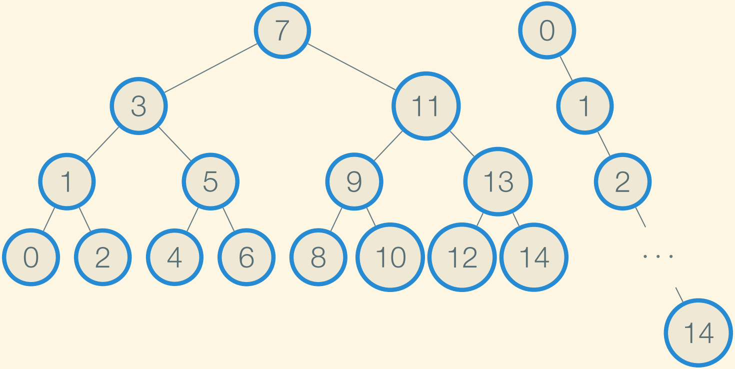
Random Binary Search Trees

IPD

The necessity of balance



The necessity of balance

n	$\lceil \lg n \rceil$
10	4
100	7
1,000	10
10,000	14
100,000	17
1,000,000	20
10,000,000	24
100,000,000	27
1,000,000,000	30

Leaf insertion

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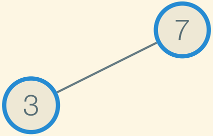

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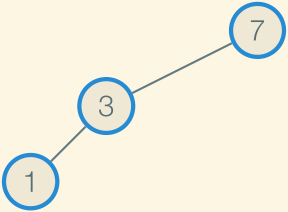

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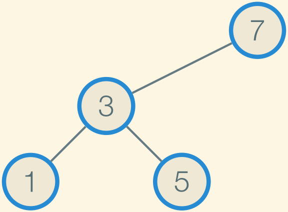

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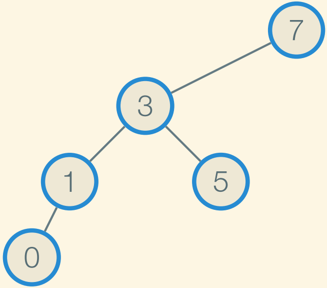

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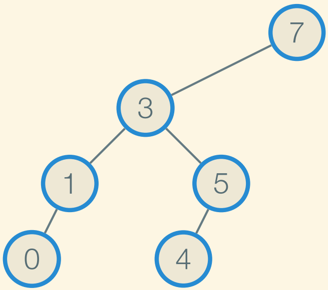

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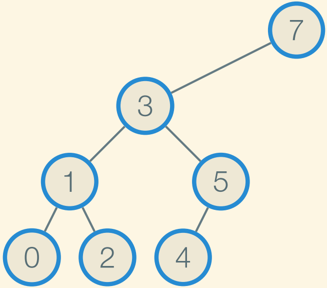

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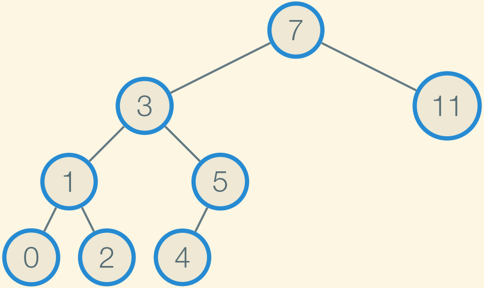

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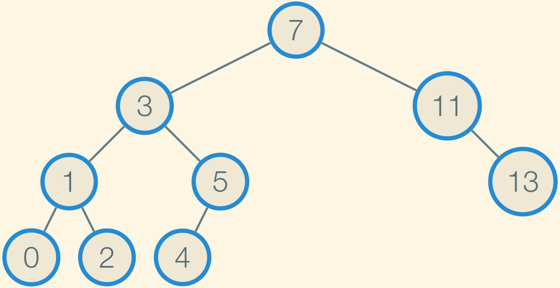

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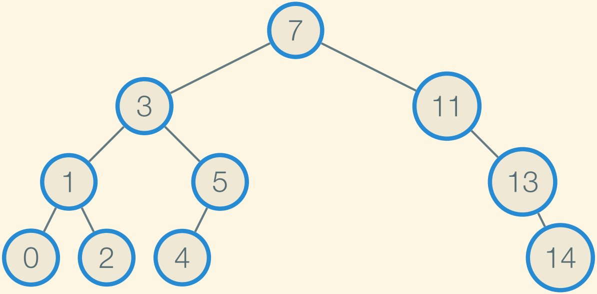

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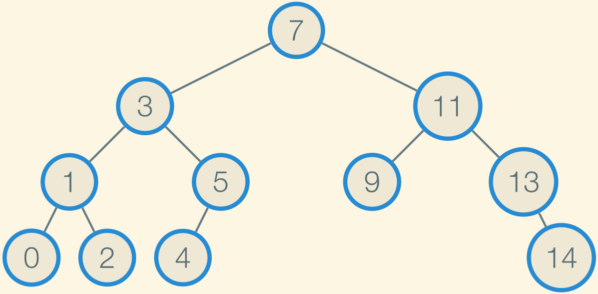

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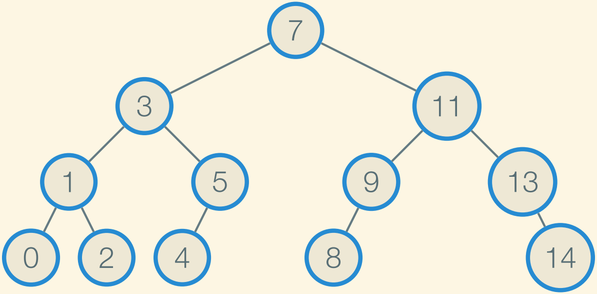

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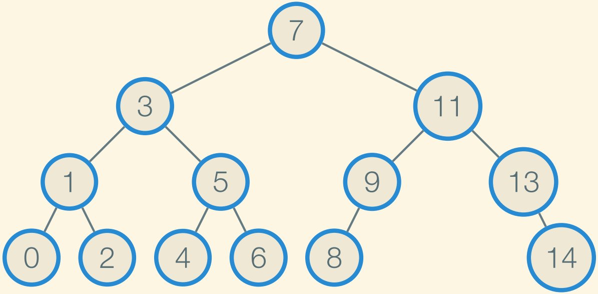

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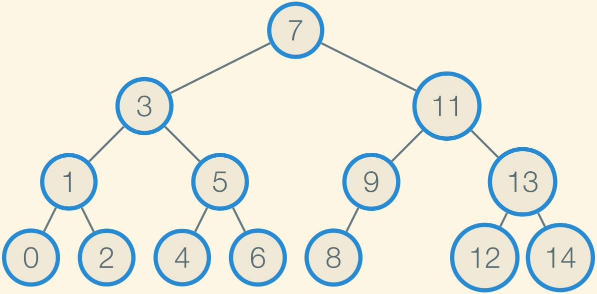

The permutation distribution

Can we characterize how sequences of insertions produce (un)balanced trees?

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In fact, the only sequence to produce the right-branching degenerate tree is 0, …, 14

There are 21,964,800 sequences that produce the same perfectly balanced tree

A random BST tends to be balanced

If you generate a tree by leaf-inserting a random permutation of its elements, it will probably be balanced

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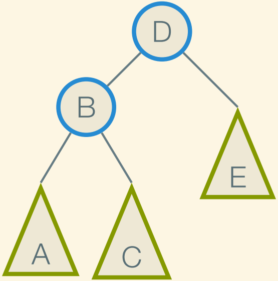
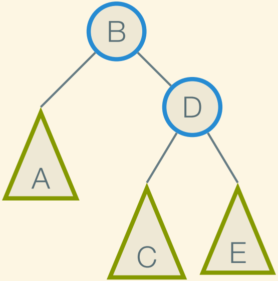
Unfortunately, we usually can't do that, but we can simulate it

A tool: tree rotations





Note that order is preserved

Root insertion

Using rotations, we can insert at the root:

- To insert into an empty tree, create a new node
- To insert into a non-empty tree, if the new key is greater than the root, then root-insert (recursively) into the right subtree, then rotate left
- By symmetry, if the key belongs to the left of the old root, root insert into the left subtree and then rotate right

Randomized insertion

We can now build a randomized insertion function that maintains the random shape of the tree:

- Suppose we insert into a subtree of size k, so the result will have size $k + 1$
- If the tree were random, the new element would be a the root with probability $\frac{1}{k+1}$
- So we root insert with that probability, and otherwise recursively insert into a subsubtree

Deletion idea

To delete a node, we join its subtrees recursively, randomly selecting which contributes the root (based on size):