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Fall 2020

Mathematical Optimization – Problem set 9

`https://moodle-app2.let.ethz.ch/course/view.php?id=12839`

Problem 1: Relaxation of the matching polytope

Let $G = (V, E)$ be a simple undirected graph. We are interested in studying a polytope P that may be a candidate for the matching polytope $P_{\mathcal{M}}$. Recall that the family of all matchings is

$$\mathcal{M} = \{M \subseteq E \colon e_1 \cap e_2 = \emptyset \text{ for all } e_1, e_2 \in M, e_1 \neq e_2\} \enspace ,$$

and that the matching polytope is, by definition, $P_{\mathcal{M}} \coloneqq \operatorname{conv}(\{\chi^M \colon M \in \mathcal{M}\})$. Consider the polytope

$$P \coloneqq \{x \in \mathbb{R}^E_{\geq 0} \colon x(\delta(v)) \leq 1 \text{ for all } v \in V\} \enspace .$$

Show that there are graphs G for which P does not describe the matching polytope, i.e., $P \neq P_{\mathcal{M}}$.

Problem 2: Relaxation of the vertex cover polytope

Let $G = (V, E)$ be a (not necessarily bipartite) undirected graph. Define the polytope

$$P \coloneqq \{x \in \mathbb{R}^V \colon 0 \leq x(v) \leq 1 \text{ for all } v \in V,\ x(u) + x(v) \geq 1 \text{ for all } \{u, v\} \in E\} \enspace .$$

(a) Show that P is a relaxation of the vertex cover polytope of G, i.e., prove that $P \cap \{0, 1\}^V$ is the set of all incidence vectors of vertex covers of G.

(b) Prove that any vertex y of P satisfies $y \in \{0, \frac{1}{2}, 1\}^V$, i.e., the vertices of P are half-integral.

(c) Give a 2-approximation for the minimum weight vertex cover problem, where we are given non-negative vertex weights $w \colon V \to \mathbb{Z}_{\geq 0}$, and the goal is to find a vertex cover $S \subseteq V$ minimizing $w(S)$. Recall that a 2-approximation for a minimization problem is an efficient algorithm that returns a solution of value within twice the value of an optimal solution.

    Hint: You may assume that linear programs over P can be solved efficiently.

Problem 3: Incidence Matrices and Total Unimodularity

Let $G = (V, E)$ be an undirected graph and let A be its incidence matrix. Prove that if A is totally unimodular, then G is bipartite.

Remark: This is the backward direction of Theorem 5.13 in the script.

Programming exercises

Work through the notebook `extremalMinCuts.ipynb`, where you prove uniqueness of inclusion-wise maximal and minimal minimum s-t cuts, characterize them in terms of components of the residual graph, and and implement algorithms for finding these cuts.