1. ***Problem 1: The algorithm of Ford-Fulkerson and the value of a flow***

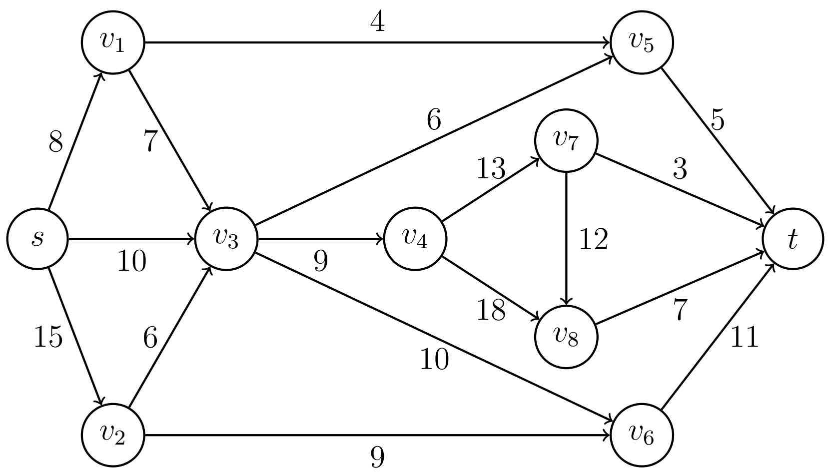
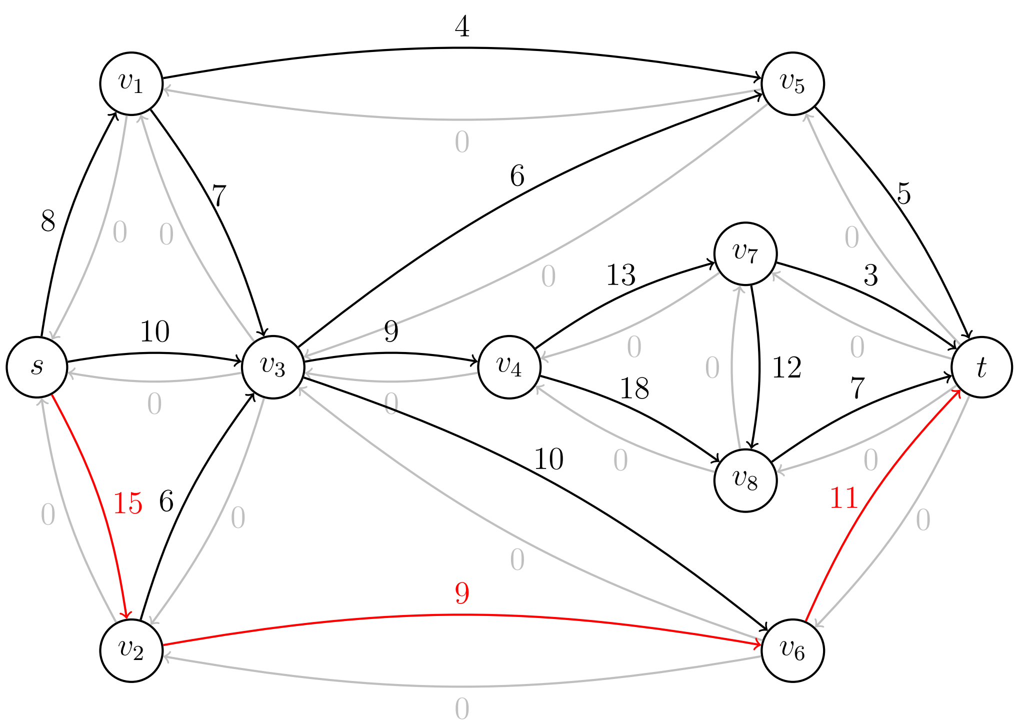
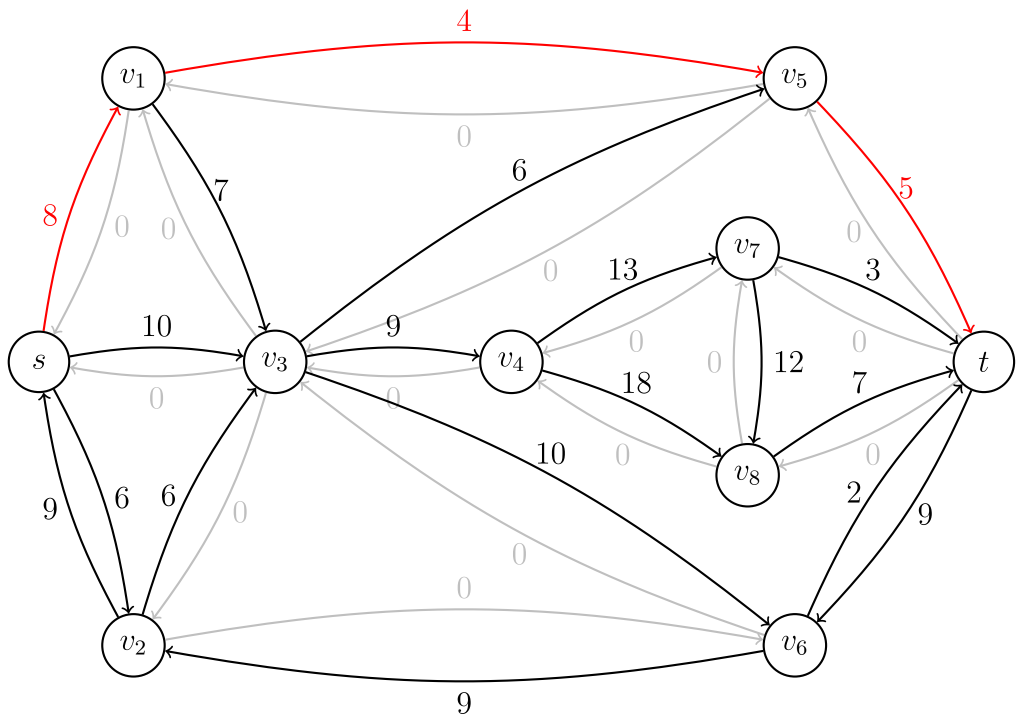


Figure 1: A digraph $G = (V, A)$ with edge capacities $u : A \to \mathbb{Z}_{\geq 0}$

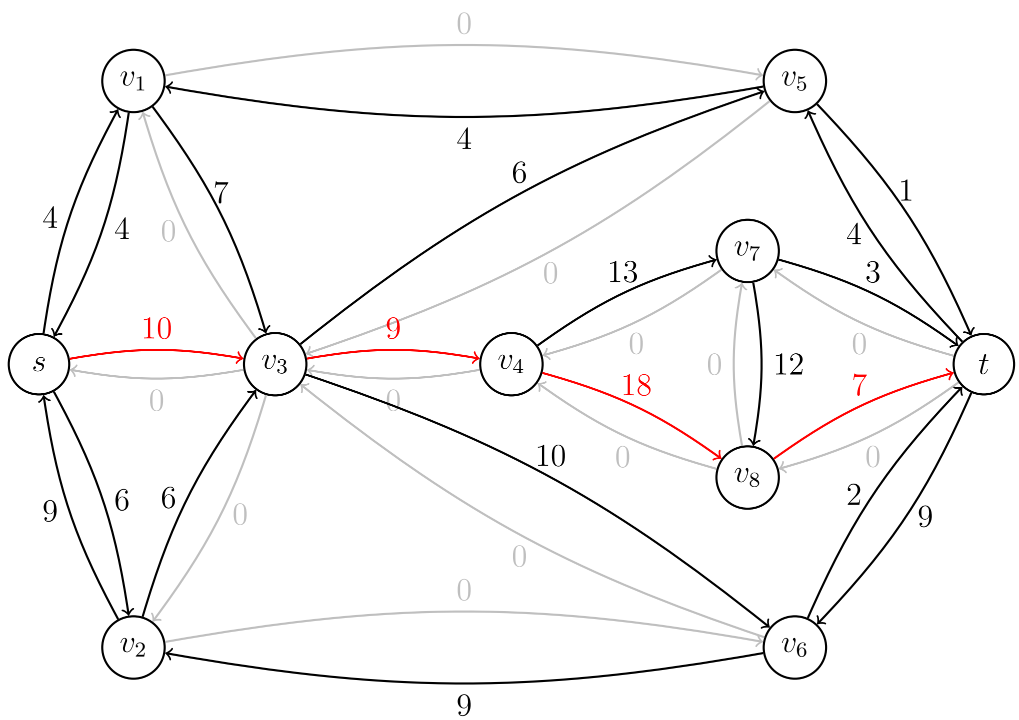
1.a *Consider the graph $G = (V, A)$ with edge capacities $u : A \to \mathbb{Z}_{\geq 0}$ given in Figure 1. Apply the algorithm of Ford and Fulkerson to obtain a maximal s-t flow. In every iteration, provide the current flow and its value, the corresponding residual graph, as well as an augmenting s-t path together with its increment value γ (or a certificate that there is no augmenting s-t path).*



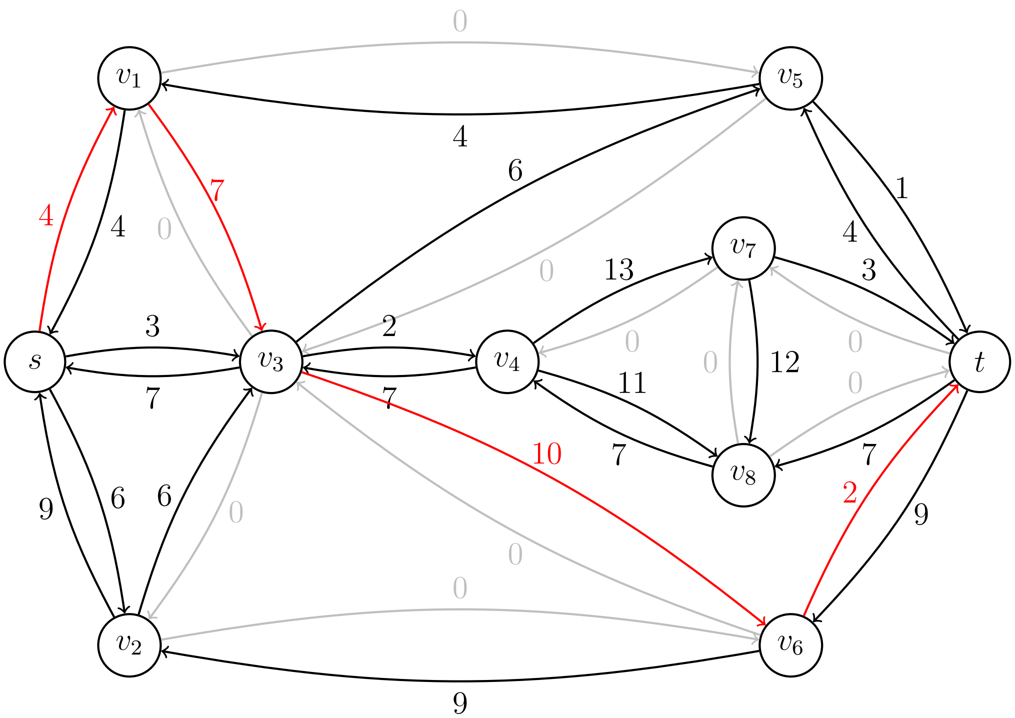
what	value
f	0
γ	9



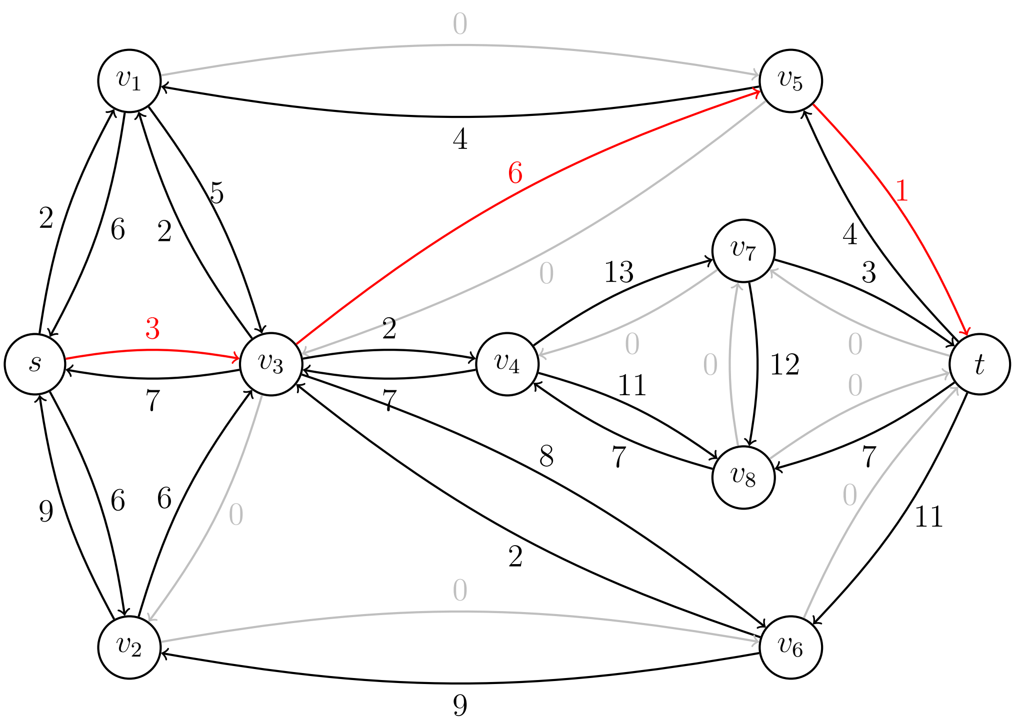
what	value
f	9
γ	4



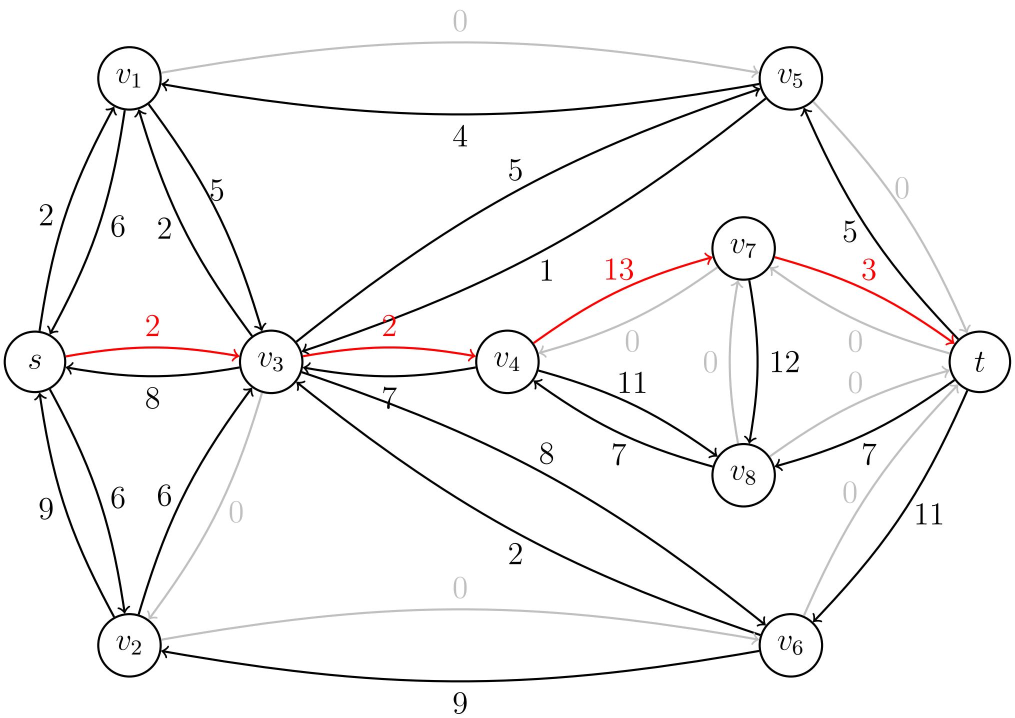
what	value
f	13
γ	7



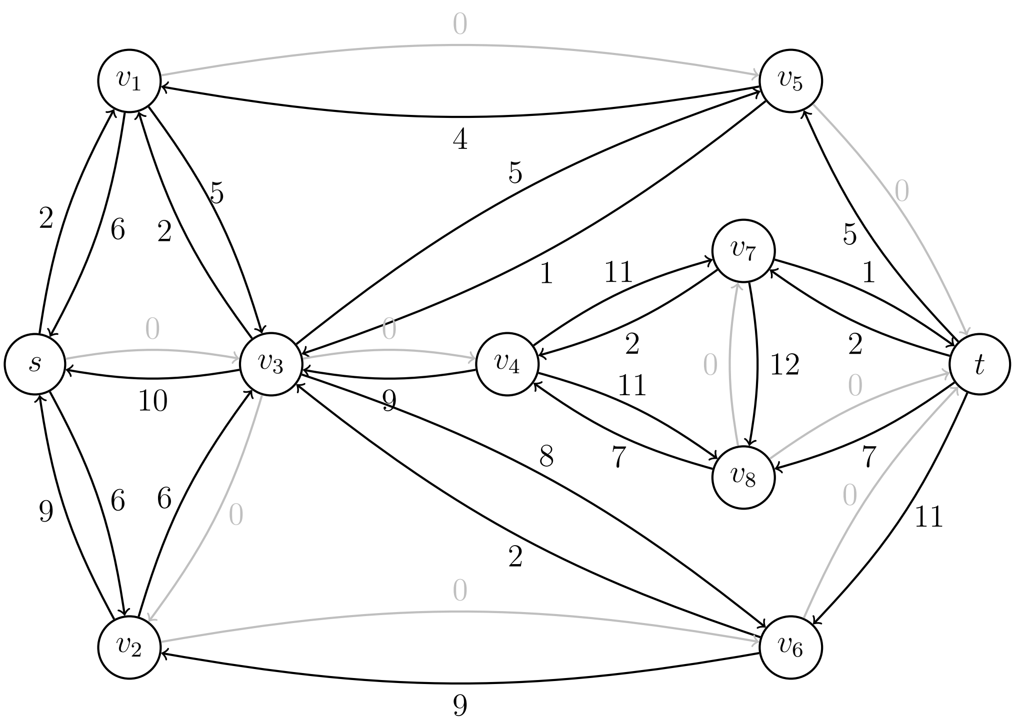
what	value
f	20
γ	2



what	value
$f(s,t)$	22
γ	1



what	value
f	23
γ	2



what	value
f	25
γ	0

There exists no augmenting s-t path. The nodes reachable from s are v_1, v_2, v_3, v_5, v_6.

1.b *Show that the value of any s-t flow f is equal to the difference between the inflow into t and the outflow at t, i.e.,$v(f) = f(\delta^-(t)) - f(\delta^+(t))$*

By Lemma 4.3 a flow can be expressed via an s-t cut as the following: $v(f) = f(\delta^+(C)) - f(\delta^-(C))$. Let C be a cut where on one side are all vertices except t ($C = V \setminus \{t\}$). The inflow into t is precisely the outflow of the defined cut ($\delta^-(t) = \delta^+(C)$). Therefore the flow can be defined as $v(f) = f(\delta^-(t)) - f(\delta^+(t))$.

2. *Problem 2: Flow through intermediate vertices*

2.a *Let $G = (V, A)$ be a directed graph with arc capacities $u : A \to \mathbb{Z}_{\geq 0}$, and let $s_1, s_2, \dots, s_l \in V$ be distinct. Assume that for every $i \in \{1, \dots, l-1\}$, there is an $s_i - s_{i+1}$ flow f_i with value $v(f_i) \geq k$ for some $k \in \mathbb{Z}_{\geq 0}$. Prove that there exists an s_1-s_l flow f with value $v(f) \geq k$.*

if there exists a path s_1-s_2 with flow $f_1 \geq k$ and a path s_2-s_3 with flow $f_2 \geq k$, there exists a path s_1-s_3 with a flow $f_{1,3} \geq k$. With this definition there exists an s_1-s_l path with $f \geq k$.

3. *Problem 3: Max-flow min-cut via duality I*

$$
\begin{aligned}
\max \quad & \nu \\
\text{s.t.} \quad & \textstyle\sum_{a\in\delta^+(v)} f_a - \sum_{a\in\delta^-(v)} f_a = \begin{cases} \nu & \text{if } v = s \\ -\nu & \text{if } v = t \\ 0 & \text{if } v \in V \setminus \{s,t\} \end{cases} \\
& f_a \leq u_a \quad \forall a \in A \\
& f_a, v \in \mathbb{R}_{\geq 0}
\end{aligned} \tag{1}
$$

3.a *Prove that the optimal value of Equation 2 equals the value of a maximum s-t flow in G.*

To prove this, we need to show that the Equation 2 is equal to the definition 4.1. (i) of the definition states the following: *Capacity constraints:* $f(a) \leq u(a) \quad \forall a \in A$. This is precisely what the second constraint in the LP states. The (ii) part states the following: *Balance constraints: for* $v \in V,\ f(\delta^+(v)) - f(\delta^-(v)) \begin{cases} = 0 & \textit{if } v \in V \setminus \{s,t\} \\ \geq 0 & \textit{if } v = s \\ \leq 0 & \textit{if } v = t \end{cases}$. There are three parts to prove.

It's easy to see that $\sum_{a\in\delta^+(v)} f_a - \sum_{a\in\delta^-(v)} f_a = 0$ is equal to $f(\delta^+(v)) - f(\delta^-(v)) = 0$.

ν is the value of the flow, so it corresponds to the outflow of s. Therefore the sum of all outflows of s must equal ν

With the same argument, the inflow for t has to equal the value of ν.

3.b *Write down the dual of the linear program Equation 2, using variables y_v for $v \in V$ and z_a for $a \in A$.*

$$
\begin{aligned}
\min \quad & \textstyle\sum_{a\in A} u_a z_a \\
\text{s.t.} \quad & \textstyle\sum_{a\in\delta^+(v)} f_a \geq \\
& \textstyle\sum_{a\in\delta^-(v)} f_a \\
& f_a \leq u_a \quad \forall a \in A \\
& f_a, v \in \mathbb{R}_{\geq 0}
\end{aligned} \tag{2}
$$