Mathematical Physics III

Lab Report for Assignment #1

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1 Theory

Q1. *What is Taylor series representation of a function? What do you mean by radius of convergence of series? What is MacLaurin Series?*

- The main purpose of series is to write a given complicated quantity as an infinite sum of simple terms; and since the terms get smaller and smaller, we can approximate the original quantity by taking only the first few terms of the series.

  Taylor Series: In the Taylor series representation of a function, the function is expanded into an infinite sum of terms that are expressed in terms of the function's derivatives at a single point.
  Let $f(x)$ be a function which is analytic at $x = a$.
  Then we can write $f(x)$ as the following power series , called the **Taylor Series of $f(x)$ at $x = a$**

  $$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \end{aligned}$$

  where $f^{(n)}(a)$ denotes the n^{th} derivative of f evaluated at point a.

  Radius Of Convergence: When a sequence converges, that means that as you get further and further along the sequence, the terms get closer and closer to a specific limit (usually a real number).
  A series is a sequence of sums. So for a series to converge, these sums have to get closer and closer to a specific limit as we add more and more terms up to infinity.
  A power series $\sum_{k=0}^{\infty} C_k x^k$ will converge only for certain values of x.
  For instance $\sum_{k=0}^{\infty} x^k$ converges for $-1 < x < 1$.
  In general, there is always an interval $(-R, R)$ in which a power series converges, and the number R **is called the radius of convergence** . The quantity R is called the radius of convergence because, in the case of a power series with complex coefficients,the values of x with $|x| > R$ form an open disk with radius R.

Mac Laurin Series: A Maclaurin series is a power series that allows one to calculate an approximation of a function $f(x)$ for input values close to zero, given that one knows the values of the successive derivatives of the function at zero.

A Maclaurin series can be used to approximate a function, find the antiderivative of a complicated function, or compute an otherwise uncomputable sum. Partial sums of a Maclaurin series provide polynomial approximations for the function.

A Maclaurin series is a special case of a Taylor series, obtained by setting $x_0 = 0$. The Maclaurin series of a function f is therefore the series

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^k(0)}{k!}x^k + ...$$

Q2. *Write down the Taylor series for a function of two variables.?*

- The Taylor Series for $f(x,t)$ is :

$$f(x,t) = f(a,b) + f_x(a,b) * (x-a) + f_t(a,b) * (t-b) + \frac{1}{2}f_{xx}(a,b) * (x-a)^2 +$$

$$\frac{1}{2}f_{xt}(a,b)(x-a)(t-b) + \frac{1}{2}f_{tx}(a,b)(x-)(t-b) + \frac{1}{2}f_{tt}(a,b)(t-b)^2 + ...$$

Q3. *Write the Maclaurin series representation for trignometric functions sin x, cos x and exp(x). Discuss the radius of convergence for each of them.*

- **For Sin(x):** We know,

$$f(0) = sin(0) = 0$$

$$f'(0) = cos(0) = 1$$

$$f''(0) = -sin(0) = 0$$

$$f'''(0) = -cos(0) = -1$$

$$f^4(0) = sin(0) = 0$$

$$f^5(0) = cos(0) = 1$$

Thus the Maclaurin series for sin(x) is

$$\sum_{k=0}^{\infty} f^{(k)}(0)\frac{(x-0)^k}{k!} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ...$$

$$= 0 + (\frac{1}{1!}x) + 0 + (\frac{-1}{3!}x^3) + 0 + (\frac{1}{5!}x^5).....$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$$

<u>Radius Of Convergence:</u> The ratio test gives us,

$$\lim_{k\to\infty} \frac{\frac{(-1)^{(k+1)}}{(2(k+1)+1)!}x^{2(k+1)+1}}{\frac{(-1)^{(k)}}{(2k+1)!}x^{2k+1}} = \lim_{k\to\infty} \frac{(2k+1)!}{(2k+3)!}|x|^2$$

$$= \lim_{k\to\infty} \frac{1}{(2k+3)(2k+2)}|x|^2 = 0$$

Because this limit is zero for all real values of x, the radius of convergence of the expansion is the set of all real numbers

- **For cos(x):**

$$cos(x) = \frac{d}{dx}sin(x)$$

$$= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

$$= \frac{d}{dx}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Radius of Convergence:The ratio test gives us,

$$\lim_{k\to\infty} \frac{\frac{(-1)^{(k+1)}}{(2(k+1))!} x^{2(k+1)}}{\frac{(-1)^{(k)}}{(2k!} x^{2k}} = \lim_{k\to\infty} \frac{(2k)!}{(2k+2)!} |x|^2$$

$$\lim_{k\to\infty} \frac{1}{(2k+1)(2k+2)} |x|^2 = 0$$

Because this limit is zero for all real values of x, the radius of convergence of the expansion is the set of all real numbers.

- **For e^x:**

$$\frac{d^k}{dx^k} f(x)|_{x=0}$$

for k= 0,1,2,3,....

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} ...... = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Radius Of Convergence: The ratio test gives us,

$$\lim_{k\to\infty} \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} = \lim_{k\to\infty} \frac{|x|}{k+1} = 0$$

Because this limit is zero for all real values of x, the radius of convergence of the expansion is the set of all real numbers.

2 Algorithm

Algorithm 1 Taylor Series e^x

function MYEXP(x, n)
▷ x is an array of values for which we need to find value of exponential function
▷ n is number of terms of taylor series we need to consider

l1=[] ▷ An empty list

for v **do** in x: ▷ Loop to take values from x
$exp = 0$
for i **do** in range(n)
$exp = exp + ((v^j)/(j)!)$ ▷ Taylor series for exponential function
$l1.append(exp)$ ▷ Appending value of exponent function to list l1
return $l1$ ▷ Returning the list l1

Algorithm 2 Taylor Series $sin(x)$

```
function MYSINSERIES(x, n)
        ▷ x is an array of values for which we need to find value of sin function
        ▷ n is number of terms of taylor series we need to consider

    l1=[]                                         ▷ An empty list

    for v do in x:                                ▷ Loop to take values from x
        sin = 0
        for i do in range(n)
            sin+ = (-1)^i * v^((2*i)+1)/((2i) + 1)!   ▷ Taylor series for sine function
            l1.append(sin)          ▷ Appending value of sin function to list l1
    return l1                                     ▷ Returning the list l1
```

Algorithm 3 Taylor Series $cos(x)$

```
function MYCOSSERIES(x, n)
        ▷ x is an array of values for which we need to find value of cosine function
        ▷ n is number of terms of taylor series we need to consider

    l1=[]                                         ▷ An empty list

    for v do in x:                                ▷ Loop to take values from x
        cos = 0
        for i do in range(n)
            cos+ = (-1)^i v^(2*i)/(2 * i)!        ▷ Taylor series for sin function
            l1.append(cos)          ▷ Appending value of cos function to list l1
    return l1                                     ▷ Returning the list l1
```

3 Programming

```
import math
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd

def MySinSeries(x,n):
    l1=[]
    for v in x:
        sin=0
        for i in range(n):
            sin+=(-1)*i*v*((2*i)+1)/math.factorial((2*i)+1)
        l1.append(sin)
    return l1

def MyCosSeries(x,n):
    l1=[]
    for v in x:
        cos=0
        for i in range(n):
            cos+=(-1)*i*v*(2*i)/math.factorial(2*i)
        l1.append(cos)
    return l1



x=np.linspace(-2*np.pi,2*np.pi,200)
sin_in=np.sin(x)
cos_in=np.cos(x)
n_a=[1,2,5,10,20]
lis=[]
lis1=[]
for n in n_a:
    lis.append(MySinSeries(x,n))
```

```
    lis1.append(MyCosSeries(x,n))
x0=[np.pi/4]
s=np.arange(2,21,2)
y0_sin=[]
y0_cos=[]
for n in s:
    y0_sin.append(MySinSeries(x0,n))
    y0_cos.append(MyCosSeries(x0,n))
d1=np.array([np.sin(x0)]*len(s))
d2=np.array([np.cos(x0)]*len(s))

fig, (ax1, ax2) = plt.subplots(2)
ax1.plot(x,lis[0],label="n=1",ls='--',marker=".")
ax1.plot(x,lis[1],label="n=2",ls='--',marker=".")
ax1.plot(x,lis[2],label="n=5",ls='--',marker=".")
ax1.plot(x,lis[3],label="n=10",ls='--',marker=".")
ax1.plot(x,lis[4],label="n=20",ls='--',marker=".")
ax1.plot(x,sin_in,label="inbulit",c="black")
ax1.set(xlabel="x",ylabel="sin(x)")
ax1.legend()
ax1.grid()
ax2.plot(s,d1,label="sin(pi/4)")
ax2.plot(s,y0_sin,marker="*")
ax2.set(xlabel="n",ylabel="sin(x)")
ax2.legend()
ax2.grid()
plt.show()

fig, (ax1, ax2) = plt.subplots(2)
ax1.plot(x,lis1[0],label="n=1",ls='--',marker=".")
ax1.plot(x,lis1[1],label="n=2",ls='--',marker=".")
ax1.plot(x,lis1[2],label="n=5",ls='--',marker=".")
ax1.plot(x,lis1[3],label="n=10",ls='--',marker=".")
ax1.plot(x,lis1[4],label="n=20",ls='--',marker=".")
ax1.plot(x,cos_in,label="inbulit",c="black")
```

```
ax1.set(xlabel="x",ylabel="cos(x)")
ax1.legend()
ax1.grid()
plt.plot(s,d2,label="cos(pi/4)")
plt.plot(s,y0_cos,marker="*")
ax2.set(xlabel="n",ylabel="cos(x)")
ax2.legend()
ax2.grid()
plt.show()

def mfun(x,tol):
    e=0
    n=0
    lis=[]
    while True:
        e=e+(-1)*n*x*((2*n)+1)/math.factorial((2*n)+1)
        lis.append(e)
        n+=1
        if len(lis)>=2:
            if lis[-2]==0:
                break
            else:
                err = abs((lis[-1]-lis[-2])/lis[-2])
            if err <= tol:
                break
    return e,n

def mysin(x_a):
    tol=float(input("Enter the tolerance value :"))
    r_a=[];n_a=[]
    for x in x_a:
        h=mfun(x,tol)
        r_a.append(h[0]);n_a.append(h[1])
    return r_a,n_a
```

```
x=np.arange(0,9*np.pi/8,np.pi/8)
sin_in=np.sin(x)
d=mysin(x)
data={"x":x,"sin(x)_calc":d[0],"n":d[1],"sin(x)_inbuilt":sin_in}
print(pd.DataFrame(data))

fig, (ax1) = plt.subplots(1)
plt.plot(x,d[0],label="calculated",marker="o")
plt.plot(x,sin_in,label="inbuilt")
plt.xlabel("x")
plt.ylabel("sin x")
plt.legend()
plt.grid()
plt.show()
```

4 Discussion

We get the plot of $sin(x)$ for different values of n from define function and comparing it with inbuilt function in the range $[-2\pi, 2\pi]$. From the plot we can see that as the value of n increases then the points lie on the curve of inbuilt function. For $n = 1, 2, 5$ the curve shows much deviation from the actual curve. And at $x = 0$ we can also see the converging nature of all curves.

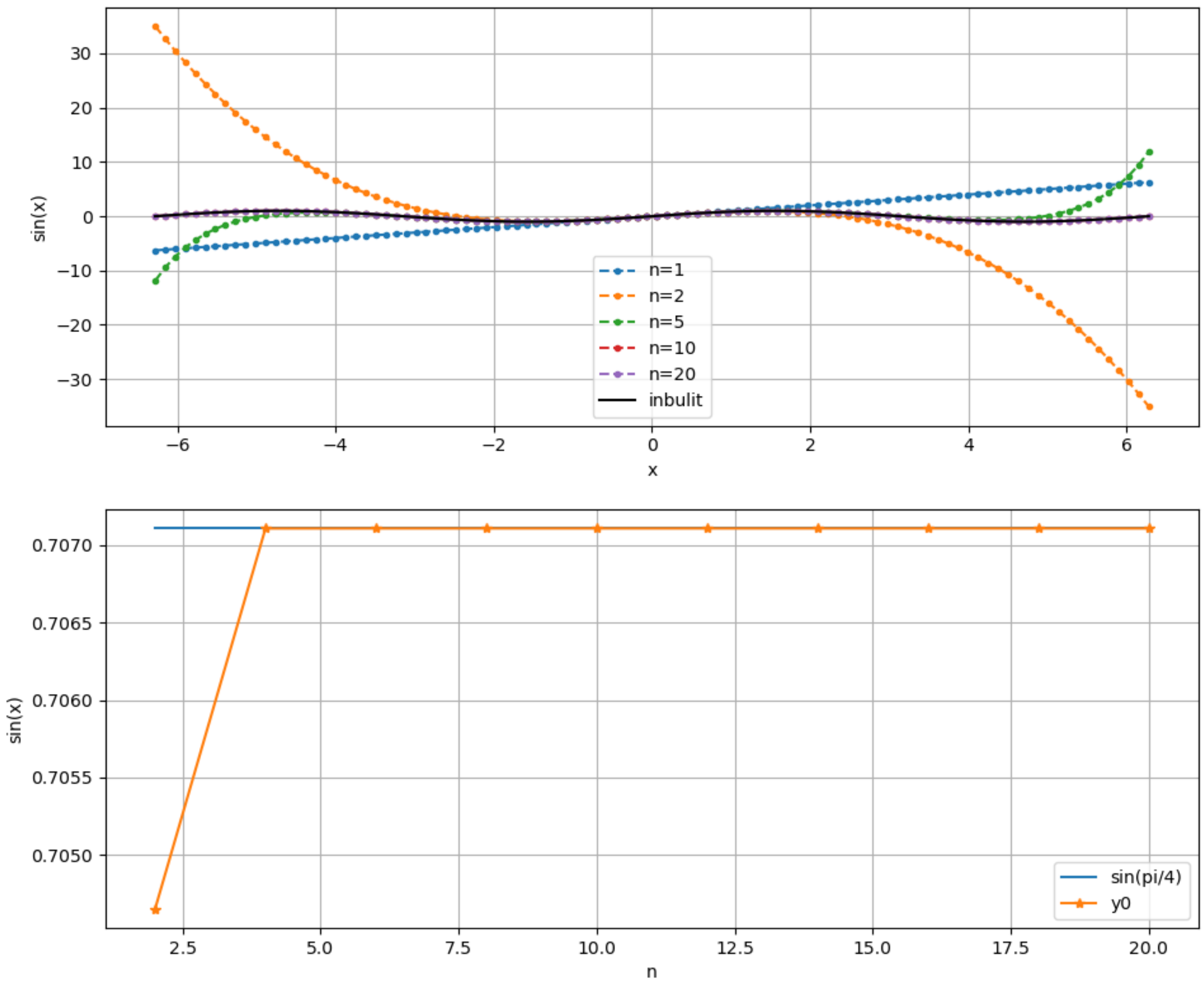

Figure 1: For sine

The second plot is of $sin(\frac{\pi}{4})$ plotted as straight line from inbuilt function and y_0 plotted by defined function with increasing value of n. From the plot we can see that for $n < 5$ the values obtained by defined function shows deviation from the exact values but as n approaches to 5 the value is same as the that of inbuilt function. So we can say that if n is large then the error in the values is minimum. Similar interpretation can be made to the plot of $cos(x)$.

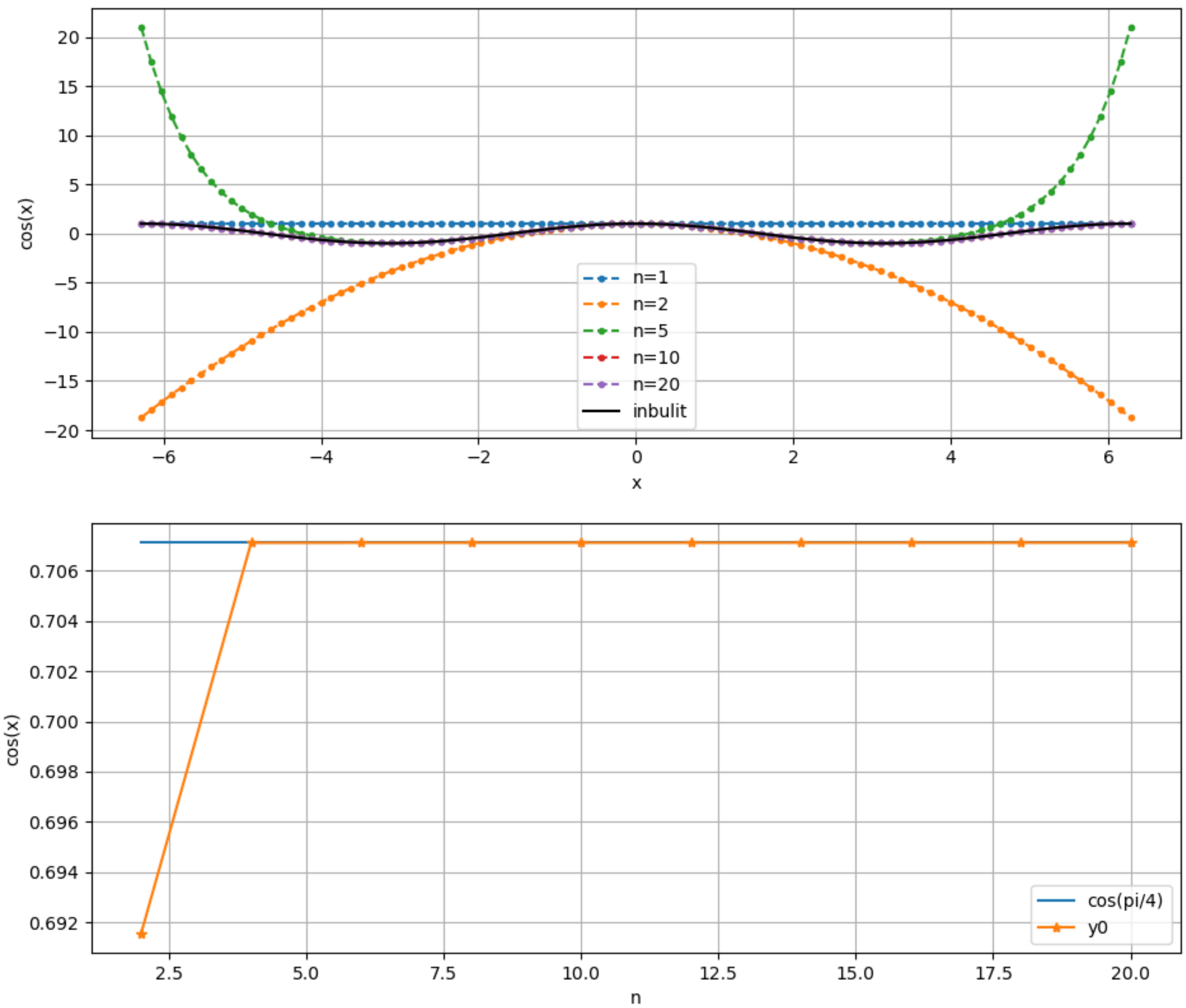

Figure 2: For cosine

This is the plot of $sin(x)$ by inbuilt python function and the points are the calculated by defined function upto 3 significant places in the given range of $[0, \pi]$ with step size $\frac{\pi}{8}$.