Mathematical Physics III

Lab Assignment #7

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Contents

1 Theory

1.1 Dirac-delta function

1.1.1 Definition of Dirac delta function

The Dirac delta function a function defined on the real line denoted by δ that is zero everywhere except at the origin where it's functional value is infinitely high and area under the curve is equal to one .

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

the practical use for this delta function lies in the physical systems or models that concerns impulse behaviour i.e effects that are concentrated within a very small volume or time .

1.1.2 Why δ function is more of a distribution rather than a function

We can not define the delta function rigorously this way. For any function that is equal to 0 for any $x \neq 0$, and is ∞ at $x = 0$, it can be shown that the integration of such function on $\mathbb{R}$ line will give 0.

Thus, instead of defining $\delta(x)$ as a function in conventional sense, we (in layman terms) define it as a limit of sequence of functions. In a rigorous setting, this is called a General function or a Distribution.

Hence, $\delta(x)$ is a distribution, not a function!

1.2 Representations of Dirac Delta function $\delta(t)$ as a limit of sequence of functions:

$$\lim_{\epsilon \to 0^+} f_\epsilon(t) = \delta(t)$$

where $f_\epsilon(t)$ is an absolutely integrable function on $\mathbb{R}$ such that :

$$\int_{-\infty}^{\infty} f_\epsilon(t) dt = 1$$

1. **As limit of sequence of Rectangles :**

$$f_\epsilon(t) = \begin{cases} 0 & t < -\epsilon/2 \\ \epsilon/2 & -\epsilon/2 < t < \epsilon/2 \\ 0 & t > \epsilon/2 \end{cases}$$

   here the width of the rectangle is ϵ and height is $1/\epsilon$ so the area of the rectangle is 1 , also note that the ϵ is an arbitrary number in the $\mathbb{R}$. So,

$$\delta(t) = \lim_{\epsilon \to 0} f_\epsilon(t)$$

2. **As limit of sequence of isosceles triangles :**

$$f_\epsilon(t) = \begin{cases} 0 & |t| > \epsilon \\ (1 - \frac{|t|}{\epsilon})/\epsilon & t > \epsilon/2 \end{cases}$$

   hence the base of the triangle is 2ϵ and height is $1/\ \epsilon$ and hence the area under the curve(triangle) is unity irrespective of our choice of ϵ . hence,

$$\delta(t) = \lim_{\epsilon \to 0} f_\epsilon(t)$$

3. **As limit of sequence of Gaussian Function:**

$$f_\epsilon(t) = \frac{1}{\epsilon\sqrt{\pi}e^{-x^2/\epsilon}}$$

   The area under the Normal Gaussian curve is well evaluated to be equal to unity , and the peak value at t = 0 is $1/\epsilon\sqrt{\pi}$.

4. **As the limit of sequence of exponential function :**

$$\delta(t) = \lim_{\epsilon \to 0^+} f_\epsilon(t)$$

   where ,

$$f_\epsilon(t) = e^{-|x|/\epsilon}/2\epsilon$$

   here the area under the curve is unity and the peak value at x = 0 is $1/2\epsilon$

5. **As the limit of sequence of Lorentzian :**

$$\delta(t) = \lim_{\epsilon \to 0} f_\epsilon(t)$$

   where ,

$$f_\epsilon(t) = \frac{1}{\pi\epsilon}\frac{\epsilon^2}{\epsilon^2 + t^2}$$

   the area under the curve is unity and the peak value at x = 0 is $\frac{1}{\pi\epsilon}$

1.3 Properties of Dirac Delta Function $\delta(x-a)$ and its 3-dimensional version $\delta^3(\vec{r}-\vec{a})$

1. $\displaystyle\int_{-\infty}^{+\infty} f(x)\delta(x-a)\,dx = f(a)$

   3-D version

   $$\int_{-\infty}^{+\infty} f(\vec{r})\delta^3(\vec{r}-\vec{a})\,d\vec{r} = f(\vec{r})$$

   Here, $d\vec{r} = (dx\hat{i} + dy\hat{j} + dz\hat{k})$

   and, $\vec{r}-\vec{a} = (x-a_1)\hat{i} + (y-a_2)\hat{j} + (z-a_3)\hat{k}$

2. $\displaystyle\int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1$

   3-D version

   $$\int_{-\infty}^{+\infty} \delta^3(\vec{r}-\vec{a})\,d\vec{r}$$

   OR

   $$\iiint\limits_{-\infty}^{+\infty} \delta(x-a_1)\delta(y-a_2)\delta(z-a_3)\,dx\,dy\,dz = 1$$

3. $\delta\left[\alpha(x-a)\right] = \dfrac{1}{|\alpha|}\delta(x-a)$

   3-D version

   $$\delta^3\left[\alpha(\vec{r}-\vec{a})\right] = \frac{1}{|\alpha|^3}\delta^3(\vec{r}-\vec{a})$$

1.4 Evaluating some examples

1. $\displaystyle\int_{-\infty}^{+\infty} \delta(x-2)(x+1)^2\,dx$

   Ans.) By comparing with $\displaystyle\int_{-\infty}^{+\infty} \delta(x-a)f(x)\,dx$

   $f(x) = (x+1)^2$ and $a = 2$

   So,

$$\int_{-\infty}^{+\infty} \delta(x-2)(x+1)^2\,dx = f(x)|_{x=2}$$

$$= (x+1)^2|_{x=2}$$

$$= 9$$

2. $\displaystyle\int_{-\infty}^{+\infty} 9x^2\delta(3x+1)\,dx$

Ans.) $= \displaystyle\int_{-\infty}^{+\infty} 9x^2\delta[3(x+\frac{1}{3})]\,dx$

$= \displaystyle\int_{-\infty}^{+\infty} \frac{9x^2}{3}\delta(x+\frac{1}{3})\,dx$ $\qquad [\because \ \ \delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)]$

$= 3\displaystyle\int_{-\infty}^{+\infty} x^2\delta(x+\frac{1}{3})\,dx$

Now, $f(x) = x^2$ and $a = \frac{-1}{3}$

So,

$$3\int_{-\infty}^{+\infty} x^2\delta(x+\frac{1}{3})\,dx = 3.f(x)|_{x=-1/3}$$

$$= 3.(x^2)|_{x=-1/3}$$

$$= 3.\left(\frac{-1}{3}\right)^2$$

$$= \frac{1}{3}$$

3. $\displaystyle\int_{-\infty}^{+\infty} 5e^{t^2}\cos(t)\delta(t-3)\,dt$

Ans.) $f(t) = 5e^{t^2}\cos(t)$ and $a = 3$

So,

$$\int_{-\infty}^{+\infty} 5e^{t^2}\cos(t)\delta(t-3)\,dt = f(t)|_{x=3}$$

$$= 5e^{t^2}cos(t)|_{x=3}$$

$$= 5e^9cos(3)$$

2 Programming

2.1 2020PHY1221_A7.py

```
import MyIntegration as mi
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from sympy import var,sympify
from sympy.utilities.lambdify import lambdify

# Name => Ishmeet Singh   Roll no. => 2020PHY1221
# Patner's Name => Sarthak Jain   Roll no. => 2020PHY1201

def func1(eps, x0 = 0):
    result = "exp(-(x-{:})**2/(2*({:})**2)) / (2*pi*({:})**2)**(1/2)".
    format(x0, eps, eps)
    return(result)

def func2(eps, x0 = 0):
    result = "{:}/(pi*((x-{:})**2 + ({:})**2))".format(eps, x0, eps)
    return((result))

if __name__ == '__main__':
    x_axis = np.linspace(0, 2, 1000)
    epsilon = [0.4/2**i for i in range(1, 6)]

    """Dirac-delta Plots: 2(a) i"""
    for j in epsilon:
        plt.figure("Gaussian")
        x = var("x")
        expr1 = sympify(func1(eps = j, x0 = 1))
        function1 = lambdify(x,expr1)
        plt.plot(x_axis, function1(x_axis), label = "$\epsilon = ${:.3e}".
    format(j))
        plt.figure("Lorentzian")
        expr2 = sympify(func2(eps = j, x0 = 1))
        function2 = lambdify(x,expr2)
        plt.plot(x_axis, function2(x_axis), label = "$\epsilon = ${:.3e}".
    format(j))

    plt.figure("Gaussian")
```

```
    plt.ylabel("$\delta_{\epsilon}(x-1)$")
    plt.xlabel("x Axis")
    plt.title("Dirac-delta function as an approximation of Gaussian Distribution\ncentered at $a = 1$")
    plt.legend()

    plt.figure("Lorentzian")
    plt.ylabel("$\delta_{\epsilon}(x-1)$")
    plt.xlabel("x Axis")
    plt.title("Dirac-delta function as an approximation of Lorentz/Cauchy Distribution\ncentered at $a = 1$")
    plt.legend()

    """Dirac-delta Properties: 2(a) ii"""
    # Analytic Results
    x_axis2 = np.linspace(-2, 2, 1000)
    res1 , res2 , res3 = np.ones(len(epsilon)) , np.ones(len(epsilon)), np.ones(len(epsilon))
    # Dirac-delta convolutions using Simpsons 1/3
    Integral1 = [[] for i in range(4)]
    Integral2 = [[] for i in range(4)]
    Integral3 = [[] for i in range(4)]
    for j in epsilon:
        """Integral 1"""
        int11 = mi.MySimp(func1(eps = j),2,-2,1000)
        int12 = mi.MySimp(func2(eps = j),2,-2,1000)
        Herm_int11 = mi.MyHermiteQuad(expression = "exp(x**2)*"+func1(eps = j), n = 50)
        Herm_int12 = mi.MyHermiteQuad(expression = "exp(x**2)*"+func2(eps = j), n = 50)
        Integral1[0].append(int11)
        Integral1[1].append(int12)
        Integral1[2].append(Herm_int11)
        Integral1[3].append(Herm_int12)

        """Integral 2"""
        int21 = mi.MySimp("((x+1)**2)*"+func1(eps = j),2,-2,1000)
        int22 = mi.MySimp("((x+1)**2)*"+func2(eps = j),2,-2,1000)
        Herm_int21 = mi.MyHermiteQuad("exp(x**2)*((x+1)**2)*"+func1(eps = j), n = 50)
        Herm_int22 = mi.MyHermiteQuad("exp(x**2)*((x+1)**2)*"+func2(eps = j), n = 50)
```

```
        Integral2[0].append(int21)
        Integral2[1].append(int22)
        Integral2[2].append(Herm_int21)
        Integral2[3].append(Herm_int22)

        """Integral 3"""
        int31 = mi.MySimp("((3*x)**2)*"+func1(eps = j, x0 = -1/3),2,-2,1000)
        int32 = mi.MySimp("((3*x)**2)*"+func2(eps = j, x0 = -1/3),2,-2,1000)
        Herm_int31 = mi.MyHermiteQuad("exp(x**2)*((3*x)**2)*"+func1(eps = j), n = 50)
        Herm_int32 = mi.MyHermiteQuad("exp(x**2)*((3*x)**2)*"+func2(eps = j), n = 50)
        Integral3[0].append(int31)
        Integral3[1].append(int32)
        Integral3[2].append(Herm_int31)
        Integral3[3].append(Herm_int32)

    plt.figure("SHI1")
    plt.plot(epsilon, res1, color = "black", label = "Analytical Result")
    plt.scatter(epsilon, Integral1[0], label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Simpson 1/3", zorder = 5)
    plt.scatter(epsilon, Integral1[1], label = "$\delta_{\epsilon}(x)$ as Lorentzian Distribution - Simpson 1/3", zorder = 5)
    plt.scatter(epsilon, Integral1[2], marker = "*", label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Gauss-Hermite", zorder = 5)
    plt.scatter(epsilon, Integral1[3], marker = "*", label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Gauss-Hermite", zorder = 5)
    plt.xlabel("$\epsilon$")
    plt.ylabel("y Axis")
    plt.title("INTEGRAL 1\n")
    plt.legend()

    plt.figure("SHI2")
    plt.plot(epsilon, res2, color = "black", label = "Analytical Result")
    plt.scatter(epsilon, Integral2[0], label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Simpson 1/3", zorder = 5)
    plt.scatter(epsilon, Integral2[1], label = "$\delta_{\epsilon}(x)$ as Lorentzian Distribution - Simpson 1/3", zorder = 5)
    plt.scatter(epsilon, Integral2[2], marker = "*", label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Gauss-Hermite", zorder = 5)
```

```python
    plt.scatter(epsilon, Integral2[3], marker = "*", label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Gauss-Hermite", zorder = 5)
    plt.xlabel("$\epsilon$")
    plt.ylabel("y Axis")
    plt.title("INTEGRAL 2\n")
    plt.legend()

    plt.figure("SHI3")
    plt.plot(epsilon, res3, color = "black", label = "Analytical Result")
    plt.scatter(epsilon, Integral3[0], label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Simpson 1/3", zorder = 5)
    plt.scatter(epsilon, Integral3[1], label = "$\delta_{\epsilon}(x)$ as Lorentzian Distribution - Simpson 1/3", zorder = 5)
    plt.scatter(epsilon, Integral3[2], marker = "*", label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Gauss-Hermite", zorder = 5)
    plt.scatter(epsilon, Integral3[3], marker = "*", label = "$\delta_{\epsilon}(x)$ as Gaussian Distribution - Gauss-Hermite", zorder = 5)
    plt.xlabel("$\epsilon$")
    plt.ylabel("y Axis")
    plt.title("INTEGRAL 3\n")
    plt.legend()

    plt.show()

    """Dirac-delta Data Tables: 2(b)"""
    print("\nIntegral 1\n")
    int1 = {"Epsilon": epsilon,
            "Gaussian Dist. - Simpson 1/3": Integral1[0],
            "Lorentzian Dist. - Simpson 1/3": Integral1[1],
            "Gaussian Dist. - Gauss-Hermite": Integral1[2],
            "Lorentzian Dist. - Gauss-Hermite": Integral1[3]
            }
    df1 = pd.DataFrame(int1)
    df1.to_csv(r'F:\Ishu\Dirac delta\Integral1.csv')
    print(df1)

    print("\nIntegral 2\n")
    int2 = {"Epsilon": epsilon,
            "Gaussian Dist. - Simpson 1/3": Integral2[0],
            "Lorentzian Dist. - Simpson 1/3": Integral2[1],
            "Gaussian Dist. - Gauss-Hermite": Integral2[2],
            "Lorentzian Dist. - Gauss-Hermite": Integral2[3]
```

```
            }
    df2 = pd.DataFrame(int2)
    df2.to_csv(r'F:\Ishu\Dirac delta\Integral2.csv')
    print(df2)

    print("\nIntegral 3\n")
    int3 = {"Epsilon": epsilon,
            "Gaussian Dist. - Simpson 1/3": Integral3[0],
            "Lorentzian Dist. - Simpson 1/3": Integral3[1],
            "Gaussian Dist. - Gauss-Hermite": Integral3[2],
            "Lorentzian Dist. - Gauss-Hermite": Integral3[3]
            }
    df3 = pd.DataFrame(int3)
    df3.to_csv(r'F:\Ishu\Dirac delta\Integral3.csv')
    print(df3)
```

3 Results

3.1 Behaviour of $\delta_\epsilon(x-a)$ as $\delta \to 0$

3.1.1 When a = 0, i.e., centered at 0

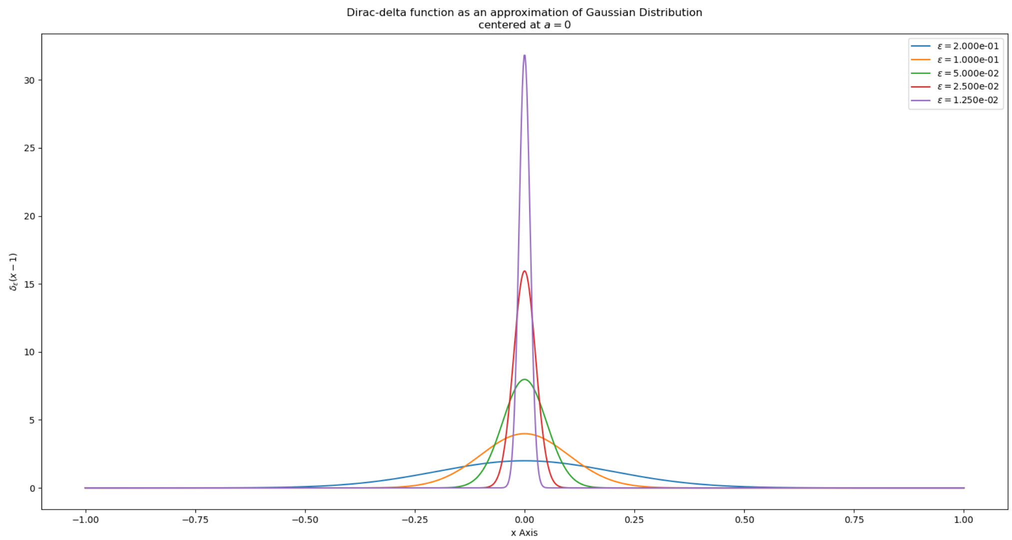

Figure 1: Gaussian Distribution

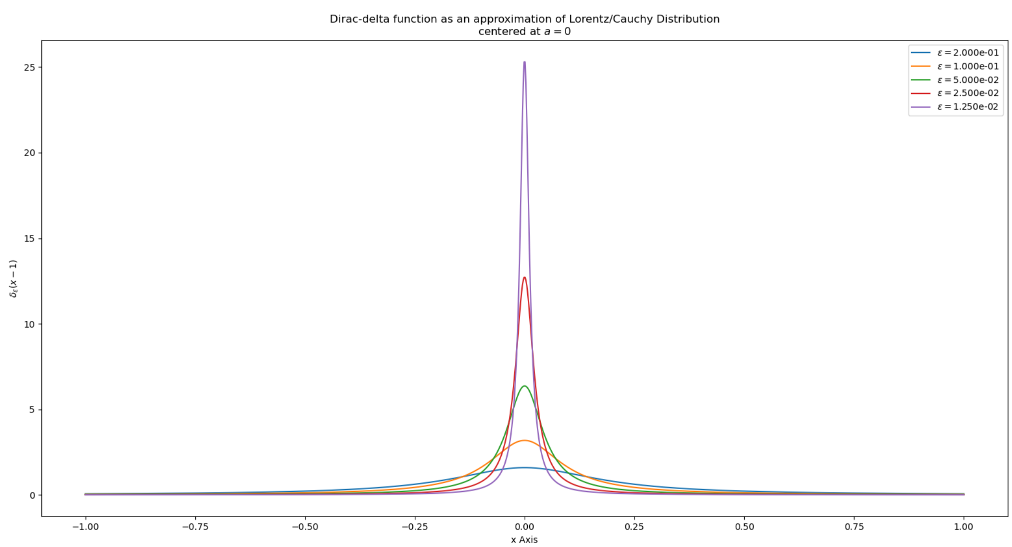

Figure 2: Lorentzian Distribution

3.1.2 When a = 1, i.e., centered at 1

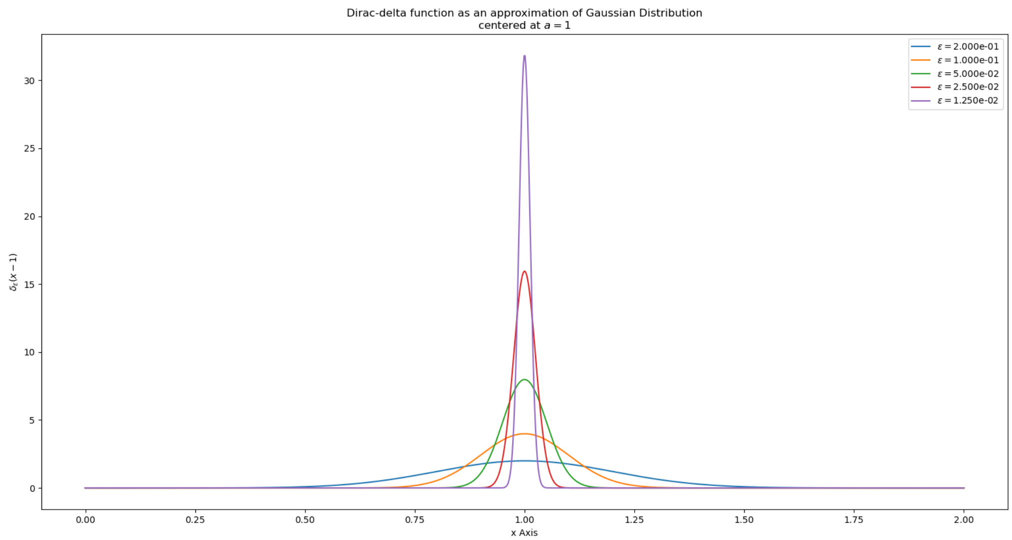

Figure 3: Gaussian Distribution

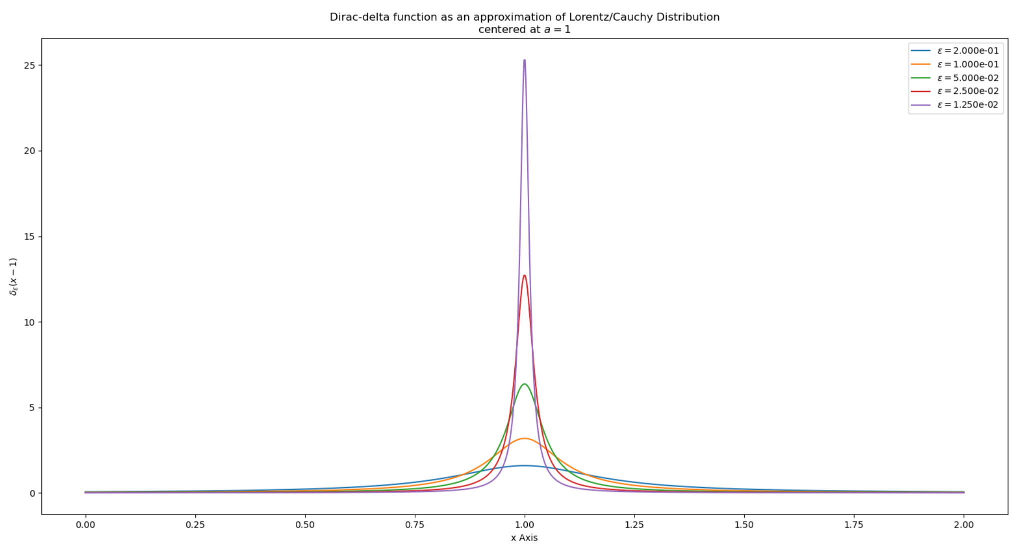

Figure 4: Lorentzian Distribution

3.2 Evaluating given integrals as per Q2(a) ii.

Integral 1

$$\int_{-\infty}^{\infty} \delta(x)dx$$

Integral 2

$$\int_{-\infty}^{\infty} \delta(x)(x+1)^2dx$$

Integral 3

$$\int_{-\infty}^{\infty} \delta(3x+1)9x^2dx$$

Simpson 1/3 Method and **Gauss-Hermite Quadrature** were used to numerically approximate the above integrals.
The following plots will show the comparison between the two methods:

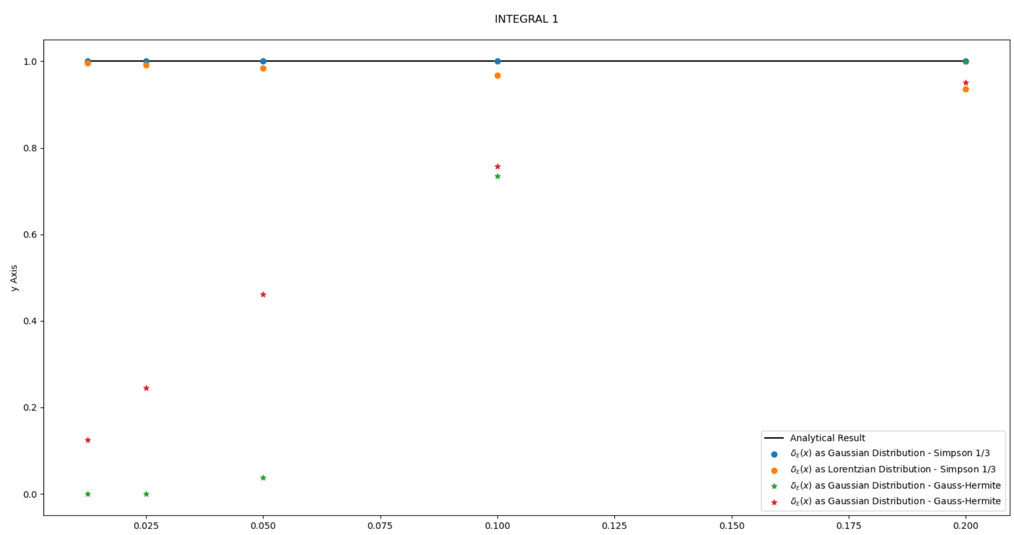

Figure 5: Integral 1

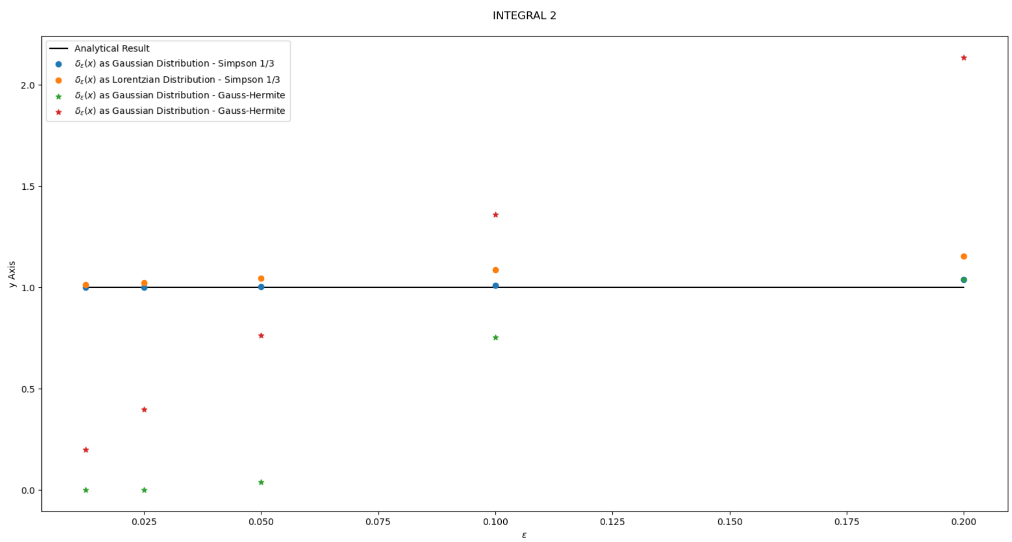

Figure 6: Integral 2

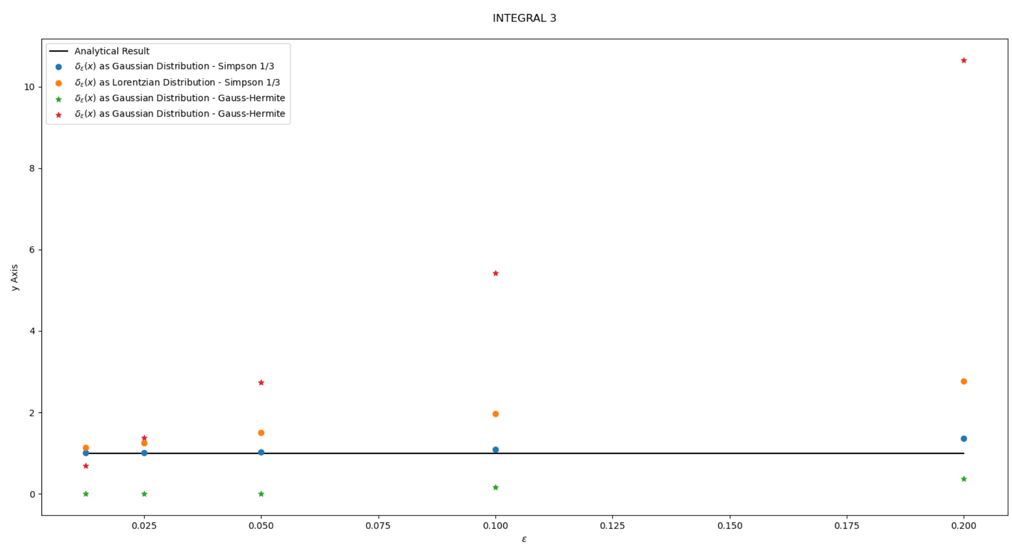

Figure 7: Integral 3

3.3 Tabulated Data

Integral 1

	Epsilon	Gaussian Dist. - Simpson 1/3	Lorentzian Dist. - Simpson 1/3	Gaussian Dist. - Gauss-Hermite	Lorentzian Dist. - Gauss-Hermite
0	0.2000	1.0	0.936549	9.993896e-01	0.951469
1	0.1000	1.0	0.968195	7.353506e-01	0.757063
2	0.0500	1.0	0.984088	3.767004e-02	0.460770
3	0.0250	1.0	0.992043	3.243477e-08	0.244438
4	0.0125	1.0	0.995985	2.228333e-33	0.124140

Integral 2

	Epsilon	Gaussian Dist. - Simpson 1/3	Lorentzian Dist. - Simpson 1/3	Gaussian Dist. - Gauss-Hermite	Lorentzian Dist. - Gauss-Hermite
0	0.2000	1.040000	1.153735	1.039762e+00	2.134047
1	0.1000	1.010000	1.085837	7.533238e-01	1.359810
2	0.0500	1.002500	1.045290	3.859034e-02	0.764777
3	0.0250	1.000625	1.023254	3.322717e-08	0.396864
4	0.0125	1.000156	1.011745	2.282772e-33	0.200411

Integral 3

	Epsilon	Gaussian Dist. - Simpson 1/3	Lorentzian Dist. - Simpson 1/3	Gaussian Dist. - Gauss-Hermite	Lorentzian Dist. - Gauss-Hermite
0	0.2000	1.360000	2.762892	3.633517e-01	10.643196
1	0.1000	1.090000	1.962057	1.617581e-01	5.424727
2	0.0500	1.022500	1.502350	8.282676e-03	2.736064
3	0.0250	1.005625	1.256653	7.131575e-09	1.371841
4	0.0125	1.001406	1.129730	4.899533e-34	0.686433

Figure 8: Data for each integral

4 Discussion

Since for the same epsilon the peak value of Gaussian distribution was higher as compared to Lorentzian Distribution , we can say that Gaussian distribution represents the δ function's impulsive behaviour more accurately.

We can also see through the results that the Simpson 1/3 rule performs better than the Gauss-Hermite Quadrature.