3.2 Velocity-based motion model

In the remainder of this chapter we will describe two probabilistic motion models for planar movement: the **velocity motion model** and the **odometry motion model**, the former being the main topic of this section. Remember that when a movement command is given to a robot, there are different factors that affect such movement (*e.g.* wheel slippage, unequal floor, inaccurate calibration, etc.), adding uncertainty to the actual move done. This results in a need for characterizing the robot motion in *probabilistic terms*, that is:

$$p(x_t|u_t, x_{t-1})$$

being:

- x_t the robot pose at time instant t,
- u_t the motion command (also called control action) at t, and
- x_{t-1} the robot pose at the previous time instant $t-1$.

So basically this probability models the probability distribution over robot poses when executing the motion command u_t, having the robot the previous pose x_{t-1}. In other words, we are considering a function $g(\cdot)$ that performs $x_t = g(x_{t-1}, u_t)$ and outputs $x_t \sim p(x_t|u_t, x_{t-1})$.

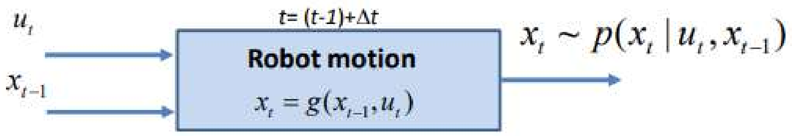


Fig. 1: Inputs and outputs of a probabilistic motion model.

Different definitions for the $g(\cdot)$ function lead to different probabilistic motion models, like the velocity motion model explored here.

3.2.1 The model

The *velocity motion model* is mainly used for motion planning, where the details of the robot's movement are of importance and odometry information is not available (*e.g.* no wheel encoders are available).

This motion model is characterized by the use of two velocities to control the robot's movement: **linear velocity** v and **angular velocity** w. Therefore, during the following sections, the movement commands will be of the form:

$$u_t = \begin{bmatrix} v_t \\ w_t \end{bmatrix}, \quad u_t \sim N(\overline{u}, \Sigma_{u_t})$$

The velocity motion model defines the function $g(\cdot)$ as:

$$g(x_{t-1}, u_t) = x_{t-1} \oplus \Delta x_t, \quad x_{t-1} \sim N(\overline{x}_{t-1}, \Sigma_{x_{t-1}})$$

being $\Delta_{x_t} = [\Delta_{x_t}, \Delta_{y_t}, \Delta_{\theta_t}]$ (assuming w and v constant):

- $\Delta x_t = \frac{v}{w}\sin(w\Delta t)$

- $\Delta y_t = \frac{v}{w}[1 - \cos(w\Delta t)]$
- $\Delta\theta_t = w\Delta t$

Note that $g(x_{t-1}, u_t) = x_{t-1} \oplus \Delta x_t$ **is not a linear operation!**

In this way, this motion model is characterized by the following equations, depending on the value of the angular velocity w (note that a division by zero would appear in the first case with $w = 0$):

- If $w \neq 0$:

$$\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{bmatrix} + \begin{bmatrix} -R\sin\theta_{t-1} + R\sin(\theta_{t-1} + \Delta\theta) \\ R\cos\theta_{t-1} - R\cos(\theta_{t-1} + \Delta\theta) \\ \Delta\theta \end{bmatrix}$$

- If $w = 0$:

$$\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{bmatrix} + v \cdot \Delta t \begin{bmatrix} \cos\theta_{t-1} \\ \sin\theta_{t-1} \\ 0 \end{bmatrix}$$

with:

- $v = w \cdot R$ *(R is also called the curvature radius)*
- $\Delta\theta = w \cdot \Delta t$

In [1]:
```
%matplotlib widget

# IMPORTS
import numpy as np
from numpy import random
import matplotlib.pyplot as plt
from IPython.display import display, clear_output
import time


import sys
sys.path.append("..")
from utils.DrawRobot import DrawRobot
from utils.PlotEllipse import PlotEllipse
```

ASSIGNMENT 1: The model in action

Modify the following `next_pose()` function, used in the `VelocityRobot` class below, which computes the next pose x_t of a robot given:

- its previous pose x_{t-1},
- the velocity movement command $u = [v, w]^T$, and
- a lapse of time Δt.

Concretly you have to complete the if-else statement that takes into account when the robot moves in an straight line so $w = 0$. *Note: you don't have to modify the* `None` *in the function header nor in the* `if cov is not None:` *condition.*

Remark that at this point **we are not taking into account uncertainty in the system**: neither from the initial pose ($\Sigma_{x_{t-1}}$) nor the movement (v, w) (Σ_{u_t}).

Example

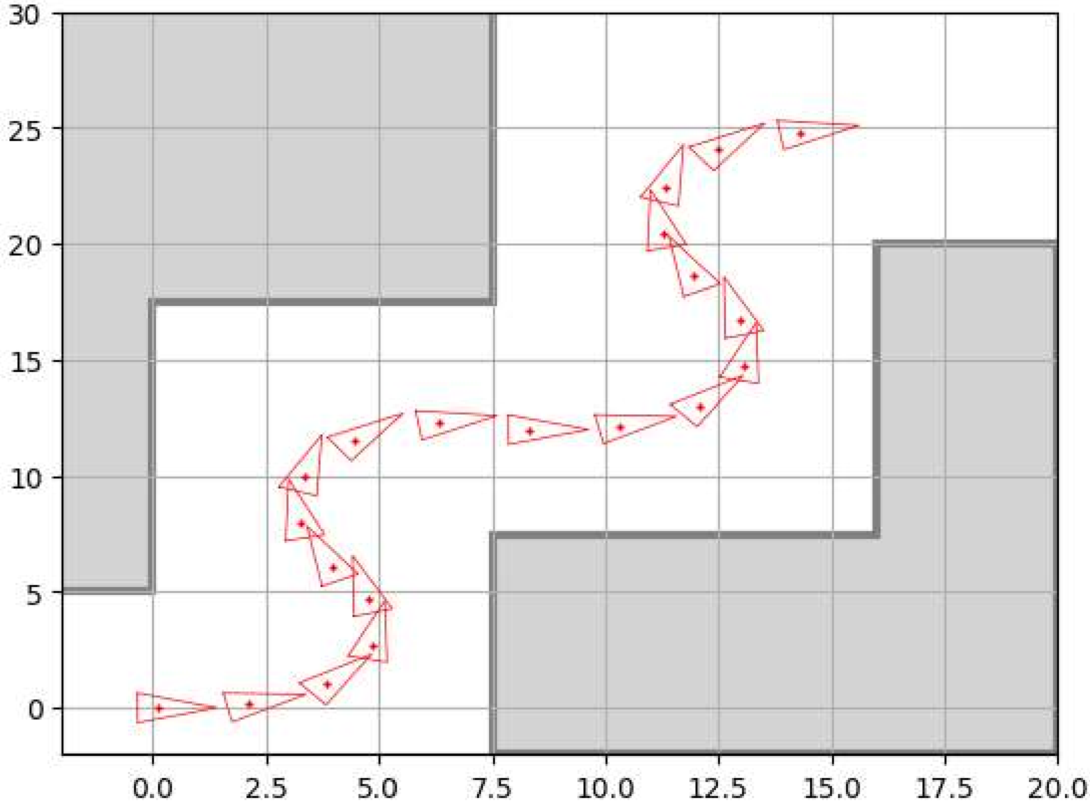


Fig. 2: Route of our robot.

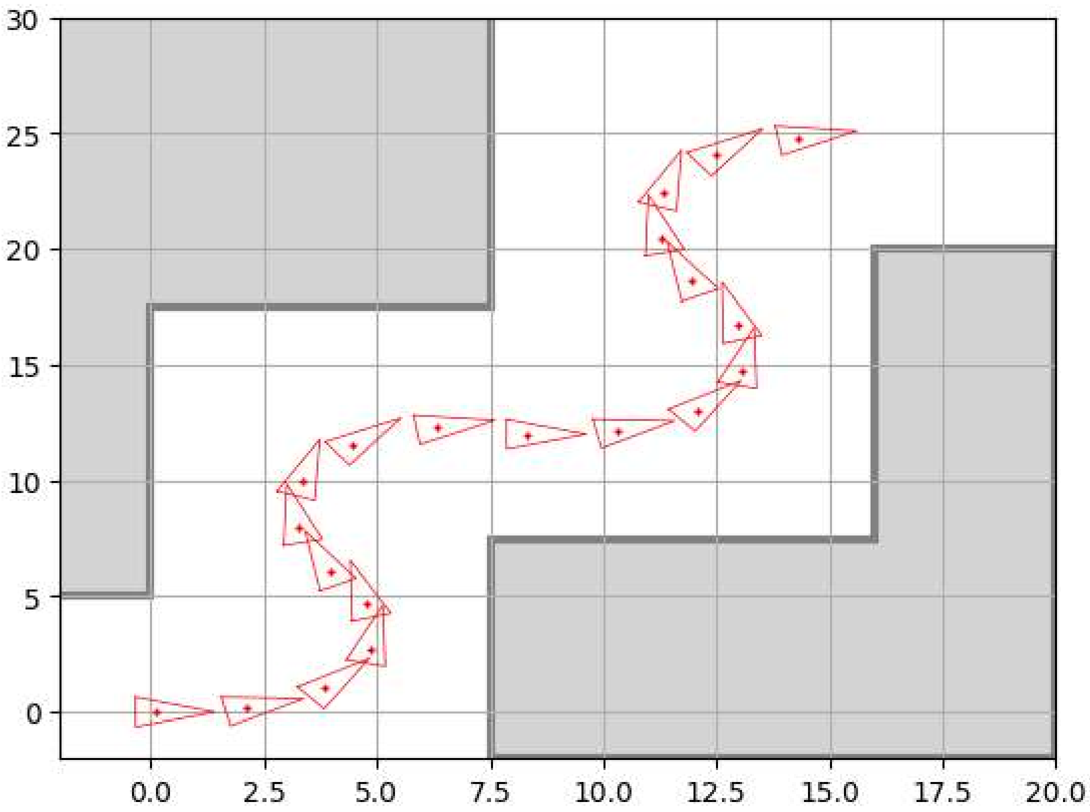
In [2]:
```
def next_pose(x, u, dt, cov=None):
    ''' This function takes pose x and transform it according to the motion
        applying the differential drive model.

        Args:
            x: current pose
            u: differential command as a vector [v, w]'
            dt: Time interval in which the movement occurs
            cov: covariance of our movement. If not None, then add gaussian
    '''
    if cov is not None:
        u += np.sqrt(cov) @ random.randn(2, 1)
        #u = np.random.multivariate_normal(u.flatten(),cov)

    theta = x[2]

    if u[1] == 0: #linear motion w=0
        # theta = x[2]
        # Calculamos la siguiente pose usando: linear motion equations
        next_x = np.vstack([x[0] + u[0] * dt * np.cos(theta),
                            x[1] + u[0] * dt * np.sin(theta),
                            x[2]])

    else: #Non-linear motion w=!0
        R = u[0]/u[1] #v/w=r is the curvature radius
        # theta = x[2]
        delta = u[1]*dt
        # Calculamos la siguiente pose usando: non-linear motion equations
        next_x = np.vstack([x[0] - R*np.sin(theta) + R*np.sin(theta + delta)
                            x[1] + R*np.cos(theta) - R*np.cos(theta + delta)
                            x[2] + delta])

    return next_x
```

In [3]:
```
class VelocityRobot(object):
    """ Mobile robot implementation that uses velocity commands.

        Attr:
            pose: expected pose of the robot in the real world (without taki
            dt: Duration of each step in seconds
    """
    def __init__(self, mean, dt):
        self.pose = mean
        self.dt = dt

    def step(self, u):
        self.pose = next_pose(self.pose, u, self.dt)

    def draw(self, fig, ax):
        DrawRobot(fig, ax, self.pose)
```

Test the movement of your robot using the demo below.

In [4]:
```python
def main(robot, nSteps):

    v = 1 # Linear Velocity
    l = 0.5 #Half the width of the robot

    # MATPLOTLIB
    fig, ax = plt.subplots()
    plt.ion()
    fig.canvas.draw()
    plt.xlim((-2, 20))
    plt.ylim((-2, 30))
    plt.fill([7.5, 7.5, 16, 16, 20, 20],[-2, 7.5, 7.5, 20, 20, -2],
             facecolor='lightgray', edgecolor='gray', linewidth=3)
    plt.fill([-3, 0, 0, 7.5, 7.5, -3],[5, 5, 17.5, 17.5, 32, 32],
             facecolor='lightgray', edgecolor='gray', linewidth=3)

    plt.grid()

    # MAIN LOOP
    for k in range(1, nSteps + 1):
        #control is a wiggle with constant linear velocity
        u = np.vstack((v, np.pi / 10 * np.sin(4 * np.pi * k/nSteps)))

        robot.step(u)

        #draw occasionally
        if (k-1)%20 == 0:
            robot.draw(fig, ax)
            clear_output(wait=True)
            display(fig)
            time.sleep(0.1)

    plt.close()
```

In [5]:
```
# RUN
dT = 0.1 # time steps size
pose = np.vstack([0., 0., 0.])

robot = VelocityRobot(pose, dT)
main(robot, nSteps=400)
```



3.2.2 Propagating uncertainty

In the previous section we introduced how to compute the robot pose x at time instant t by applying a control action u_t. However, as we know, this process has different sources of uncertainty that need to be modeled someway.

To deal with this we will consider two Gaussian distributions:

- the **robot pose** modeled as $x_t \sim (\overline{x}_t, \Sigma_{x_t})$ at time t. Similarly, for the **previous pose** at $t-1$ we have $x_{t-1} \sim (\overline{x}_{t-1}, \Sigma_{x_{t-1}})$,
- and the **movement command** as $u_t \sim (\overline{u}_t, \Sigma_{u_t})$, being applied during an interval of time Δt.

In this way, after a motion command we can retrieve the probability distribution x_t modeling the new robot pose as:

- **Mean:**

$$\overline{x}_t = \overline{x}_{t-1} \oplus \overline{u}_t = g(\overline{x}_{t-1}, \overline{u}_t)$$

- **Covariance:**

$$\Sigma_{x_t} = \frac{\partial g}{\partial x_{t-1}} \cdot \Sigma_{x_{t-1}} \cdot \frac{\partial g}{\partial x_{t-1}}^T + \frac{\partial g}{\partial u_t} \cdot \Sigma_{u_t} \cdot \frac{\partial g}{\partial u_t}^T$$

where $\partial g / \partial x_{t-1}$ and $\partial g / \partial u_t$ are the jacobians of our motion model evaluated at the previous pose x_{t-1} and the current command u_t, and the covariance matrix of this movement (Σ_{u_t}) is defined as seen below. Typically, it is constant during robot motion:

OPTIONAL

Write a Markdown cell containing the Jacobians ecuations aforementioned.

END OF OPTIONAL PART

ASSIGNMENT 2: Adding uncertainty

Now we will include uncertainty to the previous assignment, changing the behavior of the robot class `VelocityRobot()` you have implemented.

In contrast to the noisy robot `NoisyRobot()` in notebook 3.1, we will use the equations of the velocity motion model and their respective Jacobians to keep track of how confident we are of the robot's pose (i.e. the robot's pose x_t now is also a gaussian distribution).

Consider the following:

- the expected robot pose $\overline{x}_t$ is stored in `self.pose`.
- the covariance matrix of the robot pose Σ_{x_t} is named `P_t` in the code,
- the covariance matrix of the robot motion Σ_{u_t} is `Q`, and
- the jacobians of our motion model $\partial g / \partial x_{t-1}$ and $\partial g / \partial u_t$ are `JacF_x` and `JacF_u`.

First Complete the following code calculating the covariance matrix Σ_{x_t} (`P_t`). That is, you have to:

- Implement the jacobians `JacF_x` and `JacF_u`, which depend on the angular velocity w, and
- Compute the covariance matrix `P_t` using such jacobians, the current covariance of the pose `P`, and the covariance of the motion `Q`.

In [6]:
```python
def next_covariance(x, P, Q, u, dt):
    ''' Compute the covariance of a robot following the velocity motion mode

        Args:
            x: current pose (before movement)
            u: differential command as a vector [v, w]''
            dt: Time interval in which the movement occurs
            P: current covariance of the pose
            Q: covariance of our movement.
    '''
    # Aliases
    v = u[0, 0]
    w = u[1, 0]

    sx, cx = np.sin(x[2, 0]), np.cos(x[2, 0]) #sin and cos for the previous
    si, ci = np.sin(u[1, 0]*dt), np.cos(u[1, 0]*dt) #sin and cos for the hea
    R = u[0, 0]/u[1, 0] #v/w Curvature radius

    # cos(z + dt ) = cos(z)cos(dt) - sin(z)sin(dt)
    # sen(z + dt ) = cos(z)sen(dt) + sin(z)cos(dt
    if u[1, 0] == 0:  #linear motion w=0 --> R = infinite. Caso mas sencillo
        #TODO JACOBIAN HERE.
        # Jacobiano con respecto a la pose previa x_{t-1}
        JacF_x = np.array([
            [1, 0, -v*dt*sx],
            [0, 1, v*dt*cx],
            [0, 0, 1]
        ])
        # Jacobiano con respecto al movimiento u_t
        JacF_u = np.array([
            [dt*cx, 0],
            [dt*sx, 0],
            [0, 0]
        ])
    else: #Non-linear motion w=!0. Caso mas complejo.
        # TODO JACOBIAN HERE
        JacF_x = np.array([
            [1, 0, R*(cx*ci - cx - sx*si)],
            [0, 1, R*(cx*si + sx*ci - sx)],
            [0, 0, 1]
        ])

        JacF_u = (
            np.array([
                [sx*ci - sx + cx*si, R*(cx*ci - sx*si)],
                [sx*si + cx* (1- ci), R*(sx*ci + cx*si)],
                [0, 1]
            ])@
            np.array([
                [1/w, -v/w**2],
                [0, dt]
            ])
        )
    #prediction steps
    Pt = (JacF_x @ P @ np.transpose(JacF_x)) + (JacF_u @ Q @ np.transpose(Ja
```

```
    return Pt
```

Then, complete the methods:

- `step()` to get the true robot pose (ground-truth) using the Q matrix (recall the `next_pose()` function you defined before and its fourth input argument),
- and the `draw()` one to plot an ellipse representing the uncertainty about the robot pose centered at the expected robot pose (`self.pose`) as well as marks representing the ground truth poses.

Example

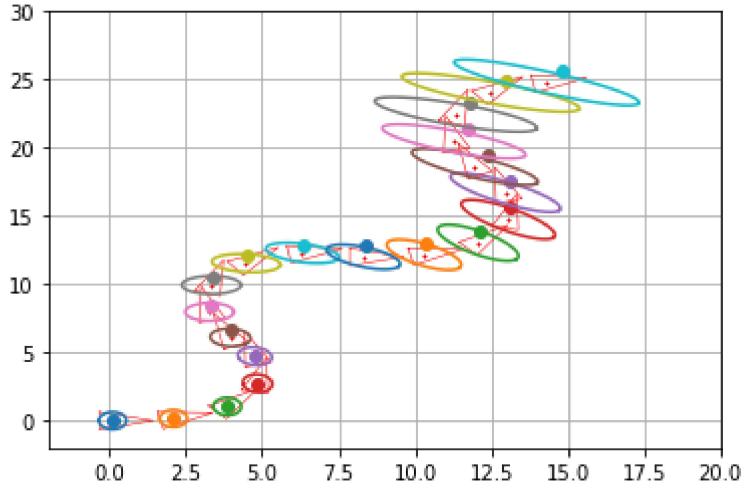
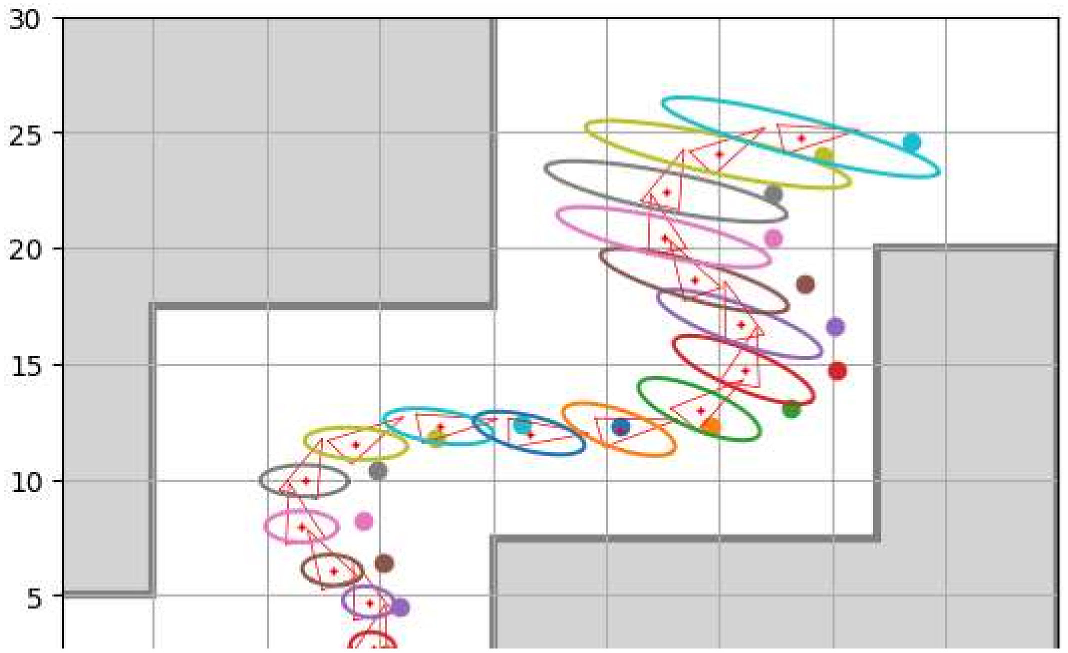


Fig. 3: Movement of a robot using velocity commands.
Representing the expected pose (in red), the true pose (as dots) and the confidence ellipse.

In [7]:
```python
class NoisyVelocityRobot(VelocityRobot):
    """ Mobile robot implementation that uses velocity commands.

        Attr:
            [...]: Inherited from VelocityRobot
            true_pose: expected pose of the robot in the real world (noisy)
            cov_pose: Covariance of the pose at each step
            cov_move: Covariance of each movement. It is a constant

    """

    def __init__(self, mean, cov_pose, cov_move, dt):
        super().__init__(mean, dt)
        self.true_pose = mean
        self.cov_pose = cov_pose
        self.cov_move = cov_move

    def step(self, u): # What this function
        self.cov_pose = next_covariance(self.pose, self.cov_pose, self.cov_m

        super().step(u)
        self.true_pose = next_pose(self.true_pose, u, self.dt, cov = self.cc

    def draw(self, fig, ax):
        super().draw(fig, ax)
        el = PlotEllipse(fig, ax, self.pose, self.cov_pose)
        ax.plot(self.true_pose[0], self.true_pose[1], 'o', color=el[0].get_c
```

Now, **try your implementation!**

In [8]:
```
# RUN
dT = 0.1 # time steps size

SigmaV = 0.2 #Standard deviation of the linear velocity.
SigmaW = 0.1 #Standard deviation of the angular velocity
nSteps = 400 #Number of motions

P = np.diag([0.2, 0.4, 0.]) #pose covariance matrix 3x3
Q = np.diag([SigmaV**2, SigmaW**2]) #motion covariance matrix 2x2

robot = NoisyVelocityRobot(np.vstack([0., 0., 0.]), P, Q, dT)
main(robot, nSteps=nSteps)

# Triangulo: pose ideal
# Punto: true pose
# Elipse: covarianza de la pose (incertidumbre)
```



Thinking about it (1)

Now that you have some experience with robot motion and the velocity motion model, **answer the following questions**:

- Why do we need to consider two different cases when applying the $g(\cdot)$ function, that is, calculating the new robot pose?

  We need to consider two different cases because the robot's motion can be either linear or non-linear. The robot can move straight and can rotate. It depends on the value of the angular velocity w. When w = 0, the robot is moving in a straight line, and the motion is linear. When w != 0, the robot follows a curved path, and the motion is non-linear.

- How many parameters compound the motion command u_t in this model?

  Two parameters: - v_t --> Linear velocity - w_t --> Angular velocity

- Why do we need to use Jacobians to propagate the uncertainty about the robot pose x_t?

Jacobian capture how small changes in the previous pose x_{t-1} affect the new pose x_t of a non-linear function. It approximates this non-linear function as a linear one within a small range, which is important for estimating the uncertainty associated with the robot's pose. When we apply Jacobian to the covariance matrix of x_{t-1} and u_t we can update the covariance of the new pose x_t. It shows how changes in the inputs affect uncertainty.

- What happens if you modify the covariance matrix Σ_{u_t} modeling the uncertainty in the motion command u_t? Try different values and discuss the results.

  It affects the degree of confidence we have in the motion command. If we increase (decrease) the values in Sigma_{u_t} it indicates higher (lower) uncertainty in the motion command. This means we have less (more) confidence in the accuracy of the control actions. As a result, when you use a more uncertain (certain) motion command, the propagated uncertainty in the new pose x_t will also increase (decrease). The