Decidability and generation for the variety of distributive ℓ-pregroups

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Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{G} = (G, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(G, \wedge, \vee)$ is a lattice,
- $(G, \cdot, ^{-1}, 1)$ is a group $(x^{-1}x = 1 = xx^{-1})$ and
- multiplication preserves the order. (eqv: it distributes over join/over meet.)

A *pregroup* (Lambek, Buzskowski) is a structure $(L, \cdot, 1, ^{\ell}, ^{r}, \leqslant)$, where $(L, \cdot, 1)$ is a monoid, multiplication preserves the order and:

$$x^{\ell}x \leqslant 1 \leqslant xx^{\ell} \text{ and } xx^{r} \leqslant 1 \leqslant x^{r}x.$$

A *lattice-ordered pregroup*, or *ℓ-pregroup*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, ^{\ell}, ^{r}, 1)$ such that

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- $(L, \cdot, 1, ^{\ell}, ^{r}, \leqslant)$ is a pregroup

We denote by LG the variety of ℓ-groups, and by DLP the variety of ℓ-pregroups for which the lattice reduct is distributive.

- *Distributive ℓ-pregroups: Generation and decidability*, with N. Galatos, Journal of Algebra, Volume 648, 2024, Pages 9-35.

Examples of ℓ-groups

Examples:

- $(\mathbb{Z}, min, max, +, -, 0)$, $(\mathbb{Q}, min, max, +, -, 0)$, $(\mathbb{R}, min, max, +, -, 0)$.
- The order-preserving permutations (aka automorphisms) $\mathbf{Aut}(C, \leqslant)$ on a totally-ordered set $(C, \leqslant)$, under functional composition and pointwise order. For example, the *symmetric* ℓ-groups: $\mathbf{Aut(n)}$, $\mathbf{Aut}(\mathbb{N})$, $\mathbf{Aut}(\mathbb{Z})$, $\mathbf{Aut}(\mathbb{R})$.

Holland's embedding theorem. Every ℓ-group can be embedded in a symmetric ℓ-group: $\mathbf{G} \hookrightarrow \mathbf{Aut(C)}$, for some chain $\mathbf{C}$.

Holland's generation theorem. The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$: $\mathsf{LG} = \mathsf{V}(\mathbf{Aut}(\mathbb{R}))$.

Theorem. (Holland - McCleary) The equational theory of ℓ-groups is decidable.

Examples of distributive ℓ-pregroups and representation

Definition: Given functions $f : \mathbf{P} \to \mathbf{Q}$ and $g : \mathbf{Q} \to \mathbf{P}$ between posets, we say that g is a *residual* for f, or that f is a *dual residual* for g, or that (f, g) form a *residuated pair*, if

$$f(p) \leqslant q \;\Leftrightarrow\; p \leqslant g(q), \text{ for all } p \in P, q \in Q.$$

Fact: The residual of f, when it exists, is unique and we denote it by f^r. The dual residual of f, when it exists, is unique and we denote it by f^ℓ. Also,

$$f^\ell(y) = \bigwedge\{x : y \leqslant f(x)\} \qquad \text{and} \qquad f^r(y) = \bigvee\{x : f(x) \leqslant y\}.$$

Given a chain $\mathbf{C}$, $\mathbf{F}(\mathbf{C})$ denotes the set of all functions from $\mathbf{C}$ to $\mathbf{C}$ that have residuals and dual residuals of all orders.

Fact (Galatos - Jipsen - Kowalski - Ono, 2007): For every chain $\mathbf{C}$, $\mathbf{F}(\mathbf{C})$ is a distributive ℓ-pregroup, under functional composition and pointwise order.

Theorem (Galatos - Horcik, 2013): Every distributive ℓ-pregroup can be embedded in $\mathbf{F}(\mathbf{C})$, for some chain $\mathbf{C}$.

Generation

It turns out that $\mathbf{F}(\mathbb{R}) = \mathbf{Aut}(\mathbb{R})$ (all bi-infinitely residuated maps on $\mathbb{R}$ are bijections), as we see below.

On the other hand the only invertible elements (bijections) in $\mathbf{F}(\mathbb{Z})$ are the translations: $x \mapsto nx$, for $n \in \mathbb{Z}$. It is thus surprising that:

Theorem (G.-Galatos): DLP is generated by $\mathbf{F}(\mathbb{Z})$.

In particular, if an equation fails in $\mathbf{Aut}(\mathbb{R})$, then it fails in $\mathbf{F}(\mathbb{Z})$.

A chain $\mathbf{\Omega}$ is called *integral* it is isomorphic to a lexicographic product $I \overset{\rightarrow}{\times} \mathbb{Z}$.

Theorem (G.-Galatos): Every distributive ℓ-pregroup can be embedded in $\mathbf{F}(\mathbf{\Omega})$ for some *integral* chain $\mathbf{\Omega}$.

Limit points

Given a chain $\mathbf{\Omega}$, we define:

$$\Omega^- := \{a \in \Omega : \; a = \bigvee_{b<a} b\}, \quad \Omega^+ := \{a \in \Omega : \; a = \bigwedge_{a<b} b\} \text{ and } \Omega^{\pm} := \Omega^- \cup \Omega^+,$$

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Maps that are residuated and dually residuated on a chain $\mathbf{\Omega}$ preserve and reflect certain limit points:

Lemma. If $\mathbf{\Omega}$ is a chain, $a \in \Omega$, $f \in F(\mathbf{\Omega})$ and $|f^{-1}[f(a)]| = 1$, then $a \in \Omega^-$ iff $f(a) \in \Omega^-$; and $a \in \Omega^+$ iff $f(a) \in \Omega^+$.

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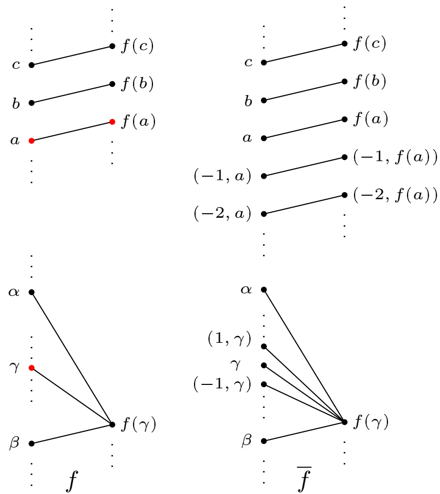
Lemma. If $\mathbf{\Omega}$ is a chain, $a \in \Omega$, $f \in F(\mathbf{\Omega})$ and $|f^{-1}[a]| > 1$, then $a \notin \Omega^\pm$, $f^\ell(a) \notin \Omega^-$ and $f^r(a) \notin \Omega^+$.

Embedding $\mathbf{F}(\mathbf{\Omega})$ in $\mathbf{F}(\overline{\mathbf{\Omega}})$, where $\overline{\mathbf{\Omega}}$ an integral chain



If $a, \gamma \in \Omega^-$ and $\gamma \in \Omega^+$, then $\overline{\Omega}$ will contain

$$\Omega \cup \{(-n, a) : n \in \mathbb{Z}^+\} \cup \{(n, \gamma) : n \in \mathbb{Z}^*\}$$

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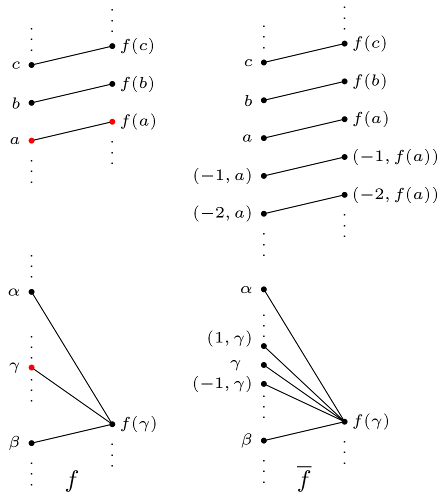

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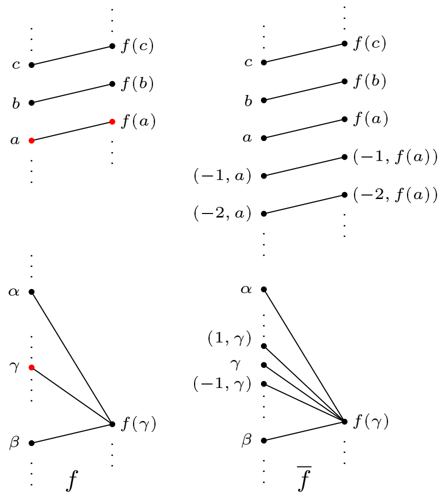

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In general:

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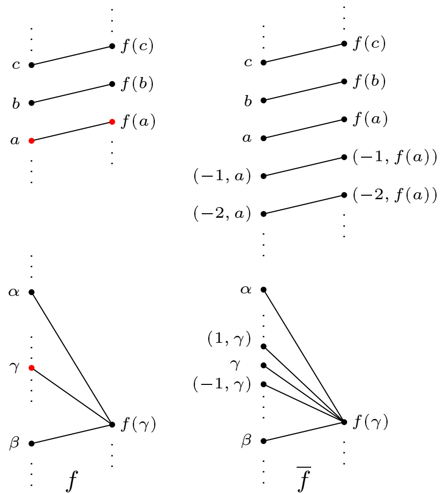

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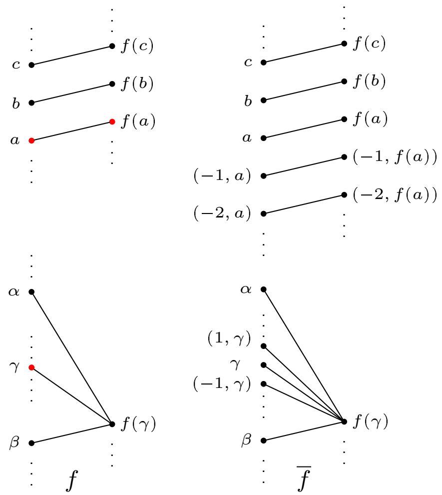

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If $f \in F(\mathbf{\Omega})$ and $kb \in \overline{\Omega}$, we define

$$\overline{f}(kb) = f(b), \text{ if } |f^{-1}[f(b)]| > 1 \text{ or } k = 0.$$

$$\overline{f}(kb) = kf(b), \text{ if } |f^{-1}[f(b)]| = 1 \ \& \ k \neq 0.$$

Improved representation

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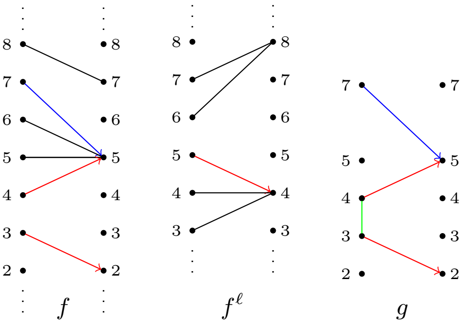
Theorem: Every distributive ℓ-pregroup can be embedded in $\mathbf{F}(\mathbf{\Omega})$ for some integral chain $\mathbf{\Omega}$.

Corollary: If an equation fails in DLP, then it fails in $\mathbf{F}(\mathbf{\Omega})$, for some integral chain $\mathbf{\Omega}$.

Example of an ℓ-pregroup diagram

The equation $1 \leqslant x^{\ell} x$ fails in $\mathbf{F}(\mathbb{Z})$, because $f^{\ell} f(7) = 4 < 7 = id_{\mathbb{Z}}(7)$.



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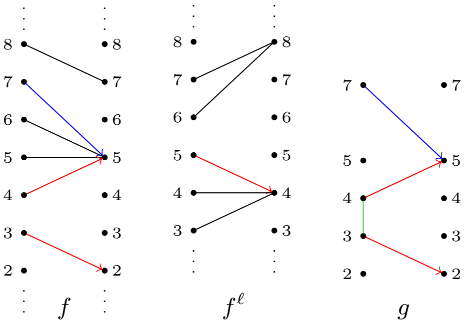

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We restrict f and f^ℓ to partial functions

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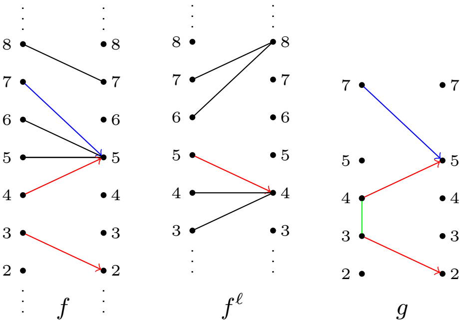

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To make sure that $g^{[\ell]}(5) = 4$ is computed correctly, we need to

- include more elements in the chain
- define g on some of these elements
- mark some covers: $3 \prec 4$.

Definitions

A *c-chain* is a triple $(\Delta, \leqslant, \prec)$, consisting of a finite chain $(\Delta, \leqslant)$ and a subset $\prec \subseteq \prec$ of the covering relation, i.e. if $a \prec b$, then a is covered by b.

Given a partial function g on a c-chain $\mathbf{\Delta} = (\Delta, \leqslant, \prec)$, we define the relation $g^{[\ell]}$ on Δ by: for all $x, b \in \Delta$, $(x, b) \in g^{[\ell]}$ iff

$$b \in Dom(g) \text{ and } \exists a \in Dom(g) \text{ such that } a \prec b \text{ and } g(a) < x \leqslant g(b).$$

Also, we define the relation $g^{[r]}$ on Δ by: for all $x, a \in \Delta$, $(x, a) \in g^{[r]}$ iff $a \in Dom(g)$, there exists $b \in Dom(g)$, such that $a \prec b$ and $g(a) \leqslant x < g(b)$.

Given a c-chain $\mathbf{\Delta}$, and an order-preserving partial function g on $\mathbf{\Delta}$, we define $g^{[n]}$, for all $n \in \mathbb{N}$, recursively by: $g^{[0]} = g$ and $g^{[k+1]} := (g^{[k]})^{[\ell]}$. Also, we define $g^{[-n]}$, for all $n \in \mathbb{N}$, recursively by $g^{[-(k+1)]} := (g^{[k]})^{[r]}$.

A *diagram* $(\mathbf{\Delta}, f_1, \ldots, f_n)$, consists of a finite c-chain $\mathbf{\Delta}$ and order-preserving partial functions $f_1, \ldots, f_n$ on $\mathbf{\Delta}$, where $n \in \mathbb{N}$.

Notation: if a has a lower cover, we denote it by $-a$ or $(-1)a$ or $a - 1$; $+a$, $1a$ and $a + 1$ denote the upper cover of a, when it exists. Also, we write $0a$ for a.

Diagrams from syntax

Given $\varepsilon : 1 \leqslant w_1 \vee \ldots \vee w_k$ (intentional form), where $w_1, \ldots, w_k$ are of the form $x_1^{(m_1)} x_2^{(m_2)} \ldots x_i^{(m_i)}$ where $x_1, \ldots, x_i$ are variables (not necessarily distinct), $i \in \mathbb{N}$ and $m_1, \ldots, m_i \in \mathbb{Z}$; $x^{(0)} := x$ and for all $m \in \mathbb{Z}^+$, $x^{(m)} := x^{\ell^m}$, $x^{(-m)} := x^{r^m}$.

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We define the set of *final subwords* of ε

$$FS_\varepsilon := \{u : w_1 = vu \text{ or ... or } w_k = vu, \text{ for some } v\}$$

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For a variable x among the variables $x_1, \ldots, x_n$ of ε, $m \in \mathbb{N}$ and $v \in FS_\varepsilon$ we define:

$$\Delta^v_{x,m} := \{v\} \cup \bigcup_{j=0}^{m} \{\sigma_j x^{(j)} \ldots \sigma_m x^{(m)} v : \sigma_j, \ldots, \sigma_m \in \{-1, 0\}, \sigma_0 = 0\}$$

$$\Delta^v_{x,-m} := \{v\} \cup \bigcup_{j=0}^{m} \{\sigma_j x^{(-j)} \ldots \sigma_m x^{(-m)} v : \sigma_j, \ldots, \sigma_m \in \{1, 0\}, \sigma_0 = 0\}$$

$$S_\varepsilon := \{(i, m, v) : i \in \{1, \ldots, n\}, m \in \mathbb{Z}, v \in FS_\varepsilon \text{ and } x_i^{(m)} v \in FS_\varepsilon\}$$

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Observe that, $FS_\varepsilon \subseteq \Delta_\varepsilon$ is finite, hence also Δ_ε is finite.

Compatible surjections

Given an equation $\varepsilon(x_1,\ldots,x_n)$ in intentional form, a *compatible surjection* for ε is an onto map $\varphi:\Delta_\varepsilon\to\mathbb{N}_q$, where $\mathbb{N}_q=\{1,\ldots,q\}$ is an initial segment of $\mathbb{Z}^+$ under the natural order (and $q\leqslant|\Delta_\varepsilon|$), such that:

(i) The relation $g_i:=\{(\varphi(u),\varphi(x_iu))\mid u,x_iu\in\Delta_\varepsilon\}$ on $\mathbb{N}_q$ is an order-preserving partial function for all $i\in\{1,\ldots,n\}$.

(ii) The relation $\prec:=\{(\varphi(v),\varphi(+v))\mid v,+v\in\Delta_\varepsilon\}\cup\{(\varphi(-v),\varphi(v))\mid v,-v\in\Delta_\varepsilon\}$ on $\mathbb{N}_q$ is contained in the covering relation $\prec$ of $\mathbb{N}_q$.

(iii) $\varphi(x_i^{(m)}u)=g_i^{[m]}(\varphi(u))$, when $i\in\{1,\ldots,n\}$, $m\in\mathbb{Z}$ and $u,x_i^{(m)}u\in\Delta_\varepsilon$.

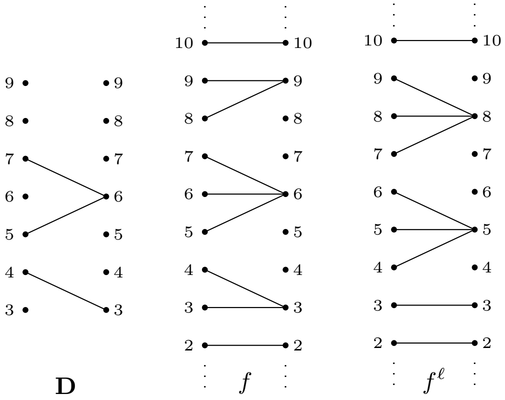
By the first two conditions, $\mathbf{D}_{\varepsilon,\varphi}:=(\mathbb{N}_q,\leqslant,\prec,g_1,\ldots,g_n)$ is a diagram; in (iii), $g_i^{[m]}$ is calculated in this diagram.

We say that the equation $1\leqslant w_1\vee\ldots\vee w_k$ in intentional form *fails* in a compatible surjection φ, if $\varphi(w_1),\ldots,\varphi(w_k)<\varphi(1)$.

Theorem: If an equation ε fails in DLP, then it fails in some compatible surjection for ε.

Theorem: If an equation ε fails in a compatible surjection, then it fails in a diagram based on a chain of size up to $|\Delta_\varepsilon|$.

From a diagram to $\mathbf{F}(\mathbb{Z})$



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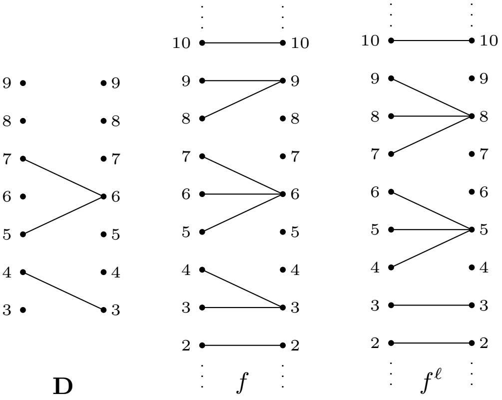

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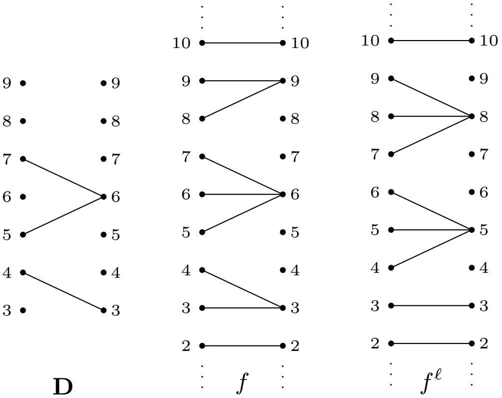

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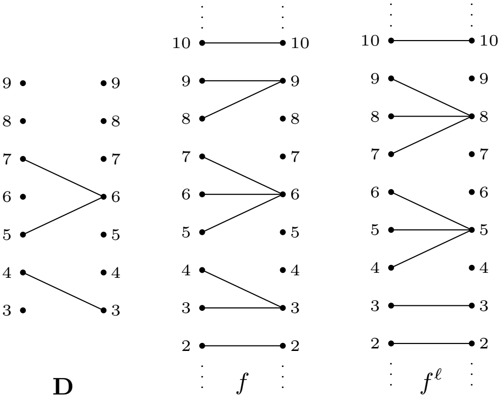

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Theorem: Every equation in the language of ℓ-pregroups that fails in a diagram also fails in $\mathbf{F}_{\mathsf{fs}}(\mathbb{Z})$, the subalgebra of $\mathbf{F}(\mathbb{Z})$ consisting of the functions of finite support.

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Corollary: An equation ε holds in DLP iff it holds in all compatible surjections for ε iff it holds in all finite diagrams for ε of size up to $|\Delta_\varepsilon|$ iff it holds in $\mathbf{F}_{\mathsf{fs}}(\mathbb{Z})$ iff it holds in $\mathbf{F}(\mathbb{Z})$.

Generation and decidability

Theorem: Every equation in the language of ℓ-pregroups that fails in a diagram also fails in $\mathbf{F}_{\mathsf{fs}}(\mathbb{Z})$, the subalgebra of $\mathbf{F}(\mathbb{Z})$ consisting of the functions of finite support.

Corollary: An equation ε holds in DLP iff it holds in all compatible surjections for ε iff it holds in all finite diagrams for ε of size up to $|\Delta_\varepsilon|$ iff it holds in $\mathbf{F}_{\mathsf{fs}}(\mathbb{Z})$ iff it holds in $\mathbf{F}(\mathbb{Z})$.

Corollary: The equational theory of DLP is decidable.

Example of failure

For $1 \leqslant x^\ell x$ we consider the sets $\Delta^1_{x,0} = \{1, x\}$ and $\Delta^x_{x,1} = \{x, x^\ell x, -x^\ell x, xx^\ell x, x - x^\ell x\}$. Their union is the set

$$\Delta_\varepsilon = \{1, x, x^\ell x, -x^\ell x, xx^\ell x, x - x^\ell x\}.$$

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The compatible surjection is $\varphi : \Delta_\varepsilon \to \{1, 2, 3, 4\}$ where:

$$\varphi(x) = \varphi(xx^{\ell}x) = 4 \quad \varphi(1) = \varphi(x-x^{\ell}x) = 3$$

$$\varphi(x^{\ell}x) = 2 \quad \varphi(-x^{\ell}x) = 1$$

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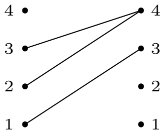

The resulting diagram is $(\mathbb{N}_4, \leqslant, \prec, g)$, where $\leqslant$ is induced by $\mathbb{Z}$, $\prec = \{(1,2)\}$ and $g = \{(3,4), (2,4), (1,3)\}$.

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