Summations and Recurrences

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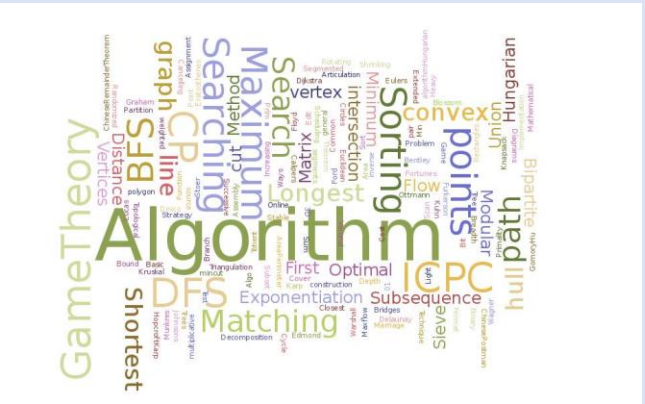

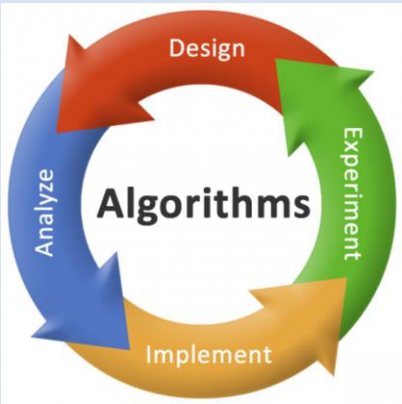

Summations Refresher

- Arithmetic Series
  - $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \Theta(n^2)$
- Sum of Squares
  - $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$
- Sum of Cubes
  - $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = \Theta(n^4)$
- Geometric Series
  - $\sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1}-1}{x-1} = \frac{1-x^{n+1}}{1-x} = \Theta(x^n)$

What is a Recurrence?

- Recurrence – equation or inequality that express the value of a function in terms of its value on smaller input size.
- Example
  - $T(n)\begin{cases}1 & n=1\\ 2T\left(\frac{n}{2}\right)+n & n>1\end{cases}$

How can we solve a recurrence?

- Recursion Tree & Substitution Method
- Master Theorem

Recursion Tree and Substitution

- Step 1: Use a recursion tree to find a solution of the recurrence
- Step 2: Use substitution (mathematical induction) to check that the solution is correct and to find constants
- Important Observation
  - If we want to compute an asymptotic notation, then we need to show only the inductive step in the induction. It will always exist with a base case that works.

Example

$$\begin{cases} T(n) = T(n-1) + n \\ \quad T(1) = \Theta(1) \end{cases}$$

Find T(n) = ? using Θ-notation

Master Theorem

- Provides a “cookbook” for solving recurrences in the form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where a ≥ 1 and b > 1 are constant and f(n) is an asymptotically positive function

Three Cases of Master Theorem

- Case 1 If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$
- Case 2 If $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \lg n\right)$
- Case 3 If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some constant $\epsilon > 0$ and Regularity Condition $(af\left(\frac{n}{b}\right) \leq cf(n)\ for\ some\ constant\ c < 1)$ then $T(n) = \Theta\left(f(n)\right)$

Examples