The 2-point Gauss - Hermite Integration Rule

We have
$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$

Gauss - Hermite Quadrature Formula

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = \sum_{i=1}^{n} w_i f(x_i)}$$

For 2-point Gauss - Hermite (n = 2)

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = w_1 f(x_1) + w_2 f(x_2)}$$

where x_1 and x_2 are the abscissas and w_1 & w_2 are the weights.

The abscissa for n-point rule are the roots of the hermite function of degree n

We have,

$H_2(x) = 4x^2 - 2$. The roots of $H_2(x) = 0$ are the abscissas for 2-point Gauss - Hermite rule

$$\Rightarrow 4x^2 - 2 = 0$$
$$4x^2 = 2$$
$$x^2 = \frac{1}{2}$$
$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

$$\Rightarrow x_1 = \frac{1}{\sqrt{2}}$$
$$\& \quad x_2 = -\frac{1}{\sqrt{2}}$$

To find the w_1 and w_2 weights, we use $H_0(x)$ and $H_1(x)$ to find the relationship eqⁿ

Using $H_0(x) = 1$

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2)$$

$$\int_{-\infty}^{\infty} e^{-x^2} (1)\,dx = w_1 + w_2$$

$w_1 + w_2 = \sqrt{\pi}$ [Integration ~~of~~ on the next page] (I_1)

— ①

Using $H_1(x) = 2x$

$$\int_{-\infty}^{\infty} e^{-x^2} (2x)\,dx = w_1 f(1/\sqrt{2}) + w_2 f(-1/\sqrt{2})$$

$w_1(\sqrt{2}) + w_2(-\sqrt{2}) = 0$ [Integration on next page] (I_2)

— ②

Solving eqⁿ ① and eqⁿ ②, we get

$w_1 + w_2 = \sqrt{\pi}$ $(\times \sqrt{2})$

$$\begin{aligned} \sqrt{2}w_1 + \sqrt{2}w_2 &= \sqrt{2}\,\pi \\ + \quad \sqrt{2}w_1 - \sqrt{2}w_2 &= 0 \\ \hline 2\sqrt{2}\,w_1 &= \sqrt{2}\sqrt{\pi} \end{aligned}$$

$$\boxed{w_1 = \frac{\sqrt{\pi}}{2}}$$

$w_1 + w_2 = \sqrt{\pi}$ $\Rightarrow$ $w_2 = \sqrt{\pi} - \dfrac{\sqrt{\pi}}{2}$ $\Rightarrow$ $\boxed{w_2 = \dfrac{\sqrt{\pi}}{2}}$

$\Rightarrow \dfrac{\sqrt{\pi}}{2} + w_2 = \sqrt{\pi}$

* Integration of $\int_{-\infty}^{\infty} e^{-x^2} dx$ (I_1)

$$I_1 = \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dy \quad \text{(dummy variables)}$$

$$I_1^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

Convert to polar coordinates, $r = \sqrt{x^2+y^2}$, element of area $dx \cdot dy = r \cdot dr \cdot d\theta$
Limits are $r = 0$ to ∞, $\theta = 0$ to 2π

$$I_1^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} \cdot r \cdot dr \cdot d\theta = 2\pi \int_0^{\infty} r \cdot e^{-r^2} dr$$

Substitute $t = r^2 \cdot dt = 2 \cdot r \cdot dr$. Limits of $t = 0$ to ∞

$$I_1^2 = 2\pi \int_0^{\infty} \frac{e^{-t}}{2} dt = \pi\left[-(e^{-\infty} - e^{-0})\right] = \pi$$

$$I_1^2 = \pi \Rightarrow \boxed{I_1 = \sqrt{\pi}}$$

* Integration of $\int_{-\infty}^{\infty} e^{-x^2} (2x)\, dx$ (I_2)

$$I_2 = \int_{-\infty}^{\infty} e^{-x^2} (2x)\, dx$$

Put $x^2 = t$
$2x\,dx = dt$
Limits stay same

$$I_2 = \int_{-\infty}^{\infty} e^{-t} dt = \left[-e^{-t}\right]_{-\infty}^{\infty}$$

$$I_2 = \left[-(e^{-\infty} - e^{\infty})\right] = 0$$

$$\Rightarrow \boxed{I_2 = 0}$$

$\therefore$, 2 - point Gauss - Hermite Quadrature Formula

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = \frac{\sqrt{\pi}}{2}\left[f\left(\frac{1}{\sqrt{2}}\right) + f\left(\frac{-1}{\sqrt{2}}\right)\right]}$$