12/12/2020

Successive Differentiation

nth derivative of Some elementary Functions:

① Power fun $(ax+b)^m$:

Let $y = (ax+b)^m$

$y_1 = m\,a\,(ax+b)^{m-1}$

$\vdots$

$y_n = m(m-1)(m-2)\cdots(m-n+1)a^n(ax+b)^{m-n}$

Case(i): When m is +ve integer, then

$$y_n = \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)\cdots 3\cdot 2\cdot 1\; a^n\,(ax+b)^{m-n}}{(m-n)\cdots 3\cdot 2\cdot 1}$$

$$y_n = \frac{d^n}{dx^n}(ax+b)^m$$

$$\boxed{y_n = \frac{m!}{(m-n)!}\,a^n\,(ax+b)^{m-n}}$$

Case(ii): When $m = n =$ +ve integer

then, $y_n = \frac{n!}{0!}a^n(ax+b)^0 = n!\,a^n$

Case(iii) → when $m = -1$, then

$Y = (ax+b)^{-1}$

$Y = \frac{1}{ax+b}$

$Y_n = (-1)(-2)(-3)\cdots a^n(ax+b)^{-1-n}$

$$\boxed{Y_n = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}}$$

② Logarithm Case :

When $y = \log(ax+b)$, then

$y_1 = \frac{a}{ax+b}$

$\vdots$

$$\boxed{y_n = \frac{a^n(-1)^{n-1}(n-1)!}{(ax+b)^n}}$$

③ Exponential fun :

(i) Consider $y = a^{mx}$

$y_1 = m\cdot a^{mx}\log_e a$

$y_2 = m^2 a^{mx}(\log_e a)^2$

$\vdots$

$$\boxed{y_n = m^n a^{mx}(\log_e a)^n}$$

(iii) Consider $y = e^{mx}$

then, $y_1 = me^{mx}$

$y_2 = m^2 e^{mx}$

$\boxed{y_n = m^n e^{mx}}$

④ <u>Trigonometric fu^n</u> $\cos(ax+b)$ or $\sin(ax+b)$

Let $y = \cos(ax+b)$, then

$y_1 = -a\sin(ax+b) = a\cos(ax+b+\frac{\pi}{2})$

$y_2 = -a^2\cos(ax+b) = a^2\cos(ax+b+2\times\frac{\pi}{2})$

$y_3 = +a^3\sin(ax+b) = a^3\cos(ax+b+\frac{3\pi}{2})$

$\boxed{y_n = a^n\cos(ax+b+n\frac{\pi}{2}) = a^n\sin(ax+b+n\frac{\pi}{2})}$

⑤ <u>Product fun</u> : $e^{ax}\sin(bx+c)$ or $e^{ax}\cos(bx+c)$

Let $y = e^{ax}\sin(bx+c)$, then

$\therefore y_1 = e^{ax}b\cos(bx+c) + a e^{ax}\sin(bx+c)$

$= e^{ax}[b\cos(bx+c) + ae^{ax}\sin(bx+c)]$

To re write this in the form of sin, put

$a = r\cos\phi$, $b = r\sin\phi$, we get

$\boxed{y_1 = e^{ax}\cdot r\sin(bx+c+\phi)}$

Where $\boxed{r = \sqrt{a^2+b^2}}$ and $\boxed{\phi = \tan^{-1}(b/a)}$

$y_2 = rae^{ax}\sin(bx+c+\phi) + rbe^{ax}\cos(bx+c+\phi)$

Substituting for a & b

$y_2 = r^2 e^{ax} r\cos\phi\cdot\sin(bx+c+\phi) + re^{ax}\cdot r\sin\phi\cos(bx+c$

$= r^2e^{ax}[\cos\phi\sin(bx+c+\phi) + \sin\phi\cos(bx+c+\phi)]$

$= r^2e^{ax}\sin(bx+c+2\phi)$

similarly $y_3 = r^3e^{ax}\sin(bx+c+3\phi)$

$\boxed{y_n = r^n e^{ax}\sin(bx+c+n\phi)}$

In the similar way for $e^{ax}\cos(bx+c)$

$\boxed{y_n = r^n e^{ax}\cos(bx+c+n\phi)}$

Ex 1 :- ① Find n^{th} Derivative of $\frac{1}{1-5x+6x^2}$

Sol^n:- $y = \frac{1}{(2x-1)(3x-1)}$

$$y = \frac{2}{2x-1} - \frac{3}{3x-1}$$

$$y_n = 2\frac{d^n}{dx^n}\frac{1}{(2x-1)} - 3\frac{d^n}{dx^n}\frac{1}{(3x-1)}$$

By use of formula

$$\boxed{y_n = \frac{2(-1)^n n!\, 2^n}{(2x-1)^{n+1}} - \frac{3\cdot(-1)^n n!\, 3^n}{(3x-1)^{n+1}}}$$

Ex 2 :- $y = e^{ax}\cos^2 x \sin x$

Sol^n, $y = e^{ax}\left(\frac{1+\cos 2x}{2}\right)\sin x \Rightarrow \frac{e^{ax}\sin x}{2} + \frac{e^{ax}}{4}(\sin 3x - \sin x)$

$$y = \frac{e^{ax}\sin x}{4} + \frac{e^{ax}}{4}\cdot\sin 3x$$

$$y_n = \frac{1}{4}\left[r^n e^{ax}\sin(x+n\phi)\right] + \frac{1}{4}\left[r_1^n e^{ax}\sin(3x+n\theta)\right]$$

where $r = \sqrt{a^2+1}$; $\tan\phi = \frac{1}{a}$

$r_1 = \sqrt{a^2+9}$; $\tan\theta = \frac{3}{a}$

H.W

Ex ③ $y = \frac{x^2}{(x-a)(x-b)}$

sol^n $y = 1 - \frac{x(a+b)}{(x-a)(x-b)} + \frac{ab}{(x-a)(x-b)}$

$$y = 1 - \left\{\frac{a}{b-a}\frac{1}{(x-a)} + \frac{b}{(b-a)}\frac{1}{(x-b)}\right\} + \frac{1}{a-b}\left\{\frac{1}{(x-a)} - \frac{1}{x-b}\right\}$$

$$y_n = -\left\{\frac{a}{b-a}\frac{(-1)^n n!}{(x-a)^{n+1}} + \frac{b}{b-a}\frac{(-1)^n n!}{(x-b)^{n+1}}\right\} + \frac{1}{a-b}\left\{\frac{(-1)^n n!}{(x-a)^{n+1}} - \frac{(-1)^n n!}{(x-b)^{n+1}}\right\}$$

Ans

Ex 4 $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, $x = \tan\theta$

Sol^n, $y = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\theta}{1-\tan\theta\tan\frac{\pi}{4}}\right)$

$$y = \frac{\pi}{4} + \theta$$

$$y = \frac{\pi}{4} + \tan^{-1}x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

$$\frac{d^n y}{dx^n} = \frac{1}{2i}\left\{\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right\}$$

Ans

Ex 5 → $y = \sin^3 x$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\sin^3 x = \frac{3\sin x}{4} - \frac{\sin 3x}{4}$

sol^n $y = \frac{3}{4}\sin x - \frac{\sin 3x}{4}$

$$y_n = \frac{3}{4}\sin\left(x + \frac{n\pi}{2}\right) - \frac{3}{4}\times 3^n\left(3x + \frac{n\pi}{2}\right)$$

$$\boxed{y_n = \frac{3}{4}\sin\left(x+\frac{n\pi}{2}\right) - \frac{3^{n+1}}{4}\left(3x+\frac{n\pi}{2}\right)}$$ Ans

Ex 6 → $y = \log x^3$

$y = 3\log x$

$$\boxed{y_n = \frac{3(-1)^{n-1}(n-1)!}{x^n}}$$ Ans

Ex 7 → $y = e^x \sin^2 x$

$\cos 2x = 2\sin^2 x - 1$

$\frac{\cos 2x + 1}{2}$

$$y = e^x\left(\frac{\cos 2x + 1}{2}\right)$$

$$y = \frac{e^x \cos 2x}{2} + \frac{e^x}{2}$$

$$\boxed{y_n = \frac{r^n e^x \cos(2x + n\phi)}{2} + \frac{e^x}{2}}$$

where $r = \sqrt{1^2 + 2^2}$

$\boxed{r = \sqrt{5} \;;\; \tan\phi = 2}$ Ans

14/12/2020

Leibnitz's Theorem

statment : If u & v be any two fu^n of x, then

$$D^n(u\cdot v) = {}^nC_0 D^n u \cdot v + {}^nC_1 D^{n-1}u \cdot Dv + {}^nC_2 D^{n-2}u \cdot Dv^2 + \cdots + {}^nC_n u \cdot D^n v$$

∴ u = successative Deff^n fun

ex(1) n^{th} Derivative of $e^x \log x$

Sol^n $u = e^x$ & $v = \log x$

$D^n u = e^x$ $\quad D^n v = \frac{(-1)^n (n-1)!}{x^n}$

By leibnitz theorem, we have

$$D^n(e^x \log x) = (D^n e^x)\log x + n_{C_1} D^{n-1}e^x D(\log x) + \cdots + e^x D^n \log x$$

$$= e^x \log x + n\cdot e^x \cdot \frac{1}{x} + \frac{n(n-1)}{2!} e^x\left(-\frac{1}{x^2}\right) + \cdots + e^x D^n \log x$$

$$= e^x\left[\log x + \frac{n}{x} - \frac{n(n-1)}{2x^2} + \cdots + \frac{(-1)^n (n-1)!}{x^n}\right]$$ Ans

(2) $y = x^2 \sin 3x$

Solⁿ→ Let $u = \sin 3x$ & $v = x^2$

$\therefore D^n u = 3^n \sin\left(3x + \frac{n\pi}{2}\right)$ $\therefore Dv = 2x$

$D^2 v = 2$

$D^3 v = 0$

$\therefore D^3 v$ is zero, we only take upto $D^2 v$

$$D^n(\underset{u}{\sin 3x} \cdot \underset{v}{x^2}) = 3^n \sin\left(3x + \frac{n\pi}{2}\right)x^2 + {}^nC_1 3^{n-1} \sin\left(3x + \frac{n-1}{2}\pi\right)2x + {}^nC_2 3^{n-2} \sin\left(3x + \frac{n-2}{2}\pi\right)2$$

$$= 3^n \sin\left(3x + \frac{n\pi}{2}\right)x^2 + n 3^{n-1} \sin\left(3x + \frac{n-1}{2}\pi\right)2x + \frac{n(n-1)}{\cancel{2}}3^{n-2}\sin\left(3x + \frac{n-2}{2}\pi\right)\cancel{2}$$

Ans

(3) If $y = a\cos(\log x) + b\sin(\log x)$, show that

$x^2 y_2 + x y_1 + y = 0$ &

$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$

Solⁿ→ $y_1 = -\frac{a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$

$x y_1 = -a\sin(\log x) + b\cos(\log x)$

Rediffⁿ

$x y_2 + y_1 = -\frac{a\cos\log x)}{x} - \frac{b\sin(\log x)}{x}$

$x^2 y_2 + y_1 x = -y \Rightarrow \boxed{x^2 y_2 + y_1 x + y = 0}$ Ans (i)

Differentating (i) n times by Leibnitz theorem

$$x^2 y_{n+2} + n y_{n+1} \cdot 2x + \frac{n(n-1)}{\cancel{2}} y_n \cdot \cancel{2} + y_{n+1} x + n y_n + y_n = 0$$

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2 - \cancel{n} + \cancel{n} + 1) y_n = 0$$

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$$ Proved

$$\boxed{x^2 y_{n+2} + (2n+1)x y_{n+1}}$$

(4) $y = x^{n-1} \log x$ at $x = \frac{1}{2}$

$y_1 = x^{n-1} \times \frac{1}{x} + \log x (n-1) x^{n-2}$

$x y_1 = x^{n-1} + (n-1) y$

Diffⁿ (n−1) times by Newtons Leibnitz theoreme

$x y_n + {}^{n-1}C_1 \cancel{y_{n-1}} = \cancel{(n-1)} y_{n-1} + (n-1)!$

$y_n = \frac{(n-1)!}{x}$

at $x = 1/2$

$\boxed{y_n = 2(n-1)!}$ Ans

Ex If $y=(x^2-1)^n$ show that

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}+n(n+1)P_n=0$$

Solⁿ, $y=(x^2-1)^n$

$y_1=n(x^2-1)^{n-1}\times 2x$

$y_1=2nx\frac{(x^2-1)^n}{x^2-1}$

$y_1(x^2-1)=2nxy \quad \text{—(i)}$

Rediffⁿ $(x^2-1)y_2+2xy_1=2nxy_1+2ny$

nth diffⁿ

$(x^2-1)y_{n+2}+n\cdot y_{n+1}(2x)+\frac{2n(n-1)}{2!}y_n+2xy_{n+1}+2ny_n$

$=2nxy_{n+1}+2n^2y_n+2ny_n$

$(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$

Second part

Let $y=(x^2-1)^n$

$y_n=\frac{d^ny}{dx^n}=P_n$ (Let)

∴ P.T. $\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}=-n(n+1)P_n$

L.H.S $\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}\Rightarrow\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}y_n\right\}$

$\Rightarrow\frac{d}{dx}\left\{(1-x^2)y_{n+1}\right\}$

$=(1-x^2)y_{n+2}-2xy_{n+1}$

$=-\left[(x^2-1)y_{n+2}+2xy_{n+1}\right]$

$=-n(n+1)y_n\Rightarrow$ R.H.S Proved ##

To find $(y_n)_0$ i.e. nth differential coffⁿ of y when $x=0$

Ex Find $(y_n)_0$ where $y=e^{m\cos^{-1}x}$

Solⁿ $y_1=e^{m\cos^{-1}x}\,m\left(\frac{-1}{\sqrt{1-x^2}}\right) \quad \text{—(a)}$

$\Rightarrow\sqrt{1-x^2}\,y_1=-e^{m\cos^{-1}x}\cdot m\Rightarrow -y\cdot m$

$\Rightarrow\sqrt{1-x^2}\,y_1=-ym$

Squaring both sides

$(1-x^2) y_1^2 = m^2 y^2$

Differentiating again

$(1-x^2) 2 y y_1 - 2 x y_1^2 = 2 m^2 y y_1$

$(1-x^2) y_2 - x y_1 = m^2 y$ — (ii)

Using Leibnitz theorem we get after differentiating n times w.r.t x

$(1-x^2) y_{n+2} - 2 n x y_{n+1} - 2 \frac{n(n-1)}{2!} y_n - x y_{n+1} - n y_n = m^2 y_n$

or $(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+m^2) y_n = 0$

putting $x = 0$

$y_{n+2}(0) - (n^2+m^2) y_n(0) = 0$

$y_{n+2}(0) = (n^2+m^2) y_n(0)$ — (iii)

$n \to (n-2)$ $y_n(0) = [(n-2)^2 + m^2] y_{n-2}(0)$ — (iv)

in eqn (iii) $n \to (n-4)$ $y_{n-2}(0) = [(n-4)^2 + m^2] y_{n-4}(0)$

Put in eqn (iv)

$y_n(0) = [(n-2)^2 + m^2] [(n-4)^2 + m^2] y_{n-4}(0)$

or $y_n(0) = [(n-2)^2 + m^2] [(n-4)^2 + m^2] - \cdots - y_1(0)$ or $y_2(0)$

Case(i): when n is odd :-

$y_n(0) = \{(n-2)^2 + m^2\} \{(n-4)^2 + m^2\} - \cdots (1^2 + m^2) y_1(0)$ — (i)

The last term obtained by putting n = 1 in eqn (iii)

Now, we have $y_1(0) = -m e^{\pi/2}$ (by using eqn (1))

So $\boxed{y_n(0) = -\{(n-2)^2 + m^2\} \{(n-4)^2 + m^2\} + \cdots (1^2+m^2) m e^{\frac{m\pi}{2}}}$

Ans

Case(ii) :- when n is even :-

$y_n(0) = \{(n-2)^2 + m^2\} \{(n-4)^2 + m^2\} + \cdots (2^2 + m^2) y_2(0)$

(the last term is obtain by putting n = 2 in eqn (iii))

$\because y_2(0) = m^2 (y_0) = m^2 e^{m\pi/2}$ (from ii)

$\boxed{y_n(0) = \{(n-2)^2 + m^2\} \{(n-4)^2 + m^2\} \cdots (2^2+m^2) m^2 e^{\frac{m\pi}{2}}}$

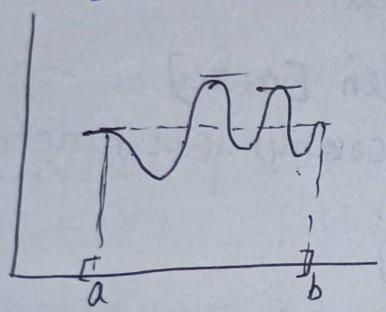
when n is even

Ans

Mean Value Theorms

Rolle's Theorms :- $f(x)$ is countinious in the close interval $[a,b]$ and diff^n in the open interval (a,b) $\{f'(x)$ exists for every value of x in the open interval $(a,b)\}$ and if $f(a)=f(b)$ then these exists at least one pt. $c \in (a,b)$ such that $f'(c)=0$

Geometrically :- Under the given Condition these is at least one pt in (a,b) such that tangent at the pt is $\parallel$el to x-axis.



Ques → Verify R.T. $f(x) = 2x^3+x^2-4x-2$

Sol^n → To find the interval

$f(x)=0 = 2x^3+x^2-4x-2$

$(x^2-2)(2x+1)=0 \Rightarrow x=\pm\sqrt{2}, x=-1/2$

$f(\sqrt{2}) = f(-\sqrt{2}) = 0$.

Consider interval $(-\sqrt{2},\sqrt{2})$, $f(x)$ is a polynomial in x $f(x)$ is countinuous in $[-\sqrt{2},-\sqrt{2}]$, diff^n for all value of x in $(-\sqrt{2},\sqrt{2})$

$\Rightarrow$ all the conditions of Rolle's theorem are satisfied in the interval.

$\Rightarrow$ there is at least one value of x in $(-\sqrt{2},\sqrt{2})$ for which $f'(x)=0$

$6x^2+2x-4=0 \Rightarrow x = -1, 2/3$

Since both those pt. belongs to $(-\sqrt{2},\sqrt{2})$

$\Rightarrow$ Rolle's theorms is verified.

Ex2 $f(x) = |x|$ in $[-1, 1]$

Sol^n $f(x) = |x| = -x, \quad -1 \le x \le 0$

$\qquad\qquad x, \quad 0 \le x \le 1$

(a) $f(x)$ is continuous in $[-1, 1]$

(b) for differentiability

R.H.D. $= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{h-0}{h} = 1$

L.H.D. $= \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$

$= \frac{-(h)-0}{-h} = -1$

$\Rightarrow f(x)$ is not differenciable at $x=0$

$\Rightarrow$ Rolle's theorem is not applicable.

Ex3 → $f(x) = \frac{\sin x}{e^x} = e^{-x}\sin x$ in $[0, \pi]$

Sol^n → $f(x)$ is a product of two continuous functions

$\Rightarrow f(x)$ is a continuous fun.

(b) $f(x)$ for diff^n in $[0, \pi]$

$f'(x) = e^{-x}(\cos x - \sin x)$

exists for every value of x in $(0, \pi)$.

$\Rightarrow f(x)$ is diff^n in $(0, \pi)$

(c) $f(0) = f(\pi) = 0$

$\Rightarrow f(x)$ satifies all the condetions of Rolle's theorem

$\Rightarrow \exists\, c \in (0, \pi)$ such that $f'(c) = 0$

$e^{-c}(\cos c - \sin c) = 0 \qquad e^{-c} \ne 0$

$\tan c = 0$

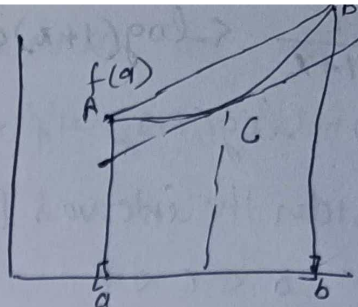
$\boxed{c = n\pi + \frac{\pi}{4},}$ where n is an integer

Roll'es theorem is applicable.

Lagranges Mean Value theorem :-

If $f(x)$ is continuous in $[a, b]$ diff^n in (a, b) then there exist at least one pt $c \in (a, b)$ such that $\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$

Geometrically ÷ Under the given Condition ∃ at least one pt $c \in (a, b)$ such that tangent at this pt is ||el to the chord AB



Slope of chord AB is $\dfrac{f(b)-f(a)}{b-a}$

and slope of tangent at c is $f'(c)$

Ques) Find c of lagranges M.V.T. if $f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

Sol^n) $f(x)$ is polynomial fu^n is continuous and diff^n for each value of x.

by LMVT. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$f(x) = x^3 - 3x^2 + 2x$$

$$f'(x) = 3x^2 - 6x + 2$$

$$3c^2 - 6c + 2 = \frac{f(1/2) - f(0)}{1/2}$$

$$c = 1 \pm \frac{\sqrt{21}}{6}$$

$\boxed{\text{only } c = 1 - \frac{\sqrt{21}}{6} \text{ lies in the interval}}$ Ans

(2) P.T. if $(0 < a < b < 1)$

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

if $f(x) = \tan^{-1} x$

$$f'(x) = \frac{1}{1+x^2}$$

by LMVT $c \in (a, b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$$

$a < c < b$

$$1 + a^2 < 1 + c^2 < 1 + b^2$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

$$\boxed{\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}}$$

Ans

③ P.T $\frac{x}{1+x} < \log(1+x) < x \quad x>0$

$f(x) = \log(1+x) \quad f'(x) = \frac{1}{1+x}$

Consider the interval $[0, x]$

$0 < c < x$

$1 < 1+c < 1+x$

$\frac{1}{1+x} < \frac{1}{1+c} < 1$

by L.M.V.T.

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{1}{1+c} = \frac{f(x)-f(0)}{x-0}$

$\frac{1}{1+c} = \frac{\log(1+x)}{x}$

$\frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$

$\boxed{\frac{x}{1+x} < \log(1+x) < x}$ Ans

④ Verify Rolle's theorem

i) $f(x) = (x-a)^m (x-b)^n$ in $[a,b]$
m & n are +ve integers

Sol^n $f(x) = (x-a)^m (x-b)^n$

$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)]$

$= (x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)]$

exists for every value of x in (a,b)

$\Rightarrow f(x)$ is diff^n in (a,b)

Rolle's theorem $\Rightarrow$ there exists at least one pt c in (a,b) such that $f'(c)=0$

$c = \frac{mb+na}{m+n}$

where c represent a pt that divides the interval $[a,b]$ internally in the ratio $m:n$

Thus c lies with (a,b). hence theorems is verified.

Cauchy's Mean Value theorem :-

(i) → If $f(x)$ and $g(x)$ be two continuous function in the the closed interval $[a,b]$

(ii) → and $f'(x)$ and $g'(x)$ exists in (a,b) {differentiable in the open interval (a,b)}

(iii) $g'(x) \neq 0$ for any value of x in (a,b) then there exists at least one point $c \in (a,b)$ such that

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

Ques(i) If in Cauchy's Mean value theorem, we write $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = \sqrt{x}$, then show c is G.M. b/w a & b by cauchy's mean value theorem.

Solⁿ

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$f(x) = \frac{1}{\sqrt{x}} \qquad f'(x) = -\frac{1}{2}x^{-3/2}$$

$$g(x) = \sqrt{x} \qquad g'(x) = \frac{1}{2}x^{-1/2}$$

$$\frac{-\frac{1}{2}c^{-3/2}}{\frac{1}{2}c^{-1/2}} = \frac{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}{\sqrt{b}-\sqrt{a}} \Rightarrow -\frac{1}{c} = \frac{-1}{\sqrt{ab}} \qquad \boxed{c = \sqrt{ab}} \text{ Proved}$$

(ii) if $f(x) = \frac{1}{x}$ $g(x) = \frac{1}{x^2}$ show c is the H.M. b/w a & b.

Solⁿ

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{-\frac{1}{c^2}}{-\frac{2}{c^3}} = \frac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{b^2}-\frac{1}{a^2}}$$

$$\frac{c}{2} = \frac{ab}{a+b}$$

$$\boxed{c = \frac{2ab}{a+b}} \text{ Proved}$$

(iii) If $f(x) = e^x$, $g(x) = e^{-x}$ show that by using C.M.V.T. c is the A.M. b/w a & b.

Solⁿ by C.M.V.T.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b-e^a}{e^{-b}-e^{-a}} \Rightarrow +e^{-2c} = +e^{a+b}$$

$$\boxed{c = \frac{a+b}{2}}$$

Proved

Physical Meaning of Lagrange's Mean Value Theorem :- The instantaneous rate of change of f at some point b/w a & b is equal to the average rate of change of f on [a, b].

Another form of Lagrange's Mean Theorem

If $b-a=h$, then $b=a+h$, $h>0$ then $c\in(a,b) \Rightarrow c=a+\theta h, 0<\theta<1$

a c a+h

Lagrange's Mean value theorem is

$$f'(a+\theta h)=\frac{f(a+h)-f(a)}{h}$$

$$f(a+h)=f(a)+h\,f'(a+\theta h),\quad 0<\theta<1$$

Ques → Using L.M.V.T. P.T. $\tan^{-1}\beta-\tan^{-1}\alpha<\beta-\alpha$, where $\beta>\alpha$

Solⁿ → $f(x)=\tan^{-1}x$, $x\in[\alpha,\beta]$ clearly $[\alpha,\beta]\subset\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

∵ $\tan^{-1}x$ is contⁿ on $[\alpha,\beta]$ and $f'(x)=\frac{1}{1+x^2}$. $f(x)$ is contⁿ on $[\alpha,\beta]$ and diffⁿ in (α,β)

So the condition of L.M.V.T. are satisfied

∴ by L.M.V.T., there exists $c\in(\alpha,\beta)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$a=\alpha$, $b=\beta$

$$\frac{1}{1+c^2}=\frac{\tan^{-1}\beta-\tan^{-1}\alpha}{\beta-\alpha} \Rightarrow \tan^{-1}\beta-\tan^{-1}\alpha=\frac{\beta-\alpha}{1+c^2}$$

since $0<\frac{1}{1+c^2}<1$ and $\beta-\alpha>0$, we have $0<\frac{\beta-\alpha}{1+c^2}<\beta-\alpha$

$$\boxed{\tan^{-1}\beta-\tan^{-1}\alpha<\beta-\alpha} \text{ Proved}$$

* If $\alpha=0$, $\beta=x$, then $\tan^{-1}x-\tan^{-1}0<x-0$

$$\Rightarrow \boxed{\tan^{-1}x<x,\ \text{if } x>0}$$

Sequence & Series

Convergent Series: A series $U_1 + U_2 + U_3 + \cdots + U_n + \cdots = \sum_{n=1}^{\infty} U_n$ is converges if the sequence (S_n) of the partial sum converges i.e. if $\lim_{n\to\infty} S_n$ exists. Also if $\lim_{n\to\infty} S_n = S$ then S is called as the sum of the given series.

Divergent Series:
If $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$

Ex show that the Geometric Series $\sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 - \cdots$ where $r > 0$ is convergent if $r < 1$ and diverges if $r \geq 1$

Sol^n: Let us define $S_1 = 1$ $\quad S_2 = 1 + r$ $\quad S_3 = 1 + r + r^2$

$$S_n = 1 + r + r^2 \cdots r^{n-1}$$

Case ① $r < 1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - r^n}{1 - r} = \frac{1}{1-r} - \lim_{n\to\infty} \frac{r^n}{1-r}$$

$(\lim_{n\to\infty} r^n = 0$ if $|r| < 1)$

$$\boxed{\lim_{n\to\infty} S_n = \frac{1}{1-r}}$$

finite

Since $\lim_{n\to\infty} S_n$ is finite $\therefore$ the sequence of partial sums i.e. (S_n) converges and hence the given series converges

Case ② $r > 1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^n - 1}{r - 1} = \lim_{n\to\infty} \frac{r^n}{1-r} - \frac{1}{r-1} \to \infty$$

(As $r^n \to \infty$ if $r > 1$)

so diverges

Case ③ $r = 1$ $\quad S_n = 1 + r + r^2 \cdots r^{n-1}$

$$= 1 + 1 + \cdots 1 = n$$

$$\boxed{\lim_{n\to\infty} S_n = \infty}$$

Since (S_n) diverges and hence the given series diverges

Positive term Series :- An infinite series whose all terms are positive is called a positive term series.

Ex - 1, -1, 2, -3, 4, 5 ---- (+ve) ∞
"no effect"

P Series :- An infinite series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} - - \frac{1}{n^p}$ --- $(p>0)$

is called 'p series'.

It converges if $p>1$ & diverges if $p \le 1$.

Example :-

① $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + ------$ Converges $(p = 3 > 1)$

② $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \frac{1}{1^{1/3}} + \frac{1}{2^{1/3}} + \frac{1}{3^{1/3}} + ----$ diverges $(p = 1/3 < 1)$

1) Limit form Test :-

Let $\sum_{n=1}^{\infty} U_n$ & $\sum_{n=1}^{\infty} V_n$ be two positive term series such that

$\lim_{n\to\infty} \frac{U_n}{V_n} = l$ (where l is a finite non zero number)

Then $\sum_{n=1}^{\infty} U_n$ & $\sum_{n=1}^{\infty} V_n$ behave in the same manner i.e. either both converge or both diverge.

Ex ① Test the convergence of the series $\frac{1}{3 \cdot 7} + \frac{1}{4 \cdot 9} + \frac{1}{5 \cdot 11} + ---$

Soln :- Here $U_n = \frac{1}{(n+2)(2n+5)}$

Let $V_n = \frac{1}{n^2}$ Now Consider $\frac{U_n}{V_n} = \frac{n^2}{2n^2+9n+10}$

$\Rightarrow \lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{2}$ (finite & non zero number)

Hence by limit form test, $\sum_{n=1}^{\infty} U_n$ & $\sum_{n=1}^{\infty} V_n$ behave similarly

since $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \frac{1}{(n+2)(2n+5)}$ also converges Ans

(ii) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

$$U_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \times \frac{(\sqrt{n+1}+\sqrt{n-1})}{(\sqrt{n+1}+\sqrt{n-1})} = \frac{2}{n(\sqrt{n+1}+\sqrt{n-1})}$$

$$U_n = \frac{2}{n(\sqrt{n+1}+\sqrt{n-1})}$$ so now consider $\frac{U_n}{V_n} = \frac{2\sqrt{n}}{n(\sqrt{n+1}+\sqrt{n-1})}$

$$V_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$ $\lim_{n\to\infty} \frac{U_n}{V_n} = 1$ (finite & non zero)

both behave similarly

$\therefore \lim_{n\to\infty} V_n = \frac{1}{n^{3/2}}$ is converges (as $P = 3/2 > 1$)

so $U_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ also converges

Ans

(iii) $\sum_{n=1}^{\infty} [(n^3+1)^{1/3} - n]$

$$U_n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$$

$$= n\left[1 + \frac{1}{3n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^6} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\cdot\frac{1}{n^3}\right] - n$$

$$U_n = \frac{1}{3n^2} - \frac{1}{9n^5}$$

Let $V_n = \frac{1}{n^2}$

$$\boxed{\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{3}}$$

so $\lim_{n\to\infty} U_n$ is converges because V_n is also converges ($P = 2 > 1$)

Ans

(2) Comparision Test :-

Let $\sum_{n=1}^{\infty} u_n$ & $\sum_{n=1}^{\infty} v_n$ be two positive term series such that $u_n \le k v_n$ ∀ n (where k is a positive number)

Then (i) if $\sum_{n=1}^{\infty} v_n$ converges then $\sum_{n=1}^{\infty} u_n$ also converges.

No need $\boxed{\text{(ii) if } \sum_{n=1}^{\infty} u_n \text{ diverges then } \sum_{n=1}^{\infty} v_n \text{ also diverges.}}$

Ques→ Test the convergence of the following series →

(1) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Solⁿ→ $u_n = \frac{1}{n^n}$ We know that $n^n > 2^n$ for $n > 2$

Hence $\frac{1}{n^n} < \frac{1}{2^n}$ for $n > 2$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a Geometric series $\left(\frac{1}{2} + \frac{1}{2^2} + \cdots\right)$ whose common ratio ½,

since $\frac{1}{2} < 1$ ∴ $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series. so $\frac{1}{n^n}$ is also convergent

(ii) $\sum_{n=2}^{\infty} \frac{1}{\log n}$

Solⁿ→ $\frac{1}{\log n} > \frac{1}{n}$ for $n \ge 2$

so Now $\sum_{n=1}^{\infty} \frac{1}{n}$ is a ~~divergent~~ series. Thus by comparing test $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is also divergent

(written above the struck word: divergent, ~~convergent~~)

(iii) $\sum_{n=1}^{\infty} \frac{1}{2^n + x}$ ∀ $x > 0$

$\frac{1}{2^n + x} < \frac{1}{2^n}$

$\frac{1}{2^n}$ is convergent so $\frac{1}{2^n + x}$ is also convergent

③ D' Alembert's Ratio Test :

Let $\sum_{n=1}^{\infty} U_n$ be a positive term series such that

$$\boxed{\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = l}$$

Then the series $\sum U_n$ is convergent $l > 1$

and divergent $l < 1$

and if $l = 1$ then Ratio Test fail.

Note

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = 1$ then $\sum U_n$ convergent if $l < 1$

divergent if $l > 1$, & fail if $l = 1$

Ques.

$$U_n = \sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$$

Solⁿ.

$$U_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$$

$$\frac{U_n}{U_{n+1}} = \frac{n!\,2^n}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!\,2^{n+1}} = \frac{1}{2}\left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{e}{2} > 1$$

$\therefore$ by De' Alembert's Ratio Test $\sum U_n$ is convergent.

Necessary Condition for Convergence:- If a positive term series ΣU_n is convergent, then $\boxed{\lim_{n\to\infty} U_n = 0}$

Let $S_n = U_1 + U_2 + U_3 + \text{------} + U_n$. Since ΣU_n is given to be Convergent

$\lim_{n\to\infty} S_n =$ a finite quantity(k) Also $\lim_{n\to\infty} S_{n-1} = k$

$U_n = S_n - S_{n-1}$ $\therefore \boxed{\lim_{n\to\infty} U_n = \lim_{n\to\infty} (S_n - S_{n-1}) = 0}$

*Obs 1. It is Important to note that the converse of this result is not true.

Ex $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \text{----} + \frac{1}{\sqrt{n}} + \text{---} + \infty$

since the term go on descending

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \text{---} + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \text{---} \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} \to \infty$$

Thus the series is divergent even through $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

so $\lim_{n\to\infty} U_n = 0$ is a necessary but not sufficient condition for convergence of ΣU_n.

*Obs 2 → The above result leads to a simple test for divergence

* If $\lim_{n\to\infty} U_n \neq 0$, the series ΣU_n must be divergent.

Qus → $\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \left(\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}\right)^2 + \text{-----}$

Soln → $U_n = \left[\frac{1\cdot 2\cdot 3\cdot 4 \text{ ---- } n}{3\cdot 5\cdot 7\cdot 9 \text{ ---- } (2n+1)}\right]^2$ $U_{n+1} = \left[\frac{1\cdot 2\cdot 3 \text{ --- } n(n+1)}{3\cdot 5\cdot 7\cdot 9 \text{ --- } (2n+1)(2n+3)}\right]^2$

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \left(\frac{n+1}{2n+3}\right)^2 = \frac{1}{4} < 1$ $\Rightarrow \boxed{\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = 4 > 1}$

given series U_n is converges

(5) Logarithmic Test :

Let $\sum_{n=1}^{\infty} U_n$ be a positive term series such that

$$\boxed{\lim_{n\to\infty} n \log \frac{U_n}{U_{n+1}} = l}$$

Then

(I) $\sum_{n=1}^{\infty} U_n$ Converges if $l > 1$

(II) $\sum_{n=1}^{\infty} U_n$ diverges if $l < 1$

Ques:(i) Test the convergenees of the series

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \text{---}$$

Sol^n

$$U_n = \frac{n^n x^n}{n!}$$

$$U_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = e \cdot x$$

Hence by Ratio Test, the given series converges if $ex < 1$; $\boxed{x < \frac{1}{e}}$ and diverges if $ex > 1$, $\boxed{x > \frac{1}{e}}$

fail if $ex = 1$, i.e. $\boxed{x = \frac{1}{e}}$

Since $\frac{U_{n+1}}{U_n}$ involves e $\therefore$ applying logarithmic test

$$\frac{U_n}{U_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n x}$$

$\therefore$ for $x = \frac{1}{e}$, $\frac{U_n}{U_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$

$$\log \frac{U_n}{U_{n+1}} = \log e - n \log\left(1 + \frac{1}{n}\right)$$

$$= \log e - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \text{---}\right)$$

$$= \cancel{1} - \cancel{1} + \frac{1}{2n} - \frac{1}{3n^2} + \text{---}$$

$$\lim_{n\to\infty} n \log\left(\frac{U_n}{U_{n+1}}\right) = \frac{1}{2} < 1 \quad \text{Diverges}$$

Hence the given series converges for $x < \frac{1}{e}$ & diverges for $x \geq \frac{1}{e}$

Ans

⑥ Cauchy's Integral Test :

Improper Integral

If $U(x)$ is non negative, integrable and monotonically ↓ing funⁿ such that $(n) = U_n$, then if $\int_1^\infty U(x)dx$ converges then the series $\sum_{n=1}^{\infty} U_n$ also converges.

Ex ① $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solⁿ: $U_n = \frac{1}{n^2+1}$

$U(x) = \frac{1}{x^2+1}$

so $U(x)$ is non negative & monotonocly ↓ing funⁿ

Consider $\int_1^\infty \frac{1}{x^2+1} dx = [\tan^{-1}x]_1^\infty$

$= \frac{\pi}{4}$ finite

Hence $\int_1^\infty \frac{1}{x^2+1} d(x)$ Converges So

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also Converges.

Diff form Other

(II) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)}$

$U_n = \frac{1}{n \log n}$

$U(x) = \frac{1}{x \log x}$

non negative ↓ing funⁿ

Consider $\int_2^\infty \frac{1}{x \log x} dx = \infty$ (infinite)

so $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

③ Cauchy's nth Root test :

Let $\sum_{n=1}^{\infty} U_n$ be a +ve term series such that

$$\boxed{\lim_{n\to\infty} (U_n)^{1/n} = l}$$

(I) $\sum_{n=1}^{\infty} U_n$ Converges if $l < 1$

(II) $\sum_{n=1}^{\infty} U_n$ diverges if $l > 1$

(III) fail if $l = 1$

Ex $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$

Solⁿ: $U_n = \frac{1}{n^n}$

$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$

Converges

(II) $U_n = 5^{-n-(-1)^n}$

$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} 5^{-\frac{[n+(-1)^n]}{n}}$

$= \lim_{n\to\infty} 5^{-1}$

$= \frac{1}{5} < 1$

Converges

Improper Integral

Improper Integral of first kind (Range of integration is infinite) :-

(I) $\int_a^{\infty} f(x)\,dx = \lim_{P\to\infty} \int_a^P f(x)\,dx$

If limit exists and is finite, say equal to 'l_1', then the improper integral converges, otherwise the improper integral diverges.

(2) $\int_{-\infty}^{b} f(x)\,dx = \lim_{P\to-\infty} \int_P^b f(x)\,dx$

If limit exists and is finite, say equal to l_2, then improper integral, converges at has the value of l_2; other-wise the improper integral diverges.

(III) $\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\lim_{a\to-\infty} \int_a^c f(x)\,dx}_{l_3} + \underbrace{\lim_{b\to\infty} \int_c^b f(x)\,dx}_{l_4}$

where 'c' is any finite constant including zero.

l_3 & l_4 exists + finite then Converges

l_3 & l_4 does not exists or are infinite then diverges

Ques→ Evaluate the following improper integrals, if they exists.

(I) $\int_0^{\infty} \frac{dx}{a^2+x^2}$, $a>0$

Solⁿ, $\lim_{P\to\infty} \int_0^P \frac{dx}{a^2+x^2}$

$= \lim_{P\to\infty} \frac{1}{a}\left[\tan^{-1}\frac{x}{a}\right]_0^P$

$= \lim_{P\to\infty} \frac{1}{a}\left[\tan^{-1}\frac{P}{a} - 0\right] = \frac{\pi}{2a}$ (finite)

Converges

(II) $\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

$= \lim_{P\to\infty} \int_1^P \frac{dx}{x\sqrt{x^2-1}}$

$= \lim_{P\to\infty} \left[\sec^{-1}x\right]_1^P$

$= \lim_{P\to\infty} \left(\sec^{-1}P - \sec^{-1}(1)\right)$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$ finite

Converges Ans

Comparison Test 1:

If $\boxed{0 \le f(x) \le g(x)}$ for all x, then

(I) $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.

If $\boxed{0 \ge f(x) \ge g(x)}$ for all x, then

(II) $\int_a^\infty f(x)\,dx$ diverges if $\int_a^\infty g(x)\,dx$ diverges

Comparison Test 2:

Suppose that $f(x)$ and $g(x)$ are positive fun^n such and let

$$\lim_{x \to \infty} \left[\frac{f(x)}{g(x)}\right] = l \quad , \quad 0 < l < \infty$$

Thus, the improper integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converges and diverges together.

Example:

(1) $\int_1^\infty e^{-x^2}\,dx$

Sol^n: $e^{-x^2} < e^{-x}$ for all $x \ge 1$

Consider $\int_1^\infty e^{-x}\,dx = \lim_{b \to \infty} \int_1^b e^{-x}\,dx = \lim_{b \to \infty} [1 - e^{-b}] = 1$

therefore $\int_1^\infty e^{-x}dx$ is convergent, then $\int_1^\infty e^{-x^2}$ also convergent Ans

Improper Integral of second kind ÷

① $\int_0^1 \frac{dx}{\sqrt{x}}$ ⇒ '0' is the only pt of infinite discontinuity (lower limit discontⁿ)

Solⁿ: $\lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} 2\{x^{1/2}\}$

$= \lim_{\varepsilon \to 0} 2\{1 - \varepsilon^{1/2}\} = 2$ finite

so $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent ## Ans

② $\int_1^3 \frac{x}{\sqrt{x-1}} dx$

'1' is the pt of infinite discontinuity

$\lim_{\varepsilon \to 0} \int_{1+\varepsilon}^{2} \frac{x}{\sqrt{x-1}} dx$ $\quad x - 1 = t$

$= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{1+t}{\sqrt{t}} dt$

$= \lim_{\varepsilon \to 0} \left(\frac{2}{3} + 2\right) - \left(\frac{2}{3}\varepsilon^{3/2} + 2\varepsilon\right) = \frac{8}{3}$ finite

so convergent ## Ans

④ $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ (upper limit discontⁿ)

here 2 is the pt of infinite discontinuity

$\lim_{\varepsilon \to 0} \int_0^{2-\varepsilon} \frac{dx}{\sqrt{4-x^2}}$

$= \lim_{\varepsilon \to 0} \left[\sin^{-1}\frac{2-\varepsilon}{2} - \sin^{-1}(0)\right]$

$= \frac{\pi}{2}$ finite convergent ## Ans

③ $\int_0^{1/e} \frac{dx}{x(\log x)^2}$

$\lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{1/e} \frac{dx}{x(\log x)^2}$

$\lim_{\varepsilon \to 0} \int_{\log \varepsilon}^{\log \frac{1}{e}} \frac{1}{t^2} dt$

$= \lim_{\varepsilon \to 0} \left\{ -\frac{1}{\log \frac{1}{e}} + \frac{1}{\log \varepsilon} \right\}$

$= 1$ finite Convergent ## Ans

⑤ $\int_{-1}^{1} \frac{dx}{x^2}$ (b/w pt discontⁿ)

$= \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$

$= \lim_{\varepsilon_1 \to 0} \int_{-1}^{0-\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2 \to 0} \int_{0+\varepsilon_2}^{1} \frac{dx}{x^2}$

$= \lim_{\varepsilon_1 \to 0} \left[\frac{1}{\varepsilon_1} - 1\right] + \lim_{\varepsilon_2 \to 0} \left[-1 + \frac{1}{\varepsilon_2}\right]$

$= \infty$ (infinite)

so $\int_{-1}^{1} \frac{dx}{x^2}$ is divergent Ans ##

Ques+. Examine the convergences of $\int_0^{4} \frac{dx}{x(4-x)}$ (two pt discont^n)

sol^n: This fun^n is discont^n at both the endpt 0 & 4.

$$\int_0^4 \frac{dx}{x(4-x)} = \int_0^{1} \frac{dx}{x(4-x)} + \int_1^4 \frac{dx}{x(4-x)}$$ ~Take any pt

$$= \lim_{\varepsilon_1 \to 0} \int_{0+\varepsilon_1}^{1} \frac{dx}{x(4-x)} + \lim_{\varepsilon_2 \to 0} \int_1^{4-\varepsilon_2} \frac{dx}{x(4-x)}$$

$$= \lim_{\varepsilon_1 \to 0} \frac{1}{4} \int_{\varepsilon_1}^{1} \left(\frac{1}{x} + \frac{1}{4-x}\right) dx + \lim_{\varepsilon_2 \to 0} \frac{1}{4} \int_1^{4-\varepsilon_2} \left(\frac{1}{x} + \frac{1}{4-x}\right) dx$$

$= \infty$ so divergent ## Ans

* The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ is converges iff $n<1$

Case (I) if $n = 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^1} = \lim_{\varepsilon \to 0} \left[\log(x-a)\right]_{a+\varepsilon}^{b}$$

$$= \lim_{\varepsilon \to 0} \left[\log(b-a) - \log \varepsilon\right]$$

$$= \log[(b-a) - \log 0] = \infty$$

$\Rightarrow \int_a^b \frac{dx}{(x-a)^n}$ is divergent at $n=1$

Case (II) if $n \neq 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \left[\frac{(x-a)^{-n+1}}{-n+1}\right]_{a+\varepsilon}^{b}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{1-n}\left[(b-a)^{-n+1} - \varepsilon^{-n+1}\right]$$

Now if $n > 1 \Rightarrow n-1 > 0$

then

$$= \lim_{\varepsilon \to 0} \frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right]$$

$$= \infty$$

$\Rightarrow \int_0^b \frac{dx}{(x-a)^n}$ diverges if $n>1$

Now if $n<1 \Rightarrow n-1<0$
then,

$$\lim_{\varepsilon \to 0} \frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right]$$

$$= \frac{1}{1-n}(b-a)^{1-n} \quad \text{finite value}$$

$$\Rightarrow \int_a^b \frac{dx}{(x-a)^n} \text{ is Convergent if } n<1 \quad \text{\# \# Ans}$$

<u>μ-Test</u> ÷ Let $f(x)$ be bounded and integrable in integral $(0, \infty)$ where $a>0$

* If there is a number $\mu>1$ such that $\lim_{x\to\infty} x^{\mu} f(x)$ exists and non zero then $\int_a^{\infty} f(x)\,dx$ is convergent.

* If there is a number $\mu \le 1$, such that $\lim_{x\to\infty} x^{\mu} f(x)$ exists and non zero then $\int_a^{\infty} f(x)\,dx$ is divergent.

① Test the Convergences of $\int_1^{\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}$

<u>solⁿ</u> → $f(x) = \frac{1}{x^{1/3}(1+x^{1/2})} = \frac{1}{x^{5/6}\left(1+\frac{1}{x^{1/2}}\right)}$

$f(x)$ is bounded in the interval $[1, \infty)$

μ = Dumnator (heghest power) − Num (heghest power)

$$\mu = \frac{5}{6} - 0 = \frac{5}{6}$$

so $$\lim_{x\to\infty} \cancel{x^{5/6}} \times \frac{1}{\cancel{x^{5/6}}\left(1+\frac{1}{x^{1/2}}\right)} = \lim_{x\to\infty} x^{\mu} f(x)$$

$= 1$ finite non zero number

~~So Converge~~

$\mu = \frac{5}{6} < 1$ so given integral is divergent # # Ans

Expansion of function of one variable-

① TAYLOR'S Theorem :-

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) \text{ --- } + \frac{h^n}{n!}f^n(x) + \text{---}$$

another form of Taylor's theorem :-

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \text{ ------ } + \frac{x^n}{n!}f^n(0) \text{---}$$

↳ Maclaurin's Theorem

① Expand $\sin x$ in ascending power of $(x - \frac{\pi}{2})$

Sol^n:

$f(x) = \sin x$ $\qquad \therefore f(\frac{\pi}{2}) = 1$

$f'(x) = \cos x$ $\qquad f'(\frac{\pi}{2}) = 0$

$f''(x) = -\sin x$ $\qquad f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x$ $\qquad f'''(\frac{\pi}{2}) = 0$

$f^{iv}(x) = \sin x$ $\qquad f^{IV}(\frac{\pi}{2}) = 1$ and so on.

$$\sin x = f(\tfrac{\pi}{2}) + (x - \tfrac{\pi}{2})f'(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^2}{2!}f''(\tfrac{\pi}{2}) \text{ ------}$$

$$\boxed{\sin x = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} + \text{ ------}}$$

Ans