1. *Finding a Chebychev center of a polyhedron*

$$\begin{aligned} \max \quad & r \\ \text{s.t.} \quad & ya_i + |a_i| r \leq b_i \quad \forall i \in [m], \\ & y_i \geq r \quad \forall i \in [n] \end{aligned} \tag{1}$$

Example given the polytope defined by this lp:

$$\begin{aligned} -x_1 + x_2 &\leq 0 \\ x_1 + x_2 &\leq 8 \\ x_i &\geq 0 \quad \forall i \in \{1, 2\} \end{aligned}$$

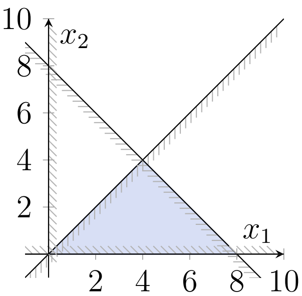

Figure 1: Sample polytope

The lp to solve for the chebychev center is

$$\text{maximize} \quad r \tag{2a}$$

$$\text{subject to} \quad -1(y_1 + \frac{-1}{\sqrt{2}} r) + (y_2 + \frac{1}{\sqrt{2}} r) \leq 0, \tag{2b}$$

$$(y_1 + \frac{-1}{\sqrt{2}} r) + (y_2 + \frac{1}{\sqrt{2}} r) \leq 8, \tag{2c}$$

$$y_i \geq r \quad \forall i \in \{1, 2\}, \tag{2d}$$

$$r \geq 0 \tag{2e}$$

The optimal objective value is 1.6585 with $y = \begin{pmatrix} 4 \\ 1.65685 \end{pmatrix}$

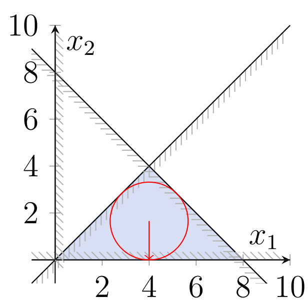

Figure 2: Sample polytope with chebychev center

1.a *Assume that P has a Chebychev center. Write a linear program for the problem of finding such a Chebyshev center and the radius of the corresponding ball, and prove that your formulation is correct.*

See Equation 1

1.b *Can the linear program that you found in* 1.a *help deciding whether a Chebychev center exists at all?*

Yes, if there is no Chebychev center at all, the lp becomes unbounded.

2. ***Existence of vertices in full-rank polyhedra***

Let $A \in \mathbb{R}^{m \times n}$ have full column rank, let $b \in R^m$, and consider the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$

2.a *For $v, w \in \mathbb{R}^n$ with $w \neq 0$, the set $L(v, w) := \{v + \lambda w : \lambda \in \mathbb{R}\}$ is called a line. Prove that P does not contain a line, i.e., there are no $v, w \in \mathbb{R}^n$ with $w \neq 0$ such that $L(v, w) \subseteq P$.*

Informal: The Polyhedron is constrained in all variables (because it is a full column rank constrained polyhedron). So there is no possible line since λ can be chosen to violate a constraint.

2.b *Prove that precisely one of the following two statements is true.*

(i) P is empty

(ii) P has a vertex

Informal: If the polyhedron is not empty, it is at least 2-dimensional. Therefore the two constraints have to intersect somewhere and form a vertex at the intersection. The same is true for the higher dimensional ones, at least two constraints intersect with each other and form a vertex.