Graph Algorithm Applications: Maximum Flow

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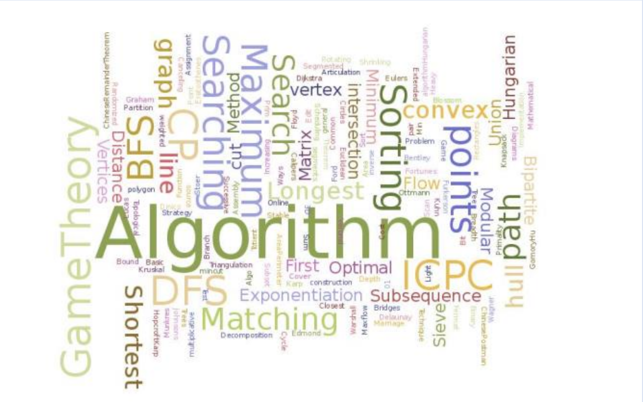
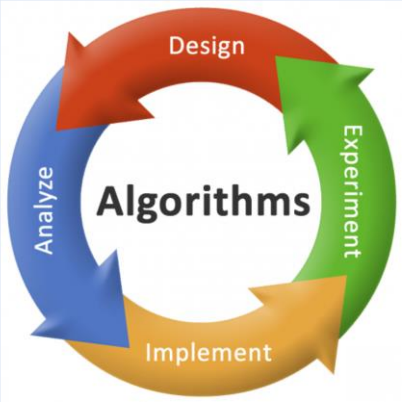


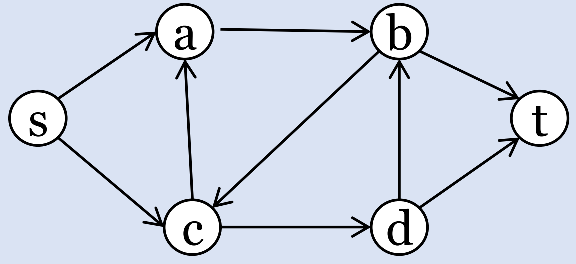


The Applications behind Flow Networks

- Graphs can assist with visual representation of transportation networks
  - Each edge on the traffic map shows limitations to carrying on a path
  - Vertices can act as some sort of switch
- Examples:
  - Highway System – edges are roadways and vertices are interchanges
  - Computer Networks – edges represent links to carry packets and vertices are switches
  - Fluid Networks – edges are pipes that carry fluids and vertices are where pipes are connected.
  - And much more…

What is a Flow Network?

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$.
- If $(u, v) \in E$, then $(v, u) \notin E$
- If $(u, v) \notin E$, then $c(u, v) = 0$
- No Self Cycles (Loops!)
- Source Vertex s is the starting point of a flow network
- Sink Vertex t is the destination point of a flow network



The Formal Definition of Flow

- A flow in G is a real-valued function $f : V \text{ x } V \rightarrow \mathbb{R}$ that satisfies the following properties:
  - Capacity Constraint: For all $u, v \in V$, we require $0 \leq f(u, v) \leq c(u, v)$
  - Flow Conservation: For all $u \in V - \{s, t\}$ we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

When $(u, v) \notin E$, there can be no flow from u to v, the $f(u, v) = 0$

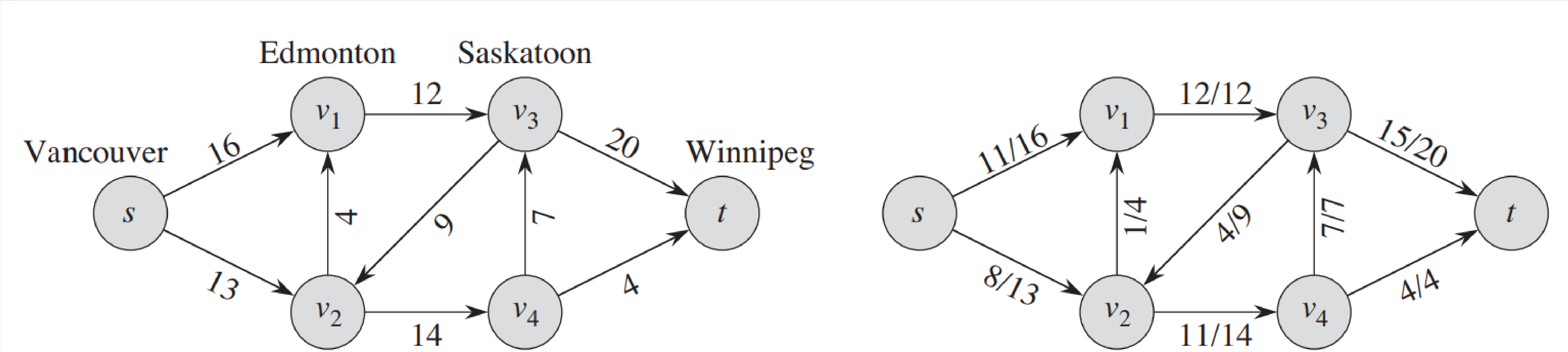
- *f(u,v)* – the flow from u to v
- The value of a flow $|f|$ of flow is defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

The Maximum Flow Problem

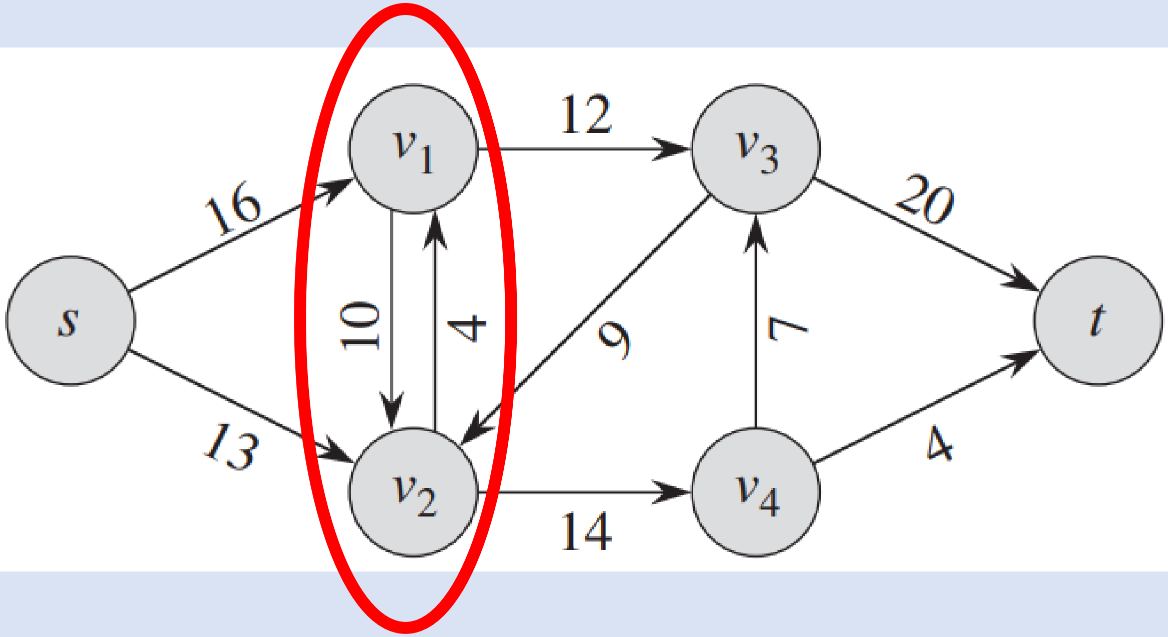
- Given a flow network G with a source s and sink t, find the maximum value.

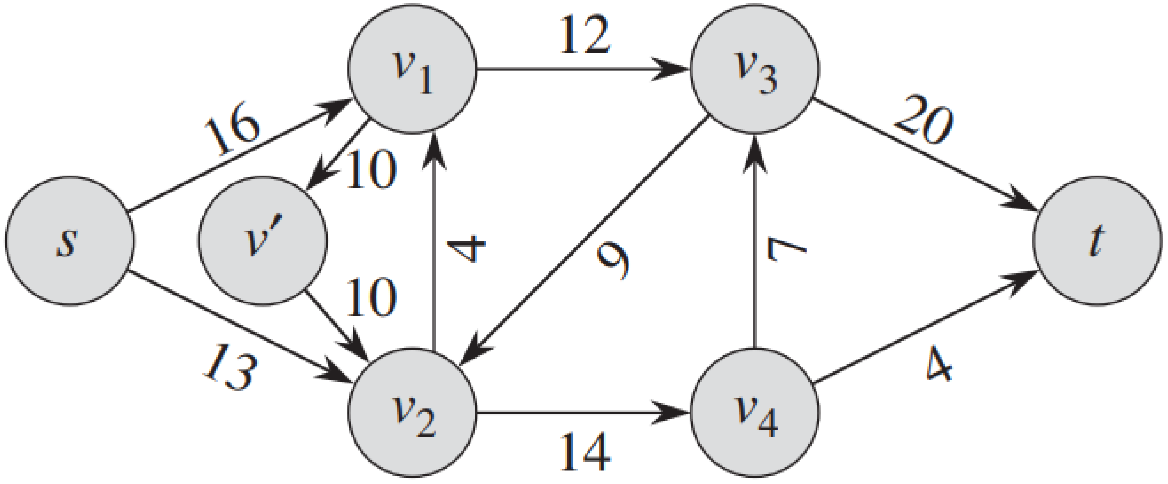
Example



Flow is 19

Another Example with antiparallel edges





Ford-Fulkerson Method

- There exists many several implementations with a different running time analysis
- Major Idea
  - Flow starts with 0
  - For each iteration, increase the flow by finding an “augmenting path” in the “residual network”
  - Keep repeating until there are no more augmenting paths in the residual network

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FORD-FULKERSON-METHOD(G, s, t)
initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
    augment flow f along p
return f
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Residual Network

- We have a flow network with a source and sink.
- f is the flow and $u, v \in V$
- Residual Capacity $c_f(u, v)$ is defined as

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & if (u,v) \in E \\ f(v,u) & if (v,u) \in E \\ 0 & otherwise \end{cases}$$

- Given a flow network $G(V, E)$ and a flow f, the residual network of G induced by f is $G_f = \left(V, E_f\right)$, where

$$\mathrm{E_f} = \left\{(u,v) \in V \text{ x } V : c_f(u,v) > 0\right\}$$

Augmenting Paths

- Let f be a flow in the flow network G
- Let f ’ be a flow in the residual network G_f
- $f \uparrow f'$: V x V $\rightarrow$ R
- $f \uparrow f' = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & if (u,v) \in E \\ 0 & otherwise \end{cases}$
- f ’(v,u) is known as cancellation since we are pushing a reverse edge in the residual network

A Lemma with Flow Augmentation

- Let G be a flow network with source and sink
- Let f be a flow in G
- Let G_f be the residual network of G induced by f
- Let f ’ be a flow in G_f
- Then $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$

Augmenting Paths cont.

- Augmenting path is a path from the source to sink in the residual network
- Let p be an augmenting path in the residual network
- Then the residual capacity of p, which can be represented as $c_f(p)$ is defined as $c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on p}\}$
- Lemma
  - Let G = (V,E) be a flow network, let f be a flow in G, and let p be an augmenting path in G_f.Define a function
    $$f_p(u,v) = \begin{cases} c_f(p) & if\,(u,v)\ is\ on\ p \\ 0 & otherwise \end{cases}$$
  - Then $\mathrm{f_p}$ is a flow in $\mathrm{G_f}$ with value $|\mathrm{f_p}| = \mathrm{c_f(p)} > 0$

A Corollary on Augmenting Paths

- Let G be a flow network
- Let f be the flow in G
- Let p be the augmenting path in G_f
- f_p defined as in the previous lemma
- Suppose we augmented f by f_p
- Then $f \uparrow f_p = |f| + |f_p|$

Cuts

- The Ford-Fulkerson method repeats augmenting paths until a max flow is found.
- How does the method know when to stop?
- The answer is very interesting: Max-Flow Min-Cut Theorem
- A cut (S,T) of a flow network G=(V,E) is a partition of V into S and $T = V - S$, such that $s \in S$ and $t \in T$

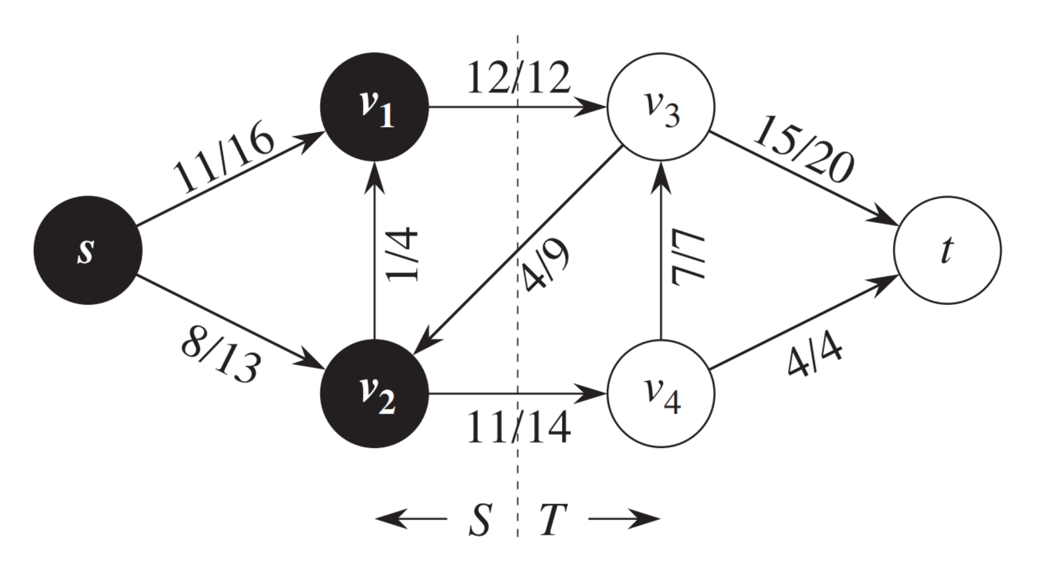
Understanding Cuts

- If f is a flow, then the net flow f(S,T) across the cut (S,T) is:

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

- The capacity of the cut (S,T) is:

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$



Max-Flow Min-Cut Theorem

- If f is a flow in a flow network $G = (V,E)$ with source s and sink t, then the following conditions are equivalent:
  - f is the maximum flow in G
  - The residual network contains no augmenting paths
  - $|f| = c(S,T)$ for some cut (S,T) of G

Ford-Fulkerson Algorithm

FORD-FULKERSON(G, s, t)

for each edge $(u, v) \in G.E$
    $(u, v).f = 0$
while there exists a path p from s to t in the residual network G_f
    $c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
    for each edge (u, v) in p
        if $(u, v) \in E$
            $(u, v).f = (u, v).f + c_f(p)$
        else $(v, u).f = (v, u).f - c_f(p)$

Example

Running Time Analysis of Ford-Fulkerson

- Depends on how the augmenting paths are selected
- In reality situations, capacities are integer numbers.
- Rational numbers results in scaling transformation to make them integral
- The while loop will execute at most $|f^*|$ times, where f^* is the maximum flow
- How are paths augmented?
  - Let G' be the graph to store the residual network
  - $|E'| \leq 2|E|$
  - BFS/DFS to augment a path $O(V + E') = O(E)$
- The total running time can be expressed as $O(E\,|f^*|)$