2014-02-21.sagews

February 21, 2014

Contents

1 Lecture on Continued Fractions (Feb 21, 2014)

- Note: there is an old English proof of the cont frac of e here (thanks to Victor Miller for telling me):

  The Simple Continued Fraction Expansion of e Author(s): C. D. Olds Source: The American Mathematical Monthly, Vol. 77, No. 9 (Nov., 1970), pp. 968-974

- on whiteboard: review of last time; proof that rational and real number has a continued fraction representation.

```
# Sage has a continued fraction command:

v = continued_fraction(939/391917); v
```

```
[0, 417, 2, 1, 1, 1, 7, 5]
```

```
continued_fraction(pi)
```

```
[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14]
```

```
continued_fraction(pi, bits=200)
```

```
[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3,
13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, 2, 1, 1]
```

```
@interact
def f(x=pi, bits=(20,(5..100))):
    try:
        v = continued_fraction(x,bits=bits)
        show(v)
        show(v.convergents())
    except:
        print "enter something valid..."
```

1.1 Fibonacci Numbers

$$f_0 = 1, f_1 = 1, f_2 = 1 + 1, , f_n = f_{n-1} + f_{n-2}$$

```
f = [1,1]
for i in range(2,20):
    f.append(f[i-1]+f[i-2])
print f
```

```
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765]
```

```
continued_fraction((1+sqrt(5))/2)
```

```
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1]
```

1.2 What well do next

Theorem A infinite (simple) continued fraction is eventually repeating if and only if it is a quadratic irrational.

```
continued_fraction(sqrt(7))
```

```
[2, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1]
```

```
continued_fraction((sqrt(2014)+1)/3)
```

```
[15, 3, 2, 2, 1, 1, 3, 2, 44, 2, 3, 1, 1, 2, 2]
```

```
continued_fraction((sqrt(2014)+1)/3,bits=300)
```

```
[15, 3, 2, 2, 1, 1, 3, 2, 44, 2, 3, 1, 1, 2, 2, 3, 29, 1, 1, 1, 2, 17, 1, 1, 2, 1, 4, 3,
1, 2, 4, 2, 1, 3, 4, 1, 2, 1, 1, 17, 2, 1, 1, 1, 29, 3, 2, 2, 1, 1, 3, 2, 44, 2, 3, 1, 1,
2, 2, 3, 29, 1, 1, 1, 2, 17, 1, 1, 2, 1, 4, 3, 1, 2, 4, 2, 1, 3, 4, 1, 2, 1, 1, 17, 2, 1,
1, 1, 29, 3, 2, 2, 1, 1, 3, 2, 44]
```

An algebraic number of degree d is any root of an irreducible polynomial of degree d with integer coefficients.

Open Problem Richard Guy (see [Guy94, pg. 260])

- Is there a (real) algebraic number of degree greater than two whose simple continued fraction has unbounded a_n?
- Does every such number have unbounded a_n?

```
continued_fraction(2^(1/3),bits=300)
```

```
[1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1, 3, 4, 1, 1, 2, 14,
3, 12, 1, 15, 3, 1, 4, 534, 1, 1, 5, 1, 1, 121, 1, 2, 2, 4, 10, 3, 2, 2, 41, 1, 1, 1, 3,
7, 2, 2, 9, 4, 1, 3, 7, 6, 1, 1, 2, 2, 9, 3, 1, 1, 69, 4, 4, 5, 12, 1, 1, 5]
```

```
R.<x> = PolynomialRing(RealField(200))
v = (x^7 + 13*x^3 + 5).roots(); v
```

```
[(-0.72223233566084097778185658243928964671385522861651662214416, 1)]
```

```
continued_fraction(v[0][0])
```

```
[-1, 3, 1, 1, 1, 1, 304, 1, 1, 3, 7, 1, 12, 1, 4, 2, 1, 4, 7, 1, 1, 2, 72, 1, 1, 24, 1, 1,
1, 2, 81, 7, 212, 1, 5, 2, 1, 9, 1, 5, 2, 1, 1, 1, 5, 1, 1, 1, 5, 20, 3, 1, 7, 3, 2]
```