Unit – 4.

Complex Variable (Differentiation)

Complex Number:- A number z is said to be complex number if defined as $\underset{R.P}{x} + i\underset{I.P.}{y}$, i.e. $z = x+iy$ $\forall$ $x, y \in R$ and $i = \sqrt{-1}$

Complex Conjugate:- $\bar{z} = x - iy$

$$z = x+iy \quad -(1)$$
$$\bar{z} = x-iy \quad -(2)$$

Adding (1) & (2)

$$z+\bar{z} = 2x$$

R.P. of z $\quad x = \frac{1}{2}(z+\bar{z})$

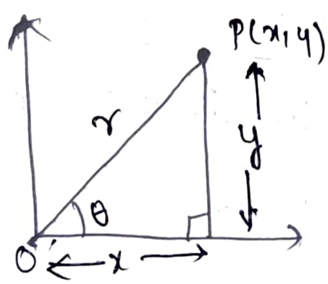
(1) – (2)

$$z-\bar{z} = 2iy$$
$$y = \frac{1}{2i}(z-\bar{z})$$

I.p. of z $\quad y = \frac{1}{2i}\left(z + \frac{1}{z}\right)$

Polar form.

$P(x,y)$



$$\sin\theta = \frac{y}{r}$$
$$y = r\sin\theta.$$
$$\cos\theta = \frac{x}{r} \qquad x = r\cos\theta.$$

$$z = x+iy$$
$$= r\cos\theta + ir\sin\theta$$
$$= r[\cos\theta + i\sin\theta]$$

$$\boxed{z = re^{i\theta}} \qquad \boxed{\because e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\boxed{\bar{z} = re^{-i\theta}}$$

Function of Complex Variable

The function of complex variable is defined as

$$W = f(z) = u(x,y) + iv(x,y)$$

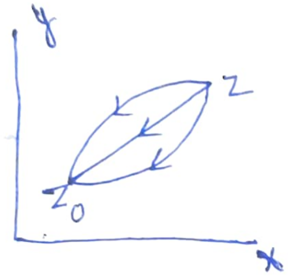
Eg Let $f(z) = z^2$

$$W = f(z) = z^2$$
$$= (x+iy)^2$$
$$= x^2 + 2ixy - y^2$$
$$= \underbrace{x^2-y^2}_{u(x,y)} + \underbrace{i(2xy)}_{v(x,y)}$$

$u(x,y) = x^2 - y^2$; $v(x,y) = 2xy$.

Limit of $f(z)$

$$\lim_{z \to z_0} f(z) = l \text{ (Finite)}$$



Continuity of $f(z)$: A single valued function $f(z)$ is said to be continuous at $z = z_0$ if

1. $f(z_0)$ defined
2. $\lim_{z \to z_0} f(z)$ exist
3. $\lim_{z \to z_0} f(z) = f(z_0)$

~~Diff~~ Derivative of $f(z)$

$$W = f(z)$$

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}.$$

Analytic function

A function is said to be analytic at point z_0 if it is and differentiable not only at z_0 but at every point of z_0.

A function $f(z)$ [which is analytic at x may be differentiable in a domain except a finite no. of points. These point are called singular point or singularities.

Necessary and sufficient condⁿ for f(z) to be analytic.

$w = f(z) = u(x,y) + iv(x,y)$ to be analytic in R, are

i) $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ are continuous in R

ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (Cauchy Riemann Equation) CR Eq.

$u_x = v_y$ $\quad u_y = -v_x$

Cauchy Riemann Eqⁿ in Polor form :-

Let (r, θ) be the polor coordinate of the point whose cartesian coordinate are (x, y).

$x = r\cos\theta$
$y = r\sin\theta$

$z = (x+iy) = r\cos + ri\sin\theta$

$z = re^{i\theta}$

$u + iv = f(z) = f(re^{i\theta})$ —(1)

diff (1) w.r.t. 'r'

$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})e^{i\theta}$ —(2)

diff. (1) w.r.t 'θ'.

$\frac{\partial u}{\partial \theta} + \frac{i\partial v}{\partial \theta} = f'(re^{i\theta})\, ri e^{i\theta}$ —(3)

$\frac{\partial u}{\partial \theta} + \frac{i\partial v}{\partial \theta} = ir\left[\frac{\partial u}{\partial r} + \frac{i\partial v}{\partial r}\right]$ from (2)

$\frac{\partial u}{\partial \theta} + \frac{i\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - \frac{r\partial v}{\partial r}$

Equating real and imaginary part:

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \& \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

Derivative of w or $f'(z)$ in polar for.

$$\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$$

$$\frac{dw}{dz} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}$$

Harmonic function

A function $u(x,y)$ is said to be harmonic if satisfy the laplace eqn.

i.e. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Theorem:- If $f(z) = u + iv$ is analytic fun then u and v are both harmonic function.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

NOTE:- u and v are called conjugate harmonic fun.

Milne's Thomson Method

(I) If $u(x,y)$ is given

(1) $\phi_1(x,y) = \frac{\partial u}{\partial x}$, $\phi_2(x,y) = \frac{\partial u}{\partial y}$

(2) Replace x by z & y by "0".

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)}, \quad \phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{z=0}$$

(3) $$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c.$$

(4) v = I.P of $f(z)$

(II) If $v(x,y)$ is given

(1) $\psi_1(x,y) = \frac{\partial v}{\partial y}$ $\quad \psi_2(x,y) = \frac{\partial v}{\partial x}$

(2) Replace x by z & y by "0".

$\varphi_1(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)}$, $\varphi_2(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$.

(3) $f(z) = \int [\varphi_1(z,0) + i\varphi_2(z,0)]\,dz + c$.

(4) u = R.P. of $f(z)$.

(III) If $(u-v)$ is given

let $f(z) = u + iv$ — (1)

$if(z) = iu - v$ — (2)

(1) + (2) $(1+i)f(z) = (u-v) + i(u+v)$

$f(z) = U + iV$.

$U = u - v$ (given)

$\phi_1 = \frac{\partial u}{\partial x}$, $\phi_2(x,y) = \frac{\partial U}{\partial y}$

Replace x by z & y by 0

$\phi_1(z,0) = \left(\frac{\partial U}{\partial x}\right)_{(z,0)}$ & $\phi_1(z,0) = \left(\frac{\partial U}{\partial y}\right)_{(z,0)}$.

By Milne Thomson Method.

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + C$.

$(1+i)f(z) =$

$f(z) = \frac{1}{1+i} \int \phi_1(z,0) + i\phi_2(z,0)]\,dz + \frac{c}{1+i}$.

(IV) If $(u+v)$ is given, $f(z)$

let $f(z) = u + iv$ — (1)

$if(z) = iu - v$ — (2)

(1) + (2) $(1+i)f(z) = (u-v) + i(u+v)$

$f(z) = U + iV$.

$V = u + v$ (given)

$\varphi_1 = \frac{\partial V}{\partial y}$, $\varphi_2(x,y) = \frac{\partial V}{\partial x}$.

Replace x by z & y by 0

$\varphi_1(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)}$ & $\varphi_2(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$

Q. Find the regular fun^n in term of z whose imag. part is $e^x(x\sin y + y\cos y)$ — (1)

Sol: Diff (1) w.r.t 'y'

$\varphi_1(x,y) = \frac{\partial v}{\partial y} = e^x[x\cos y + 1\cos y - y\sin y]$

diff. (1) w.r.t 'x'.

$\varphi_2(x,y) = \frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x \sin y$ — (3)

Replace x by z & y by 0 in (2) & (3)

$\varphi_1(z,0) = e^z[z+1]$

$\varphi_2(z,0) = e^z(0) + e^z[0]$.

By Milne Thomson Method

$f(z) = \int[\varphi_1(z,0) + i\,\varphi_2(z,0)]\,dz + C$

$= \int (z+1)e^z\,dz + C$

$f(z) = (z+1)e^z - e^z + c$

$= (z+1-1)e^z + c$

$= ze^z + c$

Ans

$f(z) = (x+iy)e^{x+iy} + C.$

$= (x+iy)\,e^x e^{iy} + C$

$(x+iy)\, e^{x}(\cos y + i \sin y) + c.$

$= e^{x}[x \cos y + i \sin y + iy \cos y - y \sin y] + c.$

$u = R.P$ of $f(z) = e^{x}[x \cos y - y \sin y] + c.$

$v = e^{x}[x \sin y + y \cos y]$

$\frac{\partial v}{\partial x} = e^{x}[x \sin y + y \cos y] + e^{x} \sin y.$

$\frac{\partial^2 v}{\partial x^2} = e^{x}[x \sin y + y \cos y] + e^{x} \sin y + e^{x} \sin y.$

$\frac{\partial v}{\partial y} = e^{x}[x \sin y + \cos y - y \sin y]$

$\frac{\partial^2 v}{\partial y^2} = e^{x}[-x \cos y - \sin y - \sin y - y \cos y].$

$e^{x} x \sin y + \cancel{y e^{x} \cos y} + e^{x} \cancel{\sin y} + \cancel{e^{x} \sin y} - x e^{x} \cos y - \cancel{e^{x} \sin y} - e^{x} \cancel{\sin y} - \cancel{y e^{x} \cos y}.$

Ques: Determine the analytic funⁿ $f(z) = u + iv$ where $u = (x-y)(x^2 + 4xy + y^2)$

Sol: $\phi_1(x,y) = \frac{\partial u}{\partial x} = 1 \cdot (x^2 + 4xy + y^2) + (x-y)(2x+4y)$

$\phi_1(z,0) = z^2 + 2z^2 = 3z^2$

$\phi_2(x,y) = \frac{\partial u}{\partial y} = (-1)(x^2 + 4xy + y^2) + (x-y)(4x+2y)$

$\phi_2(z,0) = -z^2 + 4z^2 = 3z^2,$

By Milne theorem Method.

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c.$

$= \int (3z^2 - i\, 3z^2)\, dz + c.$

$= 3(1-i) \int z^2\, dz + c \Rightarrow \cancel{3}(1-i) \frac{z^3}{\cancel{3}} + c$

$f(z) = (1-i) z^3 + c$ Ans

Q: If $(u+v) = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ & $f(z) = u+iv$ is an analytic function of $z = x+iy$. find $f(z)$ in terms of $f(z)$.

Sol:- $u+v = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ — (1)

we know that

$$\cosh 2y = \frac{e^{-2y}+e^{2y}}{2}.$$

$$u+v = \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x}.$$

we ~~know~~ have $f(z) = u+iv$. — (2)

$if(z) = iu - v$ — (3)

(2) + (3) $(1+i)f(z) = u+iv + iu - v$

$\Rightarrow (u-v) + i(u+v)$

$F(z) = U+iV$

$F(z) = (1+i)f(z)$, $U = u-v$

$$V(x,y) = (u+v) = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\varphi_1(x,y) = \frac{dV}{dy} = \frac{0(\cosh 2y - \cos 2x) - \sin 2x\, 2\sinh 2y}{(\cosh 2y - \cos 2x)^2}.$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$\varphi_1(z,0) = 0$

$$\varphi_2(x,y) = \frac{\partial V}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\sin 2x \cdot \sin 2x}{(\cosh 2y - \cos 2x)^2}.$$

$$\varphi_2(x,y) = \frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}.$$

$$\varphi_2(x,y) = \frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}.$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}.$$

$$\psi_2(z,0) = \frac{2\cos 2z \cosh 2(0) - 2}{(\cosh(2)(0) - \cos 2z)^2}.$$

$$\psi_2(z,0) = \frac{+2\cos 2z - 2}{(1-\cos 2z)^2} \Rightarrow -\frac{2\cancel{(-\cos 2z+1)}}{(1-\cos 2z)^{\cancel{2}}}.$$

$$\psi_2(z,0) = \frac{-2}{(1-\cos 2z)}$$

By Milne's Theorem

$$f(z) = \int[\psi_1(z,0) + i\psi_2(z,0)]\, dz + c.$$

$$\Rightarrow \int 0 + \frac{i(-2)}{(1-\cos 2z)}\, dz.$$

$$\Rightarrow -2i \int \frac{1}{1-\cos 2z}\, dz + C$$

$$\Rightarrow -2i \int \frac{1}{1-(1-2\sin^2 z)}\, dz + C.$$

$$\Rightarrow -\cancel{2}i \int \frac{1}{\cancel{2}\sin^2 z}\, dz + C$$

$$\Rightarrow -i\int \operatorname{cosec}^2 z\, dz + C$$

$$\Rightarrow i\cot z + C.$$

$$(i+1) f(z) = i\cot z + C$$

$$f(z) = \frac{i}{(1+i)}\cot z + \frac{C}{(1+i)}$$

Q. If $f(z) = u + iv$ is an analytic function of z & $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$

Prove that $f(z) = \frac{1}{2}\left[1 - \cot\frac{z}{2}\right]$ when $f\left(\frac{\pi}{2}\right) = 0$.

Sol:- We know $f(z) = u + iv$ — (1)

$i f(z) = iu - v$ — (2)

(1) + (2)

$$(1+i) f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV.$$

where $F(z) = (1+i) f(z)$

$U = u - v$ & $V = u + v$

$$U(x,y) = u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$$

$$\phi_1(x,y) = \frac{\partial}{\partial x} U(x,y).$$

$$= \frac{(-\sin x + \cos x)(2\cos x - 2\cosh y) + 2\sin x(\cos x + \sin x - e^{-y})}{(2\cos x - 2\cosh y)^2}.$$

$$\cancel{f(z,0) = -(\cos z + \sin x - e^{-y})} \quad (-2\sin x).$$

$$\phi_1(z,0) = \frac{(-\sin z + 1)(2\cos z - 2) + (-2\sin z)(\cos z + \sin z - 1)}{(2\cos z - 2)^2}$$

$$= \frac{-2\sin z\cancel{\cos z} + 2\cancel{\sin z} + 2\cos^2 z - 2 + (2\sin z\cos z + 2\sin^2 z - 2\sin z)}{(2\cos z - 2)^2}$$

$$= \frac{2 - 2\cos z}{(2\cos z - 2)^2} = \frac{-2(-1+\cos z)}{4(\cos z - 1)^2} = \frac{-1}{2}\frac{1}{(\cos z - 1)}$$

$$U(x,y) = u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$$

$$\phi_2(x,y) = \frac{-\partial U}{\partial y}.$$

$$\frac{e^{-y}(2\cos x - 2\cosh y) - (\cos x + \sin x - e^{-y})(-2\sinh y)}{(2\cos x - 2\cosh y)^2}.$$

$$\phi_2(z,0) = \frac{(2\cos z - 2)}{(2\cos z - 2)^2} = \frac{-1}{2(1-\cos z)}$$

By Milne Thomson Method.

$$F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + C.$$

$$F(z) = \int \left[\frac{-1}{2(\cos z - 1)} - i\frac{1}{2(\cos z - 1)}\right] dz + C.$$

$$F(z) = \int \left[\frac{1}{2(1-\cos z)} + i\frac{1}{2(1-\cos z)}\right] dz + C$$

$$f(z) = \frac{1+i}{2} \int \frac{1}{1-\cos z} \, dz + c.$$

$$f(z) = \frac{1+i}{2} \int \frac{1}{1-\left(1-2\sin^2\frac{z}{2}\right)} \, dz + c.$$

$$= \frac{1+i}{2(+2)} \int \frac{1}{\sin^2\frac{z}{2}} \, dz + c.$$

$$= \frac{1+i}{2 \times (+2)} \int \operatorname{cosec}^2 \frac{z}{2} \, dz + c.$$

$$= \frac{1+i}{\cancel{4}\, 2} \left[\frac{-\cot z/2}{1/\cancel{2}} \right] + c$$

$$(1+i) f(z) = -\frac{1+i}{2} \cot z/2 + c.$$

$$f(z) = \frac{-1}{2} \cot \frac{z}{2} + \frac{c}{1+i} \quad \text{— (A)}$$

$$f\left(\frac{\pi}{2}\right) = 0$$

$$0 = \frac{-1}{2} \cot\left(\frac{\pi}{4}\right) + \frac{c}{1+i}$$

$$0 = \frac{-1}{2} + \frac{c}{1+i}$$

$$\frac{c}{1+i} = \frac{1}{2}.$$

Put the value of $\frac{c}{1+i}$ in eq (A)

$$f(z) = \frac{-1}{2} \cot \frac{z}{2} + \frac{1}{2} \quad \text{Ans.}$$

$W = \phi + i\psi \longrightarrow$ Stream function (Flux function)

$W \downarrow$ Complex Potential Function

$\phi \searrow$ Velocity Potential function or scalar Potential function

Q: If potential funⁿ is $\log(x^2+y^2)$. Find the flux function & complex potential function.

Sol: Let $\phi(x,y) = \log(x^2+y^2)$

$\psi = 0$ (Flux function)

u = complex potential function

$$\phi_1(x,y) = \frac{\partial\phi}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\phi_2(x,y) = \frac{\partial\phi}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\phi_1(z,0) = \frac{2z}{z^2} \Rightarrow \frac{2}{z}$$

$$\phi_2(z,0) = 0$$

By Milne Method

$$w = \int[\phi_1(z,0) - i\phi_2(z,0)]\,dz + c$$

$$\Rightarrow \int\frac{2}{z}\,dz + c \quad \Rightarrow \quad w = 2\log z + c.$$

$$w = 2\left[\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right] + c.$$

$$= \log(x^2+y^2) + i\,\cancel{2}\tan^{-1}\frac{y}{x} + c$$

Flux funⁿ $\psi = 2\tan^{-1}\frac{y}{x}$,

Q: If $w = \phi + i\psi$ represents a complex potential for an electric field where $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$.

Sol: $$\psi_1(x,y) = \frac{\partial\psi}{\partial y} = -2y - \frac{2yx}{(x^2+y^2)^2}.$$

$$\psi_2(x,y) = \frac{\partial\psi}{\partial x} = 2x + \frac{x^2+y^2 - 2x^2}{(x^2+y^2)^2} \Rightarrow 2x + \frac{\cancel{3}x^2+y^2}{(x^2+y^2)^2}$$

$$\psi_1(z,0) = 0.$$

$$\psi_2(z,0) = 2z + \frac{3z^2}{z^4} \Rightarrow 2z + \frac{3}{z^2} \Rightarrow \cancel{\frac{3}{z^2}}$$

By Milne thomson method.

$$w = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz + c.$$

$$w = \int 0 + i\left(2z - \frac{1}{z^2}\right) dz + c$$

$$w = +i \int \left(2z - \frac{1}{z^2}\right) dz + c \Rightarrow +i\left[\frac{\cancel{2}z^2}{\cancel{2}} + \frac{1}{z}\right] + c$$

$$w = i\left[z^2 + \frac{1}{z}\right] + c.$$

$$= i\left[z^2 + \frac{1}{z}\right] + c \Rightarrow i\left[(x+iy)^2 + \frac{\bar{z}}{z\bar{z}}\right] + c.$$

$$\Rightarrow i\left[x^2 - y^2 + 2ixy + \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}\right] + c.$$

$$\Rightarrow ix^2 - iy^2 - 2xy + \frac{xi}{x^2+y^2} + \frac{y}{x^2+y^2} + ci$$

$$\boxed{\phi = -2xy + \frac{y}{x^2+y^2}}$$

Q. Determine a, b, c, d ~~show~~ so that the fun^n $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.

Sol^n Here, $u = x^2 + axy + by^2$.

$v = cx^2 + dxy + y^2$.

$$\frac{\partial u}{\partial x} = 2x + ay \qquad \frac{\partial u}{\partial y} = ax + 2by.$$

$$\frac{\partial v}{\partial x} = 2cx + dy \qquad \frac{\partial v}{\partial y} = dx + 2y.$$

C R. eq^n.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$2x + ay = dx + 2y \qquad d = 2 \quad a = 2.$$

$$ax + 2by = -(2cx + dy).$$

$a = -2c$

$2 = -2c$

$\boxed{c = -1}$

$2b = -d$

$2b = -2$

$\boxed{b = -1}$

Ques: Prove that the funⁿ $\sinh z$ is analyatic & find its derivative.

Sol: $u + iv = f(z)$

$= \sinh z$

$= \sinh(x+iy)$

$= \sinh x \cosh iy + \cosh x + \sinh iy$

$= \sinh x \cos y + i \cosh x \sin y.$

$\Rightarrow u = \sinh x \cos y$

$v = \cosh x \sin y.$

$\frac{\partial u}{\partial x} = \cosh x \cos y \quad ; \quad \frac{\partial u}{\partial y} = -\sinh x \sin y.$

$\frac{\partial v}{\partial x} = \sinh x \sin y \quad ; \quad \frac{\partial v}{\partial y} = -\cosh x \cos y$

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow$ CR eq. are satisfied.

$\sin y$, $\cos y$, $\sinh x$, $\cosh x$ are continuous, so

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous.

Hence, $f(z)$ is analyatic everywhere b/c both condⁿ are satisfied.

$\frac{\partial u}{\partial x}$ ~~$\frac{\partial u}{\partial y}$~~

$= \cosh x$

Q. Show that $f(z) = \log z$ is analytic everywhere in the complex plane ~~and~~ ~~the~~ except at the origin and that its derivative is $\frac{1}{z}$.

Sol:- $f(z) = \log z$.

$f(z) = \log(x+iy)$

$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

Here $u = \frac{1}{2}\log(x^2+y^2)$; $v = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$.

$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$.

$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{(x^2+y^2)}$

$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2}$

continuous

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

except at $x=0$ & $y=0$ becoz. $x^2+y^2 \neq 0$

also its derivative are continuous except $x=0$ except $(0,0)$

Hence $f(z)$ is analytic

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{(-y)}{x^2+y^2}$

$= \frac{x-iy}{x^2+y^2} = \frac{\cancel{x-iy}}{(x+iy)\cancel{(x-iy)}} = \frac{1}{x+iy} = \frac{1}{z}$

Ques:- Determine constant 'b' so that $u = e^{bx}\cos 5y$ is harmonic.

Sol:- $u = e^{bx}\cos 5y$

$\frac{\partial u}{\partial x} = b e^{bx}\cos 5y$; $\frac{\partial^2 u}{\partial x^2} = b^2 e^{bx}\cos 5y$.

$\frac{\partial u}{\partial y} = -5e^{bx}\sin 5y$, $\frac{\partial^2 u}{\partial y^2} = -25e^{bx}\cos 5y$.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = b^2e^{bx}\cos 5y - 25e^{bx}\cos 5y = 0$

$e^{bx}\cos 5y(b^2 - 25) = 0$

$b^2 = 25$

$\boxed{b = \pm 5}$

Q. Show that the funⁿ defined by $f(z) = \sqrt{|xy|}$ is not regular at the origin although C.R eqⁿ are satisfy there.

Sol: Here, $u + iv = f(z) = \sqrt{|xy|}$

$u = \sqrt{|xy|}$, $v = 0$.

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{0+0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0$$

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

CR eqⁿ is satisfied at $(0,0)$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$\lim_{z \to 0} \frac{\sqrt{xy} - 0}{x+iy}$$

let $y = mx$

$$\lim_{\substack{y \to mx \\ x \to 0}} \frac{\sqrt{|xy|}}{x+iy} \Rightarrow \lim_{x \to 0} \frac{\sqrt{x\,mx}}{x+imx} \Rightarrow \frac{\cancel{x}\sqrt{m}}{\cancel{x}(1+im)}$$

$$= \frac{\sqrt{m}}{(1+im)}$$

Now this limit is not unique since it depend on m. Hence $f'(0)$ doesnot exist.

Hence the function $f(z)$ is not regular at the origin

Q Examine the nature of the funⁿ $f(z) = \begin{cases} \dfrac{x^2y^5(x+iy)}{x^4+y^{10}} & z \neq 0 \\ 0 & z = 0 \end{cases}$ in the region including the origin.

Sol: $u + iv = f(z) = \dfrac{x^2y^5(x+iy)}{x^4+y^{10}}$

$$= \frac{x^3y^5}{x^4+y^{10}} + \frac{ix^2y^6}{x^4+y^{10}}$$

Here, $u = \dfrac{x^3y^5}{x^4+y^{10}}$ $\quad v = \dfrac{x^2y^6}{x^4+y^{10}}$

$f(0) = 0 \Rightarrow u(0,0) = 0, \; v(0,0) = 0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} \Rightarrow \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y}, \; \dfrac{\partial v}{\partial x} = -\dfrac{\partial v}{\partial y}$ CR eqⁿ is satisfied at $(0,0)$

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{x^2y^5\cancel{(x+iy)}}{(x^4+y^{10})\cancel{(x+iy)}}$$

$$\lim_{z\to 0} \frac{x^2y^5}{x^4+y^{10}}$$

a) Take a path $y = mx$

$$f'(0) = \lim_{x\to 0} \frac{x^2m^5x^5}{x^4+m^{10}x^{10}} \Rightarrow \lim_{x\to 0} \frac{x^3m^5}{1+m^{10}x^6} = \frac{0}{1+0} = 0$$

b) Take a path $y^5 = x^2$

$$f'(0) \lim_{x\to 0} \frac{x^2x^2}{x^4+x^4} \Rightarrow \frac{x^4}{2x^4} \Rightarrow \frac{1}{2}$$

Here $f'(0)$ donot exist hence $f(z)$ is not analytic at origin although CR eqⁿ are satisfied there.

Q. ~~The~~ $f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ The CR eq^n is satisfied at origin does $f'(z)$ exist?

Sol: $f(z) = \frac{(x-iy)^2}{x+iy} = \frac{[x^2 - y^2 - 2ixy][x-iy]}{(x+iy)(x-iy)}$

$= \frac{x^3 - xy^2 - 2ix^2y - ix^2y + iy^3 - 2xy^2}{x^2+y^2}$

$f(z) = (u+iv) = \frac{(x^3 - xy^2 - 2xy^2)}{x^2+y^2} + i\frac{(-2x^2y + y^3 - x^2y)}{x^2+y^2}$

$u = \frac{x^3 - xy^2 - 2xy^2}{x^2+y^2}$; $v = \frac{(-2x^2y + y^3 - x^2y)}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1.$

$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} \quad \lim_{x\to 0} \frac{0-0}{x} = 0$

$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{y-0}{y} = 1.$

Here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow$ C.R. eq^n are satisfied at $(0,0)$

$f'(0) = \lim_{z\to 0} \frac{f(z) - 0}{z} = \lim_{z\to 0} \frac{(\bar{z})^2}{z^2}$

$f'(0) = \lim_{z\to 0} \frac{(x-iy)^2}{(x+iy)^2}$

Take a path $y = mx$.

$\lim_{x\to 0} \frac{(x-imx)^2}{(x+imx)^2} \Rightarrow \lim_{x\to 0} \frac{\cancel{x^2}(1-im)^2}{\cancel{x^2}(1+im)}$

$= \frac{(1-im)^2}{(1+im)^2}$

$f'(0)$ donot exist at $(0,0)$ b/c different value of m gives different value of $f'(0)$.

Q. Show that the function $f(z) = \begin{cases} \frac{2xy(x+iy)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ are CR eqn satisfied at origin but derivative of $f(z)$ doesnot exist at origin.

Sol: $f(z) = \frac{2xy(x+iy)}{x^2+y^2} \Rightarrow \frac{2x^2y}{x^2+y^2} + \frac{i\,2xy^2}{x^2+y^2}$

~~u+iv=~~

$u = \frac{2x^2y}{x^2+y^2}$ $v = \frac{2xy^2}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$

$\frac{\partial u}{\partial y} \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$\frac{\partial v}{\partial x} \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$

$\frac{\partial v}{\partial y} \lim_{x\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

Here $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y}$

$\Rightarrow$ C.R. eqn are satisfied at $(0,0)$.

$f'(0) = \lim_{z\to 0} \frac{f(z) - 0}{z} = \lim_{z\to 0} \frac{2xy\cancel{(x+iy)}}{(x^2+y^2)\cancel{(x+iy)}}$

$\lim_{z\to 0} \frac{2xy}{x^2+y^2}$

Take a path $y = mx$.

$\lim_{z\to 0} \frac{2mx^2}{x^2+m^2x^2} \Rightarrow \lim_{z\to 0} \frac{2m\cancel{x^2}}{\cancel{x^2}(1+m^2)} \Rightarrow \lim_{z\to 0} \frac{2m}{1+m^2}$

$\frac{2m}{1+m^2}$

$f'(0)$ doesnot exist at $(0,0)$ b/c diff. value of m gives diff value of $f'(0)$.

Ques: If $f(z)$ is a regular function of z, Prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Sol:-

$$f(z) = u+iv$$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f(z)|^2 = u^2+v^2 = \phi(x,y) \quad -(1)$$

Partial diff. of (1) w.r.t x.

$$\frac{\partial\phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

Again p. diff w.r.t x

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] + 2\left[v\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial x}\right] \quad -(2)$$

P. diff of (1) w.r.t y.

$$\frac{\partial\phi}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$$

Again P diff. w.r.t y.

$$\frac{\partial^2\phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right] + 2\left[v\frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right] \quad -(3)$$

(2) + (3)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left\{\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial u}{\partial x}\right)^2\right\} + v\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right) + \left\{\left(\frac{\partial v}{\partial x}\right)^2\left(\frac{\partial v}{\partial y}\right)^2\right\}\right]$$

$\because$ $f(z)$ is regular

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left[0 + \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{-\partial v}{\partial x}\right)^2\right\} + 0 + \left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\}\right]$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad -(4)$$

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \quad -(5)$$

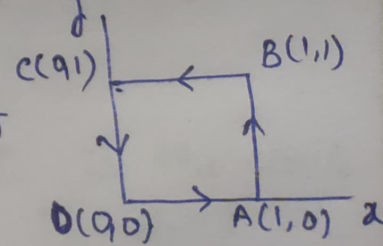
from (4) & (5)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$$

Unit 5

Ques:- Show that

$\oint_C (z+1)\,dz = 0$ where C is boundary of square whose vertices are at the point $z=0, z=1, z+1+i, z=i$



Sol: $$\int_C (z+1)dz = \int_{OA}(z+1)\,dz + \int_{AB}(z+1)\,dz + \int_{BC}(z+1)\,dz + \int (z+1)\,dz.$$