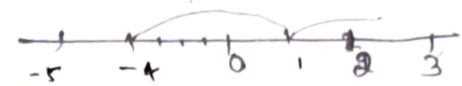
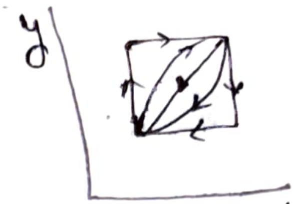
Unit - V

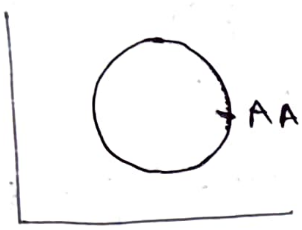
Complex Integration

$$\int_{x=a}^{x=b} f(x)\,dx = f_1(b) - f_1(a)$$



$$\int_{z=a}^{z=b} f(z)\,dz = f_1(b) - f_1(a)$$





$\oint f(z)\,dz$

Contour Integral.

Ques: Evaluate $\int_0^{1+i} (x^2 + iy)\,dz$. along the paths (a) $y = x$. (b) $y = x^2$

Sol: $\int_0^{1+i} (x^2+iy)\,(dx + i dy) = \int_0^{1+i} (x^2 dx + x^2 i dy - iy dx + i y dy)$ — (1)

a) along $y = x$ $\quad dy = dx$

$$\int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 dx + i x^2 dx - ix\,dx + x\,dx)$$

$$= \int_0^1 (x^2 - ix + ix^2 + x)\,dx = \left[\frac{x^3}{3} - \frac{ix^2}{2} + \frac{ix^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{3} - \frac{i}{2} + \frac{i}{3} + \frac{1}{2} \Rightarrow \frac{5}{6} - \frac{i}{6} \quad \text{Ans.}$$

b) $y = x^2$.

$dy = 2x\,dx$

$$\int_0^{1+i} (x^2 dx + x^2 i dy - iy dx + y dy) \Rightarrow \int_0^1 x^2 dx + i x^2\, 2x\,dx - i x^2 dx + x^2\, 2x\,dx$$

$$\int_0^1 x^2 dx + i\, 2x^3 dx - i x^2 dx + 2x^3 dx$$

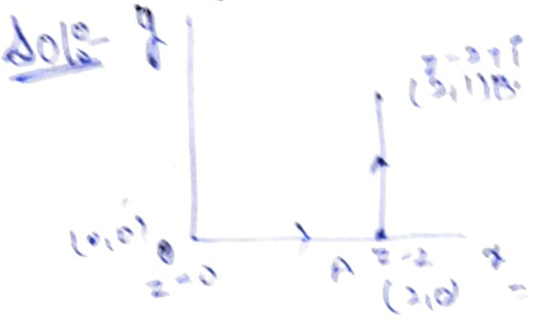
$$\Rightarrow \left[\frac{x^3}{3} + \frac{\cancel{2} x^4 i}{\cancel{4}_2} - \frac{ix^3}{3} + \frac{2x^4}{4}\right]_0^1 \Rightarrow \frac{1}{3} + \frac{i}{2} - \frac{i}{3} + \frac{1}{2}.$$

$$\boxed{\frac{5}{6} + \frac{i}{6}} \quad \text{Ans}$$

Ques: Evaluate $\int_0^{2+i} (\bar{z})^2 \, dz$ along (a) the real axis from $z=0$ to $z=2$ & then along a line parallel to y-axis from $z=2$ to $z=2+i$

b) along a line $2y = x$.

Sol:



$$\int_0^{2+i} (\bar{z})^2 \, dz = \int_0^{2+i} (x-iy)^2 (dx + i\,dy)$$

$$= \int_0^{2+i} (x^2 - y^2 - 2ixy)(dx + i\,dy)$$

$$= \int_0^{2+i} x^2dx - y^2dx - 2ixy\,dx + ix^2dy - iy^2dy + 2xy\,dy \quad \text{--- (1)}$$

(a)

$$\int_0^{2+i} (\bar{z})^2 dz = \int_{OA} x^2dx - y^2dx - 2ixy\,dx + ix^2dy - iy^2 + 2xy\,dy +$$

$$\int_{AB} x^2dx - y^2dx - 2ixy\,dx + ix^2dy - iy^2dy + 2xy\,dy.$$

Along OA $\quad y=0 \quad dy=0 \quad x = 0 \to 2.$

$$\int_0^2 x^2dx - 0 - 0 + ix^2(0) + 0 \Rightarrow \int_0^2 x^2\,dx \Rightarrow \left[\frac{x^3}{3}\right]_0^2 \Rightarrow \frac{8}{3}$$

Along AB $\quad x=2 \quad dx=0 \quad y: 0 \to 1.$

$$= \int_0^1 0 - y^2(0) - 2i2y(0) + i(4)dy - iy^2dy + 4y\,dy.$$

$$= \int_0^1 4i\,dy - iy^2dy + 4y\,dy$$

$$\Rightarrow \left[4iy - \frac{iy^3}{3} + \frac{4y^2}{2}\right]_0^1 \Rightarrow 4i - \frac{i}{3} + 2 \Rightarrow \frac{11i}{3} + 2.$$

$$\int_0^{2+i} (\bar{z})^2 \, dz = \frac{8}{3} + \frac{11i}{3} + 2 \Rightarrow \frac{11i}{3} + \frac{14}{3}$$

(b) along $2y = x$.

$2dy = dx$.

$$\int_0^{2+i} (\bar{z})^2 dz = \int_0^{i} 4y^2(2dy) - y^2(2dy) - 2i(2y)y(2dy) + i(4y^2)dy - iy^2dy + 2(2y)ydy.$$

$$\Rightarrow \int_0^1 8y^2dy - 2y^2dy - 8iy^2dy + 4y^2idy - iy^2dy + 4y^2dy$$

$$\Rightarrow \left[\frac{8y^3}{3} - \frac{2y^3}{3} - \frac{8iy^3}{3} + \frac{4iy^3}{3} - \frac{iy^3}{3} + \frac{4y^3}{3}\right]_0^1$$

$$\Rightarrow \frac{-8i}{3} + \frac{2i}{3} - \frac{8}{3} + \frac{4}{3} - \frac{1}{3} - \frac{4}{3}$$ ✗

$$\left[\frac{10y^3}{3} - \frac{5iy^3}{3}\right]_0^1$$

$$\Rightarrow \frac{10}{3} - \frac{5i}{3}$$

$$\Rightarrow \frac{-10i}{3} - \frac{5}{3}$$ Ans

Ques: Evaluate $\int_0^{4+2i} \bar{z}\,dz$ along the curve $z = t^2 + it$

$z = t^2 + it$.

$t^2 = 0$

$t = 0$

$t^2 = 4$

$t = \pm 2$.

$t = 2$

Sol:- $z = t^2 + it$.

$$\int_0^{4+2i} \bar{z}\,dz = \int_0^{2} (t^2 - it)(2t\,dt + i\,dt)$$

$$= \int_0^2 2t^3dt - i2t^2dt + it^2dt + tdt.$$

$$= \int_0^2 (2t^3 - i2t^2 + it^2 + t)dt = \left(\frac{2t^4}{4} - \frac{i2t^3}{3} + \frac{it^3}{3} + \frac{t^2}{2}\right)_0^2.$$

$$\left(2^3 - \frac{i16}{3} + \frac{i8}{3} + \frac{4}{2}\right) \Rightarrow 8 - \frac{8i}{3} + \frac{4}{2} \Rightarrow 10 - \frac{8i}{3}$$

$$\Rightarrow 10 - \frac{8i}{3}$$

Simply and Multiple connected Domain.

A domain in which every closed curve can be shrunk to a point without passing out of the region is called a simply connected domain otherwise multiple connected domain.

Simply and Multiple connected region

A curve is called simple close curve if it doesnot cross itself otherwise is called multiple curve.

A region is called simply connected region if every closed curve in the region encloses point of region only.

A region which is not simply connected is called a multiple connected region.

First M.C curve after S.C. curve.

S. C. curve

M.C. curve

* Cauchy Integral Theorem

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple close curve C then $\int_C f(z)\,dz = 0$

let R be the region bounded by C.

***Proof:**

$$\int_C f(z)dz = \int_C (u+iv)\,(dx+i\,dy)$$

$$= \int_C u\,dx + ui\,dy + iv\,dx - v\,dy$$

$$= \int_C u\,dx - v\,dy + i\int v\,dx + u\,dy.$$

$\because f'(z)$ is continuous so, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial u}$ & $\frac{\partial v}{\partial y}$ are continuous in R.

So by Green's theorem

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy.$$

$\because f(z)$ is analytic,

$\Rightarrow$ CR eq. is satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

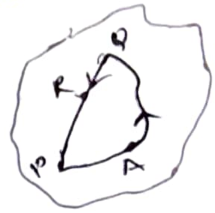
$$\int_c f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$= 0$

Hence, Proved.

Corollary.

If $f(z)$ is analytic in region R and P and Q are two point in region R then $\int_P^Q f(z)\,dz$ is independent of the path joining P and Q ~~done~~ lying entirely in R.



Proof: By Cauchy Integral Theorem

$$\int_{PAQBP} f(z)\,dz = 0.$$

$$\int_{PAQ} f(z)\,dz + \int_{QBP} f(z)\,dz = 0$$

$$\int_{PAQ} f(z)\,dz - \int_{PBQ} f(z)\,dz = 0$$

$$\Rightarrow \int_{PAQ} f(z) = \int_{PBQ} f(z)\,dz. \quad \text{H.P.}$$

Extension of Cauchy integral Theorem for Multiple connected region.

If $f(z)$ is analytic in region R b/w two simple closed curve C_1 and C_2 then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ where integral along each curve is taken in anticlockwise direction.

Proof. By Cauchy Integral Theorem.

$$\int_c f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{CD} f(z)\,dz = 0$$

there, integral along AB and CD are equal and opposite so they cancel out each other.

Reversing the direction of C_1

$$\int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz = 0 \Rightarrow \boxed{\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz}$$

Ques: Evaluate the Integral $\oint_C \frac{z^2 - z + 1}{z-1}\,dz$, where C is circle $|z| = \frac{1}{2}$.

Sol: for singularities.

$z - 1 = 0$

$z = 1$

Here $z = 1$ outside the circle ~~kies~~ $|z| = \frac{1}{2}$

So by CIT

$$\oint_C \frac{z^2 - z + 1}{z - 1}\,dz = 0.$$

Ques: $\oint_C \frac{z^3 + z + 1}{z^2 - 3z + 2}\,dz$ where $C = 4x^2 + 9y^2 = 1$.

Sol: for singularities.

$z^2 - 3z + 2 = 0$ $\quad z = 1, 2.$

$4x^2 + 9y^2 = 1$

$\frac{x^2}{(1/2)^2} + \frac{y^2}{(1/3)^2} = 1$

$z = 1$ & 2 outside the circle $4x^2 + 9y^2 = 1$

So, by CIT

$$\oint_C \frac{z^3 + z + 1}{z^2 - 3z + 2}\,dz = 0.$$

Ques: Evaluate $\frac{1}{2\pi i}\oint_C \frac{z^2 - z + 1}{z - 2}\,dz.$

$|z - 1| = \frac{1}{2}$

$r = \frac{1}{2}$ $\quad$ centre $(1, 0)$

Sol: for singularities.

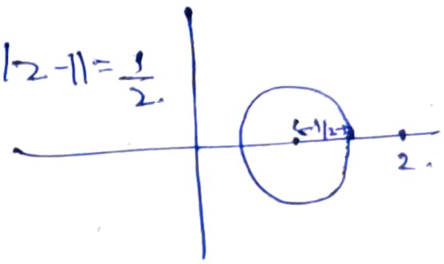
$z - 2 = 0$

$z = 2.$

Here $z = 2$ outside the circle $|z - 1| = \frac{1}{2}$.

So, By CIT

$$\frac{1}{2\pi i}\oint \frac{z^2 - z + 1}{z - 2}\,dz = \underline{0}.$$



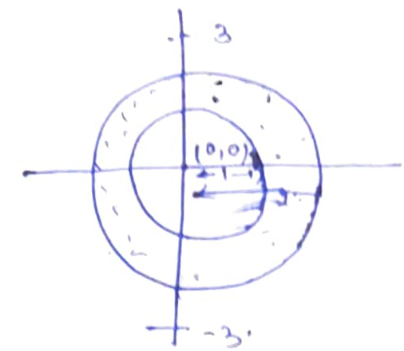
Q. $\oint \frac{1}{z^2(z^2+9)} dz$. $C = 1 < |z| < 2$.

$r = 1$.
$r = 2$.

Sol.:- For singularities.

$z^2(z^2+9) = 0$

$z = 0, 0, \pm 3$.

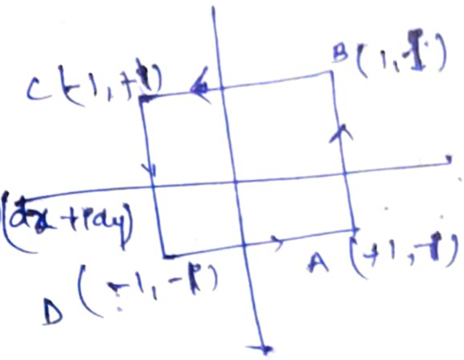


0, 0 ~~lies~~

$z = 0, 0, \pm 3$ doesnot lie in $1 < |z| < 2$.

then, $\oint \frac{1}{z^2(z^2+9)} dz = 0$.

Ques:- Verify C.I.T. by integrating z^3 along the boundary of square whose vertices at $1+i, 1-i, -1+i, -1-i$.



Sol.:- $z^3 = (x+iy)^3$.

$= x^3 - iy^3 + 3x^2 iy - 3xy^2$.

~~Integ~~ $\int_C z^3 dz = \int_C (x^3 - iy^3 + 3x^2 iy - 3xy^2)(dx + idy)$

$\Rightarrow \int [x^3 dx - iy^3 dx + 3x^2 iy dx - 3xy^2 dx + x^3 i dy + y^3 dy - 3x^2 y dy - 3ixy^2 dy]$.

Along AB. $y = i$ $dy = 0$. $x: -1$ to 1.

$\int_{-1}^{1} x^3 dx - dx - 3x^2 dx + 3x dx \Rightarrow \left[\frac{x^4}{4} - x - \frac{\cancel{3}x^3}{\cancel{3}} + \frac{3x^2}{2}\right]_{-1}^{1}$

$\frac{1}{4} - 1 - 1 + \frac{3}{2} - \left(\frac{1}{4} + 1 + 1 + \frac{3}{2}\right)$.

$\Rightarrow \cancel{\frac{1}{4}} - 1 - 1 + \cancel{\frac{3}{2}} - \cancel{\frac{1}{4}} - 1 - 1 - \cancel{\frac{3}{2}} = -4$.

Along BC. $x = 1$ $dx = 0$ $y: i \to -i$

Along ~~AB~~ BC. $y = 1$ $dy = 0$ $x: -1$ to 1.

$\int_{-1}^{1} x^3 dx - i dx + 3x^2 i dx - 3x dx \Rightarrow \left[\frac{x^4}{4} - ix + \frac{3x^3 i}{3} - \frac{3x^2}{2}\right]_{-1}^{1}$

$\left(\cancel{\frac{1}{4}} - \cancel{i} + \cancel{i} - \cancel{\frac{3}{2}}\right) - \left(\cancel{\frac{1}{4}} + \cancel{i} - \cancel{i} - \cancel{\frac{3}{2}}\right)$

$\Rightarrow 0$.

Along ABC $\quad x : 1 \quad dx = 0 \quad y : -1$ to 1.

$$\int_{-1}^{1} i\,dy + y^3 dy - 3y\,dy - 3iy\,dy \Rightarrow \left[iy + \frac{y^4}{4} - \frac{3y^2}{2} - \frac{3iy^2}{2} \right]_{-1}^{1}$$

$$\left[i + \frac{1}{4} - \frac{3}{2} - \frac{3i}{2} \right] - \left[-i + \frac{1}{4} - \frac{3}{2} - \frac{3}{2}i \right]$$

$$\Rightarrow i + \frac{1}{4} - \frac{3}{2} - \frac{3i}{2} + i - \frac{1}{4} + \frac{3}{2} + \frac{3}{2}i \Rightarrow 2i$$

Along AD $\quad y : -i \quad dy = 0 \quad x : 1 \to -1$.

$$\int_{1}^{-1} (x^3 + i - 3ix^2 - 3x)\,dx \Rightarrow \left[\frac{x^4}{4} + xi - \cancel{\frac{3ix^3}{3}} - \frac{3x^2}{2} \right]_{1}^{-1}$$

$$\left[\cancel{\frac{1}{4}} - \cancel{i} + \cancel{i} - \cancel{\frac{3}{2}} \right] - \left[\cancel{\frac{1}{4}} + \cancel{i} - \cancel{i} - \cancel{\frac{3}{2}} \right]$$

Along DC. $\quad x : -1 \quad dx = 0 \quad y : -1$ to 1.

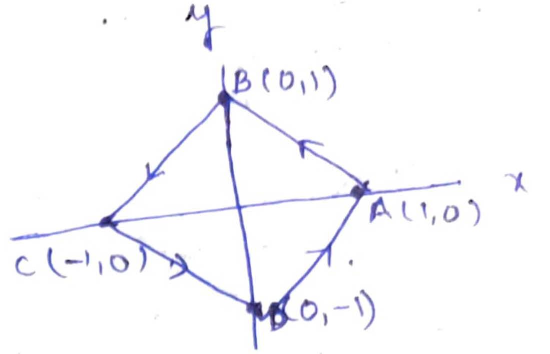
$$\int_{1}^{-1} -i\,dy + y^3 dy - 3y\,dy + 3iy^2 dy$$

$$\Rightarrow \left[-iy + \frac{y^4}{4} - \frac{3y^2}{2} + \frac{3iy^3}{3} \right]_{1}^{-1}$$

$\Rightarrow$

Ques:- State Cauchy Integral theorem for an analytic funⁿ. Verify this Theorem by integrating the funⁿ z^3+iz along the boundary of rectangle with vertices, $+1$, -1, i, $-i$.

Sol: $f(z) = (x+iy)^3 + i(x+iy)$



By C.I.T.

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz = 0$$

(1) Along AB. $x+y=1$; $y = 1-x$ $dy = -dx$.

$dz = dx+i\,dy = dx - i\,dx = (1-i)\,dx$ $x : 1 \to 0$

$$\int_1^0 \left([x+i(1-x)]^3 + i[x+i(1-x)]\right)(1-i)\,dx$$

$$= \int_1^0 \left\{[x+i-xi]^3 + i[x+i-ix]\right\}(1-i)\,dx$$

$$= \int_1^0 \left\{[i+(1-i)x]^3 + i[i+(1-i)x]\right\}(1-i)\,dx$$

$$= \left[\frac{[i+(1-i)x]^4}{4\cancel{(1-i)}} + \frac{i[i+(1-i)x]^2}{2\cancel{(1-i)}}\right]_1^0 \cancel{(1-i)}$$

$$= \left[\frac{[i+(1-i)x]^4}{4} + \frac{i[i+(1-i)x]^2}{2}\right]_1^0$$

$$= \left(\frac{(i+(1-i)0)^4}{4} + \frac{i[i+(1-i)0]^2}{2}\right) - \left[\frac{[\cancel{i}+(1-\cancel{i})]^4}{4} - \frac{i[i+(1-i)]^2}{2}\right]$$

$$= \frac{i^4}{4} + \frac{i^3}{2} - \frac{1}{4} - \frac{i}{2} = \cancel{\frac{1}{4}} - \frac{i}{2} - \cancel{\frac{1}{4}} - \frac{i}{2}$$

$$= -i$$

(2) Along BC. $-x+y=1$.

$x = y-1$ $dx = dy$

$dz = (dx+i\,dy) = (dy+i\,dy) = (1+i)\,dy$.

$$\int_1^0 [(y-1+iy)^3 + i(y-1+iy)](1+i)\,dy.$$

$$\int_1^0 \left[\{(1+i)y-1\}^3 + i\{(1+i)y-1\}\right](1+i)\,dy$$

$$\Rightarrow \left[\frac{[(1+i)y-1]^4}{4\cancel{(1+i)}} + \frac{i[(1+i)y-1]^2}{2\cancel{(1+i)}}\right]_1^0 \cancel{(1+i)}$$

$$\Rightarrow \frac{[(1+i)0-1]^4}{4} + \frac{i[(1+i)0-1]^2}{2} - \frac{[(\cancel{1}+i)1\cancel{-1}]^4}{4} - \frac{i[(\cancel{1}+i)1\cancel{-1}]^2}{2}$$

$$\Rightarrow \frac{1}{4} + \frac{i}{2} - \frac{i^4}{4} - \frac{i^3}{2} \Rightarrow \cancel{\frac{1}{4}} + \frac{i}{2} - \cancel{\frac{1}{4}} + \frac{i}{2}$$

$$\Rightarrow \underline{\underline{i}}$$

(3) Along CD $\quad -x-y=1$

$x = -1-y.$

$dx = -dy.$

$dz = dx + i\,dy$

$dz = -dy + i\,dy$

$(i-1)dy.$

$x: -1 \to 0.$

$$\int^0 (-1-y+iy)^3 + i(-1-y+iy)\,(-1+i)\,dy.$$

$$\Rightarrow \int_{-1}^0 \left\{[y(i-1)-1]^3 + i[y(i-1)-1]\right\}(i-1)\,dy.$$

$$\Rightarrow \left[\frac{[y(i-1)-1]^4}{4\cancel{(i-1)}} + \frac{i[y(i-1)-1]^2}{2\cancel{(i-1)}}\right]_{-1}^0 \cancel{(i-1)}$$

$$\Rightarrow \frac{[0(i-1)-1]^4}{4} + \frac{i[0(\cancel{i-1})\cancel{-1}]^2}{2} - \frac{[-1(i\cancel{-1})\cancel{-1}]^4}{4} - \frac{i[-1(i\cancel{-1})\cancel{-1}]^2}{2}.$$

$$\Rightarrow \frac{1}{4} + \frac{i}{2} - \frac{i^4}{4} - \frac{i^3}{2} \Rightarrow \cancel{\frac{1}{4}} + \frac{i}{2} - \cancel{\frac{1}{4}} + \frac{i}{2}.$$

$$\Rightarrow i$$

Ques:- Verify Cauchy theorem $f(z) = 4z^2 + iz - 3$ along the positively oriented square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$

Sol:-

Ques:- Verify Cauchy's theorem along the boundary with vertices $(1+i), (1-i), (-1+i), (-1,-i)$

Sol: ① if $f(z) = z^2$

② $f(z) = 3z^2$

③ $f(z) = z^2 + 3z + 2$.

Q. * Cauchy Integral formula

If $f(z)$ is analytic function within and on closed curved C and a is any point within C then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

Proof: Consider the function $\frac{f(z)}{z-a}$, which is analytic every point in C except $z=a$. Draw a small circle C_1 with 'a' as centre and radius ρ such that C_1 lies entirely inside C. The $\frac{f(z)}{z-a}$ is analytic in the region b/w C & C_1

So by Cauchy Theorem

$$\oint_C \frac{f(z)}{z-a} dz = \int_{C_1} \frac{f(z)}{z-a} dz$$

For circle C_1

$$|z-a| = \rho$$

$$z-a = \rho e^{i\theta}$$

$$z = a+\rho e^{i\theta} \Rightarrow dz = \rho e^{i\theta} i d\theta$$

$$\int_C \frac{f(z)}{(z-a)} dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}} \times \rho e^{i\theta} i d\theta$$

$$\int_C \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a+\rho e^{i\theta}) d\theta$$

In the limiting form the circle C_1 shrink to a such that $\rho \to 0$,

$$\int_C \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a) d\theta \Rightarrow i f(a) [\theta]_0^{2\pi}$$

$$\Rightarrow i f(a) \cdot 2\pi$$

$$\boxed{\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)} \quad \text{or} \quad \boxed{f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz}$$

Cauchy Integral formula for derivative

$$f^n(a) = \frac{\lfloor n}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz$$

If pole outside the circle then 0

or

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i f^n(a)}{\lfloor n}$$

$$f(a) = \frac{1}{2\pi i} \int \frac{f(z)}{(z-a)} dz$$

Ques: Using Cauchy Integral formula. Evaluate

$$\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz.$$ where $c = |z| = 3$

Sol: for pole (singularites)

$(z-1)(z-2) = 0$ $z = 1, 2$

$c = |z| = 3$ $z = 1$ & 2 lies in circle $|z| = 3$

By CIF.

$$\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_{c_1} \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}}{(z-1)} dz + \oint_{c_2} \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}}{(z-2)} dz$$

$$= 2\pi i \left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right]_{z=1} + 2\pi i \left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right]_{z=2}.$$

$$\Rightarrow 2\pi i \left[\frac{0-1}{-1}\right] + 2\pi i \left[\frac{0+1}{1}\right] \Rightarrow 2\pi i + 2\pi i \Rightarrow 4\pi i$$

Ques: Using Cauchy integral formula to evaluate

$$\oint \frac{e^{2z}}{(z+1)^4} dz,$$ where C is the circle $|z| = 3$.

Sol: for pole $(z+1)^4 = 0$

$z = -1$ of order 4.

$z = -1$ lies in the circle $|z| = 3$

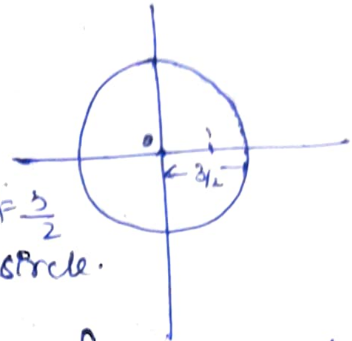
By CIF.

$$\oint \frac{e^{2z}}{(z+1)^{3+1}}\, dz = \frac{2\pi i}{\lfloor 3} \left[\frac{d^3 e^{2z}}{dz^3}\right]_{z=-1}$$

$$= \frac{2\pi i}{6} \left(8e^{2z}\right)_{z=-1} = \frac{\pi i}{3}\, 8e^{-2}.$$

Ques: $\int \frac{4-3z}{z(z-1)(z-2)}\, dz.$ $\quad c = |z| = 3/2.$

Sol:- for poles
$z(z-1)(z-2) = 0$
$z = 0, 1, 2.$



$z = 0$ & 1 lie inside the circle $|z| = \frac{3}{2}$
and $z = 2$ lie outside the circle.

By CIF

$$\int \frac{4-3z}{z(z-1)(z-2)}\, dz = \int \frac{\frac{4-3z}{(z-1)(z-2)}}{z}\, dz + \int \frac{\frac{4-3z}{z(z-2)}}{(z-1)}\, dz + \int \frac{\frac{4-3z}{z(z-1)}}{(z-2)} \rightarrow 0.$$

$$= 2\pi i \left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i \left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

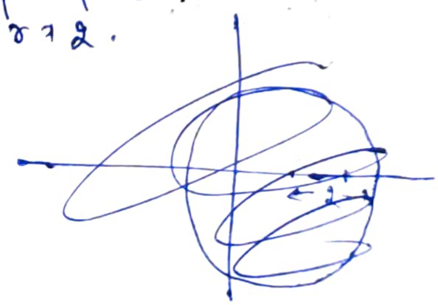
$$= 2\pi i \left[\frac{\cancel{4}\, 2}{\cancel{2}}\right] + 2\pi i \left[\frac{1}{-1}\right]$$

$$= 4\pi i - 2\pi i = 2\pi i.$$

Ques: $\int \frac{(z-1)}{(z+1)^2 (z-2)}\, dz$ where $|z - i| = 2.$

Sol:- for pole $(z+1)^2 (z-2) = 0$
$z = -1$ of order 2
$z = 2$ simple pole.

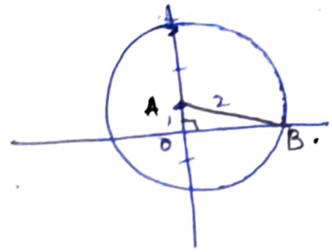
$|z - i| = 2$ $\quad$ centre $\rightarrow (0, 1)$ $\quad r = 2.$



$AB^2 = OA^2 + OB^2$

$4 = 1 + OB^2$

$OB = \sqrt{3} = 1.732$



$z = -1$ of order 2 lies in circle.

$z = 2$ lies outside the circle.

$$\int \frac{z-1}{(z+1)^{1+1}(z-2)}\,dz = \left[\frac{2\pi i}{1!}\frac{d}{dz}\frac{(z-1)}{(z-2)}\right]_{z=-1}$$

$$= 2\pi i\left[\frac{(z-2)-(z-1)}{(z-2)^2}\right]_{z=-1} = 2\pi i\left[\frac{-3+2}{9}\right] = \frac{-2\pi i}{9} \text{ Ans}$$

Ques: $\int_c \frac{(1+z)\sin z}{(2z-3)^2}\,dz$ $\quad C = |z-i| = 2$.

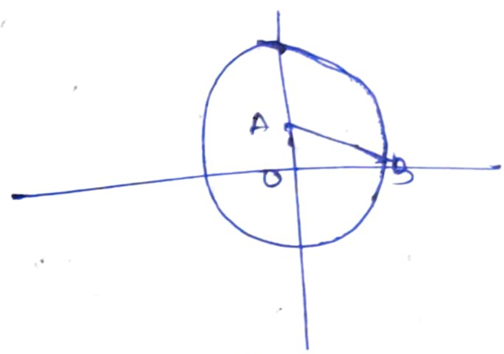
Sol: $(2z-3)^2 = 0$ $\quad z = \frac{3}{2}$ of order 2.

centre $(0,1)$ radius 2.

$OA^2 + OB^2 = AB^2$

$1 + OB^2 = 4$

$OB = 1.732$.



$z = \frac{3}{2}$ lies in circles $|z-i| = 2$.

$$\int \frac{(1+z)\sin z}{(2z-3)^2}\,dz = \frac{1}{4}\left[\frac{2\pi i}{1!}\left[\frac{d\,((1+z)\sin z)}{dz}\right]\right]_{z=3/2}$$

$(2z-3)^2 = 4(z-3/2)^2$.

$$= \frac{2\pi i}{4}\left[\sin z + (1+z)\cos z\right]_{z=3/2}$$

$$= \frac{\pi i}{2}\left[\sin\frac{3}{2} + \frac{5}{2}\cos\frac{3}{2}\right]$$

Ques: $\int_c \frac{1}{(z^2+4)^2}\,dz$. where $c = |z-i| = 2$.

Sol: $\int_c \frac{1}{(z^2+4)^2}\,dz = \int_c \frac{1}{[(z+2i)(z-2i)]^2}\,dz$

For poles $(z-2i)^2(z+2i)^2=0$

$z=2i, -2i$ of order 2.

$z=2i$ lies in circle $|z-i|=2$.

$z=-2i$ lies outside the circle

$$\int_C \frac{1}{[(z-2i)(z+2i)]^2}dz = \int_{C_1} \frac{\frac{1}{(z-2i)^2}}{(z+2i)^2} dz + \int_{C_2} \frac{\frac{1}{(z+2i)}}{(z-2i)^2} dz$$

$(z+2i)^2 \to 0$. $\to 0$.

$$\Rightarrow \frac{2\pi i}{L1}\left[d\left(\frac{1}{(z+2i)^2}\right)\right]_{z=+2i}$$

$$\Rightarrow \left[2\pi i(-2)(z+2i)^{-3}\right]_{z=+2i} \Rightarrow \frac{2\pi i(-2)}{(+4i)^3} \Rightarrow \frac{\cancel{4}\pi \cancel{i}}{16\times \cancel{4}_2 \cancel{i}}$$

$$\Rightarrow \frac{\pi\times \cancel{2}}{\cancel{32}} \Rightarrow \frac{\pi}{16}\ \underline{\text{Ans}}$$

Ques: Evaluate $\int \frac{\sin^2 z}{(z-\frac{\pi}{6})^3} dz$. circle $|z|=1$.

Sol: $(z-\frac{\pi}{6})^3=0$ $\quad z=\frac{\pi}{6}$ of order 3. lie inside the circle $|z|=1$.

$$\int_C \frac{\sin^2 z}{(z-\frac{\pi}{6})^3} = \int_{C_1} \frac{\sin^2 z}{(z-\frac{\pi}{6})^{2+1}}$$

$$\Rightarrow \frac{2\pi i}{L2}\left[\frac{d^2(\sin^2 z)}{dz^2}\right]_{z=\frac{\pi}{6}} \Rightarrow \frac{2\pi i}{2}\left[\frac{d\,2\sin z\cos z}{dz^2}\right]_{z=\frac{\pi}{6}}$$

$$\Rightarrow \frac{\cancel{2}\times 2\pi i}{\cancel{2}}\left[\cos z\cos z+\sin z(-\sin z)\right]_{\pi/6} \Rightarrow 2\pi i\left[\cos^2 z-\sin^2 z\right]_{\pi/6}$$

$$\Rightarrow 2\pi i\left[\cos^2\frac{\pi}{6}-\sin^2\frac{\pi}{6}\right] \Rightarrow 2\pi i\left[\frac{1}{4}-\frac{3}{4}\right]$$

$$\Rightarrow \cancel{2}\pi i\left[\frac{\cancel{2}}{\cancel{4}}\right] \Rightarrow \underline{\pi i}$$

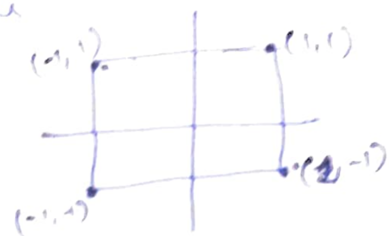
Ques:- $\oint_C \frac{e^z}{(z-\log 2)^4} dz$. where C is square with vertices $\pm 1 \pm i$

Sol: Pole $(z-\log 2)^4=0$

$z=\log 2$ of order 4

$=0.301.$

$z = \log 2$ of order 4 lies inside the circle.



$$\oint_C \frac{e^z}{(z-\log 2)^4} = \int_{C_1} \frac{e^z}{(z-\log 2)^{3+1}}$$

$$\Rightarrow \frac{2\pi i}{\lfloor 3} \left[\frac{d^3 e^z}{dz^3}\right]_{z=\log 2.}$$

$$\Rightarrow \frac{2\pi i}{6} \left[e^z \right]_{0.301} \Rightarrow \frac{2\pi i \cancel{2} (\cos 84)}{\cancel{6}\, 3} \Rightarrow \frac{2\pi i}{3} \text{ Ans.}$$

Ques:- $\oint \frac{e^z}{z(1-z)^3}\, dz$ where ① $|z| = \frac{1}{2}$ ② $|z-1| = \frac{1}{2}$ ③ $|z| = 2$.

Sol: for poles

$$z(1-z)^3 = 0$$

$z = 0$ and $z = 1$ order 3.

① $|z| = \frac{1}{2}$. $z = 0$ lies in circle $|z| = \frac{1}{2}$.

By CIF.

$$\int \frac{e^z}{z(1-z)^3}\, dz = \int \frac{e^z/(1-z)^3}{z}\, dz. = 2\pi i$$

$$= 2\pi i \left[\frac{e^z}{(1-z)^3}\right]_{z=0}$$

② circle $|z-1| = 1/2$. $z = 1$ of order 3 lies in the circle $|z-1| = 2$.

By CIF.

$$\int \frac{e^z}{z(1-z)^3}\, dz = -\int_{C_1} \frac{e^z/z}{(-1+z)^3}\, dz \Rightarrow \frac{2\pi i}{\lfloor 2} \left[\frac{d^2}{dz^2} \frac{e^z}{z}\right]_{z=1.}$$

$$= -\pi i \left[\frac{d}{dz}\left[\frac{e^z z - e^z}{z^2}\right]\right]_{z=1} \Rightarrow -\pi i \left[\frac{d}{dz}\left\{\frac{e^z(z-1)}{z^2}\right\}\right]_{z=1.}$$

$$\Rightarrow \pi i \left[\frac{\{e^z(z-1) + e^z\} z^2 - e^z (z-1) 2z}{z^4}\right]_{z=1.}$$

Ques: $\oint \frac{z^2-2z}{(z+1)^2(z^2+4)} dz$. where c is circle $|z|=3$

Sol: for poles $(z+1)^2(z^2+4)=0$

$z=-1$ of order 2.

$z=\pm 2i$.

$z=-1$ lie inside the circle $|z|=3$

$z=\pm 2i$ lies inside the circle $|z|=3$

$$\oint_c \frac{z^2-2z}{(z+1)^2(z+2i)(z-2i)} dz = \int_{c_1} \frac{\frac{z^2-2z}{(z+2i)(z-2i)}}{(z+1)^2} dz +$$

$$\int_{c_2} \frac{\frac{z^2-2z}{(z+1)^2(z-2i)}}{(z+2i)} dz + \int_{c_3} \frac{\frac{z^2-2z}{(z+1)^2(z+2i)}}{(z-2i)}$$

$$\Rightarrow \frac{2\pi i}{\lfloor 1} \left[\frac{d}{dz}\left(\frac{z^2-2z}{(z+2i)(z-2i)}\right)\right]_{z=-1} + 2\pi i \left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]_{z=-2i} +$$

$$2\pi i \left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]_{z=2i}$$

$$\Rightarrow 2\pi i \left[\frac{(2z-2)[z^2+4] - (2z)(z^2-2z)}{(z^2+4)^2}\right]_{z=-1} + 2\pi i \left[\frac{-4+4i}{(-2i+1)^2(-4i)}\right]$$

$$+ \cancel{2}\pi \cancel{i} \left[\frac{-4+4i}{(2i+1)^2(\cancel{4}i)}\right]$$

$$\Rightarrow 2\pi i \left[\frac{(-2-2)(1+4) - (-2)(1+2)}{(5)^2}\right] + \cancel{2}\pi \cancel{i} \left[\frac{\cancel{4}i-\cancel{4}}{(1-4+4i)-\cancel{4}\cancel{i}}\right]$$

$$+ \pi \left[\frac{\cancel{4}(i-1)}{(1-4+4i)\cancel{2}}\right]$$

$$\Rightarrow 2\pi i \left[\frac{-20+6}{25}\right] + \pi \left[\frac{2(i-1)}{(1-2i)^2}\right] + \pi \left(\frac{-2(i+1)}{(2i+1)^2}\right)$$

$$\Rightarrow 2\pi i \left[\frac{-14}{25}\right] + \pi \left[\frac{2(i-1)}{(1-2i)^2}\right] + \pi \left(\frac{-2(i+1)}{(2i+1)^2}\right)$$

Don't remove i

$$= 2\pi i\left(\frac{-14}{25}\right) + 2\pi i\left(\frac{i+1}{3+4i}\right) - 2\pi i\left[\frac{-1+i}{3-4i}\right]$$

$$\Rightarrow 2\pi i\left(\frac{-14}{25}\right) - 2\pi i\left[\frac{(-1+i)\ (3+4i)}{(3^2+4^2)}\right] + 2\pi i\left[\frac{(1+i)(3-4i)}{3^2+4^2}\right]$$

$$\Rightarrow \frac{2\pi i}{25}\left[-14-(-1+i)(3+4i)+(1+i)(3+4i)\right].$$

$$= \frac{2\pi i}{25}\left[-14-\{-3-4i+3i-4\}+\{3+4i+3i-4\}\right]$$

$$= \frac{2\pi i}{25}\left[-14+3+\cancel{4i}-\cancel{3i}+\cancel{4}+3+4i+\cancel{3i}-\cancel{4}\right]$$

$$= 0 \text{ Ans.}$$

Ques: $\displaystyle\int_c \frac{\exp(i\pi z)}{(2z^2-5z+2)}\,dz.$ where c is circle $|z|=1$.

Sol: for pole $2z^2-5z+2$

$$\Rightarrow 2z^2-4z-z+2=0$$

$$\Rightarrow 2z(z-2)-1(z-2)=0$$

$$\Rightarrow (2z-1)\ (z-2)=0 \Rightarrow z=\frac{1}{2},\ 2.$$

$z=\frac{1}{2}$ lies inside the circle $|z|=1$

$z=2$ lies outside the circle $|z|=1$.

$$\int_c \frac{e^{i\pi z}}{(2z-1)(z-2)}\,dz = \frac{1}{2}\left[\int_{c_1}\frac{\frac{e^{i\pi z}}{2(2z-1/2)}}{(z-2)}\,dz + \int\frac{\frac{e^{i\pi z}}{(z-2)}}{(z-1/2)}\,dz\right].$$

$$\Rightarrow \frac{1}{2}\left(\left[2\pi i\left[\frac{e^{i\pi z}}{z-1/2}\right]\right]_{z=2} \to 0 + 2\pi i\left[\frac{e^{i\pi z}}{z-2}\right]_{z=1/2}\right.$$

$$\Rightarrow \frac{1}{2}\left[2\pi i\,\frac{e^{2i\pi}}{-3/2} \to 0 + \frac{2\pi i\ e^{\frac{i\pi}{2}}}{-3/2}\right]$$

$$\Rightarrow \frac{2\pi i}{\cancel{2}} \times \frac{2}{-3}\left[\cancel{e^{2i\pi}} + e^{i\pi/2}\right] \Rightarrow \frac{-2}{3}e^{i\pi/2}$$

$$\frac{-2}{3}\pi i\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) \Rightarrow \frac{-2}{3}\pi i(0+i)$$

$$\Rightarrow \frac{2}{3}\pi. \text{ Ans}$$

Ques: $\int_C \frac{z+4}{z^2+2z+5}\, dz \qquad C = |z+1-i| = 2.$

Sol: for pole $z^2+2z+5=0$

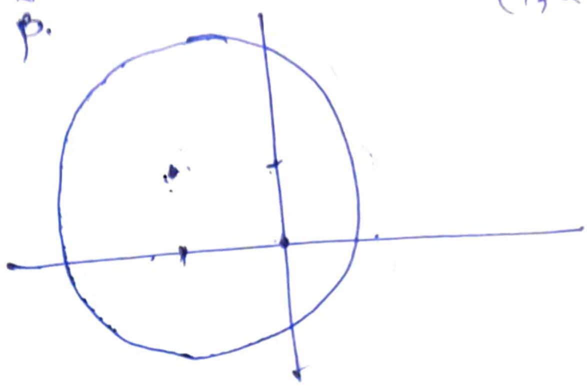
$\cancel{z = -1-2i}$ of ~~degree 2~~ order.

$z = \underset{\alpha}{-1-2i}, \quad \underset{\beta}{-1+2i}$ $\qquad (-1,-2)$ $\quad (-1,2)$.

Centre → $(-1, +1)$

radius → 2.

$-1+2i$ lie inside the circle.



$$\int_C \frac{z+4}{(z-\alpha)(z-\beta)}\, dz =$$

$$\int_{C_1} \frac{\left(\frac{z+4}{z-\alpha}\right)}{z-\beta}\, dz \Rightarrow 2\pi i\left[\frac{z+4}{z-\alpha}\right]_{z=\beta} \Rightarrow 2\pi i\left[\frac{\beta+4}{\beta-\alpha}\right]$$

$$\Rightarrow 2\pi i\left[\frac{-1+2i+4}{\cancel{-1}+2i+\cancel{1}+2i}\right] \Rightarrow 2\pi i\left[\frac{3+2i}{4i}\right] \Rightarrow \pi\left[\frac{3+2i}{2}\right]$$

Ques: Evaluate $\oint \frac{z^2+1}{z^2-1}\, dz.$ where (i) $|z| = 3/2$ (ii) $|z-1| = 3/2$ (iii) $|z| = \frac{1}{2}$

Sol: pole $z^2-1=0$

$z = \pm 1$

$z = \pm 1$ lie inside the circle $|z| = \frac{3}{2}$

$$\oint_C \frac{z^2+1}{(z+1)(z-1)}\, dz = \int_{C_1} \frac{\frac{z^2+1}{z+1}}{(z-1)}\, dz + \int_{C_2} \frac{\frac{z^2+1}{z-1}}{z+1}\, dz.$$

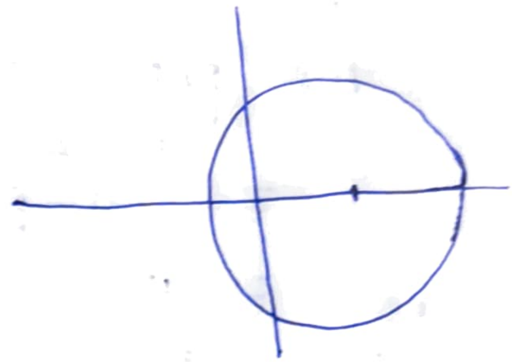
$$\Rightarrow 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} + 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1}$$

$$\Rightarrow 2\pi i\left[\frac{\cancel{2}}{\cancel{2}}\right] + 2\pi i\left[\frac{\cancel{2}}{-\cancel{2}}\right] \Rightarrow 0.$$

(iii) $|z-1| = \frac{3}{2}$.

centre $\rightarrow$ $(1, 0)$

radius $= \frac{3}{2}$.

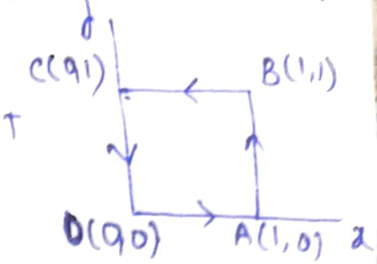


$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2.$$

Unit 5

Ques: Show that

$\oint_C (z+1)\,dz = 0$ where C is boundary of square whose vertices are at the point $z=0, z=1, z+1+i, z=i$.



Sol: $\int_C (z+1)\,dz = \int_{OA} (z+1)\,dz + \int_{AB} (z+1)\,dz + \int_{BC} (z+1)\,dz + \int_{CO} (z+i)\,dz.$

$\int (z+1)\,dz = \int (x+iy+1)(dx+i\,dy)$

$= \int x\,dx + iy\,dx + dx + ix\,dy - y\,dy + i\,dy$

(1) Along OA $\quad y=0, \; dy=0 \quad x: 0\to 1$

$\int_0^1 x\,dx + dx = \left(\frac{x^2}{2}+x\right)_0^1 = \frac{1}{2}+1 = \frac{3}{2}.$

(2) Along AB. $\quad x=1 \quad dx=0 \quad y: 0\to 1.$

$\int_0^1 (i\,dy - y\,dy + i\,dy) = \left[iy - \frac{y^2}{2} + iy\right]_0^1$

$\Rightarrow \boxed{2i - \frac{1}{2}}$

(3) Along BC $\quad y=1 \quad dy=0 \quad x: 1\to 0.$

$\int_1^0 (x\,dx + i\,dx + dx) = \left[\frac{x^2}{2} + ix + x\right]_1^0 \Rightarrow -\frac{1}{2} - ix - 1 \Rightarrow -\frac{3}{2} - i\cancel{x}$

(a) · Along CO.

$x = 0 \quad dx = 0 \qquad y : 1 \to 0.$

$$\int_1^0 -y\,dy + i\,dy = \left[-\frac{y^2}{2} + iy\right]_1^0 \to -\frac{1}{2} - i$$

$$= \cancel{\frac{3}{2}} - \cancel{\frac{1}{2}} + \cancel{2i} + \cancel{\frac{1}{2}} - \cancel{i} - \cancel{\frac{3}{2}} - \cancel{i} = 0$$

Ques: Evaluate $\oint_C |z|^2\, dz$ around the square whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$

Sol: $\int |z|^2\, dz = \int (x^2+y^2)(dx + i\,dy)$

$$= \int x^2\,dx + x^2 i\,dy + y^2\,dx + y^2 i\,dy$$

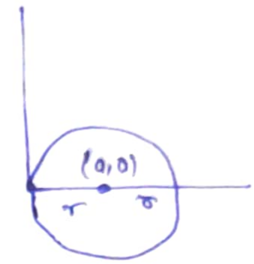
Eqn of circle.

$|z - a| = r.$ $\quad |x+iy-a| = r$

$|(x-a) + iy| = r.$

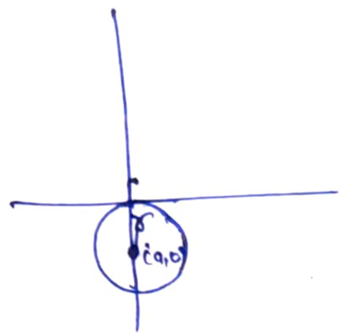
$\sqrt{(x-a)^2 + y^2} = r$ $\quad (x+a)^2 + y^2 = r^2$

$C \to (a,0), r$



$|z + ia| = r.$

$|z - (-ia)| = r$



Ques: Prove that

① $\oint_C \frac{dz}{z-a} = 2\pi i$

② $\oint_C (z-a)^n \, dz = 0$, $n \neq -1$ where C is the circle $|z-a| = r.$

Sol: $|z-a| = r.$

$z - a = re^{i\theta}.$

$z = a + re^{i\theta}.$

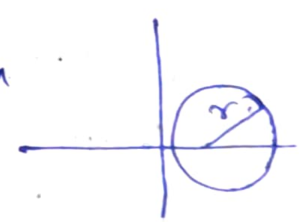
$dz = re^{i\theta} i \, d\theta.$

① $\oint \frac{1}{z-a} dz = \oint \frac{\cancel{re^{i\theta}} i \, d\theta}{\cancel{re^{i\theta}}}$

$\int_0^{2\pi} i \, d\theta.$

$i \int_0^{2\pi} d\theta = i[\theta]_0^{2\pi}$

$= 2\pi i$



② $\oint_C (z-a)^n \, dz.$

$= \int_0^{2\pi} (re^{i\theta})^n \cdot re^{i\theta} i \, d\theta \Rightarrow i \int_0^{2\pi} r^{n+1} e^{i\theta n} e^{i\theta} \, d\theta.$

$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta \Rightarrow \frac{\cancel{i} r^{n+1}}{\cancel{i}} \left(\frac{e^{i(n+1)\theta}}{(n+1)} \right)_0^{2\pi}.$

$= \frac{r^{n+1}}{(n+1)} \left[e^{i(n+1)2\pi} - 1 \right]$

$= \frac{r^{n+1}}{(n+1)} \left[\{ \cos(n+1)2\pi + i\sin(n+1)2\pi \} - 1 \right]$

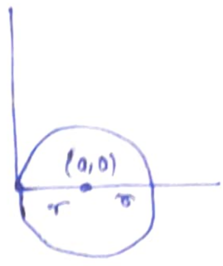
Eqn of circle.

$|z - a| = r$ $\quad |x + iy - a| = r$

$|(x-a) + iy| = r$

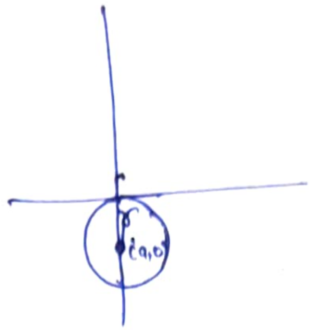
$\sqrt{(x-a)^2 + y^2} = r$ $\quad (x+a)^2 + y^2 = r^2$

$C \to (a,0), r$



$|z + ia| = r$

$|z - (-ia)| = r$



Ques: Prove that

① $\oint_C \frac{dz}{z-a} = 2\pi i$

② $\oint_C (z-a)^n \, dz = 0$, $n \neq -1$ where C is the circle $|z-a| = r$.

Sol: $|z-a| = r$

$z - a = re^{i\theta}$

$z = a + re^{i\theta}$

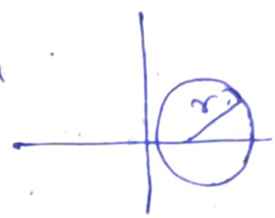
$dz = re^{i\theta} i \, d\theta$

① $\oint \frac{1}{z-a} dz = \oint \frac{\cancel{r}e^{i\theta} i \, d\theta}{\cancel{re^{i\theta}}}$

$\int_0^{2\pi} i \, d\theta$

$i \int_0^{2\pi} d\theta = i[\theta]_0^{2\pi}$

$= 2\pi i$



② $\oint_C (z-a)^n \, dz$

$= \int_0^{2\pi} (re^{i\theta})^n \cdot re^{i\theta} i \, d\theta \Rightarrow i \int_0^{2\pi} r^{n+1} e^{i\theta n} e^{i\theta} \, d\theta$

$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta \Rightarrow \frac{\cancel{i} r^{n+1}}{\cancel{i}} \left(\frac{e^{i(n+1)\theta}}{(n+1)} \right)_0^{2\pi}$

$= \frac{r^{n+1}}{(n+1)} \left[e^{i(n+1)2\pi} - 1 \right]$

$= \frac{r^{n+1}}{(n+1)} \left[\{ \cos(n+1)2\pi + i\sin(n+1)2\pi \} - 1 \right]$

$= \frac{r^{n+1}}{n+1} [\cancel{1} + 0 - \cancel{1}] \Rightarrow 0, \quad n \neq -1.$

Ques: Evaluate the integral $\int_c \log z \, dz$ where $c = |z| = 1$

Sol:

Ques:- Evaluate the integral $\int_c (z - z^2) dz$ where C is the upper half of the circle $|z-2| = 3$. What is the value of the integral if C is the lower half circle.

Sol:- $|z-2| = 3$

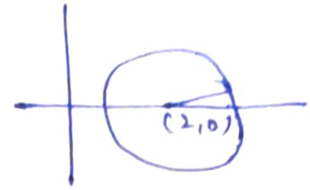
$(z-2) = 3e^{i\theta} \Rightarrow z = 2 + 3e^{i\theta} \Rightarrow dz = 3ie^{i\theta} d\theta.$

$\int_c (z - z^2) dz = \int_c (2+3e^{i\theta}) - (2+3e^{i\theta})^2 (3ie^{i\theta} d\theta)$

$= \int_c (2 + 3e^{i\theta} - 4 - 9e^{2i\theta} - 12e^{i\theta})(3ie^{i\theta} d\theta)$

$i \int_c (-2 - 9e^{i\theta} - 9e^{2i\theta})(3e^{i\theta}) d\theta$

$-3i\int_0^\pi (2e^{i\theta} + 9e^{2i\theta} + 9e^{3i\theta})\, d\theta.$



$\Rightarrow -3\cancel{i}\left[\frac{2e^{i\theta}}{\cancel{i}} + \frac{9e^{2i\theta}}{2\cancel{i}} + \frac{9e^{3i\theta}}{3\cancel{i}}\right]_0^\pi$

$-3\left[2e^{i\pi} + \frac{9e^{2i\pi}}{2} + \frac{9e^{3i\pi}}{3} - 2 - \frac{9}{2} - \frac{\cancel{9}^{3}}{\cancel{3}}\right]$

$\Rightarrow -3\left[2e^{i\pi} + \frac{9}{2}e^{2i\pi} + 3e^{3i\pi} - \frac{19}{2}\right]$

$\Rightarrow -3\left[\{+2(-1) + \frac{9}{2}(1) + 3\} - \frac{19}{2}\right]$

$\Rightarrow -3\left[\frac{10}{2} - \frac{19}{2}\right]$,

$\Rightarrow 3\left[\{2 - \frac{9}{2} - 3(-1)\} - \{-2 - \frac{9}{2} - 3\}\right]$

$\Rightarrow 3\left[2 - \cancel{\frac{9}{2}} + 3 + 2 + \cancel{\frac{9}{2}} + 3\right] = \underline{30}$

for lower half.

$\int_\pi^0 (z+1)dz. = -\int_0^\pi (z+1)\, dz.$

$= -30.$

Ques:- Prove that $\int_C \frac{1}{z}\, dz = -\pi i$ or πi acc. to C is semicircular arc $|z| = 1$ from $z = -1$ to $z = +1$ above or below the real axis.

Ques:- Integrate $f(z) = Re(z)$ from $z = 0$ to $z = 3+i$ along the real axis $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 3+i$

Q. $\int_{1-i}^{2+i} (2x+iy+1)\, dz$ along the straight line joining $(1,-1)$ to $(2+i)$

Sol: $(1,-1)$ $(2,1)$
x_1 y_1 x_2 y_2.

Eq of line passing through two point

$$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y+1 = \frac{1+1}{2-1}(x-1) \Rightarrow y+1 = 2(x-1)$$

$$y = 2x-3 \Rightarrow dy = 2dx$$

$$\int_{1-i}^{2+i} (2x+iy+1)(dx+idy) \Rightarrow \int_1^2 (2x+2x-3+1)(dx+i2dx)$$

$$\int_1^2 (4x-2)(dx-i2dx) \Rightarrow \int_1^2 4x\,dx - 8i\,dx - 2dx + 4i\,dx$$

$$\Rightarrow \left[\frac{\cancel{4}x^2}{\cancel{2}} - 8ix - 2x + 4ix\right]_1^2$$

$$\Rightarrow \left[2x^2 - 2x - 4ix\right]_1^2 \Rightarrow [8-4-8i-\cancel{2}+\cancel{2}+4i]$$

$[4-4i]$ Ans

$\Rightarrow \boxed{4(1+i)}$