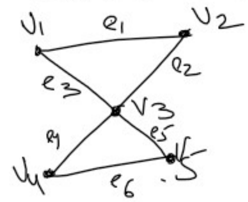
Eulerian Graph.

A circuit in a connected graph is an Euler circuit if it contains every edges of the graph exactly once.
A connected graph with an Eulers circuit is called an Eulerian graph / Euler graph.

$v_0 - e_1 - v_1 - e_2 - v_3 - e_4 - v_0$

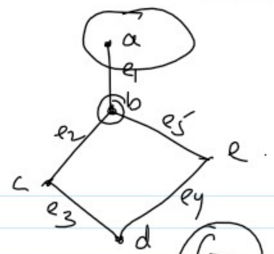
Thm :- A non-empty connected graph G is Eulerian if and only if its vertices are all of even degree.



$v_1 - e_3 - v_3 - e_5 - v_5 - e_6 - v_4 - e_4 - v_3 - e_2 - v_2 - e_1 - v_1$.

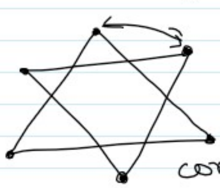
Eulerian Trail.

$a \xrightarrow{e_1} b \xrightarrow{e_2} c \xrightarrow{e_3} d \xrightarrow{e_4} e \xrightarrow{e_5} b$

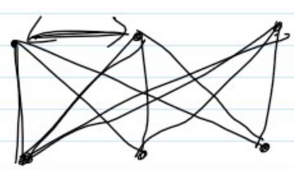


G

Thm :- A connected graph contains an Euler trail but not an Euler circuit iff it has exactly two vertices of odd degree.



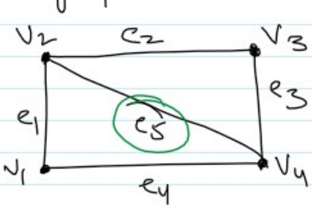
connected - NO
Euler graph Yes | NO



Not an Euler graph.

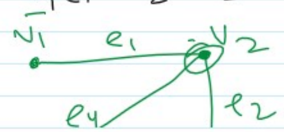
Hamiltonion Graph.

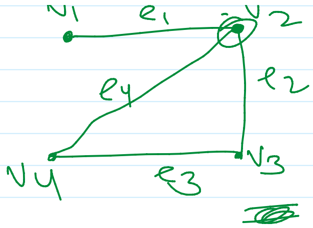
A circuit in a graph G that contains each vertex in G exactly once, except for the starting and the ending vertex that appears twice is known as Hamiltonian cycle.
A graph with Hamiltonion cycle is called Hamiltonian Graph.



→ Hamiltonian Graph.

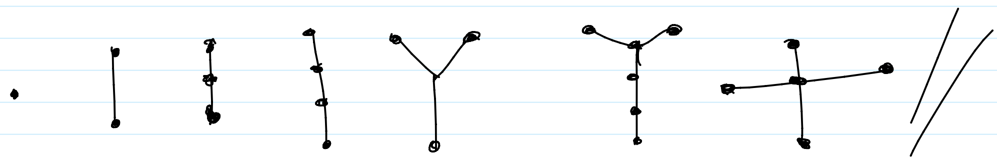
$v_1 e_1 - v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_1$.





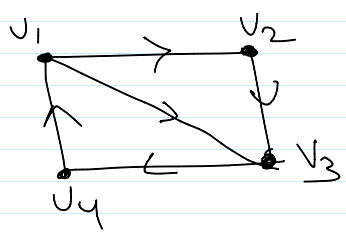
→ Not a Hamiltonian Graph.

Tree :— A tree is connected acyclic graph (a connected graph with no cycle)
Its edges are known as branches.



A collection of tree is known as Forest.

Matrices of a graph :-



Adjacency Matrix

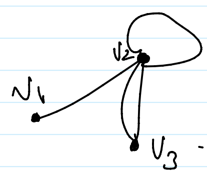
Non-direct.

	v_1	v_2	v_3	v_4
v_1	0	1	1	1
v_2	1	0	1	0
v_3	1	1	0	1
v_4	1	0	1	0

Directed graph.

	v_1	v_2	v_3	v_4
v_1	0	1	1	0
v_2	0	0	1	0
v_3	0	0	0	1
v_4	1	0	0	0

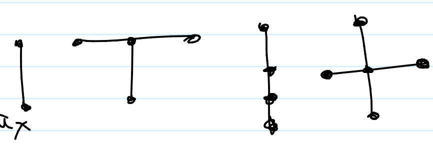
Adjacency Matrix



14/06/21

Tree

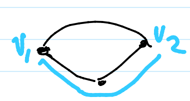
A tree is connected acyclic graph.
It edges are called branches.



Trivial tree :- A tree with only one vertex otherwise it is non-trivial.

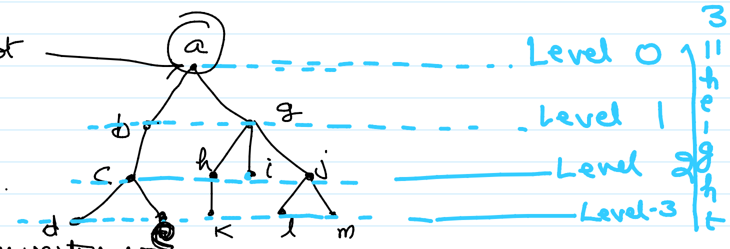
Thm 1 The is one and only path b/w every pair of vertices in a tree T.

Thm 2 A tree T with n-vertices has n-1 edges.



Thm 3 For any +ve integer n, if G is a connected graph with n-vertices and n-1 edges, then G is a tree.

Rooted Tree.



Level :- Level of a vertex is the number of edges along the unique path b/w it & the root.

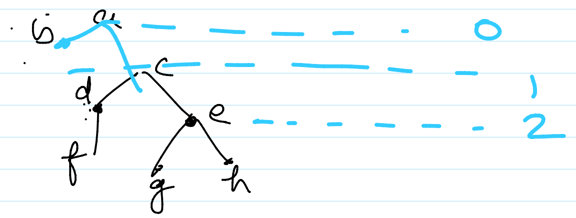
Height :- The max level to any vertex of Tree.

depth :- It is the length of the path from root to the vertex

Children :- Children of V are all those vertices that are adjacent to v and that one level away from the root than V.

Leaf (Pendant | terminal Vertex). If a vertex u has no children, then u is called Leaf.

Descendants :- Descendants of a vertex u is set consisting all the children of u together with the descendant of those children.



What is root of T
Ans a

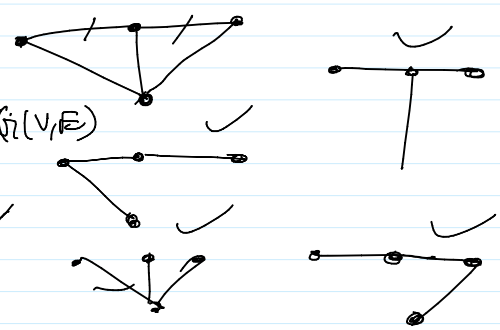
Q) What are leaf of c & e.

Q Descendant of 'a' are b, c, d, e, f, g, h.

" " c d, e, f, g, h.

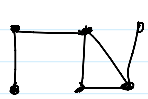
level of c = 1
" " e = 2.

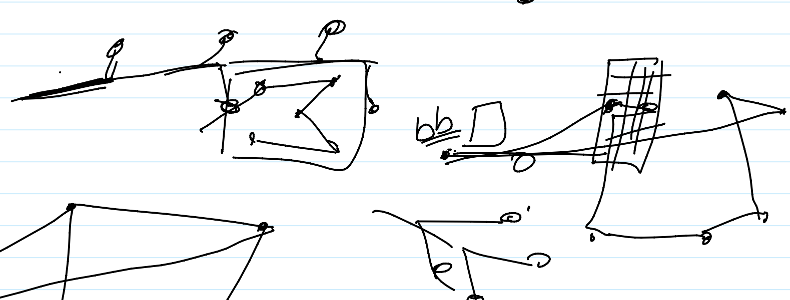
Spanning Tree.



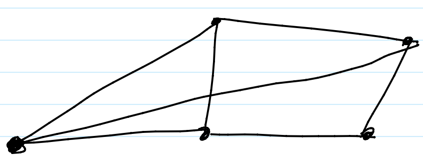
A subgraph T of a connected graph G(V,E) is called a spanning tree if

1) T is a tree ✓
2) T includes every vertices of G.

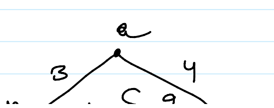




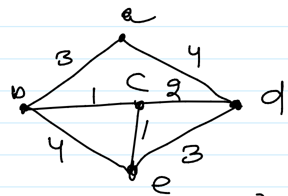
Minimal spanning tree.



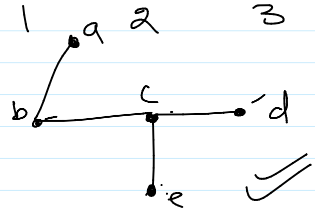
1) Kruskal Algorith..



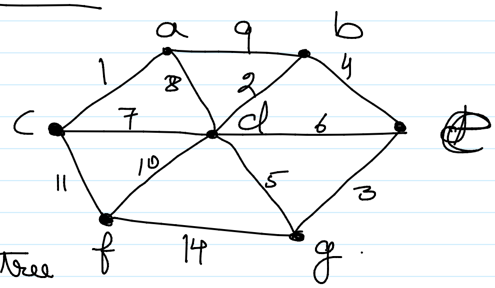
1) Kruskal Algorith..



Edges	(b,c)	(c,e)	(c,d)	(a,b)	(e,d)	(a,d)	(b,e)
Weight	1	1	2	3	3	4	4



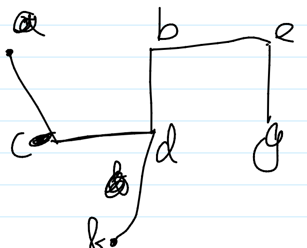
15-06-21 Kruskal's Algorithm

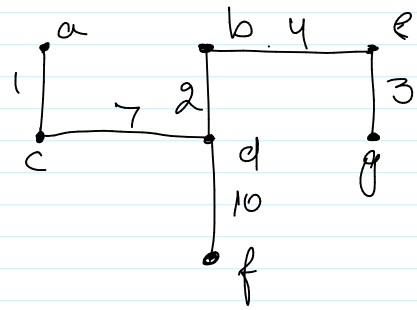


Q. Find minimal spanning tree with the help of Kruskal Algorithm.

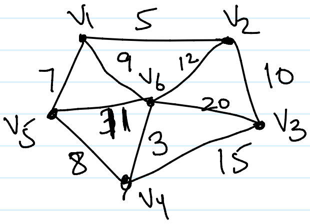
Sol^n step

Edge:	(a,c)	(b,d)	(e,g)	(b,e)	(d,g)	(d,e)	(d,c)	(a,d)	(a,b)	(f,d)	(c,f)	(f,g)
Weight:	1	2	3	4	5	6	7	8	9	10	11	14



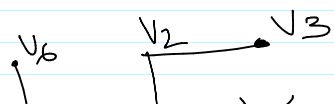


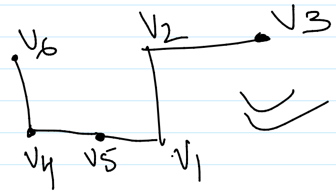
Ex



Step I

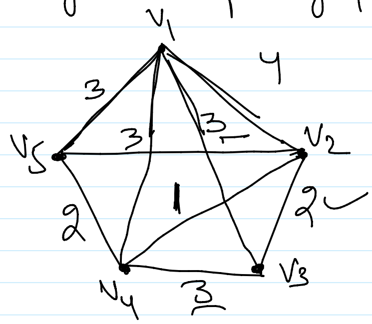
Edges	(V4,V6)	(V1,V2)	(V1,V5)	(V5,V4)	(V1,V6)	(V2,V3)	(V5,V6)	(V2,V6)	(V3,V4)	(V3,V6)
Weights	3	5	7	8	9	10	11	12	15	20

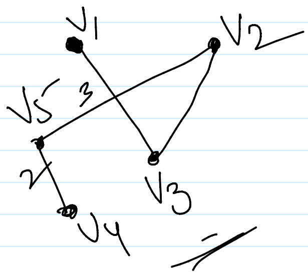


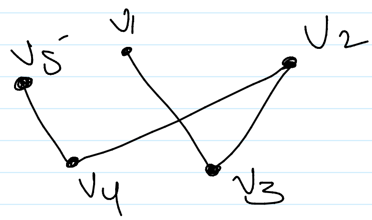


Prim's Algorithm.

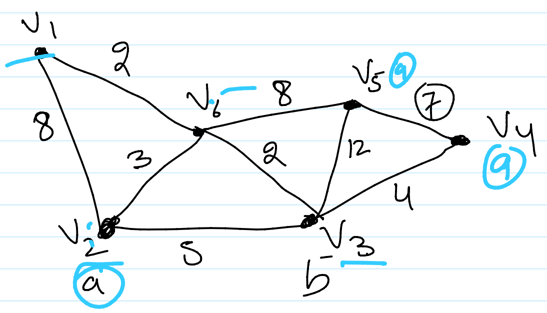
Find minimal spanning tree from graph using Prim's Algo.

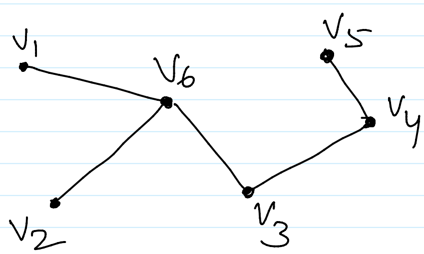






(2)



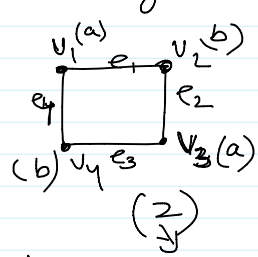


Graph Coloring.

Graph coloring is an assignment of colours to element of a graph subject to certain constraints

The starting point of graph coloring is vertex colouring.

Vertex Coloring -

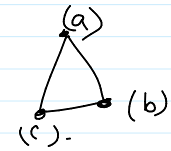


A graph in which every vertex has been assigned a colour according to a proper colouring is called properly coloured graph.

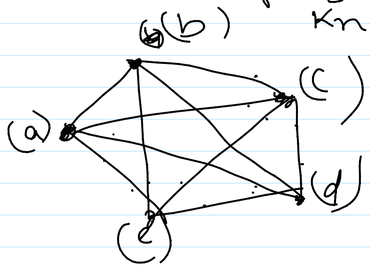
Chromatic Number - It is the minimum number of colours needed for a proper colouring i.e the min. number of colours needed to assign colours to each vertex of graph G s.t no

Chromatic Number - It is the minimum number of colours needed for a proper colouring i.e the min. number of colours needed to assign colours to each vertex of graph G s.t no two adjacent vertices are of same colour.
It is denoted by $\chi(G)$

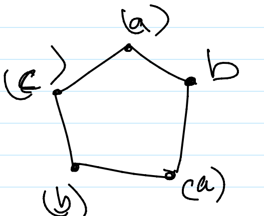
* In graph colouring, we consider the colouring of simple connected graphs only.
* The chromatic number of null graph is 1.
* A graph consisting of only isolated vertices is 1-chromatic



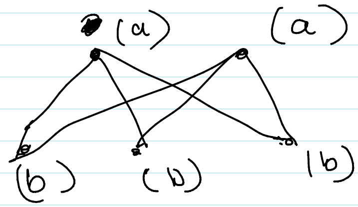
Chromatic number of K_5 = 5.
K_n = n.



Chromatic number of a Bi-Partite graph ($K_{m,n}$)



$\chi(\ ) = 3.$

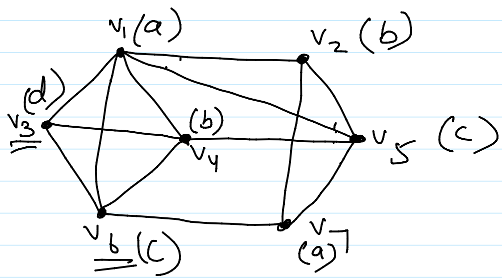


$K_{2,3}$

$\{a, b\}$

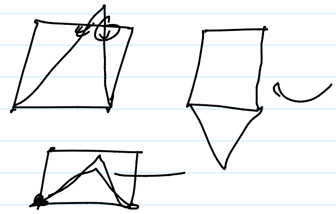
Welch – Powell Algo.

Q. For the given graph Use Welch Powell Algorithm to colour G and find $\chi(G)$



Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
degree	5	4	4	4	3	3	3
Colour.	a	b	c	c	b	d	a

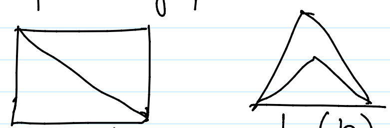
Planar Graph.



A graph G is said to be planar if there exists some geometric representation of G which can be drawn on the plane s.t no two edges intersect except only at the common vertex. The point of intersection are called crossover.

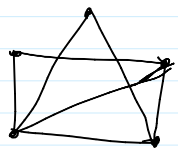
A graph that cannot be drawn on a plane without a crossover b/w its edges is called a non-planar graph.

A graph that cannot be drawn on a plane without a crossover b/w its edges is called a non-planar graph.

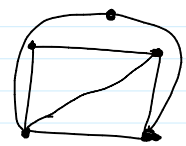


(a) (b)
planar graph.

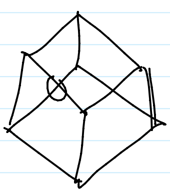
<u>Embedding</u>. A drawing of geometric represent. of a graph on any surface s.t no edges intersect is called embedding

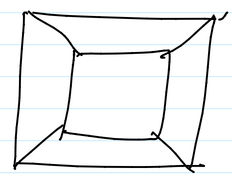


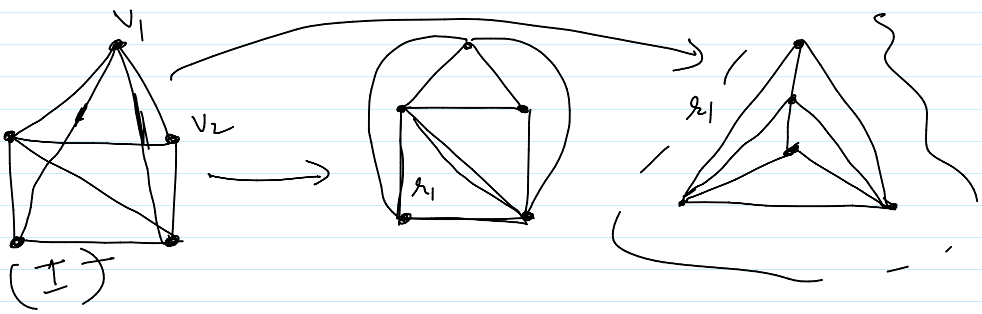
non-planar (a)
(G)



(b)
planar representation of G.







19/06/21.

<u>Region of a Graph</u>.

A region of a planar graph is defined as the area of the plane that is bounded by edges and is not further divided into subareas.

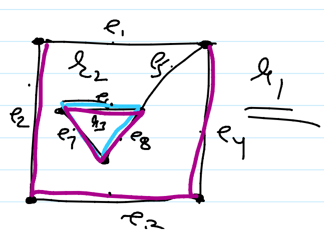
The set of edges which bounds a region is called its boundaries.

If the area of the region is finite then it's called a finite region.
" " region is infinite it is called infinite region.

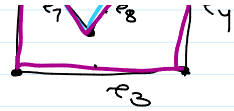
<u>Degree of region</u>

It is the number of encounters with edges in a walk round the boundary

G has three region namely r_1, r_2 & r_3
of which two are finite i.e. r_2 & r_3.
One is infinite i.e. r_1.



of which two are finite i.e. r_2 & r_3.
One is infinite i.e r_1.



The infinite region r_1 is characterised by $\boxed{G}$
the set $\{e_1, e_2, e_3, e_4\}$, hence it has degree 4.

Region r_3 has degree = 3.
" r_2 " degree = 9.

Euler's Formula.

Statement: If a connected planar graph G has n-vertices, e edges and r regions then $\boxed{n - e + r = 2}$.

Ex A connected plane graph has 10 Vertices each of degree 3. In how many regions does a the representation of this planar graph splits the plane.

Sol^n Here $n = 10$ and degree of each vertex is 3.

$$\Sigma \deg(V) = 3 \times 10 = 30.$$

But $\Sigma \deg(V) = 2e$.

$$\Rightarrow e = 15$$

By Euler's Formula

$$n - e + r = 2$$

$$10 - 15 + r = 2$$

$$-5 + r = 2$$

$$r = 7.$$

Group Theory.

Binary Operations.

*

Let G be an non-empty set.

then $G \times G = \{(a,b) ; a \in G, b \in G\}$.

If $f : G \times G \to G$ then 'f' is said to be binary operations on G

Thus binary operator is fun that assign each ordered pairs of elements of G a unique element of G

Algebraic Structure

A non-empty set together with one or more than one binary operator is called algebraic structure.

$(N, +)$, $(Z, +)$, $(R, +, \cdot)$ are all algebraic structure

Associative Law.

$$a * (b * c) = (a * b) * c.$$

Commutative Law.

$$a * b = b * a.$$

Identity Element.

An element $e \in S$ is called an identity elm. w.r.t $*$ if for any element $a \in S$

$$a * e = e * a = a.$$

Inverse Element.

$$a * b = b * a = e.$$

b is inverse of a. where $a \in S$, $b \in S$.

21/06/21 Group

Let $(G, *)$ be an algebraic structure, $*$ is binary operator, G is non-empty set

$(G, *)$ is called a group.

1.) Closure Property. $\forall\ a, b \in G$, $a * b \in G$.

2.) Associative Property $a * (b * c) = (a * b) * c$.

3.) Identity element $e * a = a * e = a$ $\forall\ a \in G$, $e \in G$

4.) Inverse element $\rightarrow$ $a * b = b * a = e$ $\forall\ a, b \in G$, $e \in G$

b is called inverse of 'a'.

Abelian Group

Any Group 'G' is called abelian if it satisfy commutative laws i.e $a * b = b * a$ $\forall\ a, b \in G$.

A group with addition binary operator is known as additive group

" " multiplication " " " " " multiplicative "

Ex 1.) R (set of real number) is a group with addition as binary operator

$(R, +)$ where 0 is the identity element

$-a$ " inverse "

Q. Prove that the fourth root of unity form abelian multipli. group.

Sol^n^ Let $G = \{1, -1, i, -i\}$.

To show $(G, \times)$ is an abelian Group.

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	-1	-1	1
$-i$	$-i$	i	1	-1

i.

$$\begin{array}{c|cccc} -1 & -1 & 1 & -i & i \\ i & i & -1 & -1 & 1 \\ -i & -i & i & 1 & -1 \end{array}$$

$i.$

$a \in G$
$b \in G$
$a \times b \in G$ - (closure)

1) Closure Property :- since all elements of composition tables belongs to G
$\therefore$ G is closed w.r.t multiplication.

b) Commutative Law. : $ab = ba \quad \forall\, a, b \in G$

From composition table it is clear that elements in each row are same as in the corresp. columns. So $ab = ba$.

c) Associative Law - : $a(bc) = (ab)c \quad \forall\, a, b, c \in G$.

$1[(-1)i] = -i = [1(-1)]i \quad \forall\, 1, -1, i \in G$

d) Identity element - $\Rightarrow$ 1 is the identity element.
as $1 \cdot a = a \cdot 1 = a$.

$a * b = e$

e) Inverse element.

$1 \cdot 1 = 1$	$\Rightarrow$	1 is ~~the~~ idnt. elmt		1
$i \cdot (-i) = 1$		$-i$ //	// //	i
$-i\ (i) = 1$		-1 //	// //	-1
$(-1)(-1) = 1$		i //	// //	$-i$

$\therefore$ G is an abelian multi. group.

Q. Show that the matrices

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A}, \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{B}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{C}, \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{D}.$$

forms multi. abelian group.

$\times$	A	B	C	D
A	A	B	C	D
B				
C				
D				

Groupoid

An algebraic structure $(G, *)$ is called groupoid.

Semi-Group.

A algebraic struc. $(G, *)$ is called semi-group.
1) Closure $\quad a * c$
2) Associative $\quad a*(b*c) = (a*b)*c \quad \forall\, a, b, c \in G$

$S = \{0, 1, 2, 3, 4\}$

*2) Associative $a*(b*c) = (a*b)*c \ \forall a,b,c \in G$

$S = \{0,1,2,3,4\}$

is a abelian group addition modulo 5

$+_5$

$2 +_5 4 = 1$

$2 +_5 3 = 0$

$3 +_5 4 = 2$

$4 +_5 4 = 3$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Q. $a * b = e$

$a +_5 b = 0$

$1 +_5 \_ = 0$

i) Closure :— All elements of composition table belong to S

$\therefore$ S is closed.

ii) Commutative : $a +_5 b = b +_5 a$

$3 +_5 4 = 2$

$4 +_5 3 = 2$

iii) Identity element :— 0 is the identity element

iv) Inverse elemts : inverse of

1 is 4

0 is 0

2 is 3

3 is 2

4 is 1

Hence $(G, +_5)$ is an abeligean group

Q. Check whether $G = \{1,2,3,4,5\}$ form a group with binary oper. addition modulo 6 $(+_6)$

$+_6$	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

(1) Closure ;— $0 \notin G$.

$a +_6 b \in G \ \forall a \in G, b \in G$

22/06/21.

Order of an element.

Order of an element in a group G is the smallest positive integer 'm' s.t $g^m = e$

For $G = \{1, -1, i, -i\}$ be a multiplicative group.

sol^n We know identity element of G is 1.

1) $1^1 = 1 \Rightarrow o(1) = 1$

2) $(-1)^2 = 1 \Rightarrow o(-1) = 2$.

3) $(i)^4 = 1 \Rightarrow o(i) = 4$

4) $(-i)^{4} = 1 \Rightarrow o(-i) = 4$.

Sub-group.

Let $(G, *)$ be a group and H is a subset of G.

$(H, *)$ is said to be subgroup of G if $(H, *)$ is also group by itself.

Ex $\underbrace{(H = \{1, -1\}, \times)}_{\text{Subgroup of } G}$, $(G = \{1, -1, i, -i\}, \times)$

Thm :- The necessary and sufficient condition for a non-empty subset H of a group $(G, *)$ to be a sub-group is

$$a \in H, b \in H \Rightarrow a * b^{-1} \in H.$$

~~b*a=b~~

Coset.

Let H be a subgroup of a group G and let $a \in G$

then the set $\{a * h : h \in H\}$ is called left coset generated by a & is denoted by aH

" " " $\{h * a : h \in H\}$ " " Right Coset generated by 'a', denoted by Ha

aH & Ha are subsets of G.

If 'e' be the identity element of G, then $e \in H$ and $He = H = eH$

$\therefore$ H is right as well as left coset.

In general $aH = Ha$ but in abelian group each left coset coincides with corresp. right coset.

Ex Let G be an additive group ie

$$G = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

Let H be a subgroup of G obtained by mult. each element of G by 3

$$H = \{\cdots, -9, -6, -3, 0, 3, 6, 9, \cdots\}$$

Since the group G is abelian, any right coset will be equal corresp. left coset

$$0 \in G$$

$$H = H+0 = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$$

Again $1 \in G$, $H+1 = \{\dots, -8, -5, -2, 1, 4, 7, 10, \dots\}$

$2 \in G$, $H+2 = \{\dots, -7, -4, -1, 2, 5, 8, 11, \dots\}$

$H, H+1, H+2$ are distinct.

$3 \in G$, $H+3 = \{\dots -9, -6, -3, 0, 3, 6, 9, 12, \dots\} = H$

$$H+3 = H$$

$4 \in G$, $H+4 = H+1$.

$\therefore$ There exist three disjoint right coset namely $H, H+1, H+2$

$$G = H \cup (H+1) \cup (H+2)$$

index of H is $\underline{\underline{3}}$

2) 24-06-21. Find the order of each element in the multiplicative Group. $G = \{1, \omega, \omega^2\}$.

Sol: The identity of the given multiplication group G is 1

$\omega^3 = 1$ $\quad o(\omega) = 3$, $\quad o(\omega^2) = 3$

$1^1 = 1$ $\quad o(1) = 1$

$(\omega^2)^3 = \omega^6 = \omega^3 \cdot \omega^3 = 1$

Cyclic Group.

A group G is said to be cyclic if all its elements can be generated by a single element.

In otherwords, a group G is cyclic if some $a \in G$, every element $x \in G$ is of the form a^n, where n is some integer

The element 'a' is called the generator of G.

Ex1 $G = \{1, \omega, \omega^2\}$ is a cyclic group.

generator $= \omega$ & ω^2. $\quad \boxed{1+\omega+\omega^2=0}$

$\omega^3 = 1$, $\omega^2 = \omega^2$ $\quad \omega =$

$(\omega^2)^3 = 1$, $\omega^4 = \omega^3 \cdot \omega = \omega$ $\quad \omega =$

$\omega^3 = 1$, $\omega^2 = \omega^2$ $\qquad \omega =$

$(\omega^2)^3 = 1$, $\omega^4 = \omega^3 \cdot \omega = \omega$ $\qquad \omega =$

Ex-2. $G = \{1, -1, -i, i\}$

generators :- $i, -i$

$(i)^2 = -1$ ✓	$(-i)^2 = -1$ ✓
$(i)^3 = -i$ ✓	$(-i)^3 = i$ ✓
$(i)^4 = 1$ ✓	$(-i)^4 = 1$ ✓

Thm If $a^2 = e$ then G is an abelian group. $(\forall a \in G)$

Sol^n Let a & b be two elements the group G.

then ab is also an element of G

$\therefore (ab)^2 = e$.

$(ab)(ab) = e$ $\qquad \boxed{a \cdot b = e}$. b is inverse 'a'.

$\Rightarrow (ab)^{-1} = ab$.

$\Rightarrow b^{-1} a^{-1} = ab$ —— $(*)$

$b \cdot a = ab$.

But $a^2 = e \Rightarrow a \cdot a = e \Rightarrow a^{-1} = a$

Similarly $b^{-1} = b$. $\qquad \Big] (**)$

$\Rightarrow ab = ba$. ||

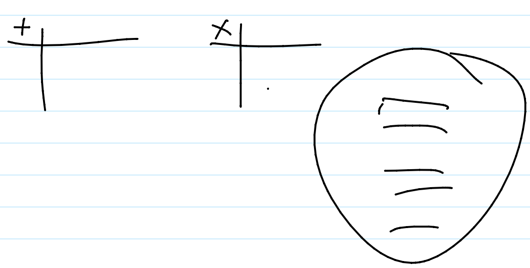
$\Rightarrow G$ is an abelian group.

Q. Show that group $\{\{1, 2, 3, 4\}, \times_5\}$ is cyclic.

$2 \times_5 4 = 3$

$2 +_5 4 = 1$

$2 \times_5 4$



(2) Check whether $\{(\ \ -), \times_5\}$ is an abelian group.

(1) Group

(2) Commut.

Normal Subgroup.

A subgroup H of a group G is said to be a normal subgroup if $Ha = aH$ $\forall a \in G$

Thm :- A subgroup H of a group G is normal if and only if

Thm:- A subgroup H of a group G is normal if and only if $g^{-1}hg \in H \;\forall\; h \in H, g \in G$.

Ring

A non-empty set 'R' together with two binary operations $+$ (addition) & $\cdot$ (multiplication) is called a ring if it satisfy the following conditions

1) $(R, +)$ is an abelian group — closure prop., Associative, Ident., Inverse, commutative.
2) $(R, \cdot)$ is semi-group — Closure, Associat.
3) The operation is distributive over the operation $+$.

$$a\cdot(b+c) = a\cdot b + a\cdot c \quad \forall\, a,b,c \in R.$$
$$(b+c)\cdot a = b\cdot a + c\cdot a.$$

Commutative Ring.

A ring R is commut. ring if $a\cdot b = b\cdot a \;\forall\; a,b \in R$.

Ring with unity.

A ring R is said to be a ring with unity if it satisfy.
$e\cdot a = a\cdot e = a \;\forall\, a \in R$.

Ex(1) $(\mathbb{Z}, +, \cdot)$ is a commutative ring

2) $(2\mathbb{Z}, +, \cdot)$ " " "

3) $\mathbb{Z}_n = \{0,1,2,3, \ldots, n-1\}$ under addit. & multi. modulo is commut. ring with unity 1.

Field

$(R, +, \cdot)$ is a field if
1) $(R, +)$ is an abelian group
2) $(R, \cdot)$ " " commut. group
3) Distributive.

1) $(\mathbb{Q}, +, \cdot)$ is a field
2) $(\mathbb{R}, +, \cdot)$ is a field.
3) '$\mathbb{C}$' - complex num. is field.

an integral domain.