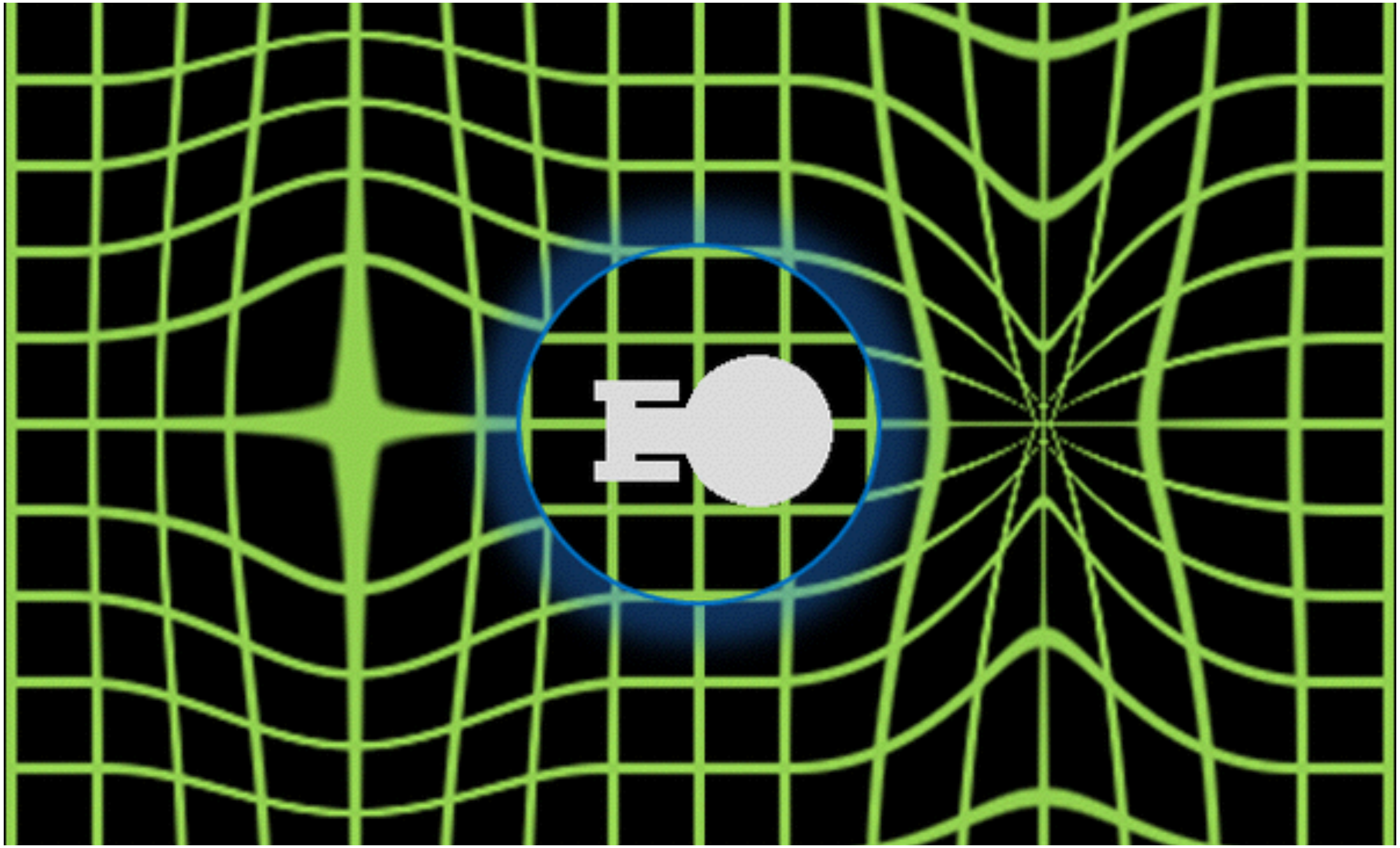


# 2D Image Transforms

16-385 Computer Vision (Kris Kitani)

**Carnegie Mellon University**


KRAFT
Macaroni & Cheese
DINNER
Family Size
ORIGINAL FLAVOR
NET WT
14.5 OZ (411g)

Extract features from an image …


what do we do next?

# Feature matching

(object recognition, 3D reconstruction, augmented reality, image stitching)


*How do you compute the transformation?*

Given a set of matched feature points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$ ← set of point correspondences

point in one image

point in the other image

and a transformation

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

transformation function

parameters

Find the best estimate of

$$\boldsymbol{p}$$

What kind of transformation functions are there?

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

# 2D Transformations

translation

rotation

aspect

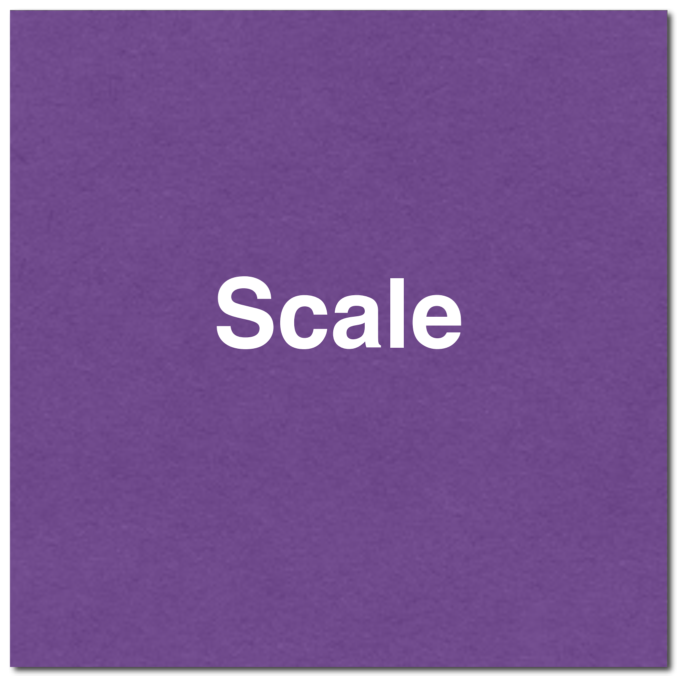
affine

perspective

cylindrical

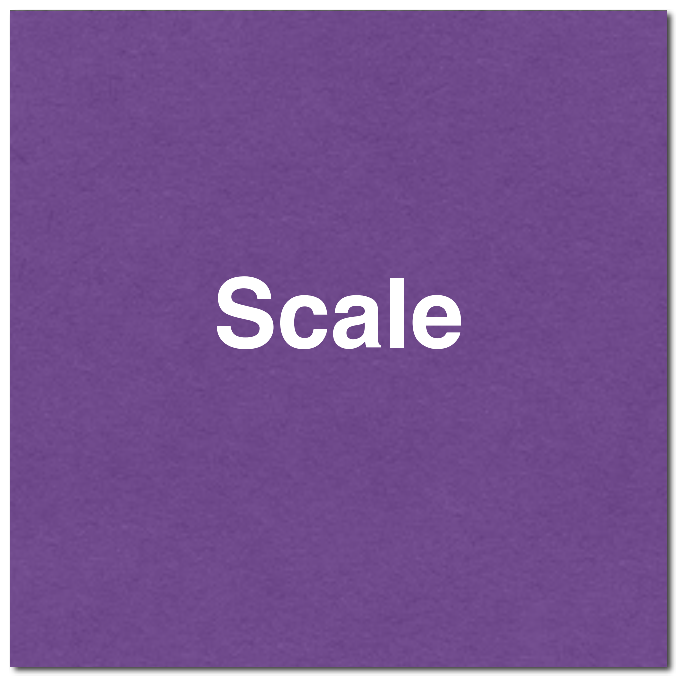
# 2D Planar Transformations

# 2D Planar Transformations




- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

# 2D Planar Transformations



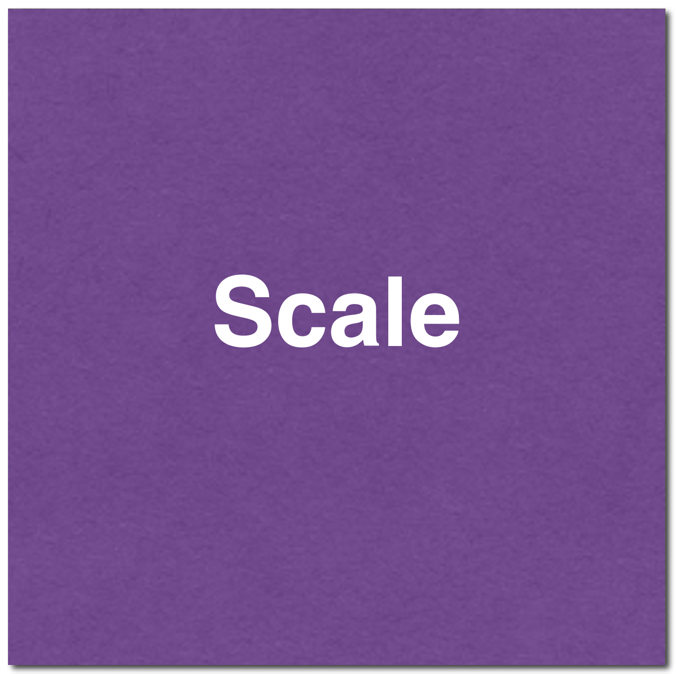

**Scale**

$$x' = ax$$

$$y' = by$$

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

# 2D Planar Transformations

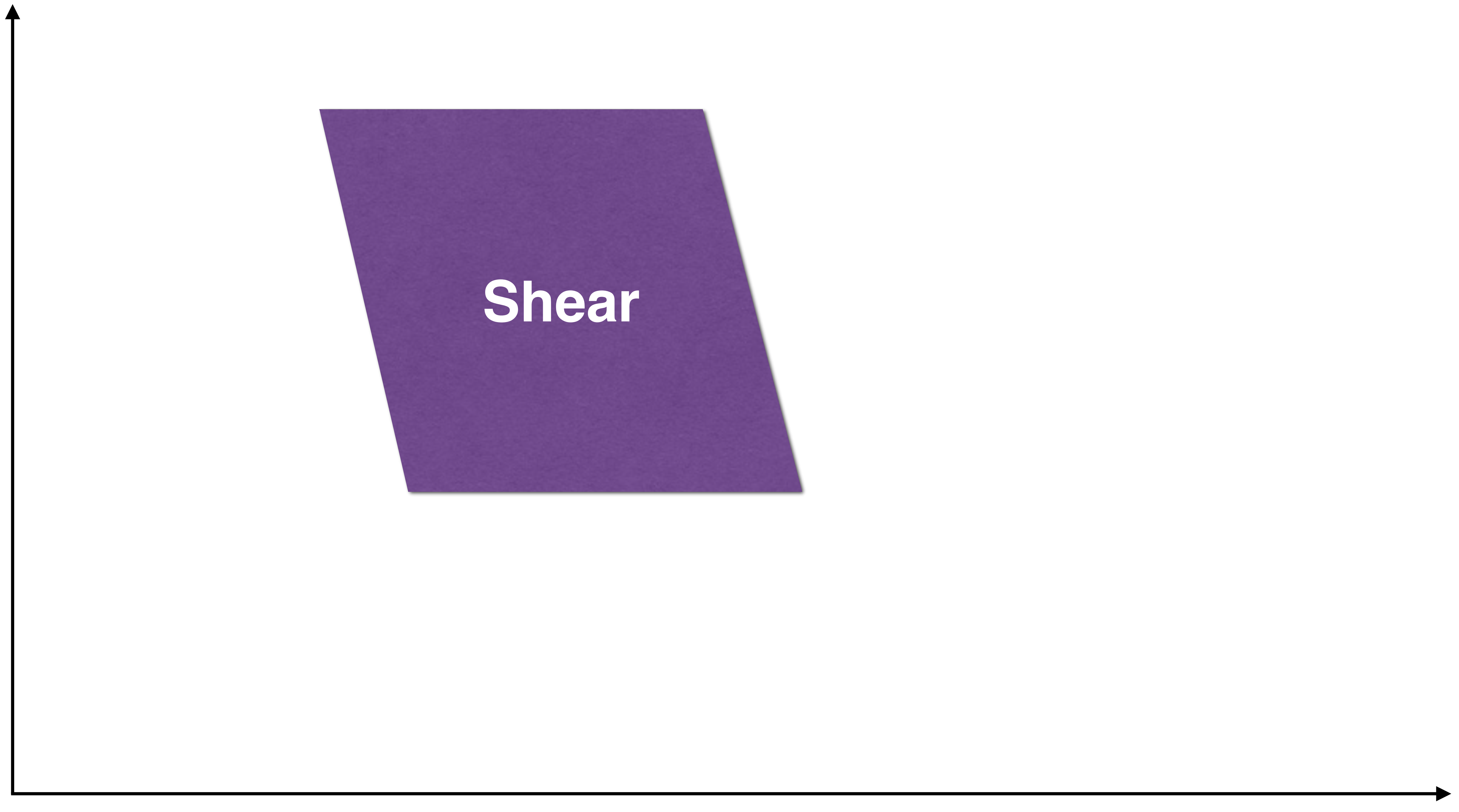



## Scale

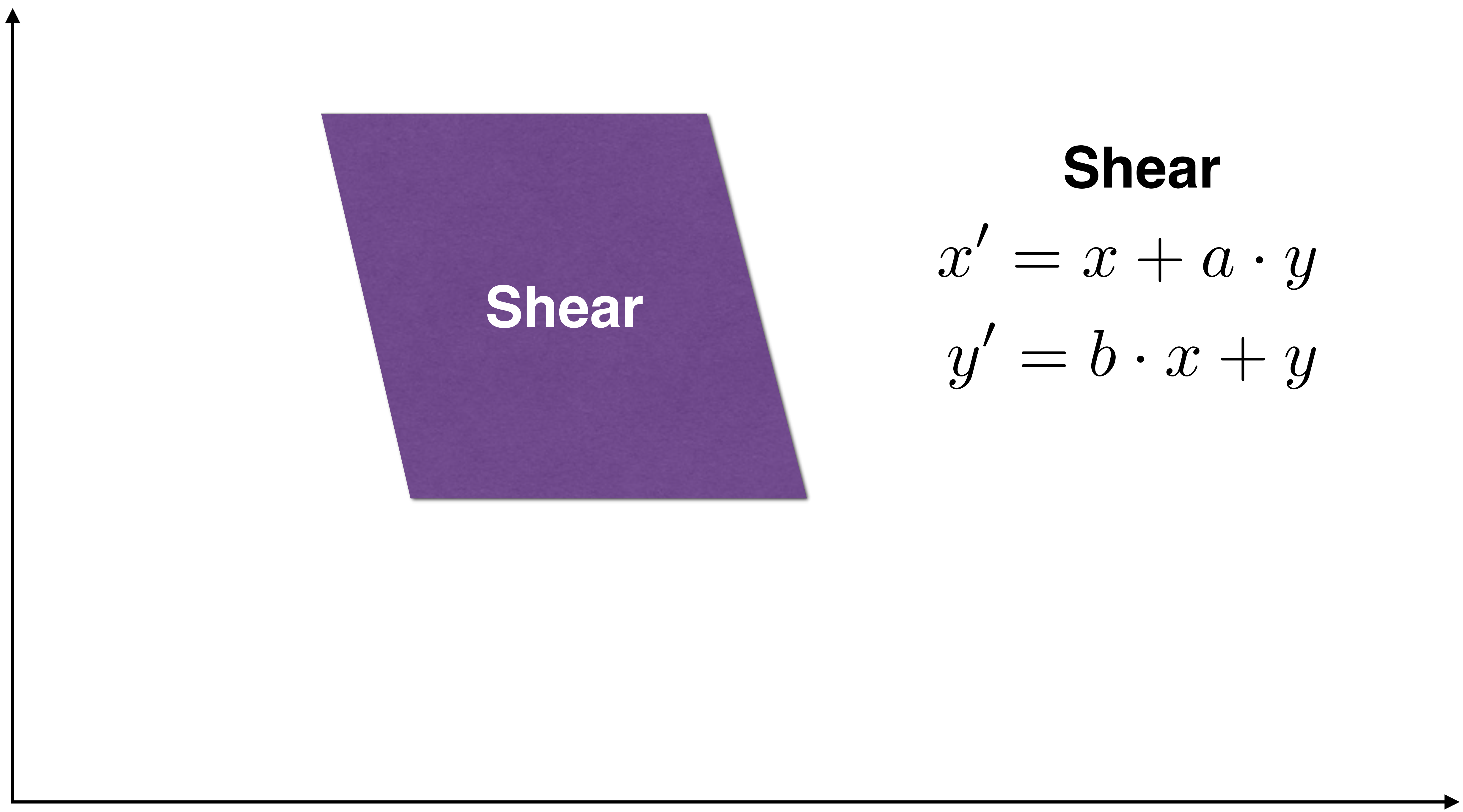
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textit{scaling matrix S}} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

# 2D Planar Transformations



# 2D Planar Transformations

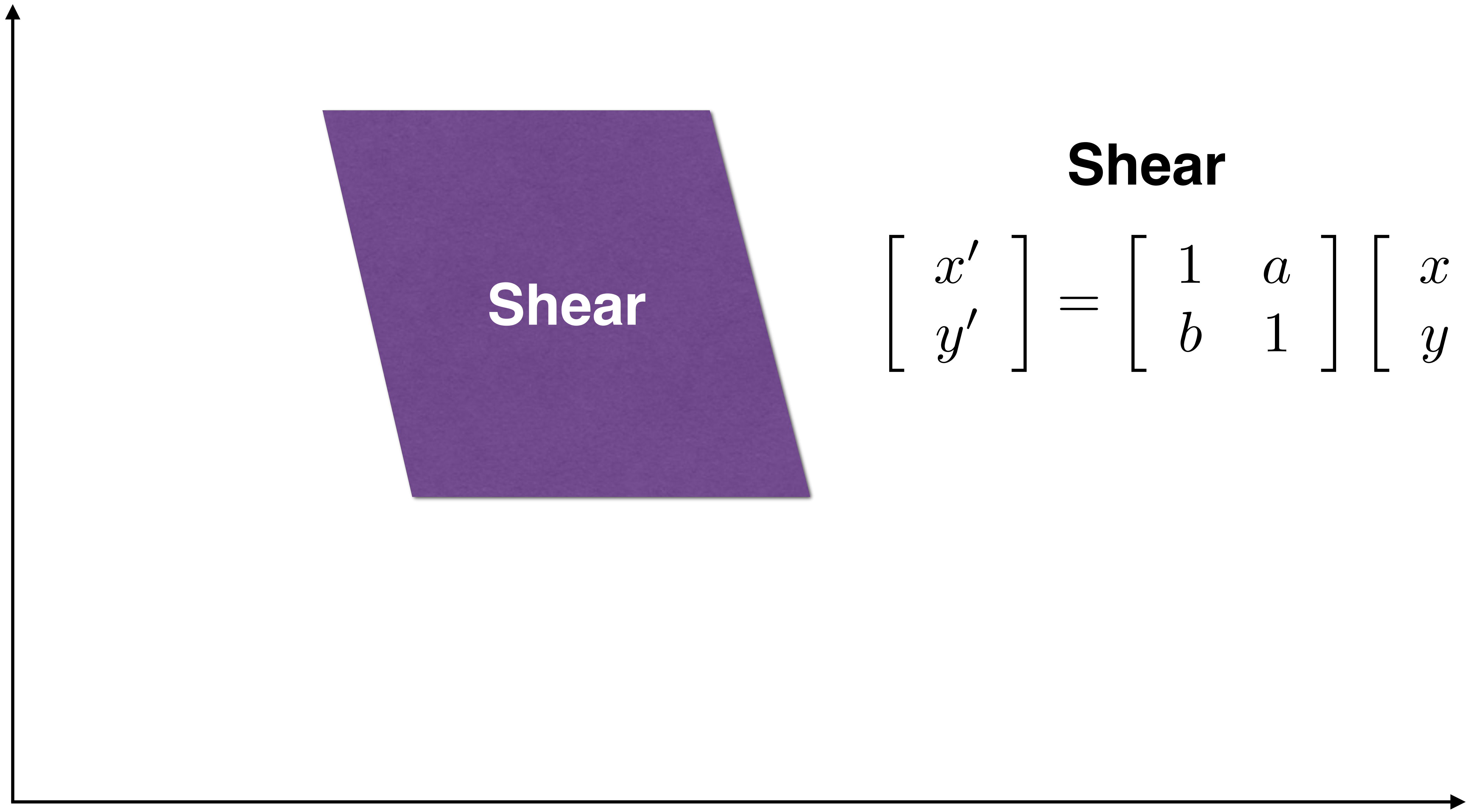



**Shear**

$$x' = x + a \cdot y$$

$$y' = b \cdot x + y$$

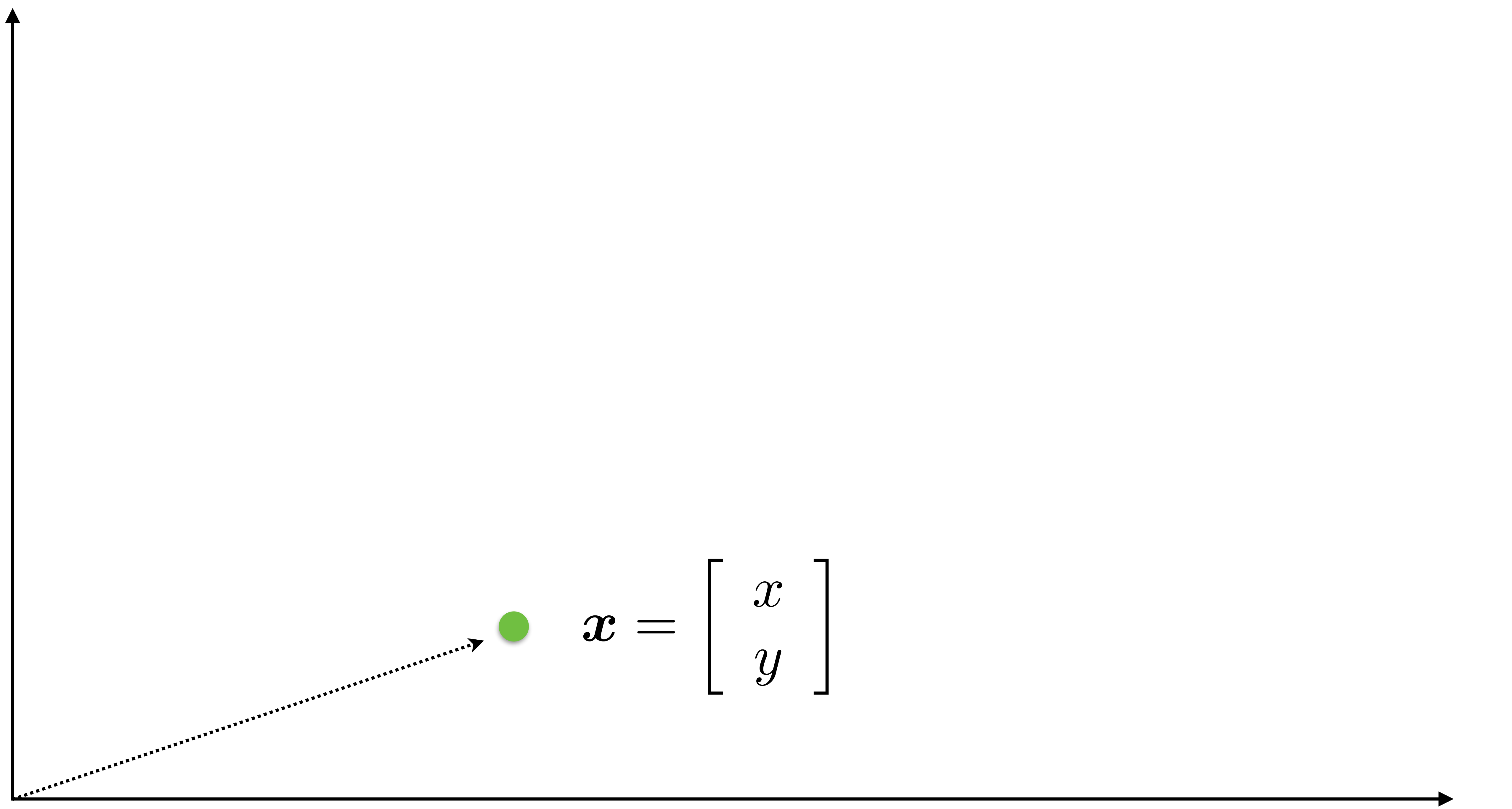
# 2D Planar Transformations




**Shear**

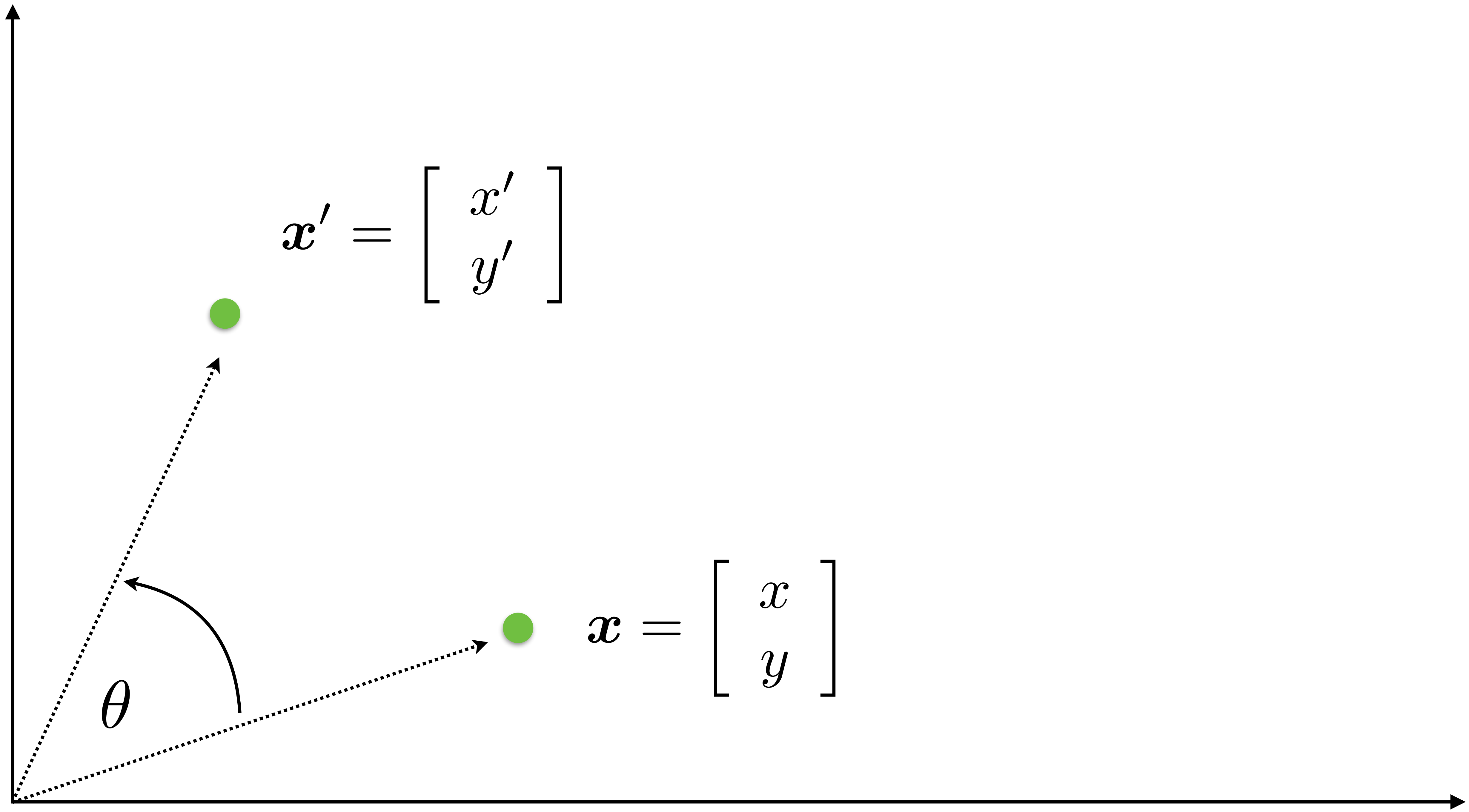
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

# 2D Planar Transformations

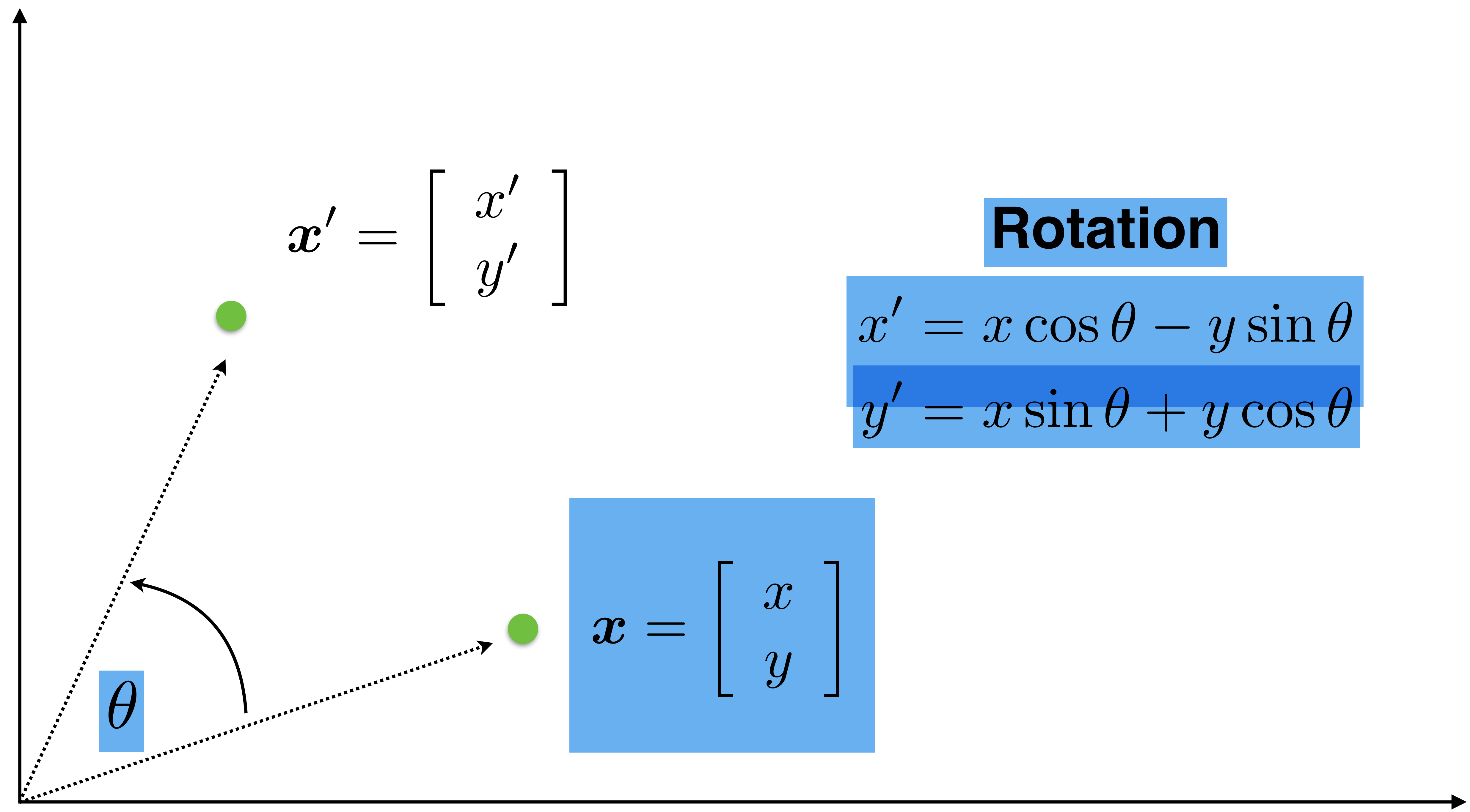


$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

# 2D Planar Transformations



# 2D Planar Transformations



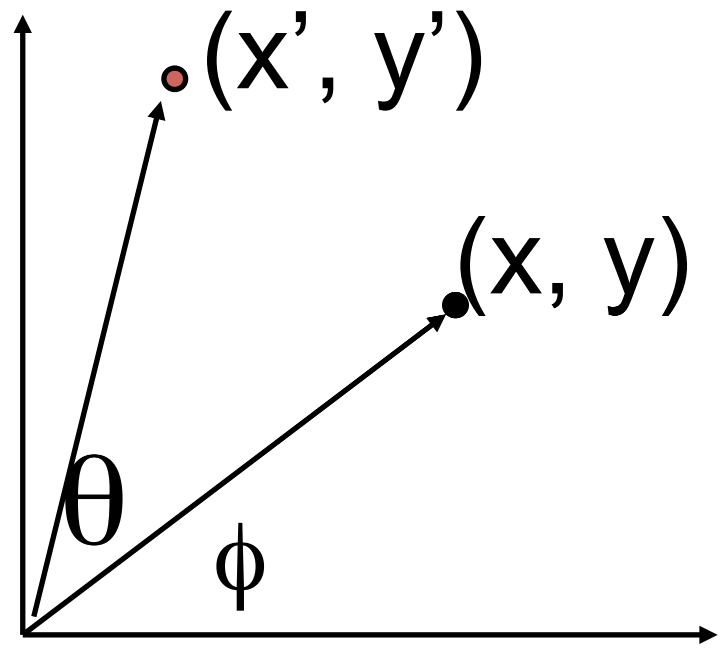

$$\boldsymbol{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

**Rotation**

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$




Polar coordinates…

$x = r \cos(\phi)$

$y = r \sin(\phi)$

$x' = r \cos(\phi + \theta)$

$y' = r \sin(\phi + \theta)$

Trig Identity…

$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$

$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$

Substitute…

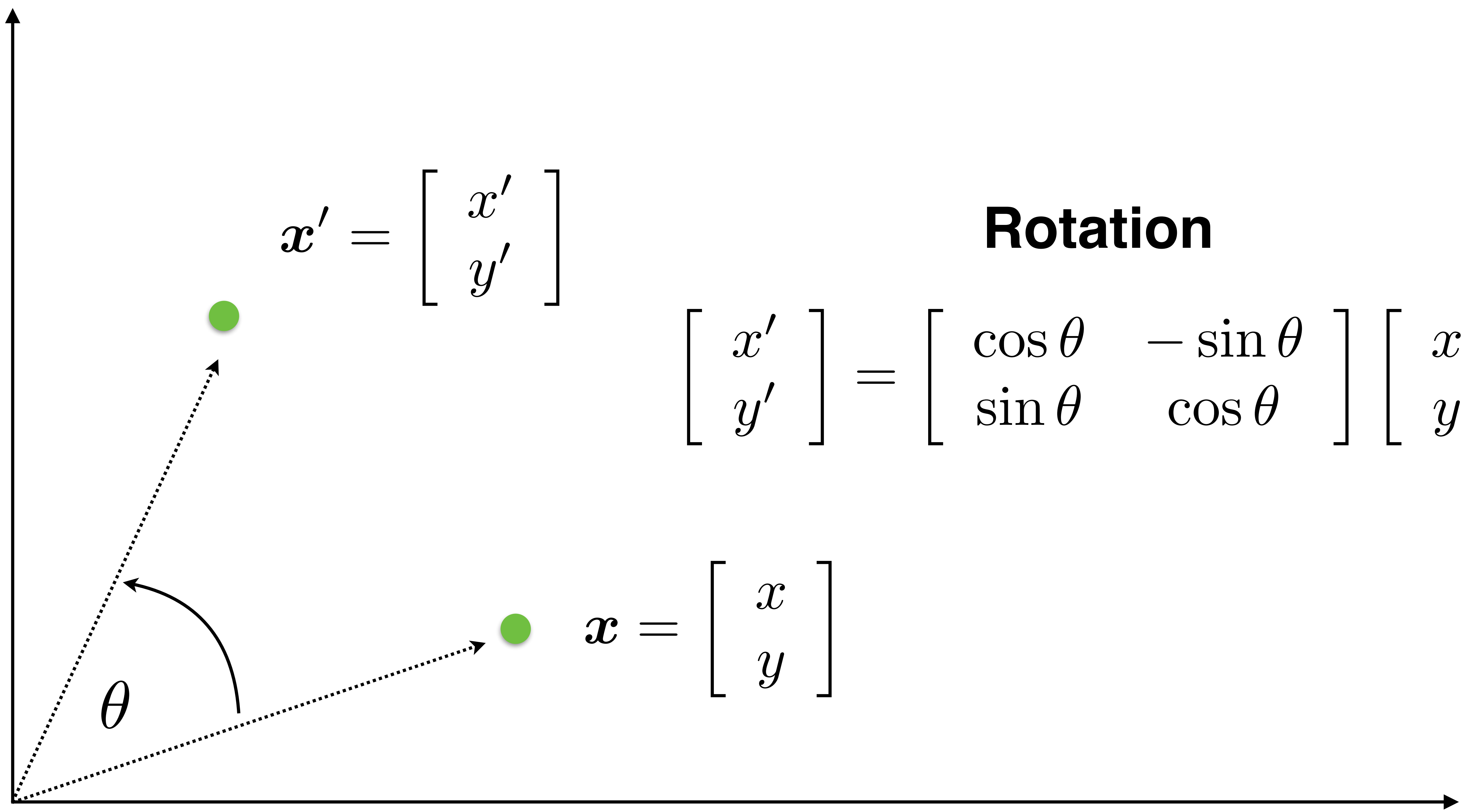
$x' = x \cos(\theta) - y \sin(\theta)$

$y' = x \sin(\theta) + y \cos(\theta)$

# 2D Planar Transformations

## Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$




$$\boldsymbol{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

# 2D **linear** transformation

(can be written in matrix form)

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x};\boldsymbol{p})$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

parameters $\boldsymbol{p}$

point $\boldsymbol{x}$

Scale

$$\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Flip across y

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rotate

$$\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Flip across origin

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

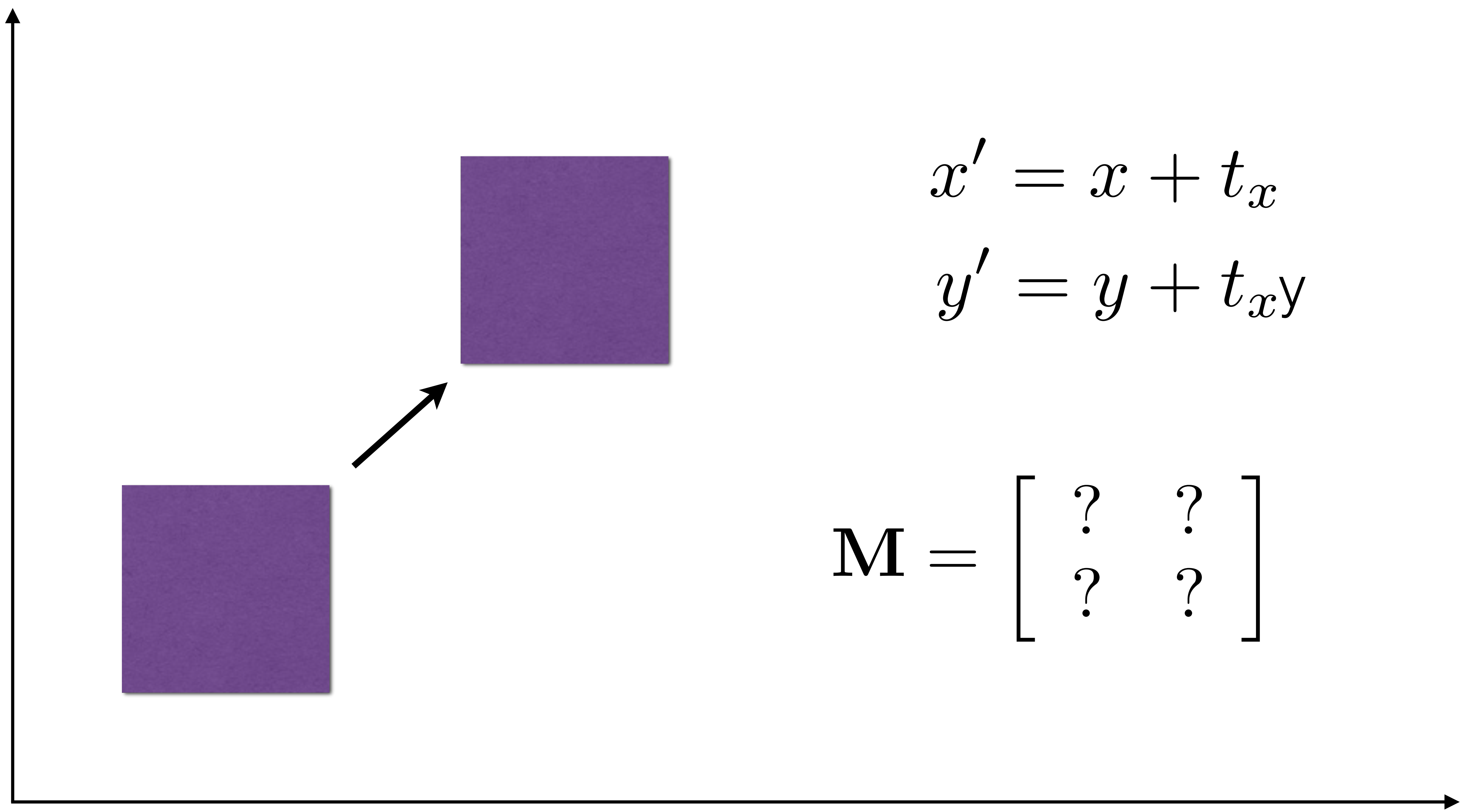
Shear

$$\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$$

Identity

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

How do you represent translation with a 2 x 2 matrix?

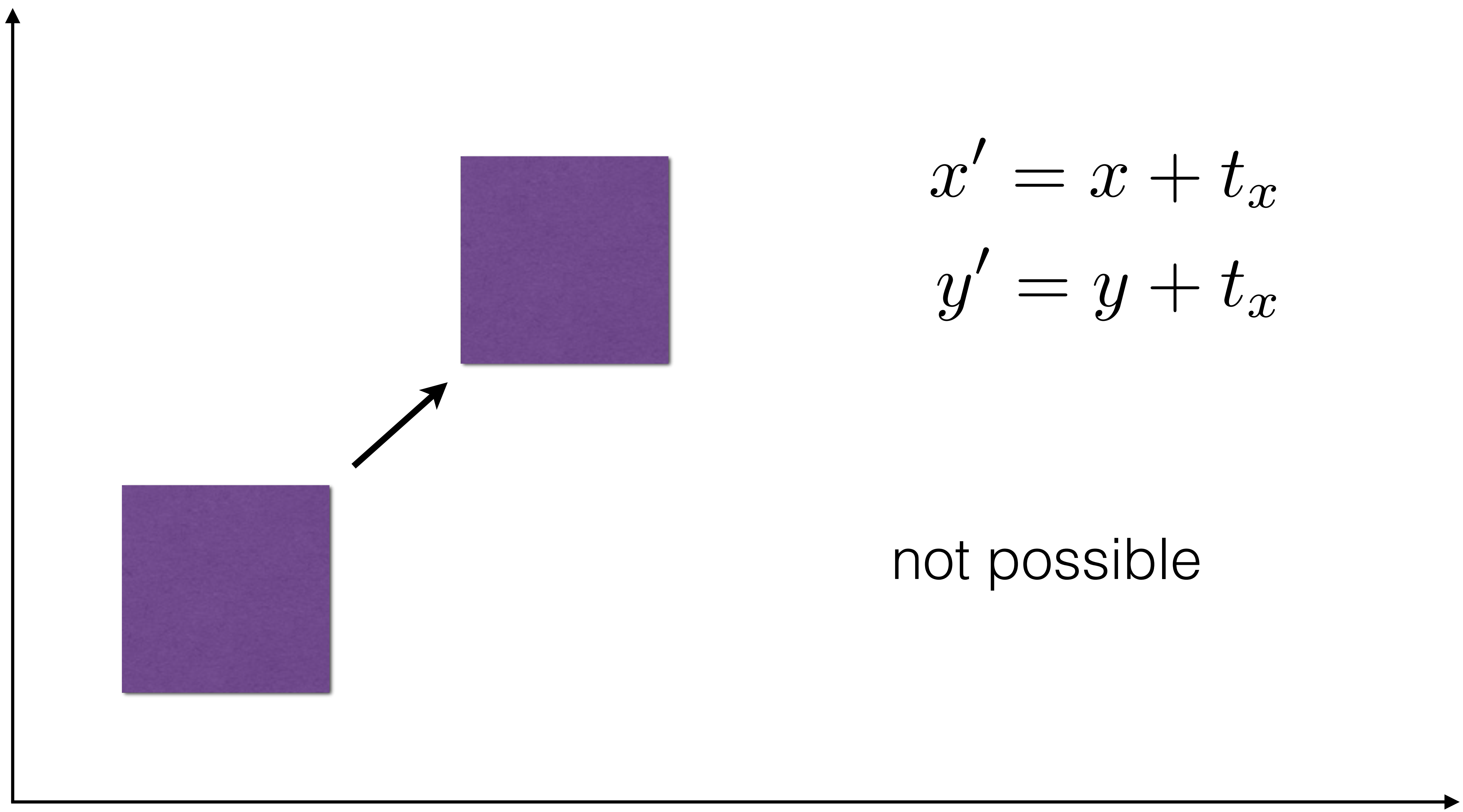


$$x' = x + t_x$$

$$y' = y + t_x\text{y}$$

$$\mathbf{M} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

# How do you represent translation with a 2 x 2 matrix?



$$x' = x + t_x$$
$$y' = y + t_x$$

not possible

*Q: How can we represent translation in matrix form?*

$$x' = x + t_x$$
$$y' = y + t_y$$

# Homogeneous Coordinates

**add a one here**

$$\begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

inhomogenous
coordinates

**homogenous**
coordinates

Represent 2D point with a 3D vector

*Q: How can we represent translation in matrix form?*

$$x' = x + t_x$$

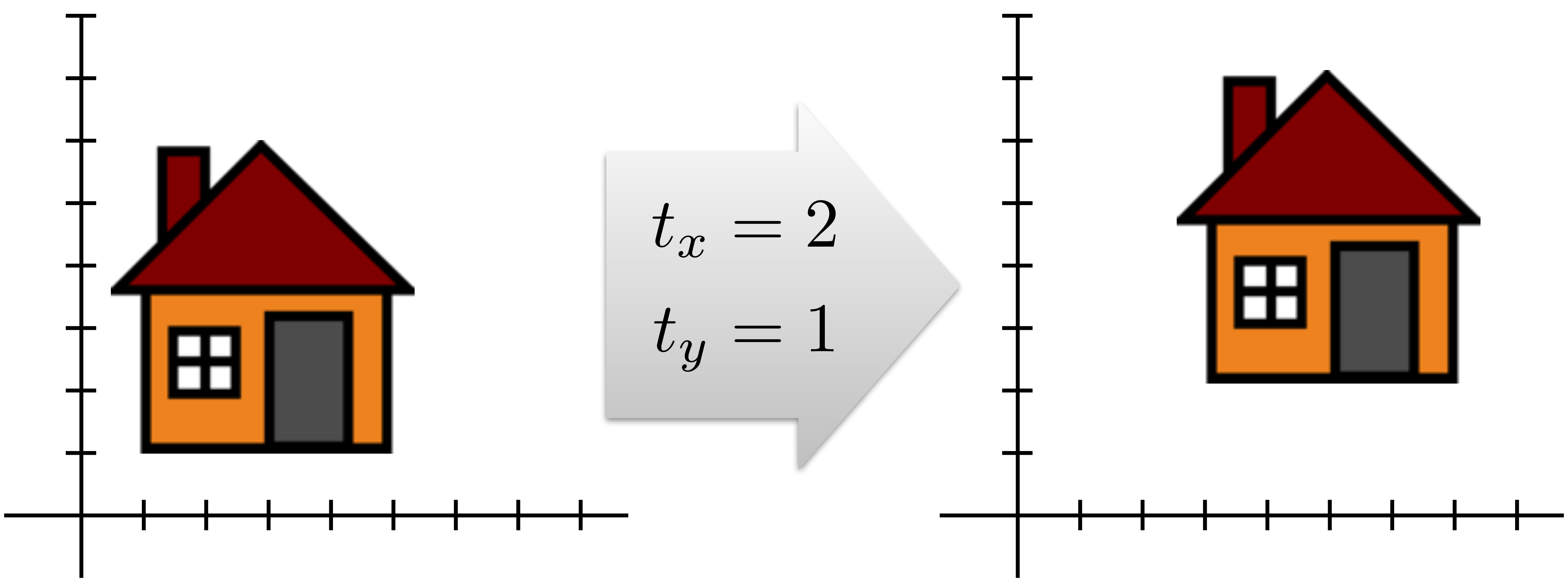
$$y' = y + t_y$$

A: append 3rd element and append 3rd column & row

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



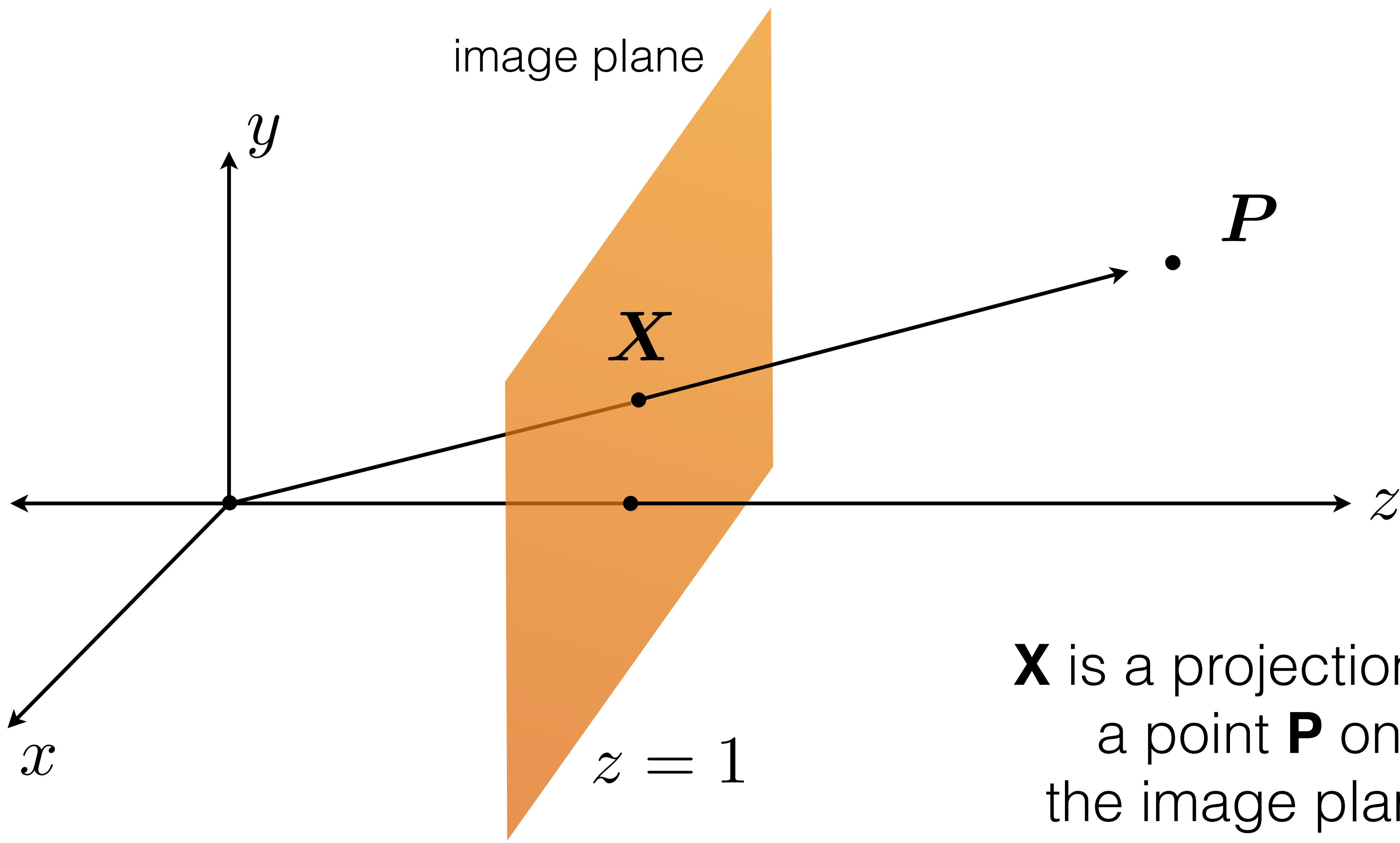
A 2D point in an image can be represented as a 3D vector

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Longleftrightarrow \quad \boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $$x = \frac{x_1}{x_3} \qquad y = \frac{x_2}{x_3}$$

*Why?*

# Think of a point on the image plane in 3D




**X** is a projection of
a point **P** on
the image plane

You can think of a conversion to homogenous coordinates as a conversion of a **point** to a **ray**

**Conversion:**

- 2D point → homogeneous point $\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

  append 1 as 3rd coordinate

- homogeneous point → 2D point $\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$

  divide by 3rd coordinate

**Special Properties**

- Scale invariant $\begin{bmatrix} x & y & w \end{bmatrix}^\top = \lambda \begin{bmatrix} x & y & w \end{bmatrix}^\top$
- Point at infinity $\begin{bmatrix} x & y & 0 \end{bmatrix}$ coz x / 0 and y / 0 is undefined
- Undefined $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ 0 / 0 is undefined

# Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_{\boldsymbol{x}} & 0 & 0 \\ 0 & \boldsymbol{s}_{\boldsymbol{y}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

# Matrix Composition

Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left( \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = T($t_x$,$t_y$) R($\Theta$) S($s_x$,$s_y$) p

*Does the order of multiplication matter?*

# 2D transformations

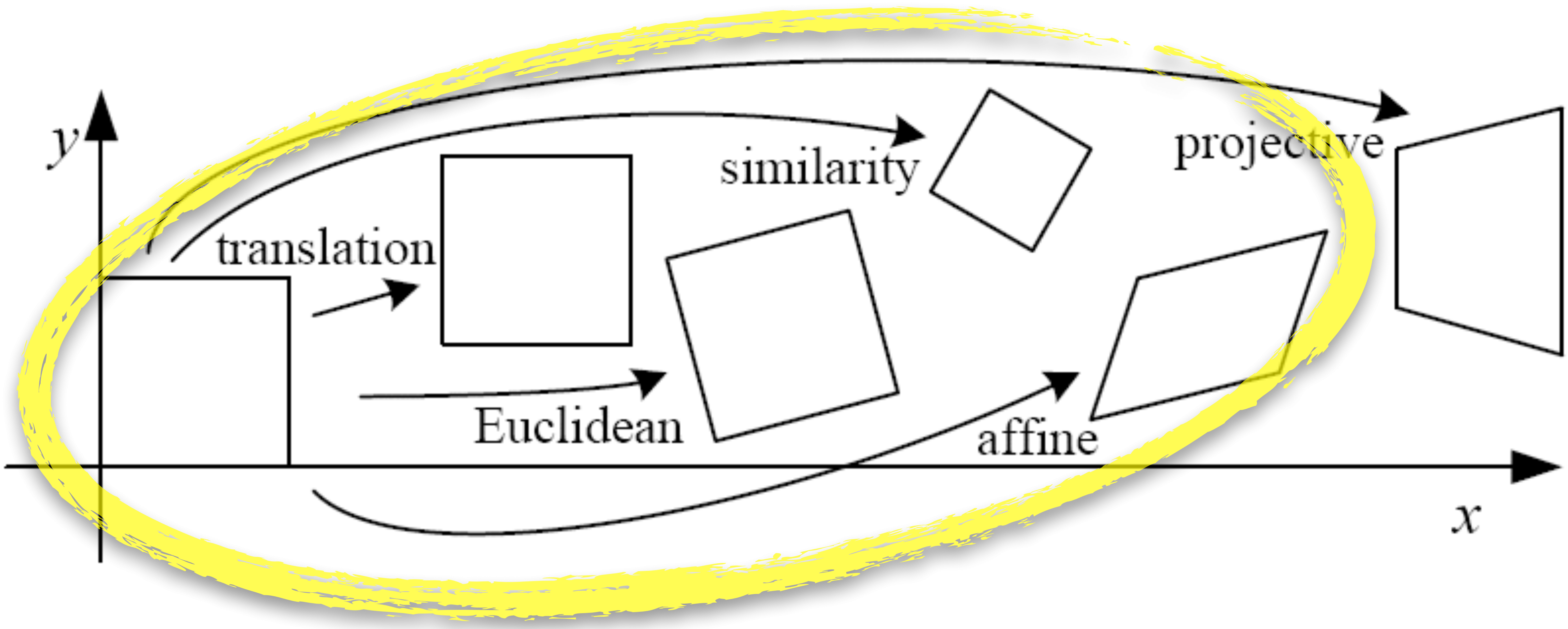



Figure 1: Basic set of 2D planar transformations

| Name | Matrix | # D.O.F. |
|---|---|---|
| translation | $\begin{bmatrix} \boldsymbol{I} & \vert & \boldsymbol{t} \end{bmatrix}_{2\times 3}$ | 2 |
| rigid (Euclidean) | $\begin{bmatrix} \boldsymbol{R} & \vert & \boldsymbol{t} \end{bmatrix}_{2\times 3}$ | 3 |
| similarity | $\begin{bmatrix} s\boldsymbol{R} & \vert & \boldsymbol{t} \end{bmatrix}_{2\times 3}$ | 4 |
| affine | $\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$ | 6 |
| projective | $\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$ | 8 |

# Affine Transformation

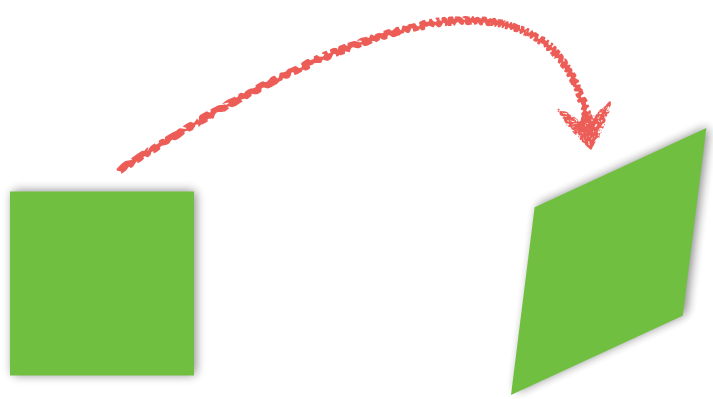
Affine transformations are combinations of

- Linear transformations, and
- Translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition (affine times affine is affine)



*Will the last coordinate w ever change?*

Coming soon…

# Projective Transform

Projective transformations are combos of

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition
- Models change of basis
- Projective matrix is defined up to a scale (8 DOF)