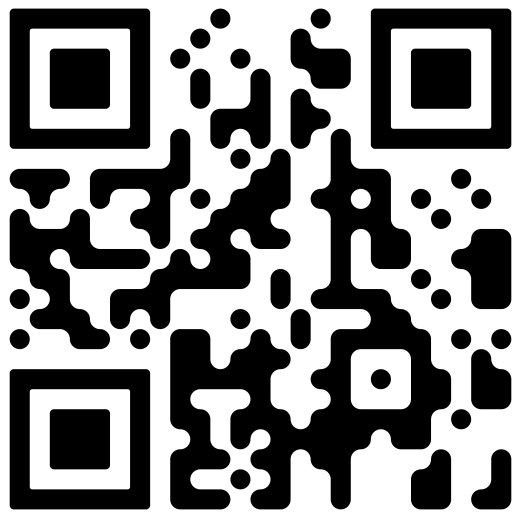
Finite-valued Streaming String Transducers

Emmanuel Filiot, Ismaël Jecker, Christof Löding,
Anca Muscholl, Gabriele Puppis, Sarah Winter



ismaeljecker.github.io

Finite-valued regular relations form an expressive set of binary relations with good algorithmic properties

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Theorem: We can decide in polynomial space whether a given SST defines a **finite-valued regular relation**

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**

Transducers

Transducers are abstract machines that recognise **relations**

$$R \subseteq \Sigma^* \times \Gamma^*$$

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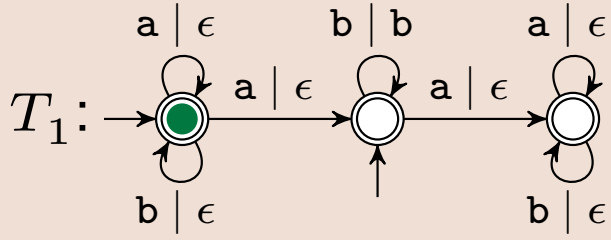

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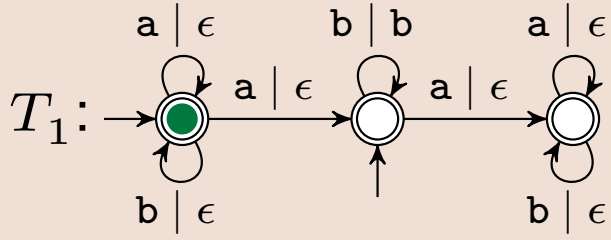

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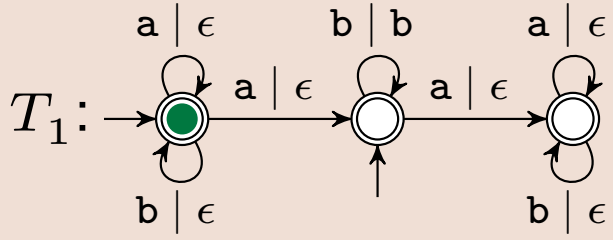

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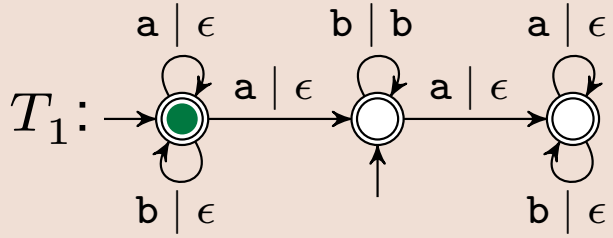

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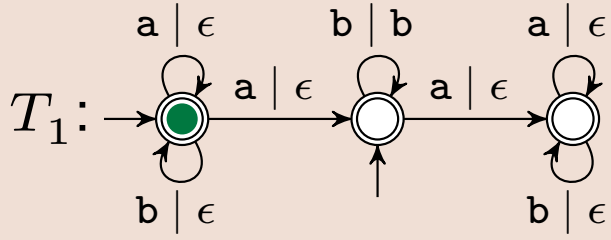

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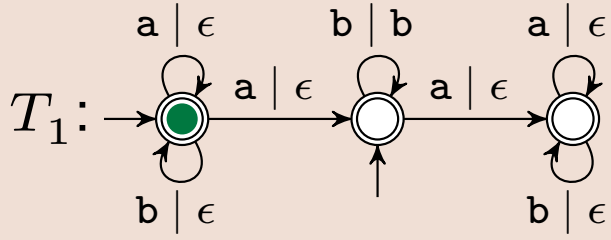

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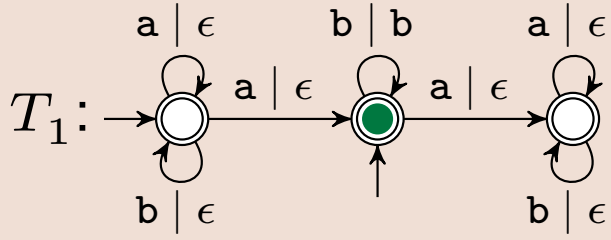

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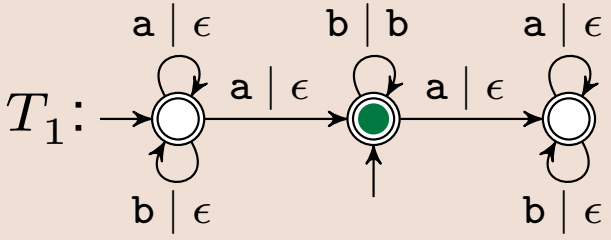
Input: a b b a a a b b b a b a a a

Output: b



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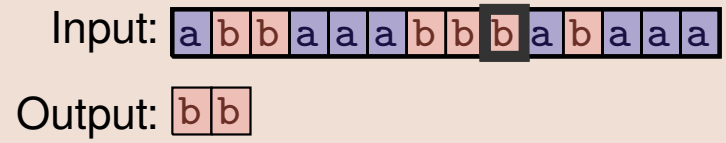

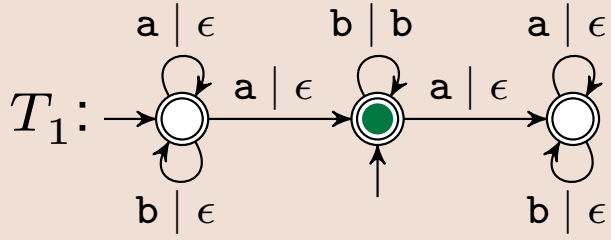

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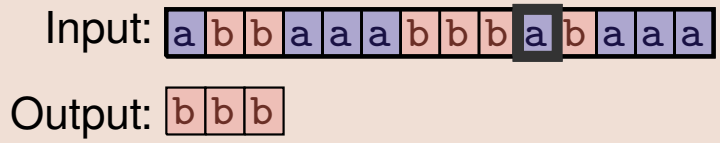

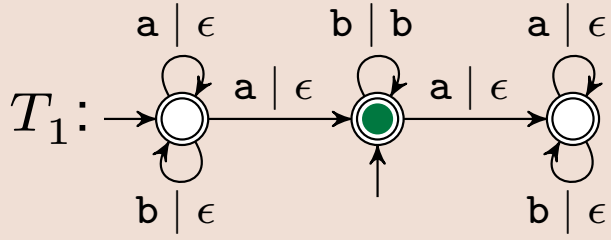

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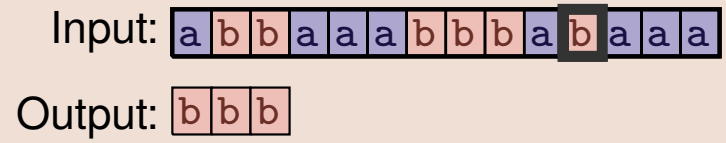

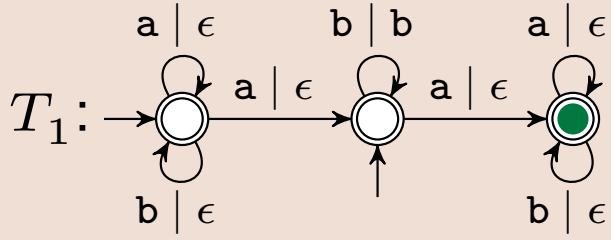

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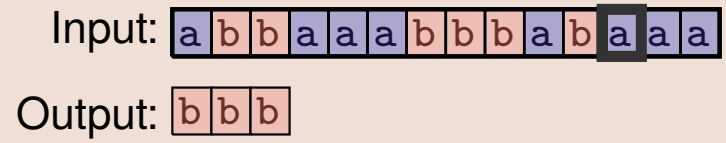

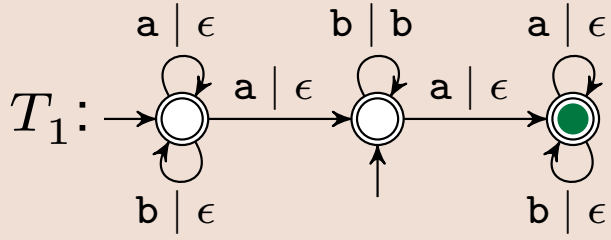

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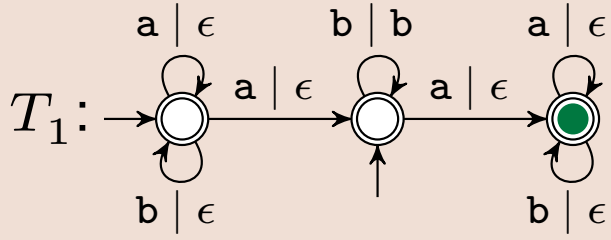

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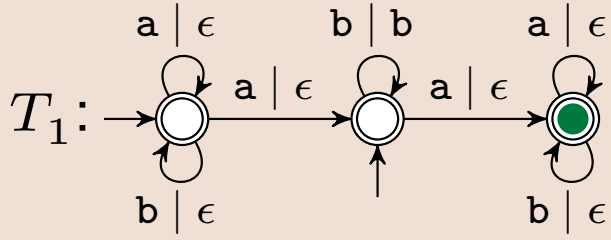

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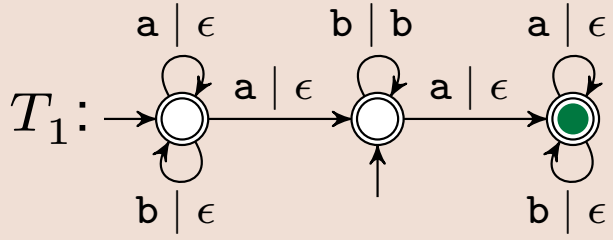

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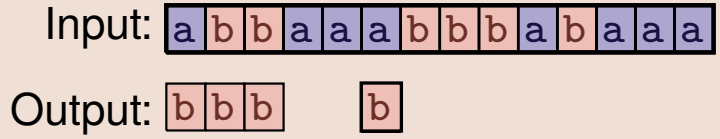

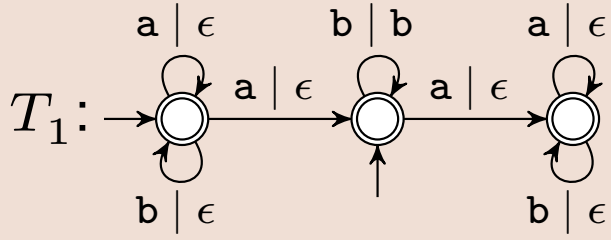

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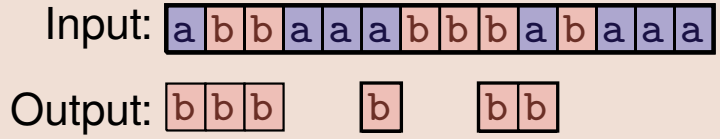

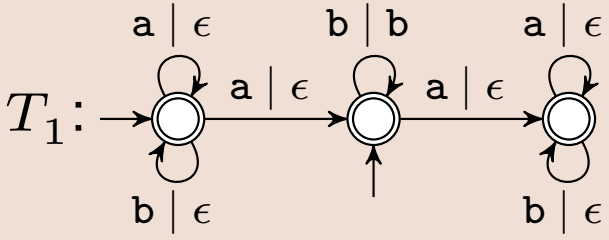

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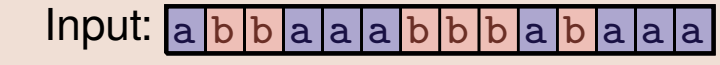

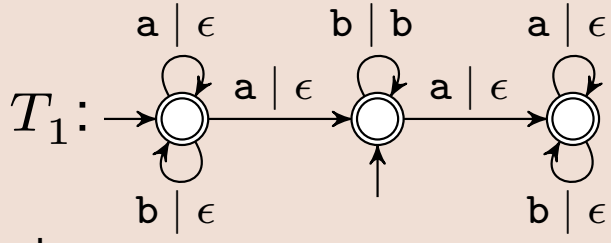

T_1 associates a word to each of its b-blocks

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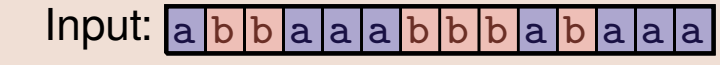

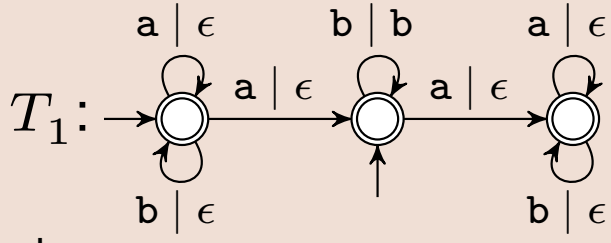

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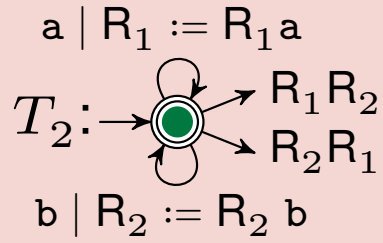
R_2:



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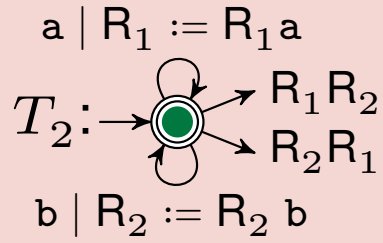

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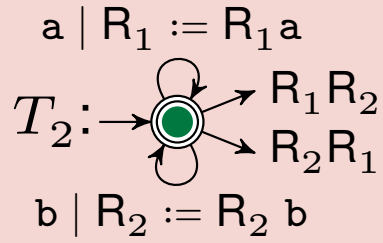

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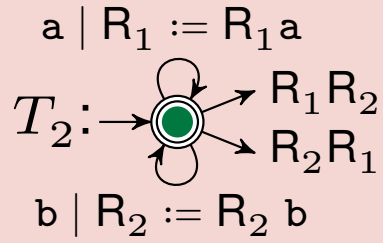

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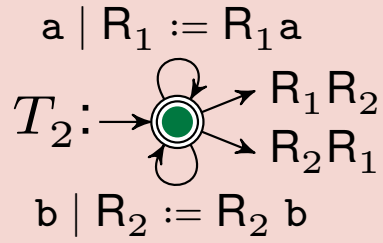

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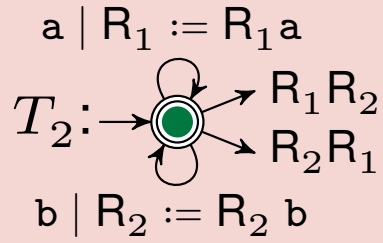

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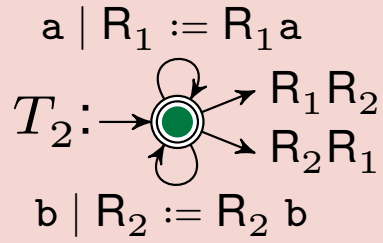

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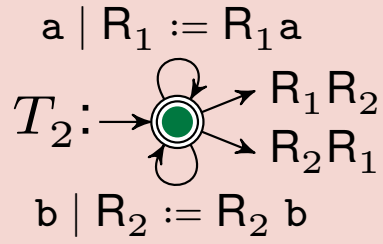

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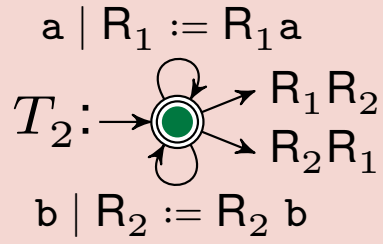

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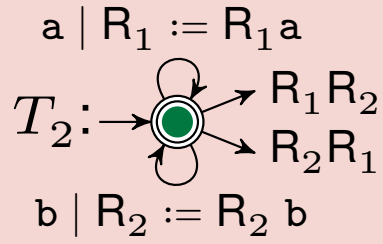

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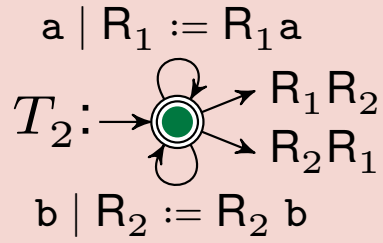

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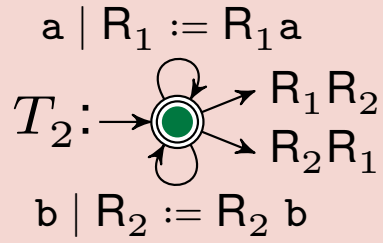

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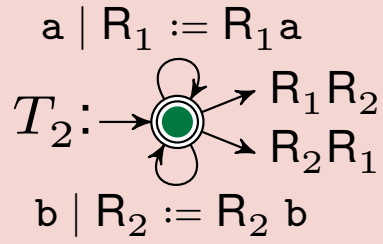

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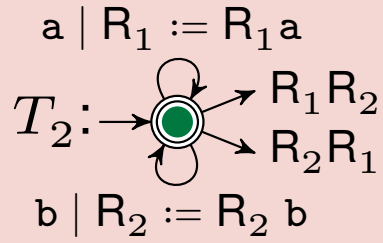

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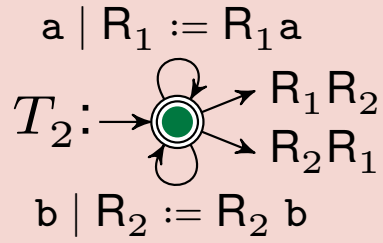


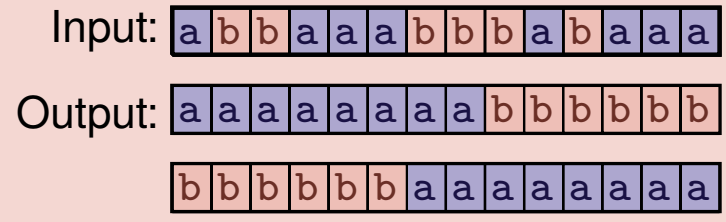
Rational relations recognised by finite state transducers

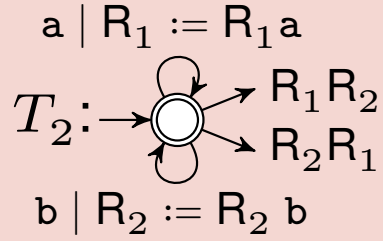
Regular relations recognised by streaming string transducers

Transducers

Transducers are abstract machines that recognise **relations**

Streaming string transducer (SST)





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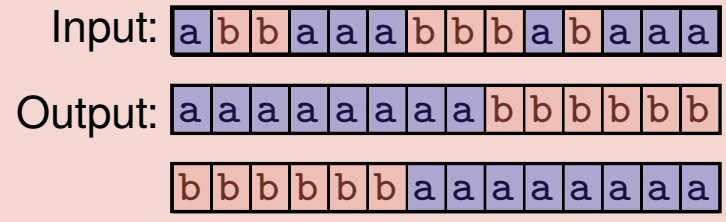

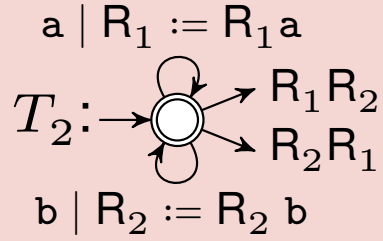

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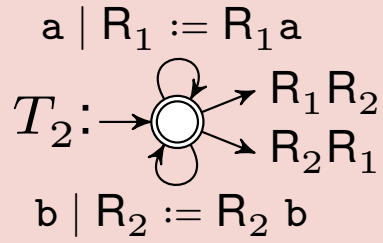
Input: a b b a a a b b b a b a a a

Output: a a a a a a a a b b b b b b

b b b b b b a a a a a a a a

T_2 groups the a's and the b's of its input



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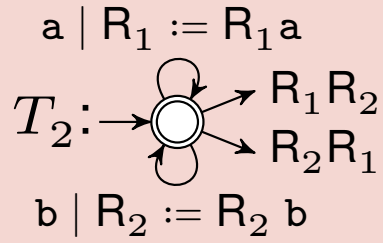

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Transducers

Equivalence of **rational relations** is **undecidable**

Rational relations recognised by finite state transducers

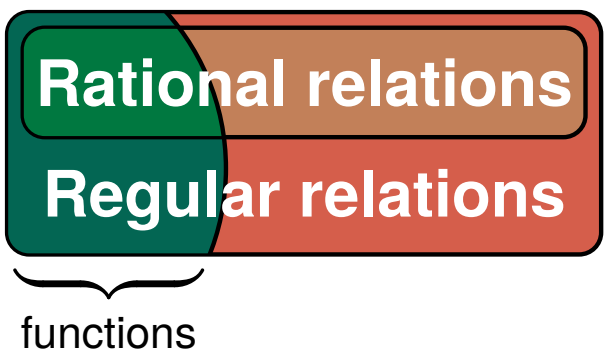
Regular relations recognised by streaming string transducers

Transducers

Equivalence of **rational relations** is **undecidable**

Equivalence of **regular functions** is **decidable**

↳ each input is mapped to at most 1 output



recognised by finite state transducers

recognised by streaming string transducers

Transducers

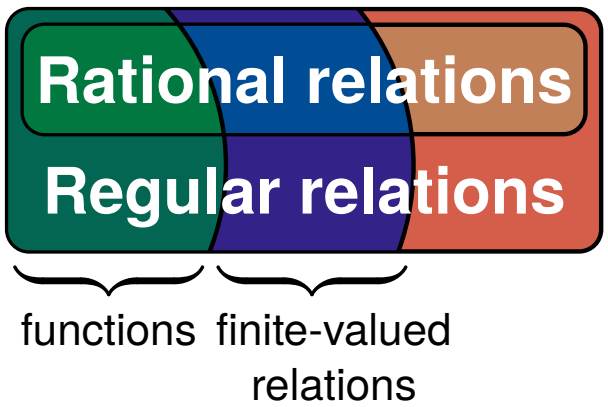
Equivalence of **rational relations** is **undecidable**

Equivalence of **regular functions** is **decidable**

↳ each input is mapped to at most 1 output

Equivalence of **finite-valued regular relations** is **decidable**

↳ $\exists \mathbf{k} \in \mathbb{N}$ s.t. each input is mapped to at most **k** outputs



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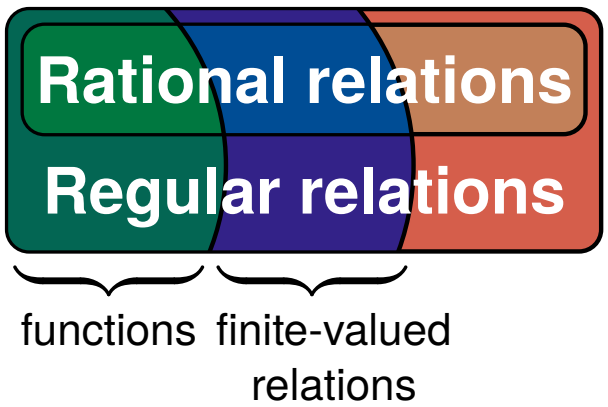
Theorem: We can decide in polynomial space whether a given SST defines a **finite-valued regular relation**

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**



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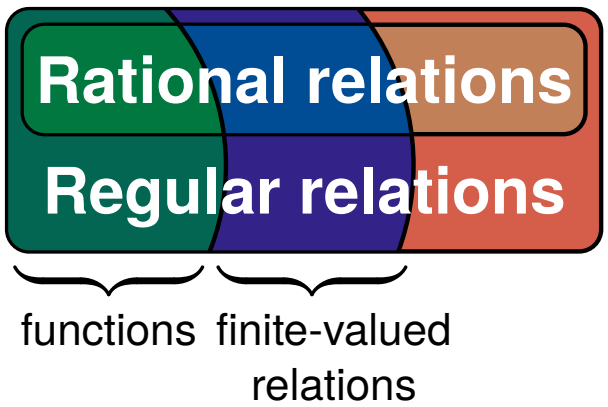
Raised as open problems by [2011. Alur, Deshmukh]

Known to hold for FST [1989. Weber], [1993. Weber]

and for SST with a single register [2017. Gallot et al.]



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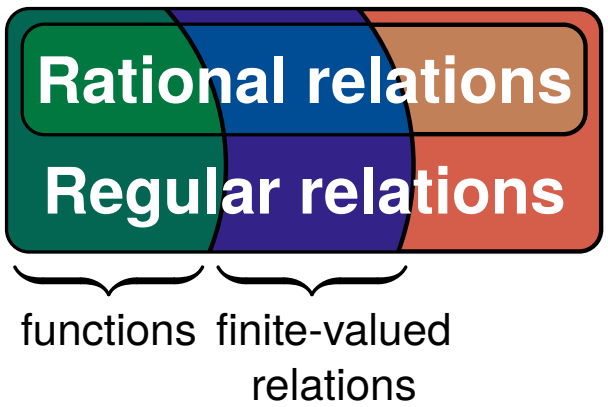

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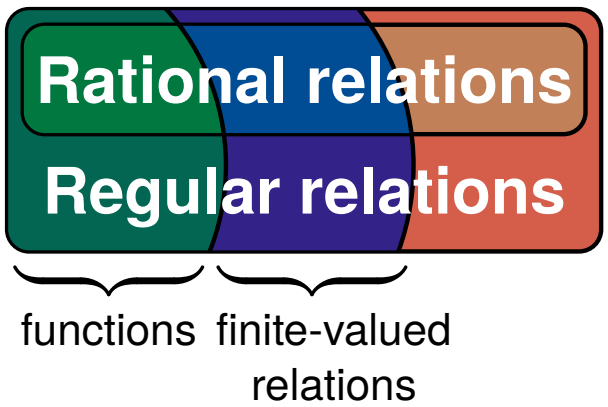
TRANSDUCERS

Theorem: We can decide in polynomial space whether a given SST defines a **finite-valued regular relation**

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**

$\Rightarrow$ The equivalence problem for finite-valued SST is in ELEMENTARY

$\Rightarrow$ Finite-valued **2-way** FST are as expressive as finite-valued SST



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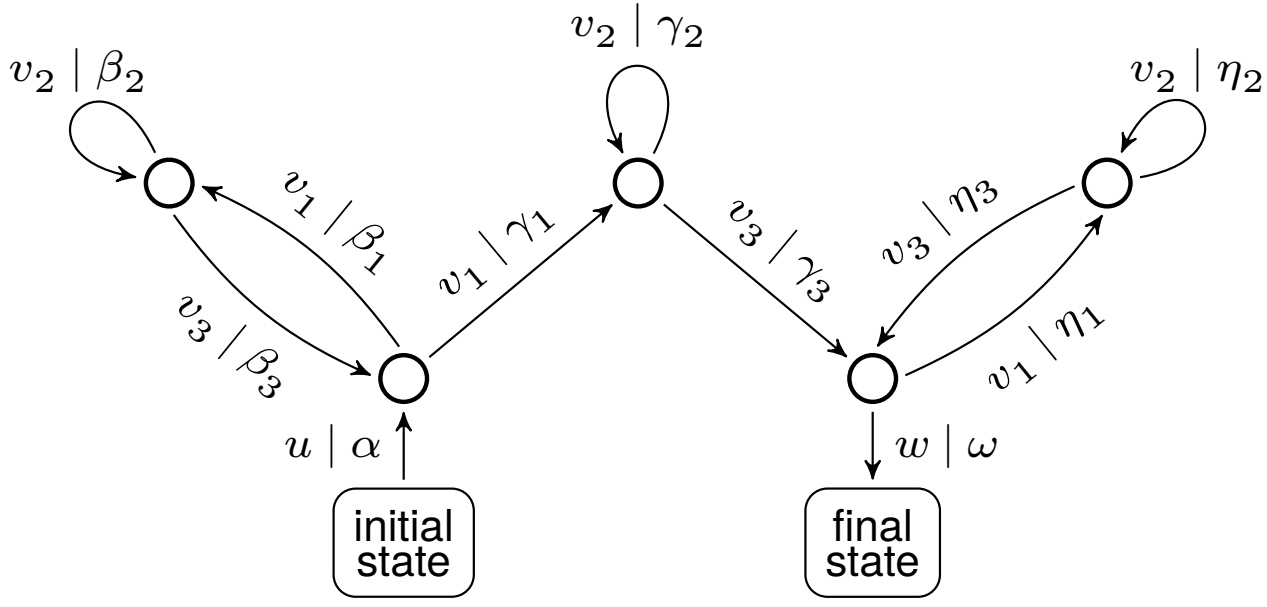
Deciding Finite Valuedness

Theorem: We can decide in polynomial space whether a given SST defines a **finite-valued regular relation**

Forbidden pattern: (inspired by [2008. De Souza])



- The relation recognised by the pattern is not 1-valued
- The substitutions produced on the loops have idempotent structure

The Decomposition Theorem

Theorem: Every **finite-valued regular relation** can be decomposed into a finite union of **regular functions**

Proof: (inspired by [2008. Sakarovitch, de Souza] and relying on [2023. FJLW])

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$\mathcal{T}$ that associates at most **k** outputs to each input

$\mathcal{T}_1, \ldots, \mathcal{T}_k$ that all associate at most **1** output to each input

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$\mathcal{T}$ that associates at most **k** outputs to each input

$\mathcal{T}'$ that has at most **k** runs on each input

$\mathcal{T}'_1, \ldots, \mathcal{T}'_k$ that all have at most **1** run on each input

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$\mathcal{T}'_i$ copies the ith lexicographically smallest run of $\mathcal{T}'$

$\mathcal{T}'_1, \ldots, \mathcal{T}'_k$ that all have at most **1** run on each input

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$\mathcal{T}'$ only keeps the runs of $\mathcal{T}$ that are **far** from each other

$\mathcal{T}'$ that has at most **k** runs on each input

$\mathcal{T}'_1, \ldots, \mathcal{T}'_k$ that all have at most **1** run on each input

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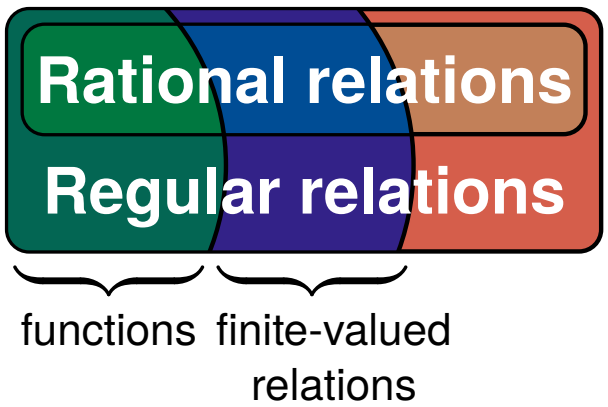

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