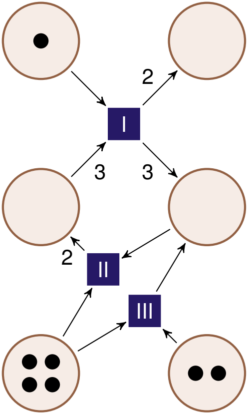
# How hard is the reachability problem for Petri nets?

Ismaël Jecker

# Petri net

A Petri net is a directed graph composed of

- **Places**, that contain tokens
- **Transitions**, that can transfer tokens



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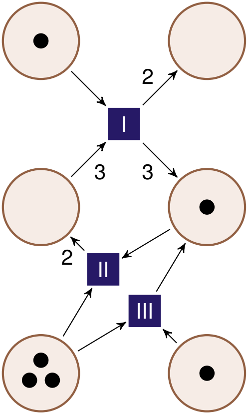

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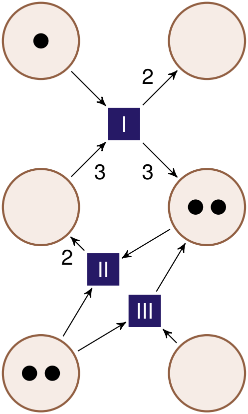

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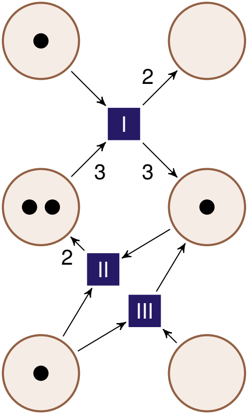

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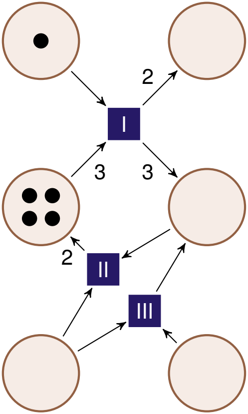

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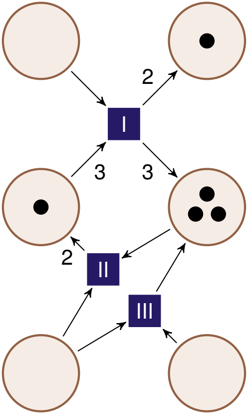


They are used to model:

- distributed systems
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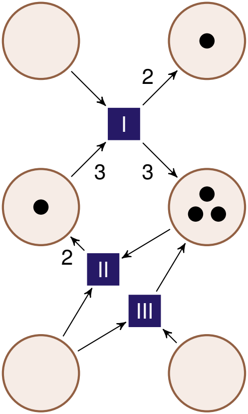

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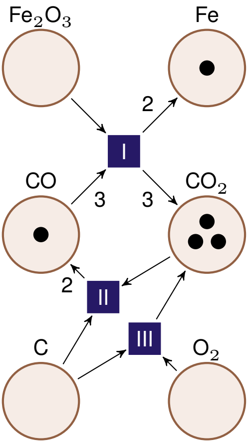
- network protocols
- chemical processes

| | |
|---|---|
| I | $Fe_2O_3 + 3CO \rightarrow 2Fe + 3CO_2$ |
| II | $CO_2 + C \rightarrow 2CO$ |
| III | $O_2 + C \rightarrow CO_2$ |



| REACHABILITY | **Input**: Two configurations $s$, $t$ of a Petri Net $P$<br>**Question**: Starting from $s$, can $P$ reach $t$? |
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Reachability

**Input**: Two configurations s, t of a Petri Net P
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ExpSpace-hard

1976: Lipton

**REACHABILITY**

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1982: Kosaraju
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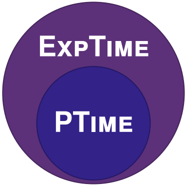

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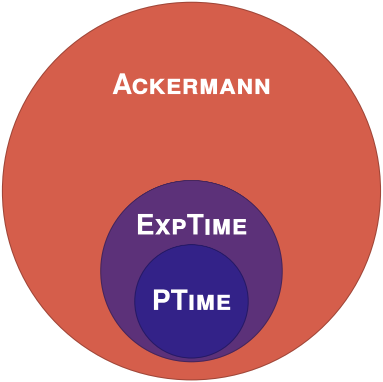

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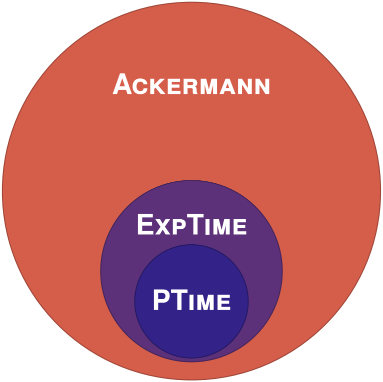
**Fast growing hierarchy:**

$$
\begin{aligned}
f_{i+1}(n) &= \Big(\underbrace{f_i \circ \ldots \circ f_i}_{n \text{ times}}\Big)(1) \\
f_1(n) &= 2n
\end{aligned}
$$




$f_1(1) = 2$ $f_1(2) = 4$ $f_1(3) = 6$ $f_1(4) = 8$ $f_1(5) = 10$

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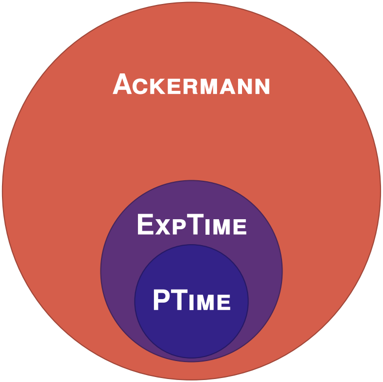


$f_2(1) = 2$ $\quad f_2(2) = 4$ $\quad f_2(3) = 8$ $\quad f_2(4) = 16$ $\quad f_2(5) = 32$

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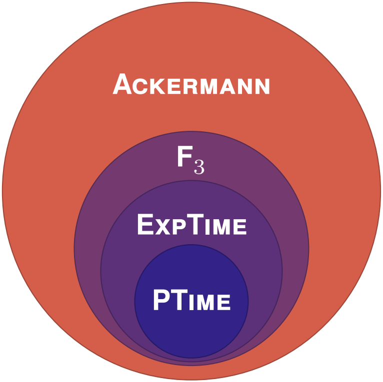


$f_3(1) = 2 \qquad f_3(2) = 4 \qquad f_3(3) = 16 \qquad f_3(4) = 65536 \qquad f_3(5) \simeq 2 \cdot 10^{19728}$

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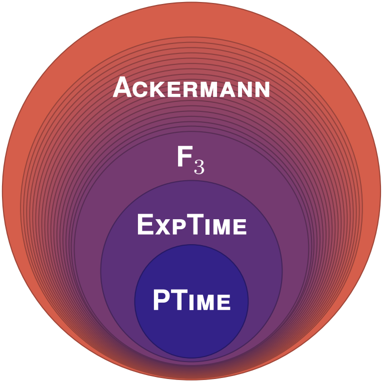

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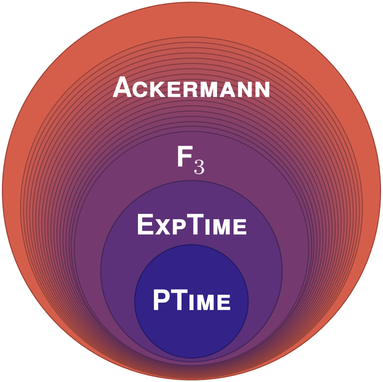

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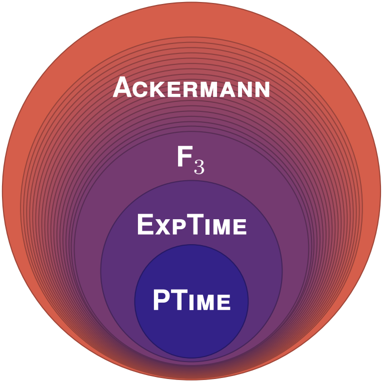

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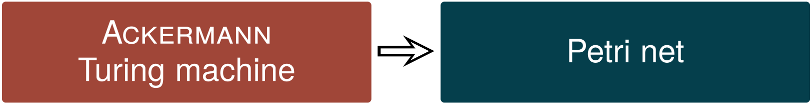
**ACKERMANN-complete**

## Reachability for Petri Nets is Ackermann-hard



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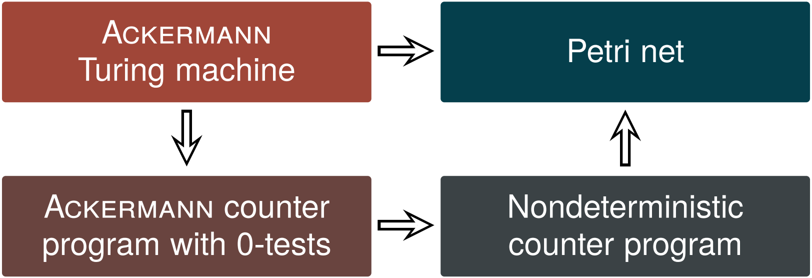

# Counter program with 0-tests (CP0)

```
double(x,y)
 1  x -= 1
 2  y += 2
 3  if x == 0 then goto 4 else goto 1
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```

**Basic instructions**:

- Increment counter `x += n`
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# Counter program with 0-tests (CP0)

name list of variables storing **non-negative** integers

list of instructions

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Indirectly, these instructions can be used to define:

- Multiplication `x *= n`
- Integer division `x /= n`

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This is sufficient to simulate runs of Turing Machines! [1967. Minsky]

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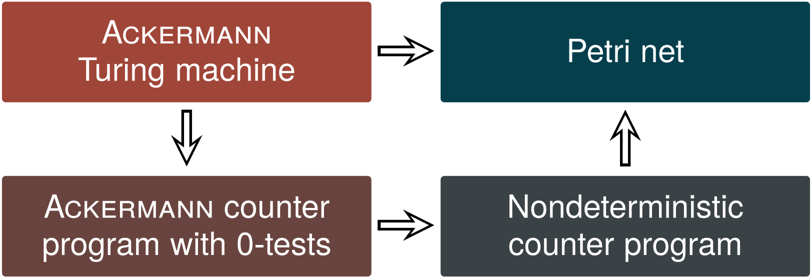

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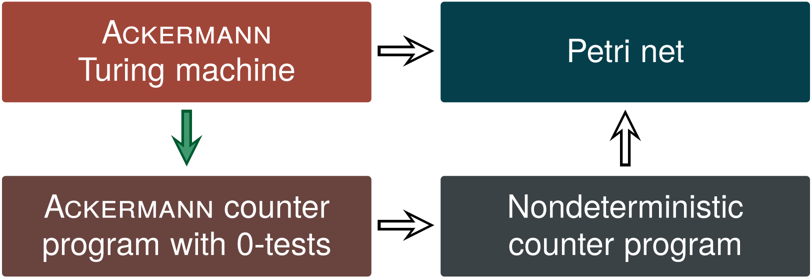

# Nondeterministic counter program (NCP)

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erraticDouble(x,y)
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```

**Basic instructions**:

- Increment counter `x += n`
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# Nondeterministic counter program (NCP)

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**Basic instructions**:

- Increment counter `x += n`
- Decrement counter `x -= n`
- Stop the program `end`

**Advanced instruction**:

- Jump nondeterministically `goto i or goto j`

# Nondeterministic counter program (NCP)

```
erraticDouble(x,y)                    (3,0)
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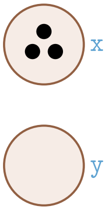


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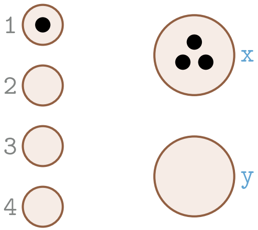


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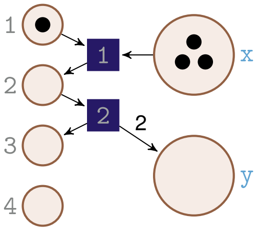


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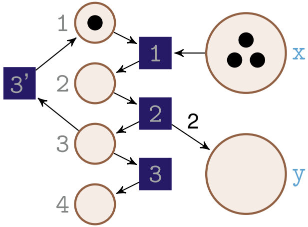


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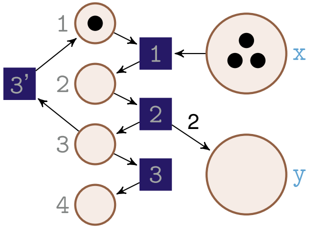


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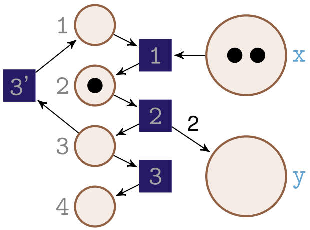


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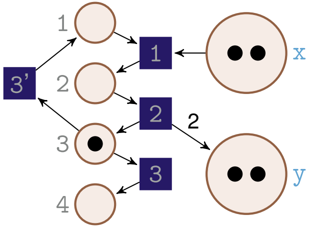


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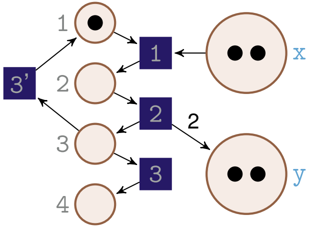


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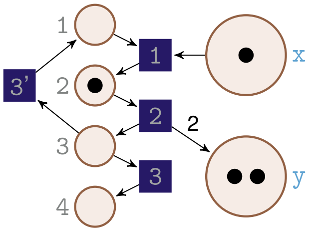


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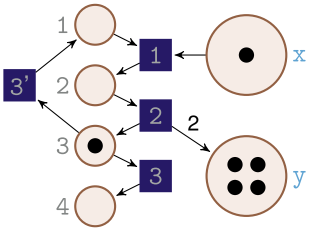


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Nondeterministic counter program ⇒ **Petri Net**

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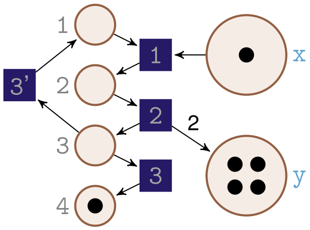


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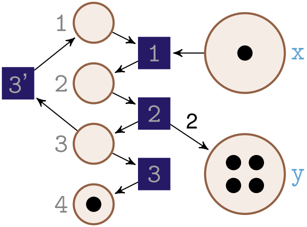


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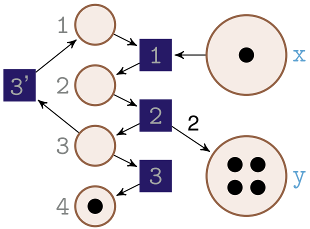


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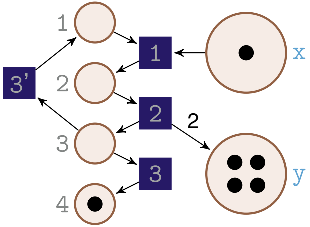


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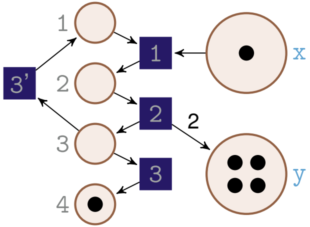


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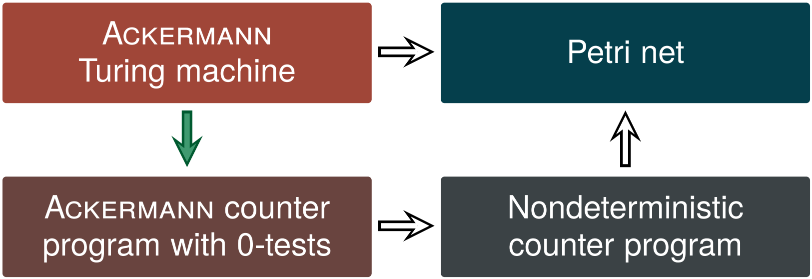
# Reachability for Petri Nets is Ackermann-hard



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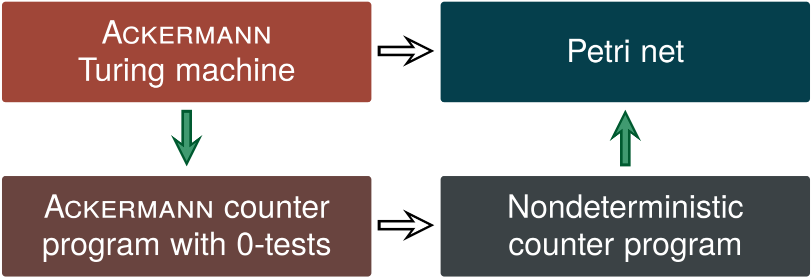

Ackermann counter program with 0-tests ⇒ Nondeterministic counter program

```
double(x,y)
1  x -= 1
2  y += 2
3  if x == 0 then goto 4 else goto 1
4  end
```

?

Ackermann counter program with 0-tests ⟹ Nondeterministic counter program

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Ackermann counter program with 0-tests

Nondeterministic counter program

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end
```

How can we control that the correct path was chosen?

Ackermann counter program with 0-tests $\Rightarrow$ Nondeterministic counter program

**Key technique:** Simulating zero tests with invariants

E:
```
2 * (x + x1) * x2 == x3
```

B:
```
2 * (x + x1) * x2 <  x3
```

Ackermann counter program with 0-tests $\Longrightarrow$ Nondeterministic counter program

**Key technique:** Simulating zero tests with invariants

**E**: `2 * (x + x1) * x2 == x3`

**B**: `2 * (x + x1) * x2  <  x3`

- **E** can be preserved iff `x` is 0:

  `E && x == 0` $\xrightarrow{\exists \text{ run of } \texttt{xTest}}$ **E**

  `E && x != 0` $\xrightarrow{\forall \text{ run of } \texttt{xTest}}$ **B**

- **E** cannot be repaired:

  **B** $\xrightarrow{\forall \text{ run of } \texttt{xTest}}$ **B**

```
xTest(x,x1,x2,x3)
1    x1 -= 1
2    x += 1
3    x3 -= 1
4    goto 5 or goto 1
5    x -= 1
6    x1 += 1
7    x3 -= 1
8    goto 9 or goto 5
9    x2 -= 1
10   end
```

ACKERMANN counter program with 0-tests ⇒ Nondeterministic counter program

**Key technique:** Simulating zero tests with invariants

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**Difficulty:** The initialisation requires to compute $\mathcal{A}(n)$

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For all $n \in \mathbb{N}$ we can build a **NCP** with $\mathcal{O}(n)$ lines that **needs** $\mathcal{A}(n)$ steps to go from $\vec{0}$ to $\vec{0}$ and uses the following number of counters:

| $6n$ | $4n + 5$ | $3n + 2$ | $2n + 3$ |
|---|---|---|---|
| 2021 | 2021 | 2022 | 2023 |
| Czerwiński & Orlikowski | Leroux | Lasota | Czerwiński, J., Lasota et al. |

ACKERMANN counter program with 0-tests ⇒ Nondeterministic counter program

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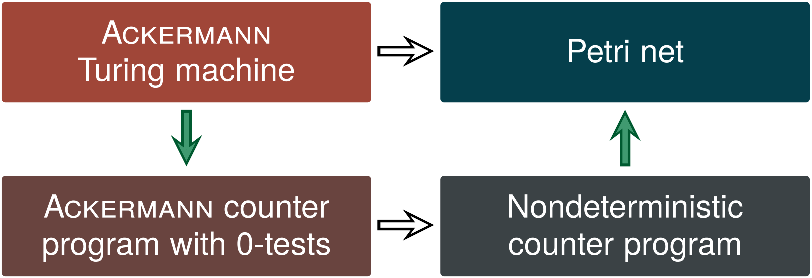
# Reachability for Petri Nets is Ackermann-hard



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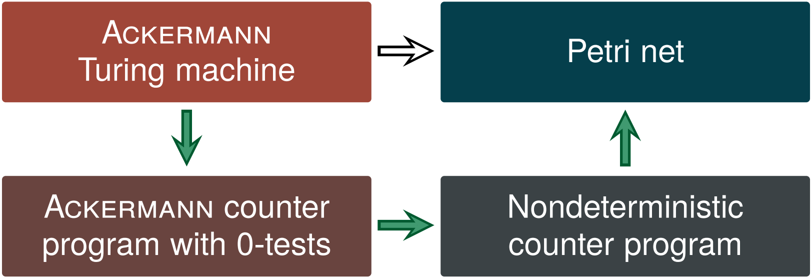

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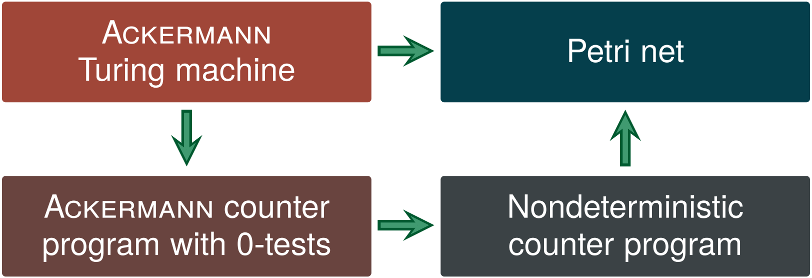

| REACHABILITY | |
|---|---|
| **Input:** | An NCP P and two valuations s, t |
| **Question:** | Can P(s) end with the valuation t? |

**in ACKERMANN**

2015: Leroux & Schmitz
2019: Leroux & Schmitz

**ACKERMANN-hard**

2021: Czerwiński & Orlikowski / Leroux
2022: Lasota
2023: Czerwiński, J., Lasota et al.

→ **ACKERMANN-complete**

| REACHABILITY | |
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→ **ACKERMANN-complete**

The reachability problem for counter programs **with $n$ counters** is:

| in $\mathbf{F}_{n+4}$ | $\mathbf{F}_{\lfloor \frac{n}{6} \rfloor}$-hard | $\mathbf{F}_{\lfloor \frac{n-2}{3} \rfloor}$-hard | $\mathbf{F}_{\lfloor \frac{n-3}{2} \rfloor}$-hard |
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| | |
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| REACHABILITY | **Input:** An NCP $\mathtt{P}$ and two valuations $\mathsf{s}$, $\mathsf{t}$<br>**Question:** Can $\mathtt{P}(\mathsf{s})$ end with the valuation $\mathsf{t}$? |

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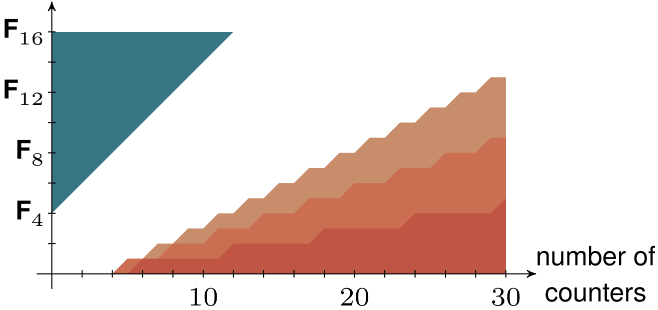


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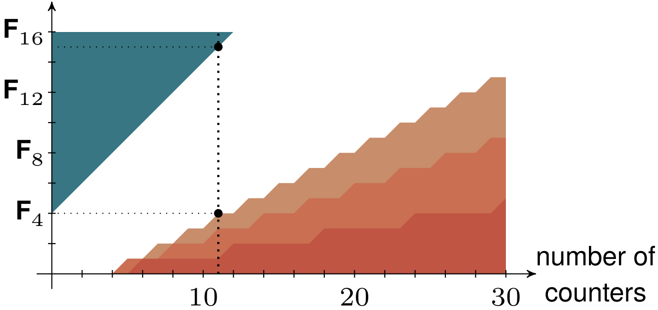


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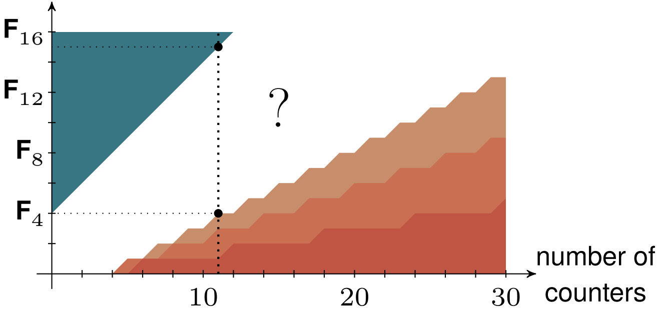


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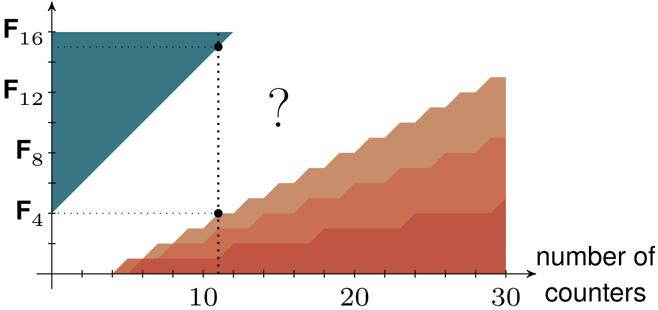


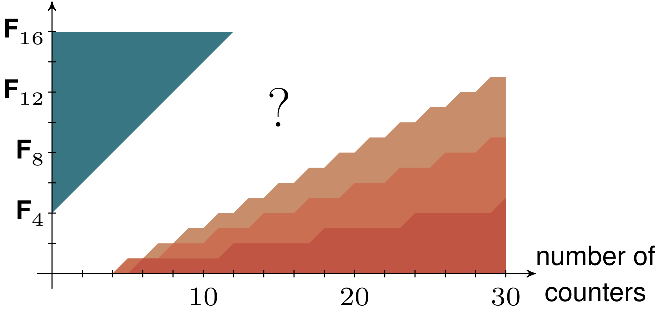

**Open question**

What is the complexity of Reachability for **NCP** parametrised by the number of counters?

**Reachability**

**Input:** An NCP $\mathtt{P}$ and two valuations $\mathsf{s}$, $\mathsf{t}$

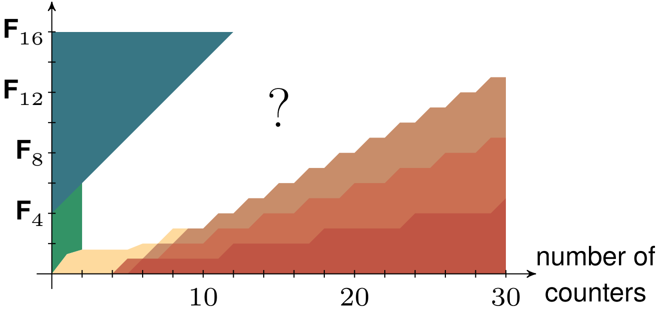
**Question:** Can $\mathtt{P}(\mathsf{s})$ end with the valuation $\mathsf{t}$?




**Open question**

What is the complexity of Reachability for **NCP** parametrised by the number of counters?

| REACHABILITY | |
|---|---|
| | **Input:** An NCP P and two valuations s, t |
| | **Question:** Can P(s) end with the valuation t? |




| Open question | What is the complexity of REACHABILITY for **NCP** parametrised by the number of counters? |
|---|---|

| REACHABILITY | |
|---|---|
| | **Input:** An NCP P and two valuations s, t |
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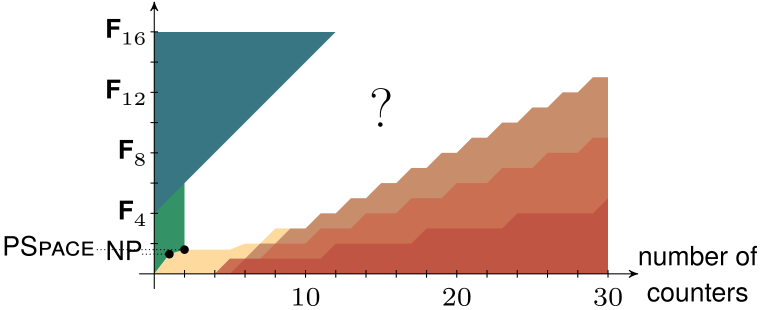
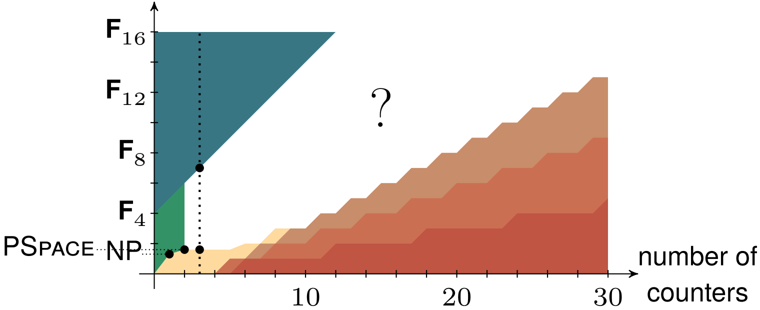



| Open question | What is the complexity of REACHABILITY for **NCP** parametrised by the number of counters? |
|---|---|

**Reachability**

**Input:** An NCP P and two valuations s, t

**Question:** Can P(s) end with the valuation t?




**Open question** What is the complexity of Reachability for **NCP** parametrised by the number of counters?

**Open question** What is the complexity of Reachability for 3-counters **NCP**?

# Pushdown nondeterministic counter program

name stack variables

list of instructions

```
program(s,x,y)
1 …
⋮
```

**Basic instructions**:

- Increment counter `x += n`
- Decrement counter `x -= n`
- Stop the program `end`

**Advanced instructions**:

- Jump nondet. `goto i or goto j`
- Push `push(s,n)`
- Pop `pop(s)`
- Pop-test `if pop(s)==n then goto i else goto j`

## Pushdown nondeterministic counter program

| REACHABILITY | |
|---|---|
| **Input:** | A PNCP $\mathtt{P}$ and two valuations $\mathsf{s}$, $\mathsf{t}$ |
| **Question:** | Can $\mathtt{P}(\mathsf{s})$ end with the valuation $\mathsf{t}$? |

**Basic instructions**:

- Increment counter
```
x += n
```
- Decrement counter
```
x -= n
```
- Stop the program
```
end
```

**Advanced instructions**:

- Jump nondet.
```
goto i or goto j
```
- Push
```
push(s,n)
```
- Pop
```
pop(s)
```
- Pop-test
```
if pop(s)==n then goto i else goto j
```

# Pushdown nondeterministic counter program

| REACHABILITY | |
|---|---|
| **Input:** | A PNCP P and two valuations s, t |
| **Question:** | Can P(s) end with the valuation t? |

| Open question | Is REACHABILITY decidable for pushdown NCP? |
|---|---|

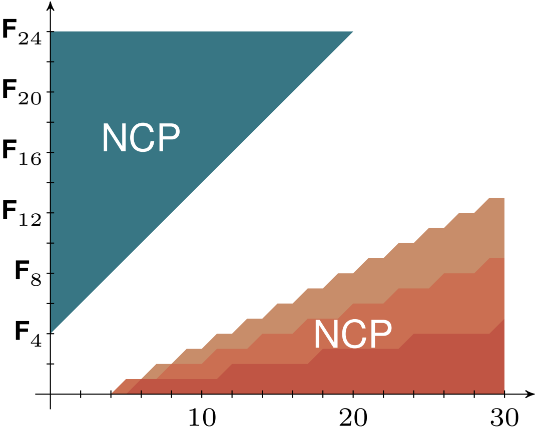
**Basic instructions**:

- Increment counter `x += n`
- Decrement counter `x -= n`
- Stop the program `end`

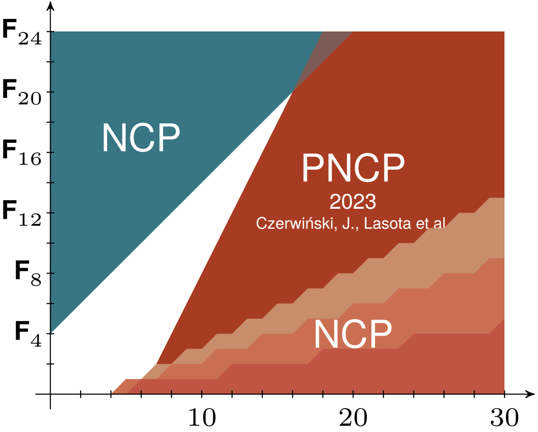
**Advanced instructions**:

- Jump nondet. `goto i or goto j`
- Push `push(s,n)`
- Pop `pop(s)`
- Pop-test `if pop(s)==n then goto i else goto j`

# Pushdown nondeterministic counter program



# Pushdown nondeterministic counter program



For all $n \in \mathbb{N}$ we can build a **NCP** with $\mathcal{O}(n)$ lines that **needs** $\mathcal{A}(n)$ steps to go from $\vec{0}$ to $\vec{0}$ and uses the following number of counters:

| $6n$ | $4n + 5$ | $3n + 2$ | $2n + 3$ |
|---|---|---|---|
| 2021 | 2021 | 2022 | 2023 |
| Czerwiński & Orlikowski | Leroux | Lasota | Czerwiński, J., Lasota et al. |

**Can YOU do better?** $n + c$ OPEN

**NCP instructions**:

- Increment counter

```
x += d
```

- Decrement counter

```
x -= d
```

- Jump nondeterministically

```
goto i or goto j
```

**Ackermann function**:

$$\mathcal{A}(n) = f_n(n)$$

$$f_1(n) = 2n$$

$$f_{i+1}(n) = \Big(\underbrace{f_i \circ \ldots \circ f_i}_{n \text{ times}}\Big)(1)$$