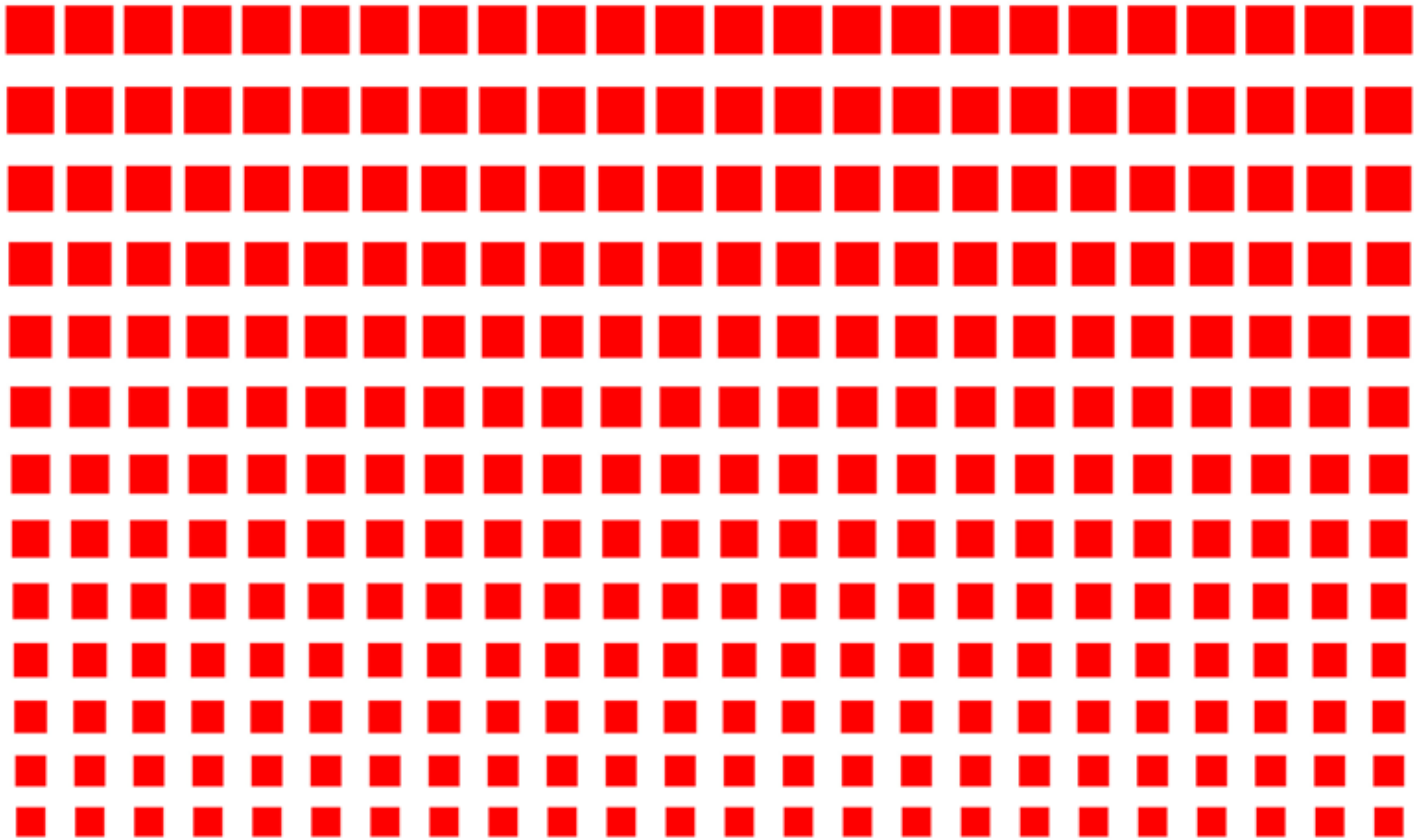


2D Alignment: Linear Least Squares

16-385 Computer Vision (Kris Kitani)

Carnegie Mellon University

Extract features from an image …

what do we do next?

Feature matching

(object recognition, 3D reconstruction, augmented reality, image stitching)

How do we estimate the transformation?

Given a set of matched feature points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

point in one image

point in the other image

and a transformation

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

transformation function

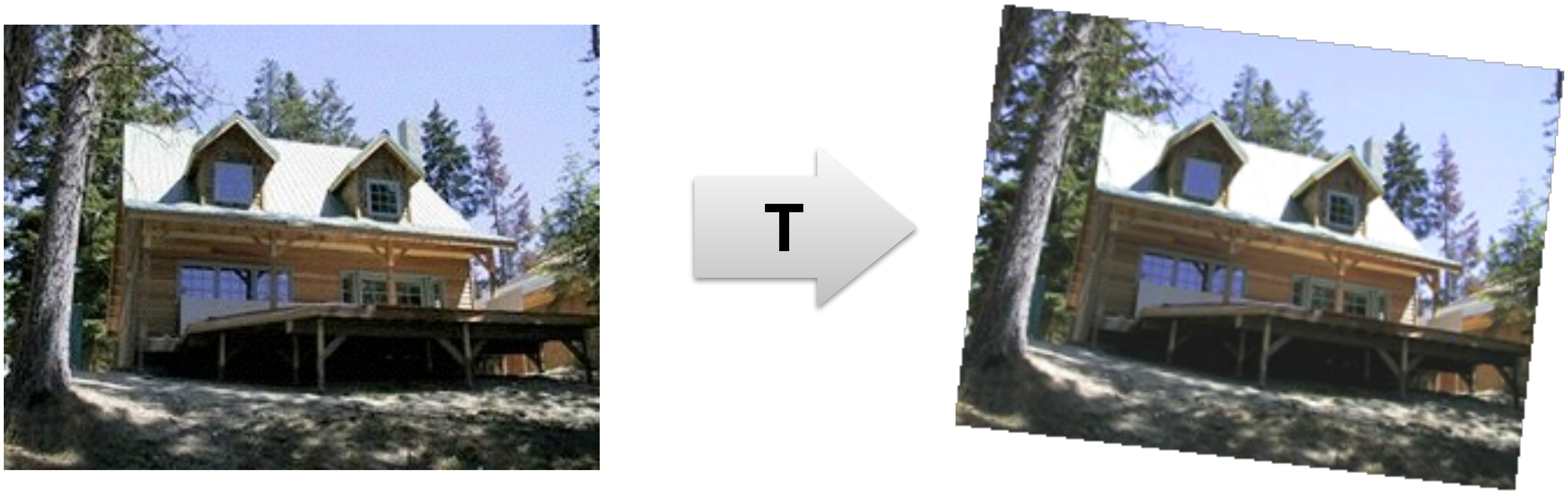
parameters

Find the best estimate of

$$\boldsymbol{p}$$

Model fitting

Recover the transformation



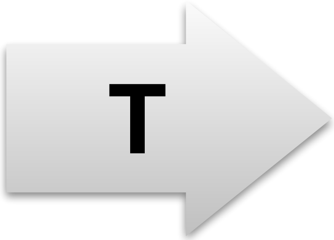
f(x,y) g(x,y)

Given f and g, how would you recover the transform T?

(user will provide correspondences)

How many do we need?

Translation



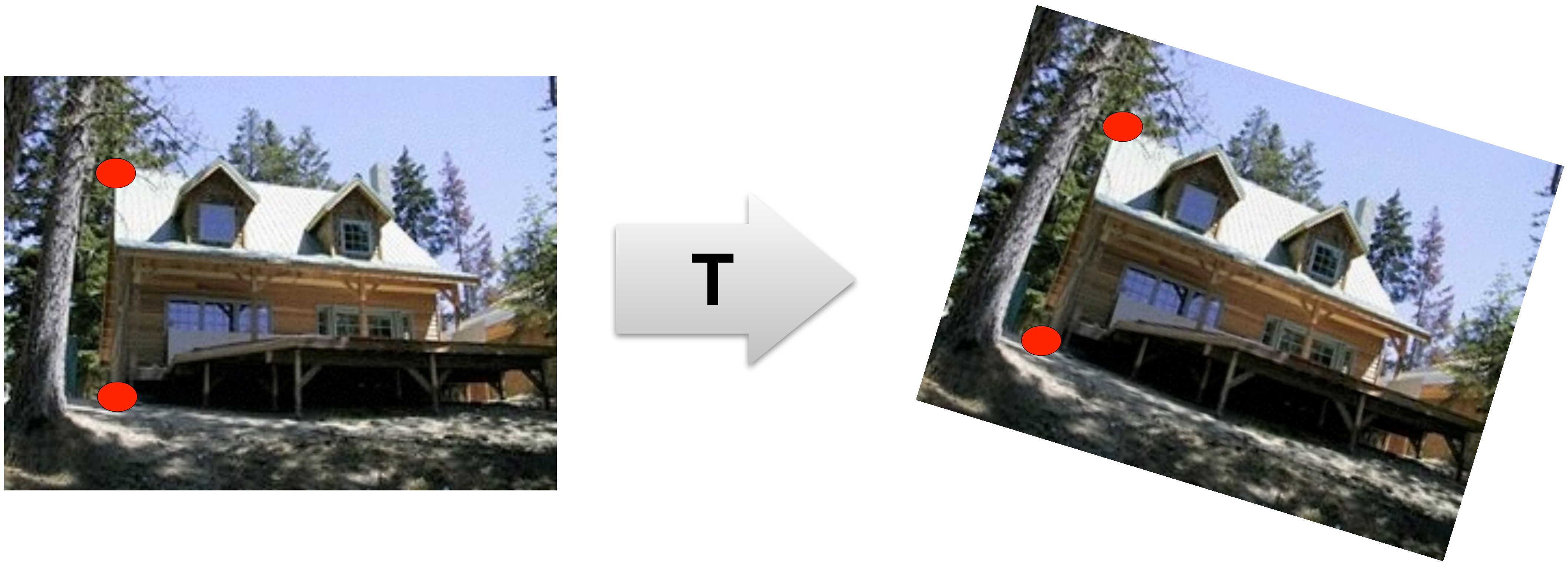
f(x,y)

g(x,y)

- *How many Degrees of Freedom?*
- *How many correspondences needed?*
- *What is the transformation matrix?*

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & p'_x - p_x \\ 0 & 1 & p'_y - p_y \\ 0 & 0 & 1 \end{bmatrix}$$

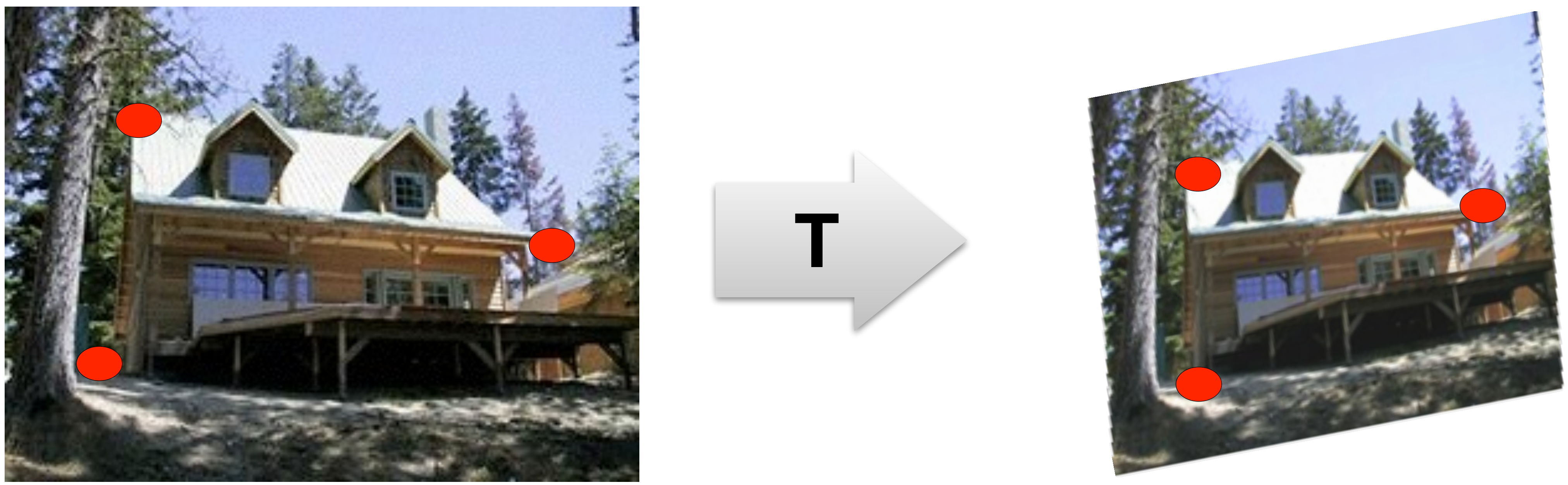
Euclidean



f(x,y) g(x,y)

- *How many Degrees of Freedom?*
- *How many correspondences needed for translation+rotation?*
- *What is the transformation matrix?*

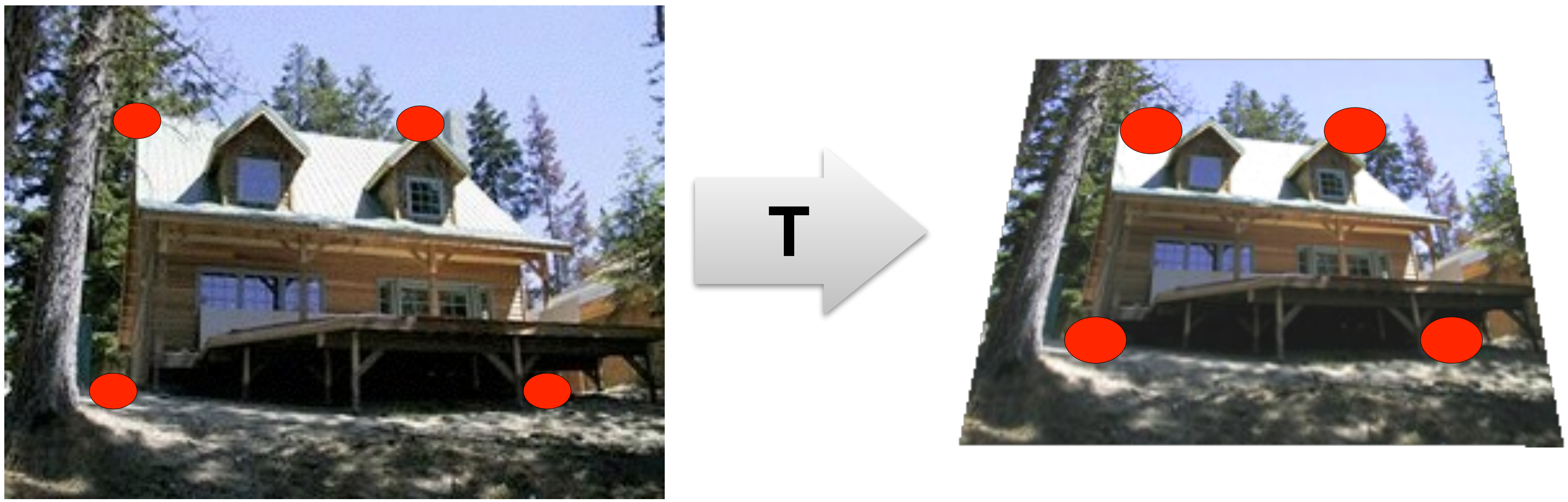
Affine



f(x,y) g(x,y)

- *How many Degrees of Freedom?*
- *How many correspondences needed for affine?*
- *What is the transformation matrix?*

Projective



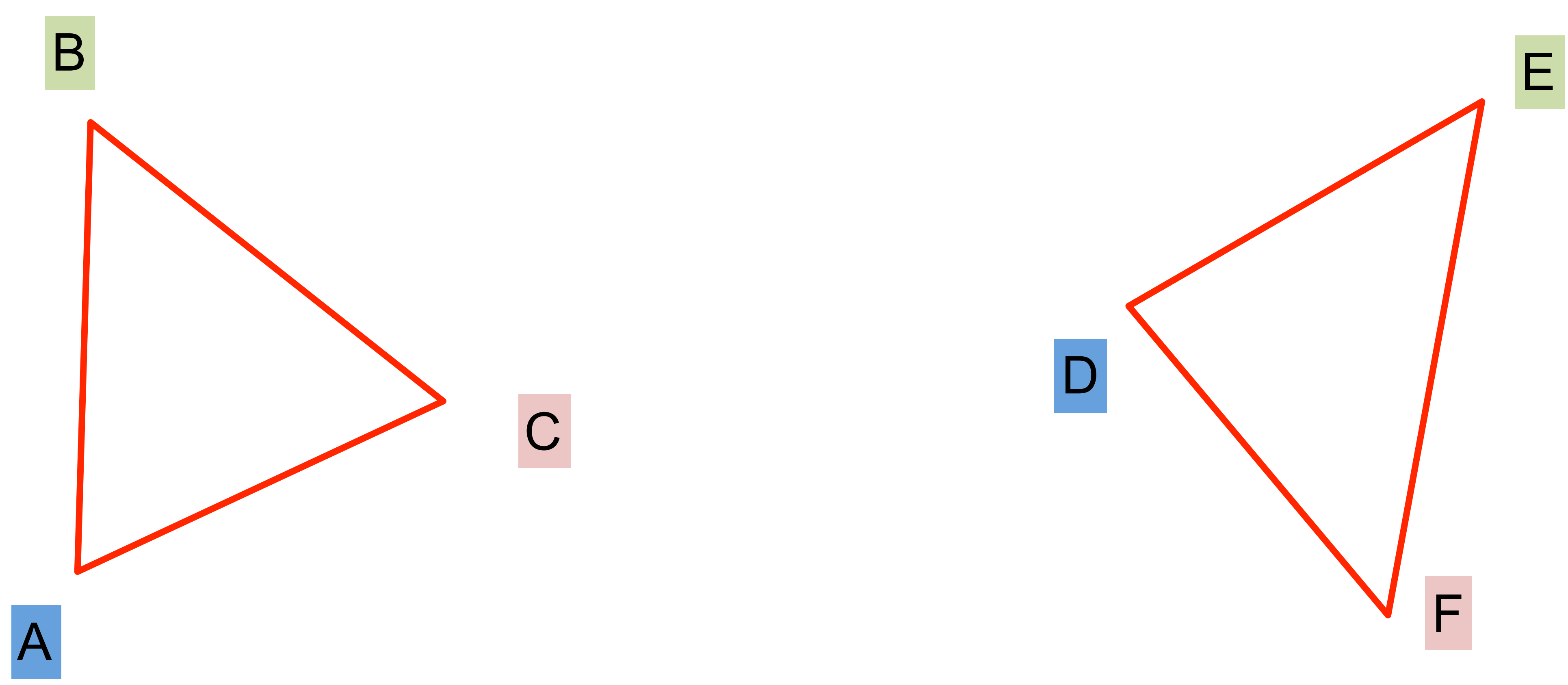
f(x,y) g(x,y)

- *How many Degrees of Freedom?*
- *How many correspondences needed for projective?*
- *What is the transformation matrix?*

Suppose we have two triangles: ABC and DEF.

What transformation will map A to D, B to E, and C to F?

How can we get the parameters?



Estimate transformation parameters using

Linear least squares

Given a set of matched feature points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

point in one image

point in the other image

and a transformation

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

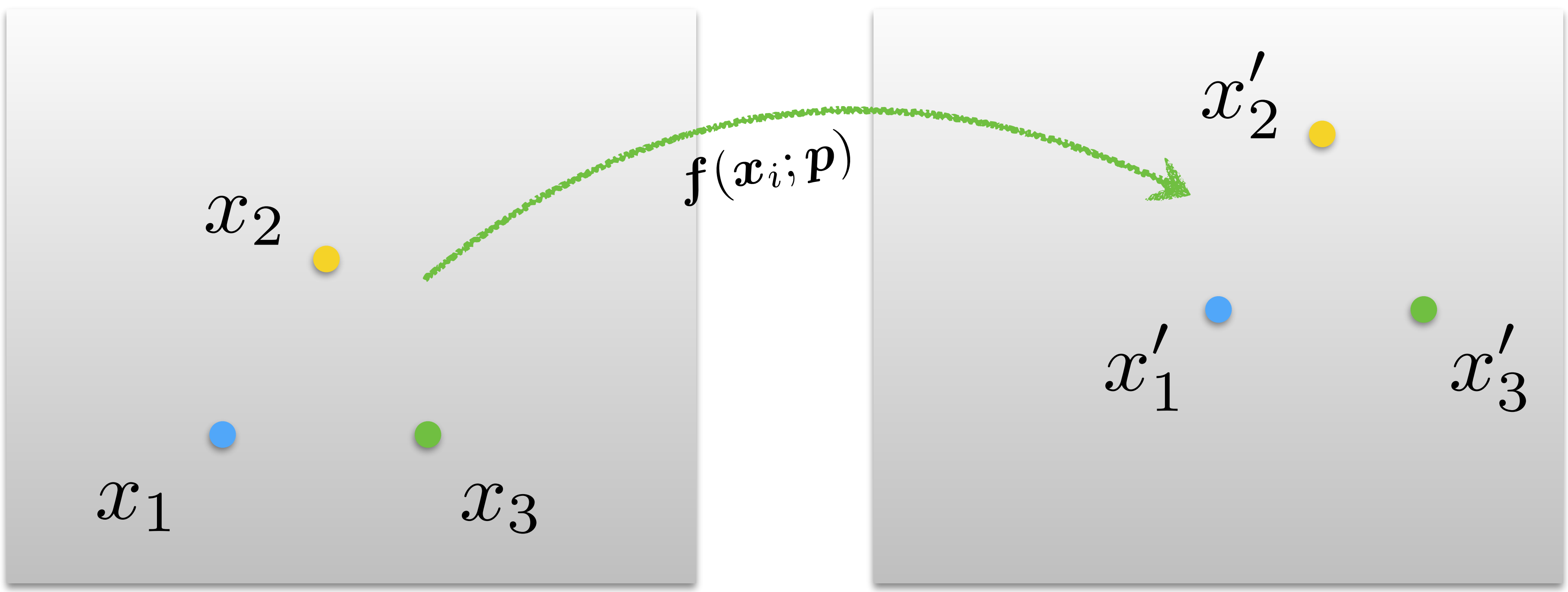
transformation function

parameters

Find the best estimate of

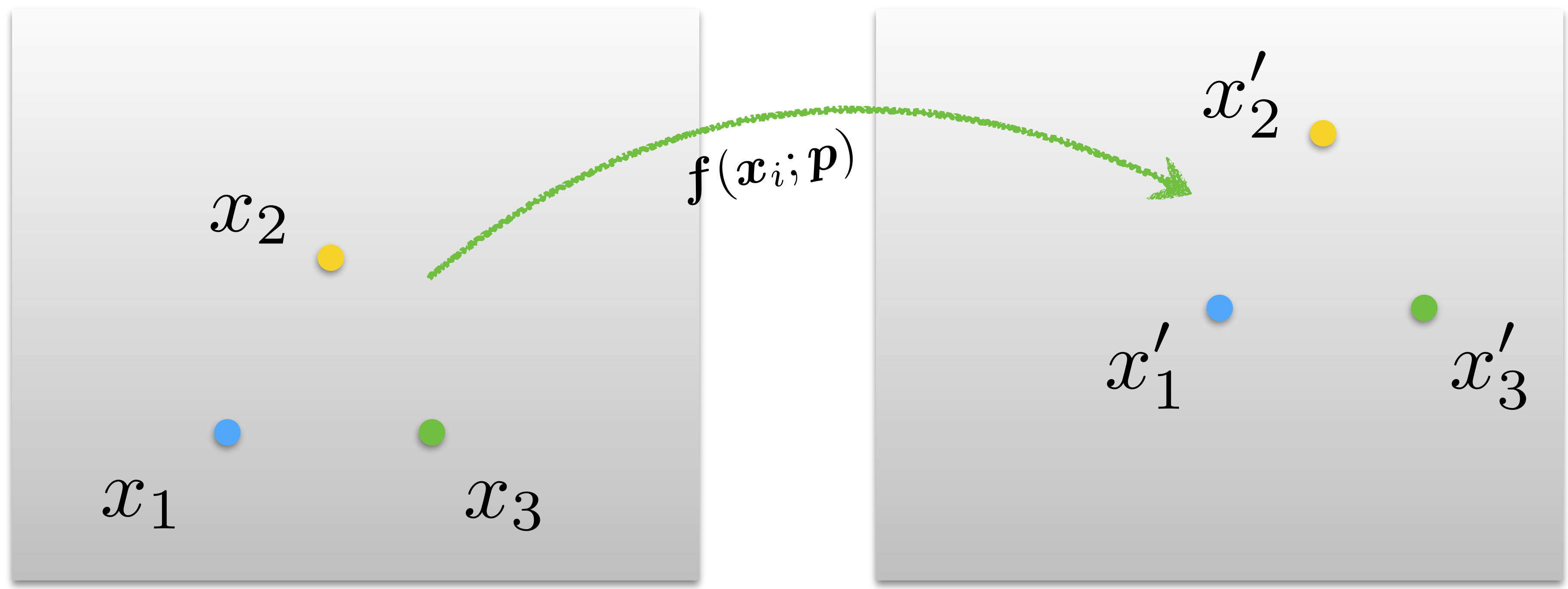
$$\boldsymbol{p}$$

Given point correspondences



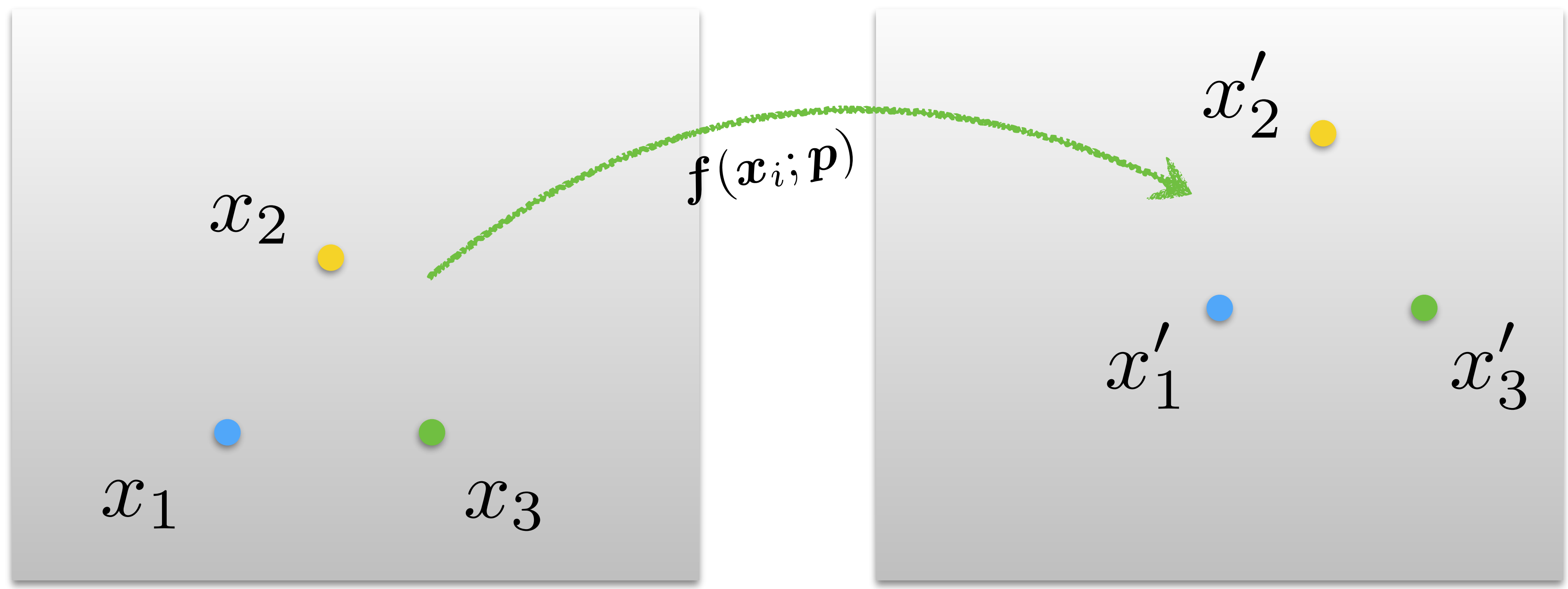
Given point correspondences

How can you solve for the transformation?



Least Squares Error

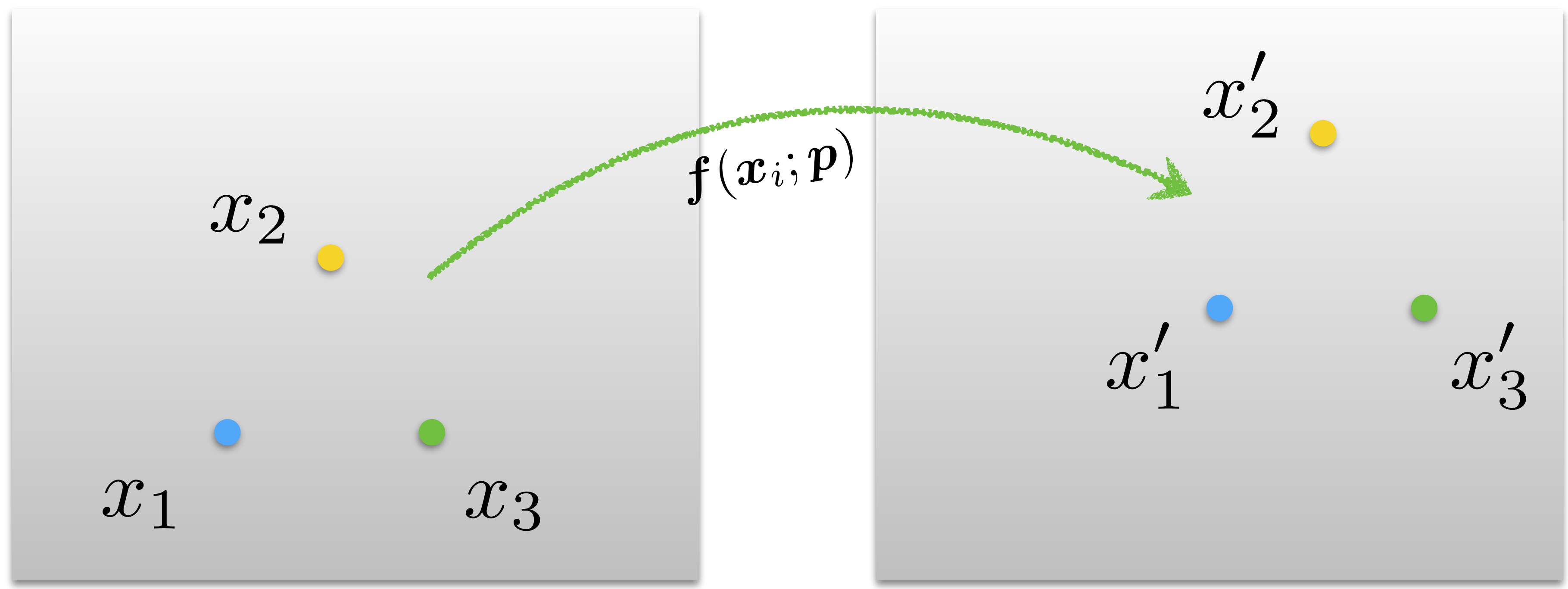
$$E_{\mathrm{LS}} = \sum_i \| \boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i \|^2$$



Least Squares Error

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

What is this? *What is this?*

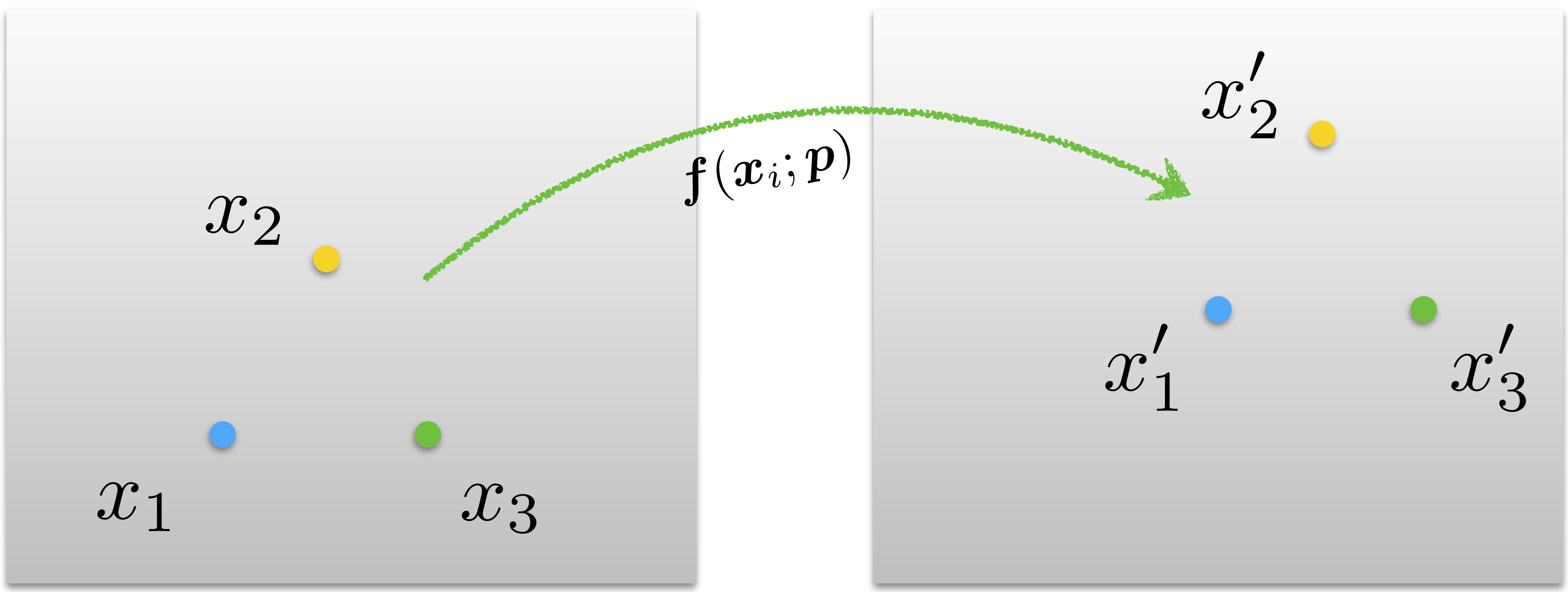


Least Squares Error

What is this?

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i;\boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

What is this? What is this?



$$||\boldsymbol{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Least Squares Error

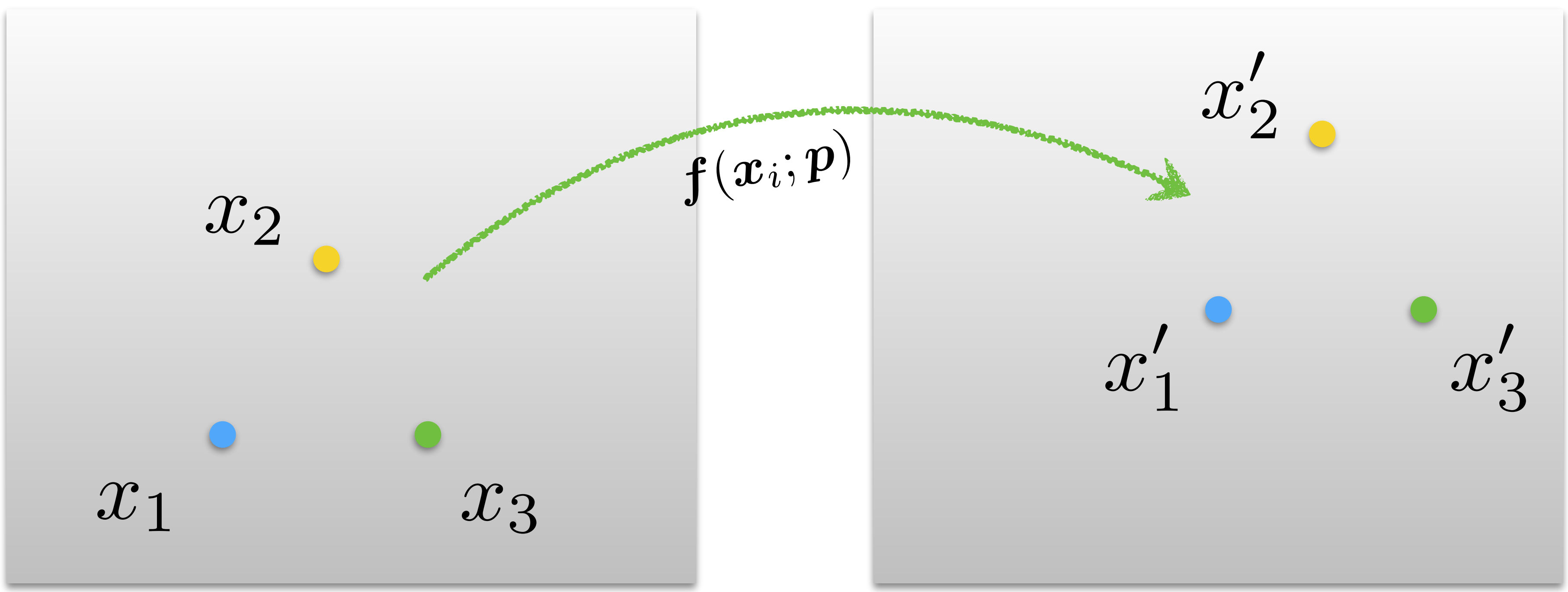
Euclidean
(L2) norm

$$E_{\mathrm{LS}} = \sum_i \| \boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i \|^2$$

squared!

predicted
location

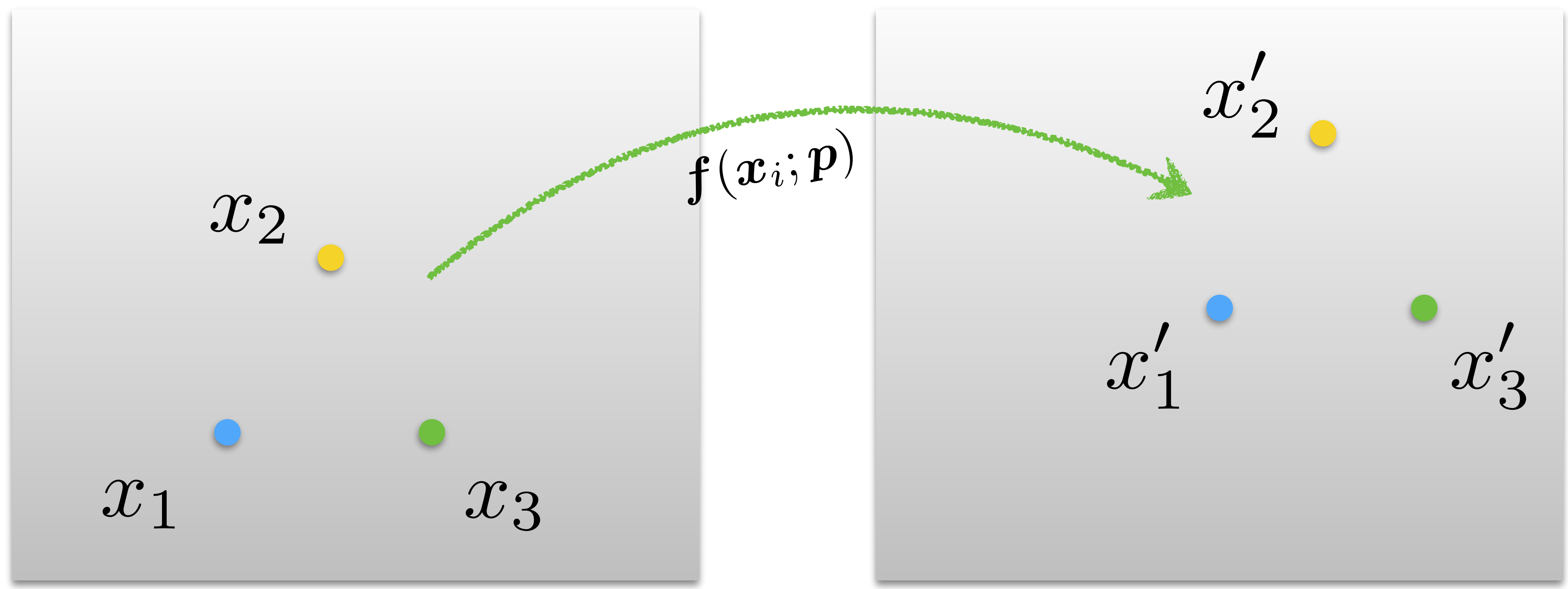
measured
location



Least Squares Error

$$E_{\mathrm{LS}} = \sum_i \underbrace{\|\boldsymbol{f}(\boldsymbol{x}_i;\boldsymbol{p}) - \boldsymbol{x}'_i\|^2}_{}$$

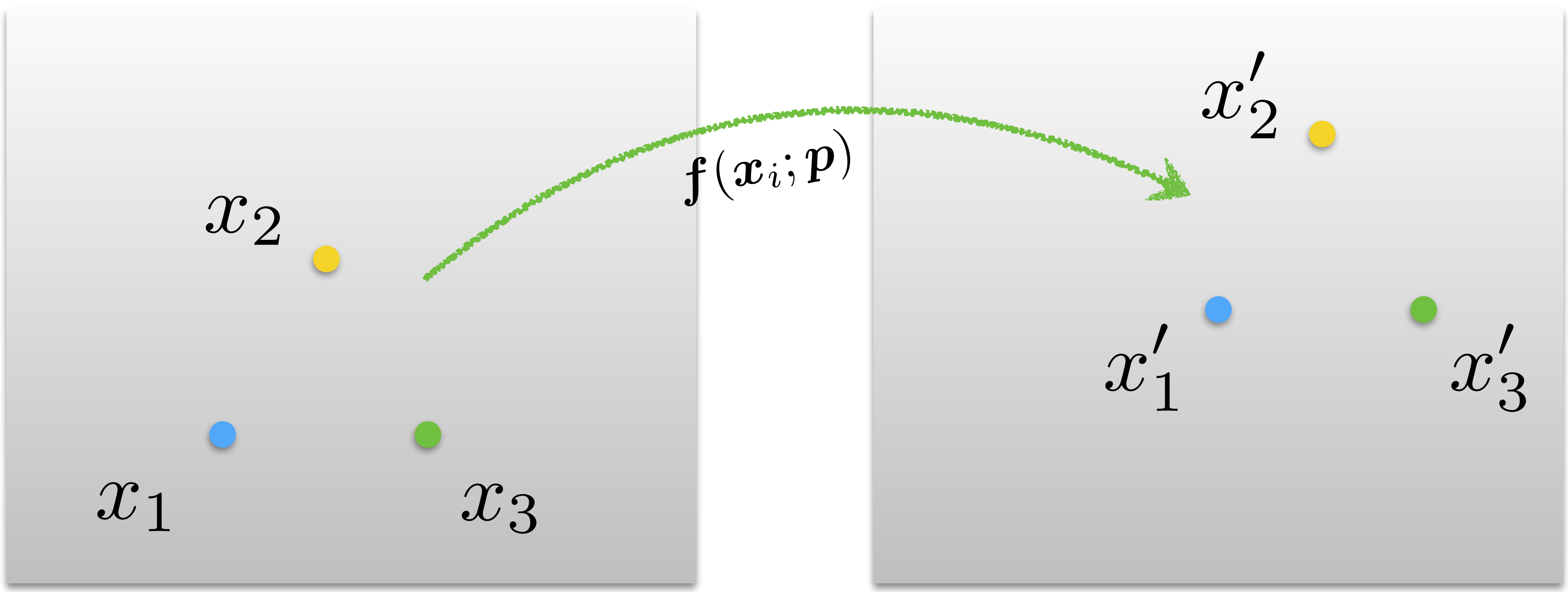
Residual
(projection error)



Least Squares Error

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i;\boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

What is the free variable?
What do we want to optimize?



Find parameters that minimize squared error

$$\hat{\boldsymbol{p}} = \arg\min_{\boldsymbol{p}} \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

General form of linear least squares

(**Warning:** change of notation. x is a vector of parameters!)

$$E_{\mathrm{LLS}} = \sum_i |\boldsymbol{a}_i \boldsymbol{x} - \boldsymbol{b}_i|^2$$

$$= \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|^2$$ (matrix form)

This function is quadratic.

How do you find the root of a quadratic?

General form of linear least squares

(**Warning:** change of notation. x is a vector of parameters!)

$$E_{\mathrm{LLS}} = \sum_i |\boldsymbol{a}_i \boldsymbol{x} - \boldsymbol{b}_i|^2$$

$$= \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|^2$$

(matrix form)

Minimize the error:

Expand

$$E_{\mathrm{LLS}} = \boldsymbol{x}^\top(\mathbf{A}^\top\mathbf{A})\boldsymbol{x} - 2\boldsymbol{x}^\top(\mathbf{A}^\top\boldsymbol{b}) + \|\boldsymbol{b}\|^2$$

Take derivative, set to zero

$$(\mathbf{A}^\top\mathbf{A})\boldsymbol{x} = \mathbf{A}^\top\boldsymbol{b}$$

(normal equation)

Solve for x

$$\boxed{\boldsymbol{x} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\boldsymbol{b}}$$

For the Affine transformation $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x};\boldsymbol{p})$

$$\boldsymbol{x}' = \mathbf{M}\boldsymbol{x} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Vectorize transformation parameters

$$\underbrace{\begin{bmatrix} x' \\ y' \\ x' \\ y' \\ \vdots \\ x' \\ y' \end{bmatrix}}_{\boldsymbol{b}} = \underbrace{\begin{bmatrix} x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \\ x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \\ & & & \vdots & & \\ x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix}}_{\boldsymbol{x}}$$

Notation in general form

Linear

least squares

estimation

only works

when the

transform function

is

?

Linear

least squares

estimation

only works

when the

transform function

is

linear!

Also

doesn't

deal well

with

outliers