On-shell recursion for Massive QED and Electroweak Theory

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Based on **2403.15538** and **2407.14587**
with Yohei Ema, Ting Gao, Wenqi Ke, Kunfeng Lyu, Zhen Liu.

Outline

- Why going on-shell?
- Recursion relation and factorization
- Momentum shifts for particles of any mass
  - All-line transverse (ALT) shift
- Application to massive QED
- Application to electroweak theory

Motivation for on-shell method

$$p_i^2 = m_i^2 \ , \ i = 1,2,\ldots,n$$

- We work with **on-shell** degrees of freedom
- For example: photon has two degrees of freedom: $\epsilon^{\pm}$
- In Lagrangian, we embed photon in $A^{\mu}(x)$ which has four DOF which leads to gauge redundancy
- Amplitudes are field redefinition invariant.

Motivation for on-shell method

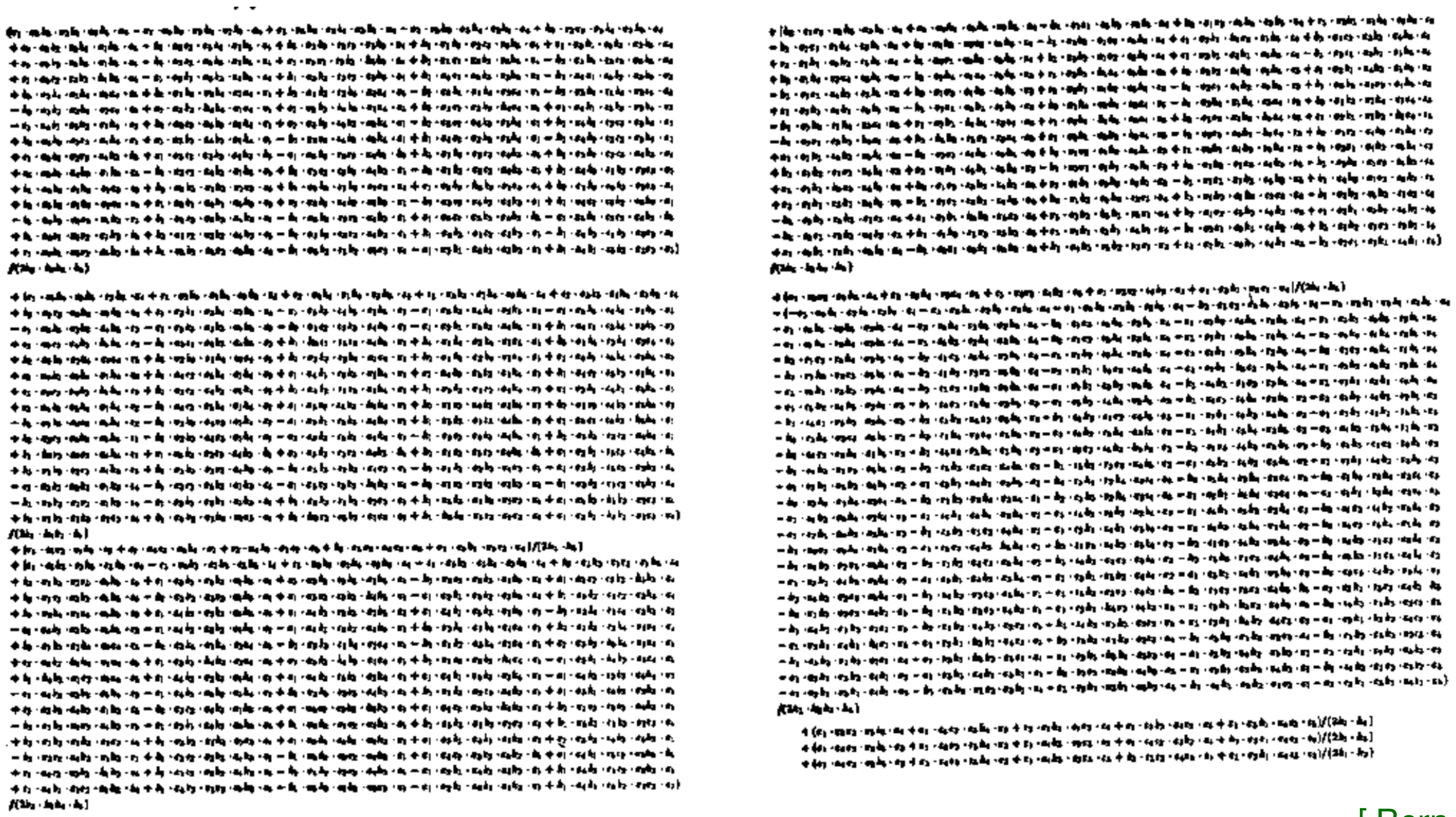
- Computations using Feynman diagram can be convoluted

$g + g \to g + g$	4 diagrams
$g + g \to g + g + g$	25 diagrams
$g + g \to g + g + g + g$	220 diagrams

- $g + g \to 8g$ one needs more than one million diagrams!

An unreadable form of the five-gluon tree amplitude

This is not all, we still have to square it and sum over helicities…



[Bern '93]

Motivation for on-shell method

- **Why the final expression simple?**

6-pt MHV gluon squared amplitude $\longrightarrow$

$$\boxed{|A_6|^2 \sim \frac{(p_1 \cdot p_2)^3}{(p_2 \cdot p_3)(p_3 \cdot p_4)(p_4 \cdot p_5)(p_5 \cdot p_6)(p_6 \cdot p_1)}}$$

[Parke, Taylor ’86]

- BCFW recursion relation can explain the simplicity

[Britto, Cachazo, Feng, and Witten ’05]

- **Efficient and practical** approach for calculating amplitude

- On-shell method may provide new insight not apparent using Feynman diagrams. Ex: double copy

Why massive formalism?

- On-shell methods well developed for massless particles

- Little group covariant massive spinor formalism introduced

[Arkani-Hamed, Huang, Huang ’17]

General recursion framework for theories with all external massive states is still lacking.

Why massive formalism?

Even the simplest $e^+e^- \to \mu^+\mu^-$ seemed puzzling

[Christensen+ ’22]

This is because we did not have any **good momentum shift**

We found a new momentum shift that resolves these ambiguities

On-shell Recursion Relation

On-shell recursion relation constructs higher-point amplitude from lower-point on-shell information

- Deform external momentum $$\hat{p}_i(z) = p_i + zq_i.$$

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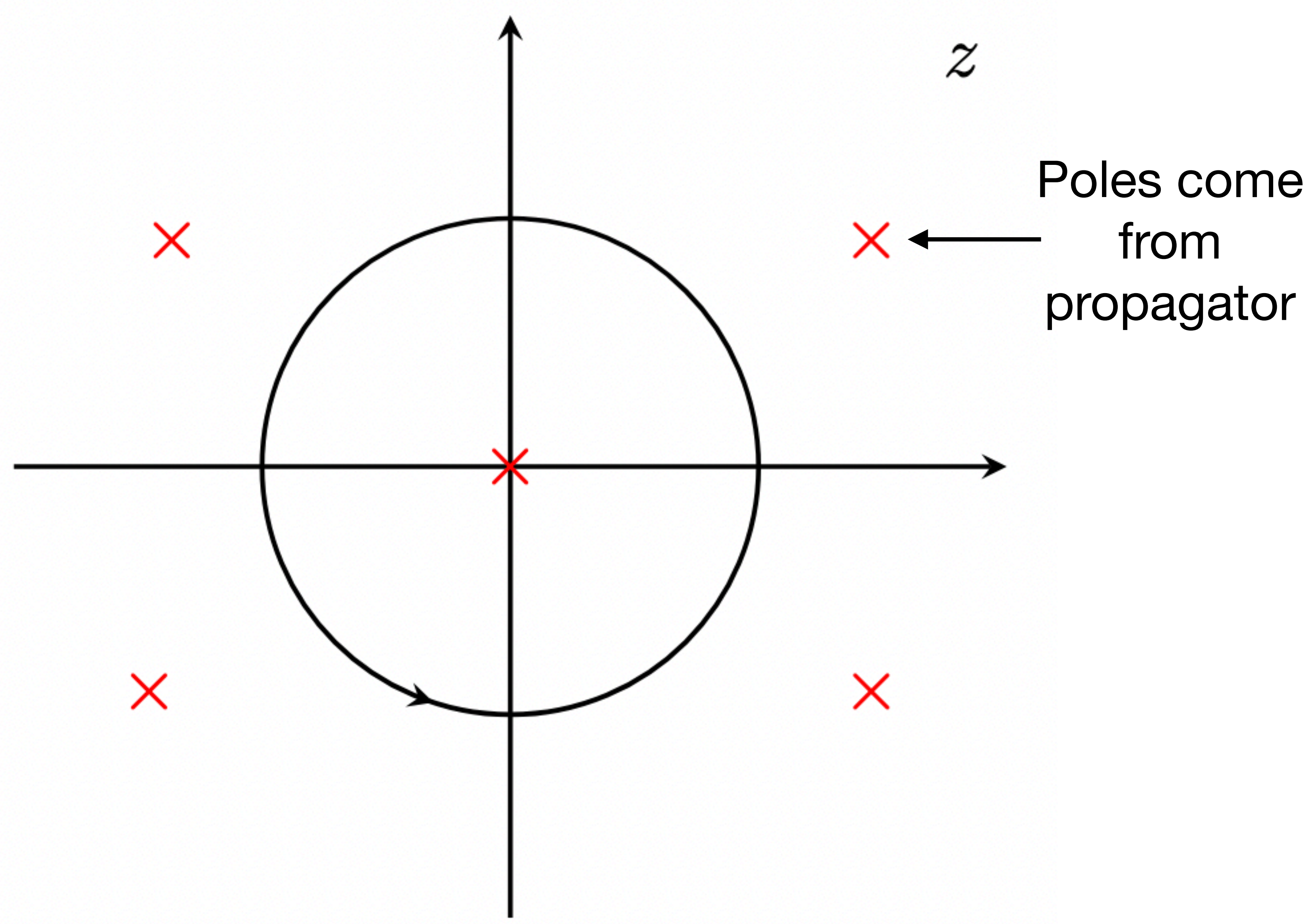
- Residue theorem

$$A_n = \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z} \hat{A}_n(z)$$



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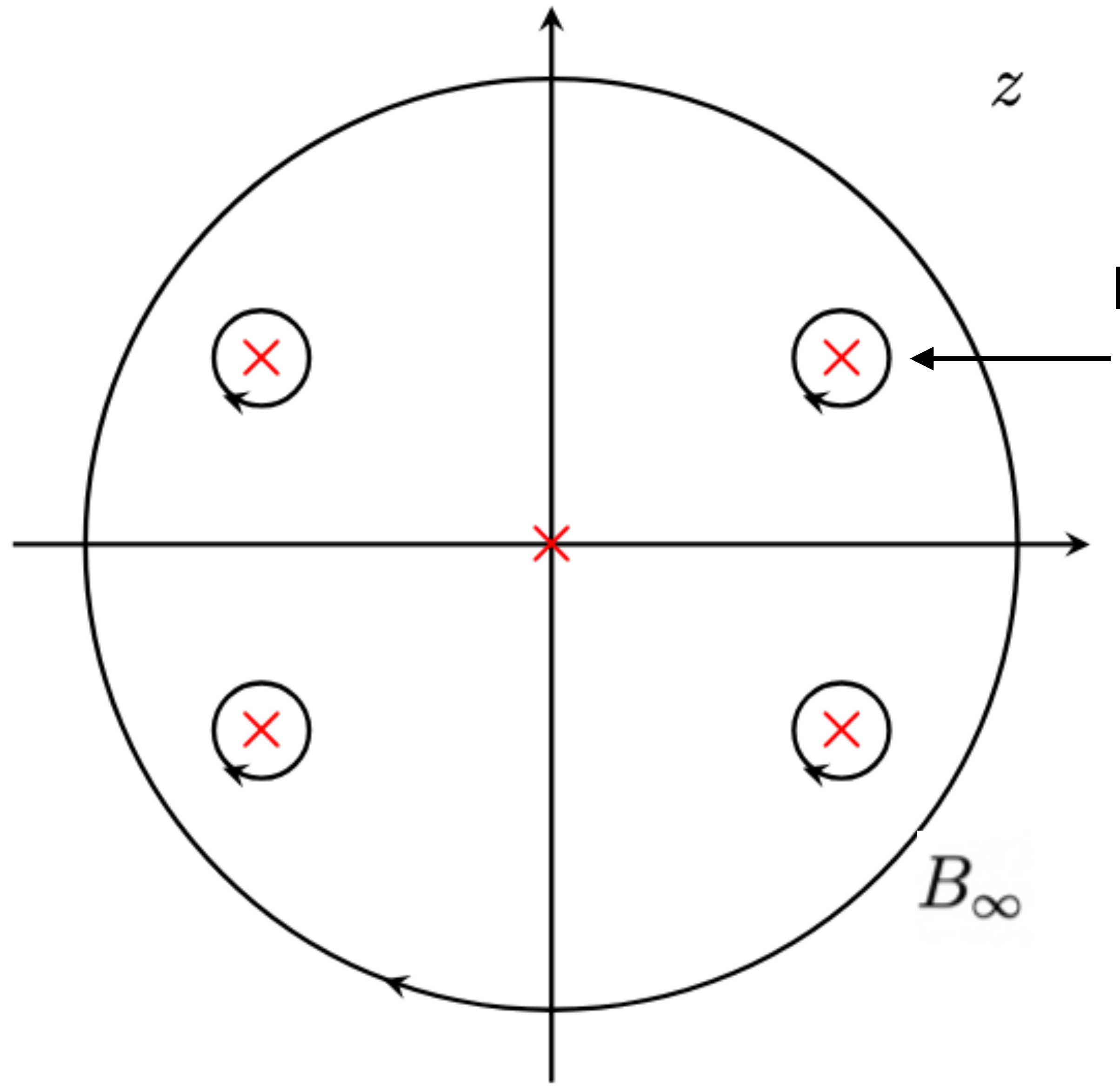
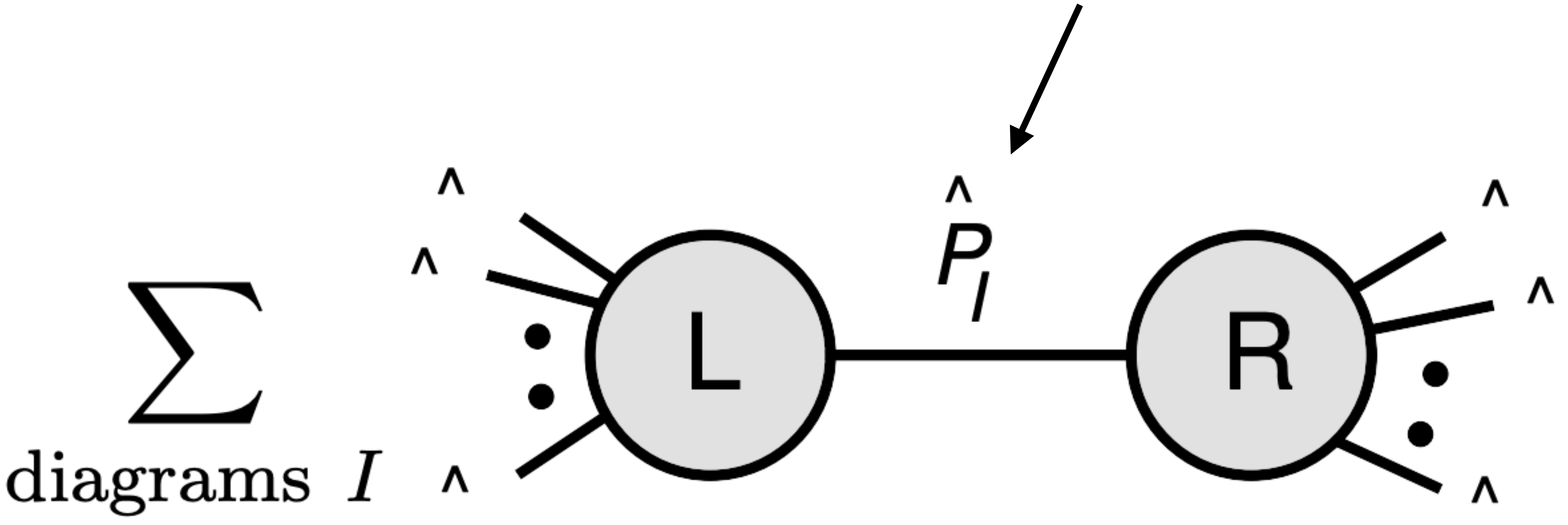
$$\begin{aligned} A_n &= \frac{1}{2\pi i}\oint_{z=0} \frac{dz}{z}\hat{A}_n(z) \\ &= -\sum_{\{z_i\}} \mathrm{Res}\left[\frac{\hat{A}_n(z)}{z}\right] + B_\infty\,. \end{aligned}$$



Factorization

$$A_n = \frac{1}{2\pi i}\oint_{z=0}\frac{dz}{z}\hat{A}_n(z) = -\sum_{\{z_I\}}\text{Res}\left[\frac{\hat{A}_n(z)}{z}\right] + B_\infty,$$

Intermediate particle go on-shell



Higher-point amplitude factorizes into lower-point on-shell amplitude

Boundary Term

$$A_n = \frac{1}{2\pi i}\oint_{z=0}\frac{dz}{z}\hat{A}_n(z) = -\sum_{\{z_I\}}\text{Res}\left[\frac{\hat{A}_n(z)}{z}\right] + \underline{B_\infty},$$

On-shell constructible if boundary term goes to zero

If $B_\infty = 0$: combine lower-point to construct higher-point

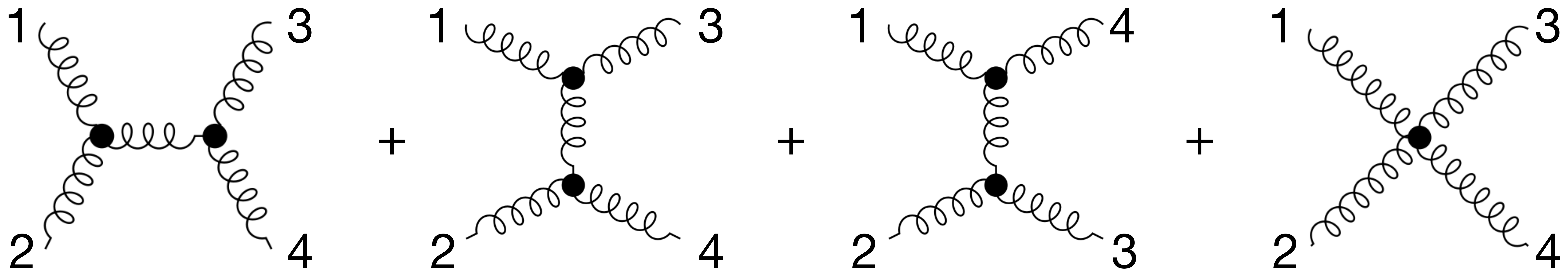
$$\lim_{z\to\infty}\hat{A}(z) = 0 \iff B_\infty = 0 \iff \text{Constructible}\,.$$

If $B_\infty \neq 0$: additional contact terms are allowed and not fixed by symmetries of lower-point amplitude

What on-shell constructibility means?

Gluon vs Scalar amplitude

$A_4^{gluon} =$



$$\frac{A_s}{s} + \frac{A_t}{t} + \frac{A_u}{u} + A_4^{contact}$$

dependent contact term

Gluon amplitude also satisfy Ward identity: $A_4^{gluon}\Big|_{\epsilon_i \to p_i} = 0$

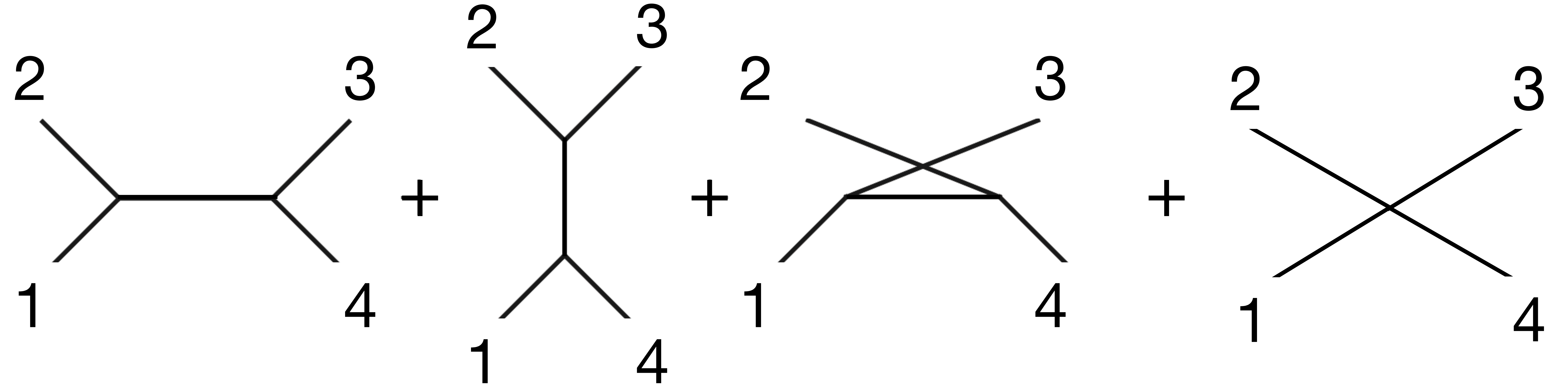
Ward identity implies four diagrams above are ***dependent***.

It can be shown only 3-point information is enough for on-shell constructibility.

What on-shell constructibility means?

Gluon vs Scalar amplitude

$A_4^{scalar} =$



$$A_4^{scalar} = \frac{A_s}{s} + \frac{A_t}{t} + \frac{A_u}{u} + A_4^{contact}$$

independent contact term

Scalar amplitude DO NOT satisfy Ward identity: $A_4^{scalar}\Big|_{\epsilon_i \to p_i} \neq 0$

Ward identity implies four diagrams above are ***independent***.

It can be shown both 3-point and 4-point information required for on-shell constructibility.

All-line Transverse Shift

[Ema, Gao, Ke, Liu, Lyu, IM 24]

- Shift external momentum by **transverse polarization**

$$\hat{p}_i = p_i + z\, c_i\, m_i\, \epsilon_i^{(\pm)}$$

- **On-shell** condition satisfied: $\hat{p}_i^2 = m_i^2$

- c_i fixed by **conservation of momentum**:

$$\sum_i \hat{p}_i = 0$$

polarization vectors

$$\epsilon_i^+ \cdot \epsilon_i^+ = \epsilon_i^- \cdot \epsilon_i^- = \epsilon_i^{(L)} \cdot \epsilon_i^\pm = 0$$

$$\epsilon_i^+ \cdot \epsilon_i^- = \epsilon_i^{(L)} \cdot \epsilon_i^{(L)} = -1$$

Large-z Behavior

$$A_n \sim \left(\sum_{\text{diag}} g \times F \right) \times \prod_{\text{vector}} \epsilon \times \prod_{\text{fermion}} u \,.$$

- n-point amplitude has mass dimension $4-n$.

$$4 - n = [g] + [F] + \frac{N_F}{2} \,.$$

- ALT: ϵ, u not shifted, picks up at most one z for each mass-dimension at $z \to \infty$.

$$\hat{A}_n(z) \sim z^\gamma, \, \gamma \leq [F] = 4 - n - [g] - \frac{N_F}{2} \,.$$

When $\gamma < 0$, an amplitude is ***on-shell constructible***

Massive QED is on-shell constructible with the shift for $n \geq 4$

Massive QED

$$A_4 = -\frac{1}{p_{12}^2}\frac{1}{z_{12}^+ - z_{12}^-}\sum_\lambda \left[z_{12}^- \left[\hat{A}_3(\mathbf{12}I^\lambda)\times\hat{A}_3(\mathbf{34}I^{-\lambda})\right]_{z_{12}^+} - (z_{12}^+ \leftrightarrow z_{12}^-)\right]$$

$$= e^2 \frac{\langle\mathbf{13}\rangle[\mathbf{24}] + \langle\mathbf{14}\rangle[\mathbf{23}] + (\mathbf{1}\leftrightarrow\mathbf{2})}{p_{12}^2}\,.$$

This correctly reproduces the Feynman diagrammatic result.

No contact term ambiguity as opposed to “gluing” without momentum shift. [Christensen+ 22; Lai+ 23]

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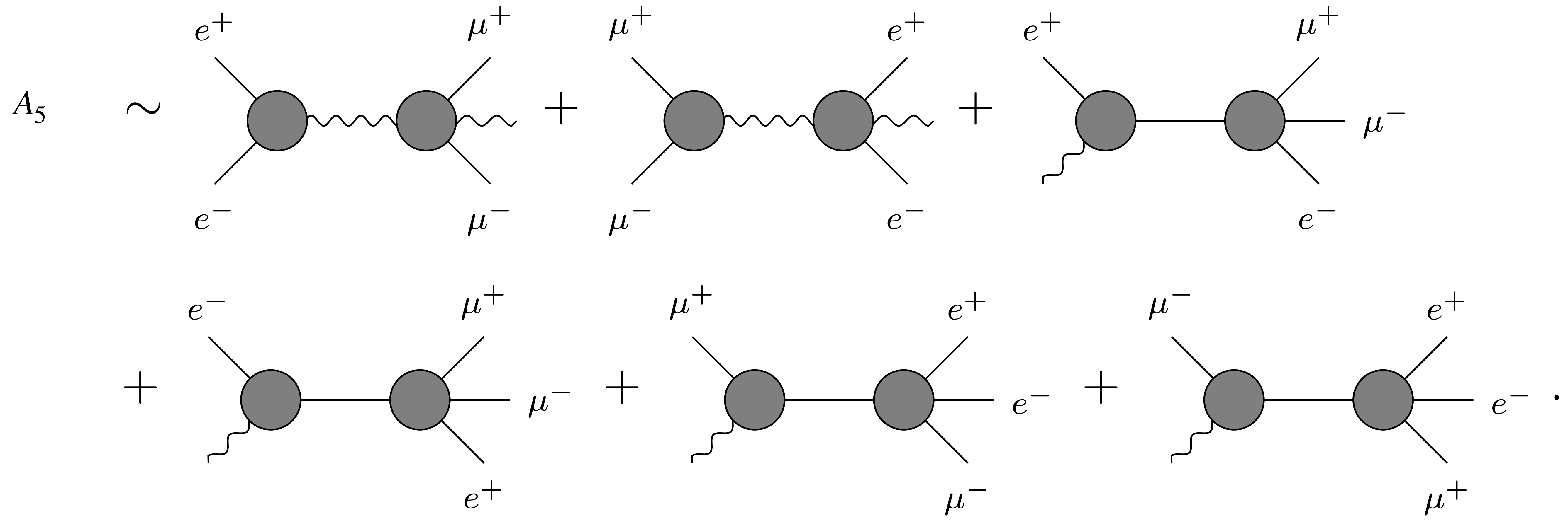
One can repeat the computation for e.g. 5pt $ee\mu\mu\gamma$ amplitude.



Massive Spin-1 Amplitude

4pt gauge boson naively leads to $\gamma \leq 4 - n - \frac{N_F}{2} = 0.$

BUT, the actual behavior better thanks to Ward identity.

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Ward Identity

$$A_4 = \epsilon^{\mu}_{1-}\epsilon^{\nu}_{2L}\epsilon^{\lambda}_{3L}\epsilon^{\sigma}_{4L}F_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$

ALT Shift: $\hat{p}_i = p_i + z\, c_i\, m_i\, \epsilon_i^{(\pm)}$

$$\lim_{z\to\infty} A_4 \sim \epsilon^{\mu}_{1-}\epsilon^{\nu}_{2L}\epsilon^{\lambda}_{3L}\epsilon^{\sigma}_{4L}F_{\mu\nu\lambda\sigma}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \longleftarrow$$

Goes to zero due to Ward identity.
Thus leading z behavior vanishes

$V_L V_L V_L V_T$ is on-shell constructible

Massive Spin-1 Amplitude

Ward Identity

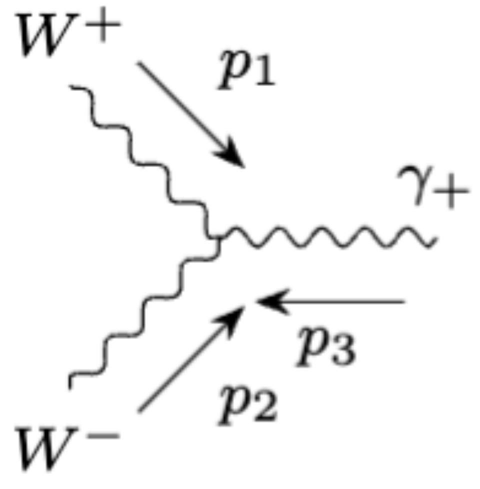
- $V_L V_L V_L V_L$ amplitude is not on-shell constructible (Similar to $\lambda\phi^4$ term)
- $V_L V_L V_L V_T$ is on-shell constructible if the amplitude satisfy **Ward identity in the massless** limit

Little group covariance connects these amplitudes

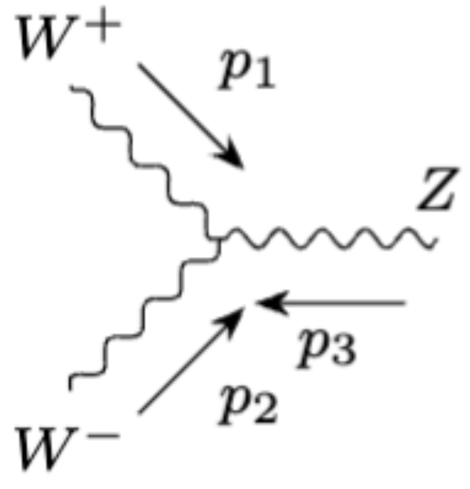
Electroweak Amplitude

- Gauge boson amplitude can be build using only three-point interaction

3-point input:



$$= \frac{g_{WW\gamma}}{m_W} x_{12}\langle\mathbf{12}\rangle^2 = \frac{g_{WW\gamma}}{\sqrt{2}m_W^2\langle 3\xi\rangle}\left[\langle\mathbf{12}\rangle[\mathbf{21}]\langle\mathbf{3}|p_1 - p_2|\mathbf{3}] + \text{cycl.}\right]$$



$$= \frac{g_{WWZ}}{\sqrt{2}m_W^2 m_Z}\left[\langle\mathbf{12}\rangle[\mathbf{21}]\langle\mathbf{3}|p_1 - p_2|\mathbf{3}] + \text{cycl.}\right]$$

- We can all-line transverse shift to construct 4-point amplitude.

Higgs interaction is not required to construct amplitude

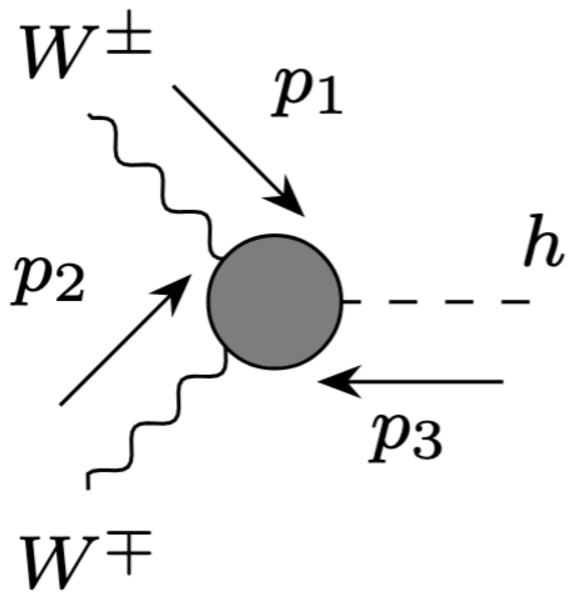
UV Completion

- These amplitudes has bad-high energy behavior.

- $W_L W_L W_L W_L$, $W_L W_L Z_L Z_L$ scale as $\mathcal{O}(E^2)$

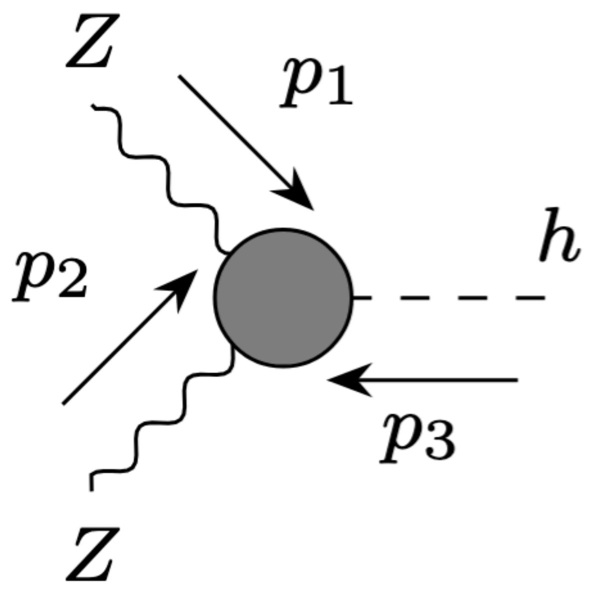
- In the simplest model, they can be controlled by introducing a neutral scalar

$A^{(\lambda_1\lambda_2)}_{WWh} =$



$= \dfrac{g_{WWh}}{m_W^2}\langle \mathbf{12}\rangle[\mathbf{21}]$

$A^{(\lambda_1\lambda_2)}_{ZZh} =$



$= \dfrac{g_{ZZh}}{m_Z^2}\langle \mathbf{12}\rangle[\mathbf{21}].$

- Demanding smooth massless limit imposes constraints on the couplings

$$g_{WWh} = g m_W, \quad g_{ZZh} = \frac{g m_Z^2}{m_W}, \quad \rho = \frac{g^2 m_W^2}{g_{WWZ}^2 m_Z^2} = 1.$$

Summary of Results

- On-shell method: alternative way of computing scattering amplitudes.
- ALT shift, works for massive scattering amplitudes.
- Applying to QED correctly reproduces amplitudes, resolving previous confusion.
- Applying to EW theory reproduces amplitudes including gauge 4pt contact term.
- **Bottom up**: broken supergravity is the unique, effective theory involving interactions of massive spin-3/2 fermions valid up to a cutoff $\Lambda >> m_{3/2}$

[Gherghetta, Ke '25]

THANK YOU!

Back up

All-line Transverse Shift

- Shift external momentum by respective transverse polarization for spin-1/2 and spin-1

$$p_i \to \hat{p}_i = p_i + z\frac{c_i m_i}{\sqrt{2}}\epsilon_i^{(I_i)},$$

- Define shift for spin-1/2 and spin-1 transverse modes

$$|i\rangle \to |\hat{i}\rangle = |i\rangle + zc_i|\eta_i\rangle \quad \text{for} \quad I_i = +,$$
$$|i] \to |\hat{i}] = |i] + zc_i|\eta_i] \quad \text{for} \quad I_i = -,$$

On-shell condition satisfied

$$\langle \hat{i}\eta_i\rangle = [\hat{i}\eta_i] = m_i$$

All-line Transverse Shift

- The shift is defined for spin-1 longitudinal modes

$$\begin{cases} |i\rangle \to |\hat{i}\rangle = |i\rangle + z\dfrac{c_i}{2}|\eta_i\rangle, \\ [\eta_i| \to [\hat{\eta}_i| = [\eta_i| - z\dfrac{c_i}{2}[i|, \end{cases} \quad \text{or} \quad \begin{cases} [i| \to [\hat{i}| = [i| + z\dfrac{c_i}{2}[\eta_i|, \\ |\eta_i\rangle \to |\hat{\eta}_i\rangle = |\eta_i\rangle - z\dfrac{c_i}{2}|i\rangle, \end{cases} \quad \text{for} \quad I_i = L,$$

- The definition can be extened for massless legs
- Shift defined in helicity basis and does not require specific spin-axis choice
- The shift exists for all possible spin configurations

Polarization tensors

Massless:

$$\epsilon^{(+)}_{a\dot{a}} = \sqrt{2}\frac{|\xi\rangle_a[i|_{\dot{a}}}{\langle i\xi\rangle}, \quad \epsilon^{(-)}_{a\dot{a}} = \sqrt{2}\frac{|i\rangle_a[\xi|_{\dot{a}}}{[i\xi]}$$

Massive:

$$\epsilon^{(+)}_{a\dot{a}} = \sqrt{2}\frac{|\eta_i\rangle_a[i|_{\dot{a}}}{m_i}, \quad \epsilon^{(-)}_{a\dot{a}} = \sqrt{2}\frac{|i\rangle_a[\eta_i|_{\dot{a}}}{m_i}, \quad \epsilon^{(L)}_{a\dot{a}} = \frac{|i\rangle_a[i|_{\dot{a}} + |\eta_i\rangle_a[\eta_i|_{\dot{a}}}{m_i}$$

Momentum:

$$(p_i)_{a\dot{a}} = |i\rangle_a[i|_{\dot{a}} - |\eta_i\rangle_a[\eta_i|_{\dot{a}}$$

Explicit form of spinors In helicity basis:

$$|i\rangle_a = \sqrt{E_i + p_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix}, \quad [i|_{\dot{a}} = \sqrt{E_i + p_i}\begin{pmatrix} -s_i & c_i \end{pmatrix}$$

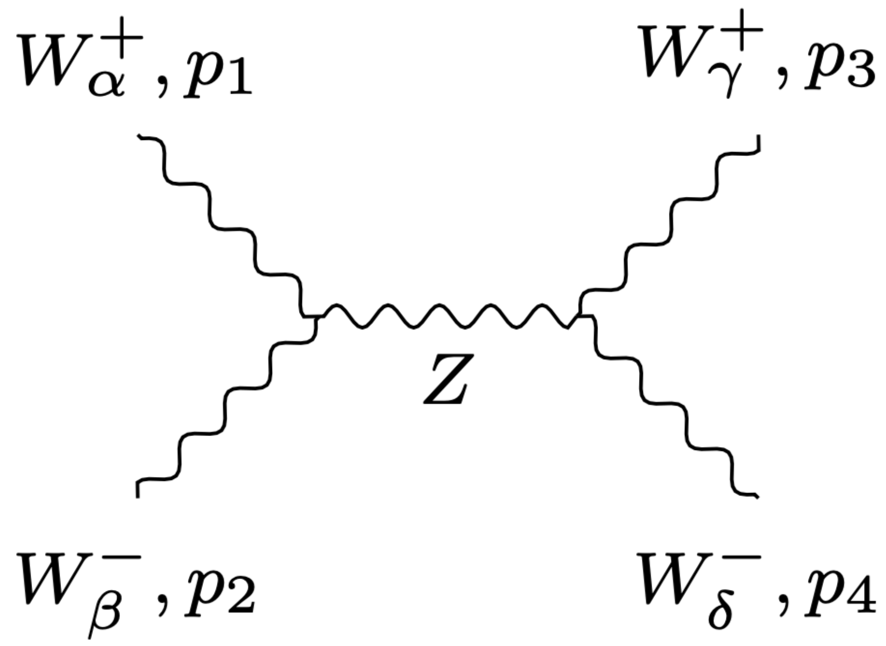
$$|\eta_i\rangle_a = \sqrt{E_i - p_i}\begin{pmatrix} c_i \\ s_i \end{pmatrix}, \quad [\eta_i|_{\dot{a}} = -\sqrt{E_i - p_i}\begin{pmatrix} c_i & s_i^* \end{pmatrix}$$

3-point QED:

$$\begin{aligned} A_3(\mathbf{1}\mathbf{2}3^+) &= \tilde{e}x_{12}\langle\mathbf{1}\mathbf{2}\rangle \\ &= -\tilde{e}\frac{\langle\mathbf{1}\xi\rangle[3\mathbf{2}] + \langle\mathbf{2}\xi\rangle[3\mathbf{1}]}{\langle\xi 3\rangle} \end{aligned}$$

$WWWW$ Calculations

Example calculation:



$$
\begin{aligned}
A_s^Z(z)\Big|_{\hat{p}_{12}^2=m_Z^2} =& \frac{g_{WWZ}^2}{m_W^4}\Big(2(\hat{p}_1-\hat{p}_2)\cdot(\hat{p}_3-\hat{p}_4)\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{3}}\hat{\mathbf{4}}\rangle[\hat{\mathbf{4}}\hat{\mathbf{3}}] \\
&-2\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{3}}|\hat{p}_1-\hat{p}_2|\hat{\mathbf{3}}]\langle\hat{\mathbf{4}}|\hat{p}_3|\hat{\mathbf{4}}]+2\langle\hat{\mathbf{1}}\hat{\mathbf{2}}\rangle[\hat{\mathbf{2}}\hat{\mathbf{1}}]\langle\hat{\mathbf{4}}|\hat{p}_1-\hat{p}_2|\hat{\mathbf{4}}]\langle\hat{\mathbf{3}}|\hat{p}_4|\hat{\mathbf{3}}]+(\hat{1},\hat{2})\leftrightarrow(\hat{3},\hat{4}) \\
&+4\langle\hat{\mathbf{1}}\hat{\mathbf{3}}\rangle[\hat{\mathbf{3}}\hat{\mathbf{1}}]\langle\hat{\mathbf{2}}|\hat{p}_1|\hat{\mathbf{2}}]\langle\hat{\mathbf{4}}|\hat{p}_3|\hat{\mathbf{4}}]-4\langle\hat{\mathbf{1}}\hat{\mathbf{4}}\rangle[\hat{\mathbf{4}}\hat{\mathbf{1}}]\langle\hat{\mathbf{2}}|\hat{p}_1|\hat{\mathbf{2}}]\langle\hat{\mathbf{3}}|\hat{p}_4|\hat{\mathbf{3}}]-(\hat{1}\leftrightarrow\hat{2})\Big)
\end{aligned}
\quad (3.4)
$$

Need to sum over two poles at this factorization channel

Ward identity and simplification

$$A_4=\epsilon_{1-}^{\mu}\epsilon_{2L}^{\nu}\epsilon_{3L}^{\lambda}\epsilon_{4L}^{\sigma}F_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)$$

As $z\to\infty$, for ALT shift $\qquad r_1^{\mu}F_{\mu\nu\lambda\sigma}(r_1,r_2,r_3,r_4)$ ← Goes to zero due to **Ward identiy.** Thus leading z behavior vanishes

Massive Spinor Variables

SU(2) little group index

$$[\mathbf{i}|^I_{\dot{a}} \qquad |\mathbf{i}\rangle^I_a$$

Chiral Index

Spin index $I = 1{,}2$ represents the little group freedom (choice of spin-axis)

We work in the *helicity basis*

$$|\mathbf{i}\rangle^I_a = |i\rangle_a \delta^I_- + |\eta_i\rangle_a \delta^I_+$$

$$[\mathbf{i}|^I_{\dot{a}} = [i|_{\dot{a}} \delta^I_+ + [\eta_i|_{\dot{a}} \delta^I_-$$

On-shell condition

$$\langle i\eta_i\rangle = [i\eta_i] = m_i$$

In the HE limit $\eta \to 0$

Massive BCFW-type Shift

[R. Franken, C. Schwinn, arXiv:1910.13407]

[C. Wu, S. H. Zhu, arXiv: 2112.12312]

Decompose momentum p_i, p_j as linear combination of two null-momentum

$$p_i = l_i + \frac{m_i^2}{2l_j \cdot l_i} l_j, \quad p_j = l_j + \frac{m_j^2}{2l_j \cdot l_i} l_i$$

Shift the spinors as

$$[\mathbf{i}, \mathbf{j}\rangle \text{ shift} : \begin{cases} [\hat{\mathbf{i}}|^2 = [\mathbf{i}|^2 - z[\mathbf{j}|^2, \\ |\hat{\mathbf{j}}\rangle^1 = |\mathbf{j}\rangle^1 + z|\mathbf{i}\rangle^1 \end{cases}$$

Good shift only exists for spin-projection $(s_i, s_j) = (1,2)$ or $(2,1)$

Need to use spin raising or lowering operator to construct amplitude

All-line Transverse Shift

- Shift external momentum by transverse polarization

$$p_i \to \hat{p}_i = p_i + z\frac{c_i m_i}{\sqrt{2}}\epsilon_i^{(I_i)},$$

- We further impose: $\hat{\epsilon}_i = \epsilon_i, \hat{u} = u, \hat{\bar{v}} = \bar{v}$

- This condition improves large-z behavior since amplitudes are linear in polarization and Dirac spinor

- Shift by same polarization as of the external state

- For longitudinal, shift by either polarization

polarization vectors

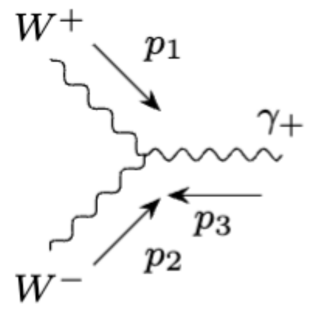
$$\epsilon_i^+ \cdot \epsilon_i^+ = \epsilon_i^- \cdot \epsilon_i^- = \epsilon_i^{(L)} \cdot \epsilon_i^{\pm} = 0$$

$$\epsilon_i^+ \cdot \epsilon_i^- = \epsilon_i^{(L)} \cdot \epsilon_i^{(L)} = -1$$

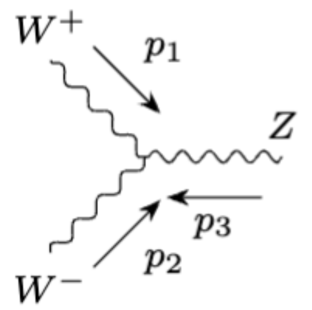
Massive Spin-1 Amplitude

- Consider $WWWW$ four-point tree level amplitude without Higgs

3-point input



$$= \frac{g_{WW\gamma}}{m_W} x_{12}\langle \mathbf{12}\rangle^2 = \frac{g_{WW\gamma}}{\sqrt{2}m_W^2\langle 3\xi\rangle}[\langle\mathbf{12}\rangle[\mathbf{21}]\langle\mathbf{3}|p_1-p_2|\mathbf{3}] + \text{cycl.}]$$



$$= \frac{g_{WWZ}}{\sqrt{2}m_W^2 m_Z}[\langle\mathbf{12}\rangle[\mathbf{21}]\langle\mathbf{3}|p_1-p_2|\mathbf{3}] + \text{cycl.}]$$

- Apply All-line transverse shift to construct 4-point amplitude

$$A_4 = \frac{A_s^Z(0)}{p_{12}^2 - m_Z^2} + \frac{A_t^Z(0)}{p_{14}^2 - m_Z^2} + \frac{A_s^\gamma(0)}{p_{12}^2} + \frac{A_t^\gamma(0)}{p_{14}^2} + A_{contact}$$

Constructible if for at least one transeverse mode

Little group covariance allows us to find all longitudinal amplitude

$WWhh$, $ZZhh$ contact term not required as input as well

$WWWW$ Contact term not required as input