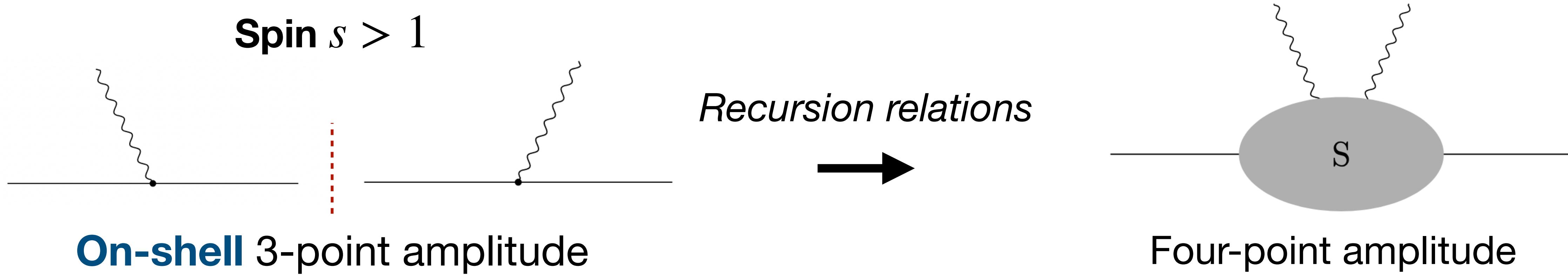


On-shell recursion relations for higher-spin Compton amplitudes

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Based on [2506.02106](#)
with Yohei Ema, Ting Gao, Wenqi Ke, Zhen Liu.

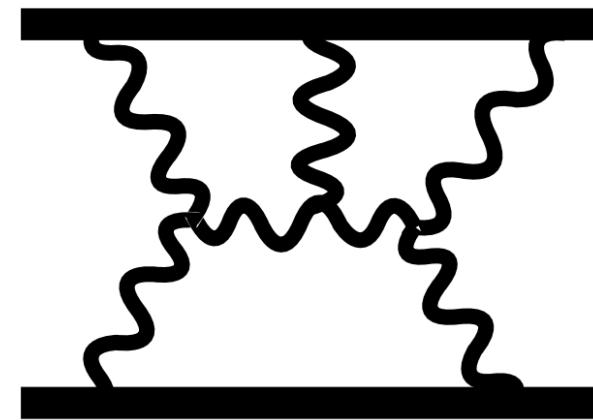
Can recursion relations fix Higher-Spin Compton Amplitudes?



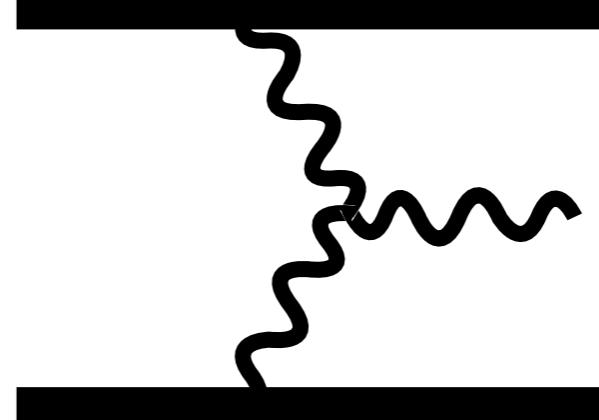
**Why on-shell methods?
Consistent higher-spin Lagrangian is difficult**

Motivations

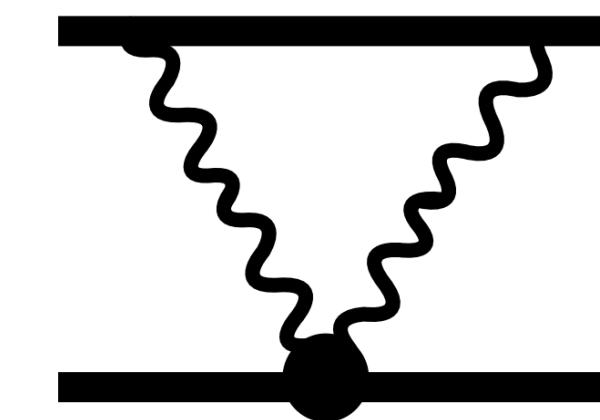
Gravitational Wave Physics



Scattering Angle, Impulse



Radiation



Finite Size Effects

[Buonanno, Khalil, O'Connell,
Roiban, Solon, Zeng,
Snowmass '22]

- The era of gravitational-wave physics requires theoretical frameworks for precision calculations.

✓ **Advances in scalar amplitudes:** increasingly high-order results powering precise computations.

✓ **5PM (four-loop)** results scattering angle and impulse of classical black hole scattering

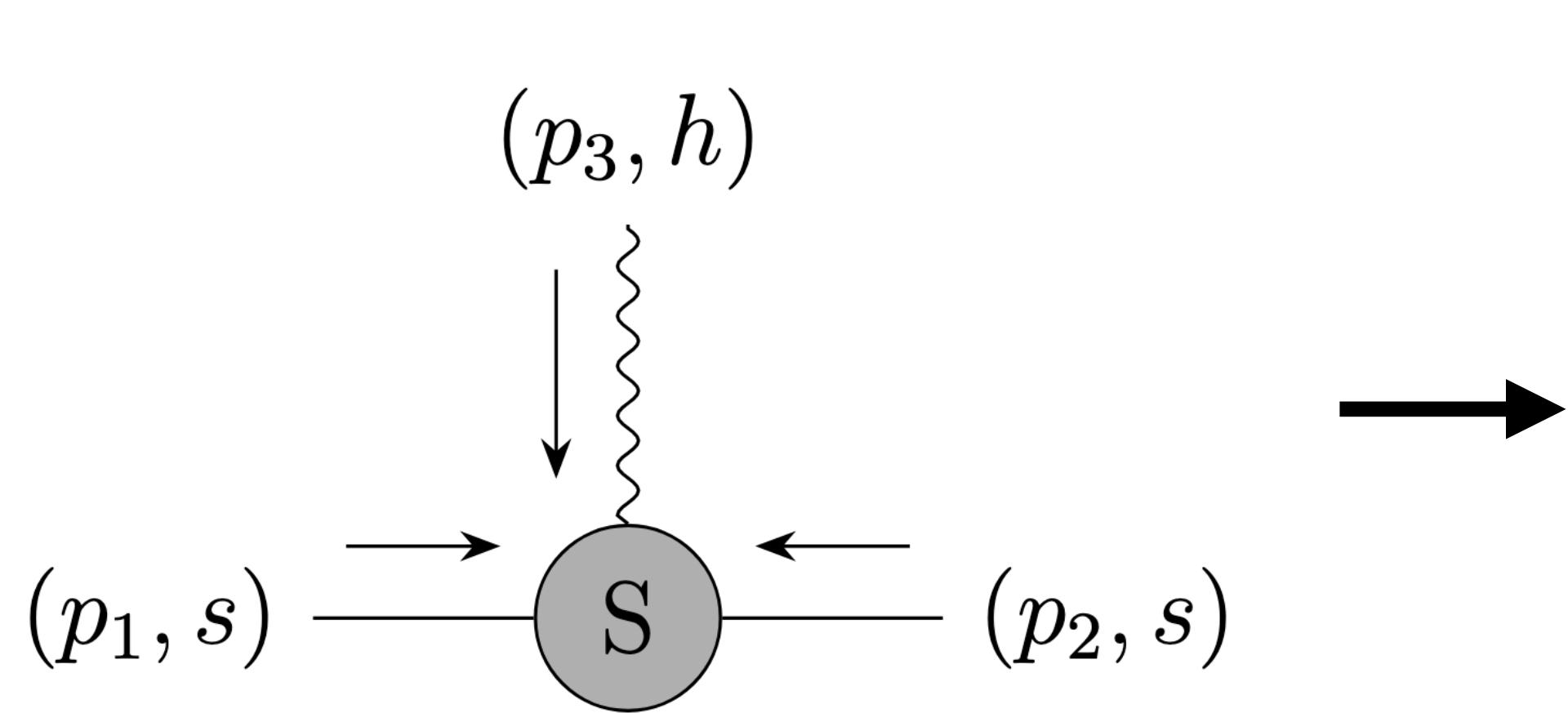
[Driesse, Jakobsen, Mogull,
Plefka, Sauer, Usovitsch '24]

Motivations

Spin and rotating objects?

- Interaction of spin degrees of freedom with graviton
- Kerr-black hole should be simplest to model since only observables depend on mass and spin
- “Roughly” the classical spin $a^\mu = \hbar s^\mu$, keeping $\hbar \rightarrow 0, s \rightarrow \infty$
- **How to write down amplitudes for higher-spin consistently**
- **Formally understand the what amplitude describes the physics of Kerr-black hole**

Three-point Amplitude



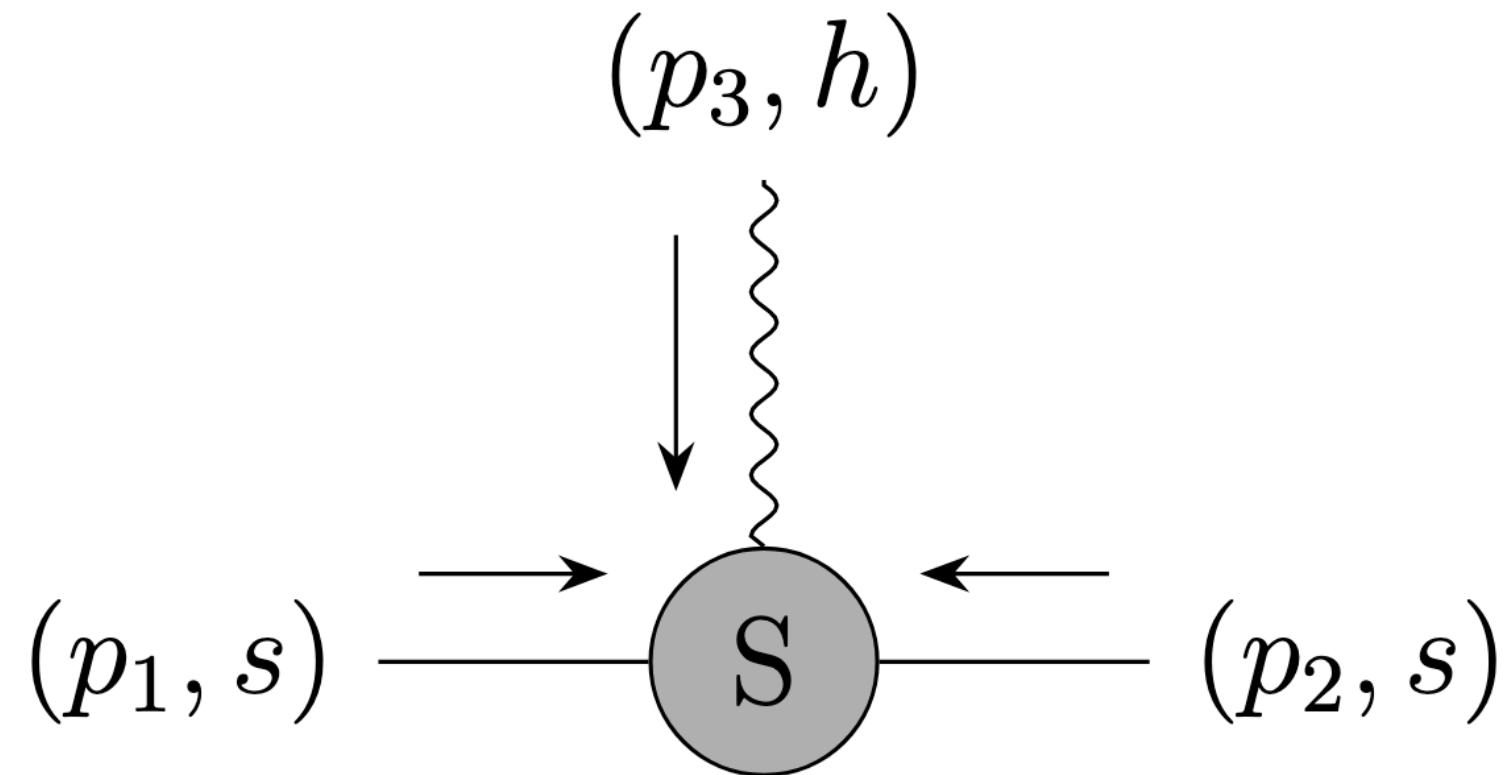
Scalar 3-pt amplitude

$$\mathcal{A}(s, s, h^+) \sim -ig_{h,s}(p_1 \cdot \epsilon_3^+)^h \langle \mathbf{12} \rangle^{2s},$$
$$\mathcal{A}(s, s, h^-) \sim -ig_{h,s}(p_1 \cdot \epsilon_3^-)^h [\mathbf{12}]^{2s}.$$

[Arkani-Hamed, Huang, Huang '17]

For $s = 0, 1/2, 1$ and $h = 1 \iff \text{SM}$

Three-point Amplitude



Scalar 3-pt amplitude

$\mathcal{A}(s, s, h^+) \sim -ig_{h,s}(p_1 \cdot \epsilon_3^+)^h \langle \mathbf{12} \rangle^{2s},$

$\mathcal{A}(s, s, h^-) \sim -ig_{h,s}(p_1 \cdot \epsilon_3^-)^h [\mathbf{12}]^{2s}.$

[Arkani-Hamed, Huang, Huang '17]

For $s = 0, 1/2, 1$ and $h = 1 \iff$ SM

Dipole moment, $g = 2$

Spin Universality

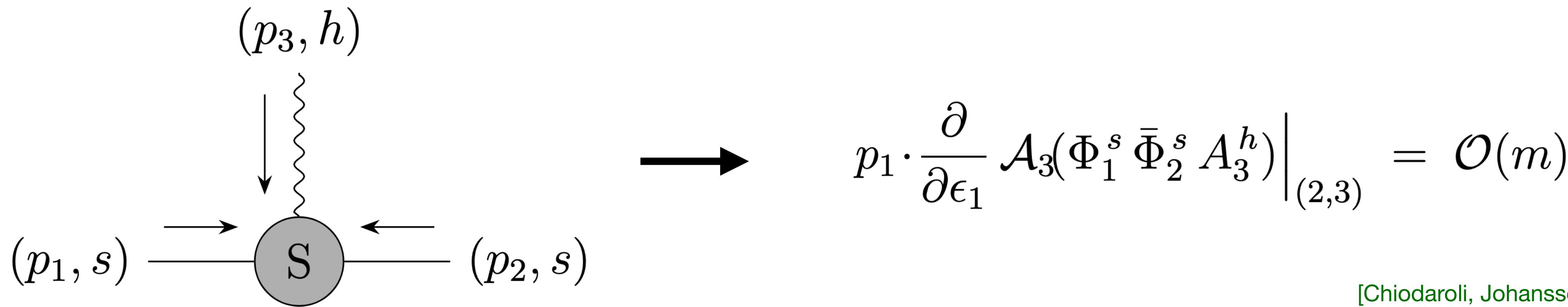
$$\mathcal{A}(2s, 2s, h = 2) \sim \mathcal{A}(s, s, h = 1)\mathcal{A}(s, s, h = 1)$$

Operator dim for photon coupling

Spin s	0-1	$\frac{3}{2}$	2	...
Dimension	4	5	6	

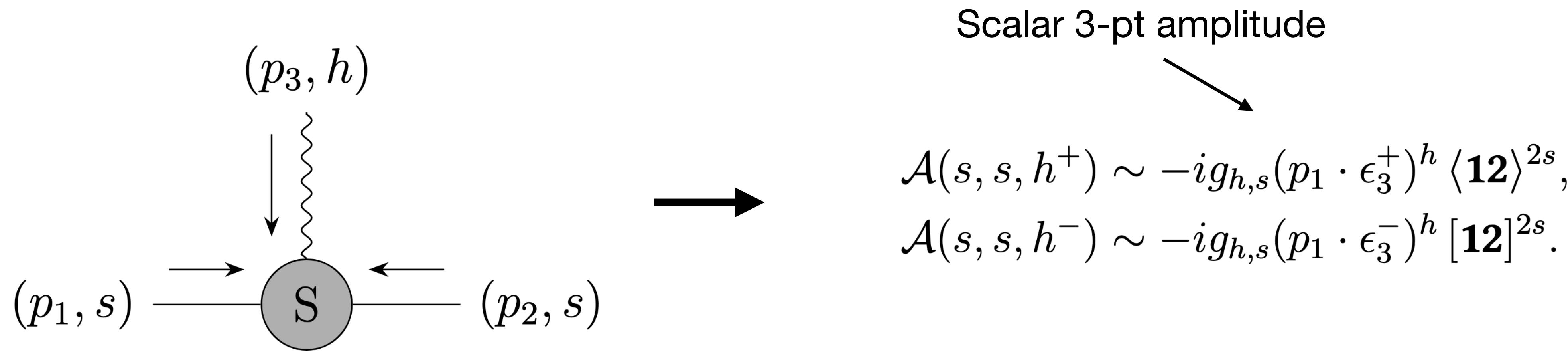
Off-shell Current Constraint

$$\partial_\mu J^\mu = \mathcal{O}(m)$$



- $\partial_\mu J^\mu = \mathcal{O}(m)$ →
- The spin- s propagator does not diverge as $m \rightarrow 0$
 - Recover Ward identity in $m \rightarrow 0$
 - Reduces $1/m$ divergences from amplitude

Classical Limit

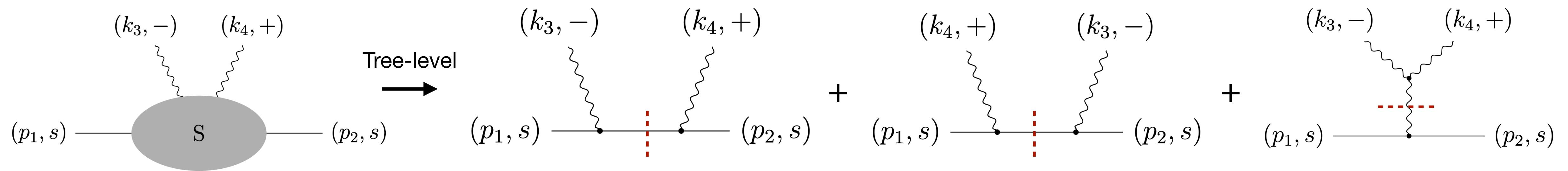


- **Describes effective amplitude for Kerr-black hole**

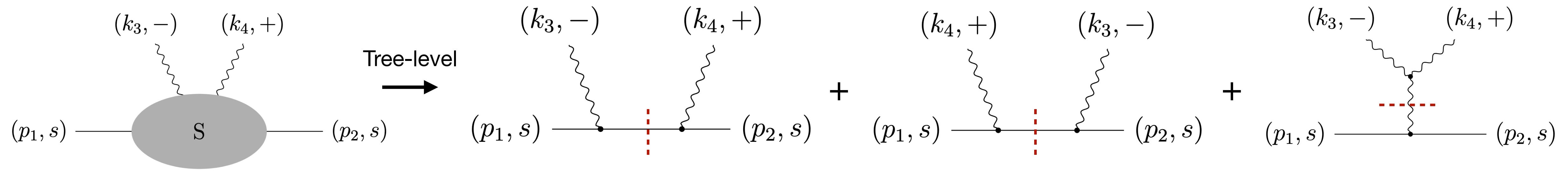
$$\mathcal{A}_{\text{cl. 3pt}} \sim \varepsilon_{k,\mu\nu} T_{\text{Kerr}}^{\mu\nu} \sim (p \cdot \varepsilon_k)^2 e^{k \cdot a}$$

[Guevara, Ochirov, Vines '17]

Higher-spin Compton Amplitude



Higher-spin Compton Amplitude



On-shell techniques: gluing, BCFW recursion relation

$$\mathcal{A}_4(\Psi_1^s \bar{\Psi}_2^s h_3^-, h_4^+) = \frac{([4\mathbf{1}]\langle 32 \rangle + [4\mathbf{2}]\langle 31 \rangle)^{2s}}{s_{12} t_{13} t_{14} [4|p_1|3]^{2s-4}}.$$

[Arkani-Hamed, Huang, Huang '17]
[Johansson, Ochirov '19]

- ✓ $s \leq 1$ (external photon)
- ✓ $s \leq 2$ (external graviton)

Spurious pole for $s > 2$ for graviton Compton amplitude

Higher-spin Compton Amplitude

How to deal with spurious poles?

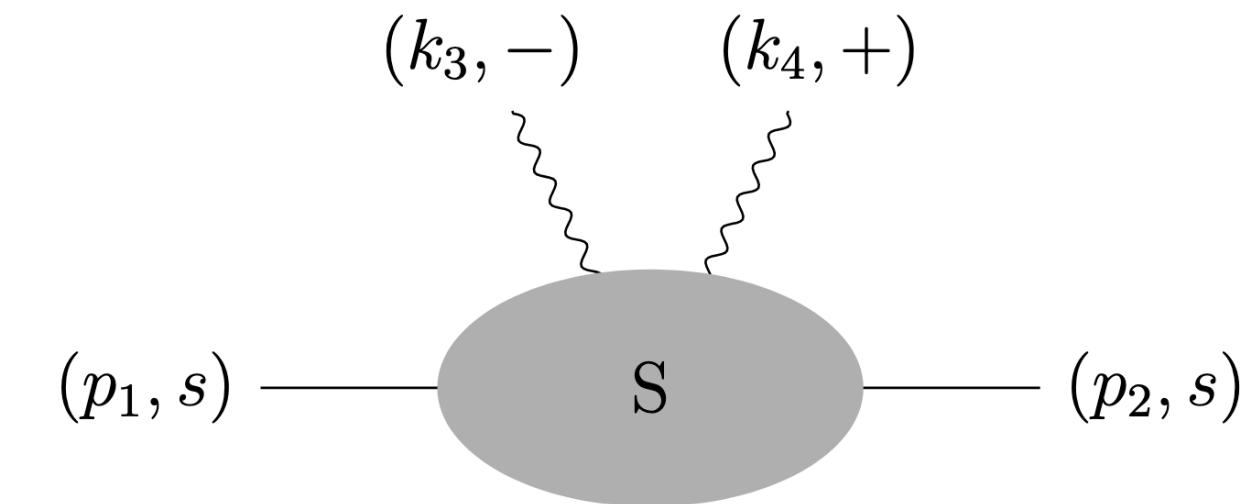
- General relativity calculations through black hole perturbation theory (BHPT) up-to $\mathcal{O}(s^8)$
 - [Bautista, Guevara, Kavanagh, Vines '22]
 - [Bautista, Bonelli, Iossa, Tanzini, Zhou '24]
- Massive gauge symmetry, current constraint to propose candidate four-point Compton amplitude for arbitrary spin
 - [Cangemi, Chiodaroli, Johansson, Ochirov, Pichini, Skvortsov '24]
 - [Bohenblust, Cangemi, Johansson, Pichini '24]
- Classical $\hbar \rightarrow 0$ limit is analyzed by requiring spin-shift symmetry
 - [Aoude, Haddad, Helset '22]
 - [Bern, Kosmopoulos, Luna, Roiban, Teng '22]

All-Line Transverse (ALT) shift

- Shift external momentum by **transverse polarization**

[Ema, Gao, Ke, Liu, Lyu, **IM '24**]

$$\hat{p}_i = p_i + z c_i m_i \epsilon_i^{(\pm)}$$



$\epsilon_i^{(\pm)}$: Polarization of external particle

z : Parameterizes the analytic continuation

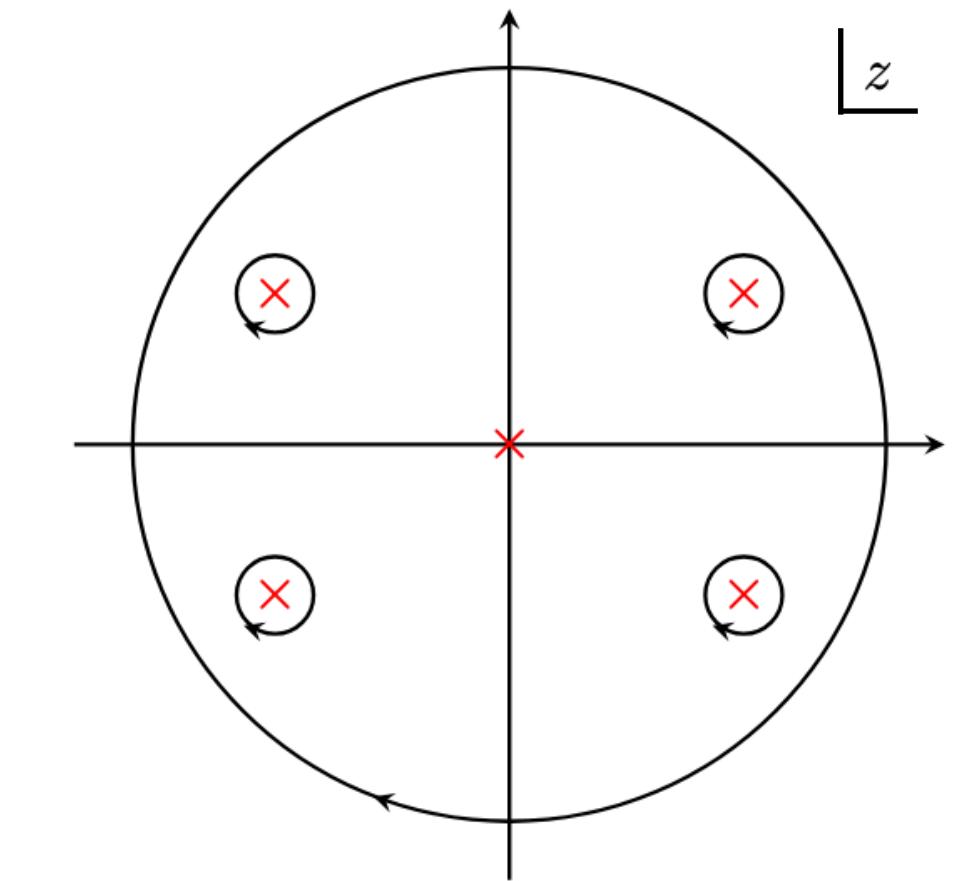
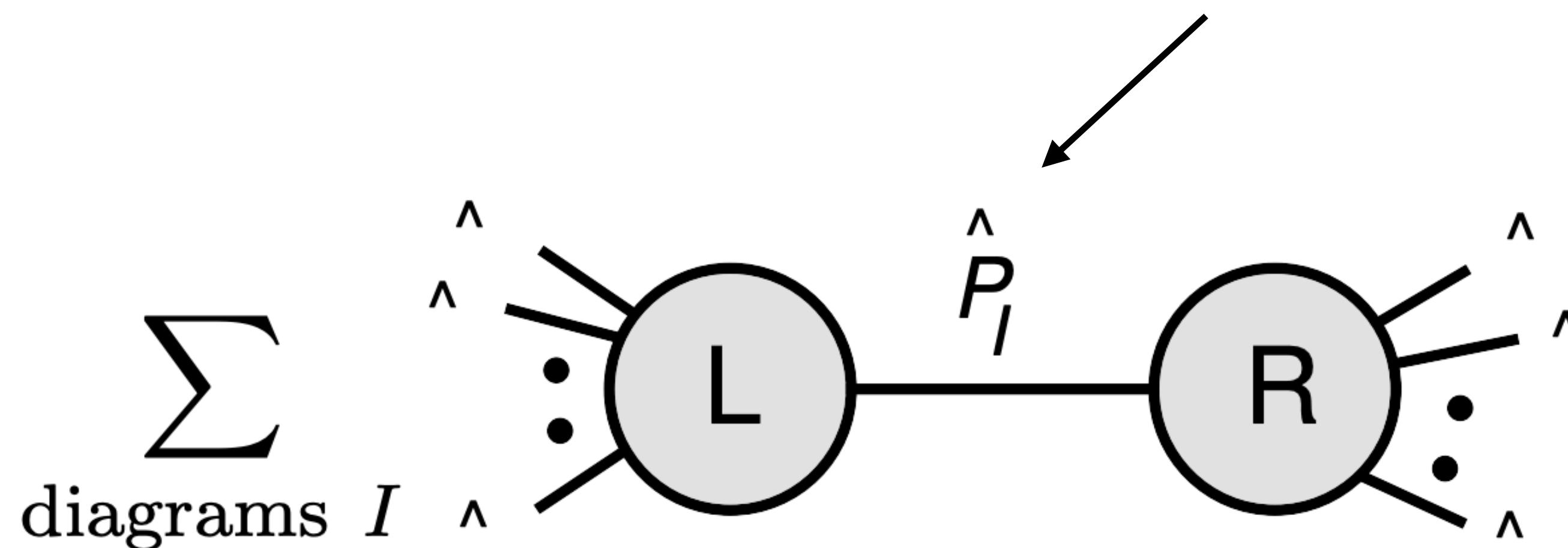
c_i : shift parameters

Since shift parameters can be arbitrary, amplitude should be independent of c_i

Factorization

$$A_n = \frac{1}{2\pi i} \oint_{z=0} dz \frac{\hat{A}_n(z)}{z} = - \sum_{\{z_I\}} \text{Res} \left[\frac{\hat{A}_n(z)}{z} \right] + B_\infty,$$

Intermediate particle go on-shell



Higher-point amplitude factorizes into lower-point on-shell amplitude

Boundary Term

$$A_n = \frac{1}{2\pi i} \oint_{z=0} dz \hat{A}_n(z) = - \sum_{\{z_I\}} \text{Res} \left[\frac{\hat{A}_n(z)}{z} \right] + B_\infty,$$

On-shell constructible

if boundary term goes to zero

If $B_\infty = 0$: combine lower-point to construct higher-point

$$\lim_{z \rightarrow \infty} \hat{A}(z) = 0 \iff B_\infty = 0 \iff \text{Constructible}.$$

If $B_\infty \neq 0$: additional contact terms are allowed and not fixed by symmetries of lower-point amplitude

Large- z analysis is performed by dimensional analysis

Higher-spin Compton Amplitude

- Off-shell current constraint
- Photon/Graviton Ward identity



- ✓ Photon Compton amplitude up-to $s = 3/2$
- ✓ Graviton Compton amplitude upto $s = 5/2$

No spurious poles



- The shift **uniquely** fixes these Compton amplitude
- Uniquely fix Compton amplitude with **multiple Photon/Graviton**
- For even higher-spin, there is still ambiguity

Spin-3/2 Photon Compton Amplitude

Opposite helicity case

$$A_{\psi_{3/2}\bar{\psi}_{3/2}\gamma\gamma}^{(\lambda_1\lambda_2-+)} = - \frac{([1|p_{13}|2\rangle + \langle 1|p_{14}|2]) ([14]\langle 23\rangle + [24]\langle 13\rangle)^2}{2m^2 t_{13} t_{14}} \\ + \frac{([14]\langle 23\rangle + [24]\langle 13\rangle)}{2m^4} \left[\frac{[24]\langle 13\rangle [1|p_{13}|2\rangle}{t_{13}} + \frac{[14]\langle 23\rangle \langle 1|p_{14}|2]}{t_{14}} \right]$$

- Free from **spurious pole**
- $1/m^4$ divergence
- Extend the computation to non-minimal case

- **Uniquely** fixed by recursion
- Matches with literature

[Chiodaroli, Johansson, Pichini '22]

Spin-5/2 Graviton Compton Amplitude

Opposite helicity case

$$\begin{aligned}
A_{\psi_{5/2}\bar{\psi}_{5/2}hh}^{(\lambda_1\lambda_2-+)} &= -\frac{1}{M_P^2} \left[([24]\langle 13 \rangle + [14]\langle 23 \rangle)^3 \frac{[\mathbf{1}|p_{13}|\mathbf{2}\rangle + \langle \mathbf{1}|p_{14}|\mathbf{2}] }{2m^2 t_{34}} \left[\frac{[24]\langle 13 \rangle + [14]\langle 23 \rangle}{t_{13}t_{14}} - \frac{[24]\langle 13 \rangle}{2m^2 t_{13}} - \frac{[14]\langle 23 \rangle}{2m^2 t_{14}} \right] \right. \\
&\quad - ([24]\langle 13 \rangle + [14]\langle 23 \rangle)^2 \left(\frac{[24]^2 \langle 13 \rangle^2 [\mathbf{1}|p_{13}|\mathbf{2}\rangle}{4m^6 t_{13}} + \frac{[14]^2 \langle 23 \rangle^2 \langle \mathbf{1}|p_{14}|\mathbf{2}] }{4m^6 t_{14}} \right) \\
&\quad \left. - \frac{([24]\langle 13 \rangle + [14]\langle 23 \rangle)([24]\langle 13 \rangle - [14]\langle 23 \rangle)^2 ([\mathbf{1}|p_{13}|\mathbf{2}\rangle [24]\langle 13 \rangle + \langle \mathbf{1}|p_{14}|\mathbf{2}\rangle [14]\langle 23 \rangle]}{4m^6 t_{34}} \right].
\end{aligned}$$

- Free from **spurious pole**
- $1/m^6$ divergence
- **Uniquely** fixed by recursion
- Matches with literature

[Chiodaroli, Johansson, Pichini '22]

Spin-3 Graviton Compton Amplitude

What goes wrong?

$$\sum_{\lambda} \hat{A}_{\psi_3 \bar{\psi}_3 h}^{(\lambda_1 \lambda -)}(z_{13}^{\pm}) \times \hat{A}_{\psi_3 \bar{\psi}_3 h}^{(\bar{\lambda} \lambda_2 +)}(z_{13}^{\pm}) = -\frac{1}{M_P^2} \left[\frac{\hat{S}^2 \mathcal{T}^4}{4m^4 \hat{t}_{14} \hat{t}_{34}} - \frac{\mathcal{T}^3 \mathcal{U} \hat{\mathcal{X}} (3\hat{\mathcal{X}} + \hat{\mathcal{Y}})}{4m^6 \hat{t}_{34}} \right],$$
$$\sum_{\lambda} \hat{A}_{\psi_3 \bar{\psi}_3 h}^{(\lambda_1 \lambda +)}(z_{14}^{\pm}) \times \hat{A}_{\psi_3 \bar{\psi}_3 h}^{(\bar{\lambda} \lambda_2 -)}(z_{14}^{\pm}) = -\frac{1}{M_P^2} \left[\frac{\hat{S}^2 \mathcal{T}^4}{4m^4 \hat{t}_{13} \hat{t}_{34}} - \frac{\mathcal{T}^3 \mathcal{V} \hat{\mathcal{Y}} (3\hat{\mathcal{Y}} + \hat{\mathcal{X}})}{4m^6 \hat{t}_{34}} \right],$$
$$\sum_{\lambda} \hat{A}_{\psi_3 \bar{\psi}_3 h}^{(\lambda_1 \lambda_2 \lambda)}(z_{12}^{\pm}) \times \hat{A}_{hh}^{(\bar{\lambda} + -)}(z_{12}^{\pm}) = -\frac{1}{M_P^2} \left[\frac{\hat{S}^2 \mathcal{T}^4}{4m^4 \hat{t}_{13} \hat{t}_{14}} - \frac{\mathcal{T}^3 \mathcal{U} \hat{\mathcal{X}} (3\hat{\mathcal{X}} + \hat{\mathcal{Y}})}{4m^6 \hat{t}_{13}} - \frac{\mathcal{T}^3 \mathcal{V} \hat{\mathcal{Y}} (3\hat{\mathcal{Y}} + \hat{\mathcal{X}})}{4m^6 \hat{t}_{14}} \right. \\ \left. + \frac{\mathcal{U} \mathcal{V} (\mathcal{U} \hat{\mathcal{X}} + \mathcal{V} \hat{\mathcal{Y}}) (2\mathcal{U} \hat{\mathcal{X}} + \mathcal{U} \hat{\mathcal{Y}} + \mathcal{V} \hat{\mathcal{X}} + 2\mathcal{V} \hat{\mathcal{Y}})}{m^8} \right],$$

- Factorized amplitude remains c_i dependent
- Need additional B_{∞} contributions to cancel these dependences

Amplitude not uniquely fixed by the all-line transverse shift

Summary of Results

- We introduce new recursion technique to provide insights for computing higher-spin Compton amplitude
- We show recursion uniquely fix the amplitudes up-to spin-5/2 for graviton
- These amplitudes are constructible even though massless limit do not exist
- Possible new methods for computing amplitudes corresponding to Kerr Black hole

THANK YOU!

BACKUP

KMOC Formalism

Change of observables

$$\Delta \hat{\mathcal{O}} = \langle \Psi_{\text{in}} | S^\dagger \hat{\mathcal{O}} S | \Psi_{\text{in}} \rangle - \langle \Psi_{\text{in}} | \hat{\mathcal{O}} | \Psi_{\text{in}} \rangle$$

Impulse: $\mathcal{O} = p,$

Radiation: $\mathcal{O} = A^\mu, h^{\mu\nu}$

After $SS^\dagger = 1, S = 1 + iT$

$$\langle \Delta \mathcal{O} \rangle_{\text{cl}} = \lim_{\hbar \rightarrow 0} \hbar^{-\beta \mathcal{O}} \left[\langle \Psi_{\text{in}} | [T, i\hat{\mathcal{O}}] | \Psi_{\text{in}} \rangle + \langle \Psi_{\text{in}} | T^\dagger [\hat{\mathcal{O}}, T] | \Psi_{\text{in}} \rangle \right]$$

Define wave-packet

$$|\Psi_{\text{in}}\rangle = \int d\Phi(p_1) d\Phi(p_2) \phi_1(p_1) \phi_2(p_2) e^{\frac{i}{\hbar}(b_1 \cdot p_1 + b_2 \cdot p_2)} |p_1, p_2\rangle,$$

Classical Limit

Prescription for classical limit

$$g \rightarrow \frac{g}{\sqrt{\hbar}},$$

Coupling

$$q^\mu \rightarrow \bar{q}^\mu \hbar,$$

Momentum of massless particle

$$w^\mu \rightarrow \bar{w}^\mu \hbar,$$

Momentum of intermediate particle

Scales

$$\lambda_C \ll \ell_s \ll r_s \ll b, \quad 1/b \sim q$$

Compton wavelength << Wave packet spread << Schwarzschild radius << Impact parameter

Definitions

$$\mathcal{U} = [\mathbf{24}]\langle\mathbf{13}\rangle, \quad \mathcal{V} = [\mathbf{14}]\langle\mathbf{23}\rangle, \quad \mathcal{X} = [\mathbf{1}|p_{13}|\mathbf{2}\rangle, \quad \mathcal{Y} = [\mathbf{2}|p_{14}|\mathbf{1}\rangle, \quad \mathcal{T} = \mathcal{U} + \mathcal{V}, \quad \mathcal{S} = \mathcal{X} + \mathcal{Y}.$$

Massless:

$$\epsilon_{a\dot{a}}^{(+)} = \sqrt{2} \frac{|\xi\rangle_a [i|_{\dot{a}}}{\langle i\xi\rangle}, \quad \epsilon_{a\dot{a}}^{(-)} = \sqrt{2} \frac{|i\rangle_a [\xi|_{\dot{a}}}{[i\xi]}$$

Massive:

$$\epsilon_{a\dot{a}}^{(+)} = \sqrt{2} \frac{|\eta_i\rangle_a [i|_{\dot{a}}}{m_i}, \quad \epsilon_{a\dot{a}}^{(-)} = \sqrt{2} \frac{|i\rangle_a [\eta_i|_{\dot{a}}}{m_i}, \quad \epsilon_{a\dot{a}}^{(L)} = \frac{|i\rangle_a [i|_{\dot{a}} + |\eta_i\rangle_a [\eta_i|_{\dot{a}}}{m_i}$$

Momentum:

$$(p_i)_{a\dot{a}} = |i\rangle_a [i|_{\dot{a}} - |\eta_i\rangle_a [\eta_i|_{\dot{a}}$$

Explicit form of spinors

In helicity basis:

$$|i\rangle_a = \sqrt{E_i + p_i} \begin{pmatrix} -s_i^* \\ c_i \end{pmatrix}, \quad [i]_{\dot{a}} = \sqrt{E_i + p_i} (-s_i \quad c_i)$$

$$|\eta_i\rangle_a = \sqrt{E_i - p_i} \begin{pmatrix} c_i \\ s_i \end{pmatrix}, \quad [\eta_i]_{\dot{a}} = -\sqrt{E_i - p_i} (c_i \quad s_i^*)$$

Massive Spinor Variables

SU(2) little group index

$$\begin{array}{c} \downarrow \qquad \downarrow \\ [\mathbf{i}]^I_{\dot{a}} \quad |\mathbf{i}\rangle^I_a \\ \uparrow \qquad \uparrow \\ \text{Chiral Index} \end{array}$$

Spin index $I = 1, 2$ represents the little group freedom (choice of spin-axis)

We work in the *helicity basis*

$$|\mathbf{i}\rangle^I_a = |i\rangle_a \delta^I_- + |\eta_i\rangle_a \delta^I_+$$

$$[\mathbf{i}]^I_{\dot{a}} = [i]_{\dot{a}} \delta^I_+ + [\eta_i]_{\dot{a}} \delta^I_-$$

On-shell condition

$$\langle i\eta_i \rangle = [i\eta_i] = m_i$$

In the HE limit $\eta \rightarrow 0$

What on-shell constructibility means?

Gluon vs Scalar amplitude

$$A_4^{gluon} = \begin{array}{c} \text{Diagram 1: Four gluons (curly lines) meeting at two vertices. Vertices 1 and 2 are black dots. Vertices 3 and 4 are white dots. Labels 1, 2, 3, 4 are at the ends.} \\ + \end{array} \begin{array}{c} \text{Diagram 2: Four gluons (curly lines) meeting at two vertices. Vertices 1 and 3 are black dots. Vertices 2 and 4 are white dots. Labels 1, 2, 3, 4 are at the ends.} \\ + \end{array} \begin{array}{c} \text{Diagram 3: Four gluons (curly lines) meeting at two vertices. Vertices 1 and 4 are black dots. Vertices 2 and 3 are white dots. Labels 1, 2, 3, 4 are at the ends.} \\ + \end{array} \begin{array}{c} \text{Diagram 4: Four gluons (curly lines) meeting at two vertices. Vertices 1 and 3 are black dots. Vertices 2 and 4 are black dots. Labels 1, 2, 3, 4 are at the ends.} \\ + \end{array}$$

$$\frac{A_s}{s} + \frac{A_t}{t} + \frac{A_u}{u} + A_4^{contact}$$

Gluon amplitude also satisfy **Ward identity**:

$$A_4^{gluon} \Big|_{\epsilon_i \rightarrow p_i} = 0$$

dependent contact term

Ward identity implies four diagrams above are **dependent**.

It can be shown only 3-point information is enough for on-shell constructibility.

What on-shell constructibility means?

Gluon vs Scalar amplitude

$$A_4^{scalar} = \begin{array}{c} \text{Diagram 1: Four external gluons (1,2,3,4) in a horizontal line.} \\ \text{Diagram 2: Three gluons (1,2,3) meeting at a central vertex, with gluon 4 below them.} \\ \text{Diagram 3: Two gluons (1,2) meeting at a central vertex, with gluons 3 and 4 crossing below them.} \\ \text{Diagram 4: Gluons 1 and 2 crossing, with gluons 3 and 4 crossing above them.} \end{array} + + + +$$

$$A_4^{scalar} = \frac{A_s}{s} + \frac{A_t}{t} + \frac{A_u}{u} + A_4^{contact}$$

Scalar amplitude DO NOT satisfy Ward identity:

$$A_4^{scalar} \Big|_{\epsilon_i \rightarrow p_i} \neq 0$$

independent contact term

Ward identity implies four diagrams above are **independent**.

It can be shown both 3-point and 4-point information required for on-shell constructibility.

On-shell constructibility

Photon

- Large- z behavior can be studied using dimensional analysis

$$\lim_{z \rightarrow \infty} \hat{A}_n \sim z^\gamma, \quad \gamma \leq 4 - n - [g] - \frac{N_F}{2}$$

n = number of external legs,
 $[g]$ = dimension of coupling,
 N_F = number of external fermions

- $[g] = -2$
- $N_F = 2 \quad \gamma = 1$
- $n = 4$

Each photon improves large- z by $1/z$

$$\gamma = -1$$

On-shell constructibility

Graviton

- Large- z behavior can be studied using dimensional analysis

$$\lim_{z \rightarrow \infty} \hat{A}_n \sim z^\gamma, \quad \gamma \leq 4 - n - [g] - \frac{N_F}{2}$$

n = number of external legs,
 $[g]$ = dimension of coupling,
 N_F = number of external fermions

- $[g] = -4$
- $N_F = 2 \quad \gamma = 3$
- $n = 4$

Each graviton improves large- z by $1/z^2$

$$\gamma = -1$$

All-line Transverse Shift

- Shift external momentum by respective transverse polarization for spin-1/2 and spin-1

$$p_i \rightarrow \hat{p}_i = p_i + z \frac{c_i m_i}{\sqrt{2}} \epsilon_i^{(I_i)},$$

- “good” shift will only deform momentum and not external polarizations
- A “good” shift exists after fixing the spin-projection of external particles
- Define shift for spin-1/2 and spin-1 transverse modes

$$\begin{aligned} |i\rangle \rightarrow |\hat{i}\rangle &= |i\rangle + z c_i |\eta_i\rangle && \text{for } I_i = +, && \text{On-shell condition satisfied} \\ |i]\rightarrow |\hat{i}] &= |i] + z c_i |\eta_i] && \text{for } I_i = -, && \langle \hat{i} \eta_i \rangle = [\hat{i} \eta_i] = m_i \end{aligned}$$