Assignment - 3

1- Find $L[e^{(a+ib)t}]$. Hence show that

$$L[e^{at}\cdot \cos bt] = \frac{s-a}{(s-a)^2+b^2}$$ &

$$L[e^{at}\cdot \sin bt] = \frac{b}{(s-a)^2+b^2}.$$

sol^n- Given-

- $$f(t) = e^{(a+ib)t}$$

$$\because L\{f(t)\} = \int_0^\infty e^{-st}\cdot f(t)\,dt$$

$$L\{e^{(a+ib)t}\} = \int_0^\infty e^{-st}\cdot e^{(a+ib)t}\,dt$$

$$= \int_0^\infty e^{-(s-(a+ib))t}\,dt$$

$$= -\left[\frac{e^{-(s-(a+ib))}}{s-(a+ib)}\right]_0^\infty$$

$$= -\left[\frac{-1}{s-(a+ib)}\right]$$

so, $$L[e^{(a+ib)t}] = 1/s-(a+ib)$$ Ans

- $L[e^{at}\cdot \cos bt]$

$$\left(\because \cos bt = \frac{e^{ibt}+e^{-ibt}}{2}\right)$$

$$L\left[e^{at}\cdot\left(\frac{e^{ibt}+e^{-ibt}}{2}\right)\right]$$

$$L\left[\frac{e^{(a+ib)t}+e^{(a-ib)t}}{2}\right]$$

$$f(t) = \frac{e^{(a+ib)t}+e^{(a-ib)t}}{2}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot \left[\frac{e^{(a+ib)t} + e^{(a-ib)t}}{2}\right] dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-[s-(a+ib)]t} + e^{-[s-(a-ib)]t}\right) dt$$

$$= \frac{1}{2}\left[\left\{\frac{e^{-[s-(a+ib)]t}}{s-a-ib}\right\}_0^\infty + \left\{\frac{e^{-[s-(a-ib)]t}}{-[s-(a-ib)]}\right\}_0^\infty\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)-ib} + \frac{1}{(s-a)+ib}\right]$$

$$= \frac{1}{2}\left[\frac{2(s-a)}{(s-a)^2+b^2}\right] \Rightarrow \frac{s-a}{(s-a)^2+b^2}$$

Hence Proved.

- $L\left[e^{at} \cdot \sin bt\right]$ $\quad \therefore \sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$$L\left[e^{at}\left(\frac{e^{ibt} - e^{-ibt}}{2i}\right)\right]$$

$$f(t) = \frac{e^{(a+ib)t} - e^{(a-ib)t}}{2i}$$

$$L\{f(t)\} = \frac{1}{2i}\int_0^\infty e^{-st}\left\{e^{(a+ib)t} - e^{(a-ib)t}\right\} dt$$

$$= \frac{1}{2i}\int_0^\infty e^{-(s-a-ib)t} dt - \int_0^\infty e^{-(s-a+ib)} dt$$

$$= \frac{1}{2i}\left[\left[\frac{e^{-(s-a-ib)t}}{-(s-a-ib)}\right]_0^\infty - \left[\frac{e^{-(s-a+ib)t}}{-(s-a+ib)}\right]_0^\infty\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-a-ib} - \frac{1}{s-a+ib}\right]$$

$$= \frac{1}{2i}\left[\frac{s-a+ib-s+a+ib}{(s-a)^2+b^2}\right]$$

$$L\{f(t)\} = \frac{b}{(s-a)^2+b^2}$$

Hence Proved.

2- Find the Laplace of the following -

- $t^3 \cdot \cos t$.

$\because f(t) = \cos t$.

$$L\{f(t)\} = \frac{s}{s^2+1}$$

so, $$L\{t^3 \cdot f(t)\} = (-1)^3 \cdot \frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$$

$$= -1\left\{\frac{d^2}{ds^2}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right\}$$

after solving,

$$= \frac{6(3s^2-1)}{(s^2+1)^4}$$ Ans

- $e^{3t} \cdot \sin t$

$\because f(t) = \sin t$

$$L\{f(t)\} = \frac{1}{s^2+1} \Rightarrow L\{e^{3t} \cdot f(t)\} = \frac{1}{(s-3)^2+1}$$

so, $$L\{e^{3t} \cdot \sin t\} = \frac{1}{s^2-6s+10}$$ Ans

- $f(t) = \begin{cases} 0 , & 0 < t < \pi \\ \sin t , & t \geq \pi \end{cases}$

<u>sol^n</u>- $L\{\sin t\} = \frac{1}{s^2+1} = f(s)$

By using 2^{nd} shifting -

$$F(t) = -\begin{cases} 0 & 0 < t \leq \pi \\ \sin(t-\pi) & t \geq \pi \end{cases}$$

$$L[f\{t\}] = -e^{-\pi s} \cdot f(s)$$

$$\Rightarrow -e^{-\pi s} \cdot \frac{1}{s^2+1}$$

- $\frac{e^{-2t} \cdot \sin 3t}{t}$

<u>sol^n</u>- $f(t) = e^{-2t} \cdot \sin 3t = e^{-2t} \cdot g(t)$

Let $g(t) = \sin 3t$

$$L\{g(t)\} = \frac{3}{s^2+9}$$

$$L\{f(t)\} = \frac{3}{(s+2)^2+9} = f(s)$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\,ds = \int_s^\infty \frac{3}{(s+2)^2+9}\,ds$$

$$= 3\left[\frac{1}{3}\tan^{-1}\left(\frac{s+2}{3}\right)\right]_s^\infty$$

$$\Rightarrow \frac{\pi}{2} - \tan^{-1}\left(\frac{s+2}{3}\right)$$

$$= \cot^{-1}\left(\frac{s+2}{3}\right)$$

Ans

- $f(t) = \begin{cases} t/a & 0 \le t \le a \\ (2a-t)/a & 0 \le t \le 2a \end{cases}$

and $f(t+2a) = f(t)$

sol:-

$\because$ Given f^n is periodic b/w 0 to 2a.

$$L\{f(t)\} = \left[\int_0^a e^{-st} \cdot f(t)\,dt\right]$$

$$\Rightarrow \left[\int_0^a e^{-st} \cdot t/a \,dt\right] + \left[\int_0^{2a} e^{-st} \cdot \frac{(2a-t)}{a}\,dt\right]$$

$$\Rightarrow \frac{1}{a}\left[\left[\frac{t \cdot e^{-st}}{(-s)}\right]_0^a - \frac{e^{-st}}{s^2}\right]_0^a + \int_0^{2a} 2 \cdot e^{-st}\,dt - \int_0^{2a} e^{-st} \cdot \frac{t}{a}\,dt$$

$$\Rightarrow \left\{\frac{e^{-as}}{-s} - \frac{e^{-as}}{as^2} + \frac{1}{as^2} + 2 \cdot \frac{e^{-st}}{(-s)}\Big|_0^{2a} - \frac{1}{a}\left[\frac{t \cdot e^{-st}}{-s}\right]_0^{2a} + \frac{e^{-st}}{s^2}\Big|_0^{2a}\right\}$$

$$\Rightarrow \left\{\frac{e^{-as}}{-s} - \frac{e^{-as}}{as^2} + \frac{1}{as^2} - \frac{2 \cdot e^{-st}}{s} + \frac{2e^{-as}}{s} + \frac{2 \cdot e^{-2as}}{s} - \frac{e^{-as}}{s} + \frac{e^{-2as}}{as^2} - \frac{e^{-as}}{as^2}\right\}$$

$$L\{f(t)\} - \frac{e^{-as}}{as^2}\left[-1-1+e^{-as}\right] + 1/as^2$$

$$\boxed{L\{f(t)\} = \frac{e^{-as}}{as^2}\left[e^{-as} - 2\right] + 1/as^2}$$

- $f(t) = t^2 \cdot u_3(t)$, where $u(\cdot)$ is Heaviside unit step f^n.

Sol:-
$$u_3(t) = \begin{cases} 0 & t<3 \\ 1 & t>3 \end{cases}$$

$$u_3(t) = u(t-3)$$

$$L\{t^2 \cdot u_3(t)\} = L\{t^2 \cdot u(t-3)\}$$
$$= e^{-3s} L\{(t+3)^2\}$$
$$= e^{-3s} \cdot L\{t^2 + 9 + 6t\}$$
$$= e^{-3s}\left[\frac{2}{s^3} + \frac{9}{s} + 6 \cdot {}^1/_{s^2}\right]$$

3- Find the inverse Laplace of following f^n's-

- $\dfrac{s-2}{s(s+3)}$

$$f(s) = \frac{s-2}{s(s+3)} \Rightarrow \frac{1}{(s+3)} - \frac{2}{s(s+3)}$$

$$L^{-1}\{f(s)\} = L^{-1}\left(\frac{1}{s+3}\right) - L^{-1}\left(\frac{2}{s(s+3)}\right)$$

$$\because L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$

$$= e^{-3t} - 2L^{-1}\frac{1}{s(s+3)}$$

$$\Rightarrow e^{-3t} - 2\int_0^t e^{-3t}\,dt$$

$$\Rightarrow e^{-3t} - 2\left[\frac{e^{-3t}}{-3}\right]_0^t$$

$$\Rightarrow e^{-3t} + \frac{2}{3} \cdot e^{-3t} - {}^2/_3$$

$$L^{-1}\left\{\frac{s-2}{s(s+3)}\right\} \Rightarrow \frac{5}{3} \cdot e^{-3t} - {}^2/_3$$

Ans

- $\frac{a}{s^2(s^2+a^2)}$

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$$

$$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$$

$$L^{-1}\left\{\frac{1}{s}\cdot\left(\frac{a}{s^2+a^2}\right)\right\} = \int_0^t \sin at\, dt$$

$$= \left[-\cos at\right]_0^t$$

$$= -\cos at + 1$$

$$L^{-1}\left\{\frac{1}{s^2}\left(\frac{a}{s^2+a^2}\right)\right\} = \int_0^t (1-\cos at)\, dt$$

$$= \left. t - \sin at \right]_0^t$$

so, $L^{-1}\left\{\frac{1}{s^2}\left(\frac{a}{s^2+a^2}\right)\right\} = t - \sin at.$ Ans

4- Find the Inverse Laplace of transformation of following f^n using convolution -

- $\frac{1}{(s^2+a^2)^2}$

$$f(s) = \frac{1}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}$$

$$F(u) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}, \quad G(u) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= \frac{\sin at}{a} \qquad\qquad = \frac{\sin at}{a}$$

by convolution -

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \int_0^t F(u)\cdot G(t-u)\, du$$

$$\Rightarrow \int_0^t \frac{\sin at}{a} \cdot \frac{\sin[at-au]}{a}\, du$$

$$\Rightarrow \frac{1}{a^2}\int_0^t \sin at\,[\sin at \cdot \cos au - \cos at \cdot \sin au]\, du$$

$$\Rightarrow \frac{\sin at}{2a^2}\int_0^t \sin 2au\, du - \frac{\cos at}{a^2}\int_0^t \sin^2 au\, du$$

$$\Rightarrow \frac{\sin at}{2a^2}\left[\frac{-\cos 2au}{2a}\right]_0^t - \frac{\cos at}{a^2}\int_0^t \frac{1-\cos 2au}{2}\, du$$

$$\Rightarrow \frac{\sin at}{2a^3}\left[\frac{1-\cos 2at}{2}\right] - \frac{\cos at}{a^2}\left[\frac{u}{2} - \frac{\sin 2au}{4a}\right]$$

$$\Rightarrow \frac{\sin^3 at}{2a^3} - \frac{t}{2a^2}\cos at + \frac{\sin 2at \cdot \cos at}{4a^3}$$

- $\frac{1}{(s-2)(s+3)}$

sol^n- $f(s) = 1/s-2 \Rightarrow L^{-1} f(s) = e^{2t} = f(t)$

$g(s) = \frac{1}{s+3} \Rightarrow L^{-1}\{g(s)\} = e^{-3t} = g(t)$

by convolution theorem -

$$L^{-1}\{f(s)\cdot g(s)\} = \int_0^t f(u)\cdot g(t-u)\, du$$

$$\Rightarrow \int_0^t e^{2u} \cdot e^{-3(t-u)}\, du$$

$$\Rightarrow \int_0^t e^{2u+3u-3t}\, du$$

$$\Rightarrow \int_0^t e^{5u-3t}\, du$$

$$\Rightarrow \left.\frac{e^{5u-3t}}{5}\right|_0^t$$

$$\Rightarrow \frac{1}{5}\left[e^{2t} - e^{-3t}\right]$$

$$10,\ L^{-1}\frac{1}{(s+3)(s-2)} = \frac{e^{2t} - e^{-3t}}{5}$$

5- Using convolution, solve the initial value problem -

$$y'' + 9y = \sin 3t,\ y(0) = 0,\ y'(0) = 0$$

sol^n - $y'' + 9y = \sin 3t$

Taking laplace on both side -

$y(0) = 0,\ y'(0) = 0$

$$s^2 L(y) - s \cdot y(0) - y'(0) + 9L(y) = \frac{3}{s^2+9}$$

$$L\{y\} = \frac{3}{(s^2+9)^2}$$

$$y = L^{-1}\left\{\frac{3}{(s^2+9)^2}\right\} = 3L^{-1}\left\{\frac{-1}{2s} \cdot \frac{d}{ds}\left(\frac{1}{s^2+9}\right)\right\}$$

$$\Rightarrow -\frac{3}{2s}\left\{-t \cdot \frac{1}{3}\sin 3t\right\}$$

$$\Rightarrow \frac{3}{2\times 3} \cdot \frac{1}{s}\left[t \cdot \sin 3t\right]$$

$$\Rightarrow \frac{1}{2}\int_0^t t \cdot \sin 3t\, dt$$

$$\Rightarrow \frac{1}{2}\left[t\left(\frac{-\cos 3t}{3}\right) - \int -\frac{\cos 3t}{3}\, dt\right]_0^t$$

$$\Rightarrow \frac{1}{2}\left[-\frac{t}{3} \cdot \cos 3t + \frac{\sin 3t}{9}\right]_0^t$$

$$y(t) = \frac{1}{18}\left[\sin 3t - 3t\cos 3t\right]$$

6- Solve the following initial value problems using Laplace transforms -

- $4y'' - 8y' + 3y = \sin t$, $y(0) = 0$, $y'(0) = 2$

sol:- Given $y(0) = 0$, $y'(0) = 2$

$$4y'' - 8y' + 3y = \sin t \Rightarrow y'' - 2y' + \frac{3}{4}y = \frac{1}{4}\sin t$$

Taking laplace on both side:

$$\Rightarrow s^2 \cdot L\{y\} - s \cdot y(0) - y'(0) - 2\left[s \cdot L\{y\} - y(0)\right] + \frac{3}{4}L\{y\} = \frac{1}{4}L\{\sin t\}$$

$$\Rightarrow s^2 L\{y\} - 2 - 2s L\{y\} + \frac{3}{4}L\{y\} = \frac{1}{4} \cdot \frac{1}{s^2+1}$$

$$\Rightarrow L\{y\}\left[s^2 - 2s + 3/4\right] = \frac{1}{4}\left(\frac{1}{s^2+1}\right) + 2$$

$$L\{y\} \cancel{\left[4s^2 - 8s + 3\right]} = \frac{1}{4(s^2+1)(s^2-2s+3/4)} + \frac{2}{(s^2-2s+3/4)}$$

$$L\{y\} = \frac{1}{4(s^2+1)(s-1/2)(s-3/2)} + \frac{2}{(s-1/2)(s-3/2)}$$

$$L(y) = \frac{1}{4}\left[\frac{A}{(s-1/2)} + \frac{B}{(s-3/2)} + \frac{Cs+D}{(s^2+1)}\right] + 2\left[\frac{E}{s-1/2} + \frac{F}{s-3/2}\right]$$

By using partial fraction: after solving eq^n -

$A = 0.3$, $B = -0.4$, $C = 0.1$, $D = -0.01$, $E = -1$, $F = 1$

$$L(y) = \frac{1}{4}\left[\frac{0.3}{(s-1/2)} - \frac{0.4}{(s-3/2)} + \frac{(0.1)s}{s^2+1} - \frac{0.01}{s^2+1}\right] + 2\left[\frac{1}{s-3/2} - \frac{1}{s-1/2}\right]$$

taking inverse of given Laplace f^n -

$$y = \frac{1}{4}\left\{L^{-1}\frac{0.3}{(S-\frac{1}{2})} - L^{-1}\frac{0.4}{S-\frac{3}{2}} + 0.1\cos t - 0.01\sin t\right\} + 2\left[L^{-1}\frac{1}{S-\frac{3}{2}} - L^{-1}\frac{1}{S-\frac{1}{2}}\right]$$

$$y = \frac{1}{4}\left[0.3 \cdot e^{-\frac{1}{2}t} - 0.4 \cdot e^{-\frac{3}{2}t} + 0.1\cos t - 0.01\sin t\right] + 2\left[e^{-\frac{3}{2}t} - e^{-\frac{1}{2}t}\right]$$

$$y = 1.6 \cdot e^{-\frac{3}{2}t} - 1.92 \cdot e^{-\frac{1}{2}t} + 0.1\cos t - 0.01\sin t$$

Ans

(●) $y'' + 2y' + 5y = \delta(t-2)$, $y(0) = 0$, $y'(0) = 0$

sol^n

Given $y(0) = 0$, $y'(0) = 0$

$$y'' + 2y' + 5y = \delta(t-2)$$

Taking Laplace on both side -

$$\Rightarrow s^2 L\{y\} - s\,y(0) - y'(0) + 2\left[s \cdot L\{y\} - y(0)\right\} + 5\,L\{y\} = L\{\delta(t-2)\}$$

$$\Rightarrow L\{y\}\left[s^2 + 2s + 5\right] = e^{-2s}$$

(∵ $\delta(t-2s)$ is a impulse f^n)

$$\Rightarrow L\{y\} = \frac{e^{-2s}}{(s^2+2s+5)}$$

$$= \frac{e^{-2s}}{(s+1)^2+2^2}$$

$$y(t) = L^{-1}\left\{\frac{e^{-2s}}{(s+1)^2+2^2}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2}$$

$$L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} = \frac{e^{-t}\cdot\sin 2t}{2}$$ (by Ist shifting)

$$L^{-1}\left\{\frac{e^{-2s}}{(s+1)^2+2^2}\right\} = \frac{e^{-(t-2)}\cdot\sin 2(t-2)}{2}$$ (by IInd shifting)

so, $$y(t) = \frac{e^{-(t-2)}\cdot\sin 2(t-2)}{2}$$

- $t\cdot y'' + 2ty' + 2y = 2$, $y(0)=1$, $y'(0)$ is arbitrary.

sol^n - taking Laplace

$$L\{t\cdot y''\} + L\{2ty'\} + L\{2y\} = L(2)$$

$$\Rightarrow -\frac{d}{ds}\{Ly''\} - 2\cdot\frac{d}{ds}[Ly'] + 2L\{y\} = 2/s$$

const^n.

$$\Rightarrow -\frac{d}{ds}[s^2Ly - sy(0) - \cancel{y'(0)}] - 2\frac{d}{ds}[s\cdot L(y) - \overset{1}{\cancel{y(0)}}] + 2L(y) = 2/s$$

Let $L(y) = \bar{y}$

$$\Rightarrow -2s\bar{y} - s^2\cdot\frac{d\bar{y}}{ds} + 1 + 0 - 2\left[s\cdot\frac{d\bar{y}}{ds} + \bar{y} - 0\right] + 2\bar{y} = 2/s$$

$$\Rightarrow -2s\bar{y} - s^2\cdot\frac{d\bar{y}}{ds} + 1 - 2s\cdot\frac{d\bar{y}}{ds} - 2\bar{y} + 2\bar{y} = 2/s$$

$$\Rightarrow -\frac{d\bar{y}}{ds}(s^2+2s) - 2s\bar{y} = 2/s - 1$$

$$\Rightarrow \frac{d\bar{y}}{ds} + \frac{2s}{(s^2+2s)}\bar{y} = \frac{s-2}{s(s^2+2s)}$$

$$\frac{d\bar{y}}{ds} + \frac{2}{(s+2)}\bar{y} = \frac{s-2}{s^2(s+2)}$$

$$I.F. = e^{\int 2/s+2\, ds} = e^{\log(s+2)^2} = (s+2)^2$$

$$\bar{Y} \cdot (s+2)^2 = \int \frac{s-2}{s^2(s+2)} \cdot (s+2)^2 \, ds$$

$$\bar{Y} (s+2)^2 = \int \frac{s^2-4}{s^2} \, ds$$

$$\bar{Y} (s+2)^2 = \frac{s^2+4}{s} + c$$

$$\bar{Y} = \frac{s^2+4-4s+4s}{s(s+2)^2} + \frac{c}{(s+2)^2}$$

$$= \frac{(s+2)^2}{s(s+2)^2} - \frac{4s}{s(s+2)^2} + \frac{c}{(s+2)^2}$$

$$\bar{Y} = \frac{1}{s} - 4/(s+2)^2 + \frac{c}{(s+2)^2}$$

$$L(y) = 1/s - \frac{4}{(s+2)^2} + \frac{c}{(s+2)^2}$$

Now taking inverse Laplace

$$y = L^{-1}\{1/s\} - 4 L^{-1} \frac{1}{(s+2)^2} + c L^{-1} \frac{1}{(s+2)^2}$$

$$y = 1 - 4 \cdot e^{-2t} \cdot t + c \cdot e^{-2t} \cdot t$$

given $y(0) = 1$, $1 = 1 - 0 + 0$

so, c is any arbitary constn {given by $y'(0) = $ arb}

$$y = 1 - 4 \cdot e^{-2t} \cdot (t) + t \cdot c \cdot e^{-2t}$$ Ans