Unit (2) Difference equations

Part (i)

Here we consider the sequence of functions,

$\{y_n\} = y_1, y_2, y_3, y_4, \ldots\ldots$ where $y_n = y(x_n)\ \forall n \in \mathbb{N}$

Forward difference operator (Δ)

Forward difference operator (denoted by Δ) is defined for a function $f(x)$

as $\Delta f(x) = f(x+h) - f(x)$.

In this unit, we'll be assuming '$h=1$'

So we use, $\boxed{\Delta f(x) = f(x+1) - f(x).}$ — (1)

eg If $f(x) = x$

$\Delta f(x) = \Delta x = (x+1) - x = 1$

If $f(x) = x^2$

$$\Delta f(x) = \Delta(x^2) = f(x+1) - f(x)$$
$$= (x+1)^2 - x^2$$
$$= 2x+1$$

$\therefore$ For the sequence $\{y_n\}$, we write,

$\boxed{\Delta y_n = y_{n+1} - y_n.}$ — (2)

So $\Delta y_2 = y_3 - y_2$ (By using ②)

$\Delta y_3 = y_4 - y_3$

$\Delta y_4 = y_5 - y_4$

$\vdots$

$\Delta y_n = y_{n+1} - y_n$

$\Delta y_{n+1} = y_{n+2} - y_{n+1}$ and so on.

Similarly, • $\Delta^2 y_2 = \Delta(\Delta y_2) = \Delta(y_3 - y_2)$

$= \Delta y_3 - \Delta y_2$

$= (y_4 - y_3) - (y_3 - y_2)$

$= y_4 - 2y_3 + y_2$

• $\Delta^2 y_n = \Delta(\Delta y_n) = \Delta(y_{n+1} - y_n)$

$= \Delta y_{n+1} - \Delta y_n$

$= (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n)$

$= y_{n+2} - 2y_{n+1} + y_n$

• $\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1}) = \Delta(y_n - y_{n-1})$

$= \Delta y_n - \Delta y_{n-1} = (y_{n+1} - y_n) - (y_n - y_{n-1})$

$= y_{n+1} - 2y_n + y_{n-1}$.

In the sequence $\{y_n\} = y_1, y_2, y_3, \ldots, y_n, \ldots$ the suffices $1, 2, 3, \ldots, n, \ldots$ are known as arguments.

Def^n: A relation b/w various differences of unknown functions (i.e., $y_1, y_2, \ldots y_n, \ldots$) at one or more values of arguments is known as difference equation.

Eg ① $\Delta y_{n+2} = 2$

② $\Delta y_n + y_{n-1} = 4^n$

③ $y_{n+1} - y_n = n.$

Note that ① can be written as.

$$y_{n+3} - y_{n+2} = 2$$

& ② can be written as

$$y_{n+1} - y_n + y_{n-1} = 4^n.$$

Formation of difference equation:

A difference equation can be formed by applying 'Δ' on both sides of given eqn of curve and eliminating arbitrary constants.

Eg: Form the difference eqn corresponding to the family of curves $y = ax+bx^2$.

Soln we have $y = ax + bx^2$ — (1)

Applying Δ on both sides of (1)

$$\Delta y = a\,\Delta x + b\,\Delta x^2$$

$$\Rightarrow \Delta y = a(x+1-x) + b\left[(x+1)^2 - x^2\right]$$

$$= a + b(2x+1). \quad -(2)$$

Applying Δ^2 on both sides of (1),

$$\Delta^2 y = \Delta(\Delta y) = \Delta\left[a + b(2x+1)\right]$$

$$= a + b(2(x+1)+1) - \left[a + b(2x+1)\right]$$

(Using $\Delta f(x) = f(x+1) - f(x)$.

$$= 2b.$$

$$\Rightarrow b = \frac{\Delta^2 y}{2} \quad -(3)$$

Using (3) in (2), $\Delta y = a + \frac{\Delta^2 y}{2}(2x+1)$

$$\Rightarrow a = \Delta y - \frac{\Delta^2 y}{2}(2x+1)$$

Substituting the values of a & b in (1)

$$y = \left(\Delta y - \frac{\Delta^2 y}{2}(2x+1)\right)x + \frac{\Delta^2 y}{2}(x^2)$$

$$\Rightarrow \Delta^2 y\left(\frac{x^2}{2} - x^2 - \frac{x}{2}\right) + \Delta y\,(x) - y = 0.$$

$$\Rightarrow \boxed{(x^2+x)\cdot\Delta^2 y - 2\cdot x\,\Delta y + 2y = 0}$$

which is the required difference eqn.

Eg form the difference eqn for,

$$y_n = a(3^n) + b(5^n) \quad \text{—} (1)$$

Soln Applying Δ on both sides of (1),

$$\Delta y_n = a\,\Delta 3^n + b\,\Delta 5^n$$

$$= a(3^{n+1} - 3^n) + b(5^{n+1} - 5^n)$$

$$= a\cdot 3^n(3-1) + b(5^n)(5-1)$$

$$= 2a\cdot 3^n + 4b\cdot 5^n \quad \text{—} (2)$$

Applying Δ^2 on both sides of (1)

$$\Delta^2 y_n =$$

$$\Delta(\Delta y_n) = \Delta(2a3^n + 4b5^n)$$

$$= 2a(\Delta 3^n) + 4b(\Delta 5^n)$$

$$= 2a(3^{n+1} - 3^n) + 4b(5^{n+1} - 5^n)$$

$$= 4a\cdot 3^n + 16b\cdot 5^n. \quad \text{—} (3)$$

Solving eqn (2) & (3) to find a & b,

$$\Delta y_n = 2a\cdot 3^n + 4b\cdot 5^n \quad \times 4$$

$$\Delta^2 y_n = 4a\cdot 3^n + 16 b\cdot 5^n$$

$$\Rightarrow 2\Delta y_n - \Delta^2 y_n = -8b5^n$$

$$\Rightarrow b = \frac{\Delta^2 y_n - 2\Delta y_n}{8(5^n)}$$

Substituting (b') in (2'),

$$a = \frac{4\Delta y_n - \Delta^2 y_n}{4(3^n)}$$

Putting a & b in (1), we get

$$y_n = \frac{4\Delta y_n - \Delta^2 y_n}{4} + \frac{\Delta^2 y_n - 2\Delta y_n}{8}$$

$$\Rightarrow \boxed{8y_n = 6\Delta y_n - \Delta^2 y_n} \quad \text{— (4)}$$

which is the required diff. eqn.

Also (4) can be written as,

$$8y_n = 6(y_{n+1} - y_n) - (y_{n+2} - 2y_{n+1} + y_n)$$

$$\Rightarrow \boxed{15 y_n = 8y_{n+1} - y_{n+2}}$$

Linear difference equations

is that in which $y_{n+1}, y_{n+2}, \ldots$ etc. occur in 1st degree only & are not multiplied together.

A linear difference eqn with constant coefficients is of the form

$$2y_{n+1} + 3y_{n+2} = 3^n$$

$$y_{n+1} + y_n = 0.$$

Though, $y_{n+1} \cdot y_n = n+1$, $y_{n+2}^2 + y_{n+1}^2 + y_n = 2^n$ are 'not' linear.

<u>In general</u>, $y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \ldots + a_r y_n = f(n)$ — (1)

is linear difference equation with constant coefficients $a_1, a_2, \ldots, a_n$. $f(n)$ is any function of n.

Eg $y_{n+1} + 2y_{n+2} + 5y_{n+3} = n + e^n$

- Solution of eqn (1) is of the form,

$y_n = C.F. + P.I$, where, C.F. = Complementary f
P.I = Particular integral

Method to find complementary function.

write eqn (1) as

$$y_{n+r} + a_1 y_{n+r-1} + \cdots + a_r y_n = 0 \quad (2)$$

<u>Step (1)</u> Put $y_{n+k} = E^k . y_n$ in (2) to obtain $(E^r + a_1 E^{r-1} + \cdots + a_r) y_n = 0$ (3)

<u>step (2)</u> write auxiliary eqn from (3) as

$$E^r + a_1 E^{r-1} + \cdots + a_r = 0 \quad (4)$$

& solve for E.

<u>step (3)</u> let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the roots of (4) then if

<u>Case (i)</u> $\lambda_1, \lambda_2, \ldots, \lambda_r$ are real & distinct then

$$C.F. = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_r \lambda_r^n.$$

<u>Case (ii)</u> If two roots are equal i.e., $\lambda_1 = \lambda_2$ then

$$C.F. = (c_1 + n c_2) \lambda_1^n.$$

<u>Case (iii)</u> If three roots are equal. i.e., $\lambda_1 = \lambda_2 = \lambda_3$ then

$$C.F. = (c_1 + n c_2 + n^2 c_3) . \lambda_1^n .$$

<u>Case (iv)</u> If 2 roots are complex i.e, $\lambda_1 = \alpha + i\beta$ & $\lambda_2 = \alpha - i\beta$ then

C.F. = $c_1(\alpha+i\beta)^n + c_2(\alpha-i\beta)^n$ and then write $\alpha+i\beta$, $\alpha-i\beta$ in polar form as $\alpha = r\cos\theta$, $\beta = r\sin\theta$, $r = \sqrt{\alpha^2+\beta^2}$

$\theta = \tan^{-1}\beta/\alpha$.

Eg. Find C.F. for $y_{n+2} + 2y_{n+1} - 8y_n = 0$ — (1)

Soln: Let $y_{n+k} = E^k y_n$

then (1) $\Rightarrow$ $(E^2 + 2E - 8)y_n = 0$

Auxilary eqn is $E^2+2E-8 = 0$

$\Rightarrow E^2 - 4E + 2E - 8 = 0$

$\Rightarrow E(E-4) + 2(E-4) = 0$

$\Rightarrow E = -2, 4$

$\therefore$ By case (i) C.F. $= c_1(-2)^n + c_2(4)^n$.

Eg. Solve $y_{n+3} - 4y_{n+2} + y_{n+1} + 6y_n = 0$ — (1)

Soln: It is of the form (2), with $f(n) = 0$, so P.I. $= 0$ so,

General soln $= y_n = CF + PI = CF + 0$
$= C.F.$

we only need to find C.F.,

let $y_{n+k} = E^k.y_n$ in (1) then

$(E^3 - 4E^2 + E + 6)y_n = 0$

Auxilary eqn is $E^3 - 4E^2 + E + 6 = 0$

$\Rightarrow \quad (E+1)(E^2-5E+6) = 0$

$\therefore \quad \Rightarrow (E+1)(E-2)(E-3) = 0$

$\Rightarrow \quad E = -1, 2, 3$

$\therefore \quad C.F. = c_1(-1)^n + c_2(2)^n + c_3(3)^n.$

General soln,

$y_n = c_1(-1)^n + c_2(2)^n + c_3(3)^n.$

(by case (i))

Eg Solve $y_{n+2} - 6y_{n+1} + 9y_n = 0$

Soln Again $f(n) = 0 \quad \therefore P.S. = 0$

General soln is $y_n = C.F. + 0.$

for C.F. let $y_{n+k} = E^k . y_n.$

$\Rightarrow (E^2 - 6E + 9)y_n = 0$

Auxilary eqn $\rightarrow \quad E^2 - 6E + 9 = 0$

$\Rightarrow (E-3)(E-3) = 0$

$\Rightarrow \quad E = 3, 3.$

$\therefore \quad C.F. = (c_1 + nc_2)(3^n)$ by case (ii).

$\therefore \quad y_n = (c_1 + nc_2)3^n.$

Eg. Solve $y_{n+2} + 2y_{n+1} + 4y_n = 0.$

Soln. let $y_{n+r} = E^r . y_n.$

then we get $(E^2 + 2E + 4)y_n = 0$

Auxilary eqn is $E^2 + 2E + 4 = 0$

$\Rightarrow E = -1 \pm i\sqrt{3}$

then, $y_n = C.F. = c_1(-1+i\sqrt{3})^n + c_2(-1-i\sqrt{3})^n$.

now, to convert it in polar form, we write, $-1 = r\cos\theta$

$\sqrt{3} = r\sin\theta$.

$\left(r = \sqrt{1+3} = 2, \quad \theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}\right)$ — (*)

$\therefore y_n = c_1[r\cos\theta + ir\sin\theta]^n + c_2[r\cos\theta - ir\sin\theta]^n$.

$= r^n[c_1(\cos n\theta + i\sin n\theta) + c_2(\cos n\theta - i\sin n\theta)]$

Using. $(\cos\theta + i\sin\theta)^m = (\cos m\theta + i\sin m\theta)$

$\Rightarrow y_n = r^n[(c_1+c_2)\cos n\theta + (c_1 i - c_2 i)\sin n\theta]$

$= r^n[A_1\cos n\theta + A_2\sin n\theta]$

where $A_1 = c_1 + c_2$

$A_2 = (c_1 - c_2)i$

$\Rightarrow y_n = 2^n\left[A_1\cos\frac{2n\pi}{3} + A_2\sin\frac{2n\pi}{3}\right]$

(using (*))

Ans.

The particular integral (P.I.)

Method to find P.I.,

Consider the linear difference eqn,

$$y_{n+r} + a_1 y_{n+r-1} + \cdots + a_r y_n = f(n) \quad \text{(1)}$$

Putting $y_{n+k} = E^k . y_n$ in (1), we write its 'symbolic form' as,

$$(E^r + a_1 E^{r-1} + \cdots + a_r) y_n = f(n)$$

Let $\phi(E) = E^r + a_1 E^{r-1} + \cdots + a_r$

then $\phi(E) . y_n = f(n)$.

So P.I. = $\boxed{y_n = \frac{1}{\phi(E)} . f(n)}$

<u>Case (i)</u> If <u>$f(n) = a^n$</u> for some const. a.

$$P.I. = \frac{1}{\phi(E)} . a^n = \frac{1}{\phi(a)} . a^n \text{ if } \underline{\phi(a) \neq 0}$$

But <u>If $\phi(a) = 0$</u> then, write $\phi(E)$ in power of $(E-a)$.

(i) If <u>$\phi(E) = (E-a)$</u>:

So $(E-a) . y_n = a^n$

$$\Rightarrow PI = \frac{1}{E-a} . a^n = n . a^{n-1}.$$

(2) If $\phi(E) = (E-a)^2$

then $\phi(E) \cdot y_n = a^n \Rightarrow (E-a)^2 \cdot y_n = a^n$

$$\Rightarrow P.I. = \frac{1}{(E-a)^2} \cdot a^n = \frac{n(n-1)}{2!} a^{n-2}.$$

(3) If $\phi(E) = (E-a)^3$

then $$P.I. = \frac{1}{(E-a)^3} \cdot a^n = \frac{n(n-1)(n-2)}{3!} a^{n-3}$$

and so on.

Eg Find P.I. for $y_{n+2} - 3y_{n+1} + 2y_n = 4^n$.

Soln symbolic form is,

$$(E^2 - 3E + 2)\, y_n = 4^n$$

$$\Rightarrow P.I. = \frac{1}{E^2 - 3E + 2} 4^n, \quad \text{let } \phi(E) = E^2 - 3E + 2$$

Since $f(n) = 4^n$

$$\therefore P.I. = \frac{1}{\phi(4)} \cdot 4^n \quad \text{as } \phi(4) = 4^2 - 3(4) + 2 = 6 \neq 0$$

$$= \frac{1}{6} \cdot 4^n \quad \text{Ans.}$$

Eg Solve $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.

Soln Soln is $y_n = C.F. + P.I.$

for C.F. Auxilary eqn. $E^2-6E+9=0$

$\Rightarrow E = 3, 3$

$\Rightarrow C.F. = (C_1 + C_2 n) 3^n.$

for P.I. symbolic form is

$(E^2 - 6E+9) \cdot y_n = 3^n.$

$P.I. = \frac{1}{E^2-6E+9} \cdot 3^n$, here, $\phi(E) = E^2-6E+9$

& $\phi(3) = 0.$

$\therefore$ we write $\phi(E)$ in powers of $(E-3)$.

$\phi(E) = E^2-6E+9 = (E-3)^2.$

$\therefore P.I. = \frac{1}{(E-3)^2} \cdot 3^n = \frac{n(n-1)}{2!} 3^{n-2}$

(by ② of case (i))

Hence General solⁿ,

$y_n = C.F. + P.I. = (C_1 + nC_2) 3^n + \frac{n(n-1)}{2!} 3^{n-2}.$

Case (ii) If $f(x) = \cos kx$

$$PI = \frac{1}{\phi(E)} \cdot \cos kx = \frac{1}{\phi(E)} \left[\frac{e^{ikx} + e^{-ikx}}{2} \right]$$

$$= \frac{1}{2}\left[\frac{1}{\phi(E)} e^{ikn} + \frac{1}{\phi(E)} e^{-ikn}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\phi(E)} (e^{ik})^n + \frac{1}{\phi(E)} (e^{-ik})^n\right].$$

$$= \frac{1}{2}\left[\frac{1}{\phi(E)} a^n + \frac{1}{\phi(E)} b^n\right]$$ where $a = e^{ik}$, $b = e^{-ik}$.

then apply case (i).

Case (iii) If $f(n) = \sin kn$.

Similar way as above, by using

$$\sin kn = \frac{e^{ikn} - e^{-ikn}}{2i}$$

Eg. Solve $u_{n+2} - 7u_{n+1} + 12u_n = \cos n$.

Soln for C.F. Auxilary eqn

$$E^2 - 7E + 12 = 0$$

$$\Rightarrow E = 3, 4$$

$$\Rightarrow C.F. = c_1 3^n + c_2 4^n \quad \text{—(1)}$$

for P.I. Symbolic form

$$(E^2 - 7E + 12) u_n = \cos n$$

$$P.I = \frac{1}{E^2 - 7E + 12} \cos n.$$ So, we apply case (ii)

$$= \frac{1}{E^2-7E+12}\left[\frac{e^{in}+e^{-in}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{E^2-7E+12}(e^{in}) + \frac{1}{E^2-7E+12}e^{-in}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\phi(e^{i})}(e^{i})^n + \frac{1}{\phi(e^{-i})}(e^{-i})^n\right]$$

where $\phi(E) = E^2-7E+12$

and $\phi(e^{i}) \neq 0$

$\phi(e^{-i}) \neq 0$

So by case (i)

$$PS = \frac{1}{2}\left[\frac{1}{e^{2i}-7e^{i}+12}(e^{i})^n + \frac{1}{e^{-2i}-7e^{-i}+12}(e^{-i})^n\right] \quad (2)$$

Hence Sol^n is $y_n = CF + PS$ where

CF & PS are given by (1) & (2)

Factorial form :->

is product of the form $x(x-1)(x-2)(x-3) \cdots (x-r+1)$ & is denoted by $[x]^r$.

eg $[x]^2 = x(x-1) \quad (r=2)$

$[x]^3 = x(x-1)(x-2) \quad (r=3)$

$[x] = x \quad (r=1)$

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- Coefficient of highest degree in a factorial form of a polynomial remains same. — (a)
- $\Delta[x]^r$ represents differentiation. — (b)

i.e., $\Delta[x]^r = r[x]^{r-1}$

Eg. write following polynomials in factorial form,

$x^2 - x + 1$

Sol: $x^2 - x + 1 = x(x-1) + 1$

$= [x]^2 + 1$

$x^2 + x - 1$

Sol: let $x^2 + x - 1 = [x]^2 + A[x] + B$.

(by (b) coeff. of $[x]^2$ remains same.)

$\Rightarrow x^2 + x - 1 = x(x-1) + Ax + B$

Putting $x = 0$ on both sides

$\Rightarrow B = -1$

Putting $x = 1$ on both sides,

$1 = A - 1 \Rightarrow A = 2$

$\therefore$ Factorial form of x^2+x-1 is
$[x]^2+2[x]-1$.

Eg. $2x^3-3x^2+3x-10$.

Sol^n Let $2x^3-3x^2+3x-10$
$= 2[x]^3+B[x]^2+C[x]+D$

$\Rightarrow 2x^3-3x^2+3x-10 = 2[x]^3+B[x]^2+C[x]+D$
$= 2x(x-1)(x-2)+Bx(x-1)$
$+Cx+D$

Putting $x=0$ on both sides,

$-10 = D$

Putting $x=1$ on both sides,

$C = 2$

Putting $x=2$ on both sides,

$B = 3$.

$\therefore$ Factorial form of given polynomial becomes $2[x]^3+3[x]^2+2[x]-10$.

Case (iv) $f(n) = n^k$

$$P.I. = \frac{1}{\phi(E)} f(n) = \frac{1}{\phi(E)} \cdot n^k$$

$$= \frac{1}{\phi(1+\Delta)} \quad (\text{factorial form of } n^k)$$

then apply binomial theorem on $[\phi(1+\Delta)]^{-1}$ till Δ^k.

Case (V) If $f(n) = a^n . F(n)$ where $F(n)$ is polynomial

$$P.I. = \frac{1}{\phi(E)} f(n)$$

$$= \frac{1}{\phi(E)} \cdot a^n . F(n) = a^n . \frac{1}{\phi(aE)} F(n).$$

then proceed as case (iv).

Eg Find P.I. for $y_{n+2} - 4y_n = n^2+n-1$.

Soln. Here $f(n) = n^2+n-1$. (a polynomial of deg. 2. so apply case (iv))

symbolic form

$$(E^2-4)y_n = n^2+n-1$$

$$\Rightarrow P.I. = \frac{1}{E^2-4}(n^2+n-1)$$

$= \dfrac{1}{(1+\Delta)^2-4}$ factorial form of (n^2+n-1)

$= \dfrac{1}{\Delta^2+2\Delta-3}\left([n]^2+2[n]-1\right)$

$= \dfrac{-1}{3}\left[1-\left(\dfrac{2\Delta}{3}+\dfrac{\Delta^2}{3}\right)\right]^{-1}\left\{[n]^2+2[n]-1\right\}$

$= -\dfrac{1}{3}\left[1+\left(\dfrac{2\Delta}{3}+\dfrac{\Delta^2}{3}\right)+\left(\dfrac{2\Delta}{3}+\dfrac{\Delta^2}{3}\right)^2+\cdots\right] \cdot \left\{[n]^2+2[n]-1\right\}$

(Using $(1-x)^{-1} = 1+x+x^2+\cdots$)

$= \dfrac{-1}{3}\left[1+\dfrac{2\Delta}{3}+\dfrac{\Delta^2}{3}+\dfrac{4\Delta^2}{9}+\cdots\right]\left\{[n]^2+2[n]-1\right\}$

↳ (expand till Δ^2 ∵ $K=2$)

$= \dfrac{-1}{3}\left[1+\dfrac{2\Delta}{3}+\dfrac{7\Delta^2}{9}+\cdots\right]\left\{[n]^2+2[n]-1\right\}$

$= \dfrac{-1}{3}\left[[n]^2+2[n]-1+\dfrac{2}{3}\Delta\left([n]^2+2[n]-1\right)+\dfrac{7\Delta^2}{9}\left([n]^2+2[n]-1\right)\right]$

$= \dfrac{-1}{3}\left[[n]^2+2[n]-1+\dfrac{2}{3}(2[n]+2)+\dfrac{7}{9}(2)\right]$

$= -\dfrac{[n]^2}{3}-\dfrac{10[n]}{9}-\dfrac{17}{27}$

(Using $\Delta[x]^r = r[x]^{r-1}$)

$$= -\frac{n(n-1)}{3} - \frac{10n}{9} - \frac{17}{27}$$

$$= -\frac{n^2}{3} - \frac{7n}{9} - \frac{17}{27}.$$

Eg. Solve $y_{n+2} - 2y_{n+1} + y_n = n^2 \cdot 2^n$.

Sol^n C.F. $= (C_1 + nC_2)(1)^n$ (can be found easily as before)

for P.I. Symbolic form,

$$(E^2 - 2E + 1) y_n = n^2 \cdot 2^n.$$

$$\Rightarrow P.I. = \frac{1}{E^2 - 2E + 1} n^2 \cdot 2^n.$$

Here $f(n) = n^2 \cdot 2^n$ is of the form, $a^n F(n)$ where $a = 2$, $F(n) = n^2$, so we apply case (v).

$$\therefore P.I. = 2^n \cdot \frac{1}{(2E)^2 - 2(2E) + 1} n^2$$

$$= 2^n \frac{1}{(2E-1)^2} n^2$$

$$= 2^n \frac{1}{[2(1+\Delta) - 1]^2} \quad \text{(factorial form of } n^2)$$

$$= 2^n \frac{1}{(1+2\Delta)^2} ([n]^2 + [n])$$

$$= 2^n [1+2\Delta]^{-2} ([n]^2 + [n])$$

$$= 2^n (1 - 4\Delta + 12\Delta^2 - \dots)([n]^2 + [n])$$

(using $(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$)

$= 2^n\left([n]^2+[n] - 4\Delta([n]^2+[n]) + 12\Delta^2([n]^2+[n])\right)$

$= 2^n\left([n]^2+[n] - 4(2[n]+1) + 24\right)$

$= 2^n\left([n]^2 - 7[n] + 20\right)$

$= 2^n\left(n(n-1) - 7n + 20\right)$

$= 2^n(n^2 - 8n + 20)$

$\therefore$ solution is $y_n = C.F. + P.I$

$= (c_1 + n c_2) + 2^n(n^2 - 8n + 20)$

Simultaneous difference equations

1) $\Delta [x]^{r}$ is same as differentiating x^{r}.

2) $\frac{1}{\Delta}[x]^{r}$ is same as integrating x^{r}.

3) $E = 1+\Delta$, $E \rightarrow$ shift operator
$\Delta \rightarrow$ Forward difference operator.

Eg. Solve the simultaneous difference eqn

$u_{n+1} - 7u_n - 10v_n = 0$ — (1)

$v_{n+1} - 4v_n - u_n = 0$ — (2) , $u_0 = 3$, $v_0 = 2$.

Sol^n Let $u_{n+k} = E^{k}.u_n$ in (1) & (2)

$v_{n+k} = E^{k}.v_n$

$\Rightarrow \quad (E-7)u_n - 10v_n = 0 \; \} \times (E-4)$

$\quad (E-4)v_n - u_n = 0 \; \} \times 10$

$\Rightarrow \quad (E-4)(E-7)u_n - 10u_n = 0 \quad \text{--- (3)}$

$\Rightarrow \quad [(E-4)(E-7) - 10]u_n = 0 \rightarrow$ Symbolic form

$\Rightarrow$ Auxiliary eqn $= (E-4)(E-7) - 10 = 0$

$\Rightarrow \quad E^2 - 11E + 18 = 0$

$\Rightarrow \quad (E-9)(E-2) = 0$

$\Rightarrow \quad E = 2, 9$

$\Rightarrow$ C.f. $= c_1 2^n + c_2 9^n$

Since R.H.S of (3) is $0 \Rightarrow$ P.I. $= 0$

$\therefore \quad u_n = c_1 2^n + c_2 9^n.$

Putting u_n in (1), we get, $v_n = \dfrac{-2^n}{2}c_1 + \dfrac{9^n c_2}{5}$

Now $u_0 = 3 \Rightarrow u_n = 3$ at $n = 0$

$\Rightarrow \quad u_0 = c_1 + c_2 = 3 \quad \text{--- (4)}$.

$v_0 = 2 \Rightarrow v_n = 2$ at $n = 0$

$\Rightarrow \quad -\dfrac{c_1}{2} + \dfrac{c_2}{5} = 2$

$\Rightarrow \quad -5c_1 + 2c_2 = 20 \quad \text{--- (5)}$

Solving (4) & (5), $c_1 = -2$, $c_2 = 5$.

Hence $u_n = -2 \cdot 2^n + 5 \cdot 9^n$,

$v_n = 2^n + 9^n$

Eg Solve $x_{n+1} - 3x_n - 2y_n = -n$ —(1)

$y_{n+1} - 2y_n - x_n = n$ —(2)

Soln let $x_{n+k} = E^k . x_n$ in (1)

$y_{n+k} = E^k y_n$ in (2)

$\Rightarrow (E-3) . x_n - 2y_n = -n$

$-x_n + (E-2) y_n = n \quad \} \times (E-3)$

$\Rightarrow [-2 + (E-3)(E-2) . y_n] = -n + n(E-3)$

$\Rightarrow [-2 + (E^2 - 5E + 6)] y_n = (En - 3n) - n.$

(Using (3) property,

$En = (1+\Delta) . n$

$= n + \Delta n$

$= n + (n+1-n)$

$= n+1$)

$\Rightarrow (E^2 - 5E + 4) y_n = 1 - 3n$ —(3)

C.F. $= c_1 + c_2 4^n .$

for P.I $\frac{1}{E^2 - 5E + 4} (1 - 3n)$

(factorial form of $(1-3n)$)

$= \frac{1}{(1+\Delta)^2 - 5(1+\Delta) + 4}$

$= \frac{1}{\Delta^2 - 3\Delta} (1 - 3[n])$

$= \frac{1}{-3\Delta (1 - \frac{\Delta}{3})} (1 - 3[n])$

$$= \frac{-1}{3\Delta}\left(1-\frac{\Delta}{3}\right)^{-1}(1-3[n])$$

$$= \frac{-1}{3\Delta}\left(1+\frac{\Delta}{3}+\frac{\Delta^2}{9}+\cdots\right)(1-3[n])$$

$$= \frac{-1}{3\Delta}\left(1-3[n]+\frac{1}{3}\Delta(1-3[n])\right)$$

$$= \frac{-1}{3\Delta}\left(1-3[n]+\frac{1}{3}(-3)\right)$$

$$= \frac{-1}{3\Delta}\left(-3[n]\right) = \frac{1}{\Delta}[n]$$

$$= \frac{[n]^2}{2} = \frac{n(n-1)}{2}.$$ {using property ②. $\frac{1}{\Delta}[n]$ represents integration}

Hence solution of ③ is given by,

$$y_n = c_1 + c_2 4^n + \frac{n(n-1)}{2}. \quad -④$$

putting in ①, we get,

$$x_n = -c_1 + 2c_2 4^n - \frac{n(n-1)}{2}. \quad -⑤$$

Again using $x_0 = 1$ & $y_0 = 0$ in ④ & ⑤, we get $c_1 = \frac{-1}{3}$, $c_2 = \frac{1}{3}$.

$$\Rightarrow y_n = \frac{-1}{3} + \frac{4^n}{3} + \frac{n(n-1)}{2}.$$

$$x_n = \frac{1}{3} + \frac{2(4)^n}{3} - \frac{n(n-1)}{2}$$ Ans.

Ex → Solve $u_{n+1} + v_n - 3u_n = n$

$3u_n + v_{n+1} - 5v_n = 4^n$.

Last topic in difference eqn →

"Application of difference eqn to deflection of a loaded string"

↳ Study article 31.8 (B.S. grewal) 42nd edn.)

& solve eg. 31.13 (P-1006).

Part (ii) $\quad Z$ - transform

Consider the sequence of functions $\{f_n\} = f_0, f_1, f_2, \ldots\ldots$ then Z-transf. of $\{f_n\}$ is defined as,

$$Z\{f_n\} = \sum_{n=0}^{\infty} f_n z^{-n} = F(z) \text{ (say)}, \quad -(1)$$

(provided the series $\sum_{0}^{\infty} f_n z^{-n}$ is convergent)

So inverse Z-transform implies

$$\{f_n\} = Z^{-1}\left(\sum_{n=0}^{\infty} f_n z^{-n}\right) = Z^{-1}(F(z))$$

• Note that $f_n = 0 \ \forall\, n<0$.

Z transform of some basic functions →

(1) $f_n = a^n \quad , n = 0, 1, 2, \ldots\ldots$

$$Z(f_n) = Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n} \quad \text{(by (1))}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \ldots\ldots$$

$$= \frac{1}{1 - a/z} = \frac{z}{z-a} \quad \text{(Sum of infinite G.P.)}$$

(2) $f_n = 1 \quad \forall\, n$.

Put $a = 1$ in $Z(a^n) = \dfrac{z}{z-a}$, we get

$$Z(1) = \frac{z}{z-1}.$$

③ $f_n = n^p$.

$$\mathcal{Z}(f_n) = \mathcal{Z}(n^p) = \sum_{0}^{\infty} n^p z^{-n}.$$

replacing p by $p-1$,

$$\mathcal{Z}(n^{p-1}) = \sum_{0}^{\infty} n^{p-1} \cdot z^{-n}$$

differentiating w.r. to z,

$$\Rightarrow \frac{d}{dz} \mathcal{Z}(n^{p-1}) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} n^{p-1} \cdot z^{-n}\right)$$

$$= \sum (-n) \cdot n^{p-1} z^{-(n+1)}$$

$$\Rightarrow \frac{d}{dz} \mathcal{Z}(n^{p-1}) = -\sum n^p z^{-(n+1)}$$

$$= -z^{-1} \sum_{0}^{\infty} n^p \cdot z^{-n}$$

$$= -z^{-1} \mathcal{Z}(n^p)$$

$$\Rightarrow \boxed{\mathcal{Z}(n^p) = -z \frac{d}{dz} \mathcal{Z}(n^{p-1})} \quad (*)$$

→ recurrence relation to find $\mathcal{Z}$-transf. of n^p.

Eg ① find $\mathcal{Z}(n)$.

Put $p = 1$ in $(*)$

$$\Rightarrow \mathcal{Z}(n) = -z \frac{d}{dz} \mathcal{Z}(n^0) = -z \frac{d}{dz} \mathcal{Z}(1)$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$$

(2) Find $Z(n^2)$

Put $p=2$ in (v).

$$Z(n^2) = -z\frac{d}{dz}Z(n)$$

$$= -z\left[\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)\right]$$

$$= \frac{-z\,(z-1-2z)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}.$$

Ex→ find $Z(n^3)$, $Z(n^4)$.

Properties of Z transform

(1) **Linearity**. $Z(au_n + bv_n) = aZ(u_n) + bZ(v_n)$

Proof $Z(au_n+bv_n) = \sum_{n=0}^{\infty}(au_n+bv_n)z^{-n}$

$$= a\sum_{n=0}^{\infty}u_n z^{-n} + b\sum_{n=0}^{\infty}v_n z^{-n}$$

$$= a\,Z(u_n) + b\,Z(v_n).$$

(2) **Damping rule**

If $Z(u_n) = U(z)$ then

a) $Z(a^{-n}u_n) = U(az)$

b) $Z(a^n u_n) = U(z/a)$.

Proof ① $Z(a^{-n} u_n) = \sum a^{-n} \cdot u_n \cdot z^{-n}$

$$= \sum_{0}^{\infty} u_n (az)^{-n}$$

$$= U(az).$$

② $Z(a^n u_n) = \sum a^n \cdot u_n \cdot z^{-n}$

$$= \sum_{n=0}^{\infty} u_n (z/a)^{-n}$$

$$= U(z/a)$$

③ **Shifting property**

If $Z(f_n) = F(z)$ then

(i) $Z\{f_{n-k}\} = z^{-k} F(z)$.

(shifting n to the left)

(ii) $Z\{f_{n+k}\} = z^{k}\left[F(z) - \sum_{n=0}^{k-1} f_n z^{-n}\right]$

Proof (i) L.H.S $= Z\{f_{n-k}\} = \sum_{n=0}^{\infty} f_{n-k} z^{-n}$

$$= \sum_{n=k}^{\infty} f_{n-k} z^{-n} \quad \left\{ \begin{array}{l} \because f_{n-k}=0 \\ \forall\, n-k<0 \\ \text{i.e., } \forall\, n<k. \end{array} \right.$$

$$= f_0 z^{-k} + f_1 z^{-(k+1)} + \cdots$$

$$= z^{-k}\left(f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots\right)$$

$$= z^{-k} \sum_{n=0}^{\infty} f_n z^{-n} = z^{-k} F(z).$$

Proof (ii) L.H.S $= z\{f_{n+k}\} = \sum_{n=0}^{\infty} f_{n+k} z^{-n}$

$$= z^{k} \left(\sum_{n=0}^{\infty} f_{n+k} z^{-n-k} \right)$$

$$= z^{k} \left(\sum_{n=0}^{\infty} f_{n+k} z^{-(n+k)} \right)$$

$$= z^{k} \left(\sum_{n=0}^{\infty} f_{n+k} z^{-(n+k)} - \sum_{n=0}^{k-1} f_n z^{-n} + \sum_{n=0}^{k-1} f_n z^{-n} \right)$$

$$= z^{k} \left[\left(f_k z^{-k} + f_{k+1} z^{-(k+1)} + \cdots \right) + \left(f_0 + f_1 z^{-1} + \cdots + f_{k-1} z^{-(k-1)} \right) - \sum_{n=0}^{k-1} f_n z^{-n} \right]$$

$$= z^{k} \left[\sum_{n=0}^{\infty} f_n z^{-n} - \sum_{n=0}^{k-1} f_n z^{-n} \right]$$

$$= z^{k} \left[F(z) - \sum_{n=0}^{k-1} f_n z^{-n} \right]$$

Eg. Find $Z(\cos n\theta)$ & $Z(\sin n\theta)$

Soln We know $Z(a^n) = \frac{z}{z-a}$

let $a = e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow$ $a^n = e^{i\theta n} = (\cos\theta + i\sin\theta)^n$

$\Rightarrow Z(e^{i\theta n}) = \frac{z}{z-e^{i\theta}}$

$\Rightarrow Z[(\cos\theta + i\sin\theta)^n] = \frac{z}{z-e^{i\theta}}$

$\Rightarrow Z[\cos n\theta + i\sin n\theta] = \frac{z}{z-e^{i\theta}}$

by linearity,

$\Rightarrow Z(\cos n\theta) + i Z(\sin n\theta) = \frac{z}{z-(\cos\theta + i\sin\theta)}$

$$= \frac{z[(z-\cos\theta) + i\sin\theta]}{(z-\cos\theta)^2 + \sin^2\theta}$$

Comparing real & imaginary parts on both sides,

$$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{(z-\cos\theta)^2 + \sin^2\theta} = \frac{z(z-\cos\theta)}{z^2+1-2z\cos\theta}$$

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2+1-2z\cos\theta}$$

Eg. Find $Z(na^n)$, $Z(n^2a^n)$.

Solⁿ By damping rule $Z(na^n) = U(z/a)$,

where $U(z) = Z(n) = \dfrac{z}{(z-1)^2}$

$$\Rightarrow U(z/a) = \frac{z/a}{(z/a-1)^2} = \frac{az}{(z-a)^2}$$

$$\Rightarrow Z(na^n) = \frac{az}{(z-a)^2}$$

Again, $Z(n^2a^n) = U(z/a)$ where

$$U(z) = Z(n^2) = \frac{z^2+z}{(z-1)^3}$$

$$\therefore U(z/a) = \frac{(z/a)^2 + z/a}{(z/a-1)^3}$$

$$= \frac{a(z^2+az)}{(z-a)^3}$$

$$\therefore Z(n^2a^n) = \frac{a(z^2+az)}{(z-a)^3}.$$

Eg. Find $Z(\cosh n\theta)$.

Solⁿ $Z(\cosh n\theta) = Z\left[\dfrac{e^{n\theta}+e^{-n\theta}}{2}\right]$

$$= \frac{1}{2}\left[\mathcal{Z}(e^{n\theta}) + \mathcal{Z}(e^{-n\theta}) \right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{\theta}+e^{-\theta})}{z^2 - z(e^{\theta}+e^{-\theta})+1} \right]$$

$$= \frac{z\,[2z - 2\cosh\theta]}{2(z^2 - 2z\cosh\theta + 1)}$$

$$= \frac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$$

Eg Find a) $\mathcal{Z}(\cos(n+1)\theta)$ b) $\mathcal{Z}(1/n!)$

c) $\mathcal{Z}\left(\frac{1}{(n+p)!}\right)$

Sol^n a) Let $f_n = \cos n\theta$

$$\mathcal{Z}(f_n) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$\mathcal{Z}(\cos(n+1)\theta) = \mathcal{Z}(f_{n+1})$, using shifting prop. ② for $k=1$,

$$= z^{1}\left[F(z) - \sum_{n=0}^{1-1} f_n z^{-n}\right]$$

$$= z\left[\mathcal{Z}(\cos n\theta) - f_0\right]$$

$$= z\left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - 1\right]$$

$$= z\left[\frac{z\cos\theta - 1}{z^2-2z\cos\theta+1}\right]$$

b) $$z(1/n!) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!}z^{-3} + \cdots$$

$$= e^{1/z} \qquad \left(\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

c) Let $f_n = \frac{1}{n!}$

$$f_{n+p} = \frac{1}{(n+p)!}$$

Applying shifting property ② for $k=p$,

$$z(f_{n+p}) = z\left[\frac{1}{(n+p)!}\right]$$

$$= z^p\left[F(z) - \sum_{n=0}^{p-1} f_n z^{-n}\right]$$

$$= z^p\left[z(1/n!) - \sum_{n=0}^{p-1} \frac{z^{-n}}{n!}\right]$$

$$= z^p\left[e^{1/z} - \sum_{n=0}^{p-1} \frac{z^{-n}}{n!}\right]$$

In particular,

for $P=1$, $\mathcal{Z}(f_{n+1}) = \mathcal{Z}\left(\frac{1}{(n+1)!}\right)$

$= z(e^{1/z}-1)$

for $P=2$, $\mathcal{Z}(f_{n+2}) = \mathcal{Z}\left(\frac{1}{(n+2)!}\right)$

$= z^2\left[e^{1/z}-(1+1/z)\right]$.

Eg find $\mathcal{Z}(e^n \cos 2n)$

Sol^n $\cancel{\mathcal{Z}}(e^n \cos 2n) = U(z/e)$ (by damping rule)

where $U(z) = \mathcal{Z}(\cos 2n)$

$= \dfrac{z(z-\cos 2)}{z^2+1-2z\cos 2}$ (done before)

$\Rightarrow U(z/e) = \dfrac{z/e\,(z/e-\cos 2)}{(z/e)^2+1-2z/e\cos 2}$

$= \dfrac{z(z-e\cos 2)}{z^2+e^2-2ze\cos 2}$

$\Rightarrow \mathcal{Z}(e^n\cos 2n) = \dfrac{z(z-e\cos 2)}{z^2+e^2-2ze\cos 2}$

Initial Value theorem :→

If $Z(u_n) = U(z)$ then $u_0 = \lim_{z\to\infty} U(z)$.

Proof $U(z) = Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$ —(1)

$$\Rightarrow \lim_{z\to\infty} U(z) = \lim_{z\to\infty} \left(\sum_{0}^{\infty} u_n z^{-n}\right)$$

$$= \lim_{z\to\infty} \left(u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots\right)$$

$$= u_0 + 0$$

$$\Rightarrow \boxed{u_0 = \lim_{z\to\infty} U(z)}$$

Next, Multiply (1) by z on both sides,

$$\Rightarrow zU(z) = z\sum_{0}^{\infty} u_n z^{-n}$$

$$= z\left(u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots\right)$$

$$\Rightarrow zU(z) - u_0 z = u_1 + \frac{u_2}{z} + \cdots$$

Take lim $z\to\infty$ on both sides,

$$\lim_{z\to\infty} [zU(z) - u_0 z] = u_1 + 0$$

$$\Rightarrow \boxed{u_1 = \lim_{z\to\infty} [zU(z) - u_0 z]}$$

Similarly, multiply eqn ① by z^2, we get

$$\boxed{u_2 = \lim_{z \to \infty} [z^2 U(z) - z^2 u_0 - z u_1]}$$

Final Value theorem :→

If $Z(u_n) = U(z)$ then $\lim_{n \to \infty} u_n = \lim_{z \to 1} (z-1) U(z)$

Proof Consider,

$Z(u_{n+1} - u_n)$

So $Z(u_{n+1} - u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$.

$\Rightarrow Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$.

$\Rightarrow z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$

$\Rightarrow (z-1) U(z) - z u_0$ (Using shifting prop. ② on $Z(u_{n+1})$ for $k=1$)

$= \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$.

$\Rightarrow (z-1) \cdot U(z) - u_0 z = [u_1 - u_0] + \frac{(u_2 - u_1)}{z} + \frac{(u_3 - u_2)}{z^2} + \cdots$

Taking $\lim z \to 1$.

$\Rightarrow \lim_{z \to 1} (z-1) U(z) - u_0 = (u_1 - u_0) + (u_2 - u_1) + \cdots$

$= \sum_{n=0}^{\infty} (u_{n+1} - u_n)$

R.H.S. is known as telescoping series which is equal to $\lim_{n\to\infty} u_n - u_0$ by defn.

$$\therefore \lim_{z\to 1} [(z-1).U(z)] - \cancel{u_0} = \lim_{n\to\infty} u_n - \cancel{u_0}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} u_n = \lim_{z\to 1} [(z-1).U(z)]}$$

Eg If $U(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$; evaluate u_1 & u_2.

Soln by initial value theor.,

$$u_0 = \lim_{z\to\infty} U(z) = \lim_{z\to\infty} \frac{2z^2+5z+14}{(z-1)^4}$$

$$= \lim_{z\to\infty} \frac{2/z^2 + 5/z^3 + 14/z^4}{(1-1/z)^4}$$

$$= 0.$$

$$u_1 = \lim_{z\to\infty} [z(U(z)-u_0)]$$

$$= \lim_{z\to\infty} \left[\frac{2z^3+5z^2+14z}{(z-1)^4}\right]$$

$$= \lim_{z\to\infty} \left[\frac{2+5/z+14/z^2}{z(1-1/z)^4}\right] = 0$$

$$u_2 = \lim_{z \to \infty} \left[z^2 \left(U(z) - u_0 - \frac{u_1}{z} \right) \right]$$

$$= \lim_{z \to \infty} \left[z^2 \frac{(2z^2 + 5z + 14)}{(z-1)^4} \right]$$

$$= \lim_{z \to \infty} \left[\frac{2 + 5/z + 14/z^2}{(1 - 1/z)^4} \right] = 2$$

Eg If $Z(u_n) = U(z) = \dfrac{z^3 - 3z^2 + 2z + 1}{(z+3)^2 (z-1)}$

find $\lim_{n \to \infty} u_n$.

Sol^n $\lim_{n \to \infty} u_n = \lim_{z \to 1} [(z-1) \cdot U(z)]$

$$= \lim_{z \to 1} \left[\frac{z^3 - 3z^2 + 2z + 1}{(z+3)^2} \right]$$

$$= \frac{1 - 3 + 2 + 1}{16} = \frac{1}{16}$$

Multiplication by n

If $\mathcal{Z}(u_n) = U(z)$ then $\mathcal{Z}(n u_n) = -z \dfrac{d}{dz} U(z)$. —(1)

Proof $\mathcal{Z}(n u_n) = \sum_{n=0}^{\infty} n \cdot u_n z^{-n}$

$$= -z \sum_{n=0}^{\infty} u_n (-n) \cdot z^{-n-1}$$

$$= -z \sum u_n \frac{d}{dz} (z^{-n})$$

$$= -z \sum_{0}^{\infty} \frac{d}{dz} (u_n z^{-n})$$

$$= -z \frac{d}{dz} \sum_{0}^{\infty} (u_n z^{-n})$$

$$= -z \frac{d}{dz} U(z).$$

In general,

$$\mathcal{Z}(n^p \cdot u_n) = -z \frac{d}{dz} \mathcal{Z}(n^{p-1} \cdot u_n) \quad \text{—(2)}$$

Eg Find $\mathcal{Z}(n^2 e^{n\theta})$

Sol^n Putting $p = 2$ in (2) and $u_n = e^{n\theta}$.

$$\mathcal{Z}(n^2 e^{n\theta}) = -z \frac{d}{dz} \mathcal{Z}(n \cdot e^{n\theta})$$

$$= -z \frac{d}{dz} \left[-z \frac{d}{dz} (e^{n\theta}) \right] \quad \text{Again using (2)}$$

$$= -z \frac{d}{dz}\left[(-z)\,\frac{(z-e^{\theta}-z)}{(z-e^{\theta})^2}\right]$$

$$= -z \frac{d}{dz}\left[\frac{ze^{\theta}}{(z-e^{\theta})^2}\right]$$

$$= -z\,\frac{[ze^{\theta}-e^{2\theta}-2ze^{\theta}]}{(z-e^{\theta})^3}$$

$$= \frac{z\,[ze^{\theta}+e^{2\theta}]}{(z-e^{\theta})^3} \quad \underline{\text{Ans}}.$$

Inverse z-transform

Eg find inverse z transform of the ~~f~~ following functions

(a) $F(z) = \dfrac{z}{(z+4)(z+5)}$

Sol^n^ $\Rightarrow$ $\dfrac{F(z)}{z} = \dfrac{1}{(z+4)(z+5)}$.

$$= \frac{1}{z+4} - \frac{1}{z+5} \quad \text{(By Partial fraction}$$

$$\Rightarrow F(z) = \frac{z}{z+4} - \frac{z}{z+5}$$

Apply z^{-1} on both sides,

$$z^{-1}(F(z)) = z^{-1}\left(\frac{z}{z+4}\right) - z^{-1}\left(\frac{z}{z+5}\right)$$

$$= (-4)^n - (-5)^n$$

$\left[\because z(a^n) = \frac{z}{z-a} \Rightarrow a^n = z^{-1}\left(\frac{z}{z-a}\right)\right.$ (*)

(b) $$F(z) = \frac{7z - 11z^2}{(z-1)(z-2)(z+3)}.$$

Soln $\Rightarrow$ $$\frac{F(z)}{z} = \frac{7-11z}{(z-1)(z-2)(z+3)}$$

$$= \frac{1}{z-1} - \frac{3}{z-2} + \frac{2}{z+3}$$ (Partial fraction)

$$\Rightarrow F(z) = \frac{z}{z-1} - \frac{3z}{z-2} + \frac{2z}{z+3}$$

$$\Rightarrow z^{-1}(F(z)) = z^{-1}\left(\frac{z}{z-1}\right) - z^{-1}\left(\frac{3z}{z-2}\right) + z^{-1}\left(\frac{2z}{z+3}\right)$$

$$= (1)^n - 3(2)^n + 2(-3)^n$$ (Again by (*))

<u>Convolution theorem</u> (Used to find inverse z transform)

If $z^{-1}(U(z)) = u_n$ and $z^{-1}(V(z)) = v_n$ then $z^{-1}(U(z)\cdot V(z)) = \sum_{m=0}^{n} u_m v_{n-m}$

$$= u_n * v_n$$

Eg. Use Convolution theorem to evaluate

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$$

Soln.

$$\frac{z^2}{(z-a)(z-b)} = \left(\frac{z}{z-a}\right)\left(\frac{z}{(z-b)}\right) \quad -(1)$$

$$\text{let } U(z) = \frac{z}{z-a}$$

$$V(z) = \frac{z}{z-b}$$

then $Z^{-1}(U(z)) = Z^{-1}\left(\frac{z}{z-a}\right) = a^n (= u_n)$

$$Z^{-1}(V(z)) = Z^{-1}\left(\frac{z}{z-b}\right) = b^n (= v_n)$$

Applying Z^{-1} on both sides of (1),

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{(z-a)}\,\frac{z}{(z-b)}\right]$$

$$= u_n * v_n$$

$$= \sum_{m=0}^{n} u_m \cdot v_{n-m}$$

$$= \sum_{m=0}^{n} a^m \cdot b^{n-m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n\left[\frac{(a/b)^{n+1}-1}{(a/b)-1}\right] = \frac{a^{n+1}-b^{n+1}}{a-b}$$

(sum of Geometric series)

Eg. Verify convolution theorem for

$f_n = 2^n$, $g_n = 3^n$.

Sol^n: ~~To show~~ let $F(z) = Z(f_n)$

$G(z) = Z(g_n)$

To show, $Z^{-1}(F(z)\cdot G(z)) = f_n * g_n$.

L.H.S. $= Z^{-1}(F(z)\cdot G(z)) = Z^{-1}\left(\frac{z}{z-2}\cdot\frac{z}{z-3}\right)$

$\left(\because F(z) = Z(2^n) = \frac{z}{z-2}\right.$

$G(z) = Z(3^n) = \frac{z}{z-3}$.

Let $K(z) = \frac{z}{z-2}\cdot\frac{z}{z-3}$

$$= \frac{z^2}{(z-2)(z-3)}$$

$$\Rightarrow \frac{K(z)}{z} = \frac{z}{(z-2)(z-3)} = \frac{-2}{z-2} + \frac{3}{z-3}.$$

$$\therefore \cancel{Z^{-1}}\,K(z) = -2\frac{z}{z-2} + \frac{3z}{z-3}$$

$$\Rightarrow Z^{-1}(K(z)) = -2(2^n) + 3(3^n)$$

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$\Rightarrow$ L.H.S. $= z^{-1}\left(\frac{z}{z-2}\cdot\frac{z}{z-3}\right) = -2(2^n)+3(3^n)$.

R.H.S $= f_n * g_n = \sum_{m=0}^{n} f_m g_{n-m}$

$= \sum_{m=0}^{n} 2^m \cdot 3^{n-m}$

$= 3^n \sum_{m=0}^{n} \left(\frac{2}{3}\right)^m$

$= 3^n \frac{\left[\left(2/3\right)^{n+1}-1\right]}{2/3-1}$

$= \frac{\cancel{3^n}\left[2^{n+1}-3^{n+1}\right]}{-1/\cancel{3}\cdot \cancel{3^{n+1}}}$

$= -2(2^n)+3(3^n)$ = L.H.S.

Convergence of z transform :$\rightarrow$

We consider the sequence $\{u_n\}$, where $-\infty<n<\infty$ so that,

$$Z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n z^{-n} = U(z) \text{ say.}$$

ROC $\rightarrow$ Region of convergence of $Z\{u_n\}$.

Eg. Find Z-transform of u_n & ROC.
where $u_n = \begin{cases} 4^n, & n<0 \\ 2^n, & n\geq 0 \end{cases}$

Sol^n. By def^n $Z(u_n) = \sum_{-\infty}^{\infty} u_n z^{-n}$

$$= \sum_{-\infty}^{-1} 4^n z^{-n} + \sum_{0}^{\infty} 2^n z^{-n}.$$

Putting $-n = m$ in 1st series,

$$= \sum_{1}^{\infty} 4^{-m} z^m + \sum_{0}^{\infty} 2^n z^{-n}.$$

$$= \underbrace{\left(\frac{z}{4} + \frac{z^2}{4^2} + \cdots\right)}_{G.P.\ 1)} + \left(1 + \underbrace{\frac{2}{z} + \frac{2^2}{z^2} + \cdots}_{G.P.\ 2)}\right)$$

we know, a G.P. $\sum_{1}^{\infty} ar^{n-1}$ is convergent if $|r| < 1$

$\therefore$ G.P. 1) is cgt if $|z/4| < 1$
i.e., $|z| < 4$. — (a)

G.P. 2) is cgt if $|2/z| < 1$
i.e., $|z| > 2$ — (b)

By (a) & (b) ROC $\rightarrow$ $\boxed{2 < |z| < 4}$

Eg. $f(n) = 2^n$, $n < 0$.

Sol^n. let $f_n = 0$ for $n \geq 0$ then

$$Z(f_n) = \sum_{-\infty}^{\infty} f_n z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n} + 0$$

let $n = -m$

$$= \sum_{m=1}^{\infty} 2^{-m} z^{m}$$

$$= \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots$$

$$= \frac{z}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots\right] \quad \text{---} (1)$$

$$= \frac{z}{2}\left[\frac{1}{1 - z/2}\right] = \frac{z}{2 - z}.$$

Again as before G.P. in (1) is cgt if $|z| < 2$.

i.e., ROC is $|z| < 2$.

Application of z- transform

Eg. Use z- transform to solve

$u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ with $u_0 = 0$, $u_1 = 1$ — (1)

Sol^n: Apply z- transform on both sides of (1)

$\Rightarrow Z(u_{n+2}) + 4Z(u_{n+1}) + 3Z(u_n) = Z(3^n)$ (2)

let $Z(u_n) = U(z)$.

Now,

By shifting property (2), $Z(u_{n+2}) = z^2\left[U(z) - u_0 - u_1/z\right]$

$$Z(u_{n+1}) = z[U(z) - u_0]$$

Also $Z(3^n) = \dfrac{z}{z-3}$

$\therefore$ (2) becomes

$$z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] + 4\left[z\left(U(z) - u_0\right)\right] + 3U(z) = \frac{z}{z-3}.$$

$$\Rightarrow U(z) = \frac{z}{z^2+4z+3} + \frac{z}{(z-3)(z^2+4z+3)}$$ Using $u_0 = 0$, $u_1 = 1$

$$\Rightarrow \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)}$$

$$= \frac{3}{8}\cdot\frac{1}{(z+1)} + \frac{1}{24(z-3)} - \frac{5}{12(z+3)}$$

(By partial fractn)

$$\Rightarrow U(z) = \frac{3z}{8(z+1)} + \frac{z}{24(z-3)} - \frac{5z}{12(z+3)}$$

$\left[\because Z(a^n) = z/(z-a) \Rightarrow a^n = Z^{-1}[z/(z-a)]\right]$

Taking inverse,

$$u_n = \frac{3}{8} Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24} Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12} Z^{-1}\left(\frac{z}{z+3}\right)$$

$$= \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n.$$

Ans

Eg #. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, $y_0 = y_1 = 0$. — (1)

Sol^n Apply z-trans., & let $z(y_n) = Y(z)$

$$z(y_{n+2}) + 6\,z(y_{n+1}) + 9\,z(y_n) = z(2^n) \quad \text{— (2)}$$

Again by shifting property,

$$z(y_{n+2}) = z^2\left[Y(z) - y_0 - y_1/z\right]$$

$$z(y_{n+1}) = z\left[Y(z) - y_0\right]$$

Also $z(2^n) = \dfrac{z}{z-2}$

$\therefore$ (2) $\Rightarrow$ $z^2\left[Y(z) - y_0 - y_1/z\right] + 6\left[z\left(Y(z) - y_0\right)\right] + 9Y(z) = z/z-2,$

$$\Rightarrow \frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

$$= \frac{1}{25}\left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2}\right] \text{ By partial fraction.}$$

$$\Rightarrow Y(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} - \frac{5z}{(z+3)^2}\right]$$

Taking inverse,

$$y_n = \frac{1}{25}\left[z^{-1}\left(\frac{z}{z-2}\right) - z^{-1}\left(\frac{z}{z+3}\right) - \frac{5}{3}z^{-1}\left(\frac{3z}{(z+3)^2}\right)\right]$$

$$= \frac{1}{25}\left[2^n - (-3)^n + \frac{5}{3}n(-3)^n\right] \quad \left\{\text{Using } z(na^n) = \frac{az}{(z-a)^2}\right.$$

Ans