Mathematical Formulation of Driver Assignment Exercise (DAX)

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1 Description of DAX Problem

We assume that we only have one plant available to do several loads originated from different (finitely many though) costumers. In other words, these costumers totally require $|I|$ number of loads to be delivered, i.e., $[I] := \{1, 2, \ldots, I\}$ loads are available. Each job is attributed to one of the costumers and therefore the information of the ith job has the costumer's number inside itself (inherently). There are $|J|$ number of drivers, i.e., $[J] := \{1, 2, \ldots, J\}$ drivers are available, and $|T|$ number of time units for delivering the loads, i.e., $[T] := \{1, 2, \ldots, T\}$ discrete times are available to start a delivery. To be precise, we have natural, medical, and regular loads, that is, $I = I_m \cup I_r \cup I_n$, where I_m, I_r and I_n denote medical, regular, and natural loads, respectively. These three sets construct a partition of I. We also assume that there are l lines in the available plant for loading trucks, which has a capacity at each time t given as p_t^l. Other information given a priori include the trip time of each job (based on the considered time unit), defined as

$$tr_i := l_i + d_i + u_i + d_i = l_i + 2d_i + u_i,$$

where l_i, d_i, and u_i are the loading, distance from the plant to the costumer, and the unloading time of the ith job. These numbers are calculated based on the related costumer. Also, each driver has some shift time given by $[s_j^l, s_j^u]$ where WLOG we assume that s_j^l and $s_j^u \in [T]$. Costumers may require gate period, that is, the gate period for the ith job is considered as $[g_i^l, g_i^u]$. In other words, we know that the ith job is for which costumer and hence its gate period is known to us. We only consider the shift and gate times inside $[T]$ for each run of the program. The remaining available times will be considered for the next run. There are other constraints that will be explained later. It must be noted that the drivers are paid by their shift time no matter the deliver a load in which. So, the objective function of this problem should try to keep them as busy as possible along with other possible practical desires.

2 Variables of Problem Formulation

To mathematically formulate this problem, we need the following variables:

$$x_{ijtl} := \begin{cases} 1 & \text{if } i\text{th job is started by } j\text{th driver at time } t \text{ in the } l\text{th line} \\ 0 & \text{otherwise;} \end{cases}$$

$$y_{ijtl} := \begin{cases} 1 & \text{if } i\text{th job is finished by } j\text{th driver at time } t \text{ in the } l\text{th line} \\ 0 & \text{otherwise;} \end{cases}$$

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and

$$w_{jt} := \begin{cases} 1 & \text{if } j\text{th driver is busy in } t\text{th time unit} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.1. *Assuming that x and y are binary variables, for mathematically formulating this problem, we are often faced with the following situations:*

i. if $x = 0$, we want $y = 0$ but we do not want to have any constraints when $x = 1$, that is, when $x = 1$, we can have $y = 0$ or 1. This can be captured by the following equations:

$$x \geq y$$

ii. if $x = 0$, we want $y = 1$ but we do not want to have any constraints when $x = 1$, that is, when $x = 1$, we can have $y = 0$ or 1. This can be captured by the following equation:

$$x + y \geq 1$$

iii. if $x = 1$, we want $y = 0$ but we do not want to have any constraints when $x = 0$, that is, when $x = 0$, we can have $y = 0$ or 1. This can be captured by the following equation:

$$x + y \leq 1$$

iv. if $x = 1$, we want $y = 1$ but we do not want to have any constraints when $x = 0$, that is, when $x = 0$, we can have $y = 0$ or 1. This can be captured by the following equation:

$$x \leq y.$$

3 Constraints of DAX Problem

Here, we use the defined variables to formulate different constraints of the problem.

3.1 One job at most can be started by a driver at a time

This can be formulated as

$$\sum_{il} x_{ijtl} \leq 1 \qquad \forall j \;\; \forall t$$

3.2 One job at most can be finished by a driver at a time

This can be formulated as

$$\sum_{i} y_{ijtl} \leq 1 \qquad \forall l \;\; \forall j \;\; \forall t$$

3.3 If a driver starts a job, they must finish it

This can be formulated as

$$y_{ij(t+tr_i)l} \geq x_{ijtl} \qquad \forall l \;\; \forall i \;\; \forall j \;\; \forall t$$

3.4 Plant capacity at a time

This can be formulated as

$$\sum_{ij} x_{ijtl} \leq p_t^l \qquad \forall l \;\; \forall t$$

3.5 Gate period of costumer

$$g_i^l x_{ijtl} \leq (t + l_i + d_i) x_{ijtl} \leq g_i^u x_{ijtl} \qquad \forall l \ \ \forall j \ \ \forall i \ \ \forall t$$

3.6 Busy time definition

Note that once the jth driver starts the ith job at time t, this driver would busy during $[t, t + tr_i]$, so we can have the following constraint:

$$\sum_{z=t}^{t+tr_i-1} w_{jz} \geq tr_i x_{ijtl} \qquad \forall l \ \ \forall j \ \ \forall i \ \ \forall t$$

3.7 Shift time of each driver

$$s_j^l w_{jt} \leq t w_{jt} \leq s_j^u w_{jt} \qquad \forall j \ \ \forall t$$

Note that the busy time definition forces w_{jt} to be 1 whenever the jth driver is actually busy. Therefore, together with this constraint, the above says that whenever the jth driver is busy at time t, the time t has to be inside $[s_j^l, s_j^u]$. Also, note that whenever $w_{jt} = 0$, this constraint vanishes without forcing an extra constraint.

3.8 One delivery at most for each $t \geq 1 + \min_i \ (l_i + d_i)$

For this constraint, we can define the variable $m_{ijtl} := x_{ij(t-[l_i+d_i])l}$ for all $t \geq 1 + \min_i \ (l_i + d_i)$ and have $\sum_i m_{ijtl} \leq 1$, which is equivalent to the following:

$$\sum_i x_{ij(t-[l_i+d_i])l} \leq 1 \qquad \forall l \ \ \forall j \ \ \forall t \geq 1 + \min_i \ (l_i + d_i)$$

3.9 Natural loads constraint

A natural load can be started only if a regular load is done right before it excluding idle times in between. In other words, the closest job started before a natural load must be regular. This can be captured as if $x_{\bar{i}_n \bar{j} \bar{t} l} = 1$ for some $\bar{i}_n \in I_n, \bar{j} \in J,$ and $\bar{t} \in T$, we must have $i^* \in I_r$ such that i^* IS the job index of the x_{ijtl} that gives the optimal solution of the following problem:

$$\text{maximize} \ \ t \quad \text{subject to} \quad x_{ijtl} = 1 \quad \text{for some } i \in I \setminus \{\bar{i}_n\}, j \in J, \text{and } t < \bar{t}$$

This is a nonlinear constraint and therefore would increase the complexity of our formulation. Thus, we replace it with the following constraint:

$$x_{i_m jtl} + x_{i_n \bar{j}(t+1)l} \leq 1 \qquad \forall l \ \ \forall t \ \ \forall i_m \in I_m \ \ \forall i_n \in I_n \ \ \forall j, \text{ and } \bar{j} \in J. \tag{1}$$

This basically says that after a medical load, we cannot have a natural load.

4 Objective Function

To define an appropriate objective function, we need to consider to facts: (1) drivers are paid daily not the number of loads they deliver in a day, (2) to deliver as many loads as possible, and (3) the length of a load must not have a negative impact on its priority in delivery time (schedule). To achieve the first goals, we have $\sum_{jt} w_{jt}$. For the second objective, we have $\sum_{ijtl} x_{ijtl}$, where the third goal suggest to modify this term with $\sum_{ijtl} tr_i x_{ijtl}$. Therefore, our objective function of the DAX problem is defined as

$$\text{maximize} \quad \sum_{jt} w_{jt} + \sum_{ijtl} tr_i x_{ijtl} \tag{2}$$

In order to have a more sophisticated objective function, one can have the follwoing:

$$\text{maximize} \quad \sum_{jt} f(j,t) w_{jt} + \sum_{ijtl} g(i,j,t,l) tr_i x_{ijtl}, \tag{3}$$

where functions $f(j,t)$ and $g(i,j,t)$ are defined to capture other favorite features. For example, assume that the more experienced a driver is, the more costly his deliveries are. Then, a nice function $f(j,t)$ can be defined to achieve this goal.

5 DAX Mathematical Formulation

Based on the previous sections, we have the following formulation for the DAX problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{jt} w_{jt} + \sum_{ijtl} tr_i x_{ijtl} \\
\text{subject to} \quad & \sum_{il} x_{ijtl} \leq 1 && \forall j \;\; \forall t \\
& \sum_{i} y_{ijtl} \leq 1 && \forall j \;\; \forall t \\
& y_{ij(t+tr_i)l} \geq x_{ijtl} && \forall i \;\; \forall j \;\; \forall t \geq 1 + \min_i \; tr_i \\
& \sum_{ij} x_{ij} \leq p_t^l && \forall l \;\; \forall t \\
g_i^l x_{ijtl} \leq & (t + l_i + d_i) x_{ijtl} \leq g_i^u x_{ijtl} && \forall l \;\; \forall j \;\; \forall i \;\; \forall t \\
& s_j^l w_{jt} \leq t w_{jt} \leq s_j^u w_{jt} && \forall j \;\; \forall t \\
& \sum_{i} x_{ij(t-[l_i+d_i])l} \leq 1 && \forall l \;\; \forall j \;\; \forall t \geq 1 + \min_i \; (l_i + d_i) \\
& \sum_{z=t}^{t+d_i-1} w_{jz} \geq d_i x_{ijtl} && \forall l \;\; \forall j \;\; \forall i \;\; \forall t \\
& x_{ijtl} \in \{0,1\} && \forall l \;\; \forall i \;\; \forall j \;\; \forall t \\
& y_{ijtl} \in \{0,1\} && \forall l \;\; \forall i \;\; \forall j \;\; \forall t \\
& w_{jt} \in \{0,1\} && \forall j \;\; \forall t.
\end{aligned} \tag{4}$$

However, note that if $x_{ijtl} = 1$, then $y_{ij(t+tr_i)l} = 1$ and conversely $y_{ijtl} = 1$ if $x_{ij(t-tr_i)} = 1$. This means that $y_{ijtl} = x_{ij(t-tr_i)l}$, so we can remove the variable y_{ijtl} and have the new efficient model:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{jt} w_{jt} + \sum_{ijtl} tr_i x_{ijtl} \\
\text{subject to} \quad & \sum_{il} x_{ijtl} \leq 1 \qquad \forall l \;\; \forall j \;\; \forall t \\
& \sum_{i} x_{ij(t-tr_i)l} \leq 1 \qquad \forall l \;\; \forall j \;\; \forall t \geq 1 + \min_i \; tr_i \\
& \sum_{ij} x_{ij} \leq p_t^l \qquad \forall l \;\; \forall t \\
g_i^l x_{ijtl} \leq & (t + l_i + d_i) x_{ijtl} \leq g_i^u x_{ijtl} \qquad \forall l \;\; \forall j \;\; \forall i \;\; \forall t \\
s_j^l w_{jt} \leq & \, t w_{jt} \leq s_j^u w_{jt} \qquad \forall j \;\; \forall t \\
& \sum_{i} x_{ij(t-[l_i+d_i])l} \leq 1 \qquad \forall l \;\; \forall j \;\; \forall t \geq 1 + \min_i \; (l_i + d_i) \\
& \sum_{z=t}^{t+d_i-1} w_{jz} \geq d_i x_{ijtl} \qquad \forall l \;\; \forall j \;\; \forall i \;\; \forall t \\
& x_{ijtl} \in \{0,1\} \qquad \forall l \;\; \forall i \;\; \forall j \;\; \forall t \\
& w_{jt} \in \{0,1\} \qquad \forall j \;\; \forall t.
\end{aligned}
\tag{5}
$$