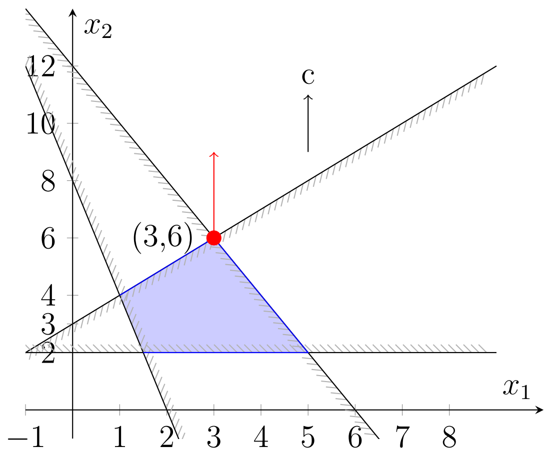
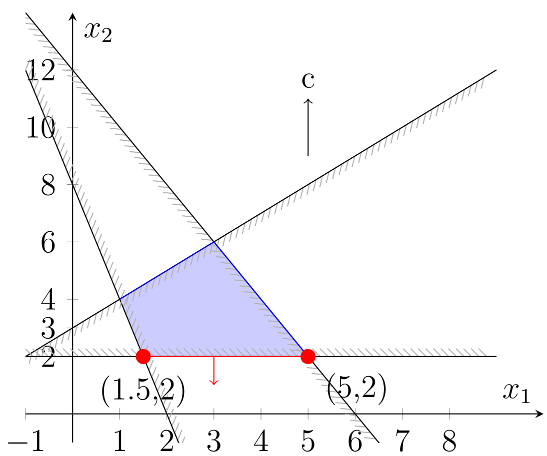
Problem 1: Graphical solution of a linear program

$$
\begin{array}{lrrrrrr}
\max & & & & x_2 & & \\
\text{s.t} & - & 4x_1 & - & x_2 & \leq & -8 \\
& - & x_1 & + & x_2 & \leq & 3 \\
& & & - & x_2 & \leq & -2 \\
& & 2x_1 & + & x_2 & \leq & 12 \\
& & x_1 & & & \geq & 0 \\
& & & & x_2 & \geq & 0
\end{array}
$$

a *Determine all optimal solutions graphically.*



b *Instead of maximizing the objective function, minimize it and graphically determine all optimal solutions of the minimization problem.*

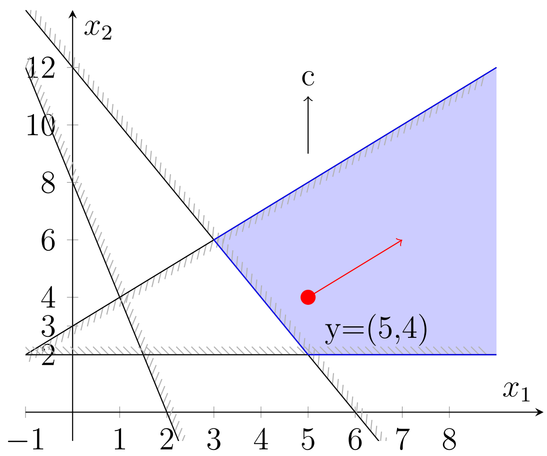


c *Change the fourth constraint $2x_1 + x_2 \leq 12$ such that*

(i) the point (5, 2) still satisfies the fourth constraint with equality, and

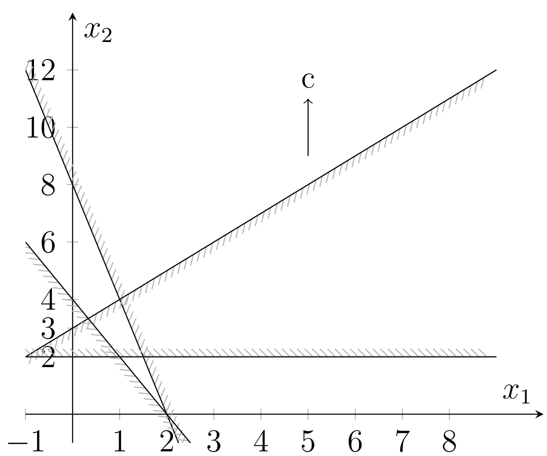
(ii) the maximization problem becomes unbounded.

A simple change would be to make the constraint unnecessary by changing $\leq$ to $\geq$. The maximization problem becomes unbounded since there is always an improved maximum value (∞).



d *Change the right-hand side of the fourth constraint $2x_1+x_2 \leq 12$ (i.e., the number 12) such that the maximization problem becomes infeasible. Is the corresponding minimization problem infeasible, as well?*

Any value < 5 will do, the min. problem will become infeasible too since there is no polyhedron.



e *Prove that the linear programs that you wrote in parts (c) and (d) are unbounded and infeasible, respectively. How would you prove optimality of the solutions you found in parts (a) and (b)?*

1. unbounded: certificate of unboundedness: there is no constraint limiting x_2 from growing. $y = (5, 4)$, $q = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
2. infeasible: if there is no polyhedron, there cannot be a solution.
3. optimality in a and : Introducing the constraint $x_2 \geq 6$ which is perpendicular to the objective reduces the polyhedron to a vertex. Therefore the dimension dropped to 0 and the unique optimum is found.
4. optimality in b: introducing the constraint $x_2 \leq 2$ which is perpendicular to the objective results in a 1-dimensional edge. Therefore the dimension dropped and the minimum is found.