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Mathematical Optimization – Problem set 2

https://moodle-app2.let.ethz.ch/course/view.php?id=12839

Problem 1: Finding a Chebychev center of a polyhedron

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ for some $m, n \in \mathbb{Z}_{>0}$. A ball B with center $y \in \mathbb{R}^n$ and radius $r \in \mathbb{R}_{\geq 0}$ is defined as the set of all points within Euclidean distance at most r from y, i.e., $B = \{x \in \mathbb{R}^n : \|x - y\|_2 \leq r\}$. We are interested in finding a ball with largest possible radius that is contained in P. The center of such a ball is called a *Chebychev center* of P.

(a) Assume that P has a Chebychev center. Write a linear program for the problem of finding such a Chebyshev center and the radius of the corresponding ball, and prove that your formulation is correct.

Hint: Given a ball B with center $y \in \mathbb{R}^n$ and radius $r \in \mathbb{R}_{\geq 0}$ that is contained in a single halfspace $\{x \in \mathbb{R}^n : a_i^\top x \leq b_i\}$, what is the point in B closest to the hyperplane $\{x \in \mathbb{R}^n : a_i^\top x = b_i\}$?

(b) Can the linear program that you found in (a) help deciding whether a Chebychev center exists at all?

Problem 2: Existence of vertices in full-rank polyhedra

Let $A \in \mathbb{R}^{m\times n}$ have full column rank, let $b \in \mathbb{R}^m$, and consider the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

(a) For $v, w \in \mathbb{R}^n$ with $w \neq 0$, the set $L(v, w) := \{v + \lambda w : \lambda \in \mathbb{R}\}$ is called a *line*. Prove that P does not contain a line, i.e., there are no $v, w \in \mathbb{R}^n$ with $w \neq 0$ such that $L(v, w) \subseteq P$.

(b) Prove that precisely one of the following two statements is true.

(i) P is empty.

(ii) P has a vertex.

Problem 3: Finite linear programming optima are attained

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a non-empty polyhedron, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ for some $m, n \in \mathbb{Z}_{>0}$. Moreover, let $c \in \mathbb{R}^n$ such that

$$\gamma := \sup\{c^\top x : x \in P\}$$

is finite. In this problem, we show that there exists a point $x^* \in P$ such that $c^\top x^* = \gamma$. To do so, we proceed by induction on $\dim(P)$.

(a) Observe that if $\dim(P) = 0$, the statement is true.

(b) Prove that if $\ker(A) \neq \{0\}$, it is enough to show the statement for $(\dim(P) - 1)$-dimensional polyhedra.

(c) In the case where $\ker(A) = \{0\}$, we propose a direct proof using the following claim.

Claim. For every $y \in P$, there exists a non-singular subsystem $A'x \leq b'$ of $Ax \leq b$ such that $y' := (A')^{-1}b' \in P$ and $c^\top y' \geq c^\top y$.

Show that the claim implies that the supremum γ is attained.

(d) Prove the above claim. To this end, start with a point $y \in P$ and consider the y-tight constraints in the system $Ax \leq b$. If this subsystem is not full-rank, try to change y so that the number of tight constraints increases, while the objective does not decrease.

(e) Combine the above points (a) to (d) to get a full proof of existence of $x^* \in P$ with $c^\top x^* = \gamma$.

Remark: The above problem shows that the optimal value of a feasible and finite linear program is always attained, which justifies writing linear programs using maxima and minima instead of suprema and infima.

Problem 4: Polytopes and vertices

(a) Prove that every non-empty polytope has a vertex.

(b) Let $P \subseteq \mathbb{R}^n$ be a polytope and let $c \in \mathbb{R}^n$. Prove that the optimum value of the linear program $\max\{c^\top x \colon x \in P\}$ is attained at a vertex of P.

(c) Let P be a polytope. Prove that $\mathrm{conv}(\mathrm{vertices}(P)) = P$.

Hint: You can use the following separation statement without proving it. If $Y \subseteq \mathbb{R}^n$ is a closed convex set and $z \in \mathbb{R}^n \setminus Y$, then there exist $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\top y < b$ for all $y \in Y$, and $a^\top z > b$.

Problem 5: Finite convex hull

Let $X = \{x_1, \ldots, x_k\} \subseteq \mathbb{R}^n$ be a finite set. In this problem, we show that $\mathrm{conv}(X)$ is a polytope. To this end, we first assume that $\mathrm{conv}(X)$ is full-dimensional, i.e., X contains $n+1$ affinely independent points. In this case, we define a polyhedron P as follows: For any n affinely independent points in X, consider the unique hyperplane $a^\top x = b$ containing all of them. If one of the halfspaces defined by $a^\top x \leq b$ and $a^\top x \geq b$ contains all the points in X, we add the corresponding inequality to the description of P.

(a) Prove that $\mathrm{conv}(X) \subseteq P$.

(b) To prove $P \subseteq \mathrm{conv}(X)$, we assume for the sake of arriving at a contradiction that there exists a point $y \in P \setminus \mathrm{conv}(X)$. Follow the steps below to reach a contradiction.

(i) Consider the polyhedron D defined by

$$D = \{d \in \mathbb{R}^n \colon (y - x_i)^\top d \geq 1 \ \forall i \in [k]\} \ .$$

Show that D has a vertex.

Hint: Exploit that y and $\mathrm{conv}(X)$ can be separated by a hyperplane, and use the characterization obtained in Problem 2(b).

(ii) Consider a vertex d of the polyhedron D. Prove that the inequality $d^\top x \leq d^\top y - 1$ appears in the description of P, and observe that this implies $y \notin P$, giving the desired contradiction.

(c) Conclude that if X contains $n+1$ affinely independent points, then $\mathrm{conv}(X)$ is a polytope.

(d) Reduce the case where $\mathrm{conv}(X)$ is not full-dimensional to the one where it is.

Hint: If $k \coloneqq \dim(\mathrm{conv}(X)) < n$, the points in X all lie in an affine subspace of dimension k. Use an affine bijection to transform X to a set in $\mathbb{R}^k \times \{0\}^{n-k}$, and argue that the convex hull of the transformed points is a polytope. To complete the proof, you may use that the affine image of a polytope is a polytope.

Programming exercise

Complete the notebook `02_shortestPath.ipynb` on finding the length of shortest s-t paths using a linear program.