B-Trees

COP 3503
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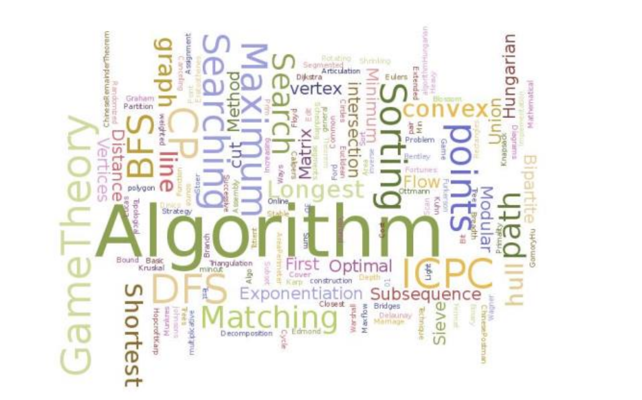
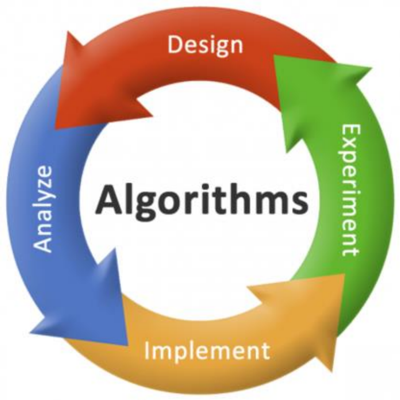
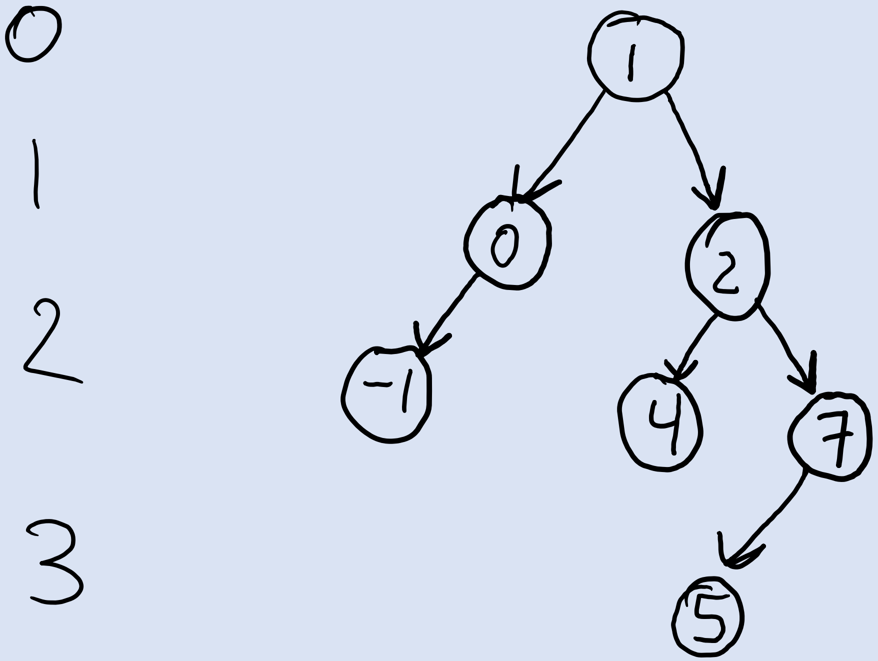
Department of Computer Science
University of Central Florida
Dr. Steinberg

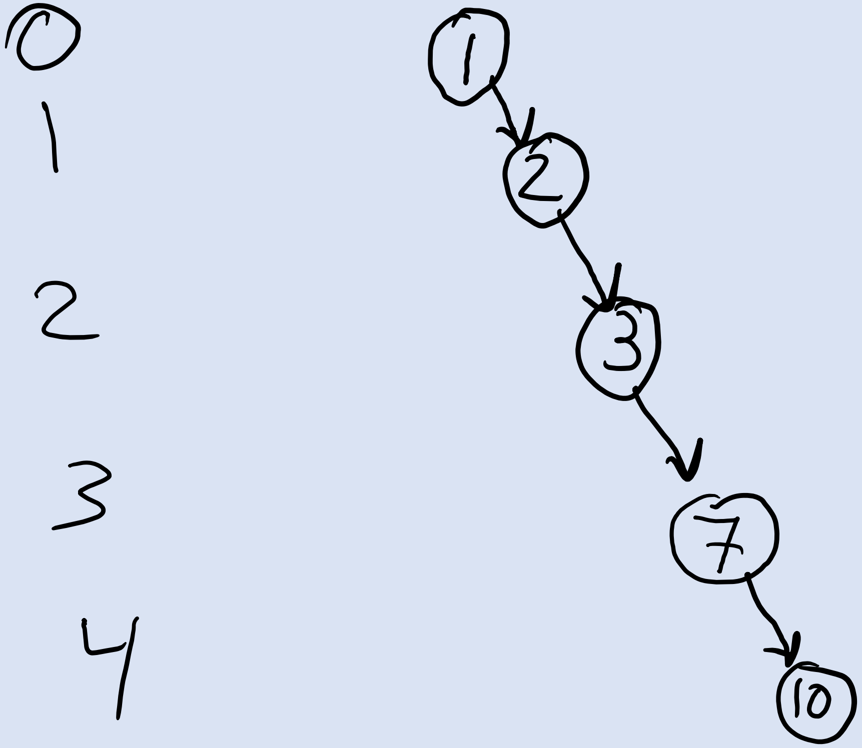




Introduction

- Binary Search Trees – Trees where a node has only at most two children
- Examples:



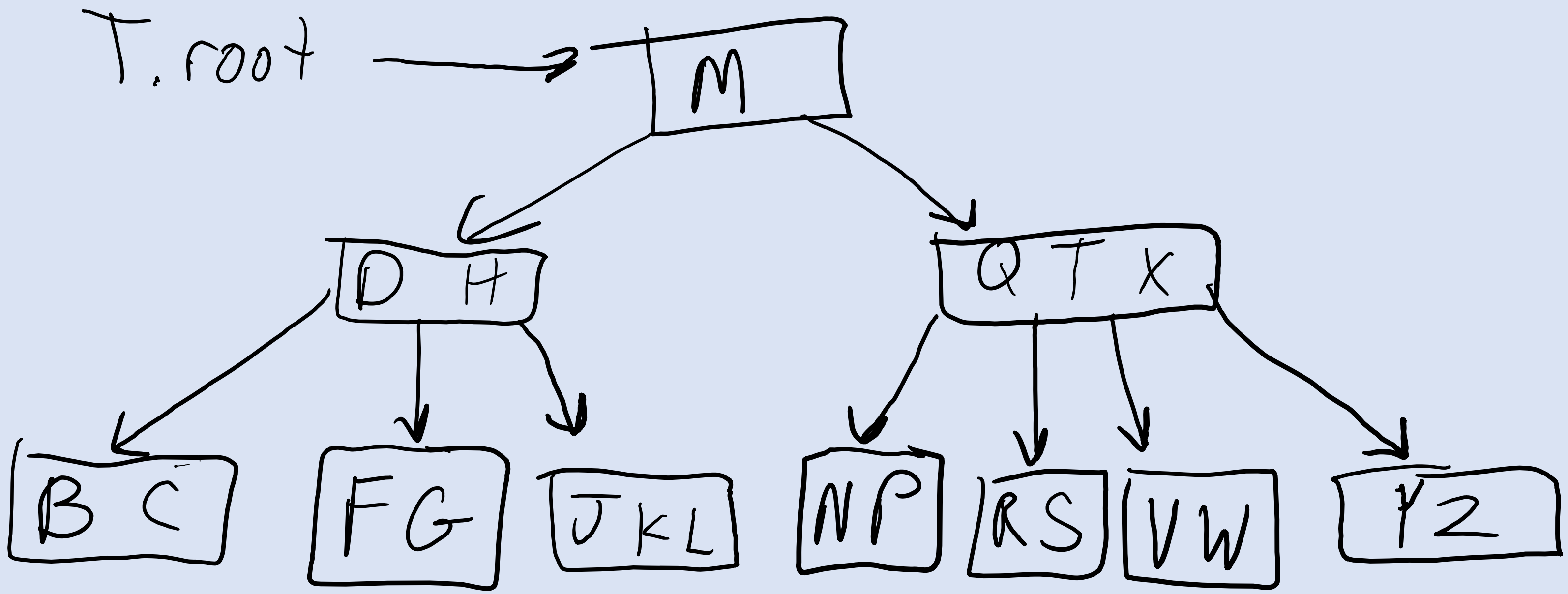


B-Trees

- Balanced Search Trees
- One con with a Binary Search Tree (BST) is that we can potentially have our algorithms run in linear time rather than height level time.
- B-Trees nodes have many children (few to even thousands!)
- This data structure is primarily used in disks and other direct-access secondary storage devices.
- Database systems use B-Trees or even variants.
- The height of a B-tree is $O(\lg n)$

B-Tree Sample

- In this sample, they keys are letters from the alphabet
- Each node has x.n keys and x.n + 1children
- How does a search work for the letter P?



Data Structures on Secondary Storage

- Primary memory (main memory) consists of silicon memory chips.
- Secondary storage consists of magnetic storage
  - Tapes
  - Disks
- Disks are cheaper and have higher capacity than the main memory.
- Disks are slower than main memory due to motion mechanical components.

Disk Drive

- The average access time for the disk ranges from 8 to 11 milliseconds.
- The average access time in main memory is about 50 nanoseconds!
- Information on a disk is divided into pages which range from $2^{11} - 2^{14}$ bytes.
- Each disk reads and/or writes on a single or multiple pages.

B-Tree Applications

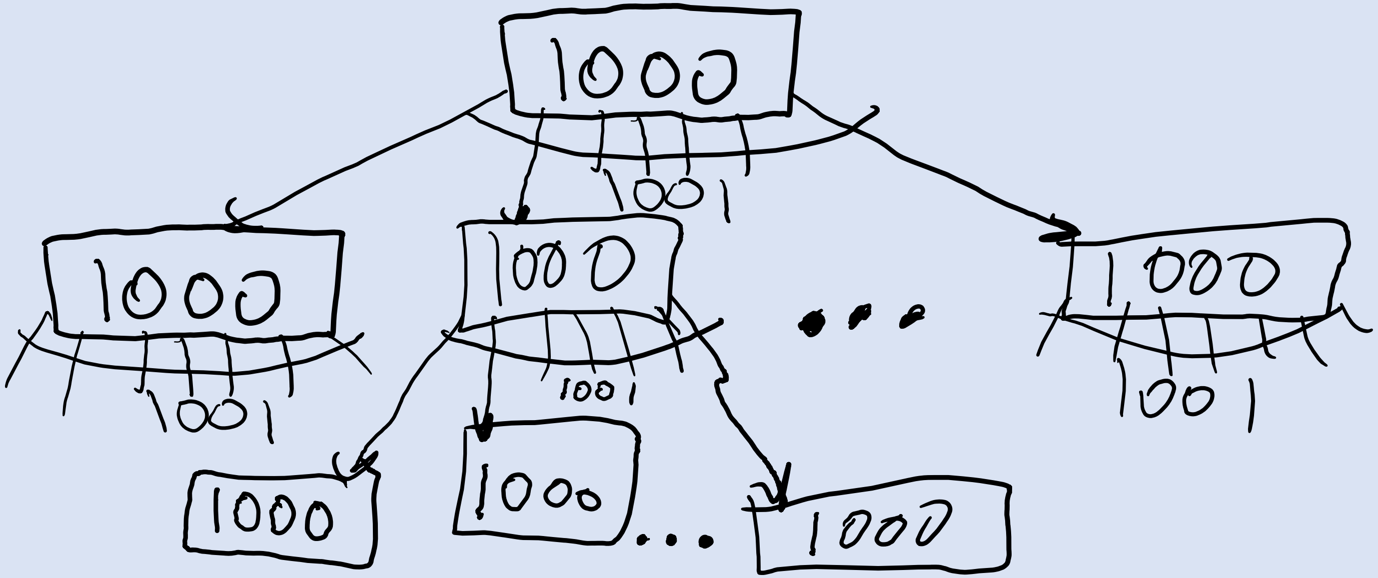
- The Whole B-Trees do not fit in the main memory!!!
- Operating Systems copies the pages from the disks into main memory. After performing tasks, the operating system writes back to the respective pages that were modified.

```
x = a pointer to some object
DISK-READ(x)
operations that access and/or modify attributes of x

DISK-WRITE(x)
other operations that access but do not modify attributes of x
```

B-Tree Example

- Branch Factor = 1001
- Height = 2



- B-Trees often have branching factors ranging from 50 -2000
- Root node is permanently in main memory in order to find any key with at most two disk accesses.

The Official B-Tree Definition

- A B-Tree is a rooted tree (where T.root is the root) with the following properties:
  1. Every node x has the following attribures
     a) x.n is the number of keys currently stored in x
     b) The keys $x.key_1, x.key_2, \ldots, x.key_{x.n}$ such that
        $$x.key_1 \leq x.key_2 \leq \ldots \leq x.key_{x.n}$$
     c) x.leaf – a Boolean value which is true if x is a leaf and false if x is an internal node
  2. Each internal node x has $x.n + 1$ pointers $x._{c1}, x._{c2}, \ldots, x._{cx.n+1}$ to its children. If x is a leaf then the pointers are undefined.
  3. If k_i is any key stored in the subtree with root $x.c_i$ then:
     $$k_1 \leq x.key_1 \leq k_2 \leq x.key_2 \leq \ldots \leq x.key_{x.n} \leq k_{x.n+1}$$

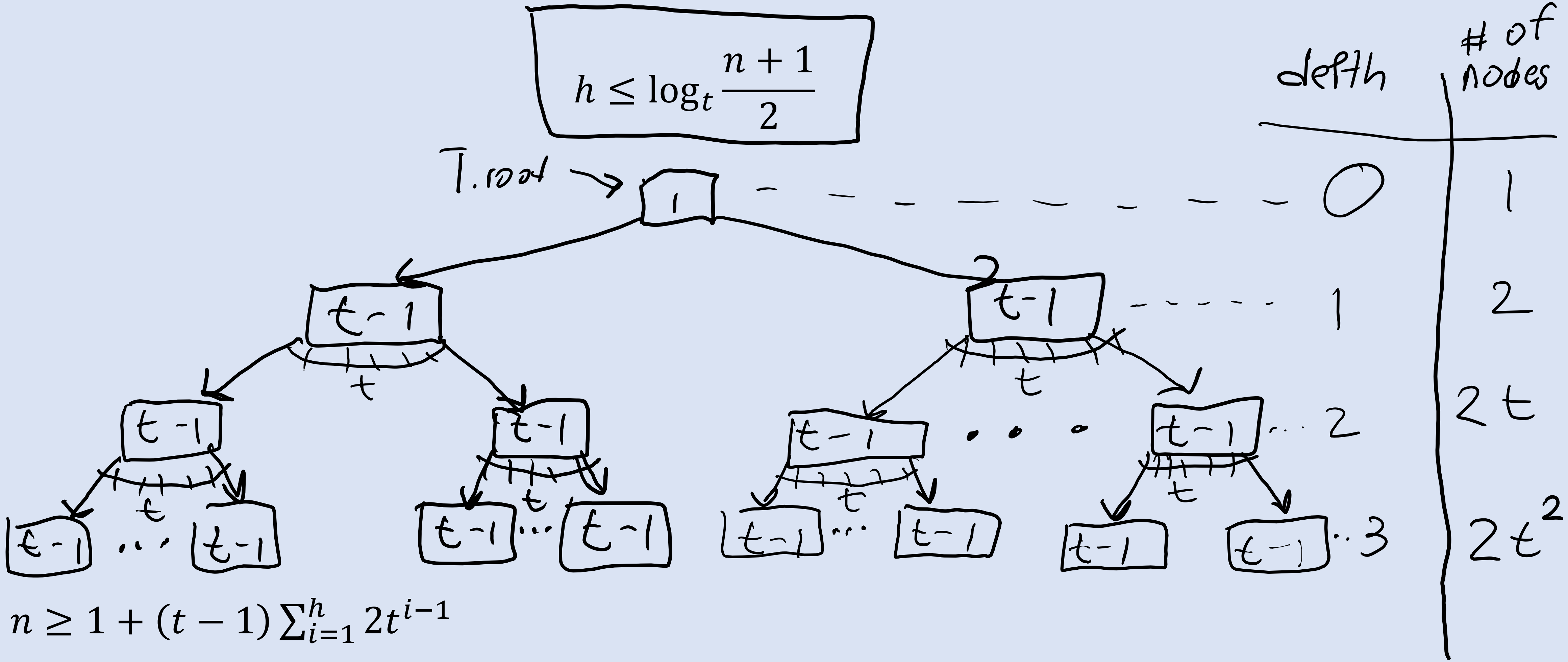
The official B-Tree Definition Continued

- All leaves have the same depth, which is the tree high h.
- The B-Tree has a minimum degree t (where t is an interger t >= 2):
  - Every node other than the root must have >= t – 1 keys and >= t children; if B-tree is nonempty, then the root has at least one key
  - Every node has $\leq 2t - 1$ keys and $\leq 2t$ children
    A node is considered full if it has 2t – 1 keys inserted.

Interesting Theorem About Height in B-Trees

- Theorem: if $n \geq 1$, then for any n-key B-tree T of height h and minimum degree t,

$$\boxed{h \leq \log_t \frac{n+1}{2}}$$



- $n \geq 1 + (t-1)\sum_{i=1}^{h} 2t^{i-1}$

Theorem cont.

- $n \geq 1 + (t-1)\sum_{i=1}^{h} 2t^{i-1}$

$$= 1 + 2(t-1)\sum_{i=1}^{h} t^{i-1}$$

$$= 1 + 2(t-1)\ \frac{t^h - 1}{t-1} = 2t^h - 1$$

$$t^h \leq \frac{n+1}{2}$$

$$h \leq \log_t \frac{n+1}{2}$$

$h = O(\log n)$

Operations We Will Observe for B-Trees

- B-Tree-Search
- B-Tree-Create
- B-Tree-Insert
- B-Tree-Delete

B-Tree Search

<u>B-Tree-Search(x, k)</u>

i = 1

while $i \leq x.n$ and $k \;\; > x$

    i = i + 1

if $i \leq x.n$ and k == $x.key_i$

    return (x, i)

else if x.leaf == True

    return NULL

else DISK-READ($x.c_i$)

    return B-Tree-Search($x.c_i, k$)

RT: $O(tlog_t n)$

B-Tree-Create

- Creating an empty tree with root node
  B-Tree-Create(T)
  x = Allocate-Node()
  x.leaf = True
  x.n = 0
  Disk-Write(x)
  T.root = x

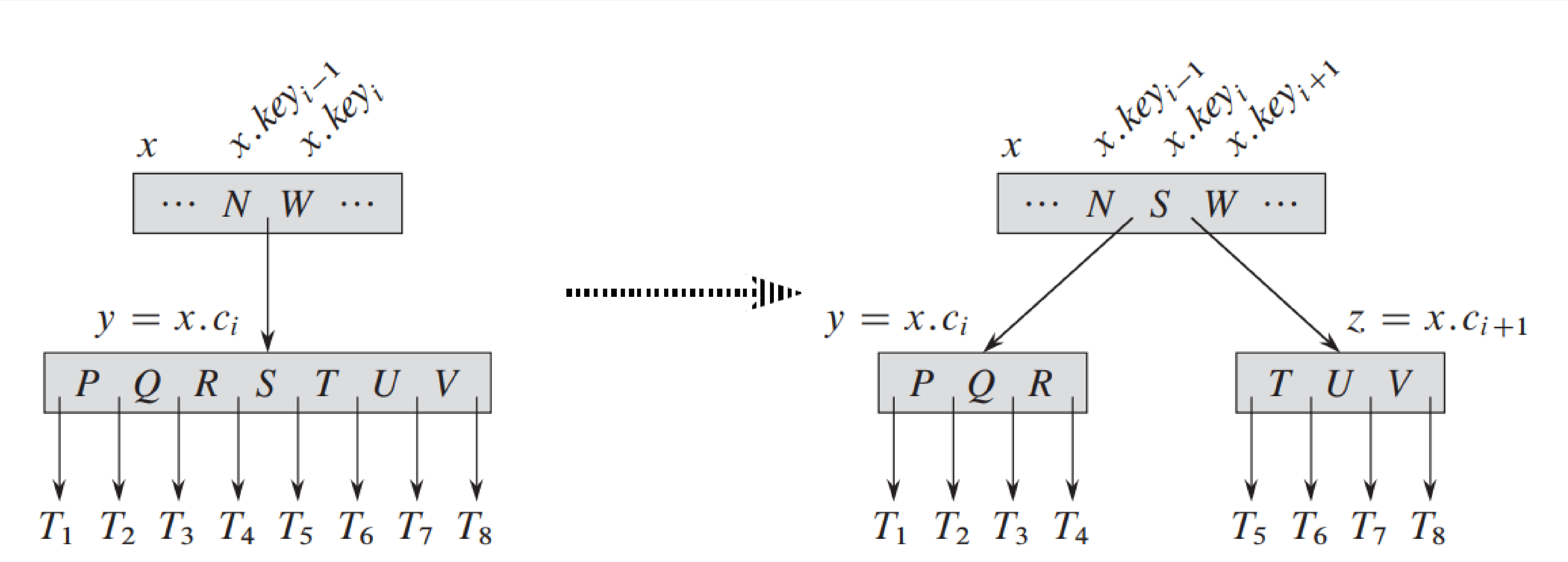
B-Tree Insert Operations

- The insert operation has 3 functions/methods we need to understand
- B-Tree-Split-Child(x,i)
- B-Tree-Insert(T,k)
- B-Tree-Insert-Nonfull(x,k)

Insert Operation and Overall Goal

- Search for a leaf where to put new key
- Inserting into an existing leaf node
  - Cannot create a new leaf
- If the leaf node is full, then split around the median key
- The overall goal is to insert they key while maintaining B-Tree rules. As the algorithms traverses down the tree, it splits each full node along the way, including the leaf.

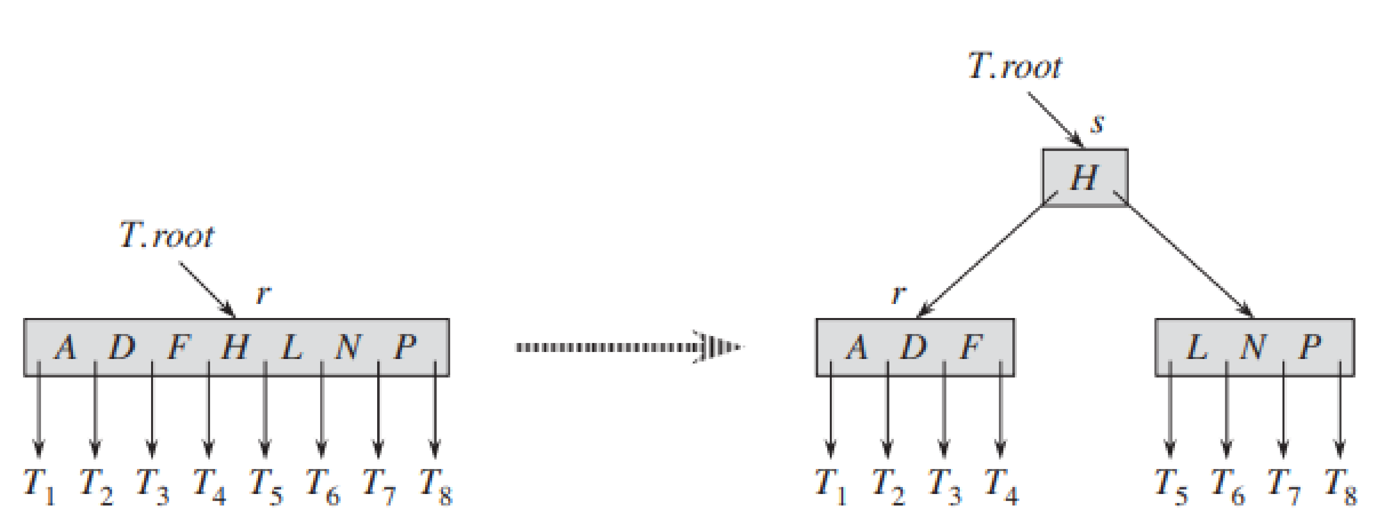
Splitting a Node



```
B-TREE-SPLIT-CHILD(x, i)
z = ALLOCATE-NODE()
y = x.c_i
z.leaf = y.leaf
z.n = t - 1
for j = 1 to t - 1
    z.key_j = y.key_{j+t}
if not y.leaf
    for j = 1 to t
        z.c_j = y.c_{j+t}
y.n = t - 1
for j = x.n + 1 downto i + 1
    x.c_{j+1} = x.c_j
x.c_{i+1} = z
for j = x.n downto i
    x.key_{j+1} = x.key_j
x.key_i = y.key_t
x.n = x.n + 1
DISK-WRITE(y)
DISK-WRITE(z)
DISK-WRITE(x)
```

B-Tree-Insert()

- t = 4
  range of keys 3-7



- If root node is full, then split the root and new node will become the root

```
B-TREE-INSERT(T, k)
r = T.root
if r.n == 2t - 1
    s = ALLOCATE-NODE()
    T.root = s
    s.leaf = FALSE
    s.n = 0
    s.c_1 = r
    B-TREE-SPLIT-CHILD(s, 1)
    B-TREE-INSERT-NONFULL(s, k)
else B-TREE-INSERT-NONFULL(r, k)
```

```
B-TREE-INSERT-NONFULL(x, k)
i = x.n
if x.leaf
    while i ≥ 1 and k < x.key_i
        x.key_{i+1} = x.key_i
        i = i - 1
    x.key_{i+1} = k
    x.n = x.n + 1
    DISK-WRITE(x)
else while i ≥ 1 and k < x.key_i
        i = i - 1
    i = i + 1
    DISK-READ(x.c_i)
    if x.c_i.n == 2t - 1
        B-TREE-SPLIT-CHILD(x, i)
        if k > x.key_i
            i = i + 1
    B-TREE-INSERT-NONFULL(x.c_i, k)
```

Insertion Examples

RT for Insert

- $O(th) = O(t log_t n)$

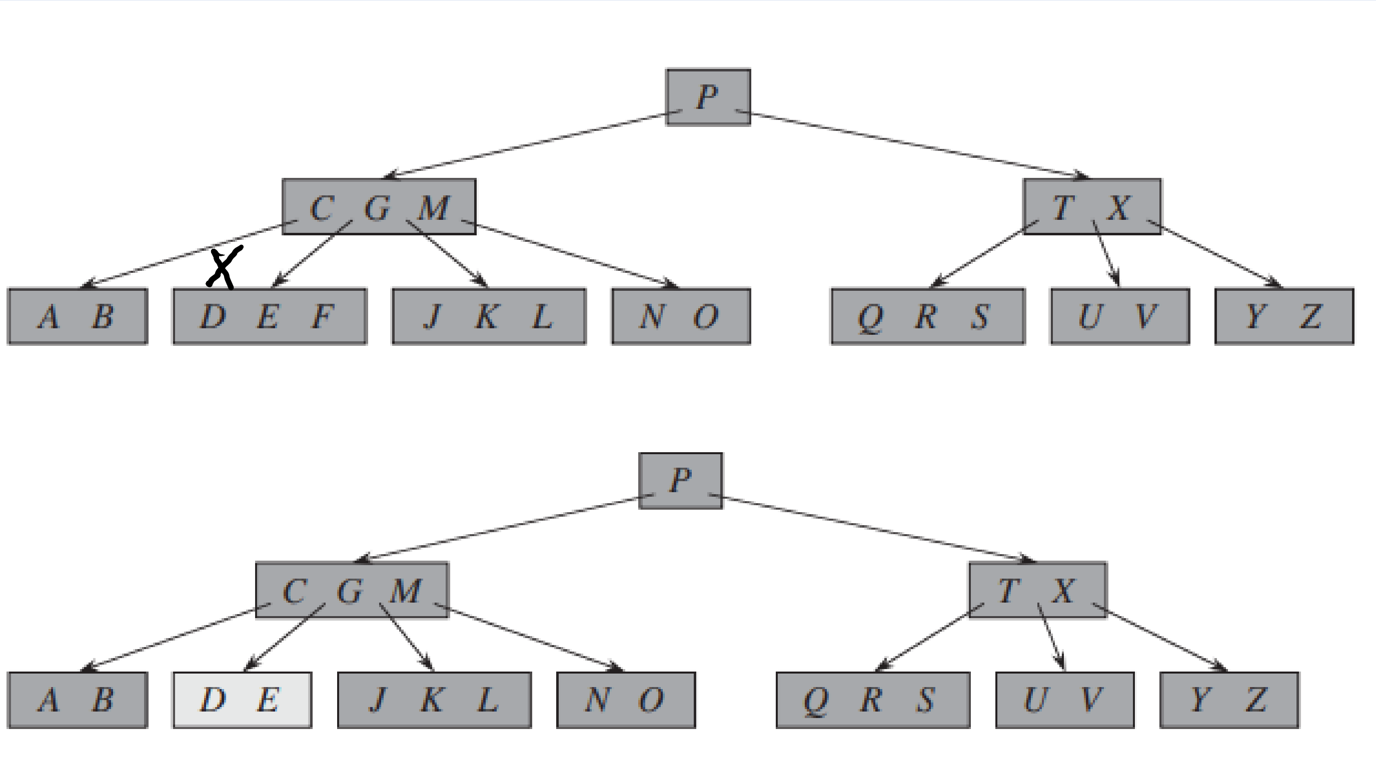
B-Tree-Delete()

- Important things to remember!
  - When a key is removed, we must rearrange the node's children!
  - Any node (EXCEPT FOR THE ROOT) cannot have fewer than t -1 keys
  - The algorithm deletes a key k from the subtree rooted at x
  - Something to consider: When delete is called on a node, we should guarantee that the number of keys in x is greater than or equal to t.
- The overall objective is to remove a key while maintaining the B-tree properties.
- There are 3 rules to consider when deleting from the B-Tree

Rule 1

- If the key *k* is part of a leaf node *x*, then just delete the key.
  t = 3

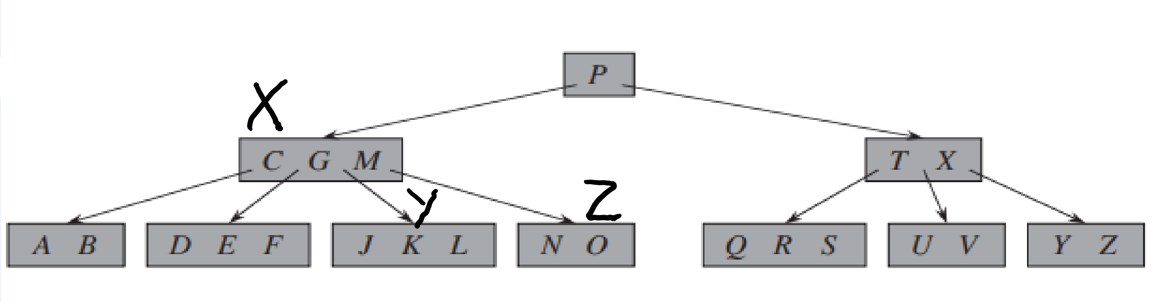
Delete F



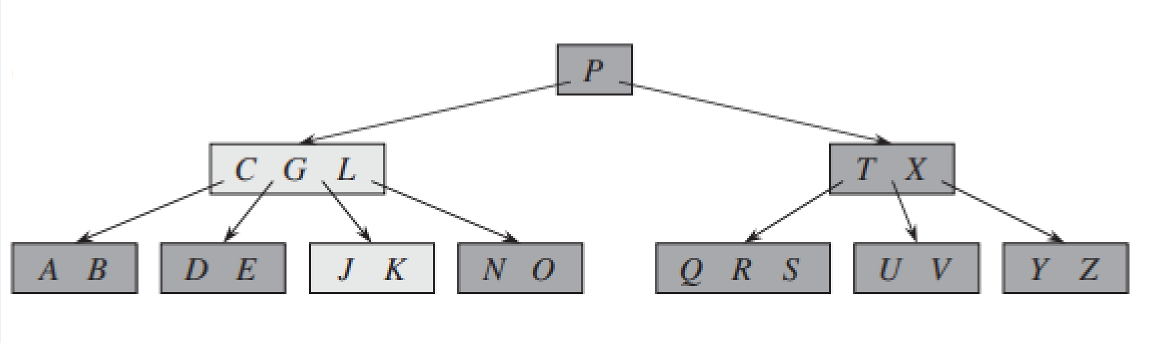
Rule 2a

- If the key *k* belongs to an internal node *x*.
- If the child *y* that precedes *k* in a node *x* has at least *t* keys, then find the predecessor *k'* of *k* in the subtree rooted at *y*. Recursively delete *k'* and replace *k* by *k'* in *x*.

t = 3



Delete M



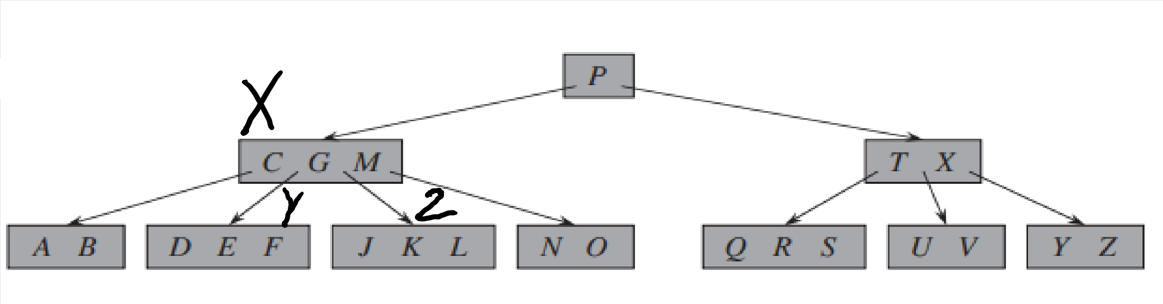
Rule 2b

- If the key k belongs to an internal node.
- If y has fewer than t keys, then, symmetrically, examine the child z that follows k in node x. If z has at least t keys, then find the successor k' of k in the subtree rooted at z. Recursively delete k' and replace k by k' in x.

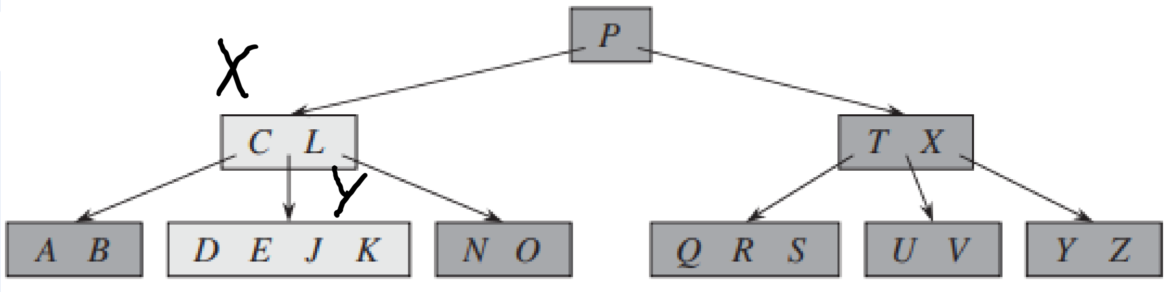
Rule 2c

- If the key *k* belongs to an internal node *x*.
- Otherwise, if both *y* and *z* have only *t* – *1* keys, merge *k* and all of *z* into *y*, so that *x* loses both *k* and the pointer to *z*, and *y* now contains *2t -1* keys. Then free *z* and recursively delete *k* from *y*.

t = 3



Delete G



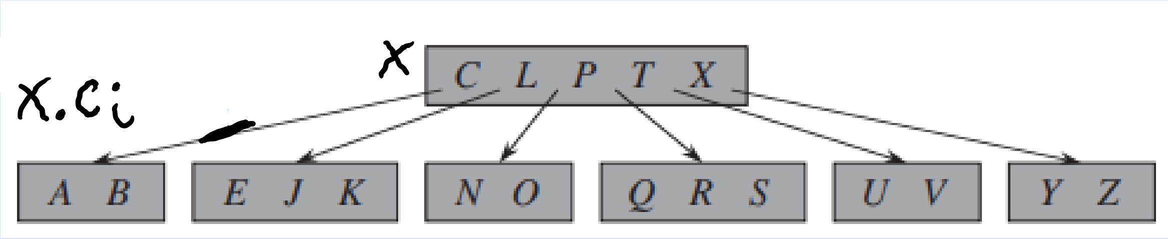
Rule 3a

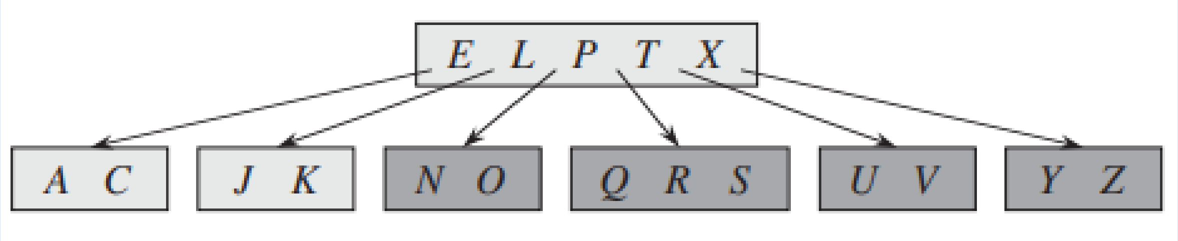
- If the key k is not part of the internal node x, take $x.c_i$ the root of the subtree that must contain k (if k is in the tree). If $x.c_i$ has only *t-1* keys, then use 3a or 3b to guarantee we descend to a node with greater than or equal to t keys.

If $x.c_i$ has an immediate silbing with greater than or equal to t keys, then give $x.c_i$ an extra key by:

  - Moving a key from x to $x.c_i$
  - Moving a key from $x.c_i$'s immediate left or right sibling up x
  - Moving the appropriate child pointer from the sibling into $x.c_i$

t = 3
Delete B

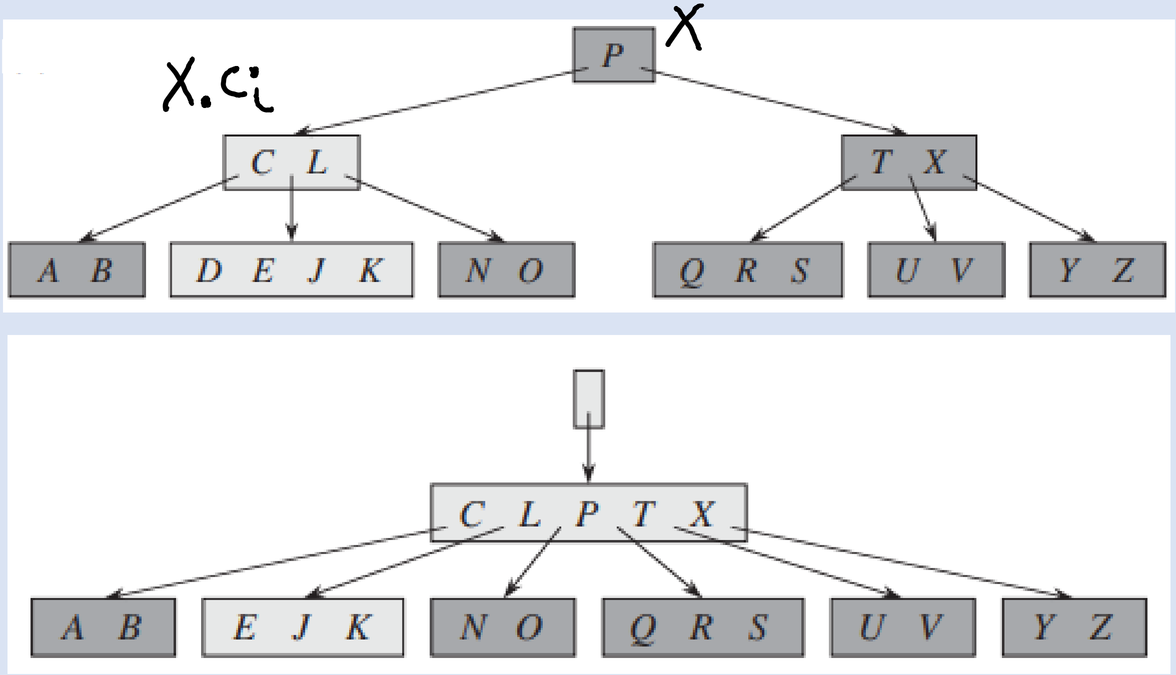




Rule 3b

- If both $x.c_i$'s immediate sibling have $t - 1$ keys, merge $x.c_i$ with one sibling, which involves moving a key from x down into the new merged node to become the median for that node

Delete D



RT for Delete

- RT is $O(th) = O(tlog_t n)$

Delete Examples