Boundary Value Problem (Finite Difference Method)

Lab Report for **Assignment No. 11**

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Course and Semester - B.Sc(Hons.) Physics Sem IV
Paper Title - Mathematical Physics III Lab
Unique Paper Code - 32221401

Due Date - 25 April 2022
Date of Submission - 25 April 2022

Theory

(a) The finite difference method for the linear second - order BVP

$$y'' = p(x)y' + q(x)y + r(x)$$

for $a \le x \le b$ with $y(a) = \alpha$ and $y(b) = \beta$

First we divide the interval $[a, b]$ into $(N+1)$ equal subintervals whose end points are the mesh points

$$x_i = a + ih \quad \text{for} \quad i = 0, 1 \dots N+1$$

where $h = \dfrac{b-a}{(N+1)}$ (step size)

At the interior mesh points x_i for $i = 1, 2 \dots N$

$$y''(x_i) = p(x_i)\, y'(x_i) + q(x_i)\,(x_i) + r(x_i)$$

Expanding y in a third Taylor Polynomial about x_i evaluated at x_{i+1} and x_{i-1}.

$$y(x_{i+1}) = y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{6} y'''(x_i) + \frac{h^4}{24} y^{(4)}(\xi_i^+) \quad \text{--- (1)}$$

for some ξ_i^+ in (x_i, x_{i+1})

$$y(x_{i-1}) = y(x_i - h) = y(x_i) - hy'(x_i) + \frac{h^2}{2} y''(x_i) - \frac{h^3}{6} y'''(x_i) + \frac{h^4}{24} y^{(4)}(\xi_i^-) \quad \text{--- (2)}$$

for some ξ_i^- in (x_{i-1}, x_i)

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Adding (1) & (2)

$$y(x_{i+1}) + y(x_{i-1}) = 2y(x_i) + h^2 y''(x_i) + \frac{h^4}{24}\left[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)\right]$$

$$\Rightarrow y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{24}\left[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)\right]$$

$$\Rightarrow y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{12} y^{(4)}(\xi_i)$$

for some & ξ_i in (x_{i-1}, x_{i+1})

→ Centered Difference formula for $y''(x_i)$.

~~Use of this centered difference formula in given BVP will result in~~

$$\cancel{y'(x_i) = \frac{1}{2h}\left[y(x_{i+1}) - y(x_{i-1})\right] - \frac{h^2}{6} y'''(\eta_i)}$$

~~for some η_i in (x_{i-1}, x_{i+1})~~

Use of the centered difference formula in given eq^n will results in

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = p(x_i)\left[\frac{y(x_{i+1}) - y(x_{i-1})}{2h}\right] + q(x_i)y(x_i) + r(x_i) - \frac{h^2}{12}\left[2p(x_i)y'''(\eta_i) - y^{(4)}(\xi_i)\right]$$

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Using this eq^n together with the boundary condition $y(a) = \alpha$ and $y(b) = \beta$ we ~~get~~ define the system of linear equations

$$w_0 = \alpha \quad , \quad w_{N+1} = \beta$$

and

$$\left(\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2}\right) + p(x_i)\left(\frac{w_{i+1} - w_{i-1}}{2h}\right)$$

$$+ q(x_i) w_i = -r(x_i) \quad , \quad \text{for each } i = 1, 2, \ldots N$$

$$\Rightarrow -\left(1 + \frac{h}{2} p(x_i)\right) w_{i-1} + \left(2 + h^2 q(x_i)\right) w_i - \left(1 - \frac{h}{2} p(x_i)\right) w_{i+1}$$

$$= -h^2 r(x_i)$$

The resulting system of eq^{ns} is expressed in $N \times N$ matrix form

$$A w = b \quad \text{where}$$

$$A = \begin{bmatrix} 2 + h^2 q(x_1) & -1 + \frac{h}{2} p(x_1) & 0 & - & - & - & 0 \\ -1 - h/2\, p(x_2) & 2 + h^2 q(x_2) & -1 + h/2\, p(x_2) & & & & \vdots \\ 0 & & & & & & 0 \\ \vdots & & & & & & -1 + h/2\, p(x_{N-1}) \\ 0 & - & - & - & 0 & -1 - \frac{h}{2} p(x_N) & 2 + h^2 q(x_N) \end{bmatrix}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

$$b = \begin{bmatrix} -h^2 r(x_1) + \left(1 + \frac{h}{2} p(x_1)\right) w_0 \\ -h^2 r(x_2) \\ \vdots \\ -h^2 r(x_{N-1}) \\ -h^2 r(x_N) + \left(1 - \frac{h}{2} p(x_N)\right) . w_{N+1} \end{bmatrix}$$

(b) $-y'' + \pi^2 y = 2\pi^2 \sin(\pi x) \quad ; \quad 0 \le x \le 1$

Boundary conditions $y(0) = y(1) = 0$

$$y'' = \pi^2 y - 2\pi^2 \sin(\pi x)$$

comparing with

$$y'' = p_i y' + q_i y + r_i$$

$p_i = 0$, $q_i = \pi^2$, $r_i = -2\pi^2 \sin(\pi x)$

$-\cancel{\left(1 + \frac{h}{2}\right)}$

$$-\left(1 + \frac{h}{2} p_i\right) w_{i-1} + (2 + h^2 q_i) w_i - \left(1 - \frac{h}{2} p_i\right) w_{i+1}$$

$$= -h^2 r_i \quad \text{—— (1)}$$

For 3 grid point $(n = 4)$

Since $q_i = \pi^2$ and, $p_i = 0$, $h = \frac{b-a}{4} = 0.25$

Putting these in eq (1)

we get

$$-w_{i-1} + [2 + 0.625\pi^2] w_i - w_{i+1} = -h^2 r_i$$

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$\pi^2 h^2 \quad 2h^2\pi^2$

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for $n = 1$

$$0 + [2 + 0.625\pi^2] w_1 - w_2 = -h^2 2\pi^2 \sin(0.25\pi)$$

for $n = 2$

$$-w_1 + [2 + 0.625\pi^2] w_2 - w_3 = -h^2 2\pi^2 \sin[\pi/2)$$

for $n = 3$

$$-w_2 + [2 + 0.625\pi^2] w_3 - w_4 = -h^2 2\pi^2 \sin[3\pi/4]$$

for $n = 4$

$$-w_3 + [2 + 0.625\pi^2] w_4 - 0 = -h^2 2\pi^2 \sin(\pi)$$

In matrix form

$$\begin{bmatrix} 1 & 0 & - & - & - & \\ -1 & (2+\pi^2h^2) & (-1) & & & \\ & -1 & (2+\pi^2h^2) & -1 & & \\ & & -1 & 2+\pi^2h^2 & -1 & \\ & & & 0 & 1 & \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = B$$

$$B = \begin{bmatrix} 0 \\ 0.87837 \\ 1.225 \\ 0.87837 \end{bmatrix}$$

Obtained output $\rightarrow$ $[0, 0.72537, 1.02562, 0.72537, 0]$

16, $y'' + y = \sin 3x$ $[0 \le x \le \pi/2]$

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Boundary conditions

$y(0) + y'(0) = -1$

$y'(\pi/2) = 1$

Comparing with $y'' = p_i y' + q_i y + r_i$

$p_i = 0$, $q_i = -1$, $r_i = \sin 3x$.

$$-\left(1 + \frac{h^2}{2} p_i\right) w_{i-1} + (2 + h^2 q_i) w_i - \left(1 - \frac{h}{2} p_i\right) w_{i+1}$$

$$= -h^2 r_i$$

For 3 grid points

$$h = \frac{b-a}{N+1} = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$$

for $n = 0$

$$0 + \left[2 + \left(\pi/8\right)^2\right] w_i - w_{i+1} = h^2 r_1$$

for $n = 1$

$$-w_0 + 1.845 w_1 - 1 w_2 = -0.1424$$

for $n = 2$

$$-w_1 + 1.845 w_2 - 1 w_3 = -0.1.0904$$

for $n = 3$

$$-w_2 + \cancel{1.1950} \; 1.845 w_3 - w_4 = 0.05901$$

for $n = 4$

$$-w_4 + 1.845 w_5 = h^2 r_4$$

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$$A = \begin{bmatrix} 1.060 & -2 & 0 & 0 & 0 \\ -1 & 1.845 & -1 & 0 & 0 \\ 0 & -1 & 1.845 & -1 & 0 \\ 0 & 0 & -1 & 1.845 & -1 \\ 0 & 0 & 0 & -2 & 1.845 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.7853 \\ -0.14247 \\ -0.10904 \\ 0.0590 \\ 0.93\cancel{6}10 \end{bmatrix}$$

Obtained values of y

$y = [-1.02367, -0.935, -0.5604, 0.0099, 0.518]$

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Algorithm

To approximate the solution of BVP

$$y'' = p(x) y' + q(x) y + r(x) \quad \text{for } a \le x \le b$$

with $y(a) = \alpha$ and $y(b) = \beta$;

INPUT end points a, b; boundary condition α, β, integer $N \geq 2$.

OUTPUT approximation w_i to $y(x_i)$ for each $i = 0, 1 \ldots, N+1$.

Step 1 set $h = (b-a)/(N+1)$;

$x = a + h$;

$a_1 = 2 + h^2 q(x)$;

$b_1 = -1 + (h/2)\, p(x)$;

$d_1 = -h^2 r(x) + (1 + (h/2)\, p(x))\alpha$

Step 2 for $i = 2 \ldots N-1$

set $x = a + ih$;

$a_i = 2 + h^2 q(x)$;

$b_i = -1 + (h/2)\, p(x)$;

$c_i = -1 - (h/2)\, p(x)$;

$d_i = -h^2 r(x)$.

Step 3 set $x = b - h$;

$a_N = 2 + h^2 q(x)$;

$c_N = -1 - (h/2)\, p(x)$;

$d_N = -h^2 r(x) + (1 - (h/2)\, p(x))\beta$.

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Step 4   Set $l_1 = a_1$;

$u_1 = b_1 / a_1$;

$z_1 = d_1 / l_1$

Step 5   For $i = 2, \ldots, N-1$

Set $l_i = a_i - c_i u_{i-1}$;

$u_i = b_i / l_i$;

$z_i = (d_i - c_i z_{i-1}) / l_i$.

~~Step 6 Set~~

Step 6   Set $l_N = a_N - c_N u_{N-1}$;

$z_N = (d_N - c_N z_{N-1}) / l_N$

Step 7   Set $w_0 = \alpha$;

$w_{N+1} = \beta$.

$w_N = z_N$

Step 8   For $i = N-1, \ldots, 1$

Set $w_i = z_i - u_i w_{i+1}$.

Step 9   For $i = 0, \ldots, N+1$

Set $x = a + ih$;

OUTPUT (x, w_i).

Step 10   STOP