QuantumWinter Hackathon

Team Quanta

Question 1.

Considering the 1-D wave equation,

$$\frac{\partial u}{\partial t} + C\,\frac{\partial u}{\partial x} = 0 \qquad \text{where } C > 0 \tag{1}$$

Where, as $C > 0$, the wave propagates in the positive x–direction. Let's take,

$$U_t = \frac{\partial u}{\partial t}\,, \quad U_x = \frac{\partial u}{\partial x}$$

So, the wave equation (1) may be written as,

$$U_t + CU_x = 0 \tag{2}$$

This partial differential equation (PDE) can be discretized in space and time by a finite difference method.

Discretizing the domain

The spatial domain $[\mathbf{0}, \mathbf{L}]$ is represented by a set of mesh points,

$$0 = x_0 < x_1 < \ldots.. < x_{N-1} < x_{Nx} = L$$

The temporal domain $[\mathbf{0}, \mathbf{T})$ is represented by a set of mesh points,

$$0 = t_0 < t_1 < \ldots.. < t_{N-1} < t_{Nt} = T$$

The mesh is therefore two-dimensional in the x–t plane.

We use a uniform distribution of mesh points.

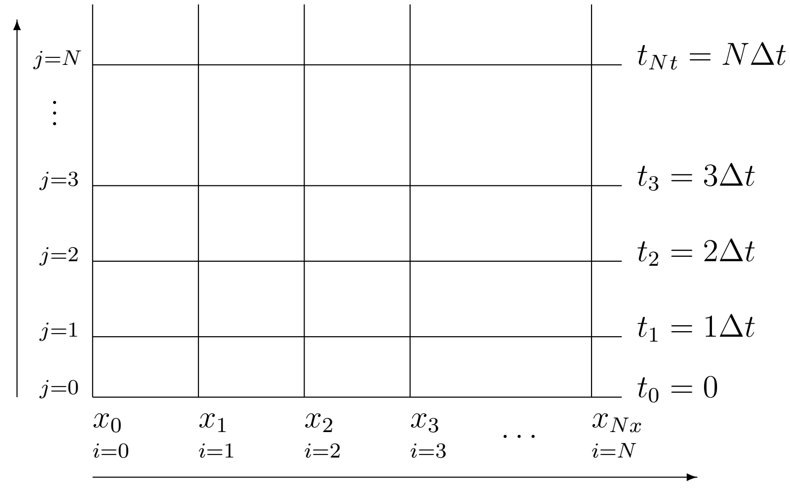

Figure 1: Grid of mesh points.

The PDE must be discretized using an implicit integration scheme with 2nd order spatial accuracy and 1st order time accuracy.

To perform this specified integration in time t, the PDE (Eq. 1) is discretized by approximating $\partial u/\partial t$ with the backward difference scheme and $\partial u/\partial x$ with the central difference scheme (BTCS).

Backward in time central in space discretization

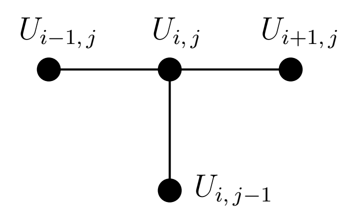

Figure 2: Computational stencil for BTCS scheme.

The temporal and spatial derivatives are approximated separately. Using the implicit form or the backward method, the difference formulas for our time and space derivatives are,

$$\left(\frac{\partial u}{\partial t}\right)_{ij} = U_{tij} \approx \frac{U_{i,j} - U_{i,j-1}}{\Delta t} \tag{3}$$

$$\left(\frac{\partial u}{\partial x}\right)_{ij} = U_{xij} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} \tag{4}$$

Where,

$$\begin{aligned} U_{i,j} &\equiv U(i\Delta x, j\Delta t) \\ x_i &\equiv i\Delta x \\ t_j &\equiv j\Delta t \end{aligned}$$

Combining (3) and (4), we obtain,

$$\frac{U_{i,j} - U_{i,j-1}}{\Delta t} + C\left(\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}\right) = 0 \tag{5}$$

or,

$$U_{i,j} = U_{i,j-1} - \frac{C\Delta t}{2\Delta x}\left(U_{i+1,j} - U_{i-1,j}\right) \tag{6}$$

Here, the Courant–Fredrich–Lewy number is,

$$\text{CFL} = \frac{C\Delta t}{\Delta x}$$

Finally, this scheme is given to us with:

- first-order in time $O(\Delta t)$
- second-order in space $O(\Delta x^2)$