1. *Problem 3: Perfect matchings in three-regular graphs*

Prove that every 3-regular bridgeless graph admits a perfect matching. To this end, recall that a *bridge* in a graph $G = (V, E)$ is an edge $e \in E$ whose removeal from the graph increases the number of connected components by one. A graph is *bridgeless* if it contains no bridge. Moreover, a graph is 3-*regular* if the degree of every vertex is three.

Solution We show that the perfect matching polytope is non-empty. The perfect matching polytope is defined by $P(G) = \left\{ x \in R^E_{\geq 0} \middle| \begin{array}{l} x(\delta(v)) = 1 \quad \forall v \in V \\ x(\delta(S)) \geq 1 \quad \forall S \subseteq V, \quad |S| \text{ odd} \end{array} \right\}$.

Wlog we assume the graph is connected.

Let x be the all $\frac{1}{3}$-vector. We need to show that x is in the polytope P. To be in the polytope, all constraints must be satisfied.

$\boldsymbol{x(\delta(v)) = 1 \quad \forall v \in V}$**:** This constraint is satisfied by the construction of x. In a 3-regular graph, every vertex has 3 edges and therefore the sum of all outgoing edges is equal to $= 1$.

$\boldsymbol{x(\delta(S)) \geq 1 \quad \forall S \subseteq V, |S|}$ **odd:** To fulfill this constraint there need to be 3 outgoing edges for every subset S. By sake of contradiction we assume there are 0, 1, or 2 outgoing edges. By disproving all of them, there are at least 3 outgoing edges.

Case 0: In the lecture we showed that there is no perfect matching in a Graph with odd number of vertices. Since we assumed that x is in P this case will never happen. $\lightning$

Case 1: If there exists only one edge outgoing of S, that edge becomes a bridge. By definition G is bridgeless. $\lightning$

Case 2: If we sum up all the vertices in G we get $3 \cdot |V|$. Expressed with respect to S we can also say we have $2 \cdot e + 2 \quad \forall e \in S$. But $3 \cdot |V|$ is odd and $2 \cdot e + 2$ is even. $\lightning$

Therefore the number of outgoing edges have to be at least 3 which fulfills the constraint.