Diagrams in involutive residuated lattices

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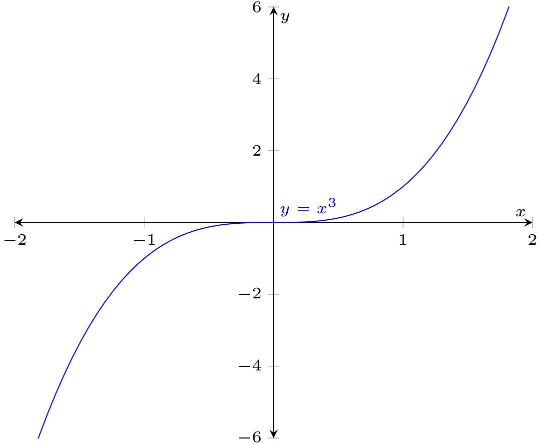
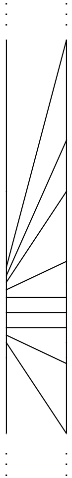
June 10, 2024

ℓ-groups: $\mathbf{Aut}(\mathbb{Q})$

The collection of the order preserving permutations of $\mathbb{Q}$ i.e. strictly increasing invertible functions from $\mathbb{Q}$ to itself forms an algebra under composition, meet, join and inverse, and we will denote it by $\mathbf{Aut}(\mathbb{Q})$. For example:





$f(x) = x^3$

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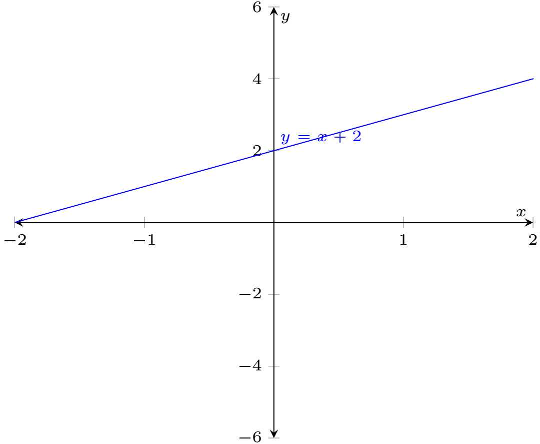

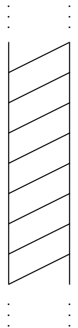

$g(x) = x + 2$

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Theorem (Holland's embedding theorem)

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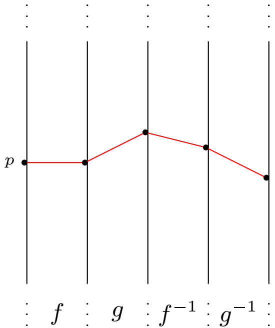

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$p = f(p)$ •

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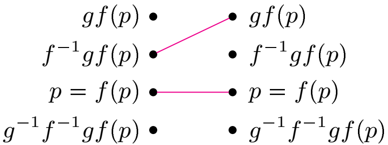

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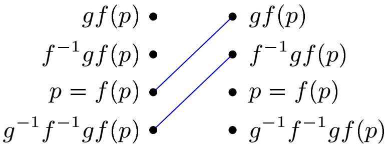

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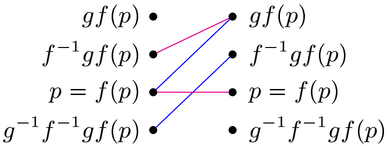

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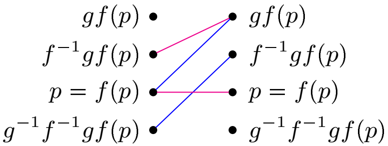

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two relations (magenta and blue) such that:

- they are order preserving partial functions and,
- they are injective.

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$$\Delta_\varepsilon = \{1, x, yx, x^{-1}yx, y^{-1}x^{-1}yx\}$$

given that $|\Delta_\varepsilon| \leqslant |\varepsilon|$ we know that $|\Delta_\varepsilon| < \infty$

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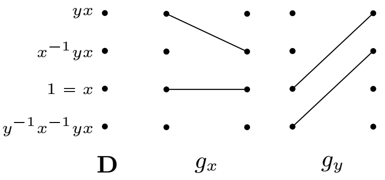

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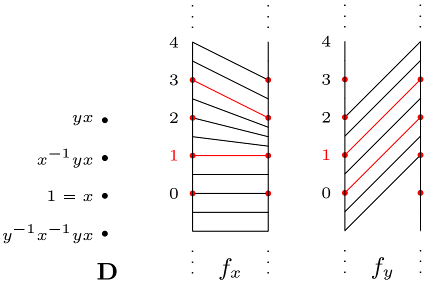
More formally $|\Delta_\varepsilon|$ with the order on the graphic, $|\Delta_\varepsilon|$ controlled, satisfying that g_x, g_y order preserving, injective, partial functions, satisfies

$$y^{-1}x^{-1}yx < 1$$

so ε fails.

From a failure in a diagram to a failure in $\mathbf{Aut}(\mathbb{Q})$



Given a diagram in which an equation ε fails, we would like to extend the injective partial functions g_x, g_y to $\mathbb{Q}$ in an order preserving and bijective manner.

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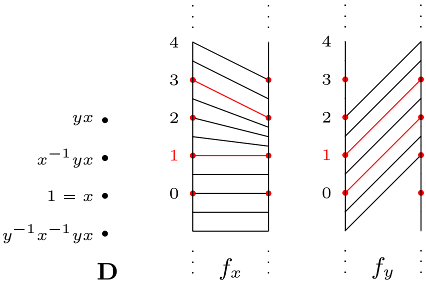

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Using the fact that $\mathbf{Aut}(\mathbb{Q})$ is n-transitive, we can extend this partial functions to order preserving bijections in $\mathbb{Q}$.

Theorem (Holland)

If an equation ε fails in an ℓ-group, it fails in a diagram of size at most $|\varepsilon|$.

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If an equation ε fails in a diagram, it fails in $\mathbf{Aut}(\mathbb{Q})$.

$$
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$$

Theorem (Holland - McCleary, 1979)

The equational class LG *is decidable*

Theorem (Holland, 1976)

The equational class LG *can be generated by* $\mathbf{Aut}(\mathbb{Q})$.

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We will focus for now on distributive ℓ-pregroups, the equational class they form is denoted by DLP.

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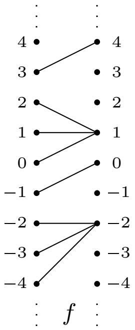

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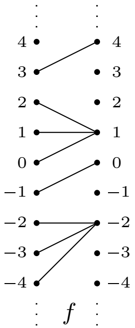

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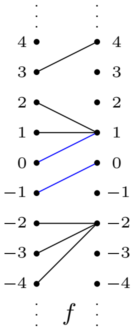

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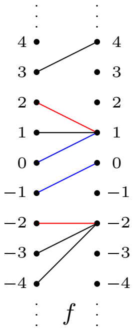

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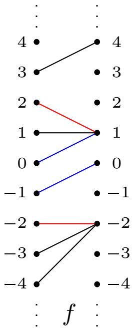

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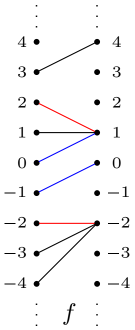

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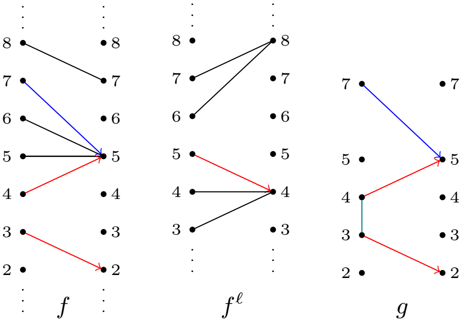
Theorem (Representation: Galatos-Horcik, 2013)

Every distributive ℓ-pregroup can be embedded in $\mathbf{F}(\mathbf{\Omega})$ for some chain $\mathbf{\Omega}$.

Example of an ℓ-pregroup diagram



The equation $1 \leqslant x^\ell x$ fails in $\mathbf{F}(\mathbb{Z})$, because $f^\ell f(7) = 4 < 7 = id_{\mathbb{Z}}(7)$.

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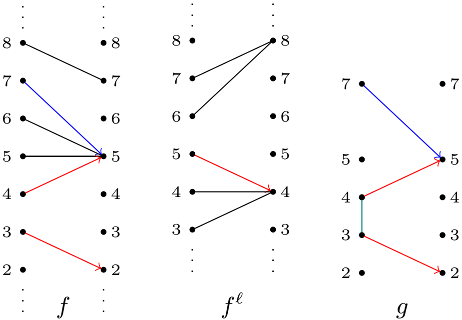

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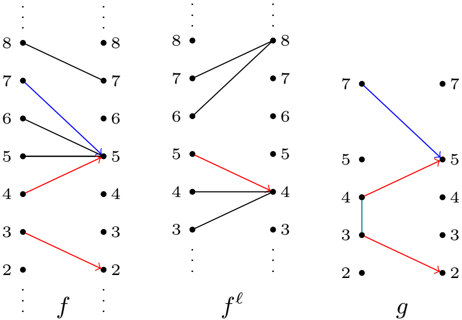

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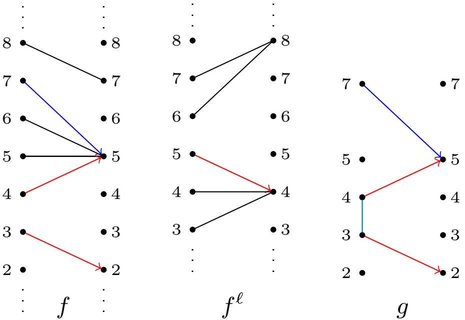

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Also, to make sure that $g^{[\ell]}(5) = 4$ is computed correctly, we need to

- include more elements in the chain
- define g on some of these elements
- mark some covers: $3 \prec 4$.

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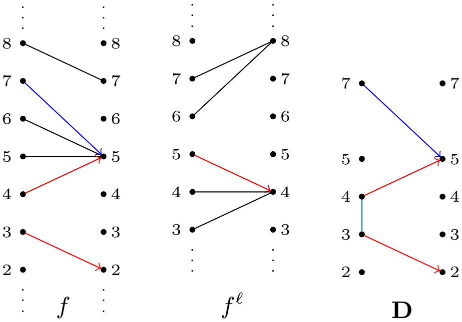

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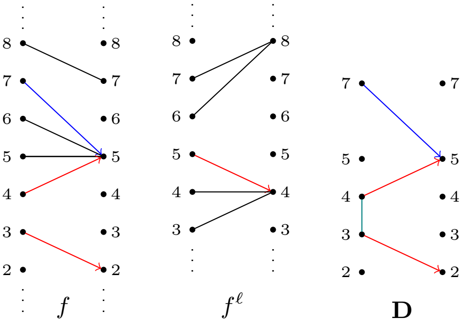

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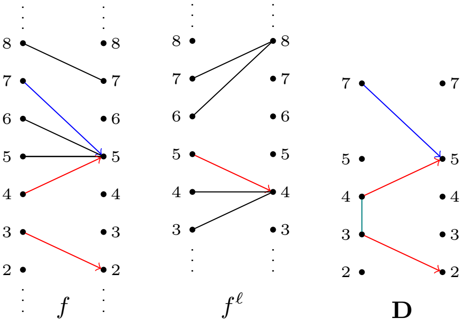

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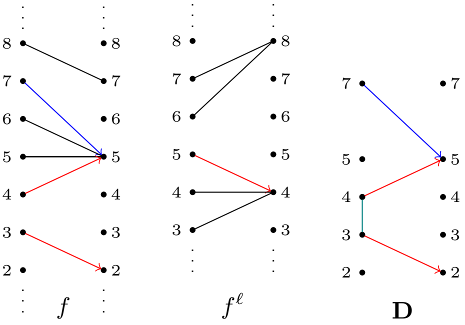

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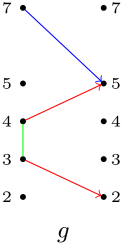
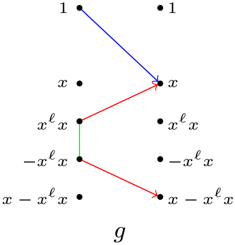
Example: How to build a diagram

Given the equation $1 \leqslant x^\ell x$





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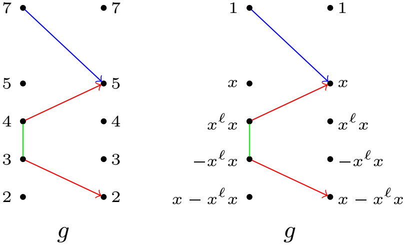

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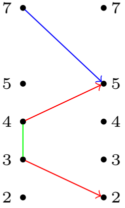

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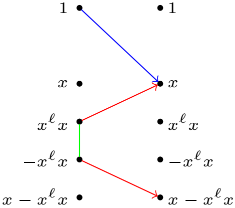

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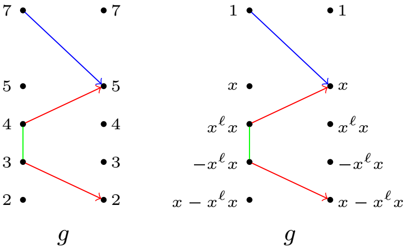

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More formally we get the diagram with set Δ_ε:

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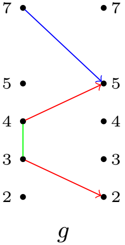

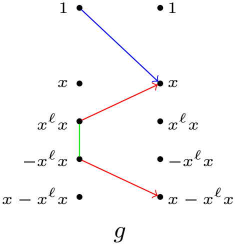

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with the ordering on the left, $|\Delta_\varepsilon|$ controlled, satisfying a set of compatibility conditions $x^\ell x < 1$, so the equation fails.

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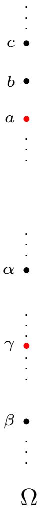

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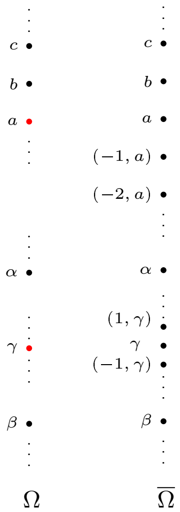

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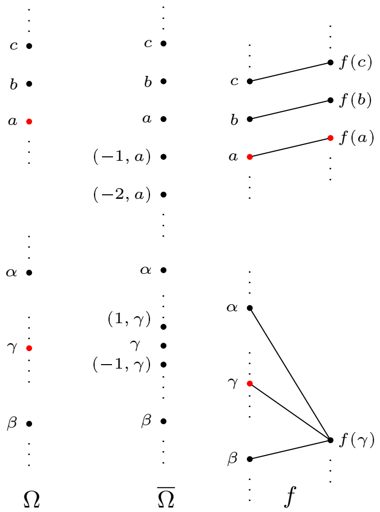

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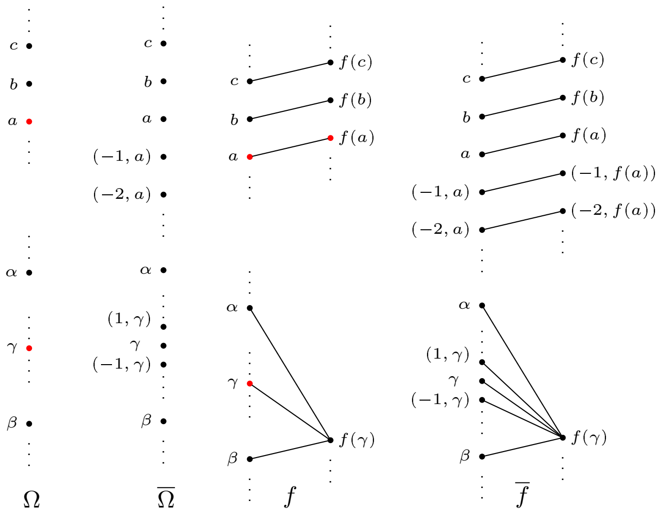

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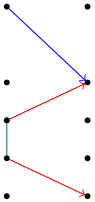

$\mathbf{D}$

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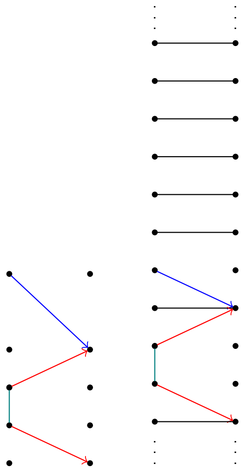

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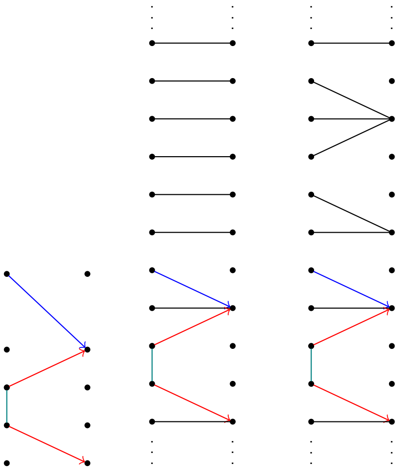

$\mathbf{D}$ $\quad$ $f \in F_{fs}(\mathbb{Z})$ $\quad$ $f \in F_5(\mathbb{Z})$

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Periodic ℓ-pregroups

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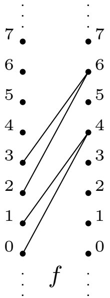

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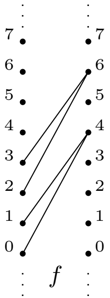

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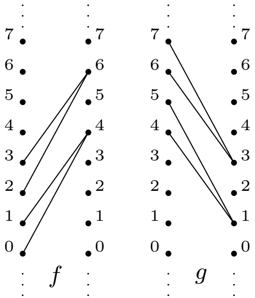

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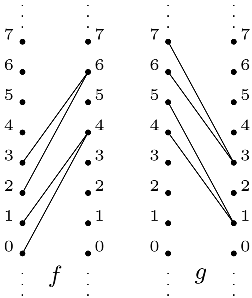
The equational class of n-periodic ℓ-pregroups will be denoted by LP_n.

Periodic ℓ-pregroups



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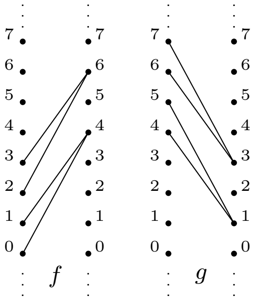

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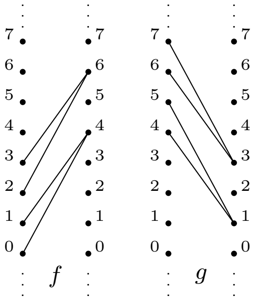

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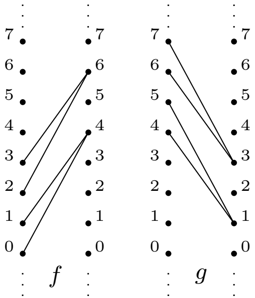

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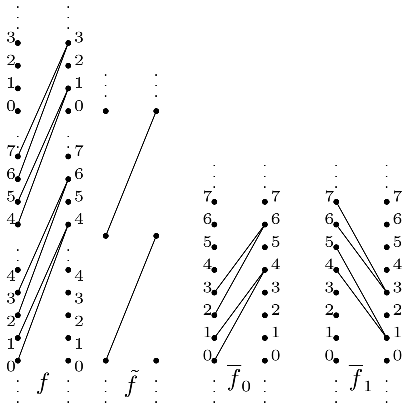
Periodic ℓ-pregroups

Theorem (Galatos - G.)

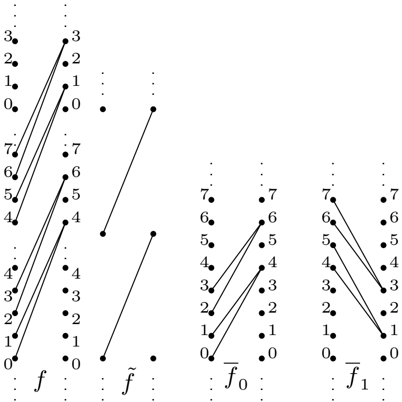
A function $f \in \mathbf{F}_n(\mathbf{J} \overrightarrow{\times} \mathbb{Z})$ decomposes into: a bijection $\tilde{f} : \mathbb{Z} \to \mathbb{Z}$ and, $\overline{f} : J \to F_n(\mathbb{Z})$, such that $f(j, r) = (\tilde{f}(j), \overline{f}_j(r))$ for all $(j, r) \in J \times \mathbb{Z}$.



Periodic ℓ-pregroups

Theorem (Galatos - G.)

For every chain $\mathbf{J}$ *and* $n \in \mathbb{Z}^+$, $\mathbf{F}_n(\mathbf{J} \overrightarrow{\times} \mathbb{Z}) \cong \mathbf{Aut}(\mathbf{J}) \wr \mathbf{F}_n(\mathbb{Z})$. *Therefore, every* n*-periodic* ℓ*-pregroup can be embedded in a wreath product of an* ℓ*-group and the simple* n*-periodic* ℓ*-pregroup* $\mathbf{F}_n(\mathbb{Z})$.



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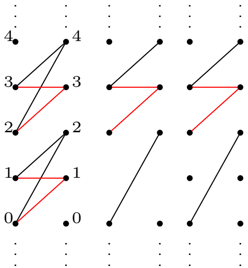

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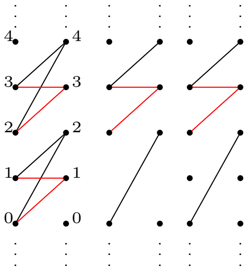

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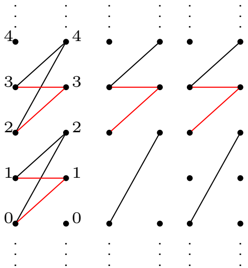

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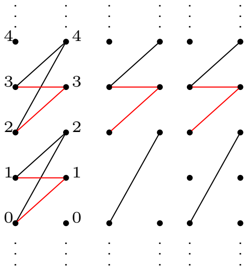

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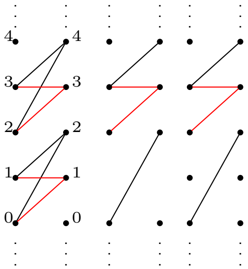

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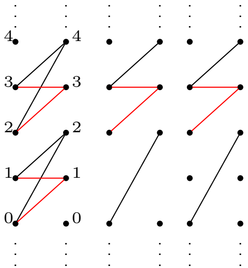

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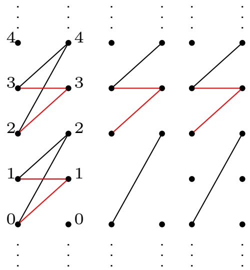

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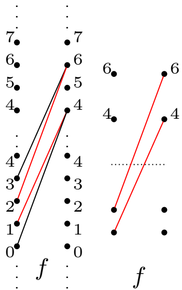
Theorem (Galatos - G.)

The equation class generated by $\mathbf{F}_n(\mathbb{Z})$ is decidable.

Partition diagrams

Because functions in $\mathbf{F}_n(\mathbf{J}\overrightarrow{\times}\mathbb{Z})$ decompose into a global and a local component, the diagram can also be decomposed into a partition of the points, where the partitions are connected in an injective way (global component).



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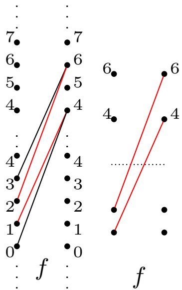

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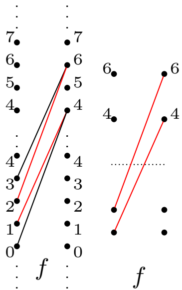

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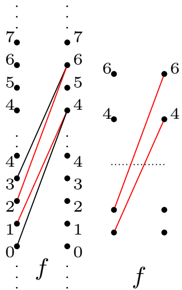

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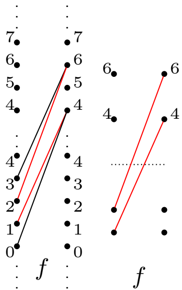
- Global component extends to a bijection on $\mathbb{Q}$,
- The local components extend to functions in $\mathbf{F}_n(\mathbb{Z})$



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Involutive Residuated Lattice

An involutive residuated lattice (InRL) is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, {}^{\ell}, {}^{r})$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and for all $a, b, c \in A$,

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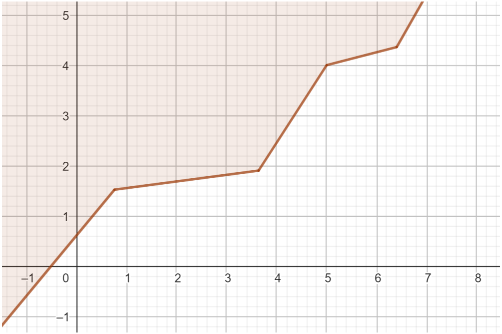
Residuated lattices include: Heyting algebras, ideal lattices of rings, power sets of monoids. Also, residuated lattices are algebraic semantics for substructural logics.

Weakening relations

Given a poset $\mathbf{P} = (P, \leqslant)$, we define the set of *weakening relations* on $\mathbf{P}$:

$$Wk(\mathbf{P}) = \{R \subseteq P \times P \mid \leqslant \circ R \circ \leqslant \subseteq R\}$$



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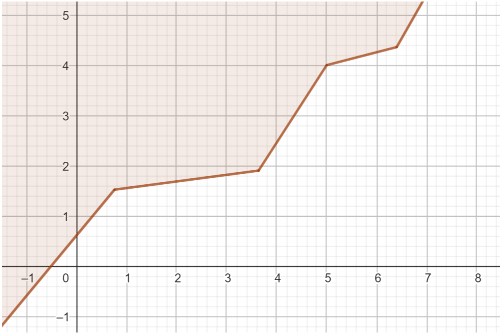

The set $Wk(\mathbf{P})$ forms a cyclic involutive residuated lattice under union, intersection, relational composition, the identity being the $\leqslant$ relation, complement-converse. The induced algebra $\mathbf{Wk}(\mathbf{P})$ is called the *full weakening relation algebra on the poset* $\mathbf{P}$.

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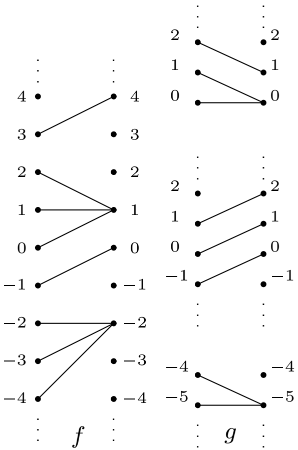
Theorem (Galatos - Jipsen, 2020)

Given a poset $\mathbf{P}$ we have $\mathbf{Wk(P)} \cong \mathbf{Res}(\mathcal{O}(\mathbf{P}^\partial))$ with the correct operations.

Where $\mathcal{O}(\mathbf{P}^\partial)$ denotes the down sets of the inverted poset P.

Weakening relations



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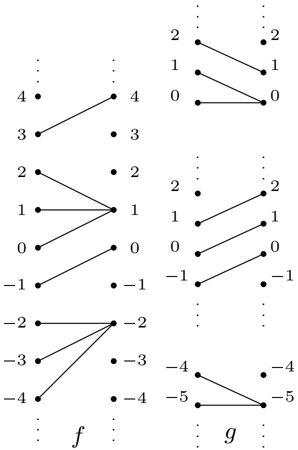

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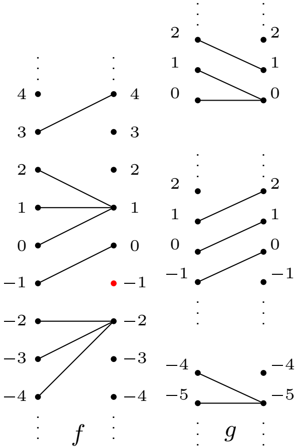

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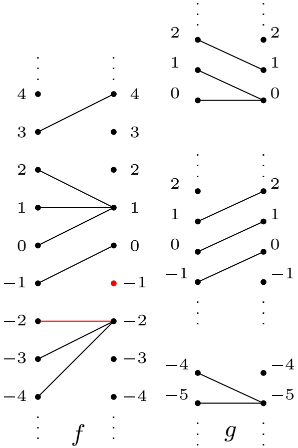

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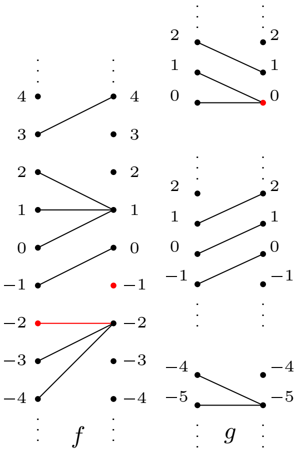

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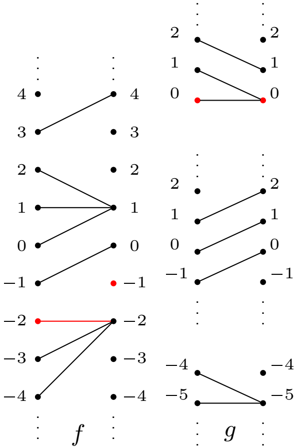

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Fortunately this behavior is unique and can be identified by the expression:

$$gg'(a) = a$$

We define V_∞ to be the equational class generated by $\{\mathbf{Res}(\mathbf{J} \overrightarrow{\times}_{\top} \mathbb{N}) : \mathbf{J}$ is a well-ordered chain$\}$.

Every equation in the language of involutive residuated lattices can be written as $1 \leqslant w_1 \vee \ldots \vee w_n$ where every w is a term using only multiplication and negation.

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Lemma

Given an equation ε in intentional form, Δ_ε is finite.

Then we define partition diagrams over Δ_ε that satisfy certain complex conditions.

Extending partition diagrams

Partition diagrams can be extended to functions in $\mathbf{Res}(\mathbf{J} \overrightarrow{\times}_{\top} \mathbb{N})$ by taking special care of the "limit points from below".

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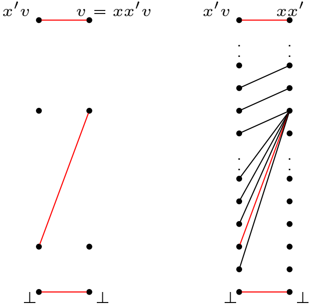

Theorem (Galatos - G.)

If an equation ε *fails in* $\mathbf{Res}(\mathbf{J}\overrightarrow{\times}_{\top}\mathbb{N})$ *where* $\mathbf{J}$ *where is a well-ordered chain, it fails in a diagram with at most* $\min\{|J|, |\Delta_\varepsilon|\}$ *partitions.*

Theorem (Galatos - G.)

If an equation ε *fails in a partition diagram, it fails in* $\mathsf{V}(\mathbf{Res}(\mathbb{N}\overrightarrow{\times}_{\top}\mathbb{N}))$*.*

$$\begin{tikzcd}
 & \text{Failure in } \mathsf{V}_\infty \arrow[dl] & \\
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The equational class V_∞ *is decidable.*

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Conjecture: The equational class consisting of $\mathbf{Res}(\boldsymbol{\Omega})$ s.t. $\boldsymbol{\Omega}$ is a perfect chain is decidable and generated by one such algebra.

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Distributivity and -pregroups

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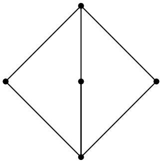
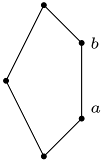


$\mathbf{M}_3$



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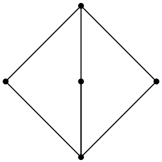

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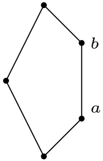

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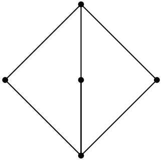

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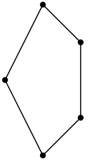

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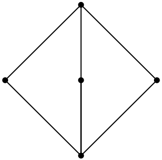
Theorem (Kinyon)

If L is an ℓ-pregroup, with a sublattice N_5, then $a = 1$ or $b = 1$.



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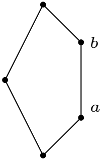

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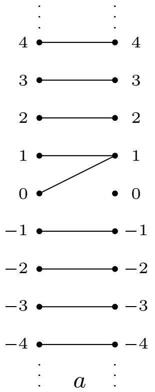

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We consider $\mathbf{F}(a, f) := \mathbf{F}_{sf}(\mathbb{Z}) * \mathbf{F}_{sf}(\mathbb{Z})$.

To do this, we consider $\mathbf{T}(a, f)$ the monoid of all the terms of over the variables $\{a, f\}$ and $\mathbf{T}(a, f)/\equiv$ were $\equiv$ is the equivalence relation obtained from the monoidal structure.

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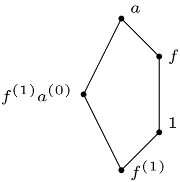

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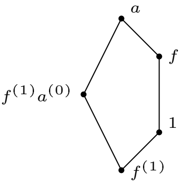

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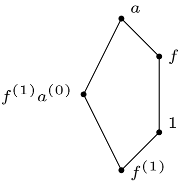

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The subset of P_0 of P is the set created only by the rules 1-4; the elements of P_0 are called obviously positive.

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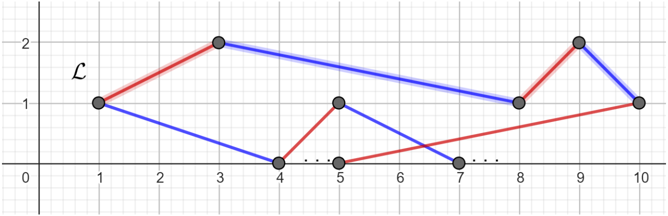

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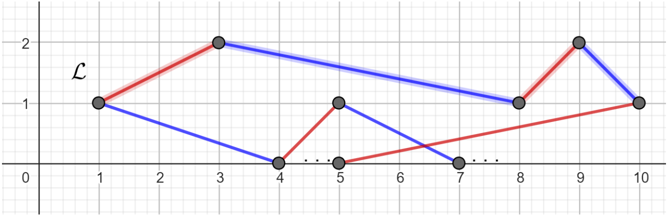

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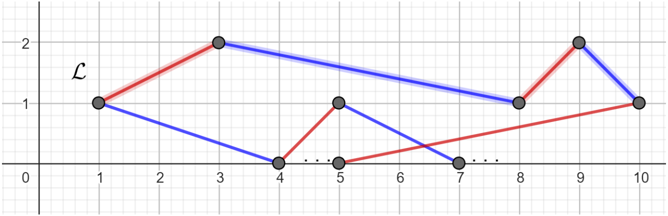

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Conjecture: The structure $\mathbf{F}(a, f)$ is a non-distributive ℓ-pregroup.

Thank you for your attention!!