# Inverted pendulum on cart - ES642 project report

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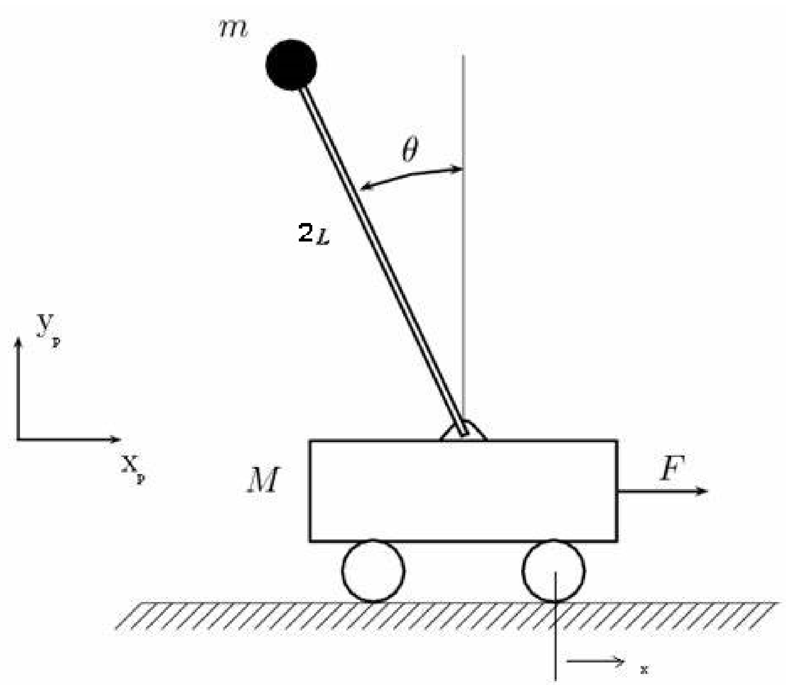
## 1 System description

In this report, we consider the cart-pole system, where, to balance a simple pendulum around its unstable equilibrium(inverted state), we can control only the horizontal force on the cart.




Inverted Pendulum on Cart

### 1.1 Equations of motion

As given in the problem statement of the inverted pendulum on a cart, the equations of motion are:

$$(m + M)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F \tag{1}$$

$$(I + ml^2)\ddot{\theta} + ml\ddot{x}\cos\theta - mgl\sin\theta = 0 \tag{2}$$

where,
$m$ is the mass of the pendulum,
$M$ is the mass of the cart,
$I$ is the moment of inertia of the pendulum rod,
$2l$ is the length of the pendulum rod and
$g$ is the acceleration due to gravity.

Extracting the acceleration terms $\ddot{x}$ *and* $\ddot{\theta}$, we obtain the expressions:

$$\ddot{x} = \frac{(F + ml\dot{\theta}^2 sin(\theta) - \frac{ml^2 g\sin(2\theta)}{2(I+ml^2)})(I + ml^2)}{(m + M)I + mMl^2 + m^2l^2\sin^2\theta}, \tag{3}$$

$$\ddot{\theta} = \frac{(m + M)(mgl\sin(\theta) - ml\cos\theta(F + ml\dot{\theta}^2\sin\theta))}{(m + M)I + mMl^2 + m^2l^2\sin^2\theta}. \tag{4}$$

### 1.2 Linearized equations of motion

The nonlinear equations of motion mentioned in the previous section can be linearised about unstable equilibrium and expressed as shown below:

$$(m + M)\ddot{x} + ml\ddot{\theta} = F, \tag{5}$$

$$(I + ml^2)\ddot{\theta} + ml\ddot{x} = 0. \tag{6}$$

## 2 Feedback linearization based control

### 2.1 Description

Feedback linearization is a differentiable coordinate transformation(whose inverse transformation is also differentiable) of a non-linear system model using non-linear state feedback input to estimate and track(to zero) the non-linear terms in the model.[1]

This provides the control system design engineer with the freedom to use well-established tools from linear systems theory to design control algorithms for non-linear systems.

Consider a non-linear system given by:

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}, \quad \mathbf{y} = h(\mathbf{x}) \qquad (7) \\ \text{where, } \mathbf{x} &\in \mathbb{D}_x(\subseteq \mathbb{R}^n), \mathbf{u} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^p \\ f &: \mathbb{D}_x \to \mathbb{R}^n, g : \mathbb{D}_x \to \mathbb{R}^{n\times m}, h : \mathbb{D}_x \to \mathbb{R}^p \end{aligned}$$

After linearization $(x, u)$ maps to $(z, v)$ satisfying:

$$\begin{aligned} \dot{\mathbf{z}} &= A\mathbf{z} + B\mathbf{v} \\ \mathbf{y} &= C\mathbf{z}, \quad \mathbf{z} \in \mathbb{D}_z(\subseteq \mathbb{R}^k), \mathbf{v} \in \mathbb{R}^m \end{aligned} \qquad (8)$$

In the general case, when $A|B$ is not full rank, we have:

$$\zeta = T(x) = \begin{bmatrix} \eta \\ z \end{bmatrix} \qquad (9)$$

$$\dot{z} = Az + Bv, \ v = \gamma(x)[u - \alpha(x)] \qquad (10)$$

$$y = Cz \qquad (11)$$

$$\underbrace{\dot{\eta} = f(\eta, z)}_{\text{(internal dynamics)}} \qquad (12)$$

### 2.2 Feedback linearizability

Consider the nonlinear system,

$$\dot{x} = f(x) + g(x)u \tag{13}$$

where, $f : \mathbb{D} \to \mathbb{R}^n$,$g : \mathbb{D} \to \mathbb{R}^{n \times p}$, being sufficiently smooth on $D \subset \mathbb{R}^n$. This is said to be feedback linearizable (or input-state linearizable) if there exists a continuously differentiable map with a continuously differentiable inverse(a diffeomorphism), $T : \mathbb{D} \to \mathbb{R}^n$, such that $D_z = T(D)$ contains the origin and the change of variables $z = T(x)$ transforms [1] the nonlinear system into the form

$$\dot{z} = Az + Bv \tag{14}$$

$$v = \gamma(T^{-1}(z)[u - \alpha(T^{-1}(z))] \implies u = \alpha + \gamma^{-1}(x)v \tag{15}$$

This also has $A, B$ being controllable,$\gamma$ being non-singular for all $x \in D$.

### 2.3 Partial feedback linearization for underactuated systems

Consider an underactuated mechanical system i.e. whose dynamics may be expressed in the form shown below:

$$M_{11}(q, \dot{q})\ddot{q}_1 + M_{12}(q, \dot{q})\ddot{q}_2 + h_1(q, \dot{q}) + g_1(q, \dot{q}) = 0 \tag{16}$$

$$M_{21}(q, \dot{q})\ddot{q}_2 + M_{22}(q, \dot{q})\ddot{q}_2 + h_2(q, \dot{q}) + g_2(q, \dot{q}) = u \tag{17}$$

These systems, due to non-actuated degrees of freedom, cannot undergo full state feedback linearization.

Hence we can choose to linearize either the actuated degree of freedom(collocated partial feedback linearization) or the non-actuated degree of freedom(non-collocated partial feedback linearization). In this report, we only discuss the collocated partial feedback linearization approach. From (17) we have,

$$\ddot{q}_1 = -M_{11}^{-1}(M_{12}\ddot{q}_2 + h_1 + g_1). \tag{18}$$

By substituting (18) in (16), we obtain the partial feedback linearizing controller $u_{lin}$:

$$u_{lin} = (M_{22} - M_{21}M_{11}^{-1}M_{12})\ddot{q}_2 + h_2 - M_{21}M_{11}^{-1}h_1 + g_2 - M_{21}M_{11}^{-1}g_1 \tag{19}$$

The resulting system is given by:

$$M_{11}\ddot{q}_1 + h_1 + g_1 = -M_{12}(u_{lin}) \tag{20}$$

$$\ddot{q}_2 = u_{lin}. \tag{21}$$

Now, the control system designer is free to use tools from linear systems theory to control the new input $u_{lin}$.

### 2.4 Energy pumping and partial feedback linearization for cart-pole system

For the cart-pole system, we obtain the following equations after partial feedback linearization:

$$\ddot{x} = \frac{u + ml\dot{\theta}^2 + 0.5mg\sin(2\theta)}{M + m\sin^2\theta} = u_{lin} \tag{22}$$

$$\ddot{\theta} = \frac{g\sin\theta - \ddot{x}\cos\theta}{l} = \frac{g\sin\theta - u_{lin}\cos\theta}{l} \tag{23}$$

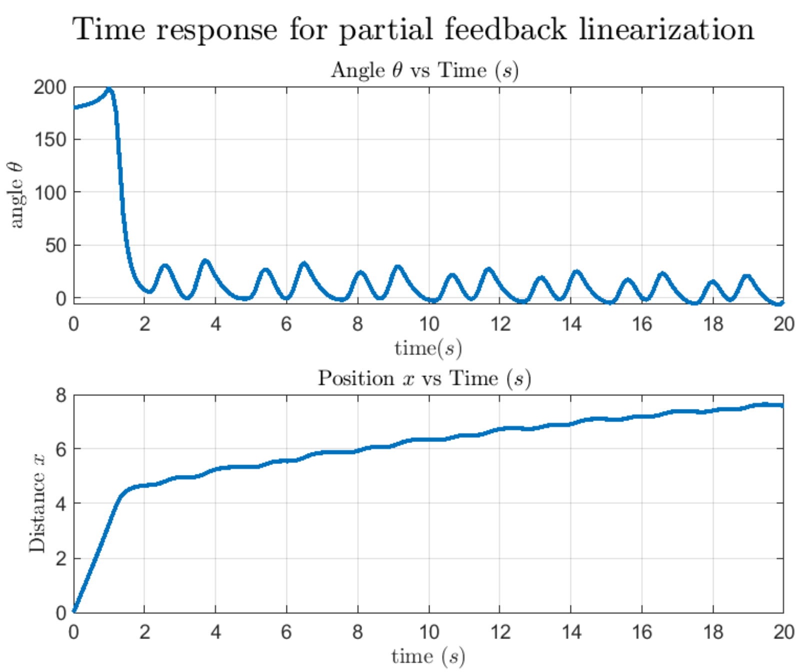
Now, for pumping energy [2] in the system, we try to match the phase change of the pendulum with the cart position, i.e. by tracking $\dot{\theta}$ with $x$. Hence we choose a simple PD feedback control strategy with cart position, given by:

$$u = u_{lin} + K_p(\dot{\theta} - x) + K_d(0 - \dot{x}), \tag{24}$$

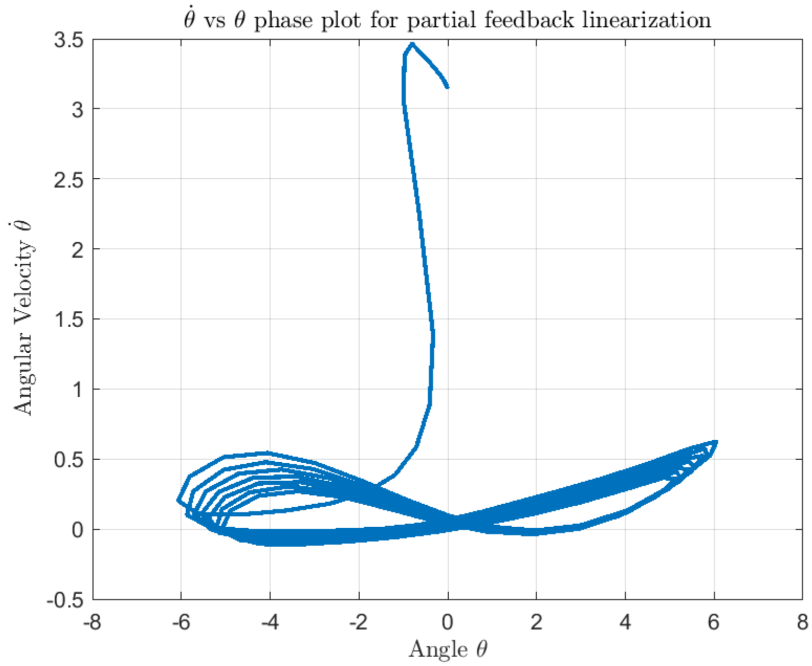
where $K_p$ and $K_d$ are the controller gains. In neighbourhood of the inverted equilibrium, we could use linear controllers(like linear quadratic regulators) with linearized system dynamics.

### 2.5 Simulation plots for partial feedback linearization

Using Matlab, derived dynamic equations, and the control input derived in the previous section, the below simulation results were obtained.




Partial feedback linearization simulation plots(a)




Partial feedback linearization simulation plots(b)

### 2.6 Discussion

This control strategy is more suited for the swing-up task and might not perform so well in the balancing task. Partial feedback linearization essentially helps virtually decouple the dynamics of the actuated and non-actuated degrees of freedom through the feedback linearizing input $u_{lin}$. In collocated partial feedback linearization, the actuated dynamics are tracked by the feedback linearizing input, thus offering direct access to the non-actuated dynamics through the coupling terms in the dynamics. However, the tracking close to unstable equilibrium is affected by the zero-dynamics and hence the pole oscillates close to the inverted position after swing-up as observed in Figure 5.

# 3 Control Lyapunov function method

## 3.1 Description

Consider the system,

$$\dot{x} = f(x, u), x, u \in \mathbb{R}^n. \tag{25}$$

A Control Lyapunov Function(CLF) $V(x, u) : \mathbb{D} \to \mathbb{R}$ is a continuously differentiable, positive definite, radially unbounded function such that,

$$\inf_{u \in \mathbb{R}^n} \left( \frac{\partial V}{\partial x} . f(x, u) \right) < 0, \quad \forall \;\; x \neq 0. \tag{26}$$

## 3.2 Artstein's Theorem

Consider the system with Lipschitz continuous $f : \mathbb{D} \to \mathbb{R}^n$

$$\dot{x} = f(x, u) x \in \mathbb{R}^n, u \in \mathbb{R}^m, f(0, 0) = 0 \tag{27}$$

If there exists a continuously differential positive definite radially unbounded function $V(x, u) : \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall \;\; x \neq 0, \quad \inf_{u \in \mathbb{R}^m} \left( \frac{\delta V}{\delta x} . f(x, u) \right) < 0 \tag{28}$$

Then there exists an input function $u = \alpha(x) : \mathbb{R}^n \to \mathbb{R}^m$ that is infinitely differentiable for all $x \neq 0$ and $\dot{x} = f(x, \alpha(x))$ is globally asymptotically stable.

### 3.3 Control Lyapunov function based input on the inverted pendulum with cart

$$\xi = x + K_p \sin\theta \tag{29}$$

$$\varphi(x) = k_p(1 - \cos\theta) - \left(k_p\dot{\theta}^2 - \frac{1}{2}\dot{x}^2\right) \tag{30}$$

The candidate Lyapunov function is

$$V(x) = \frac{k_p}{2}\xi^2 x + \frac{1}{2}\dot{\xi}^2 x + k_p\phi(x) \tag{31}$$

$$\dot{V}(x) = \frac{\dot{\xi}}{x}(k_p\xi x + \ddot{\xi}x + k_p\dot{x}) \tag{32}$$

By using equations (1),(29) and (30), we put the values in the above expression for $V$.
$\dot{V} < 0$ by substituting the control input.
The control input for the above energy function will then be:

$$u = (M^{-1}B)^{-1}(-M^{-1}C\begin{bmatrix}\dot{x}\\ \dot{\theta}\end{bmatrix} - M^{-1}G) \tag{33}$$

$$+K_p(x_{des} - \theta) + K_d(\dot{x}_{des} - \dot{\theta}) \tag{34}$$

Where $M, C, B$ and $G$ are the matrices

$$M = \begin{bmatrix} M+m & Ml\cos(\theta) \\ Ml\cos(\theta) & I + ml^2 \end{bmatrix} \tag{35}$$

$$C = \begin{bmatrix} 0 & -ml\cos(\theta) \\ 0 & 0 \end{bmatrix} \tag{36}$$

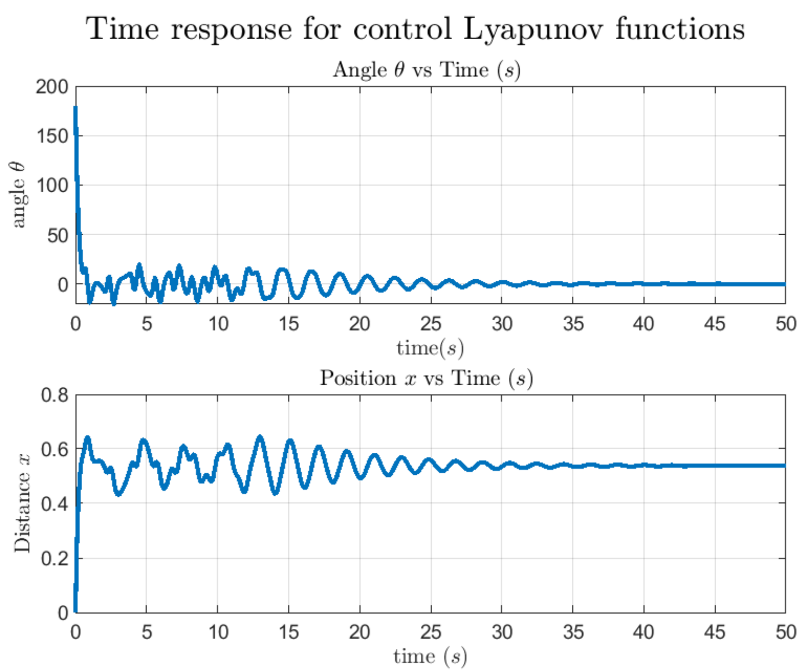
$$G = \begin{bmatrix} 0 \\ -mgl\sin(\theta) \end{bmatrix} \tag{37}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{38}$$

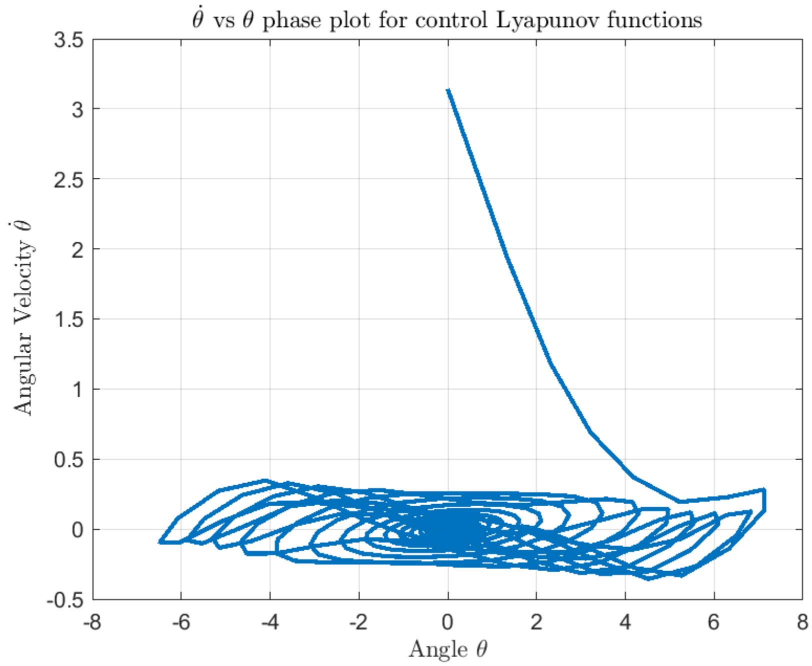
where,
$K_p$ = Proportional gain
$K_d$ = Derivative gain.

### 3.4 Simulation Results

Using Matlab, derived dynamic equations, and the control input derived in the previous section, the below simulation results were obtained.




Control Lyapunov Function method simulation plots(a)




Control Lyapunov Function method simulation plots(b)

### 3.5 Discussions

From the phase plot we observe that the sytem starts at the initial condition and gradually spirals to the unstable equilibrium point (0,0). Also, from $\theta$ vs time plot, we see that the amplitude of oscillations is gradually decaying to zero. Moreover, the position of the cart has an upper bound of approximately 0.7 units for the given initial condition.

## 4 Sontag's Universal Control Linearization

### 4.1 Description

According to Sontag's universal feedback theorem [3], For $V : \mathbb{R}^n \to \mathbb{R}$ is a candidate Lyapunov function for the single input, single output control system

$$\dot{x} = f(x) + g(x)u$$

where $f, g$ are locally Lipschitz continuous.

$$u = \alpha_s(x) = \begin{cases} -\frac{L_f V(x) + \sqrt{(L_f V(x))^2 + (L_g V(x))^4}}{L_g V(x)} & L_g V(x) \neq 0 \\ 0 & L_g V(x) = 0 \end{cases} \tag{39}$$

where $V(x)$ is the candidate Lyapunov function and $\dot{V}(x)$ can be expressed as

$$\dot{V}(x) = L_f V(x) + L_g V(x) u \tag{40}$$

where,

Taking the dynamic equations of Inverted pole on cart and candidate Lyapunov function as

$$V(x) = \frac{m\dot{\theta}^2}{2}, \tag{41}$$

we get $\dot{V}(x)$ as below,

$$\begin{aligned} \dot{V}(x) &= \dot{\theta}\ddot{\theta}m \\ &= \dot{\theta}m \left[ \frac{-u\cos\theta - ml\dot{\theta}^2\cos\theta\sin\theta - (M+m)g\sin\theta}{l(M + m\sin^2\theta)} \right] \end{aligned} \tag{42}$$

On comparing this with general form of $\dot{V}(x)$, we can express the input in terms of $\theta$ as shown below

$$L_f = \dot{\theta} m \frac{-ml\dot{\theta}^2 \cos\theta \sin\theta - (M+m)g\sin\theta}{l(M + m\sin^2\theta)} \tag{43}$$

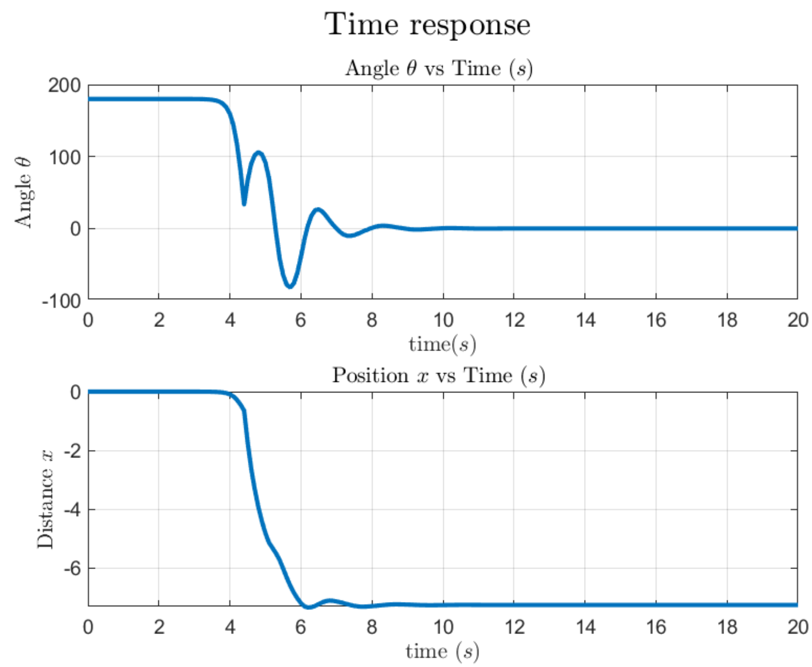
$$L_g = \dot{\theta} m \frac{-\cos\theta}{l(M + m\sin^2\theta)} \tag{44}$$

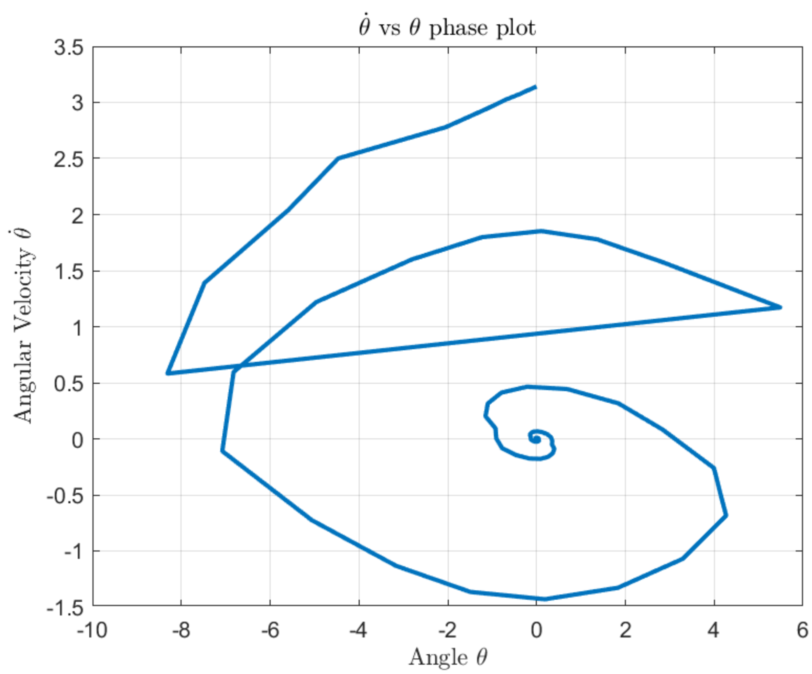
$$u = -\frac{\dot{\theta} m \frac{-ml\dot{\theta}^2 \cos\theta \sin\theta - (M+m)g\sin\theta}{l(M+m\sin^2\theta)}}{\dot{\theta} m \frac{-\cos\theta}{l(M+m\sin^2\theta)}} \tag{45}$$

$$+ \frac{\sqrt{\dot{\theta} m (\frac{-ml\dot{\theta}^2 \cos\theta \sin\theta - (M+m)g\sin\theta}{l(M+m\sin^2\theta)})^2 + \dot{\theta} m (\frac{-\cos\theta}{l(M+m\sin^2\theta)})^4}}{\dot{\theta} m \frac{-\cos\theta}{l(M+m\sin^2\theta)}} \tag{46}$$

### 4.2 Results and Simulations

Using Matlab, derived dynamic equations, and the control input derived in the previous section, the below simulation results were obtained.




Sontag's Universal control Feedback simulations

Since the control law for $u$ is only defined for a single input, i.e it should be expressed in any one of the states, it is essential that the candidate Lyapunov function is chosen wisely. In our case, as mentioned earlier, we chose $V(x)$ as $V(x) = \frac{m}{2}\dot{\theta}^2$.

### 4.3 Discussions

From the $\theta$ vs time plot, we observe that the pole crosses the unstable equilibrium without completing one full rotation before stabilizing at $0°$. Looking at the phase plot, we see that at $0°$, we observe that the system spirals to the unstable equilibrium point emphasizing asymptotic stability. Also, the cart stabilizes at a position of around $-7$ units from the initial position and doesn't reach back to its initial position.

## 5 Sliding Mode Control

### 5.1 Description

For any second-order system of the form as mentioned below,

$$\dot{x}_1 = x_2 \tag{47}$$

$$\dot{x}_2 = h(x) + g(x)u \tag{48}$$

where $h$ and $g$ are unknown nonlinear functions and $g(x) >= g_0 > 0 \ \forall \ x$. To design a state feedback law to stabilize the origin, we design a control law that constrains the system's motion to the manifold $s = a_1x_1 + x_2 = 0$. On this manifold, the equation of motion is $\dot{x_1} = -a_1x_1$. The parameter $a_1 > 0$ dictates the rate of

convergence.
We can show that by choosing a candidate Lyapunov function $V(x) = \frac{1}{2}s^2$ and taking $u = -\beta(x)sgn(s)$ where $\beta = |\frac{a_1x_2+h(x)}{g(x)}| + \beta_0$.

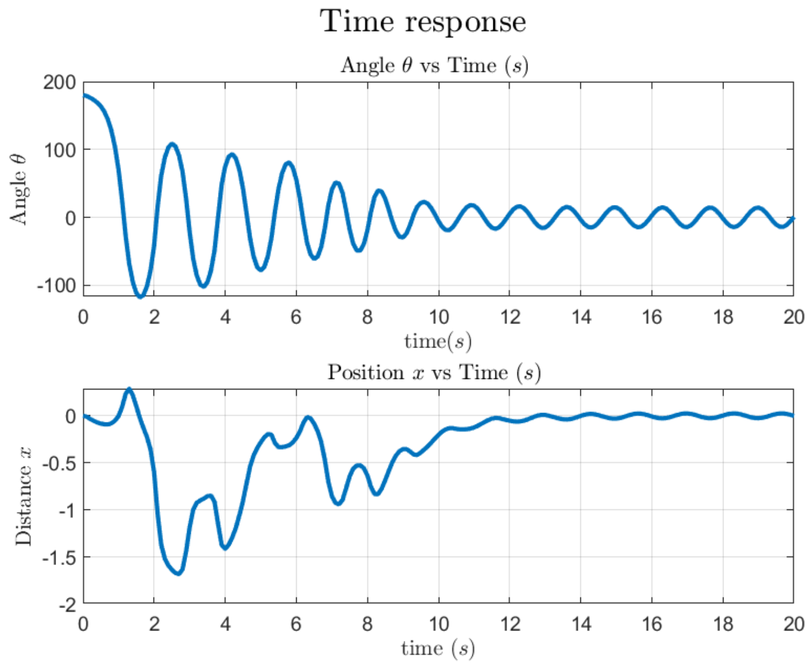
Taking the Inverted pole on cart example and expressing its dynamics in the form 47 and 48, we get the input $u$ as below

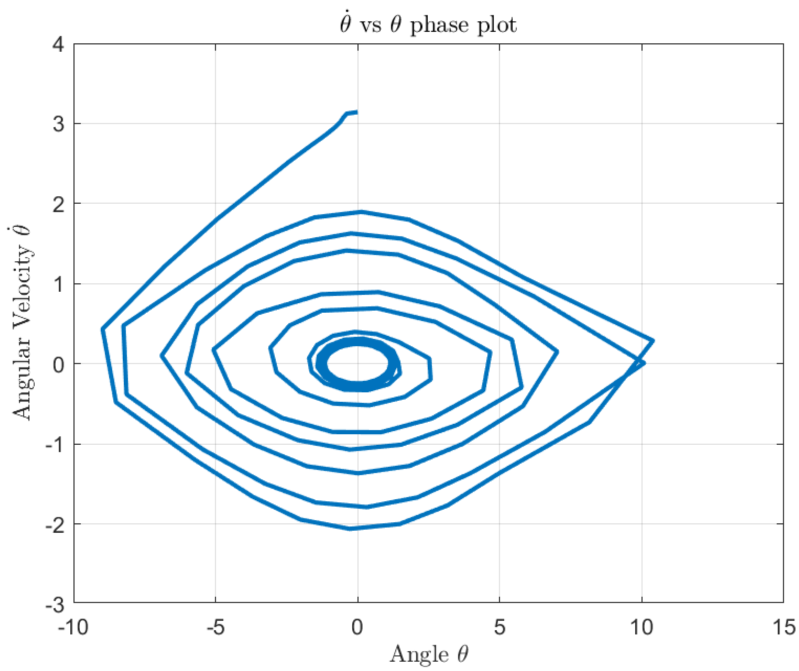
$$u = \left(\frac{a_1x_1 + ml\dot{x_3}^2\sin\theta - mgl\sin x_3}{ml} + b_0\right) sgn(a_1x_3 + x_1) \tag{49}$$

where $x_1$ and $x_3$ are displacement along x and angle of the pole $\theta$ made with the verticle axis.

### 5.2 Results and Simulations

Using Matlab, derived dynamic equations, and the control input derived in the previous section, the below simulation results were obtained.




Sliding Mode control simulations

The above simulation results were obtained after rigorous tuning of the parameters $a_1$ and $b_0$, which were finally taken as $a_1 = 0.2$ and $b_0 = 10$, respectively.

### 5.3 Discussions

We know that the control objective was to stabilize the system at the origin, that is at $\theta = 0°$ regardless of the position of the cart. However, we can see from the results obtained in the previous section that this pole orientation could be controlled by simultaneously restricting the cart position with given set of parameters. This can be confirmed from the Position vs Time graph in figure 7 where the position of the cart returns back to its initial position. Also, from the plot of Angular velocity vs Angle, we observe that at $\theta = 0°$, the system slides along parallel surfaces to reach to the origin (0,0). Finally from the plot of $\theta$ vs Time, we see that the amplitude of oscillations keeps decaying to reach zero.

## 6 Passivity based control

### 6.1 Description

Passivity is a useful tool to analyse the non-linear system behaviour in terms of sector non-linearities [4] and input-output relationships. Consider a square(equal number of inputs and outputs) dynamical system given by:

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= h(x, u), \end{aligned} \tag{50}$$

where,$f$ is Lipschitz continuous, with $f(0,0) = 0, h(0,0) = 0$. This system is considered passive if:

$$\exists\, V(x) \geq 0 \text{ such that,} \tag{51}$$

$$u^T y \geq \dot{V}(x)\ \forall\ (x,u) \in \mathbb{D}_{(x,u)}. \tag{52}$$

For designing a controller using these principles, we design an input $u$ for the system to be passive, as the origin of a passive system $\dot{x} = f(x,0)$ with $V(x) > 0$ is asymptotically stable.(Lemma 5.5) [4].

### 6.2 Passive controller design for cart-pole system

Consider a storage function candidate given by:

$$V(x) = \frac{M\dot{x}^2}{2}, \tag{53}$$

$$V(0) = 0, V(x \neq 0) > 0$$

$$\implies \dot{V} = M\dot{x}\ddot{x}$$

$$\dot{V} = M\dot{x}\left(\frac{u + m\sin\theta(l\dot{\theta}^2 + g\cos\theta)}{M + m\sin^2\theta}\right) \tag{54}$$

We then obtain a control input which makes the system passive, for $\theta = 0°$.

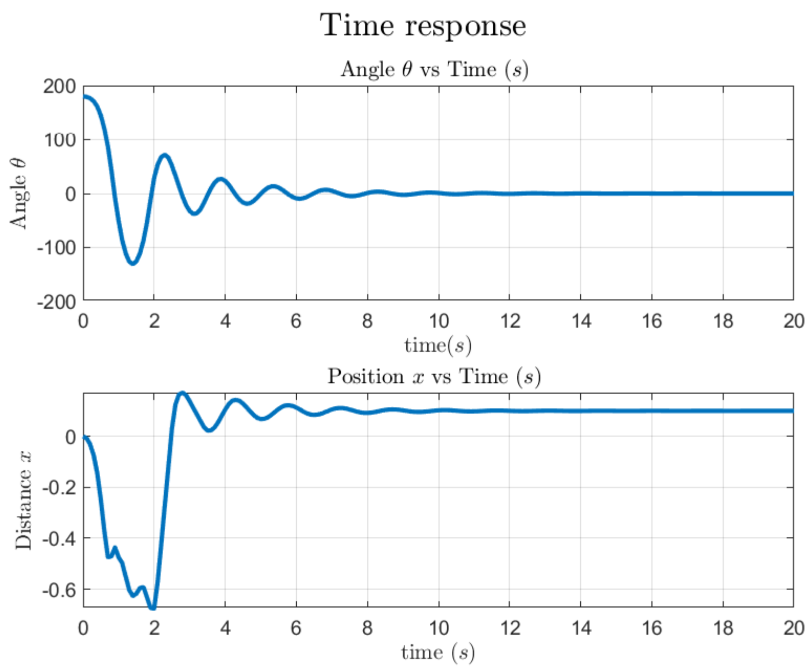
$$u\theta \geq \dot{V}(x) \tag{55}$$

$$\implies u\left(1 - \frac{M\dot{x}}{M + m\sin^2\theta}\right) \geq \frac{Mm\dot{x}\sin\theta(l\dot{\theta}^2 + g\cos\theta)}{M + m\sin^2\theta} \tag{56}$$

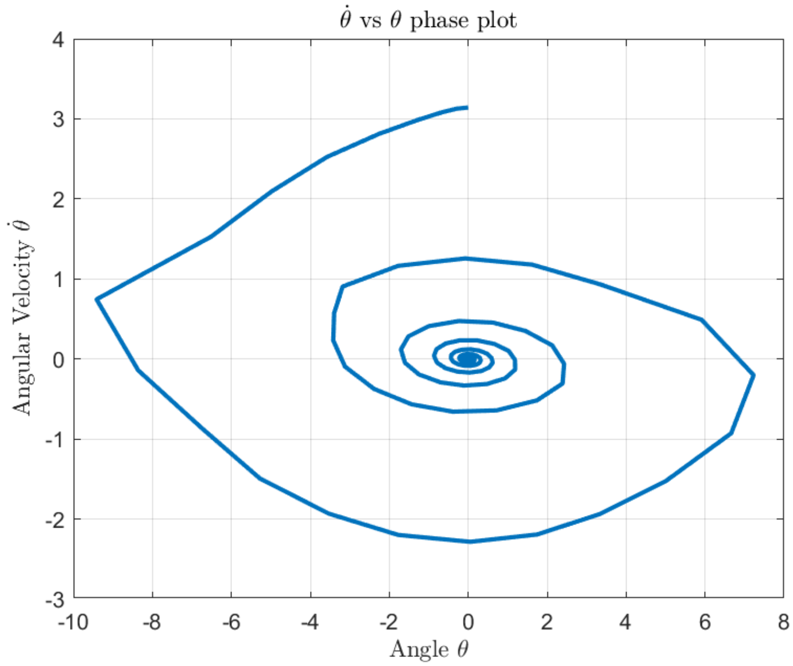
$$\implies u \geq \frac{Mm\dot{x}(l\dot{\theta} + g\cos(\theta))\sin(\theta)}{(M(1-\dot{x}) + m\sin^2\theta)} \tag{57}$$

### 6.3 Results and simulations

Using Matlab, derived dynamic equations, and the control input derived in the previous section, the below simulation results were obtained.




Passivity-based controller(a)




Passivity-based controller(b)

### 6.4 Discussion

From the phase plot, we see that the system reaches to the unstable equilibrium (0,0), i.e the angle and angular velocity both become zero. From the angle vs time plot, we see that the pole oscillates rigorously initially and then slowly decays to zero. Also, the cart moves along the negative x-axis till -0.6 units and reaches back to initial position(close to 0) to stabilize the pole.

## 7 Conclusion

The applicability and design of controllers using Feedback linearization-based control,control lyapunov function method,Sontag's universal feedback controller,Sliding mode control and Passivity-based control inverted pendulum on cart dynamics have been discussed, along with simulation results.

## References


[1] M. Vidyasagar, *Nonlinear systems analysis*. SIAM, 2002.

[2] M. W. Spong, "Energy based control of a class of underactuated mechanical systems," *IFAC Proceedings Volumes*, vol. 29, no. 1, pp. 2828–2832, 1996.

[3] E. D. Sontag, "A 'universal'construction of artstein's theorem on nonlinear stabilization," *Systems & control letters*, vol. 13, no. 2, pp. 117–123, 1989.

[4] H. K. Khalil, *Nonlinear control*. Pearson New York, 2015, vol. 406.