The 2-point Gauss - Laguerre Integration Rule

We have
$$L_0(x) = 1$$
$$L_1(x) = -x+1$$
$$L_2(x) = \frac{1}{2}(x^2-4x+2)$$

Gauss - Laguerre Quadrature

$$\int_0^\infty e^{-x} f(x)\, dx = \sum w_i\, f(x_i)$$

for 2-point gauss - laguerre

$$\int_0^\infty e^{-x} f(x)\, dx = w_1 f(x_1) + w_2 f(x_2)$$

where x_1 and x_2 are the abscissas and w_1 & w_2 are the weights.
The abscissa for n-point rule are the roots of the laguerre function of degree n.
We have,
$$L_2(x) = \frac{1}{2}(x^2-4x+2)$$
The roots of $L_2(x) = 0$ are the abscissas for 2-point gauss laguerre rule.

$$\frac{1}{2}(x^2-4x+2) = 0$$

$$x^2-4x+2=0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow x = \frac{4 \pm \sqrt{16-8}}{2}$$

$\Rightarrow 2 \pm \sqrt{2}$

$x_1 = 2 + \sqrt{2}$, $x_2 = 2 - \sqrt{2}$

To find the w_1 and w_2 weights, we use $L_0(x)$ and $L_1(x)$ to find relationship eq^n.

Using $L_0(x) = 1$

$$\int_0^\infty e^{-x} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2)$$

$$\int_0^\infty e^{-x} (1)\,dx = w_1 + w_2$$

$$[-e^{-x}]_0^\infty = w_1 + w_2$$

$$\Rightarrow w_1 + w_2 = 1 \quad \text{— } (1)$$

Using $L_1(x) = -x + 1$

$$\int_0^\infty e^{-x}(-x+1)\,dx = w_1 f(x_1) + w_2 f(x_2)$$

$$\int_0^\infty e^{-x}(-x)\,dx + \int_0^\infty e^{-x}\,dx = w_1 f(2+\sqrt{2}) + w_2 f(2-\sqrt{2})$$

$$[xe^{-x} + e^{-x}]_0^\infty + [-e^{-x}]_0^\infty = w_1(-\sqrt{2}-1) + w_2 (\sqrt{2}-1)$$

$$w_1(-\sqrt{2}-1) + w_2(\sqrt{2}-1) = 0 \quad \text{— } (2)$$

Subtracting $(\sqrt{2}-1)$ eq (1) from eq (2), we get

$$2 w_1 \sqrt{2} = \sqrt{2} - 1$$

$$w_1 = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$w_1 = \frac{2-\sqrt{2}}{4}$$

and so, $$w_2 = \frac{2+\sqrt{2}}{4}$$

2-point Gauss - Laguerre Quadrature Integration Rule is given below

$$\boxed{\int_0^\infty e^{-x} f(x)\, dx = \frac{2-\sqrt{2}}{4} f(2+\sqrt{2}) + \frac{2+\sqrt{2}}{4} f(2-\sqrt{2})}$$