[UNIT-3]

[CENTROID AND MOMENT OF INERTIA]

(1) The point, through which the whole weight of the body acts, is known as 'centre of gravity'.

(2) The point, at which the total area of a plane figure is assumed to be concentrated, is known as 'centroid' of that area. The centroid and center of gravity are at the same point.

(3) The COG of a uniform rod lies at its middle point.

(4) The COG of a △ lies at a point where the three medians of a △ meet.

(5) The COG of a ▭ where diagonals meet.

(6) The COG of a body consisting of different areas is given by -

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + \cdots}{a_1 + a_2 + \cdots}, \quad \bar{y} = \frac{a_1 y_1 + a_2 y_2 + \cdots}{a_1 + a_2 + \cdots}$$

where $\bar{x}, x_1, x_2, \cdots \bar{y}, y_1, y_2, \cdots$ are from reference axis.

(7) The MOI of an area (mass) about an axis is the product of area (or mass) and square of the distance of COG of the area (or mass) from that axis. represented by I.

(8) Moment of Inertia of a rectangular section:—

(i) about an horizontal axis passing through C.G.

$$= \frac{bd^3}{12}$$

(ii) about an horizontal axis passing through base.

$$= \frac{bd^3}{3}.$$

(9) Moment of Inertia of a circular section $= \frac{\pi D^4}{64}$

(10) MOI of a triangular section ⇒

(i) about the base $= \frac{bh^3}{12}$

(ii) about an axis passing through CoG and parallel to the base $= \frac{bh^3}{36}$

(11) The C.G of an area by integration method is given by—

$$\bar{x} = \frac{\int x^* \, dA}{\int dA} \quad \text{and} \quad \bar{y} = \frac{\int y^* \, dA}{\int dA}$$

where x^* = distance of C.G of area dA from y-axis

y^* = - - - - - - - - - - x-axis.

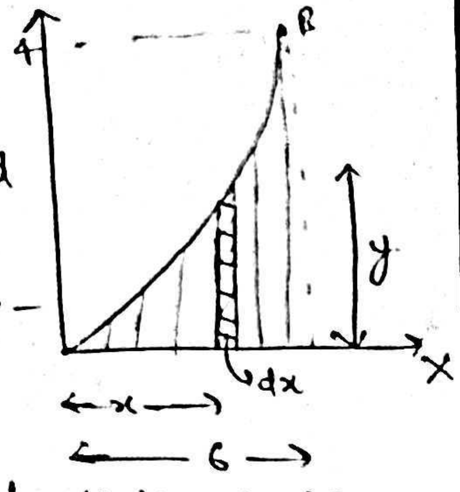
(12) The C.G of a straight or curved line —

$$\bar{x} = \frac{\int x^* \, dl}{\int dl}, \quad \bar{y} = \frac{\int y^* \, dl}{\int dl}.$$

* Determine the co-ordinates of the C.G of the area OAB shown in Fig. 9.7, if curve OB represents the eq^n of a parabola given by –

$$x^2 = 9y.$$

* Consider a strip of height y and width dx as shown in Fig. The area dA of the strip is given by–



$$dA = y\,dx.$$

The coordinates of the C.G of this area dA are x and $y/2$.

So,

$$x^* = x \quad , \quad y^* = y/2$$

$$\bar{x} = \frac{\int x^*\,dA}{\int dA} = \frac{\int_0^6 x \times y\,dx}{\int y\,dx} = \frac{\int_0^6 x \times \frac{x^2}{9}\,dx}{\int_0^6 \frac{x^2}{9}\,dx}$$

$$\boxed{\bar{x} = 4.5}$$

$$\bar{y} = \frac{\int y^*\,dA}{\int dA} = \frac{\int \frac{y}{2} \times y\,dx}{\int y\,dx} = \frac{\int_0^6 \frac{1}{2} \times \frac{x^4}{81}\,dx}{\int_0^6 \frac{x^2}{9}\,dx}$$

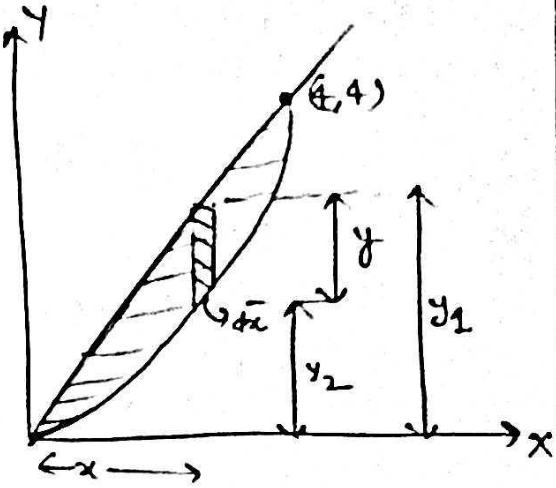
$$\bar{y} = \frac{6}{5} = 1.2$$

Hence coordinates of C.G is (4.5, 1.2)

* Determine the coordinates of C.G of the shaded area b/w $y = \frac{x^2}{4}$ and $y = x$ as shown.

* $dA = y\,dx = (y_1 - y_2)\,dx$

$$dA = \left(x - \frac{x^2}{4}\right) dx$$



$$x^* = x$$

$$y^* = \frac{y_1 - y_2}{2} + y_2 = \frac{y_1 + y_2}{2}$$

$$y^* = \frac{1}{2}\left(x + \frac{x^2}{4}\right)$$

$$\bar{x} = \frac{\int x^* dA}{\int dA} = \frac{\int_0^4 \left(x^2 - \frac{x^3}{4}\right) dx}{\int_0^4 \left(x - \frac{x^2}{4}\right) dx} = 2$$

$$\bar{y} = \frac{\int y^* dA}{\int dA} = \frac{\int_0^4 \frac{1}{2}\left(x^2 - \frac{x^4}{16}\right) dx}{\int_0^4 \left(x - \frac{x^2}{4}\right) dx} = \frac{8}{5} = 1.6$$

Hence coordinates of C.G of is (2, 1.6).

* Problems of finding Centroid or COG of line segment by Integration method -

* Determine the COG of an arc shown in fig.

1st method -

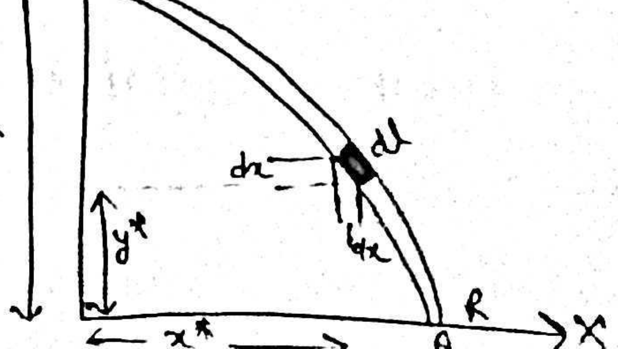
* eqn of circle -

$$x^2 + y^2 = R^2$$

differentiating the above eqn -

$$2x\,dx + 2y\,dy = 0$$

$$dy = -x\frac{dx}{y}$$



$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 + \frac{x^2 (dx)^2}{y^2}} = \frac{dx}{y}\sqrt{x^2+y^2}$$

$$dL = \frac{dx \times R}{y}$$

$$\bar{y} = \frac{\int y^* \, dL}{\int dL} = \frac{\int_0^R y \times \frac{R}{y} \, dx}{\int_0^R \frac{R}{y} \, dx} = \frac{2R}{\pi}$$

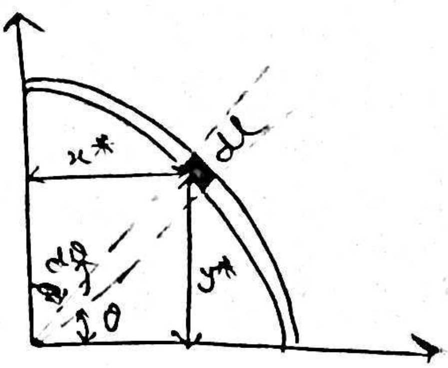
$$\bar{x} = \frac{\int x^* \, dL}{\int dL} = \frac{2R}{\pi}$$ (due to symmetry).

2nd method:—

$dL = R\, d\theta$

$y^* = R \sin\theta$

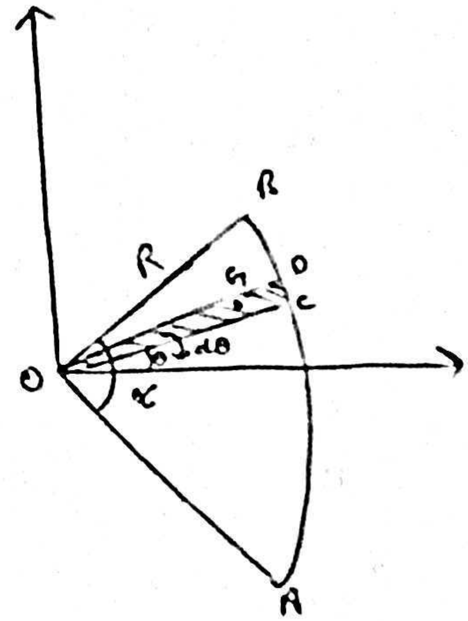
$x^* = R \cos\theta$



$$\bar{y} = \frac{\int y^* \, dL}{\int dL} = \frac{\int_0^{\pi/2} R\sin\theta \times R\, d\theta}{\int_0^{\pi/2} R\, d\theta} = \frac{2R}{\pi}$$

$$\bar{x} = \frac{\int x^* \, dL}{\int dL} = \frac{\int_0^{\pi/2} R\cos\theta \cdot R\, d\theta}{\int_0^{\pi/2} R\, d\theta} = 2R/\pi.$$

* Determine C.G of the area of circular sector OAB of radius R as shown.



* $dA = \frac{1}{2} \times R d\theta \times R = \frac{R^2}{2} d\theta$

The C.G of this triangular element is at G.

$OG = \frac{2}{3} \times OC = \frac{2}{3} R$

$x^* = OG \cos\theta = \frac{2}{3} R \cos\theta$

$y^* = \frac{2}{3} R \sin\theta$

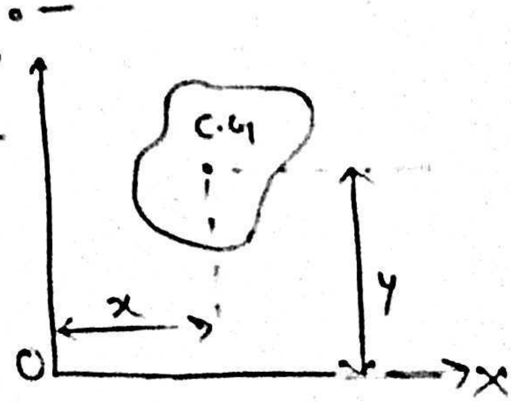
$$\bar{x} = \frac{\int_{-\alpha/2}^{\alpha/2} \frac{2}{3} R\cos\theta \times \frac{R^2}{2} d\theta}{\int_{-\alpha/2}^{\alpha/2} \frac{R^2}{2} d\theta} = \frac{4R}{3\alpha} \sin\left(\frac{\alpha}{2}\right)$$

section is symmetric about x axis so

$\bar{y} = 0$.

* Moment of inertia of Area :-

Moment of area about axis OY -



= Area × ⊥er distance of C.G from OY.

= Ax ---------- (i)

Eqn (i) is known as first moment of area about the axis OY. The first moment of area is used to determine the COG of the area.

If the moment of area given by eq^n (i) is again multiplied by the $\perp^{er}$ distance b/w the C.G of the area and axis OY, then the quantity $(Ax).x = Ax^2$ is known as Moment of the moment of area or second moment of area or moment of inertia of area about the axis OY. similarly,

MOI of area about OX axis $= Ay^2$.

* Radius of Gyration –

$$\boxed{R = \sqrt{\frac{I}{A}}}$$

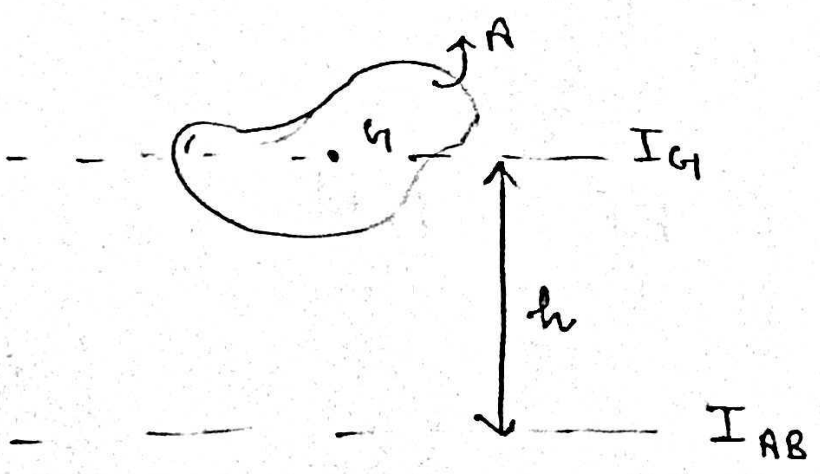
* Perpendicular axis theoram –

$$I_{zz} = I_{xx} + I_{yy}.$$

The MOI 'I_{zz}' is also known as polar MoI.

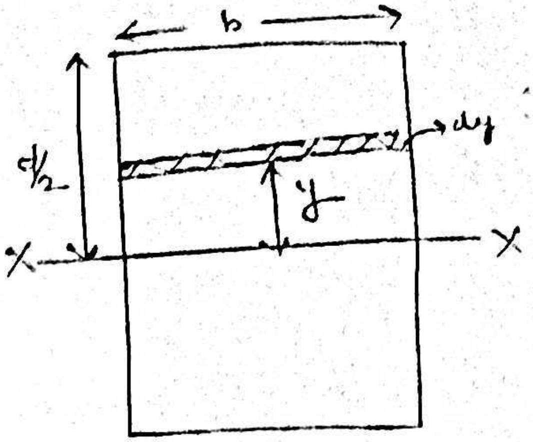
* Parallel axis theoram –

$$\boxed{I_{AB} = I_G + Ah^2}$$



* MOI of a rectangular section –

(1) XX– axis passing through the C.G.



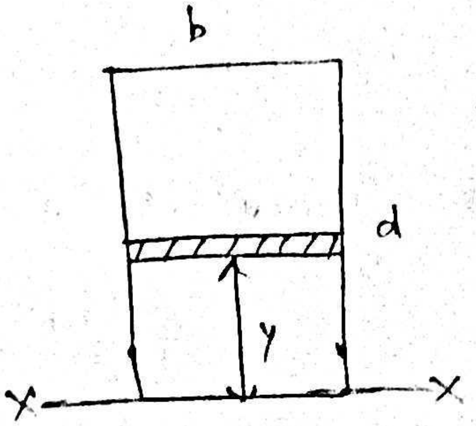
$$I_{XX} = \int_{-d/2}^{d/2} (b \times dy) \times y^2$$

$$\boxed{I_{XX} = \frac{bd^3}{12}}$$

Similarly, the MOI of rectangular section about the Y–Y axis passing through the C.G of the section is given by –

$$\boxed{I_{YY} = \frac{db^3}{12}}$$

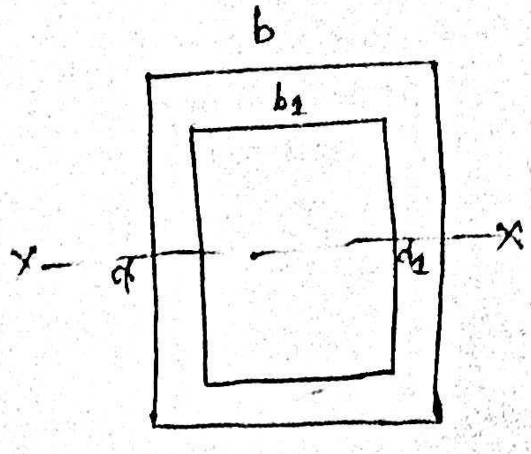
(2) MOI of the rectangular section about a line passing through base –



$$I_{XX} = \int_0^d (b \cdot dy) \times y^2$$

$$\boxed{I_{XX} = \frac{bd^3}{3}}$$

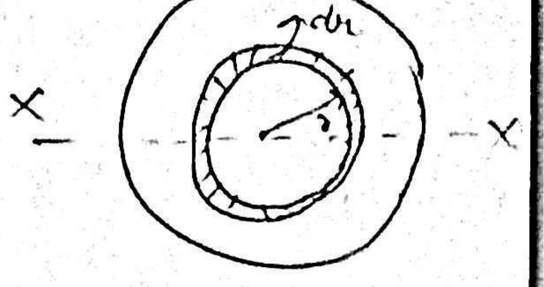
*



$$I_{XX} = \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}$$

* MOI of a circular section —

$dA = 2\pi r\, dr$



$$I_{zz} = \int_0^R (2\pi r\, dr) \times r^2$$

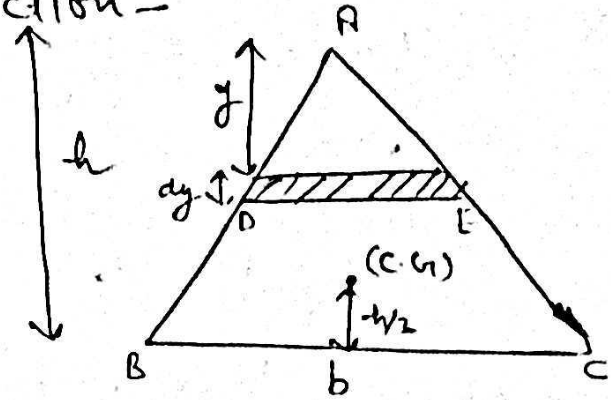
$$I_{zz} = \frac{\pi R^4}{2}$$

$I_{xx} = I_{yy} \rightarrow$ (symmetry)

$$I_{zz} = 2 I_x$$

$$\boxed{I_{xx} = \frac{\pi R^4}{4}}$$

* MOI of a triangular section —



$dA = DE \times dy$

$$\frac{DE}{BC} = \frac{y}{h}$$

$$DE = \frac{by}{h}$$

$$dA = \frac{by}{h}\, dy$$

distance of the strip from the base $= (h - y)$

$$I_{base} = \int_0^h \frac{by}{h} (h-y)^2\, dy$$

$$I_{c.g} = \int_{-h/3}^{2h/3} \frac{by}{h}\, dy\, (h-y)^2$$

$$\boxed{I_{c.g} = \frac{bh^3}{36}}$$

OR

$$I_{base} = I_{c.g} + A \times \left(\frac{h}{3}\right)^2$$

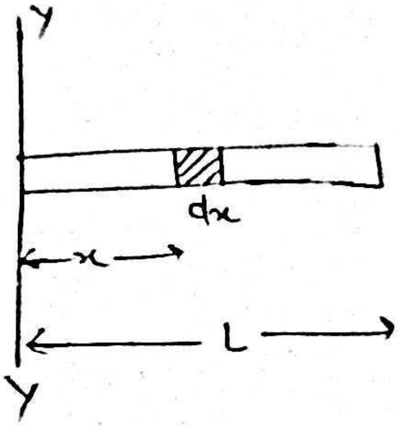
$$I_{c.g} = \frac{bh^3}{12} - \frac{b \times h}{2} \times \frac{h^2}{9}$$

$$\boxed{I_{c.g} = bh^3/36}$$

* Uniform thin rod — (MoI).

~~dA~~

$$dm = \frac{M}{L} \times dx$$



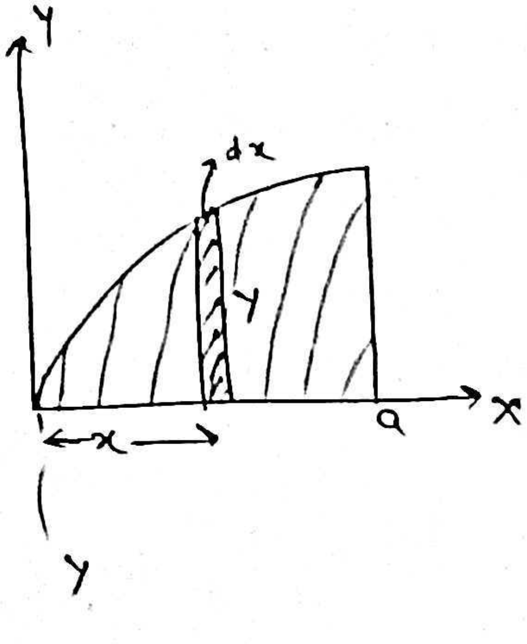
$$I_{YY} = \int_0^L \frac{M}{L}\, dx \times x^2 = \frac{ML^2}{3}.$$

* Moment of inertia of a Area under a curve of given eqⁿ —

$$x = ky^2$$

$$y = b \quad \text{when } x = a$$

* $y^2 = \frac{b^2 x}{a}$



$$dA = \frac{b}{\sqrt{a}} \sqrt{x} \times dx$$

$$I_{YY} = \int_0^a \frac{b}{\sqrt{a}} \sqrt{x}\, dx \times x^2$$

$$\boxed{I_{YY} = \frac{2}{7} b a^2}$$

$$I_{XX} = \int_0^b (x\, dy) \times y^2 = \int_0^b \frac{a}{b^2} y^2 \times y^2\, dy$$

$$\boxed{I_{XX} = \frac{2}{15} ab^3}$$

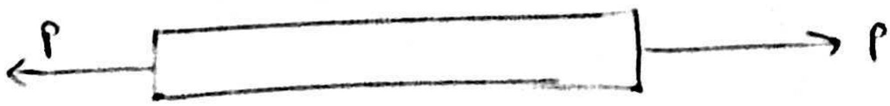
[UNIT-1]

* Stress $= \dfrac{\text{External force or load}}{\text{Cross-sectional Area}} = \dfrac{P}{A}$.

→ Normal stress :— Normal stress is further divided into tensile stress and compressive stress.

" The stress which acts in a direction perpendicular to the area represented by (σ) is known as Normal stress".

(i) Tensile stress :—



$$\boxed{\sigma = P/A}$$

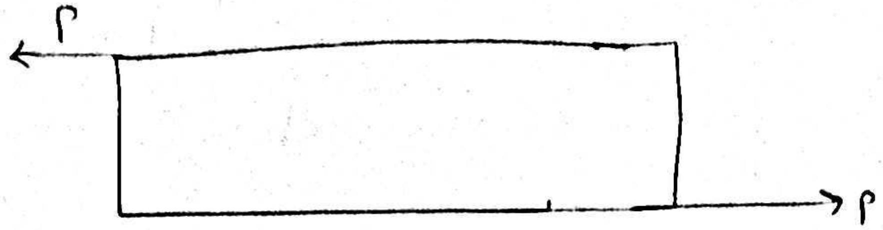
$$\boxed{\text{tensile strain } (e) = \frac{dL}{L}}$$

(ii) Compressive stress :—

$$\boxed{\sigma = P/A}$$

$$\boxed{\text{compressive strain } (e) = \frac{dL}{L}}$$

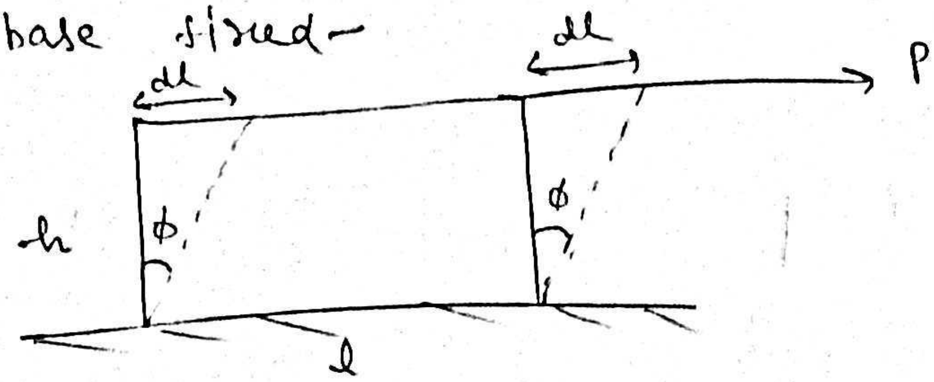
→ Shear stress :-



$$\text{Shear stress } (\tau) = \frac{\text{Shear Resistance}}{\text{Shear Area}} = \frac{R}{A}$$

* Shear stress is tangential to the area over which it acts.

→ If base fixed -



Shear strain -

$$\boxed{\phi = \frac{dL}{h}}$$

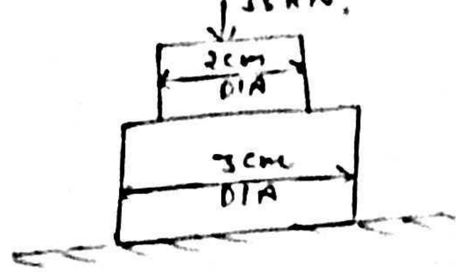
* Modulus of Elasticity (or Young Modulus) -

$$\boxed{E = \frac{\text{stress}(\sigma)}{\text{strain}(e)}}$$

* Modulus of rigidity (or shear Modulus) -

$$C \text{ (or } G \text{ or } N) = \frac{\text{Shear stress } (\tau)}{\text{Shear strain } (\gamma)}$$

* Find the maximum and minimum stre[ss] produced.



* $P = 35\ KN = 35 \times 10^3\ N$

$D_1 = 2\ cm = 20\ mm$

$A_1 = 100\pi\ mm^2$

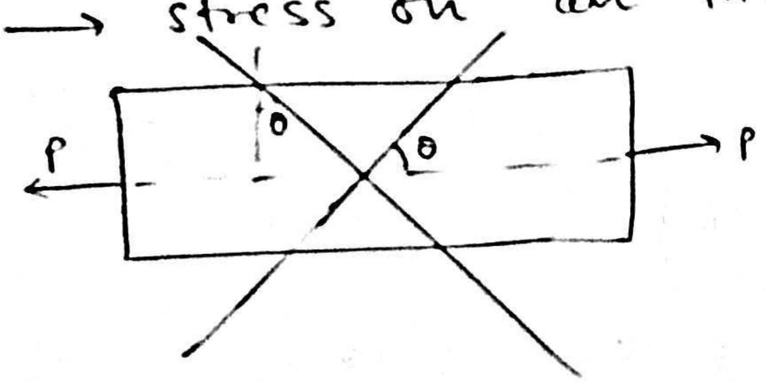
$A_2 = 225\pi\ mm^2$

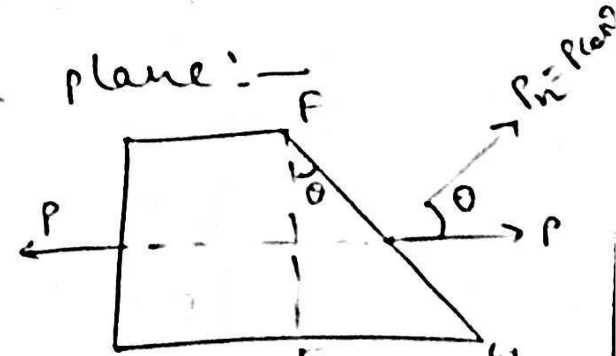
The stress is load divided by area. Load is same for both as they are in series. Hence

$$P_{max} = \frac{Load}{Area_{min}} = \frac{35 \times 10^3}{100\pi} = 111.400\ N/mm^2$$

$$P_{min} = \frac{load}{Area_{max}} = \frac{35 \times 10^3}{225\pi} = 49.5146\ N/mm^2$$

→ stress on an inclined plane:—





Area of section FG = FG × 1 (thickness = 1).

$$= \frac{EF}{\cos\theta} = \frac{A}{\cos\theta}$$

stress on FG along x-axis $= \frac{P}{A} \times \cos\theta = \sigma\cos\theta$

Normal stress and tangential stress—

Normal stress $(\sigma_n) = \frac{P\cos\theta}{A/\cos\theta} = \sigma\cos^2\theta.$

tangential (shear) stress -

$$\sigma_t = \frac{P\sin\theta}{A/\cos\theta} = \frac{\sigma}{2}\sin 2\theta .$$

* Find the diameter of a circular bar which is subjected to an axial pull of 160 kN, if the max allowable shear stress on any section is 65 N/mm².

* Maximum shear force stress $= \frac{\sigma}{2}$.

$$\frac{\sigma}{2} = 65$$

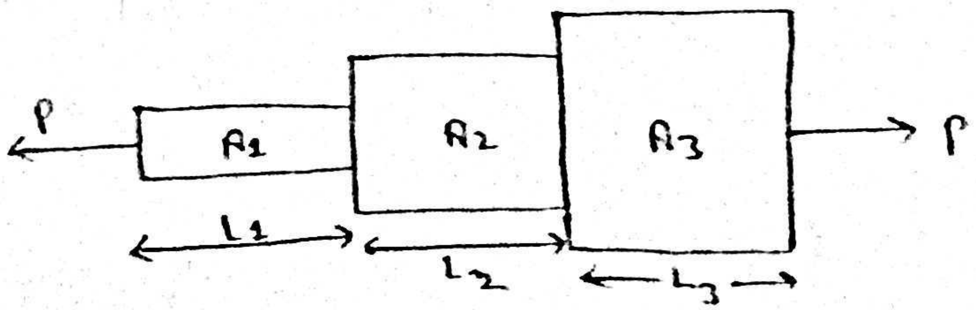
$$\sigma = 130$$

$$\frac{P}{A} = 130$$

$$\frac{160000}{\frac{\pi}{4}D^2} = 130$$

$$\boxed{D = 39.58 \text{ mm}}$$

→ Members of varying cross-sections-



$\sigma_1 = \frac{P}{A_1}$, $e_1 = \frac{P}{A_1E_1}$

$\sigma_2 = \frac{P}{A_2}$, $e_2 = \frac{P}{A_2E_2}$

$\sigma_3 = P/A_3$, $e_3 = \frac{P}{A_3E_3}$

$$e_1 = \frac{dL_1}{L_1}$$

$$dL_1 = \frac{PL_1}{A_1E_1} \qquad \left[\because e_1 = \frac{P}{A_1E_1}\right]$$

Similarly,

$$dL_2 = \frac{PL_2}{A_2E_2}, \quad dL_3 = \frac{PL_3}{A_3E_3}$$

Total change in length –

$$\boxed{dL = P\left[\frac{L_1}{A_1E_1} + \frac{L_2}{A_2E_2} + \frac{L_3}{A_3E_3}\right]}$$

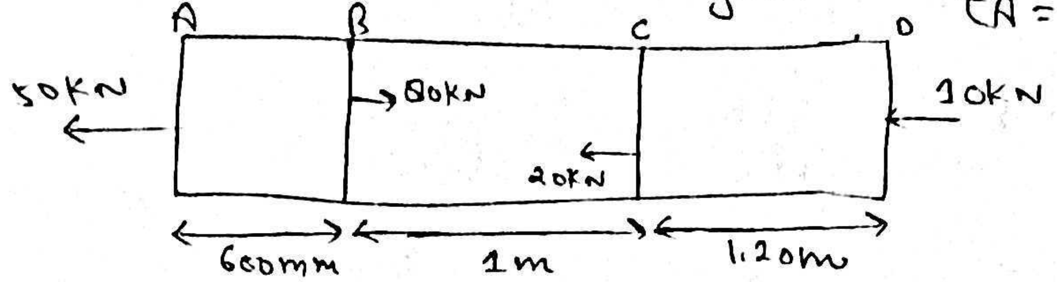
If material are same then $E_1 = E_2 = E_3 = E$

$$\boxed{dL = \frac{P}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right]}$$

* Principal of superposition –

When a number of loads are acting on a body, the resulting strain, according to principal of superposition, will be the algebric sum of strains caused by individual loads.

* Find the total elongation. ($E = 1.05 \times 10^5$ N/mm², $A = 1000$ mm²).

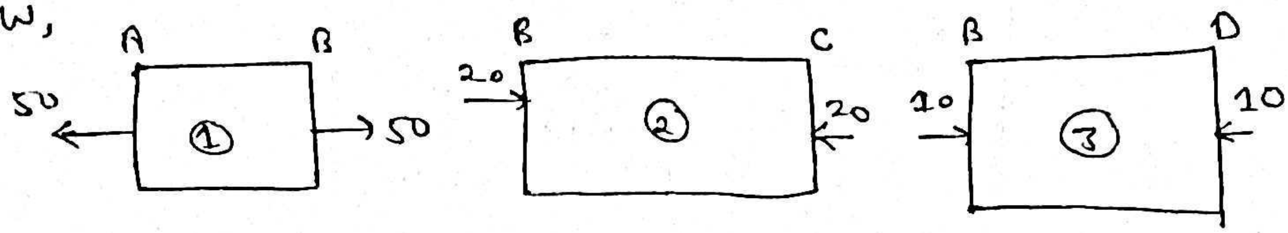


* For the eq^m of the rod -

Force towards left = Force towards right

$50 + 20 + 10 = 80$

Now,



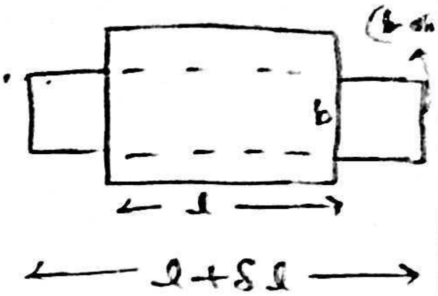
$$dL = dL_1 + dL_2 + dL_3$$

$$= \frac{P_1 L_1}{AE} + \left(\frac{-P_2 L_2}{AE}\right) + \left(\frac{-P_3 L_3}{AE}\right)$$

$$= \frac{1}{1000 \times 1.05 \times 10^5} \left[50 \times 10^3 \times 600 + (-20 \times 10^2 \times 1000) + (-10 \times 10^3 \times 2200) \right]$$

$$\boxed{dL = -0.1142 \text{ m}}$$

* Lognitudnal strain $= \frac{dL}{L}$



* Lateral strain $= \frac{db}{b}$ or $\frac{\delta d}{d}$

→ Poisson's Ratio :-

$$\boxed{\mu = \frac{\text{Lateral strain}}{\text{Lognitudnal strain}}}$$

* Lateral strain $= \mu \times$ Lognitudnal strain.

As lateral strain is opposite in sign to Lognitudnal strain - Hence,

* Lateral strain $= -\mu \times$ lognitudnal strain.

* μ varies from 0.25 to 0.33 but for Rubber it is 0.45 to 0.50.

→ Determine the change in length, breadth and thickness. $F = 30\ KN$, $L = 4m$, $b = 30mm$, $t = 20\ mm$. $E = 2 \times 10^5\ N/mm^2$, $\mu = 0.3$.

* $\frac{dL}{L}$ (lognitudnal strain) $= \frac{\text{stress}}{E} = \frac{\text{Load}}{\text{Area} \times E}$

$$\frac{dL}{L} = \frac{30000}{30 \times 20 \times 2 \times 10^5} = 0.00025$$

$$\frac{dL}{4000} = 0.00025$$

$$\boxed{dL = 1 \text{ mm}}$$

$$\because \quad +\frac{dL}{L} \times \mu = \frac{db}{b} = \frac{dt}{t}$$

$$\frac{db}{b} = 0.000075 \qquad [\because \mu = 0.3]$$

$$\frac{db}{30} = 0.000075$$

$$\boxed{db = 0.00225 \text{ mm}}$$

$$\boxed{dt = 0.0015 \text{ mm}}$$

→ volumetric strain –

$$\boxed{e_v = \frac{dV}{V} = \frac{\delta L}{L} + \frac{\delta b}{b} + \frac{\delta d}{d}}$$

$\frac{\delta L}{L}$ = Lognitudnal strain , $\frac{\delta b}{b}$ (or $\frac{\delta d}{d}$) = lateral strain

e_v = Lognitudnal strain + 2 × lateral strain

= " + 2 × (− μ × lognitudnal strain)

e_v = Lognitudnal strain (1 − 2μ)

$$\boxed{e_v = \frac{\delta L}{L}(1-2\mu)}$$

→ Volumetric strain of a Rectangular Bar subjected to three forces which are mutually perpendicular —

* $$V = xyz$$

$$\log V = \log x + \log y + \log z$$

$$\boxed{\frac{dV}{V} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}$$

$$\frac{dV}{V} = e_x + e_y + e_z \quad \text{— (1)}$$

Now, let

σ_x = Tensile stress in x-direction.

σ_y = – – – y – – –.

σ_z = – – – z – – –.

E = Young's Modulus.

μ = Poisson's ratio.

Now, σ_x will produce a tensile strain equal to $\frac{\sigma_x}{E}$ in x-direction, and a compressive strain $\frac{\mu \times \sigma_x}{E}$ in y and z direction Hence,

Similarly σ_y will produce a tensile strain

$\cancel{e_x = \frac{\sigma_x}{E} - \frac{\mu \times \sigma_y}{E}}$.

$\frac{\sigma_y}{E}$ in y-direction & $\frac{\mu\sigma_y}{E}$ in x + z direction

by σ_z,

$\frac{\sigma_z}{E}$ in z-direction & $\frac{\mu \times \sigma_z}{E}$ in x & y direction.

∴ Now tensile strain along x - direction -

$$e_x = \frac{\sigma_x}{E} - \frac{\mu \times \sigma_y}{E} - \frac{\mu \times \sigma_z}{E} = \frac{\sigma_x}{E} - \frac{\mu(\sigma_y + \sigma_z)}{E}$$

Similarly,

$$e_y = \frac{\sigma_y}{E} - \frac{\mu(\sigma_x + \sigma_z)}{E}$$

$$e_z = \frac{\sigma_z}{E} - \frac{\mu(\sigma_x + \sigma_y)}{E}$$

Putting in eqn (1) —

$$\boxed{\frac{dV}{V} = \frac{1}{E}(\sigma_x + \sigma_y + \sigma_z)(1-2\mu)}$$

* Volumetric strain of a cylindrical rod:—

d = diameter.

L = length.

$$V_i = \frac{\pi}{4} d^2 L$$

$$V_f = \frac{\pi}{4}(d - \delta d)^2 (L + \delta L)$$

$$V_f = \frac{\pi}{4}(d^2 L - 2d \times L \times \delta d + d^2 \times \delta L)$$

$$dV = V_i - V_f = \frac{\pi}{4}(d^2 \times \delta L - 2d \times L \times \delta d)$$

$$\boxed{dV = \frac{\pi}{4}(d^2 \times \delta L - 2d \times L \times \delta d)}$$

Volumetric strain –

$$e_v = \frac{dV}{V}$$

$$\boxed{e_v = \frac{\delta L}{L} - 2\frac{\delta d}{d}}$$

* ∴ Volumetric strain = strain in L – twice strain in diameter

* Bulk Modulus –

$$\boxed{K = \frac{\text{Direct stress}}{\text{Volumetric strain}} = \frac{\sigma}{\left(\frac{dV}{V}\right)}}$$

* $\boxed{E = 3K(1-2\mu)}$

* $\boxed{\mu = \frac{3K-E}{6K}}$

* Modulus of rigidity –

$$\boxed{C = \frac{E}{2(1+\mu)}}$$

* Resilience :- The total strain Energy stored in a body.

* Proof Resilience :- Max. strain. E stored in a body

* Modulus of Resilience = $\frac{\text{Proof Resilience}}{\text{Volume of body}}$