Laplace tranform :-

Definition → Let $f(t)$ be a f^n of t defined for $0 \le t \le \infty$ then Laplace tranform of $f(t)$ denoted by $L[f(t)]$ or $f(s)$ is defined by

$$L[f(t)] = f(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[1] = \int_0^\infty e^{-st} \cdot 1 \cdot dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^\infty$$

$$= \left(\frac{e^{-\infty} - e^{-0}}{-s}\right)$$

$$= \frac{1}{s}$$

$$\left\{ L(1) = \frac{1}{s} \;\middle|\; L^{-1}\left(\frac{1}{s}\right) = 1 \right\}$$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt = \frac{\Gamma n+1}{s^{n+1}} \qquad L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{\Gamma n+1}$$

$$L(e^{at}) = \frac{1}{s-a}, \quad L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$L(\sin at) = \frac{a}{s^2+a^2}, \quad L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$$

$$L(\cos at) = \frac{s}{s^2+a^2}, \quad L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L(\sinh at) = \frac{a}{s^2-a^2}, \quad L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{\sinh at}{a}$$

$$L(\cosh at) = \frac{s}{s^2-a^2}, \quad L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

first shifting theorem →

if $L[f(t)] = F(s)$

then $L[e^{at} f(t)] = F(s-a)$

① → $L(e^{2t} \sin 3t)$

$$L(\sin 3t) = 3/(s^2+9)$$

$$L(e^{2t} \sin 3t) = \frac{3}{(s-2)^2+9}$$

② → $L(e^{-3t} \cos 2t)$

$$L(\cos 2t) = \frac{s}{s^2+4}$$

$$L(e^{-3t} \cos 2t) = \frac{s+3}{(s+3)^2+4}$$

③ → $L(t^4 . e^{7t})$

$$L(t^4) = \frac{\lfloor 5}{s^5}$$

$$L(e^{7t} . t^4) = \frac{\lfloor 5}{(s-7)^5}$$

first shifting thm for inverse laplace transform →

Ifs $L^{-1}(f(s)) = f(t)$

$$L^{-1}(f(s-a)) = e^{at} f(t)$$

$$L^{-1}(f(s-a)) = e^{at} L^{-1}(f(s))$$

Q. $L^{-1}\left(\frac{3}{(s-2)^2+9}\right) = 3L^{-1}\left(\frac{1}{(s-2)^2+9}\right)$

$= 3e^{2t} L^{-1}\left(\frac{1}{s^2+9}\right)$

$= \cancel{3} e^{2t} \frac{\sin 3t}{\cancel{3}}$

$= e^{2t} \sin 3t$

Q. $L^{-1}\left(\frac{s+3}{(s+3)^2+4}\right) = e^{-3t} L^{-1}\left(\frac{s}{s^2+4}\right)$

$= e^{-3t} \cos 2t$

Q $L^{-1}\left(\frac{\sqrt{5}}{(s-7)^5}\right) = \sqrt{5}\, e^{7t} L^{-1}\left(\frac{1}{s^7}\right)$

$= \cancel{\sqrt{5}}\, e^{7t} \left(\frac{t^4}{\cancel{\sqrt{5}}}\right)$

$= t^4 e^{7t}$

Q $L^{-1}\left(\frac{1}{s^2-4s+3}\right) = L^{-1}\left(\frac{1}{s^2-4s+4+9}\right)$

$= L^{-1}\left(\frac{1}{(s-2)^2+9}\right)$

$= e^{2t} L^{-1}\left(\frac{1}{s^2+9}\right)$

$= e^{2t} \frac{\sin 3t}{3}$

Date ____
Page ____

Q. $L^{-1}\left(\frac{s}{s^2-4s+13}\right) = L^{-1}\left(\frac{s}{s^2-4s+4+9}\right)$

$= L^{-1}\left(\frac{s-2+2}{(s-2)^2+9}\right)$

$= L^{-1}\left(\frac{s-2}{(s-2)^2+9}\right) + 2\,L^{-1}\left(\frac{1}{(s-2)^2+9}\right)$

$= e^{2t}\,L^{-1}\left(\frac{s}{s^2+9}\right) + 2.e^{2t}\,L^{-1}\left(\frac{1}{s^2+9}\right)$

$= e^{2t}\cos 3t + 2.e^{2t}\frac{\sin 3t}{3}$

Second Shifting thm →

unit step fn →

$$H(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$

$L(f(t)) = f(s)$

$L(f(t-a)\,H(t-a)) = e^{-as} f(s)$

Q. $= \begin{cases} \sin(t-\pi/3) & t > \pi/3 \\ 0 & t < \pi/3. \end{cases}$

$= \sin(t-\pi/3)\,H(t-\pi/3)$

$\Rightarrow L(\sin(t-\pi/3)\,H(t-\pi/3)) = e^{-\pi/3\, s}\frac{1}{s^2+1}$

Q. $\Rightarrow \begin{cases} e^{t-a} & t > a \\ 0 & t < a \end{cases}$

$= e^{t-a}\,H(t-a)$

$\Rightarrow L(e^t) = \frac{1}{s-1}$

$\Rightarrow L(e^{t-a}\,H(t-a)) = e^{-as}\frac{1}{s-1}$

Q. $\cos\left(t - \frac{2\pi}{3}\right) \quad t > 2\pi/3$

$0 \quad t < 2\pi/3$

$= \cos\left(t - \frac{2\pi}{3}\right) \begin{cases} 1 & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$

$= \cos\left(t - \frac{2\pi}{3}\right) H\left(t - \frac{2\pi}{3}\right)$

$= L(\cos t) = \frac{s}{s^2+1}$

$L\left(\cos\left(t - 2\pi/3\right) H\left(t - \frac{2\pi}{3}\right)\right) = e^{-2\pi/3} \frac{s}{s^2+1}$

Second Shifting thm for Inverse Laplace transform

$L^{-1}\left(f(s)\right) = f(t)$

$L^{-1}\left(e^{-as} f(s)\right) = f(t-a) H(t-a)$

Q. $L^{-1}\left(e^{-\pi/3 s} \frac{1}{s^2+1}\right)$

$L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$

$L^{-1}\left(e^{-\pi/3 s} \frac{1}{s^2+1}\right) = \sin\left(t - \frac{\pi}{3}\right) H\left(t - \frac{\pi}{3}\right)$

$= \sin\left(t - \frac{\pi}{3}\right) \begin{cases} 1 & t > \pi/3 \\ 0 & t < \pi/3 \end{cases}$

$= \begin{cases} \sin\left(t - \frac{\pi}{3}\right) & t > \pi/3 \\ 0 & t < \pi/3 \end{cases}$

Date ___
Page ___

Q. $L^{-1}\left(e^{-as}\dfrac{1}{s-1}\right)$

$L^{-1}\left(\dfrac{1}{s-1}\right) = e^{t}$

$L^{-1}\left(e^{-as}\dfrac{1}{s-1}\right) = e^{t-a}\,H(t-a)$

Q $L^{-1}\left(\dfrac{1}{s^2}\,e^{-2s}\right)$

$L^{-1}\left(\dfrac{1}{s^2}\right) = t$

$L^{-1}\left(\dfrac{1}{s^2}\,e^{-2s}\right) = (t-2)\,H(t-2)$

Q $L^{-1}\left(e^{-\frac{2\pi}{3}s}\cdot\dfrac{s}{s^2+1}\right)$

$L^{-1}\left(\dfrac{s}{s^2+1}\right) = \cos t$

$L^{-1}\left(e^{-\frac{2\pi}{3}s}\,\dfrac{s}{s^2+1}\right) = \cos\left(t-\dfrac{2\pi}{3}\right) H\left(t-\dfrac{2\pi}{3}\right)$

$= \cos\left(t-\dfrac{2\pi}{3}\right)\begin{cases} 1 & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$

$= \begin{cases} \cos\left(t-\frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$

Q $L^{-1}\left(\dfrac{s}{s^2-\omega^2}\,e^{-as}\right)$

$L^{-1}\left(\dfrac{s}{s^2-\omega^2}\right) = \cosh \omega t$

$L^{-1}\left(e^{-as}\dfrac{s}{s^2-\omega^2}\right) = \cosh \omega(t-a)\,H(t-a)$

multiplication property.

$L(f(t)) = f(s)$

$L(t\cdot f(t)) = \dfrac{d}{ds} f(s)$

$L(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n} f(s)$

Date ____
Page ____

Q $L(t \cdot \sin 2t)$

$L(\sin 2t) = \frac{2}{s^2+4}$

$L(t \sin 2t) = \frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$

$L(e^{-3t} \cdot t \cdot \sin 2t)$

$\Rightarrow \frac{4(s+3)}{((s+3)^2+4)^2}$

Q. $L(t \cos 3t)$

$L(\cos 3t) = \frac{s}{s^2+9}$

$L(t \cos 3t) = -\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$

$= \frac{s^2-9}{(s^2+9)^2}$

$L(e^{5t}\, t \cos 3t)$

$= \frac{(s-5)^2-9}{((s-5)^2+9)^2}$

Q. find $\int_0^\infty e^{-3t} \cdot t \cdot \sin 2t \, dt$

$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$\int_0^\infty e^{-st}(t \sin 2t) \cdot dt = \frac{4s}{(s^2+4)^2}$

put $s=3$

$= \frac{4 \times 3}{(3^2+4)^2} = \frac{12}{169}$

= multiplicative prop. for inverse.

$L^{-1}(f(s)) = f(t)$

$L^{-1}(f'(s)) = -t f(t)$

$L^{-1}(f'(s)) = -t\, L^{-1}(f(s))$

Q $L^{-1}\left(\log \frac{(s-4)}{(s+3)}\right)$, $f(s) = \log\frac{(s-4)}{(s-3)} = \log(s-4) - \log(s-3)$

$\Rightarrow f'(s) = \left(\frac{1}{s-4}\right) - \left(\frac{1}{s-3}\right)$

$\Rightarrow L^{-1}(f'(s)) = L^{-1}\left(\frac{1}{s-4}\right) - L^{-1}\left(\frac{1}{s+3}\right)$

$-t\, L^{-1}(f(s)) = e^{4t} - e^{-3t}$

$$L^{-1}(f(s)) = \frac{(e^{-3t} - e^{4t})}{t}$$

Q $L^{-1}\left(\log\left(1+\frac{1}{s^2}\right)^{1/2}\right)$

$$f(s) = \log\left(1+\frac{1}{s^2}\right)^{1/2} = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

$$f(s) = \frac{1}{2}\left(\log(s^2+1) - \log s^2\right)$$

$$f'(s) = \frac{1}{2}\left(\frac{2s}{s^2+1} - \frac{2s}{s^2}\right)$$

$$f'(s) = \left(\frac{s}{s^2+1} - \frac{1}{s}\right)$$

$$L^{-1}(f'(s)) = L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s}\right)$$

$$-t\,L^{-1}(f(s)) = \cos t - 1$$

$$L^{-1}(f(s)) = \frac{1-\cos t}{t}$$

Q $L^{-1}\left(\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right)$

Let $f(s) = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

$$f(s) = \log(s^2+a^2) - \log(s^2+b^2)$$

$$f'(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}$$

$$L^{-1}(f'(s)) = 2\, L^{-1}\left(\frac{s}{s^2+a^2}\right) - L^{-1}\left(\frac{s}{s^2+b^2}\right)$$

$$-t\,L^{-1}(f(s)) = 2(\cos at - \cos bt)$$

$$L^{-1}(f(s)) = \frac{2(\cos bt - \cos at)}{t}$$

Laplace of $\frac{f(t)}{t}$

Date ______
Page ______

Division Prop of Laplace transform

$$L\left(f(t)\right) = f(s)$$

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s).ds$$

Q $L\left(\frac{e^{-4t} - e^{2t}}{t}\right)$

$$L\left(e^{-4t} - e^{2t}\right) = \frac{1}{s+4} \quad \frac{-1}{s-2}$$

$$L\left(\frac{e^{-4t} - e^{2t}}{t}\right) = \int_s^\infty \left(\frac{1}{s+4} \quad \frac{-1}{s-2}\right) ds$$

$$= \log(s+4) - \log(s-2)\Big|_s^\infty$$

$$= \log\frac{(s+4)}{(s-2)}\Big|_s^\infty$$

$$= 0 - \log\left(\frac{s+4}{s-2}\right)$$

$$= \log\frac{(s-2)}{(s+4)}$$

Q $L\left(\frac{1-\cos 3t}{t}\right)$ & evaluate $\int_0^\infty e^{-t}\left(\frac{1-\cos 3t}{t}\right) ds$

$$L(1-\cos 3t) = 1/s - s/s^2+9$$

$$L\left(\frac{1-\cos 3t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+9}\right).ds$$

$$= \log s - \frac{1}{2}\log(s^2+9)\Big|_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{s^2}{s^2+9}\right)\Big|_0^\infty =$$

$$L\left(\frac{1-\cos 3t}{t}\right) = \frac{1}{2}\log\left(\frac{s^2+9}{s^2}\right)$$

$$\int_0^{\infty} e^{-st}\frac{(1-\cos 3t)}{t}\cdot dt = \frac{1}{2}\log\frac{(s^2+9)}{(s^2)}$$

Put $s=1$

$$\int_0^{\infty} e^{-t}\frac{(1-\cos 3t)}{t}\,dt = \frac{1}{2}\log\left(\frac{1+9}{1}\right) = \frac{1}{2}\log 10$$

Q Prove that $L\left(\frac{\sin t}{t}\right) = \tan^{-1}\left(\frac{1}{s}\right)$

Hence Prove that ① $\int_0^{\infty}\frac{\sin t}{t}\,dt = \frac{\pi}{2}$

② $\int_0^{\infty}\frac{e^{-t}\sin t}{t}\cdot dt = \pi/4$

Solⁿ $L(\sin t) = \frac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_s^{\infty}\frac{1}{s^2+1}\cdot ds = \left(\tan^{-1}s\right)_s^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}s$$

$$= \pi/2 - \tan^{-1}s$$

$$= \cot^{-1}s$$

$$L\left(\frac{\sin t}{t}\right) = \tan^{-1}(1/s)$$

(i) $\int_0^{\infty} e^{-st}\frac{\sin t}{t}\,dt = \tan^{-1}1/s$

Put $s=0$

$$\int_0^{\infty}\frac{\sin t}{t}\,dt = \tan^{-1}\infty = \pi/2$$

(2) Put $s=1$

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \tan^{-1} 1 = \pi/4$$

Inverse Laplace of $f(s)/s$.

$$L^{-1}\left(f(s)\right) = f(t)$$

$$L^{-1}\left(\frac{f(s)}{s}\right) = \int_0^t f(t)\cdot dt$$

Q $L^{-1}\left(\frac{1}{s^3(s^2+4)}\right)$

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{\sin 2t}{2}$$

$$L^{-1}\left(\frac{1}{s(s^2+4)}\right) = \int_0^t \frac{\sin 2t}{2}\,dt$$

$$= \frac{1}{2}\left(\frac{-\cos 2t}{2}\right)_0^t$$

$$= \frac{1}{4}\left(1-\cos 2t\right)$$

$$L^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \frac{1}{4}\int_0^t (1-\cos 2t)$$

$$= \frac{1}{4}\left(t - \frac{\sin 2t}{2}\right)_0^t$$

$$= \frac{1}{4}\left(t - \frac{\sin 2t}{2}\right)$$

$$L^{-1}\left(\frac{1}{s^3(s^2+4)}\right) = \frac{1}{8}\int_0^t (2t - \sin 2t)\,dt$$

Convoluation thm →

if $L^{-1}(f(s)) = f(t)$

$L^{-1}(g(s)) = g(t)$

$$L^{-1}(f(s) * g(s)) = \int_0^t f(u)\, g(t-u)\, du$$

$$L^{-1}(f(s) * g(s)) = \int_0^t g(u) \cdot f(t-u)\, du$$

Q. $L^{-1}\left(\frac{1}{s^2(s^2+4)}\right)$

$L^{-1}\left(\frac{1}{s^2}\right) = t$, $L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{\sin 2t}{2}$

$$L^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \int_0^t \frac{\sin 2u}{2} \cdot (t-u)\, du$$

$$= \frac{1}{2}\left[(t-u)\int \sin 2u\, du - \int\left(\frac{d(t-u)}{du}\int \sin 2u\, du\right) du\right]_0^t$$

$$= \frac{1}{2}\left(-\frac{\cos 2u}{2}(t-u) + \frac{\sin 2u}{4}\right)_0^t$$

$$= \frac{1}{2}\left[\left(0 + \frac{\sin 2t}{4}\right) - \left(\frac{-1}{2} t\right)\right]$$

$$= \frac{1}{2}\left(\frac{\sin 2t}{4} + \frac{1}{4}\right)$$

$$Q\ L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)} * \frac{s}{(s^2+b^2)}\right]$$

$$2\cos A\cos B = \frac{1}{2}\cos(A+B)\cos(A-$$

$$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$$

$$L^{-1}\left(\frac{s}{(s^2+a^2)} * \frac{s}{(s^2+b^2)}\right) = \int_0^t \cos au \cdot \cos b(t-u)\cdot du$$

$$= \int_0^t \cos\left((a-b)u + bt\right) + \cos\left((a+b)u - bt\right)\Big] du$$

$$= \left[\frac{\sin\left((a-b)u+bt\right)}{a-b} + \frac{\sin\left((a+b)u - bt\right)}{a+b}\right]_0^t$$

$$= \left(\frac{\sin at}{a-b} + \frac{\sin at}{a+b}\right) - \left(\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)$$

$$= \left(\frac{1}{a-b} + \frac{1}{a+b}\right)\sin at - \left(\frac{1}{a-b} - \frac{1}{a+b}\right)\sin bt$$

$$= \frac{1}{2}\left(\frac{2a}{a^2-b^2}\sin at - \frac{2b\sin bt}{a^2-b^2}\right)$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Q $L^{-1}\left(\frac{s}{(s^2+a^2)(s^2+b^2)}\right)$

$L^{-1}\left(\frac{s}{(s^2+a^2)} * \frac{1}{(s^2+b^2)}\right)$

$\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$

$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at;$

$L^{-1}\left(\frac{1}{s^2+b^2}\right) = \frac{\sin bt}{b}$

$L^{-1}\left(\frac{s}{s^2+a^2} * \frac{1}{s^2+b^2}\right) = \int_0^t \frac{\cos au \sin b(t-u)\, du}{b}$

$= \frac{1}{2b}\int_0^t \sin((a-b)u+bt) - \sin((a+b)u-bt)\, du$

$= \frac{1}{2b}\left[\frac{-\cos(a-b)u+bt}{a-b} + \frac{\cos((a+b)u-bt)}{a+b}\right]_0^t$

$= \frac{1}{2b}\left(\left(\frac{-\cos at}{a-b} + \frac{\cos at}{a+b}\right) - \left(\frac{-\cos bt}{a-b} + \frac{\cos bt}{a+b}\right)\right)$

$= \frac{1}{2b}\left[\left(\frac{1}{a+b} - \frac{1}{a-b}\right)(\cos at - \cos bt)\right.$

$= \frac{\cos at - \cos bt}{b^2-a^2}$

Appl^n. of Laplace transform to ordinary diff. eq^n. with constant coeff.

Q ~~$D^2+1)y$~~ $y =$

$$L(D^2y) = s^2 L(y) - s\{y(0)\} - y'(0)$$

$$L(Dy) = s\,L(y) - y(0)$$

$$L(D^3y) = s^3L(y) - s^2y(0) - sy'(0) - y''(0)$$

Q. $(D^2+1)y = t$ given $y(0) = 1$, $y'(0) = -2$

$$L(D^2y) + L(y) = L(t)$$

$$s^2 L(y) - sy(0) - y'(0) + L(y) = \frac{1}{s^2}$$

$$(s^2+1)L(y) = \frac{1}{s^2} + s - 2$$

$$L(y) = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$y = L^{-1}\left(\frac{1}{s^2(s^2+1)}\right) + L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$y = t - \sin t + \cos t - 2\sin t$$

$$\boxed{y = t + \cos t - 3\sin t}$$

$$\left\{\begin{aligned} &L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t \\ &L^{-1}\left(\frac{1}{s(s^2+1)}\right) = \int_0^t \sin t\,dt = -(\cos t)_0^t = 1 - \cos t \\ &L^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \int_0^t (1-\cos t) = (t - \sin t)_0^t = t - \sin t \end{aligned}\right\}$$

Date
Page

Que $(D^2+3D+2)x = 1$

$$x(0) = 0 = x'(0)$$

$$L(D^2x) + 3L(Dx) + 2L(x) = L(1)$$

$$[s^2L(x) - s\,x(0) - x'(0)] + 3[sL(x) - x(0)] + 2L(x) = \frac{1}{s}$$

$$(s^2+3s+2)L(x) = \frac{1}{s}$$

$$L(x) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)}$$

$$x = L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$A = 1/2$

$B = -1$

$C = 1/2$

$$x = L^{-1}\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right)$$

$$x = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$\boxed{x = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}}$$

Q $(D^2+9)y = \cos 2t$, $\quad y(0) = 1$

$y'(\pi/2) = -1$

Let $y'(0) = K$.

$$L(D^2y) + 9L(y) = L(\cos 2t)$$

$$s^2L(y) - s\,y(0) - y'(0) + 9L(y) = \frac{s}{s^2+4}$$

$$(s^2+9)L(y) - s - K = \frac{s}{s^2+4}$$

$$(s^2+9)L(y) = \frac{s}{s^2+4} + s + K$$

$$L(y) = \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+9} + \frac{K}{s^2+9}$$

Q $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ given $u(0,t) = 0 = u(5,t)$

$u(x,0) = 10\sin 4\pi x$

$$L\left(\frac{\partial u}{\partial t}\right) = 2L\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$s\bar{u} - u(x,0) = 2\frac{d^2\bar{u}}{dx^2}$$

$$2\frac{d^2\bar{u}}{dx^2} - s\bar{u} = -u(x,0)$$

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{2}\bar{u} = \frac{-u(x,0)}{2}$$

$$(D^2 - s/2)\bar{u} = -5\sin 4\pi x$$

$$m^2 = s/2$$

$$m = \pm\sqrt{s}/\sqrt{2}$$

$$C.F = y = C_1 e^{\sqrt{s/2}\,x} + C_2 e^{-\sqrt{s/2}\,x}$$

$$P.I \Rightarrow \bar{v} = \frac{-5\sin 4\pi x}{D^2 - s/2}$$

$$= \frac{-5\sin 4\pi x}{-16\pi^2 - \frac{s}{2}} = \frac{10\sin 4\pi x}{s + 32\pi^2}$$

$$\bar{v} = C.f + P.I = C_1 e^{\sqrt{s/2}\,x} + C_2 e^{-\sqrt{s/2}\,x} + \frac{10\sin 4\pi x}{s + 32\pi^2}$$

Apply, $u(0,t) = 0 = u(5,t)$

$$\Rightarrow C_1 = C_2 = 0$$

$$L(u(x,t)) = \frac{10\sin 4\pi x}{s + 32\pi^2}$$

$$u(x,t) = 10\sin 4\pi x\, L^{-1}\left(\frac{1}{s + 32\pi^2}\right)$$

$$\boxed{u(x,t) = 10\sin 4\pi x\, e^{-32\pi^2 t}}$$

Q $\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial t} = 1 - e^{-t}$, $0 < x < 1$

$t > 0$

$u(x,0) = x$

$$L\left(\frac{\partial u}{\partial x}\right) - L\left(\frac{\partial u}{\partial t}\right) = L(1) - L(e^{-t})$$

$$\frac{d\bar{u}}{dx} - \{s\bar{u} - u(x,0)\} = \frac{1}{s} - \frac{1}{s+1}$$

$$\frac{d\bar{u}}{dx} - s\bar{u} + x = \frac{1}{s(s+1)}$$

$$\frac{d\bar{u}}{dx} - s\bar{u} = \frac{1}{s(s+1)} - x$$

$$(D - s)\bar{u} = \frac{1}{s(s+1)} - x$$

$m = s$

$C.F \Rightarrow ce^{sx}$

$P.I \Rightarrow$

$\hookrightarrow \bar{u} = \dfrac{\frac{1}{s(s+1)} - x}{D - s}$

$$\bar{u} = \frac{1}{s}\left(\frac{\frac{1}{s(s+1)} - x}{1 - D/s}\right)$$

$$\bar{u} = (1 - D/s)^{-1}\left[-\frac{1}{s^2(s+1)} + \frac{x}{s}\right)$$

$$\bar{u} = \left(1 + \frac{D}{s}\right)\left(-\frac{1}{s^2(s+1)} + \frac{x}{s}\right)$$

$$\bar{u} = ce^{sx} + \left(\frac{1}{s^2} - \frac{1}{s^2(s+1)} + \frac{x}{s}\right)$$

Apply $0 < x < 1$, $t > 0$

$\hookrightarrow c = 0$

Date ____
Page ____

$$L^{-1}\left(\frac{1}{s(s+1)}\right) = \int_0^t e^{-t}\,dt$$
$$= -\left(e^{-t}\right)_0^t = 1 - e^{-t}$$

$$L^{-1}\left(\frac{1}{s^2(s+1)}\right) = \int_0^t (1-e^{-t})\,dt = \left(t + e^{-t}\right)_0^t$$
$$= t + e^{-t} - 1$$

$$u(x,t) = L^{-1}\left(\frac{1}{s^2}\right) - L^{-1}\left(\frac{1}{s^2(s+1)}\right) + xL^{-1}\left(\frac{1}{s}\right)$$

$$u(x,t) = t - (t + e^{-t} - 1) + x$$

$$\boxed{u(x,t) = -e^{-t} + 1 + x}$$

Q $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad x>0,\ t>0 \qquad u(x,0) = 6e^{-3x}$

$$L\left(\frac{\partial u}{\partial x}\right) = 2\left(\frac{\partial u}{\partial t}\right) + L(u)$$

$$\frac{d\bar{u}}{dx} = 2\left(s\bar{u} - u(x,0)\right) + \bar{u}$$

$$\frac{d\bar{u}}{dx} - 2(2s+1)\bar{v} = -2u(x,0)$$

$$\cancel{D}\ (D - (2s+1))\bar{u} = -12e^{-3x}$$

$m = 2s+1$

$C.F = Ce^{(2s+1)x}$

$$P.I \Rightarrow \bar{u} = \frac{-12e^{-3x}}{D-(2s+1)}$$

$$\bar{u} = \frac{-12e^{-3x}}{-3-\cancel{s}-1}$$

$$\bar{u} = \frac{6e^{-3x}}{s+2}$$

$$\bar{u} = C.F + P.I$$

$$\bar{u} = ce^{(2s+1)x} + \frac{6e^{-3x}}{s+2}$$

Apply $x > 0,\ t > 0$
$\Rightarrow \cancel{\phi}\ [C = 0]$

$$u(x,t) = 6e^{-3x}\ \cancel{e^{-2t}}\ L^{-1}\left(\frac{1}{s+2}\right)$$

$$\boxed{u(x,t) = 6e^{-3x}\ e^{-2t}}$$

Dirac Delta f^n $\Rightarrow$

L.T.   I.L.T.

$L\{\delta(t)\}$ $L^{-1}\{e^{-as} f(a)\}$

Heaviside f^n

$L^{-1}\{e^{-as} f(s)\}$

Laplace transform of Dirac Delta f^n. $L\{\delta(t)\}$
standard formulae.

1. $L\{\delta(t)\} = 1$
2. $L\{\delta(t-a)\} = e^{-as} \cdot L\{\delta(t)\} = e^{-as}$
3. $L\{f(t) \cdot \delta(t-a)\} = e^{-as} \cdot f(a)$

Q. find $L\{t^2 \cdot \delta(t-3)\}$ $a = 3$.

$= e^{-3s}(3)^2$

$= 9e^{-3s}$

Q find $L\{\sin 2t \cdot \delta(t-2)\}$ $a = 2$

$= e^{-2s} \sin 2(2)$

$= e^{-2s} \sin 4.$

Inverse laplace of Dirac Delta f^n →

$$*L\{e^{-as} f(a)\}$$

Standard formulae →

① $L^{-1}[1] = \delta(t)$

② $L^{-1}\{e^{-as}\} = \delta(t-a)$

*③ $L^{-1}\{e^{-as} f(a)\} = \delta(t-a) f(t)$

Q find $L^{-1}\{e^{-2s} \sin 2\}$

$$= \delta(t-2) \sin t$$

Q find $L^{-1}\left\{\frac{s}{s+1}\right\}$ $\left\{ L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} \right\}$

$$L^{-1}\left\{\frac{s+1-1}{s+1}\right\}$$

$$= L^{-1}\left\{1 - \frac{1}{s+1}\right\} = L^{-1}\{1\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \delta(t) - e^{-t}$$

~~Let~~ Laplace Transformation (Periodic f^n)

Q. Let $f(t)$ be a periodic f^n with period T, then prove that

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Q find laplace transform of

$$f(t) = \begin{cases} \sin \omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Sol^n Since $f(t)$ is a periodic f^n with period $\frac{2\pi}{\omega}$

$T \Rightarrow 2\pi/\omega$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\, dt + \cancel{\int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0\, dt}^{\,0} \right]$$

using formula.

$$\hookrightarrow \int e^{at} \sin bt\, dt = \frac{e^{at}}{a^2+b^2} (a \sin bt - b \cos bt)$$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s \cdot \sin\omega t - \omega \cos\omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2} (-s \sin\pi - \omega \cos\pi) - \frac{e^0}{s^2+\omega^2} (-s \cdot \sin 0 - \omega \cdot \cos 0) \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{-e^{-s\pi/\omega}}{s^2+\omega^2} (\omega \cdot \underbrace{\cos\pi}_{-1}) + \frac{1}{s^2+\omega^2} (\omega \cdot \cos 0) \right]$$

$$= \frac{1}{1-e^{2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega} \cdot \omega + \omega}{s^2+\omega^2} \right]$$

$$\boxed{L\{f(t)\} = \frac{\omega(e^{-s\pi/\omega} + 1)}{(1-e^{-2\pi s/\omega})(s^2+\omega^2)}}$$

Date ______
Page ______

Q find the L.T of $f(t) = \begin{cases} 1, & 0 \le t < a \\ -1, & a < t \le 2a \end{cases}$

and $f(t)$ is a periodic with period $2a$.

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} (1)\, dt + \int_a^{2a} e^{-st} (-1)\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left| \frac{e^{-st}}{-s} \right|_0^a - \left| \frac{e^{-st}}{-s} \right|_a^{2a} \right]$$

$$= \frac{-1}{s(1-e^{-2as})} \left[(e^{-as} - e^{0}) - (e^{-2as} - e^{-as}) \right]$$

$$= \frac{-1}{s(1-e^{-2as})} \left[-1 + 2e^{-as} - e^{-2as} \right]$$

$$= \frac{1}{s(1-e^{-2as})} \left[1^2 - 2(1)(e^{-as}) + (e^{-as})^2 \right]$$

$$= \frac{1}{s(1-e^{-2as})} \cdot (1-e^{-as})^2 = \frac{(1-e^{-as})^2}{s[1^2 - (e^{-as})^2]}$$

$$= \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}$$