1. *Problem 1.5: Characterization of bipartite graphs*

Show that a graph is bipartite if and only if it does not contain any odd cycles, i.e., cycles with an odd number of edges.

Defintion 1. *A graph $G = (V, E)$ is bipartite if its vertex set V admits a bipartition $V = X\dot{\cup}Y$ such that every edge $e \in E$ has one endpoint in X and the other one in Y.*

Proof. It follows from the definition that for every vertex v in X all adjacent vertices u are in Y and for all vertices u in Y the adjacent vertices v are in X.

We can traverse the cycle and will always alternate between the sets X and Y. Wlog we assume we start in X. Since it is a cycle, we will end up in X again. Since there are no edges that travel from X to X and from Y to Y we have to follow an even number of edges to end up in X again (X–Y–X). This produces a contradiction since we are in an odd cycles. ∎