BS Grewal → Complex F.

unit-1

COMPLEX NUMBERS

If z is a complex variable, then w is the function of z.

$w = f(z)$
$= u + iv$.

or

$w = u(x,y) + iv(x,y)$.

$f(z) = z+3$.
$= \underbrace{(x+3)}_{u} + i\underbrace{(y)}_{v}$

Theorem :-

The necessary and sufficient condition for the derivative of $f(z)$ to exist is

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

(ii) $\left.\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right\}$ C.R eqⁿ

Cauchy - Riemann eqⁿ

Proof :- let $f'(z)$ exists

To show that :- (1), (2) holds

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$z = x + iy$

$\delta z = \delta x + i\delta y$

$f(z) = u(x,y) + iv(x,y)$

$$\cancel{= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} u(x+\delta x, y+\delta y)}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - (u(x,y) + iv(x,y))}{\delta x + i\delta y} \quad \text{—— (1)}$$

let $\delta z = 0$ along real axis, $\delta y = 0$, (1) gives :-

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - (u(x,y) + iv(x,y))}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(u(x+\delta x, y) - u(x,y)) + i(v(x+\delta x, y) - v(x,y))}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad — ②$$

let $\delta z \to 0$ along imag. axis, $\delta x = 0$. ① gives :-

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) + iv(x, y+\delta y) - (u(x,y) + iv(x,y))}{i\delta y}$$

$$= \frac{1}{i} \lim_{\delta y \to 0} \frac{(u(x, y+\delta y) - u(x,y)) + i(v(x, y+\delta y) - v(x,y))}{\delta y}$$

$$= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad — ③$$

from ② and ③

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$$

Converse :-

proof : 1, 2 holds.

To show that : $f'(z)$ exists

$$f(z+\delta z) = u(x+\delta x,\, y+\delta y) + iv(x+\delta x,\, y+\delta y)$$

$$= u(x,y) + \left(\delta x \frac{\partial u}{\partial x} + \delta y \frac{\partial u}{\partial y}\right) + \cdots$$

$$+ i\left(v(x,y) + \left(\delta x \frac{\partial v}{\partial x} + \delta y \frac{\partial v}{\partial y}\right)\right) + \cdots$$

neglecting higher power of $\delta x, \delta y$

$$= (u(x,y) + iv(x,y)) + \delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$+ \delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= \delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \delta y\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)$$

$$= \delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\delta y\left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y)$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$\delta z \to 0$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Hence, proved.

Analytic function

A function which is differentiable at every point of a region is called analytic function / Regular / Holomorphic.

→ Singular point :-
A point is called a singular point if the function is not analytic at that point.

→ Conjugate function
The real and imaginary parts of an analytic function $f(z) = u + iv$, are called conjugate functions.

→ Harmonic function
A real valued function $u(x,y)$ is called harmonic in a region R if all its 2nd order partial derivatives exist and are continuous and satisfy Laplace eqn, i.e $u_{xx} + u_{yy} = 0$.

→ Conjugate harmonic function

If real and imag. parts of an analytic fn satisfy Laplace eqn then they are called conjugate harmonic fn.

Q. Show that the real and imag parts of an analytic f^n satisfy Laplace eq^n.

Given:-

$f(z) = u + iv$, analytic

CR eq^n holds.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (2)}$$

diff (1) & (2) partially wrt x & y

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

Adding corresponding sides

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u satisfies Laplace eq^n.

diff (1) & (2) wrt y & x.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$$

$$-\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x^2}$$

Adding -

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0$$

$\therefore$ v satisfies Laplace eq^n.

Polar form of CR eq^n :-

if $z = x + iy$, let $x = r\cos\theta$, $y = r\sin\theta$

$$z = r\cos\theta + ir\sin\theta$$
$$= re^{i\theta}$$

$$\therefore f(z) = f(re^{i\theta})$$
$$u + iv = f(re^{i\theta}) \text{ ——— } ①$$

Diff partially ① wrt r & θ.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \text{ —— } ②$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial u}{\partial \theta} = f'(re^{i\theta}) \cdot ri e^{i\theta} \text{ — } ③$$

put ② in ③

$$\frac{\partial u}{\partial \theta} + \frac{i\partial v}{\partial \theta} = ri\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

$$\therefore \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

Orthogonal System

Given $f(z) = u(x,y) + i\,v(x,y)$

let $u(x,y) = C_1$
$v(x,y) = C_2$

TST: $m_1 m_2 = -1$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} = 0$$

$$\therefore \frac{\partial y}{\partial x} = -\frac{\partial u/\partial x}{\partial u/\partial y} = m_1$$

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} = 0$$

$$\therefore \cancel{\frac{\partial v}{\partial y}}\cdot\frac{\partial y}{\partial x} = \frac{-\partial v/\partial x}{\partial v/\partial y} \quad \cancel{= m_2}$$

$\because f(z)$ is analytic, using CR eq^n

$$\frac{\partial y}{\partial x} = \frac{\partial u/\partial y}{\partial u/\partial x} = m_2$$

$\therefore m_1 m_2 = -1$

$\therefore$ Real & imag parts of an analytic f^n forms an orthogonal f^n system.

Construction of analytic function.

→ Milne Thompson's Method

1. Let $f(z) = u + iv$

$$\left\{ z = x + iy \ , \ \bar{z} = x - iy \right.$$

$$\therefore x = \frac{z+\bar{z}}{2} \ ; \ y = \frac{z-\bar{z}}{2i}$$

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

put $z = \bar{z}$

$$f(z) = u(z,0) + iv(z,0)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

② ~~f u~~ If u is given then -

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Let $\phi_1(x,y) = \frac{\partial u}{\partial x}$

$\phi_2(x,y) = \frac{\partial u}{\partial y}$

$$f'(z) = \phi_1(x,y) - i\phi_2(x,y)$$

If v is given :-

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \psi_1(x,y) + i\psi_2(x,y)$$

$x \to z,\ y \to 0$

$f'(z) = \psi(z,0) + i\psi_2(z,0)$

integrate

③ Replace $x \to z$ $y \to 0$

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

on integration,

$$f(z) = \int \phi_1(z,0) - i\phi_2(z,0)\, dz + c$$

Q1. Find the analytic f^n whose real part is

$$u = e^{-x}\left((x^2-y^2)\cos y + 2xy \sin y\right)$$

Let $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = -e^{-x}\left[(x^2-y^2)\cos y + 2xy\sin y\right]$$
$$+ e^{-x}\left[2x\cos y + 2y\sin y\right] = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^{-x}\left[(x^2-y^2)(-\sin y) + (-2y)\cos y + 2x\sin y + 2xy\cos y\right]$$
$$= \phi_2(x,y)$$

Replace $x \to z$, $y \to 0$.

$$\phi_1(z,0) = -e^{-z}\left[z^2 \cdot 1 + 0\right] + e^{-z}\left[2z \cdot 1 + 0\right]$$
$$= e^{-z}\left[2z - z^2\right]$$

$$\phi_2(z,0) = e^{-z}[0]$$
$$= 0$$

$$\therefore f'(z) = \phi_1(z,0) - i\,\phi(z,0)$$

on integration,

$$f(z) = \int \underset{\text{I}}{e^{-z}}\,\underset{\text{II}}{(2z-z^2)}\,dz + c \qquad (z^2 \cdot e^{-z} + c)$$

$$= -e^{-z}(2z-z^2) - \int (2-2z)\,e^{-z}\,dz$$

$$= -e^{-z}(2z-z^2) - \int$$

Q. $u = x^3 + 3xy^2$ is harmonic (show that) find corresponding $f(z)$.

TST: $u_{xx} + u_{yy} = 0$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy \rightarrow \frac{\partial^2 u}{\partial y^2} = -6x.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is harmonic.

let $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \left(\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right)$$

let $\phi(x,y) = 3x^2 - 3y^2$ & $\phi_2(x,y) = -6xy$

Replace $x \to z$, $y \to 0$

$\phi_1(z,0) = 3z^2$
$\phi_2(z,0) = 0$

$\therefore f'(z) = 3z^2$

On integration,

$$f(z) = \int 3z^2 dz + c \quad = z^3 + c$$

Q3. PT $u = x^2 - y^2$ and $v = \frac{-y}{x^2+y^2}$ are harmonic but they are not harmonic conjugate.

$u = x^2 - y^2 \qquad v = \frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial^2 u}{\partial y^2} = -2$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \therefore u$ is harmonic.

$\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2 \cdot (-2y) - (-2xy) 2(x^2+y^2) \times 2x}{(x^2+y^2)^4}$

$= \frac{(x^2+y^2)(-2y) + 8x^2y}{(x^2+y^2)^3}$

$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$

$\frac{\partial v}{\partial y} = \frac{1(x^2+y^2) - (y) \cdot 2y}{(x^2+y^2)^2}$

$= \frac{x^2 + y^2 - 2y^2}{(x^2+y^2)^2} = \cancel{\frac{y^2 - x^2}{(x^2+y^2)^2}} \quad \frac{x^2 - y^2}{(x^2+y^2)^2}$

$\frac{\partial^2 v}{\partial y^2} = \frac{(-2y)(x^2+y^2)^2 - (x^2-y^2) 2(x^2+y^2)(2y)}{(x^2+y^2)^4}$

$= \frac{(x^2+y^2)(-2y) - (x^2-y^2)(4y)}{(x^2+y^2)^3} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \therefore v \text{ is harmonic.}$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

Since CR equations are not holding true, u, v are not harmonic conjugates.

Q4. Construct an analytic function whose real part is

$$u = e^x(x\cos y - y\sin y)$$

Let $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \cancel{e^x(x\cos y - y\sin y) + e^x(-x\sin y - y\cos y)}$$

$$= e^x(x\cos y - y\sin y) + e^x(\cos y) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = \phi_2(x,y)$$

Replace $\rightarrow x \rightarrow z,\ y \rightarrow 0$

$$\phi_1(z,0) = e^z(z) + e^z = e^z(z+1)$$

$$\phi_2(z,0) = e^z(0) = 0$$

$\Rightarrow f'(z) = e^z(1+z)$

on int,

$$f(z) = \int \underset{I}{(1+z)}\, \underset{II}{e^z}\, dz + c$$

$$= (1+z).e^z - \int 1.e^z$$

$$= (1+z).e^z - e^z + c$$

$$= z.e^z + c$$

Q5 Find the analytic f^n whose imag part is

$$v = e^x \sin y$$

Let $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = e^x \cos y = \varphi_1(x,y)$$

$$\frac{\partial v}{\partial x} = e^x \sin y = \varphi_2(x,y)$$

Replace $x \to z, y \to 0$

$\varphi_1(z,0) = e^z$

$\varphi_2(z,0) = 0$

$\therefore f'(z) = e^z$

On int:

$$f(z) = \int e^z dz$$

$$= e^z + c.$$

Q6. $v = \dfrac{x-y}{x^2+y^2}$

$f(z) = u + iv$, $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{(-1)(x^2+y^2) - (x-y)(2y)}{(x^2+y^2)^2} = \varphi_1(x,y)$$

$$\frac{\partial v}{\partial x} = \frac{1\cdot(x^2+y^2) - (x-y)(2x)}{(x^2+y^2)^2} = \varphi_2(x,y)$$

Replace $x \to z, y \to 0$

$\varphi_1(z,0) = \dfrac{-z^2}{z^4} = \dfrac{-1}{z^2}$

$\varphi_2(z,0) = \frac{z^2 - (z)(2z)}{z^4} = \frac{-z^2}{z^4} = \frac{-1}{z^2}$

$\therefore f'(z) = \frac{-1}{z^2} - i\frac{1}{z^2}$

$= \frac{1}{z^2}(-1-i)$

$f(z) = \int \frac{1}{z^2}(-1-i)\,dz + c$

$= \frac{1}{z}(1+i) + c.$

Q7. if $u - v = (x-y)(x^2+4xy+y^2)$ then find $f(z)$

$f(z) = u + iv$

$\Rightarrow if(z) = iu - v$

adding corresponding sides.

$f(z) + if(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$

$= \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$

$\frac{\partial U}{\partial x} = 1\cdot(x^2+4xy+y^2) + (x-y)(2x+4y) = \phi_1(x,y)$

$$\frac{\partial v}{\partial y} = (x-y)(4x+2y) + (-1)(x^2+4xy+y^2) = \phi_2(x,y)$$

Replace $x \to z,\ y \to 0$

$$\phi_1(z,0) = z^2 + 2\cdot zz.$$
$$= 3z^2$$

$$\phi_2(z,0) = z(4z) - (z^2)$$
$$= 3z^2$$

$$\cancel{f'(z)} \Rightarrow f'(z) = 3z^2 - i3z^2 = 3z^2(1-i).$$

On int:

$$f(z) = \int 3z^2 \cdot (1-i)\, dz + c.$$

$$= \cancel{z^3 + c}\ z^3(1-i) + c$$

$$f(z) = \frac{z^3(1-i)}{(1+i)} + \frac{c}{1+i}$$

$$= \frac{z^3(1-i)}{1+i} \times \frac{(1-i)}{(1-i)} + c'$$

$$= \frac{z^3(1-2i-1)}{2}$$

$$= \cancel{z^3(1-i) + c'}\ -iz^3 + c'$$

Q8. If $u - v = e^{x}(\cos y - \sin y)$ find $f(z)$

$$f(z) = u + iv$$
$$+ \quad if(z) = iu - v$$

$$f(z) + if(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV \quad \Rightarrow f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$\frac{\partial U}{\partial x} = e^{x}(\cos y - \sin y) = \phi_1(x,y)$$

$$\frac{\partial U}{\partial y} = -e^{x}(\sin y + \cos y) = \phi_2(x,y)$$

Replace $x \to z,\ y \to 0$

$$\phi_1(z,0) = e^{z}$$
$$\phi_2(z,0) = -e^{z}$$

$$f'(z) = e^{z} + ie^{z} = e^{z}(1+i)$$

$$F(z) = \int e^{z}(1+i)\,dz + c.$$

$$= (1+i)e^{z} + c$$

$$f(z) = \frac{(1+i)e^{z} + c}{(1+i)}$$

$$f(z) = e^{z} + c'$$

Hyperbolic function

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$, $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

☆ $d(\cosh\theta) = \sinh\theta$, $d(\overset{\sinh\theta}{\cancel{\cosh\theta}}) = \cosh\theta$

☆ $\int \cosh\theta = \sinh\theta$ $\qquad \int \sinh\theta = \cosh\theta$

☆ $\cosh^2\theta - \sinh^2\theta = 1$

$\sinh 2\theta = 2\sinh\theta \cosh\theta$

$\cosh 2\theta = 1 + 2\sinh^2\theta$
$\qquad = \cosh^2\theta + \sinh^2\theta$

$\cosh 0^\circ = 1$
$\sinh 0^\circ = 0$

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Q1. if $u+v = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ find $f(z)$

$$u+v = \frac{\cancel{2}\sin 2x}{\cancel{2}\cosh 2y - \cancel{2}\cos 2x}$$

let $f(z) = u+iv$

$if(z) = iu - v$

$f(z)(1+i) = (u-v) + i(u+v)$

$F(z) = U + iV$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$$

$$= \cancel{\frac{\partial V}{\partial y}} + \cancel{i}\cancel{\frac{\partial V}{\partial y}}$$

$$\frac{\partial V}{\partial y} = \frac{-\sin 2x \cdot 2\sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial V}{\partial x} = \frac{(\cosh 2y - \cos 2x)\cos 2x \cdot 2 - \sin 2x \cdot \sin 2x \cdot 2}{(\cosh 2y - \cos 2x)^2}$$

$\varphi_1(z,0) = 0$

$$\varphi_2(z,0) = \frac{(1-\cos 2z)\cdot 2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

$\therefore f'(z) = 0 - i\,\mathrm{cosec}^2 z$

On int:

$$f(z) = \int -i\,\mathrm{cosec}^2 z \, dz + c$$

$$(1+i) f(z) = i \cot z + c.$$

$$f(z) = \frac{i}{1+i} \cot z + \frac{c}{1+i}$$

$$= \frac{i(1-i)}{(1+i)(1-i)} \cot z + c'$$

$$= \frac{i+1}{2} \cot z + c'$$

Cauchy's Integral Theorem

If $f(z)$ is analytic and $f'(z)$ is continuous at each point within and on a closed curve C, then prove that ~~int~~

$$\oint_C f(z)\,dz = 0$$

LHS:

Let $f(z) = u + iv$, $dz = dx + i\,dy$.

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx)$$

Using Green's Theorem:

$$\left[\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy\right]$$

$$LHS = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

using CR eq^n as $f(z)$ is analytic

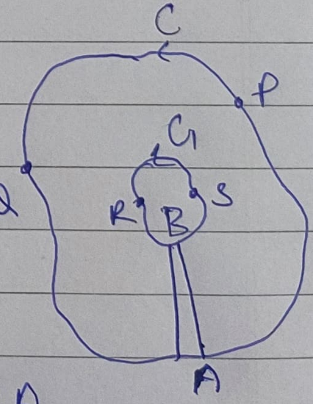
$$u_x = v_y, \quad u_y = -v_x$$

$$= 0 = RHS.$$

Corollary :-

If $f(z)$ is analytic in the region R, bounded by 2 closed curves C and C_1 then P.T.

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$



Let C_1 lie within C, draw a cross cut AB, then $f(z)$ is analytic in the region $ABRC_1SBAPCQA$ (AC_1BAC)

$\therefore$ By Cauchy's theorem

$$\oint_{AC_1BAC} f(z)\,dz = 0$$

$$\oint_{C \text{ anti clockwise}} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BA} f(z)\,dz + \oint_{C_1 \text{ clockwise}} f(z)\,dz = 0$$

$$\Rightarrow \oint_{C} f(z)\,dz = \oint_{C_1 \text{ anticlockwise}} f(z)\,dz.$$

Extension of Corollary of Cauchy's theorem

If $C_1, C_2, \ldots C_n$ be closed curves within C then

$$\oint_{C} f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz.$$