(b)(i) Eq: $-y'' + \pi^2 y = 2\pi^2 \sin(\pi x)$: $x \in [0,1]$

BC: $y(0) = y(1) = 0$ (Dirichlet BC)

$p(x) = 0 = p_i$

$q(x) = \pi^2 = q_i$

$r(x) = -2\pi^2 \sin(\pi x)$

$\therefore r_i = -2\pi^2 \sin(\pi \cdot ih)$

where $h = \frac{1-0}{n}$, n = no. of nodal points.

$\therefore$ We can solve this equation by matrix form of finite difference method.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2+(\pi/4)^2 & -1 & 0 & 0 \\ 0 & -1 & 2+(\pi/4)^2 & -1 & 0 \\ 0 & 0 & -1 & 2+(\pi/4)^2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \sqrt{2}(\pi/4)^2 \\ 2(\pi/4)^2 \\ \sqrt{2}(\pi/4)^2 \\ 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Solution: $Ax = B$

$X = A^{-1}B$

upon solving $A^{-1}B$, we get:

$$X = \begin{bmatrix} 0 \\ 0.725371 \\ 1.0259 \\ 0.725470 \\ 0 \end{bmatrix}$$

(ii) EQ: $u'' + u = \sin(3x)$, $x \in [0, \pi/2]$

BC: $u(0) + u'(0) = -1$ (Robin BC)

$u'(\pi/2) = 1$ (Neumann BC)

for 3 nodal and 2 boundary points:

$$h = \frac{\pi}{2 \times 4} = \frac{\pi}{8}$$

$p(x) = 0 = p_i$

$q(x) = -1 = q_i$

$r(x) = \sin(3x)$ $\therefore$ $r_i = \sin(3ih) = \sin(3i\pi/8)$.

Again, using matrix form of the finite diff. Metho

$$A = \begin{bmatrix} 2.060389 & -2 & 0 & 0 & 0 \\ -1 & 1.845787 & -1 & 0 & 0 \\ 0 & -1 & 1.845787 & -1 & 0 \\ 0 & 0 & -1 & 1.845787 & -1 \\ 0 & 0 & 0 & -2 & 1.845787 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.785398 \\ -0.0031706 \\ -0.00633995 \\ -0.009506575 \\ -0.012669121 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Upon Solving $X = A^{-1}B$ numerically, we get:

$$X = \begin{bmatrix} -1.023672 \\ -0.935445 \\ -0.560587 \\ -0.005657 \\ 0.599840 \end{bmatrix}$$