ES 613: Modern Control Theory
Project I

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1 Inverted Pendulum

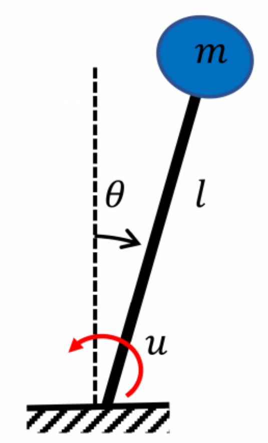

Fig.1 Inverted Pendulum

The Dynamics of the system are given as,

$$ml^2\ddot{\theta} = mgl\sin\theta - u \tag{1}$$

1.1 Stabilizing Control Input - Direct Lyapunov Method

Consider a candidate Lyapunov function as below,
$V = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\theta^2$
Upon differentiating the above, we get,

$$\begin{aligned}\dot{V} &= \dot{\theta}\ddot{\theta} + \theta\dot{\theta} \\ &= \dot{\theta}(\frac{g}{l}\sin\theta - \frac{u}{ml^2}) + \theta\dot{\theta}\end{aligned} \tag{2}$$

Let us now choose a $u = ml^2(\frac{g}{l}\sin\theta + \dot{\theta} + \theta)$ and substitute in eqn 2. Upon substitution we see,

$$\begin{aligned}\dot{V} &= \dot{\theta}(\frac{g}{l}\sin\theta - \frac{ml^2(\frac{g}{l}\sin\theta + \dot{\theta} + \theta)}{ml^2}) + \theta\dot{\theta} \\ &= \dot{\theta}(\frac{g}{l} - \frac{g}{l} - \dot{\theta} - \theta) + \theta\dot{\theta} \\ &= -\dot{\theta}^2\end{aligned} \tag{3}$$

Since $\dot{V}$ is negative for the chosen u which is evident from the eqn 3, the control input u stabilises the system.
To be able to alter the performance of the controller, a gain element k can be multiplied to $\dot{V}$ and corresponding u can be computed.

1.2 Open-loop Simulation

Solving the dynamics of the system without any input, we get the below plots for θ and $\dot{\theta}$

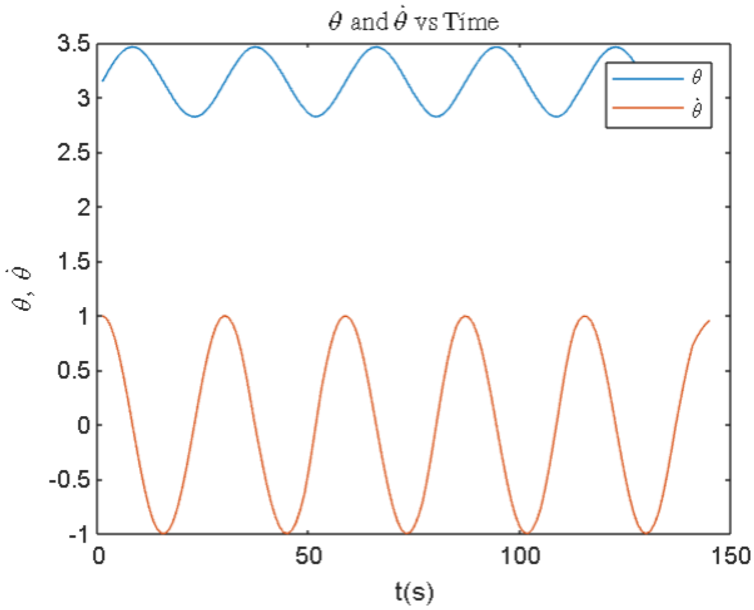

Fig.2 Inverted Pendulum

The above plot shows the variation of θ and $\dot{\theta}$ with time when the pendulum is given an initial velocity of 1 rad/s at $\theta = \pi/2$. Since there is no damping in the joints, the oscillations go on till infinite time.

1.3 Implementation of Non-Linear Controller (Previously derived)

The Control input designed in section 1.1 was implemented and the response i.e the variation of θ and $\dot{\theta}$ were observed.
Below are the plots for 3 initial conditions,

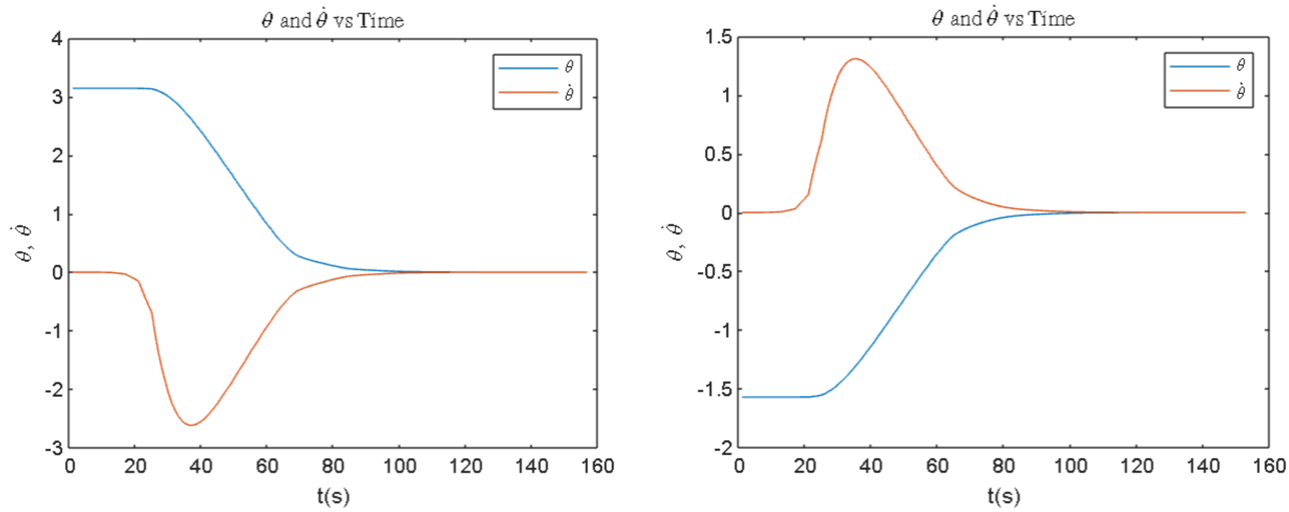

Fig.3a Simulation for initial condition $(3.14, 0)$ and $(-\pi/2, 0)$

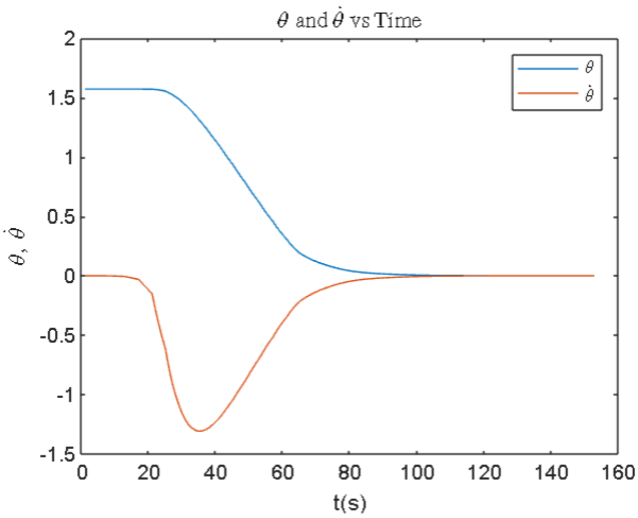

Fig.3b Simulation for initial condition $(\pi/2, 0)$

From the above plots we see that the settling time is around $100s$ and the rise time is close to $40s$. Also, the overshoot is of maximum $3rad/s$. These performance characteristics can further be altered by adding an adding gain term.

1.4 Implementation of Linear Quadratic Controller

The linearized system dynamics expressed in state-space representation are as follows

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{4}$$

Given the cost function to optimize as below,
$J = \frac{1}{2} \int_0^\infty (\theta^2 + \frac{1}{4}u^2)dt$
we get $Q = I$ and $R = 0.25$

To implement LQR controller, the algebraic Ricatti's equation was solved and matrix P was obtained as
$P(BB^T)P/R - Q - PA - A^TP = 0$ and $P = \begin{bmatrix} -19.82 & 0 \\ 0 & -6.60 \end{bmatrix}$

Using $u = PX$, below are the plots obtained for 3 initial conditions,

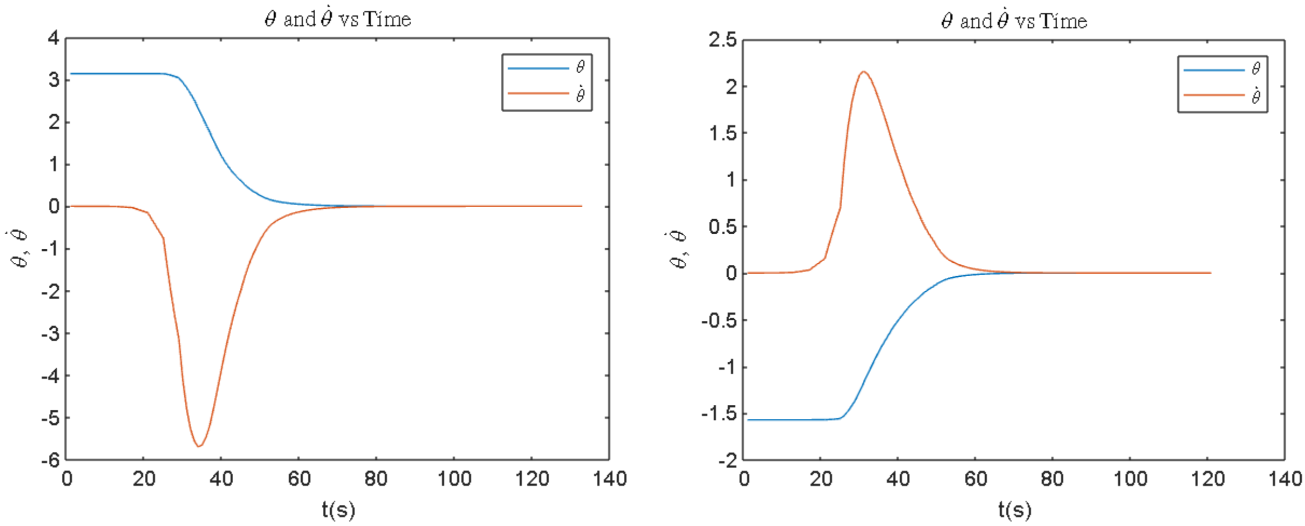

Fig.4a Simulation for initial condition $(3.14, 0)$ and $(-\pi/2, 0)$

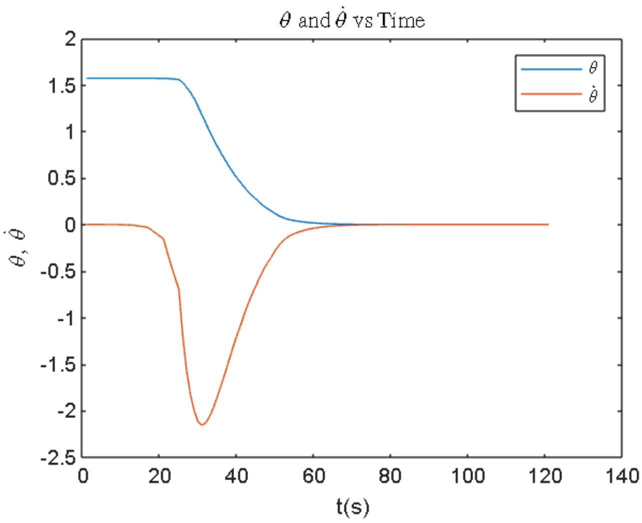

Fig.4b Simulation for initial condition $(\pi/2, 0)$

From the above plots the settling time is around $80s$ while the rise time is around $30s$. The maximum peak value is close to $6rad/s$.

1.5 Comparision of Lyapunov based controller and LQR

Both the controllers stabilize the pendulum, however the Lyapunov based controller is non-linear where as the LQR is a Linear controller. This means that the LQR stabilizes the system only when the initial conditions lie in the neighbourhood of the equilibrium point i.e (0,0).

Another major difference lies that the performance requirement is clearly met in the case of LQR but to achieve the same in Lyapunov based controller an addition gain term needs to be added. To ensure similar performance as LQR, it is not trivial to find the corresponding Lyapunov gain.
While Lyapunov based controller is robust, tunning it's parameters to meet certain performance characteristics may be difficult.

2 Quadcopter

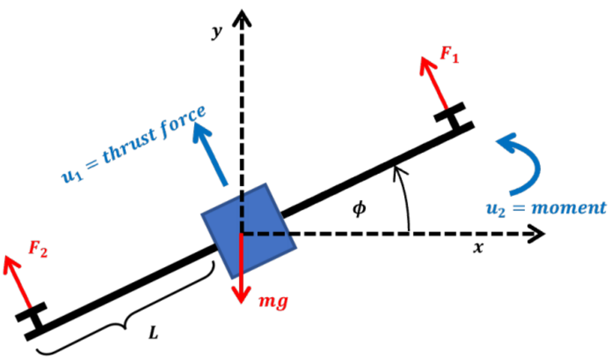

Fig.5 Quadcopter

The Dynamics of the system are given as below,

$$\begin{aligned} m\ddot{x} &= -u_1 \sin\phi \\ m\ddot{y} &= -mg + u_1 \cos\phi \\ I_{zz}\ddot{\phi} &= u_2 \end{aligned} \tag{5}$$

where, $u_1 = F_1 + F_2$ and $u_2 = (F_1 - F_2) * L$

2.1 Equilibrium Points

To find the equilibrium points, consider the states $\ddot{x}, \ddot{y}, \ddot{\phi}, \dot{x}, \dot{y}, \dot{\phi}$ in 5 equated to 0

$$\begin{aligned} m\ddot{x} &= -u_1 \sin\phi = 0, \\ m\ddot{y} &= -mg + u_1 \cos\phi = 0, \\ I_{zz}\ddot{\phi} &= u_2 \\ \sin\phi &= 0 \implies \phi = n\pi, \\ u_1 \cos\phi &= mg \implies u_1 = mg, \\ \ddot{x} &= 0 \implies \dot{x} = C_1 = 0 \implies x = C_1 x + C_2 = C_2 \\ \ddot{y} &= 0 \implies \dot{y} = C_3 = 0 \implies y = C_3 x + C_4 = C_4 \\ \ddot{\phi} &= 0 \implies \dot{\phi} = C_5 = 0 \end{aligned} \tag{6}$$

Thus, the equilibrium point for Quadcopter expressed in the states taken in the order $(x, y, \phi, \dot{x}, \dot{y}, \dot{\phi})$ are $(C_2, C_4, n\pi, 0, 0, 0)$ where C_2, C_4 are real numbers.

2.2 Design of PD Controller

In order to represent the system dynamics in state-space representation and design a controller, we first need to linearize the system about an equilibrium point[1].

$$
\begin{aligned}
\ddot{x} &= -\frac{u_1 \sin\phi}{m} = f_1 \\
\ddot{y} &= -g + \frac{u_1 \cos\phi}{m} = f_2 \\
\ddot{\phi} &= \frac{u_2}{I_{zz}} \\
\dot{X} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \phi \\ \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/m & 0 & -1 \\ 0 & 1/I_{zz} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ g \end{bmatrix}
\end{aligned} \tag{7}
$$

Now, let us consider $u = Kx$ where K is a 3×6 matrix. To simplify the computation, the 18 unknowns can be reduced to 6 unknowns corresponding to each state.

$$
K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \end{bmatrix}
$$

Substituting this in the above equation and ensuring the matrix $|A - BK|$ has eigenvalues whose real part is negative, will stabilize the system.
Intuitively the control input u_1 i.e the thrust can be imagined to drive the error in y and $\dot{y}$ to 0. While the u_2 determines the direction and can drive the error in x and $\dot{x}$ to 0.
Based on this understanding the control law should be of the form,

$$
\begin{aligned}
u_1 &= mg + p_1(y_d - y) + p_2(-\dot{y}) \\
\phi_d &= p_3(x_d - x) + p_4(-\dot{x}) \\
u_2 &= p_5(\phi_d - \phi) + p_6(\dot{\phi}_d - \dot{\phi})
\end{aligned} \tag{8}
$$

Out of the many such values, one set of values found are[1]
$p_1 = 2.5, p_2 = 7, p_3 = 0.04, p_4 = 0.56, p_5 = 1.42$ and $p_6 = 0.008$

2.3 Implementation of PD Controller

The PD controller was simulated for 3 different initial conditions with (X_d, Y_d) defined as $(0.5, 0.5)$ and the plots obtained are as below

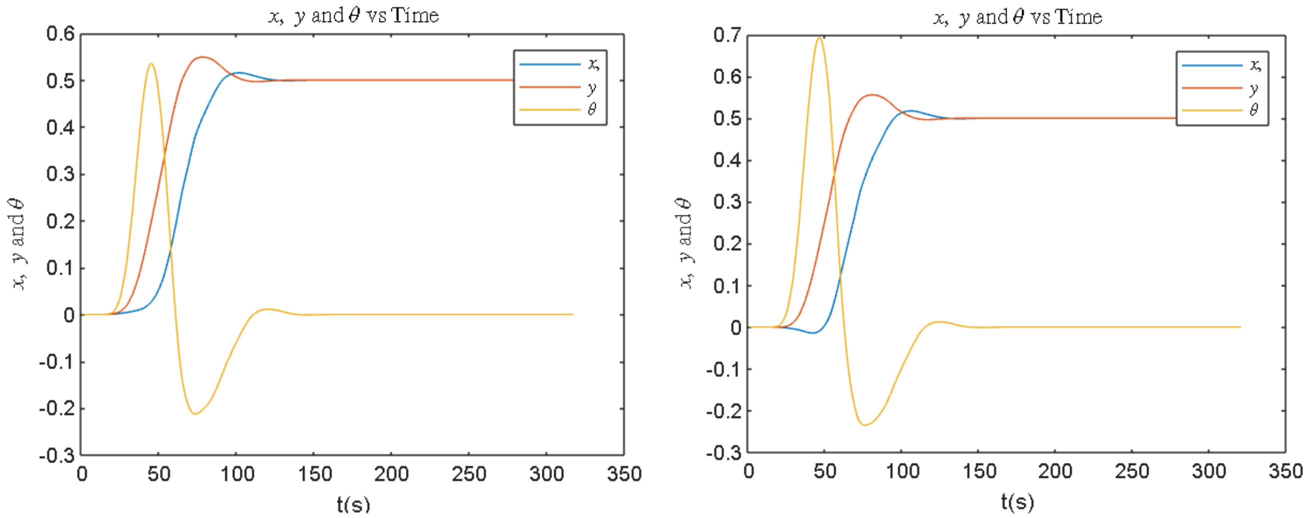

Fig.6a Simulation for initial condition $(0, 0, 0, 0.1, 0.1, 0)$ and $(0, 0, 0, -0.1, -0.1, 0)$

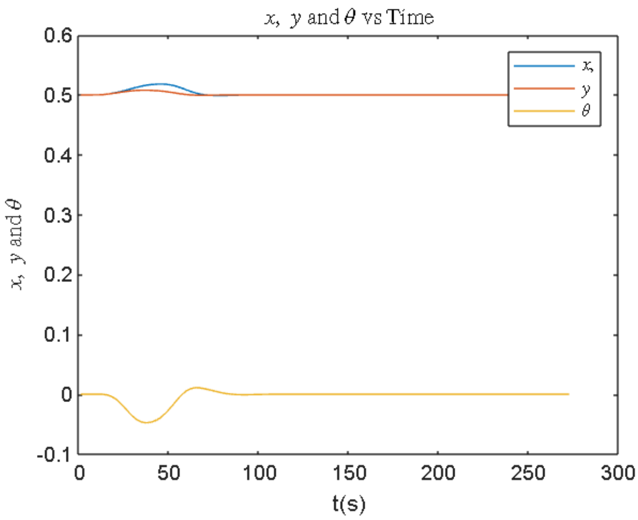

Fig.6b Simulation for initial condition $(0.5, 0.5, 0, 0.1, 0.1, 0)$

The above plots show the variation of displacement along x, y and rotation of the rotor by angle ϕ. We see that the settling time is dependent on the initial conditions i.e the initial displacement and initial velocity and hence is least $70s$ in fig.6b. The peak values are also subject to the initial conditions and are higher in case of negative initial velocities.

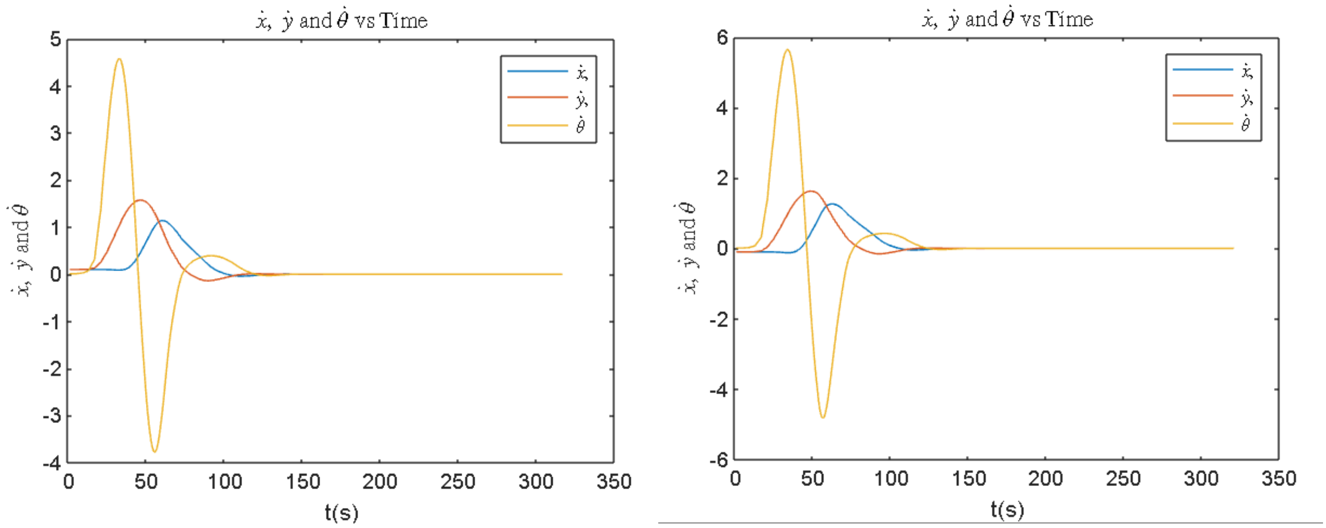

Fig.7a Simulation for initial condition $(0, 0, 0, 0.1, 0.1, 0)$ and $(0, 0, 0, -0.1, -0.1, 0)$

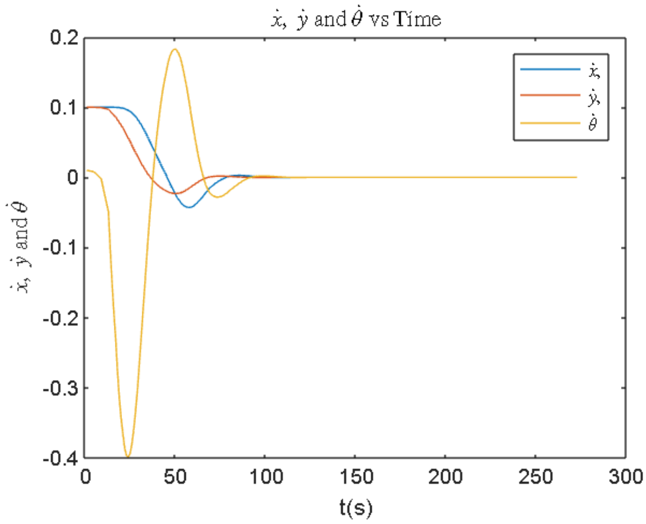

Fig.7b Simulation for initial condition $(0.5, 0.5, 0, 0.1, 0.1, 0)$

The above plots show the changes in the velocity of the rotor along the x, y direction and the angular velocity $\dot{\phi}$. As shown in the above plots, the average settling time is $150s$ and the maximum peak value is not more than $6rad/s$(subject to initial conditions).

2.4 Discussion

While the control architecture is intuitive, rigorous process to tune the gains i.e p_i was done. The procedure followed is as follows [1]
1. Increase p_1, p_2, p_3 till sustained oscillations are obtained.
2. Increase p_4, p_5, p_6 to add damping.
The proportional gains affect the peak values while the derivative gains affect the settling time. An abnormal increase in the derivative gain adds oscillations and makes it unstable.
An attempt to design Lyapunov based controller was made, however only the

velocities along x, y could be driven to zero i.e the position and orientation of the rotor are uncontrolled but the system is stabilized. (not asymptotically). Below are the plots that manifest this behaviour.

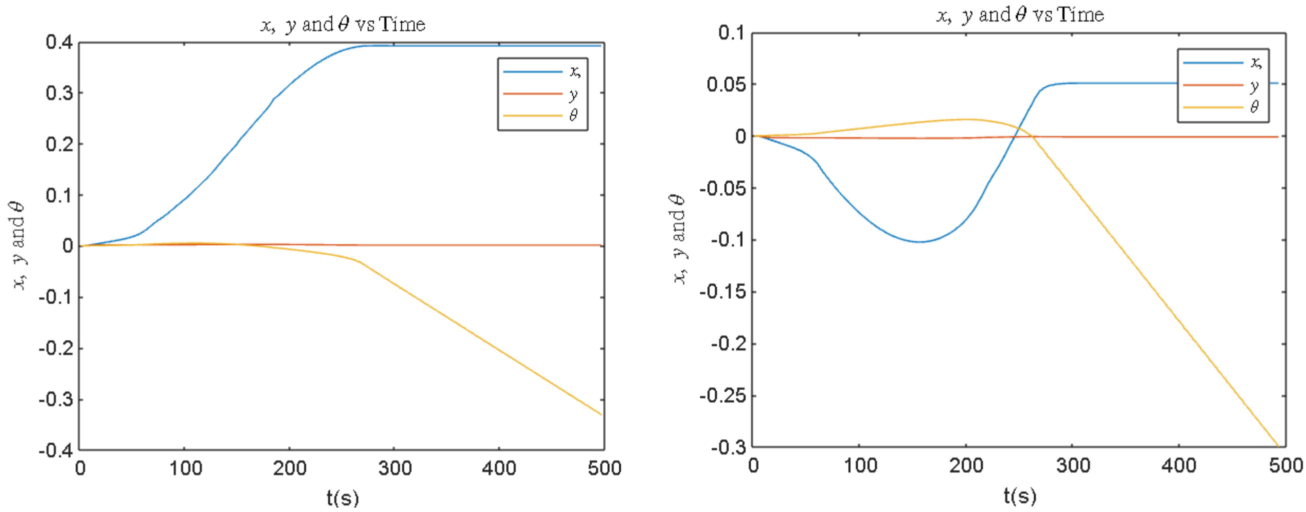

Fig.8 Simulation for initial condition $(0, 0, 0, 0.1, 0.1, 0)$ and $(0, 0, 0, -0.1, -0.1, 0)$

Finding the right Lyapunov candidate function to drive all the states to equilibrium point might be cumbersome but the Lyapunov based controller used (as mentioned in MATLAB code) stabilizes the system.

References

[1] Praveen Venkatesh, Sanket Vadhvana, and Varun Jain. "Analysis and control of a planar quadrotor". In: *arXiv preprint arXiv:2106.15134* (2021).