200106058
Riya Jaiswal

Maths Assignment (Unit-2)

(1)

1. $$\frac{x\,\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{x(2x+y)}{x^2+xy+y^2} + \frac{y(x+2y)}{x^2+xy+y^2}$$

$$= \frac{2x^2+xy+xy+2y^2}{x^2+xy+y^2}$$

$$= \frac{2(x^2+xy+y^2)}{x^2+xy+y^2} = 2$$ Hence, Proved

Ans 2) i) $u = \log\left(\frac{x^2+y^2}{xy}\right)$

$$\frac{\partial u}{\partial x} = \frac{xy}{x^2+y^2}\left(\frac{2x(xy) - (x^2+y^2)y}{x^2y^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{xy}{x^2+y^2}\left(\frac{2x^2y - x^2y - y^3}{x^2y^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{x^2y - y^3}{xy(x^2+y^2)} = \frac{x^2-y^2}{x(x^2+y^2)}$$

$$x\frac{\partial u}{\partial x} = \frac{x^2-y^2}{x^2+y^2} \quad -(1)$$

$$\frac{\partial y}{\partial x} = \frac{xy}{x^2+y^2}\left[\frac{2yxy - x(x^2+y^2)}{x^2-y^2}\right]$$

$$\frac{\partial y}{\partial x} = \frac{xy^3 - x^3}{xy(x^2+y^2)} = \frac{y^2-x^2}{y(x^2+y^2)}$$

$$y\frac{\partial y}{\partial x} = \frac{y^2-x^2}{y^2+x^2} \quad -(2)$$

(1) + (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \frac{x^2-y^2}{x^2+y^2} + \frac{y^2-x^2}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = 0$$

Hence, verified Euler's theorem

Ans 2) ii) $u = \dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$

u is a homogenous functions of degree $\left(-\frac{1}{6}\right)$

By Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{6}u$$

$$\frac{\partial u}{\partial x} = \frac{\frac{1}{3}x^{-2/3}\left(x^{1/2} + y^{1/2}\right) - \frac{1}{2}x^{-1/2}\left(x^{1/3} + y^{1/3}\right)}{\left(x^{1/2} + y^{1/2}\right)^2}$$

$$x\frac{\partial u}{\partial x} = \frac{\frac{1}{3}x^{5/6} + \frac{1}{3}x^{1/3}y^{1/2} - \frac{1}{2}x^{5/6} - \frac{1}{2}x^{1/2}y^{2/3}}{\left(x^{1/2} + y^{1/2}\right)^2} \quad -(1)$$

$$y\frac{\partial u}{\partial y} = \frac{\frac{1}{3}y^{5/6} + \frac{1}{3}y^{1/3}x^{1/2} - \frac{1}{2}y^{1/2}x^{1/3} - \frac{1}{2}y^{5/6}}{\left(x^{1/2} + y^{1/2}\right)^2} \quad -(2)$$

On adding equations ① and 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-\frac{1}{6}x^{5/6} - \frac{1}{6}y^{5/6} - \frac{1}{6}x^{1/2}y^{1/3} - \frac{1}{6}x^{1/3}\,3y^{1/2}}{(x^{1/2}+y^{1/2})^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{6}\left[\frac{x^{1/2}(x^{1/3}+y^{1/3}) + y^{1/2}(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^2}\right]$$

$$= -\frac{1}{6}\,\frac{(x^{1/2}+y^{1/2})(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^2}$$

$$= -\frac{1}{6}\left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{6}u$$ Hence, Proved

3. $x = r\cos\theta$; $y = r\sin\theta$

$u = f(r)$ = $\frac{\partial u}{\partial r} = f'(r)\frac{\partial r}{\partial x}$ & $\frac{\partial^2 u}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2}$

$$\frac{\partial u}{\partial y} = f'(r)\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 x}{\partial y^2}\right]$$

$\because$ $r^2 = x^2 + y^2$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x^2/r}{r^2} = \frac{r^2-x^2}{r^3} = \frac{y^2}{r^3}$$

$$\frac{\partial r}{\partial y} = \cancel{\frac{y}{\sqrt{x}}}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{r - y^2/r}{r^2} = \frac{r^2-y^2}{r^3} = \frac{x^2}{r^3}$$

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$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]+f'(r)\left[\frac{y^2}{r^3}+\frac{x^2}{r^3}\right]$$

$$=f''(r)+f'(r)\left(\frac{r^2}{r^3}\right)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$$ Hence, Proved

4) $z=\sec u=\frac{x^3-y^3}{x+y}$

u is not homogeneous function but $z=\sec u=\frac{x^3-y^3}{x+y}$ is homogenous with degree 2

By euler's theorem, we have

$$x\frac{\partial x}{\partial x}+y\frac{\partial z}{\partial y}=2z$$

$$x\sec v\tan v\frac{\partial u}{\partial x}+y\sec v\tan v\frac{\partial u}{\partial y}=2\sec v$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\cot v$$ ... Ist Equation

on differentiating with respect to x; we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=-2\operatorname{cosec}^2 u\frac{\partial u}{\partial x}.$$

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=(-2\operatorname{cosec}^2 u-1)x\frac{\partial u}{\partial x}$$ ... IInd Equation

differentiating equation Ist partially with respect to y.

$$xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=-2y\operatorname{cosec}^2 u\frac{\partial u}{\partial y}$$

$$xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=(-2\operatorname{cosec}^2 u-1)y\frac{\partial u}{\partial y}$$ ... IIIrd Equation

Adding IInd & IIIrd

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(-2\operatorname{cosec}^2 u-1)2\cot u$$

$= (-2\operatorname{cosec}^2 u - 1)\, 2\cot u$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + 2\cot v\,(2\operatorname{cosec}^2 u - 1) = 0$$

Ans 5) $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

Let $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$

$u = f(r, s, t)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}(2) + \frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial t}(2)$$

$$\frac{\partial u}{\partial x} = \frac{2\partial f}{\partial r} - \frac{2\partial f}{\partial t} \quad —① $$

$$\frac{6\partial u}{\partial x} = \frac{12\partial f}{\partial r} - \frac{12\partial f}{\partial t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y} = (-3)\frac{\partial f}{\partial r} + \frac{3\partial f}{\partial s} + 0\cdot\frac{\partial f}{\partial t}$$

$$\frac{4\partial u}{\partial y} = \frac{-12\partial f}{\partial r} + \frac{12\partial f}{\partial s} \quad —②$$

$$\frac{\partial u}{\partial z} = 0\cdot\frac{\partial f}{\partial r} - \frac{4\partial f}{\partial x} + \frac{4\partial f}{\partial t}$$

$$\frac{3\partial u}{\partial z} = -\frac{12\partial f}{\partial s} + \frac{12\partial f}{\partial t} \quad —③$$

~~Adding~~ I + II III

$$\frac{6\partial u}{\partial x} + \frac{4\partial u}{\partial y} + \frac{3\partial u}{\partial z} = \frac{12\partial f}{\partial r} - \frac{12\partial f}{\partial t} - \frac{12\partial f}{\partial r} + \frac{12\partial f}{\partial s} - \frac{12\partial f}{\partial s} + \frac{12\partial f}{\partial t}$$

$$\frac{6\partial u}{\partial x} + \frac{4\partial u}{\partial y} + \frac{3\partial u}{\partial z} = 0$$

Hence Proved!

Ans 6) $x = e^u \cos v$ & $y = e^v \sin v$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial u} = e^u \cos v \frac{\partial w}{\partial x} + e^v \sin v \frac{\partial w}{\partial y}$$

$$y\frac{\partial w}{\partial u} = y\left(e^u \cos v \frac{\partial w}{\partial x} + e^v \sin v \frac{\partial w}{\partial y}\right)$$

$$y\frac{\partial w}{\partial u} = e^{2u} \sin v \cos v \frac{\partial w}{\partial x} + e^{2u} \sin^2 v \frac{\partial w}{\partial y} \quad -(1)$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial v} = -\sin v\, e^u \frac{\partial w}{\partial x} + e^u \cos v \frac{\partial w}{\partial y}$$

$$x\frac{\partial w}{\partial v} = e^u \cos v\left(-\sin v\, e^v \frac{\partial w}{\partial x} + e^u \cos v \frac{\partial w}{\partial y}\right)$$

$$x\frac{\partial w}{\partial v} = -e^{2v} \sin v \cos v \frac{\partial w}{\partial x} + e^{2v} \cos^2 v \frac{\partial w}{\partial y} \quad — (2)$$

Adding (1) & (2)

$$y\frac{\partial w}{\partial v} + x\frac{\partial w}{\partial u} = e^{2u} \sin v \cos v \frac{\partial w}{\partial x} + e^{2v} \sin v \frac{\partial w}{\partial y} - e^{2u} \sin v \cos v \frac{\partial w}{\partial x} + e^{2u} \cos^2 v \frac{\partial w}{\partial y}$$

$$y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v} = e^{2v}(\sin^2 v + \cos^2 v)\frac{\partial w}{\partial y}$$

$$\boxed{y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v} = e^{2u}\frac{\partial w}{\partial y}}$$ Hence, Proved

Ans 7) $x = u\cos v$; $y = u \sin v$, $z = uv$

$$\frac{\partial w}{\partial u} = \cos v \frac{\partial w}{\partial x} + \sin v \frac{\partial w}{\partial y} + v\frac{dz}{du}$$

$$v\frac{\partial w}{\partial u} = u\cos v \frac{\partial w}{\partial x} + \sin v \frac{\partial w}{\partial y} + uv\frac{\partial z}{\partial v} \quad -(1)$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial v} = -u\sin v\frac{\partial w}{\partial x} + u\cos v\frac{\partial w}{\partial y} + u\frac{\partial w}{\partial z}$$

$$v\frac{\partial w}{\partial v} = -uv\sin v\frac{\partial w}{\partial x} + uv\cos v\frac{\partial w}{\partial y} \quad -(2)$$

(1) - (2)

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = (u\cos v + uv\sin v)\frac{\partial w}{\partial x} + (u\sin v - uv\cos v)\frac{\partial w}{\partial y}$$

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial u} = \frac{x(u\cos v + uv\sin v)}{\sqrt{x^2+y^2+z^2}} + \frac{y(u\sin v - uv\cos v)}{\sqrt{x^2+y^2+z^2}}$$

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u^2\cos^2 v + u^2\sin^2 v + u^2 v\sin v\cos v - u^2 v\sin v\cos v}{\sqrt{u^2 + u^2v^2}}$$

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u^2}{u\sqrt{1+v^2}} = \frac{u}{\sqrt{1+v^2}}$$

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}$$

Ans 8)

$$u = x + y + z$$
$$v = x^2 + y^2 + z^2$$
$$w = x^3 + y^3 + z^3 - 3xyz$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= 6\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2+zx-yz & y^2-z^2+xy-zx & z^2-xy \end{vmatrix}$$

$$= 6\begin{vmatrix} 0 & 0 & 1 \\ (x-y) & (y-z) & z \\ (x-y)(x+y+z) & (y-z)(x+y+z) & z^2-xy \end{vmatrix}$$

$$= 6(x-y)(y-z)\begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & z \\ x+y+z & x+y+z & z^2-xy \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

So, u, v & w are not independent.

$$w = x^3+y^3+z^3-3xyz$$
$$w = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$
$$w = u\left[v - \left(\frac{u^2-v}{2}\right)\right]$$
$$= u\left(\frac{2v-u^2+v}{2}\right)$$
$$2w = u(3v-u^2)$$
$$2w = 3uv - u^3$$
$$\boxed{2w + u^3 = 3uv}$$

Ans 9) $f_1 = x - v^2 - w^2$; $f_2 = y - w^2 - u^2$; $f_3 = z - u^2 - v^2$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} \quad \text{— (1)}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} 0 & -2v & -2w \\ -2w & 0 & -2w \\ -2u & -2v & 0 \end{vmatrix}$$

$$= -8uvw \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -8uvw\,(1+1)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = -16uvw)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = 1$$

from equation (1) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \dfrac{-16uvw}{1}$

$$= 16uvw \quad \text{— (2)}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-1}{-16uvw} = \frac{1}{16uvw} \quad \text{— (3)}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$$

Ans 10) $u = r^2\cos^2\theta - r^2\sin^2\theta$

$u = r^2\cos 2\theta$

$v = 2r^2\sin\theta\cos\theta$

$v = r^2\sin 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$$= 4r^3 \begin{vmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{vmatrix}$$

$$= 4r^3(\cos^2 2\theta + \sin^2 2\theta)$$

$$\boxed{\left(\frac{\partial(u,v)}{\partial(r,\theta)}\right) = 4r^3}$$

Ans 11) $f_1 = u^3 + v^3 - x - y = 0$

$f_2 = u^2 + v^2 - x^3 - y^3 = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{(-1)^2 \frac{\partial(f_1, f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$= 3(y^2 - x^2)$$

$$\frac{\partial(f_1,f_2)}{(u,v)} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= 6uv \begin{vmatrix} u & v \\ 1 & 1 \end{vmatrix}$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = 6uv(u-v)$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = (-1)^3 \frac{3(y^2-x^2)}{6uv(u-v)} \Rightarrow \frac{y^2-x^2}{2uv(u-v)} \quad \text{(Ans)}$$

Ans 12) $u(x,y) = x^2y + 3y - 2$

Taylor's expansion $f(x,y)$ in powers of $(x-a)$ & $(x-b)$ i.e

$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$

$+ \frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b)$

$+ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)] + \ldots$ (1)

Hence; $a = 1$ $b = -2$ $f(x,y) = x^2y + 3y - 2$

$f(1,-2) = -2 + 3(-2) - 2 = -10$

$f_x(1,-2) = -4$; $f_y = x^2 + 3 \Rightarrow f_y(1,-2) = +4$

$f_{xx} = 2y = -4$; $f_{xy} = 2x$; $f_{xy}(1,-2) = 2$

$f_{yy} = 0$; $f_{yy} = (1,-2) = 0$; $f_{xxx}(1,-2) = 0$

$f_{xxy}(1,-2) = 2$; $f_{xyy}(1,-2) = 0$

Substituting the values in Eq(1), we get

$u(x,y) = x^2y + 3y - 2 = -10 + [(x-1) - 4 + (y+2)4] + \frac{1}{2}[(x-1)^2 - 4 + 2(x-1)(y+2)2 + (y+2)^2 \times 0] + 1$

All the partial derivatives of higher order vanish

$u(x,y) = x^2y + 3y - 2 = \cancel{-10} -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$

Ans 13) $u(x,y) = x^4 + x^2y^2 - y^4$

$u(x,y) = f(x,y)$

here $a = 1$, $b = 1$ $f(1,1) = x^4 + 1 - 1$
$= 1$

$f_x = 4x^3 + 2xy^2$ $f_x(1,1) = 6$

$f_y = 2yx^2 - 4y^3$ $f_y(1,1) = -2$

$f_{xx} = 12x^2 + 2y^2$ $f_{xx}(1,1) = 14$

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$f_{xy} = 4xy \qquad f_{xy}(1,1) = 4$

$f_{xy} = 2x^2 - 12y^2 \quad f_{yy}(1,1) = -10$

$f_{xxx} = 4y \qquad f_{xxy}(1,1) = 4$

$f_{xyy} = 4x \qquad f_{xyy}(1,1) = 4$

$f_{yyy} = -24y \qquad f_{yyy}(1,1) = -24$

Taylor's Expansion of $f(x,y)$ in power of $(x-1)$ & $(y-1)$ i.e given by;

$f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)] + \frac{1}{2!}[(x-1)^2$
$f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)]$
$+ \frac{1}{3!}[(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1) + 3(x-1)$
$(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)] + \cdots$

$f(x,y) = 1 + [6(x-1) + 2(y-1)] + \frac{1}{2!}[14(x-1)^2 + 8(x-1)(y-1)$
$- 10(y-1)^2]$

$x^4 + x^2y^2 - y^4 = 1 + [6(x-1) - 2(y-1)] + [7(x-1)^2 + 4(x-1)(y-1)$
$- 5(y-1)^2]$

Ans 14) r = radius

h = height

$V = \pi r^2 h$

$\delta V = \frac{\delta V}{\delta r}\delta r + \frac{\delta V}{\delta h}\delta h$

$\delta V = 2\pi r h \,\delta r + \pi r^2 \delta h$

$\delta r = (2rh\,\delta r + r^2 \delta h)\pi$

$2r = 8cm \quad h = 12.5\,cm$ (given)

$\delta h = 0.05$

$\delta V = (8 \times 12.5 \times 0.05) + 4^2 \times 0.05)\pi$

$\delta V = (5 + 0.8)\pi$

$\delta V = 5.8\pi\, cm^3$

Ans 15) Power (P) = $\lambda v^3 l^3$

taking log on both sides

$\log P = \log \lambda + 3\log v + 3\log l$

on differentiating

$$\frac{\delta P}{P} = 0 + \frac{3\delta u}{u} + \frac{3\delta l}{l} \qquad \lambda = \text{constant})$$

$$100\times\frac{\delta P}{P} = 3\left(\frac{\delta v}{u}\times 100\right) + 3\left(\frac{\delta l}{l}\times 100\right)$$

$$= 3\times 3 + (3\times(-1))]$$

$$100\times\frac{\delta P}{P} = 6\%.$$

Ans 16) i) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

$$I = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$I = I_1 - I_2$ — Ist Equation

$x = \frac{t}{1-t}$ $\quad x = 0 \quad t = 0$

$\quad x = \infty \quad t = 1$

$$1+x = 1+\frac{t}{1-t} = \frac{1}{1-t}$$

$$dx = \frac{1}{(1-t)^2}\,dt$$

$$I_1 = \int_0^1 \frac{\left(\frac{t}{1-t}\right)^8 \frac{1}{(1-t)^2}\,dt}{\left(\frac{1}{1-t}\right)^{24}} = \int_0^1 t^8(1-t)^{14}\,dt$$

$$= \int_0^1 t^{9-1}(1-t)^{15-1}\,dt$$

$$I_1 = \frac{\sqrt{9}\sqrt{15}}{\sqrt{15+9}} = \frac{\sqrt{5}\sqrt{15}}{\sqrt{24}} \quad -②$$

$$I_2 = \int_0^1 \left(\frac{t}{1-t}\right)^{14}(1-t)^{24}\cdot\frac{1}{(1-t)^2}\,dt$$

$$I_2 = \int_0^1 t^{14}(1-t)^8\,dt = \int_0^1 t^{15-1}(1-t)^{9-1}\,dt = I_2 = \frac{\sqrt{15}\sqrt{9}}{\sqrt{24}} \quad ③$$

$$\boxed{\therefore I = 0}$$

ii) $\int_0^\infty x^{2n-1} e^{-ax^2}\, dx$

$ax^2 = t$

$x = \left(\frac{t}{a}\right)^{1/2} \Rightarrow dx = \frac{1}{2\sqrt{a}}\, t^{-1/2}\, dt$

$$\int_0^\infty x^{2n-1} e^{-ax^2}\, dx = \int_0^\infty \left(\frac{t}{a}\right)^{\frac{2n-1}{2}} e^{-t} \cdot \frac{1}{2\sqrt{a}}\, t^{-1/2}\, dt$$

$$= \frac{1}{2a^n} \int_0^\infty t^{\frac{2n-2}{2}} e^{-t}\, dt$$

$$= \frac{1}{2a^n} \int_0^\infty t^{n-1} e^{-t}\, dt$$

$$= \frac{\Gamma(n)}{2a^n}$$

iii) $\int_0^1 x^3 (1-x)^{4/3}\, dx$

$$\int_0^1 x^{4-1} (1-x)^{\frac{7}{3}-1}\, dx$$

$$= \frac{\Gamma(4)\,\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(4+\frac{7}{3}\right)} = \frac{\Gamma(4)\,\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{19}{3}\right)}$$

Now,

$$\int_0^1 x^3 (1-x)^{4/3}\, dx = \frac{3!\,\Gamma\left(\frac{17}{3}\right)}{\frac{16}{3}\,\frac{13}{3}\,\frac{10}{3}\,\frac{7}{3}\,\Gamma\left(\frac{7}{3}\right)}$$

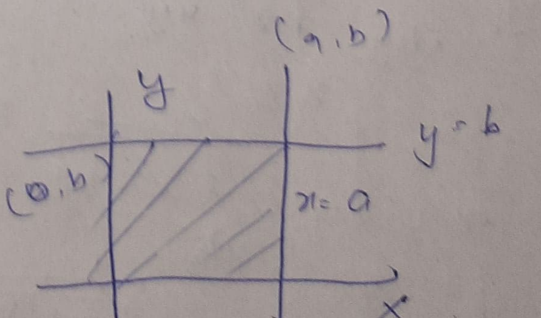
$$= \frac{243}{7280}$$

Ans 17) i) $\int_0^b \int_0^a \frac{x}{x^2+y^2}\, dy\, dx.$

$y = 0 \;;\; x = 0$

$y = b \;;\; x = a$

$$= \int_{y=0}^{b} \frac{1}{2} \int_{x=y^2}^{x=a^2+y^2} \frac{1}{t}\, dt\, dy$$



Let $x^2+y^2 = t$

$2x\,dx = y$

when $x = 0$ $t = y^2$

$x = a$ $t = a^2+y^2$

$$= \int_{y=0}^{b} \frac{1}{2} \left(\log(a^2+y^2) - \log y^2\right] dy$$

$$= \frac{1}{2}\int_0^b \log(a^2+y^2)\,dy - \frac{1}{2}\int_0^b \log y^2\,dy$$

$$= \frac{1}{2}\left\{ y\log(a^2+y^2)\Big]_0^b - \int_0^b \frac{y}{a^2+y^2}\,dy \right\} - \int_0^b \log y\,dy$$

$$= \frac{1}{2} b\log(a^2+b^2) - \frac{1}{4}\int_{a^2}^{a^2+b^2} \frac{1}{u}\,du - \Big[y\log y - y\Big]_0^b$$

$$= \frac{b}{2}\log(a^2+b^2) - \frac{1}{4}\Big[\log u\Big]_{a^2}^{a^2-b^2} - b\log b + b$$

$$= \frac{b}{2}\log(a^2+b^2) - \frac{1}{4}\log(a^2+b^2\Big] + \frac{1}{4}\log^2 - b\log b + b$$

$$= \left(\frac{b}{2} - \frac{1}{4}\right)\log(a^2+b^2) + \frac{1}{2}\log a - b\log b + b$$

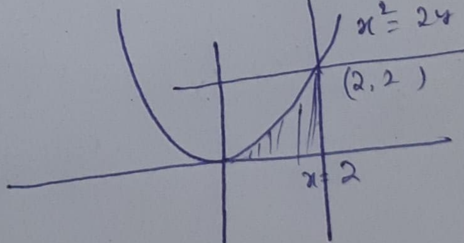
$$\therefore \boxed{\int_0^b\int_0^a \frac{x}{x^2+y^2}\,dy\,dx = \left(\frac{b}{2}-\frac{1}{4}\right)\log(a^2+b^2) + \frac{1}{2}\log a - \log b + b}$$

ii) $\int_0^2\int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4-4y^2}}\,dx\,dy$

$x = 2$ $\quad\therefore y = 0$

$x = \sqrt{2y}$ $\quad y = 2$

$x = \sqrt{2y}$ or $x^2 = 2y$



Now,

$$\int_0^2\int_{\sqrt{2y}}^2 \frac{x}{\sqrt{x^4-4y^2}}\,dx\,dy$$

$$\int_{x=0}^{2}\int_{y=0}^{x^2/2} \frac{x^2}{\sqrt{x^4-4y^2}}\,dy\,dx.$$

$$= \int_{x=0}^{2} \frac{x^2}{2}\int_{y=0}^{x^2/2} \frac{1}{\sqrt{\left(\frac{x^2}{2}\right)^2-y^2}}\,dy\,dx.$$

$$= \int_{x=0}^{2} \frac{x^2}{2}\left[\sin^{-1}\left(\frac{2y}{x^2}\right)\right]_0^{x^2/2}$$

$$= \int_{x=0}^{2} \frac{x^2}{2} \sin^{-1}(1) = \int_0^2 \frac{x^2}{2}\frac{\pi}{2} = \frac{\pi}{4}\int_0^2 x^2 dx.$$

$$= \frac{\pi}{4}\int_0^2 x^2 dx = \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^2 = \frac{\pi}{12} \times 8 = \frac{2\pi}{3}$$

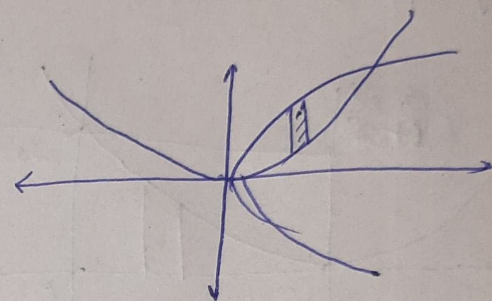
iii) $\displaystyle\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$

$x = 0$ $\qquad y = 2\sqrt{ax}$

$x = 4a$ $\qquad y = x^2/4a$

$y^2 = 4ax$ & $x^2 = 4ay$

Now, curve tracing



$$= \int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\, dy.$$

$$= \int_{y=0}^{4a} [x]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \int_0^{4a}\left(2a^{1/2}y^{1/2} - \frac{y^2}{4a}\right)dy$$

$$= \left[2a^{1/3}\frac{2}{3}y^{3/2} - \frac{y^2}{12a}\right]_0^{4a} dy$$

$$= \frac{4}{3}a^{1/2}\, 8a^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{ squares unit}$$

Ans 18) i) $\displaystyle\int_0^1\int_{y^2}^{y}(1+xy^2)\, dx\, dy$

$$= \int_{y=0}^{1}\int_{x=y^2}^{y}(1+xy^2)\, dx\, dy.$$

$$= \int_{y=0}^{1}\left[x + \frac{x^2y^2}{2}\right]_{y^2}^{y} dy.$$

$$= \int_{y=0}^{1}\left(y + \frac{y^4}{2} - y^2 - \frac{y^6}{2}\right)dy = \left[\frac{y^2}{2} + \frac{y^5}{10} - \frac{y^3}{3} - \frac{y^7}{14}\right]_0^1 = \frac{41}{210}$$ (Ans)

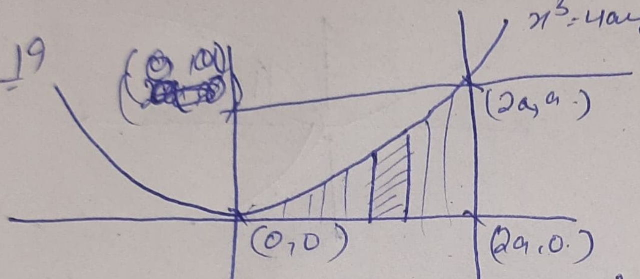
ii) $\int_0^1\int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\,dx\,dy$

$= \int_{x=0}^{1}\int_{y=0}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-y^2}}\,dy\,dx$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}} [\sin^{-1} y]_0^1\,dx.$

$= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}[\sin^{-1}x]_0^1 = \frac{\pi^2}{4}$

Ans 19



$\iint_S xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx$

$= \int_0^{2a} \left[\frac{xy^2}{2}\right]_0^{x^2/4a} dx$

$= \int_0^{2a} \frac{x}{2}\cdot\frac{x^2}{16a^2}\,dx.$

$= \int_0^{2a} \frac{x^5}{32a^2}\,dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{(2a)^6}{32\times 6a^2}$

$= \frac{1}{3}a^4$ (Ans)

Ans 20) $\iint \frac{r\,dr}{\sqrt{a^2+r^2}}$

$= \int_{\theta=\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r\,dr}{\sqrt{a^2+r^2}}\,d\theta$

Let $a^2+r^2=t$
$2r\,dr = dt$
$r=0,\ t=a^2$
$r=a\sqrt{\cos 2\theta}$
$t=2a^2\cos^2\theta$

$= \int_{\theta=\pi/4}^{\pi/4} \frac{1}{2}\int_{t=a^2}^{2a^2\cos^2\theta} \frac{1}{\sqrt{t}}\,dt\,d\theta$

$= \int_{\theta=\pi/4}^{\pi/4} \frac{1}{2}[2\sqrt{t}]_{a^2}^{2a^2\cos^2\theta} = 2a\int_0^{\pi/4} \sqrt{2}\cos\theta - 1)\,d\theta$

$= 2a(1-\frac{\pi}{4})$ (Ans)

Ans 2) $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} x^2yz\, dx\,dy\,dz$

$= \int_{x=0}^{1}\int_{y=0}^{2} \frac{3x^2}{2} y\, dy\, dy$

$= \int_{x=0}^{1} \frac{3x^2}{2} \int_{y=0}^{2} y\, dy\, dx.$

$= \int_{x=0}^{1} \frac{3x^2}{2}\left[\frac{y^2}{2}\right]_0^2 dx = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$

ii) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+3}\, dx\,dy\,dz$

$= \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{x+y} e^{x+y+3}\, dx\,dy\,dz$

$= \int_{x=0}^{a}\int_{y=0}^{x} \left[e^{x+y+3}\right]_0^{x+y} dy\, dx$

$= \int_{x=0}^{a}\int_{y=0}^{x} \left[e^{2(x+y)} - e^{x+y}\right] dy\, dx$

$= \int_{x=0}^{a} \left[\frac{1}{2} e^{2(x+y)} - e^{x+y}\right]_0^x$

$= \int_{x=0}^{a} \left(\frac{1}{2}e^{4x} - e^{2x} - \frac{1}{2}e^{2x} + e^{x}\right) dx$

$= \left[\frac{1}{8}e^{4x} - \frac{1}{2}e^{2x} - \frac{1}{4}e^{2x} + e^{x}\right]_0^a$

$= \frac{1}{8}e^{4x} - \frac{3}{4}e^{2x} + e^{x}\Big]_0^a.$

$= \frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^{a} - \frac{1}{8} + \frac{3}{4} - 1$

$= \frac{e^{4a} - 6e^{2a} + 8e^{a} - 3}{8}$

Ans 22

$$\iiint x^{1-1} y^{1-1} 3^{1-1} (1+x+y+3)^4 \, dx\, dy\, dz$$

$$= \frac{\sqrt{1}\sqrt{1}\sqrt{1}}{\sqrt{3}} \int_0^1 u^{1+1+1-1} (1+u)^4 \, du$$

$$= \frac{1}{\sqrt{3}} \int_0^1 u^2 (1+u)^4 \, du$$

Let $1+u = t$

$du = dt$

$u = 0, t = 1$

$u = 1, t = 2$

$$= \frac{1}{2} \int_1^2 (t-1)^2 t^4 \, dt$$

$$= \frac{1}{2} \int_1^2 (t^2 + 1 - 2t)\, t^4 \, dt$$

$$= \frac{1}{2} \int_1^2 (t^6 + t^4 - 2t^5) \, dt$$

$$= \frac{1}{2} \left[\frac{t^7}{7} + \frac{t^5}{5} - \frac{2t^6}{6} \right]_1^2$$

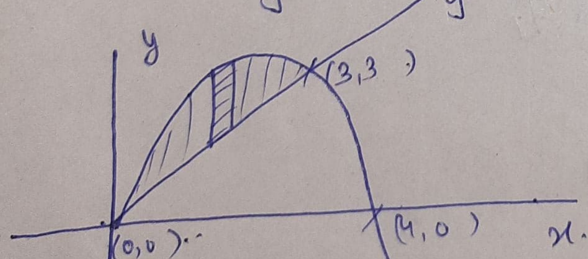
$$= \frac{1}{2} \left(\frac{2^7}{7} + \frac{2^5}{5} - \frac{2^6}{3} - \frac{1}{7} - \frac{1}{5} + \frac{1}{3} \right)$$

$$= \frac{1}{2} \left(\frac{32 \times 11}{105} - \frac{1}{105} \right)$$

$$= \frac{1}{2} \times \frac{351}{105} = \frac{351}{210}$$

Ans 23) $y = 4x - x^2$

$y = x.$



Area $\iint dy\, dx.$

$$= \int_{x=0}^{3} \int_{y=4x-x^2}^{x} dy\, dx.$$

$$= \int_{x=0}^{3} [y]_{4x-x^2}^{x} \, dx.$$

$$= \int_0^3 (x - 4x + x^3)\, dx.$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^3 = \frac{27}{3} - \frac{27}{2} = 9 - \frac{27}{2} = \frac{9}{2}$$ Area square unit.

Ans 24) $x^2 + y^2 = 4z$

$y + z = 4$ & $z = 0$

$$V = \iiint dx\,dy\,dz = \int_{x=-2}^{x=+2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{4-y} dx\,dy\,dz$$

$$V = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{4-y}\,dy\,dx$$

$$V = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx.$$

$$V = \int_{x=-2}^{2} \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

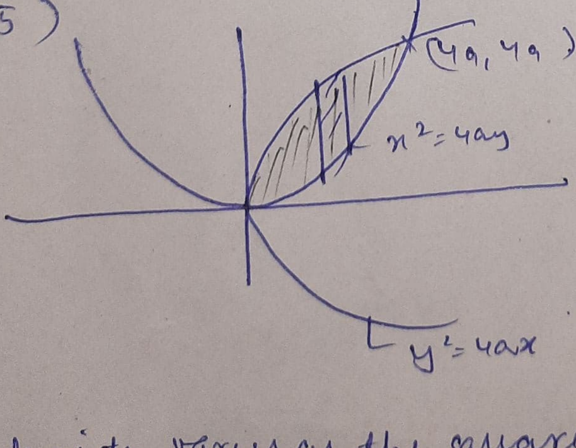
$$V = \int_{-2}^{2} 4\sqrt{4-x^2} - \frac{(4-x^2)}{2} + 4\sqrt{4-x^2} + \frac{(4-x^2)}{2}\Big)\,dx.$$

$$V = \int_{-2}^{2} 8(\sqrt{4-x^2}\,dx.$$

$$V = 16\int_0^2 \sqrt{4-x^2}\,dx = 16\left[\frac{x}{2}\cdot\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2$$

$$V = 16 \times 2 \times \frac{\pi}{2}. \qquad V = 16\pi \text{ cubic units}$$

Ans 25)



$\because$ density varies as the square of distance from the origin

$$\rho = K(x^2 + y^2)$$

$$m = K\int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} (x^2 + y^2)\,dy\,dx.$$

$$m = K \int_{x=0}^{4a} \left(x^2 y + \frac{y^3}{3} \right]_{x^2/4a}^{2\sqrt{ax}} dx.$$

$$m = K \int_0^{4a} \left(2a^{1/2} x^{5/2} + 8a^{3/2} x^{3/2} - \frac{x^4}{4a} - \frac{x^6}{92a^3} \right) dx$$

$$m = K \left[\frac{4}{7} a^{1/2} x^{7/2} + \frac{16}{15} a^{3/2} x^{5/2} - \frac{x^5}{20a} - \frac{x^2}{192 \times 7a^3} \right]_0^a$$

$$m = K \left[\frac{4}{7} a^4 + \frac{16}{15} a^4 - \frac{a^4}{20} - \frac{a^4}{1344} \right)$$

$$m = K \frac{(10667)\, a^4}{6720}$$

$$\boxed{m = 1.587\, Ka^4}$$