Dirac Delta

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Assignment # A7

Theory

(a) Dirac Delta function : A Dirac delta function or simply delta function is a generalised function on the real number line denoted by "δ" that is zero everywhere expect at zero, with an integral of one over the entire real line.

The dirac delta can be loosely thought of as a function on the real line which is zero everywhere expect at the origin, where it is infinite,

$$\delta(x) = \begin{cases} \infty & , \quad x = 0 \\ 0 & , \quad x \neq 0. \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

For lots of applications, such as those involving PDEs, one would like to have a function $\delta(x)$ whose integral is "concentrated" at the point $x = 0$. That is, one would like the function

$$\delta(x) = 0 \quad \forall \quad x \neq 0.$$

but with

$$\int \delta(x)\, dx = 1.$$

for any integration region that includes $x = 0$; this concept is called "delta function".
It is also the simplest way to consider physical effects that are concentrated within very small volumes or times, for example, think of concepts of a "point mass," a "point charge," a "kick" that suddenly imparts some momentum to an object and so on.

Now we are going to ~~answer~~ explain the statement written below:
Dirac Delta is not a function but rather a Distributions.

We think function as a map from $\mathbb{R} \to \mathbb{R}$: given an x, we get a value $f(x)$.

Informally one often sees "definitions" of $\delta(x)$ that describe it as some mysterious object that is "not quite" a function, which $= 0$ for $x \neq 0$ but is undefined at $x = 0$ and which is "only really defined inside an integral" [where it integrates to 1].
This may leave you with a queasy feeling that $\delta(x)$ is somehow not real or rigorous. For example, integration is an operation that is classically only defined for ordinary functions, so it may not even be clear (yet) what "$\int$" means when we write "$\int \delta(x)\, dx$".

A function $f(x)$ gives us a value at every point x, but this really correspond to measurable quantity in the physical universe?

for ex- To measure the velocity at one instant in time or the density at one point in fluid, the device used should be very precise, and very small and very fast, but in the end all we can ever measure are averages of $f(x)$ over a small region of space and/or time.

But average is the same thing as an integral $\int f(x)$ over the averaging region. More generally, instead of averaging $f(x)$ uniformly in some region, we could average with some weights $\phi(x)$ (e.g. our device could be more sensitive to some points that others.). Thus the only physical question that we should have in mind, is the value of integral.

$$\int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx$$

of $f(x)$ against a test function $\phi(x)$.

But if all we can ever ask in such integrals
→ why are we worrying about isolated points?
→ Why do we even define $f(x)$ to have ~~results~~ values at points at all?

The old kind of function is a map from

$\mathbb{R} \rightarrow \mathbb{R}$: given an x, we get a value $f(x)$.

As discussed above ; we should think of for an "average" value given some weight function $\phi(x)$. So make a new definition of "function" that provides this information, and only this information.

$\Rightarrow$ f is a rule that given any test function $\phi(x)$ returns a number $f\{\phi\}$.

This new definition is called a <u>generalized function</u> or <u>a distribution</u>. We are no longer allowed to ask the value ~~of~~ at a point x. This will fix all the Problems with the old function from above.

<u>Note</u> : $\phi(x)$ is an ordinary function $\mathbb{R} \rightarrow \mathbb{R}$ [not a distribution] in some set D. We require $\phi(x)$ to be infinety differentiable. We also require $\phi(x)$ to be non zero only in some finite region.

(b) Five representation of dirac delta function $\delta(x)$ as a limit of sequence of functions:

$$\delta(x) = \lim_{\epsilon \to 0^+} f_\epsilon(x).$$

where $f_\epsilon(x)$ is an absolutely integrable function on $\mathbb{R}$ s.t.

$$\int_{-\infty}^{\infty} f_\epsilon(x)\, dx = 1.$$

(1) Limit of sequence of Rectangles :

$$\delta(x) = \lim_{\epsilon \to 0} f_\epsilon(x).$$

where,

$$f_\epsilon(x) = \begin{cases} 0 & \text{for } x < -\frac{\epsilon}{2} \\ \frac{1}{\epsilon} & \text{for } -\frac{\epsilon}{2} < x < \frac{\epsilon}{2} \\ 0 & \text{for } x > \frac{\epsilon}{2}. \end{cases}$$

Here width of the rectangle is ϵ and height is $\frac{1}{\epsilon}$, so area is unity.

(2) Limit of sequence of isoceless triangle

$$\delta(x) = \lim_{\epsilon \to 0} f_\epsilon(x)$$

where,

$$f_\epsilon(x) = \begin{cases} 0 & \text{for } |x| > \epsilon \\ \dfrac{\left[1 - \frac{|x|}{\epsilon}\right]}{\epsilon} & \text{for } |x| < \epsilon \end{cases}$$

Here the base of the triangle is 2ϵ and height is $\frac{1}{\epsilon}$, so area is unity.

(3) Limit of sequence of Gaussian functions :

$$\delta(x) = \lim_{\epsilon \to 0} f_\epsilon(x).$$

where,

$$f_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-\frac{x^2}{\epsilon}}$$

This is the normalized Gaussian distribution function. The area under the ~~cr~~ curve is unity and the peak value at $x = 0$ is $\frac{1}{\epsilon\sqrt{\pi}}$

(4) Limit of sequence of Exponential function :

$$\delta(x) = \lim_{\epsilon \to 0} f_\epsilon(x).$$

where,

$$f_\epsilon(x) = \frac{1}{2\epsilon} e^{-\frac{|x|}{\epsilon}}$$

The area under the curve is unity and the peak value at $x=0$ is $\frac{1}{2\epsilon}$

(5) Limit of sequence of Lorentzian

$$\delta(x) = \lim_{\epsilon \to 0} f_\epsilon(x).$$

where, $f_\epsilon(x) = \frac{1}{\pi\epsilon} \frac{\epsilon^2}{\epsilon^2 + x^2}$

The peak value at $x = 0$ is $\frac{1}{\pi\epsilon}$.

(c) Properties of Dirac Delta function $\delta(x-a)$. and its 3-dimensionsal version $\delta^3(\vec{r}-\vec{a})$.

(1) $$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a).$$

3-d version

$$\int_{-\infty}^{\infty} f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,d\tau = f(\vec{r}).$$

Here $d\tau = \cancel{dx}\,\cancel{dy}\,\cancel{dz} \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$
$\vec{r}-\vec{a} = (x-a_1)\,\hat{i} + (y-a_2)\,\hat{j} + (z-a_3)\,\hat{k}$

(2) $$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

3-d version

$$\int_{-\infty}^{\infty} \delta^3(\vec{r}-\vec{a})\,d\tau = 1.$$

OR. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-a_1)\,\delta(y-a_2)\,\delta(z-a_3)\,dx\,dy\,dz = 1$$

3 $$\delta[\alpha(x-a)] = \frac{1}{|\alpha|}\,\delta(x-a).$$

3-d version.

$$\delta^3[\alpha(\vec{r}-\vec{a})] = \frac{1}{|\alpha|^3}\,\delta^3(\vec{r}-\vec{a}).$$

(d) Evaluate

(1) $$\int_{-\infty}^{\infty} \delta(x-2)(x+1)^2 \, dx$$

$f(x) = (x+1)^2$
$a = 2$] by comparing with $\int_{-\infty}^{\infty} \delta(x-a) f(x) dx$.

So, $$\int_{-\infty}^{\infty} \delta(x-2)(x+1)^2 dx = [f(x)]_{x=2}.$$

$$= (x+1)^2 \Big|_{x=2}.$$

$$= 9.$$

(2) $$\int_{-\infty}^{\infty} 9x^2 \, \delta(3x+1) \, dx.$$

$$= \int_{-\infty}^{\infty} 9x^2 \, \delta\left[3\left(x+\frac{1}{3}\right)\right] dx.$$

$$= \int_{-\infty}^{\infty} \frac{9x^2}{3} \, \delta\left(x+\frac{1}{3}\right) dx \qquad \left[\because \delta(ax) = \frac{1}{|a|}\delta(x)\right].$$

$$= 3\int_{-\infty}^{\infty} x^2 \, \delta\left(x+\frac{1}{3}\right) dx.$$

$f(x) = x^2 \qquad a = \frac{-1}{3}.$

$$= 3 \, . \, f(x)\Big|_{x=\frac{-1}{3}} = 3\left(\frac{-1}{3}\right)^2 = \frac{1}{3}.$$

(3) $\int_{-\infty}^{\infty} 5e^{t^2} \cos(t)\, \delta(t-3)\, dt.$

$f(t) = 5e^{t^2}\cos(t) \qquad a=3.$

So, $\int_{-\infty}^{\infty} 5e^{t^2}\cos(t)\, \delta(t-3) = f(t)\Big|_{x=3}$

$= \cancel{5}et\ 5e^{t^2}\cos(t)\Big|_{x=3}$

$= 5e^{9}\cos(3).$

1 Programming

```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import math
from scipy import integrate
from sympy import *
from sympy import simplify
import scipy
from MyIntegration import MySimp
from MyIntegration import MyHermiteQuad
from MyIntegration import new_simp
from prettytable import PrettyTable
from MyIntegration import herm
'''
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'''

x=symbols('x')
j5=lambda x: np.exp(abs(x/e))/(2*e)
j7= lambda x : np.exp(-x**2/4/e)/np.sqrt(4*np.pi*e)                    # ##
e_a=[0.4/2**1,0.4/2**2,0.4/2**3,0.4/2**4,0.4/2**5]
x_=np.linspace(-10,10,10000)
g1=[];g2=[]

fig, (ax1, ax2) = plt.subplots(1, 2)
fig.suptitle('VERIFICATION OF BEHAVIOUR OF SEQUENCES OF FUNCTIONS AS DIRAC DELTA FUNCTION',c='r')
for e in e_a:
    g1=[];g2=[]
    for x in x_:
        g2.append(j7(x))
        g1.append(j5(x))
    ax1.plot(x_,g1,label="$\u03B5={0}$".format(e))
    ax2.plot(x_,g2,label="$\u03B5={0}$".format(e))
ax1.grid()
ax1.legend()
ax1.set(xlabel="x",ylabel= "\u03B4(x)" ,title="Sequence 1")
ax2.set(xlabel="x",ylabel= "\u03B4(x)" ,title="Sequence 2")
ax2.grid()
ax2.legend()
plt.show()
e=0.4/2**5

print("Representation 1 : f_e(x)=np.exp(-(x-e)**2/(2*e))/(np.pi*2*e)**(1/2) ")
print("Representation 2 : f_e(x)=np.exp(-x**2/2/e)/np.sqrt(2*np.pi*e)")
```

```
#I1
I1_R1_s=["Rep 1"]
I1_R2_s=["Rep 2"]
I1_R1_h=["Rep 1"]
I1_R2_h=["Rep 2"]
f1=lambda x:  np.exp(x**2)
fi = lambda x : j5(x)*f1(x)
fj = lambda x : j7(x)*f1(x)
for e in e_a:
   I1_R1_s.append(new_simp(j5,10,10**8,0.1e-6)[0])
   I1_R2_s.append(new_simp(j7,10,10**8,0.1e-6)[0])
   I1_R1_h.append(herm(fi,2,350,0.1e-6)[0])
   I1_R2_h.append(herm(fj,2,350,0.1e-6)[0])

print()
print("################################################")
print("INTEGRAL I1")
ea=[""]+e_a
data1={"EPSILON":ea,"SIMP":I1_R1_s,"HERM":I1_R1_h,"SIMPSON":I1_R2_s,"HERMITE":I1_R2_h}
print(pd,pd.DataFrame(data1))


#I2
I2_R1_s=["Rep 1"]
I2_R2_s=["Rep 2"]
I2_R1_h=["Rep 1"]
I2_R2_h=["Rep 2"]
f1=lambda x:  np.exp(x**2)*(x+1)**2
f1_=lambda x: (x+1)**2

fi = lambda x : j5(x)*f1_(x)
fj = lambda x : j7(x)*f1_(x)

fii = lambda x : j5(x)*f1(x)
fjj = lambda x : j7(x)*f1(x)

for e in e_a:
   I2_R1_s.append(new_simp(fi,10,10**8,0.1e-6)[0])
   I2_R2_s.append(new_simp(fj,10,10**8,0.1e-6)[0])
   I2_R1_h.append(herm(fii,2,350,0.1e-6)[0])
   I2_R2_h.append(herm(fjj,2,350,0.1e-6)[0])


print()
print("################################################")
print("INTEGRAL I2")
data2={"EPSILON":ea,"SIMP":I2_R1_s,"HERM":I2_R1_h,"SIMPSON":I2_R2_s,"HERMITE":I2_R2_h}
print(pd,pd.DataFrame(data2))


#I3
I3_R1_s=["Rep 1"]
```

```
I3_R2_s=["Rep 2"]
I3_R1_h=["Rep 1"]
I3_R2_h=["Rep 2"]
f1=lambda x:  np.exp(x**2)*(x-1)**2/3
f1_=lambda x: (x-1)**2/3

fi = lambda x : j5(x)*f1_(x)
fj = lambda x : j7(x)*f1_(x)

fii = lambda x : j5(x)*f1(x)
fjj = lambda x : j7(x)*f1(x)

for e in e_a:
   I3_R1_s.append(new_simp(fi,10,10**8,0.1e-6)[0])
   I3_R2_s.append(new_simp(fj,10,10**8,0.1e-6)[0])
   I3_R1_h.append(herm(fii,2,350,0.1e-6)[0])
   I3_R2_h.append(herm(fjj,2,350,0.1e-6)[0])


print()
print("#################################################")
print("INTEGRAL I3")
data3={"EPSILON":ea,"SIMP":I3_R1_s,"HERM":I3_R1_h,"SIMPSON":I3_R2_s,"
   HERMITE":I3_R2_h}
print(pd,pd.DataFrame(data3))

print()
print("#################################################")
```

2 Result and Discussion

```
Representation 1 : f_e(x)=np.exp(abs(x/e))/(2*e)
Representation 2 : f_e(x)=np.exp(-x**2/4/e)/np.sqrt(4*np.pi*e)
```

INTEGRAL I1

	EPSILON	SIMP	HERM	SIMPSON	HERMITE
0		Rep 1	Rep 1	Rep 2	Rep 2
1	0.2	1.0	1.0	1.0	1.0
2	0.1	1.0	1.0	1.0	1.0
3	0.05	1.0	1.0	1.0	1.0
4	0.025	1.0	1.0	1.0	1.0
5	0.0125	1.0	1.0	1.0	1.0

INTEGRAL I2

	EPSILON	SIMP	HERM	SIMPSON	HERMITE
0		Rep 1	Rep 1	Rep 2	Rep 2
1	0.2	1.64	1.190118	1.2	1.190118
2	0.1	1.31	1.137906	1.1	1.137906
3	0.05	1.1525	1.082986	1.05	1.082986
4	0.025	1.075625	1.045538	1.025	1.045538
5	0.0125	1.037656	1.023857	1.0125	1.023857

INTEGRAL I3

	EPSILON	SIMP	HERM	SIMPSON	HERMITE
0		Rep 1	Rep 1	Rep 2	Rep 2
1	0.2	0.28	0.240258	0.4	0.240258
2	0.1	0.303333	0.278714	0.366667	0.278714
3	0.05	0.3175	0.303341	0.35	0.303341
4	0.025	0.325208	0.31755	0.341667	0.31755
5	0.0125	0.329219	0.325227	0.3375	0.325227

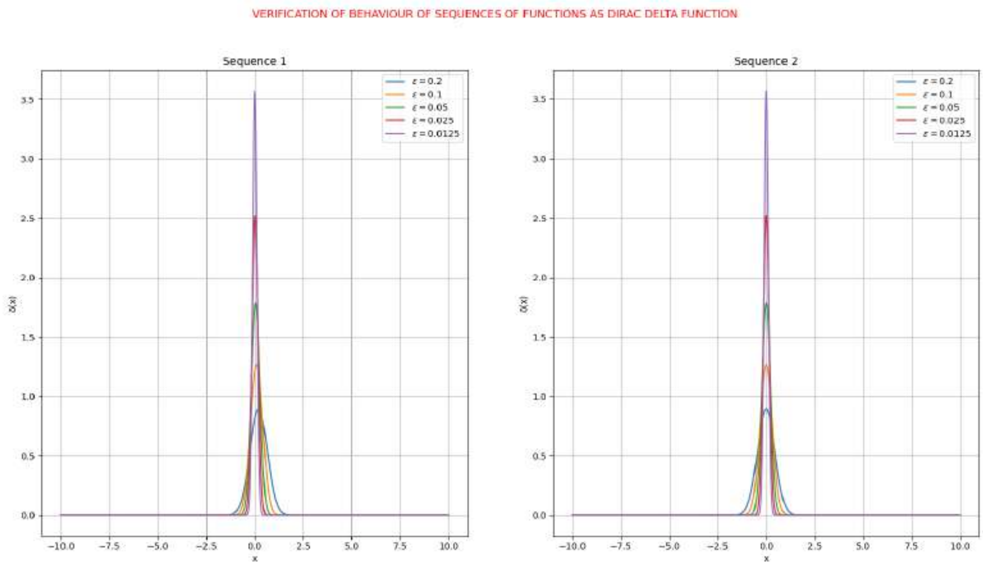

Figure 1:

In the graph above we can clearly see that at x=0 we are getting spike whereas for all other values it is zero. As we reduce epsilon the height of spike is increasing and becoming narrower.