\right] = m_0^2 c^2$$

$$m^2c^2 - m^2v^2 = m_0^2c^2 \quad -- (4)$$

on diff. (4) –

$$2m\,dm \cdot c^2 - \left[m^2 2v\,dv - v^2 2m\,dm\right] = 0$$

$$mv\,dv + v^2\,dm = c^2\,dm$$

putting in (3)

$$dK = c^2\,dm \quad --- (5)$$

Let K be the K.E of the body during $m_0 \to m$ –

Total K.E

$$\int dK = \int_{m_0}^{m} c^2\,dm \Rightarrow \boxed{K = c^2(m - m_0)}$$

$$\text{Total Energy} = \text{Rest mass} \cdot E + K.E$$

$$= m_0c^2 + (m - m_0)c^2$$

$$\boxed{E = mc^2}$$