The tractive force that a railroad locomotive can develop depends upon the frictional resistance between the drive wheels and the rails. When the potential exists for wheel slip to occur, such as when a train travels upgrade over wet rails, sand is deposited on top of the railhead to increase this friction.

CHAPTER 8

Friction

Chapter 8 Friction

8.1 Introduction
8.2 The Laws of Dry Friction. Coefficients of Friction
8.3 Angles of Friction
8.4 Problems Involving Dry Friction
8.5 Wedges
8.6 Square-Threaded Screws
8.7 Journal Bearings. Axle Friction
8.8 Thrust Bearings. Disk Friction
8.9 Wheel Friction. Rolling Resistance
8.10 Belt Friction

8.1 INTRODUCTION

In the preceding chapters, it was assumed that surfaces in contact were either *frictionless* or *rough*. If they were frictionless, the force each surface exerted on the other was normal to the surfaces and the two surfaces could move freely with respect to each other. If they were rough, it was assumed that tangential forces could develop to prevent the motion of one surface with respect to the other.

This view was a simplified one. Actually, no perfectly frictionless surface exists. When two surfaces are in contact, tangential forces, called *friction forces,* will always develop if one attempts to move one surface with respect to the other. On the other hand, these friction forces are limited in magnitude and will not prevent motion if sufficiently large forces are applied. The distinction between frictionless and rough surfaces is thus a matter of degree. This will be seen more clearly in the present chapter, which is devoted to the study of friction and of its applications to common engineering situations.

There are two types of friction: *dry friction,* sometimes called *Coulomb friction,* and *fluid friction*. Fluid friction develops between layers of fluid moving at different velocities. Fluid friction is of great importance in problems involving the flow of fluids through pipes and orifices or dealing with bodies immersed in moving fluids. It is also basic in the analysis of the motion of *lubricated mechanisms*. Such problems are considered in texts on fluid mechanics. The present study is limited to dry friction, i.e., to problems involving rigid bodies which are in contact along *nonlubricated* surfaces.

In the first part of this chapter, the equilibrium of various rigid bodies and structures, assuming dry friction at the surfaces of contact, is analyzed. Later a number of specific engineering applications where dry friction plays an important role are considered: wedges, square-threaded screws, journal bearings, thrust bearings, rolling resistance, and belt friction.

8.2 THE LAWS OF DRY FRICTION. COEFFICIENTS OF FRICTION

The laws of dry friction are exemplified by the following experiment. A block of weight **W** is placed on a horizontal plane surface (Fig. 8.1*a*). The forces acting on the block are its weight **W** and the reaction of the surface. Since the weight has no horizontal component, the reaction of the surface also has no horizontal component; the reaction is therefore *normal* to the surface and is represented by **N** in Fig. 8.1*a*. Suppose, now, that a horizontal force **P** is applied to the block (Fig. 8.1*b*). If **P** is small, the block will not move; some other horizontal force must therefore exist, which balances **P**. This other force is the *static-friction force* **F**, which is actually the resultant of a great number of forces acting over the entire surface of contact between the block and the plane. The nature of these forces is not known exactly, but it is generally assumed that these forces are due to the irregularities of the surfaces in contact and, to a certain extent, to molecular attraction.

If the force **P** is increased, the friction force **F** also increases, continuing to oppose **P**, until its magnitude reaches a certain *maximum value* F_m (Fig. 8.1*c*). If **P** is further increased, the friction force

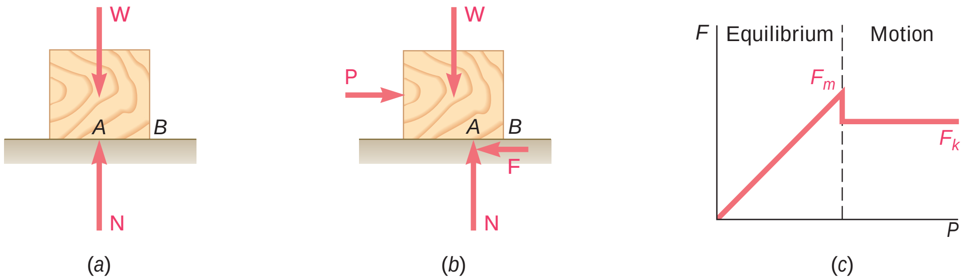


Fig. 8.1

cannot balance it any more and the block starts sliding.† As soon as the block has been set in motion, the magnitude of **F** drops from F_m to a lower value F_k. This is because there is less interpenetration between the irregularities of the surfaces in contact when these surfaces move with respect to each other. From then on, the block keeps sliding with increasing velocity while the friction force, denoted by $\mathbf{F}_k$ and called the *kinetic-friction force,* remains approximately constant.

Experimental evidence shows that the maximum value F_m of the static-friction force is proportional to the normal component N of the reaction of the surface. We have

$$\boxed{F_m = \mu_s N} \tag{8.1}$$

where μ_s is a constant called the *coefficient of static friction*. Similarly, the magnitude F_k of the kinetic-friction force may be put in the form

$$\boxed{F_k = \mu_k N} \tag{8.2}$$

where μ_k is a constant called the *coefficient of kinetic friction*. The coefficients of friction μ_s and μ_k do not depend upon the area of

†It should be noted that, as the magnitude F of the friction force increases from 0 to F_m, the point of application A of the resultant **N** of the normal forces of contact moves to the right, so that the couples formed, respectively, by **P** and **F** and by **W** and **N** remain balanced. If **N** reaches B before F reaches its maximum value F_m, the block will tip about B before it can start sliding (see Probs. 8.15 through 8.18).

the surfaces in contact. Both coefficients, however, depend strongly on the *nature* of the surfaces in contact. Since they also depend upon the exact condition of the surfaces, their value is seldom known with an accuracy greater than 5 percent. Approximate values of coefficients of static friction for various dry surfaces are given in Table 8.1. The corresponding values of the coefficient of kinetic friction would be about 25 percent smaller. Since coefficients of friction are dimensionless quantities, the values given in Table 8.1 can be used with both SI units and U.S. customary units.

TABLE 8.1 Approximate Values of Coefficient of Static Friction for Dry Surfaces

Metal on metal	0.15–0.60
Metal on wood	0.20–0.60
Metal on stone	0.30–0.70
Metal on leather	0.30–0.60
Wood on wood	0.25–0.50
Wood on leather	0.25–0.50
Stone on stone	0.40–0.70
Earth on earth	0.20–1.00
Rubber on concrete	0.60–0.90

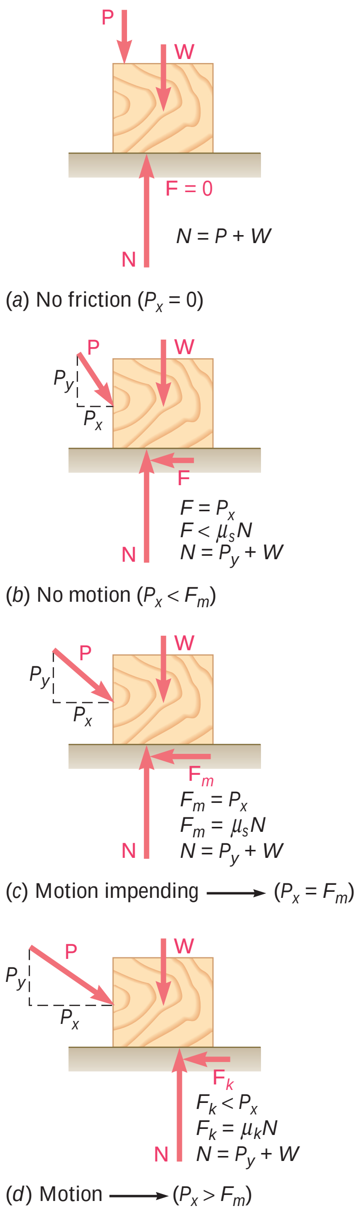


Fig. 8.2

From the description given above, it appears that four different situations can occur when a rigid body is in contact with a horizontal surface:

1. The forces applied to the body do not tend to move it along the surface of contact; there is no friction force (Fig. 8.2*a*).
2. The applied forces tend to move the body along the surface of contact but are not large enough to set it in motion. The friction force **F** which has developed can be found by solving the equations of equilibrium for the body. Since there is no evidence that **F** has reached its maximum value, the equation $F_m = \mu_s N$ *cannot be used* to determine the friction force (Fig. 8.2*b*).
3. The applied forces are such that the body is just about to slide. We say that *motion is impending*. The friction force **F** has reached its maximum value F_m and, together with the normal force **N**, balances the applied forces. Both the equations of equilibrium and the equation $F_m = \mu_s N$ *can be used*. We also note that the friction force has a sense opposite to the sense of impending motion (Fig. 8.2*c*).
4. The body is sliding under the action of the applied forces, and the equations of equilibrium do not apply any more. However, **F** is now equal to $\mathbf{F}_k$ and the equation $F_k = \mu_k N$ may be used. The sense of $\mathbf{F}_k$ is opposite to the sense of motion (Fig. 8.2*d*).

8.3 ANGLES OF FRICTION

It is sometimes convenient to replace the normal force **N** and the friction force **F** by their resultant **R**. Let us consider again a block of weight **W** resting on a horizontal plane surface. If no horizontal force is applied to the block, the resultant **R** reduces to the normal force **N** (Fig. 8.3*a*). However, if the applied force **P** has a horizontal component $\mathbf{P}_x$ which tends to move the block, the force **R** will have a horizontal component **F** and, thus, will form an angle ϕ with the normal to the surface (Fig. 8.3*b*). If $\mathbf{P}_x$ is increased until motion becomes impending, the angle between **R** and the vertical grows and reaches a maximum value (Fig. 8.3*c*). This value is called the *angle of static friction* and is denoted by ϕ_s. From the geometry of Fig. 8.3*c*, we note that

$$\tan \phi_s = \frac{F_m}{N} = \frac{\mu_s N}{N}$$

$$\boxed{\tan \phi_s = \mu_s} \qquad (8.3)$$

If motion actually takes place, the magnitude of the friction force drops to F_k; similarly, the angle ϕ between **R** and **N** drops to a lower value ϕ_k, called the *angle of kinetic friction* (Fig. 8.3*d*). From the geometry of Fig. 8.3*d*, we write

$$\tan \phi_k = \frac{F_k}{N} = \frac{\mu_k N}{N}$$

$$\boxed{\tan \phi_k = \mu_k} \qquad (8.4)$$

Another example will show how the angle of friction can be used to advantage in the analysis of certain types of problems. Consider a block resting on a board and subjected to no other force than its weight **W** and the reaction **R** of the board. The board can be given any desired inclination. If the board is horizontal, the force **R** exerted by the board on the block is perpendicular to the board and balances the weight **W** (Fig. 8.4*a*). If the board is given a small angle of inclination θ, the force **R** will deviate from the perpendicular to the board by the angle θ and will keep balancing **W** (Fig. 8.4*b*); it will then have a normal component **N** of magnitude $N = W \cos \theta$ and a tangential component **F** of magnitude $F = W \sin \theta$.

If we keep increasing the angle of inclination, motion will soon become impending. At that time, the angle between **R** and the normal will have reached its maximum value ϕ_s (Fig. 8.4*c*). The value of the angle of inclination corresponding to impending motion is called the *angle of repose*. Clearly, the angle of repose is equal to the angle of static friction ϕ_s. If the angle of inclination θ is further increased, motion starts and the angle between **R** and the normal drops to the lower value ϕ_k (Fig. 8.4*d*). The reaction **R** is not vertical any more, and the forces acting on the block are unbalanced.

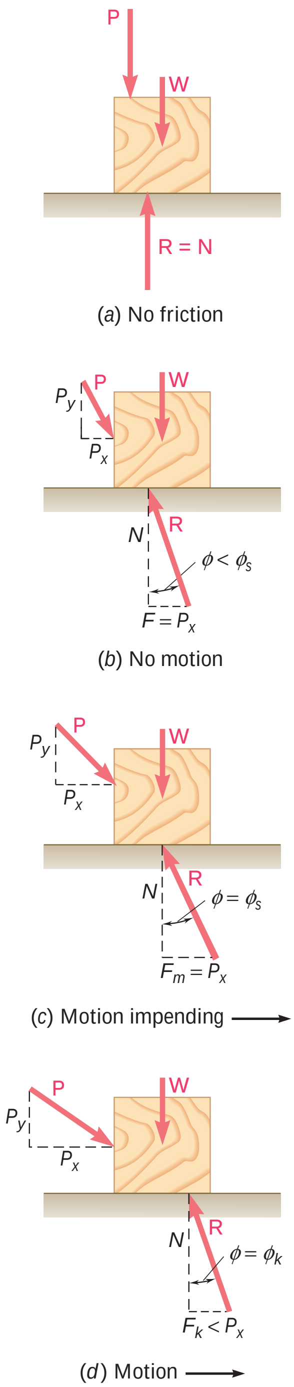
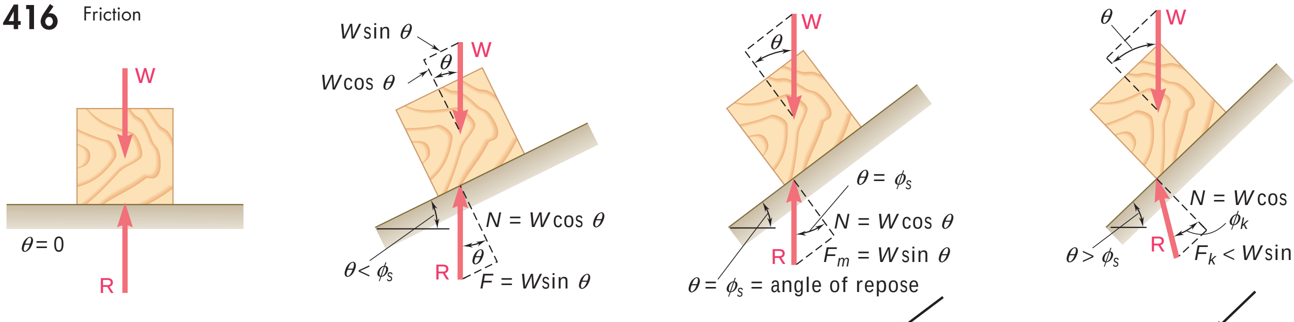


Fig. 8.3



(*a*) No friction (*b*) No motion (*c*) Motion impending (*d*) Motion

Fig. 8.4

Photo 8.1 The coefficient of static friction between a package and the inclined conveyer belt must be sufficiently large to enable the package to be transported without slipping.

8.4 PROBLEMS INVOLVING DRY FRICTION

Problems involving dry friction are found in many engineering applications. Some deal with simple situations such as the block sliding on a plane described in the preceding sections. Others involve more complicated situations as in Sample Prob. 8.3; many deal with the stability of rigid bodies in accelerated motion and will be studied in dynamics. Also, a number of common machines and mechanisms can be analyzed by applying the laws of dry friction. These include wedges, screws, journal and thrust bearings, and belt transmissions. They will be studied in the following sections.

The *methods* which should be used to solve problems involving dry friction are the same that were used in the preceding chapters. If a problem involves only a motion of translation, with no possible rotation, the body under consideration can usually be treated as a particle, and the methods of Chap. 2 used. If the problem involves a possible rotation, the body must be considered as a rigid body, and the methods of Chap. 4 should be used. If the structure considered is made of several parts, the principle of action and reaction must be used as was done in Chap. 6.

If the body considered is acted upon by more than three forces (including the reactions at the surfaces of contact), the reaction at each surface will be represented by its components **N** and **F** and the problem will be solved from the equations of equilibrium. If only three forces act on the body under consideration, it may be more convenient to represent each reaction by the single force **R** and to solve the problem by drawing a force triangle.

Most problems involving friction fall into one of the following *three groups:* In the *first group* of problems, all applied forces are given and the coefficients of friction are known; we are to determine whether the body considered will remain at rest or slide. The friction force **F** *required to maintain equilibrium* is unknown (its magnitude is *not* equal to $\mu_s N$) and should be determined, together with the normal force **N**, by drawing a free-body diagram and *solving the equations of equilibrium* (Fig. 8.5*a*). The value found for the magnitude F of the friction force is then compared with the maximum value $F_m = \mu_s N$. If F is smaller than or equal to F_m, the body remains at rest. If the value found for F is larger than F_m, equilibrium cannot be maintained and motion takes place; the actual magnitude of the friction force is then $F_k = \mu_k N$.

In problems of the *second group,* all applied forces are given and the motion is known to be impending; we are to determine the value of the coefficient of static friction. Here again, we determine the friction force and the normal force by drawing a free-body diagram and solving the equations of equilibrium (Fig. 8.5*b*). Since we know that the value found for F is the maximum value F_m, the coefficient of friction may be found by writing and solving the equation $F_m = \mu_s N$.

In problems of the *third group,* the coefficient of static friction is given, and it is known that the motion is impending in a given direction; we are to determine the magnitude or the direction of one of the applied forces. The friction force should be shown in the free-body diagram with a *sense opposite to that of the impending motion* and with a magnitude $F_m = \mu_s N$ (Fig. 8.5*c*). The equations of equilibrium can then be written, and the desired force determined.

As noted above, when only three forces are involved it may be more convenient to represent the reaction of the surface by a single force **R** and to solve the problem by drawing a force triangle. Such a solution is used in Sample Prob. 8.2.

When two bodies A and B are in contact (Fig. 8.6*a*), the forces of friction exerted, respectively, by A on B and by B on A are equal and opposite (Newton's third law). In drawing the free-body diagram of one of the bodies, it is important to include the appropriate friction force with its correct sense. The following rule should then be observed: *The sense of the friction force acting on A is opposite to that of the motion (or impending motion) of A as observed from B* (Fig. 8.6*b*).† The sense of the friction force acting on B is determined in a similar way (Fig. 8.6*c*). Note that the motion of A as observed from B is a *relative motion*. For example, if body A is fixed and body B moves, body A will have a relative motion with respect to B. Also, if both B and A are moving down but B is moving faster than A, body A will be observed, from B, to be moving up.

†It is therefore *the same as that of the motion of B as observed from A.*

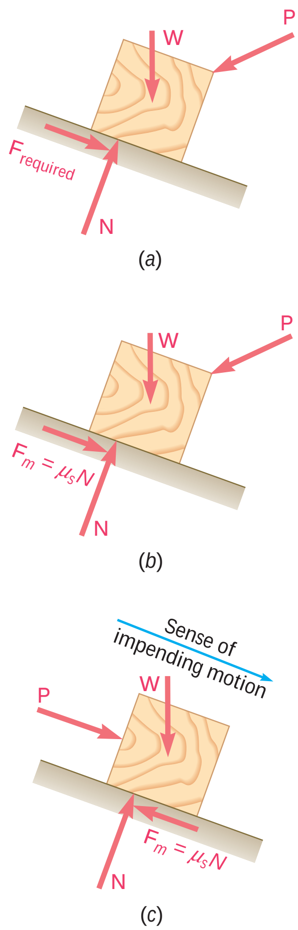


Fig. 8.5

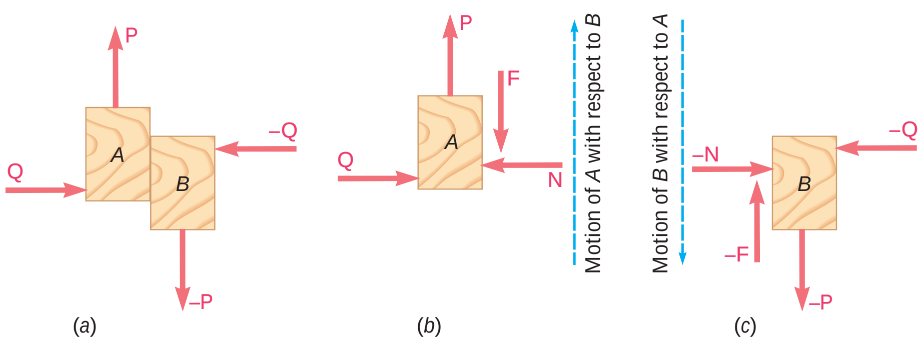
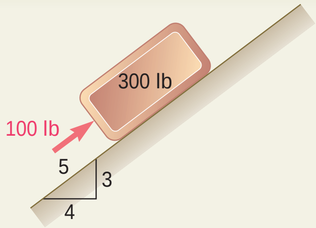


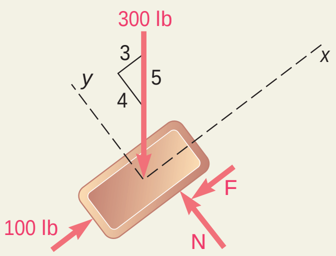
Fig. 8.6



SAMPLE PROBLEM 8.1

A 100-lb force acts as shown on a 300-lb block placed on an inclined plane. The coefficients of friction between the block and the plane are $\mu_s = 0.25$ and $\mu_k = 0.20$. Determine whether the block is in equilibrium, and find the value of the friction force.

SOLUTION



Force Required for Equilibrium. We first determine the value of the friction force *required to maintain equilibrium*. Assuming that **F** is directed down and to the left, we draw the free-body diagram of the block and write

$$+\nearrow \Sigma F_x = 0: \qquad 100 \text{ lb} - \tfrac{3}{5}(300 \text{ lb}) - F = 0$$
$$F = -80 \text{ lb} \qquad \mathbf{F} = 80 \text{ lb} \nearrow$$

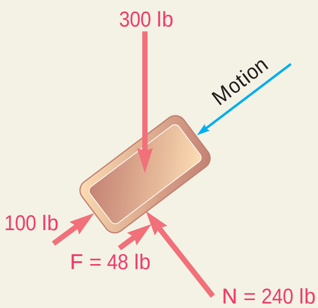
$$+\nwarrow \Sigma F_y = 0: \qquad N - \tfrac{4}{5}(300 \text{ lb}) = 0$$
$$N = +240 \text{ lb} \qquad \mathbf{N} = 240 \text{ lb} \nwarrow$$

The force **F** required to maintain equilibrium is an 80-lb force directed up and to the right; the tendency of the block is thus to move down the plane.

Maximum Friction Force. The magnitude of the maximum friction force which may be developed is

$$F_m = \mu_s N \qquad F_m = 0.25(240 \text{ lb}) = 60 \text{ lb}$$

Since the value of the force required to maintain equilibrium (80 lb) is larger than the maximum value which may be obtained (60 lb), equilibrium will not be maintained and *the block will slide down the plane.*



Actual Value of Friction Force. The magnitude of the actual friction force is obtained as follows:

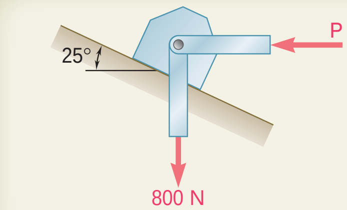
$$F_{\text{actual}} = F_k = \mu_k N$$
$$= 0.20(240 \text{ lb}) = 48 \text{ lb}$$

The sense of this force is opposite to the sense of motion; the force is thus directed up and to the right:

$$\mathbf{F}_{\text{actual}} = 48 \text{ lb} \nearrow \quad \blacktriangleleft$$

It should be noted that the forces acting on the block are not balanced; the resultant is

$$\tfrac{3}{5}(300 \text{ lb}) - 100 \text{ lb} - 48 \text{ lb} = 32 \text{ lb} \swarrow$$



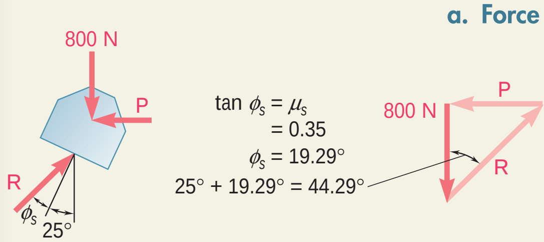
SAMPLE PROBLEM 8.2

A support block is acted upon by two forces as shown. Knowing that the coefficients of friction between the block and the incline are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine the force **P** required (*a*) to start the block moving up the incline, (*b*) to keep it moving up, (*c*) to prevent it from sliding down.

SOLUTION

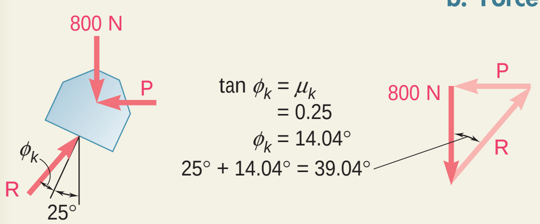
Free-Body Diagram. For each part of the problem we draw a free-body diagram of the block and a force triangle including the 800-N vertical force, the horizontal force **P**, and the force **R** exerted on the block by the incline. The direction of **R** must be determined in each separate case. We note that since **P** is perpendicular to the 800-N force, the force triangle is a right triangle, which can easily be solved for **P**. In most other problems, however, the force triangle will be an oblique triangle and should be solved by applying the law of sines.

a. Force *P* to Start Block Moving Up



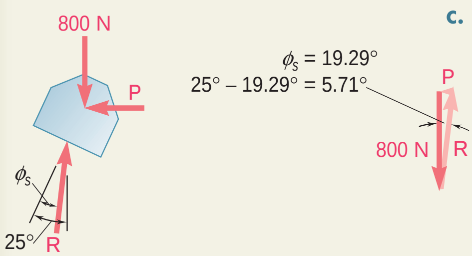
$$P = (800 \text{ N}) \tan 44.29°$$ **P** = 780 N z

b. Force *P* to Keep Block Moving Up

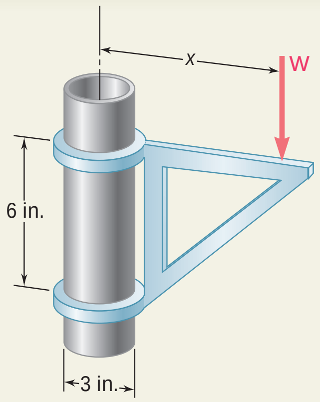


$$P = (800 \text{ N}) \tan 39.04°$$ **P** = 649 N z

c. Force *P* to Prevent Block from Sliding Down

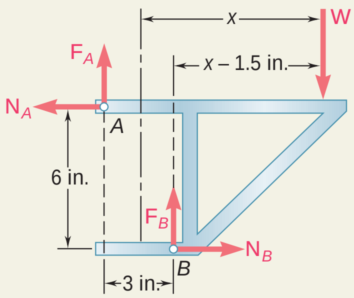


$$P = (800 \text{ N}) \tan 5.71°$$ **P** = 80.0 N z



SAMPLE PROBLEM 8.3

The movable bracket shown may be placed at any height on the 3-in.-diameter pipe. If the coefficient of static friction between the pipe and bracket is 0.25, determine the minimum distance x at which the load **W** can be supported. Neglect the weight of the bracket.



SOLUTION

Free-Body Diagram. We draw the free-body diagram of the bracket. When **W** is placed at the minimum distance x from the axis of the pipe, the bracket is just about to slip, and the forces of friction at A and B have reached their maximum values:

$$F_A = \mu_s N_A = 0.25\ N_A$$
$$F_B = \mu_s N_B = 0.25\ N_B$$

Equilibrium Equations

$\overset{+}{\rightarrow}\Sigma F_x = 0$: $\quad N_B - N_A = 0$

$$N_B = N_A$$

$+\uparrow\Sigma F_y = 0$: $\quad F_A + F_B - W = 0$

$$0.25N_A + 0.25N_B = W$$

And, since N_B has been found equal to N_A,

$$0.50N_A = W$$
$$N_A = 2W$$

$+\circlearrowleft\Sigma M_B = 0$: $\quad N_A(6\text{ in.}) - F_A(3\text{ in.}) - W(x - 1.5\text{ in.}) = 0$

$$6N_A - 3(0.25N_A) - Wx + 1.5W = 0$$
$$6(2W) - 0.75(2W) - Wx + 1.5W = 0$$

Dividing through by W and solving for x,

$$x = 12\text{ in.}$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you studied and applied the *laws of dry friction*. Previously you had encountered only (*a*) frictionless surfaces that could move freely with respect to each other, (*b*) rough surfaces that allowed no motion relative to each other.

A. In solving problems involving dry friction, you should keep the following in mind.

1. The reaction R exerted by a surface on a free body can be resolved into a component **N** and a tangential component **F**. The tangential component is known as the *friction force*. When a body is in contact with a fixed surface the direction of the friction force **F** is opposite to that of the actual or impending motion of the body.

a. No motion will occur as long as F does not exceed the maximum value $F_m = \mu_s N$, where μ_s is the *coefficient of static friction*.

b. Motion will occur if a value of F larger than F_m is required to maintain equilibrium. As motion takes place, the actual value of F drops to $F_k = \mu_k N$, where μ_k is the *coefficient of kinetic friction* [Sample Prob. 8.1].

2. When only three forces are involved an alternative approach to the analysis of friction may be preferred [Sample Prob. 8.2]. The reaction **R** is defined by its magnitude R and the angle ϕ it forms with the normal to the surface. No motion will occur as long as ϕ does not exceed the maximum value ϕ_s, where $\tan \phi_s = \mu_s$. Motion will occur if a value of ϕ larger than ϕ_s is required to maintain equilibrium, and the actual value of ϕ will drop to ϕ_k, where $\tan \phi_k = \mu_k$.

3. When two bodies are in contact the sense of the actual or impending relative motion at the point of contact must be determined. On each of the two bodies a friction force **F** should be shown in a direction opposite to that of the actual or impending motion of the body as seen from the other body.

(continued)

B. Methods of solution. The first step in your solution is to *draw a free-body diagram* of the body under consideration, resolving the force exerted on each surface where friction exists into a normal component **N** and a friction force **F**. If several bodies are involved, draw a free-body diagram of each of them, labeling and directing the forces at each surface of contact as you learned to do when analyzing frames in Chap. 6.

The problem you have to solve may fall in one of the following three categories:

1. All the applied forces and the coefficients of friction are known, and you must determine whether equilibrium is maintained. Note that in this situation the friction force is unknown and *cannot be assumed to be equal* to $\mu_s N$.

a. Write the equations of equilibrium to determine *N* and *F*.

b. Calculate the maximum allowable friction force, $F_m = \mu_s N$. If $F \leq F_m$, equilibrium is maintained. If $F > F_m$, motion occurs, and the magnitude of the friction force is $F_k = \mu_k N$ [Sample Prob. 8.1].

2. All the applied forces are known, and you must find the smallest allowable value of μ_s for which equilibrium is maintained. You will assume that motion is impending and determine the corresponding value of μ_s.

a. Write the equations of equilibrium to determine *N* and *F*.

b. Since motion is impending, $F = F_m$. Substitute the values found for N and F into the equation $F_m = \mu_s N$ and solve for μ_s.

3. The motion of the body is impending and μ_s is known; you must find some unknown quantity, such as a distance, an angle, the magnitude of a force, or the direction of a force.

a. Assume a possible motion of the body and, on the free-body diagram, draw the friction force in a direction opposite to that of the assumed motion.

b. Since motion is impending, $F = F_m = \mu_s N$. Substituting for μ_s its known value, you can express F in terms of N on the free-body diagram, thus eliminating one unknown.

c. Write and solve the equilibrium equations for the unknown you seek [Sample Prob. 8.3].

PROBLEMS

FREE BODY PRACTICE PROBLEMS

8.F1 Draw the free-body diagram needed to determine the smallest force **P** for which equilibrium of the 7.5-kg block is maintained.

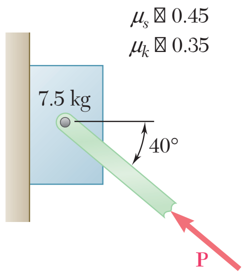


Fig. P8.F1

8.F2 Two blocks *A* and *B* are connected by a cable as shown. Knowing that the coefficient of static friction at all surfaces of contact is 0.30 and neglecting the friction of the pulleys, draw the free-body diagrams needed to determine the smallest force **P** required to move the blocks.

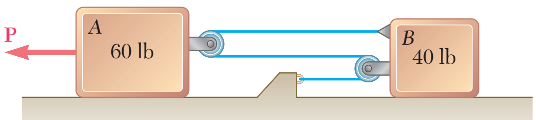


Fig. P8.F2

8.F3 The cylinder shown is of weight *W* and radius *r*, and the coefficient of static friction m_s is the same at *A* and *B*. Draw the free-body diagram needed to determine the largest couple **M** that can be applied to the cylinder if it is not to rotate.

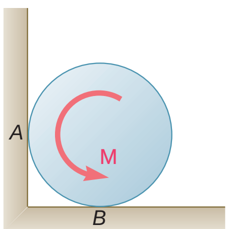


Fig. P8.F3

8.F4 A uniform crate of mass 30 kg must be moved up along the 15° incline without tipping. Knowing that the force **P** is horizontal, draw the free-body diagram needed to determine the largest allowable coefficient of static friction between the crate and the incline, and the corresponding force **P**.

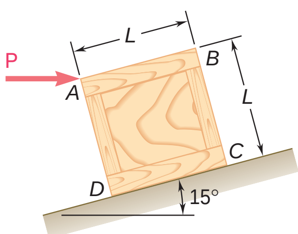


Fig. P8.F4

END-OF-SECTION PROBLEMS

8.1 Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when u = 25° and P = 750 N.

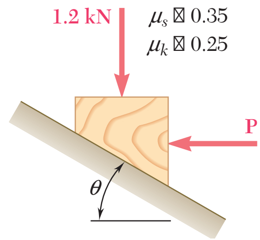


Fig. P8.1 and P8.2

8.2 Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when u = 30° and P = 150 N.

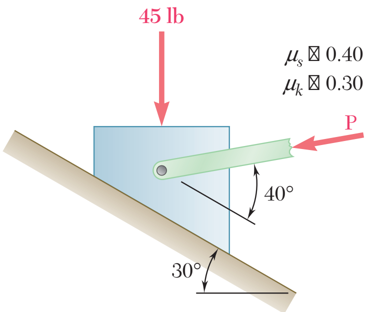


Fig. P8.3, P8.4, and P8.5

8.3 Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when P = 100 lb.

8.4 Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when P = 60 lb.

8.5 Determine the smallest value of P required to (*a*) start the block up the incline, (*b*) keep it moving up, (*c*) prevent it from moving down.

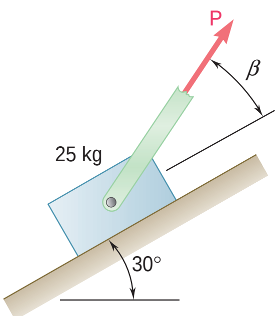


Fig. P8.6

8.6 Knowing that the coefficient of friction between the 25-kg block and the incline is m$_s$ = 0.25, determine (*a*) the smallest value of P required to start the block moving up the incline, (*b*) the corresponding value of b.

8.7 The 80-lb block is attached to link AB and rests on a moving belt. Knowing that m$_s$ = 0.25 and m$_k$ = 0.20, determine the magnitude of the horizontal force **P** that should be applied to the belt to maintain its motion (*a*) to the right, (*b*) to the left.

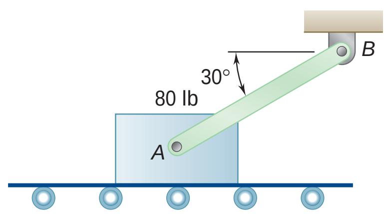


Fig. *P8.7*

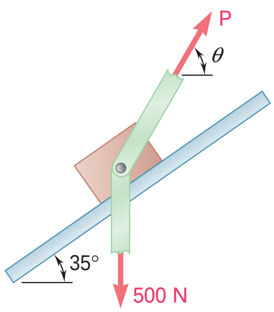


Fig. P8.8

8.8 The coefficients of friction between the block and the rail are m$_s$ = 0.30 and m$_k$ = 0.25. Knowing that u = 65°, determine the smallest value of P required (*a*) to start the block moving up the rail, (*b*) to keep it from moving down.

8.9 Considering only values of u less than 90°, determine the smallest value of u required to start the block moving to the right when (*a*) $W = 75$ lb, (*b*) $W = 100$ lb.

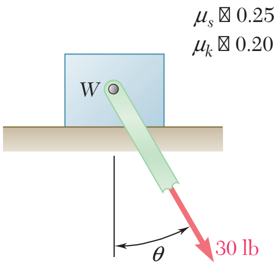


Fig. *P8.9*

8.10 Determine the range of values of P for which equilibrium of the block shown is maintained.

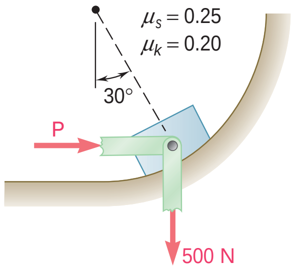


Fig. P8.10

8.11 The 20-lb block A and the 30-lb block B are supported by an incline that is held in the position shown. Knowing that the coefficient of static friction is 0.15 between the two blocks and zero between block B and the incline, determine the value of u for which motion is impending.

8.12 The 20-lb block A and the 30-lb block B are supported by an incline that is held in the position shown. Knowing that the coefficient of static friction is 0.15 between all surfaces of contact, determine the value of u for which motion is impending.

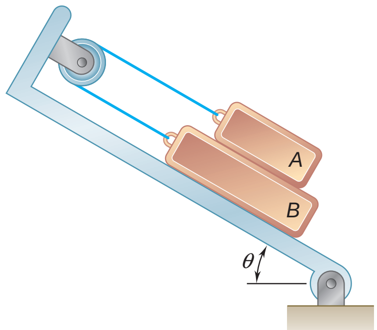


Fig. P8.11 and P8.12

8.13 and 8.14 The coefficients of friction are $m_s = 0.40$ and $m_k = 0.30$ between all surfaces of contact. Determine the smallest force **P** required to start the 30-kg block moving if cable AB (*a*) is attached as shown, (*b*) is removed.

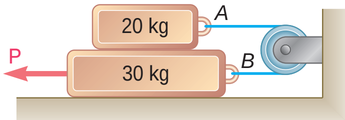


Fig. P8.13

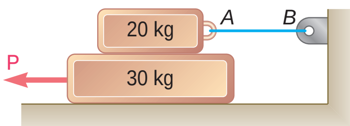


Fig. P8.14

8.15 A 40-kg packing crate must be moved to the left along the floor without tipping. Knowing that the coefficient of static friction between the crate and the floor is 0.35, determine (*a*) the largest allowable value of a, (*b*) the corresponding magnitude of the force **P**.

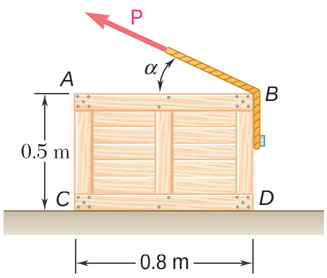


Fig. *P8.15* and *P8.16*

8.16 A 40-kg packing crate is pulled by a rope as shown. The coefficient of static friction between the crate and the floor is 0.35. If $a = 40°$, determine (*a*) the magnitude of the force **P** required to move the crate, (*b*) whether the crate will slide or tip.

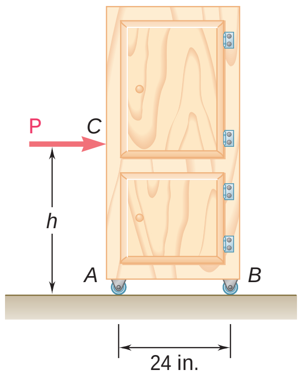


Fig. P8.17 and P8.18

8.17 A 120-lb cabinet is mounted on casters that can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.30. If $h = 32$ in., determine the magnitude of the force **P** required to move the cabinet to the right (*a*) if all casters are locked, (*b*) if the casters at *B* are locked and the casters at *A* are free to rotate, (*c*) if the casters at *A* are locked and the casters at *B* are free to rotate.

8.18 A 120-lb cabinet is mounted on casters that can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.30. Assuming that the casters at both *A* and *B* are locked, determine (*a*) the force **P** required to move the cabinet to the right, (*b*) the largest allowable value of *h* if the cabinet is not to tip over.

8.19 Wire is being drawn at a constant rate from a spool by applying a vertical force **P** to the wire as shown. The spool and the wire wrapped on the spool have a combined weight of 20 lb. Knowing that the coefficients of friction at both *A* and *B* are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine the required magnitude of the force **P**.

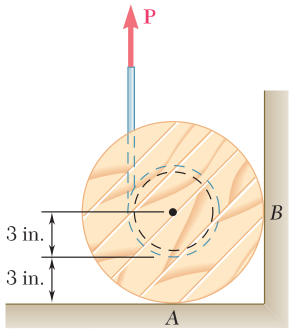


Fig. P8.19

8.20 Solve Prob. 8.19 assuming that the coefficients of friction at *B* are zero.

8.21 The hydraulic cylinder shown exerts a force of 3 kN directed to the right on point *B* and to the left on point *E*. Determine the magnitude of the couple **M** required to rotate the drum clockwise at a constant speed.

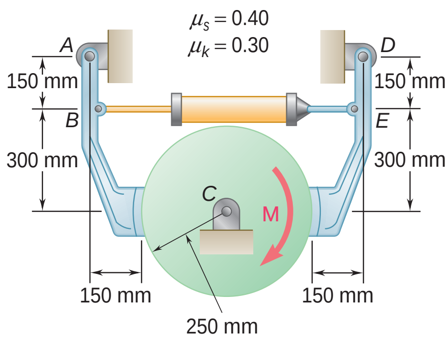


Fig. P8.21 and P8.22

8.22 A couple **M** of magnitude 100 N · m is applied to the drum as shown. Determine the smallest force that must be exerted by the hydraulic cylinder on joints *B* and *E* if the drum is not to rotate.

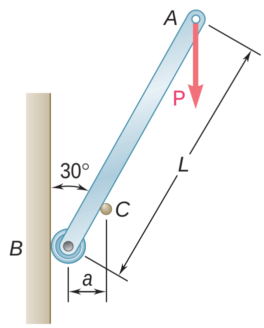


Fig. P8.23

8.23 A slender rod of length *L* is lodged between peg *C* and the vertical wall, and supports a load **P** at end *A*. Knowing that the coefficient of static friction between the peg and the rod is 0.15 and neglecting friction at the roller, determine the range of values of the ratio *L*/*a* for which equilibrium is maintained.

8.24 Solve Prob. 8.23 assuming that the coefficient of static friction between the peg and the rod is 0.60.

8.25 A 6.5-m ladder *AB* leans against a wall as shown. Assuming that the coefficient of static friction μ_s is zero at *B*, determine the smallest value of μ_s at *A* for which equilibrium is maintained.

8.26 A 6.5-m ladder *AB* leans against a wall as shown. Assuming that the coefficient of static friction μ_s is the same at *A* and *B*, determine the smallest value of μ_s for which equilibrium is maintained.

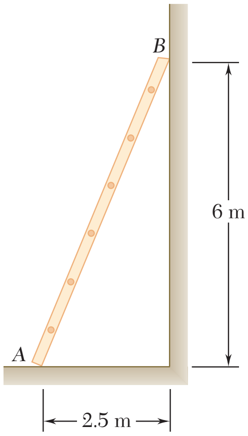


Fig. P8.25 and *P8.26*

8.27 The press shown is used to emboss a small seal at *E*. Knowing that the coefficient of static friction between the vertical guide and the embossing die *D* is 0.30, determine the force exerted by the die on the seal.

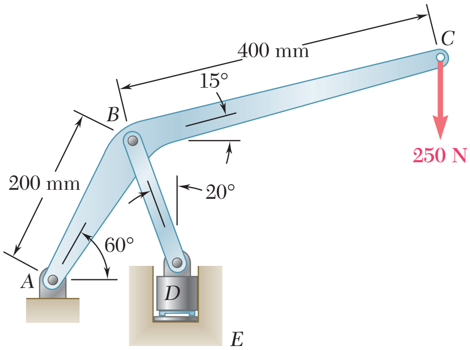


Fig. P8.27

8.28 The machine base shown has a mass of 75 kg and is fitted with skids at *A* and *B*. The coefficient of static friction between the skids and the floor is 0.30. If a force **P** of magnitude 500 N is applied at corner *C*, determine the range of values of θ for which the base will not move.

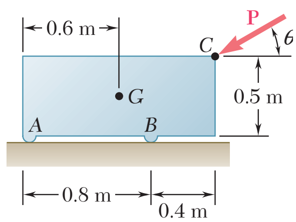


Fig. *P8.28*

8.29 The 50-lb plate *ABCD* is attached at *A* and *D* to collars that can slide on the vertical rod. Knowing that the coefficient of static friction is 0.40 between both collars and the rod, determine whether the plate is in equilibrium in the position shown when the magnitude of the vertical force applied at *E* is (*a*) $P = 0$, (*b*) $P = 20$ lb.

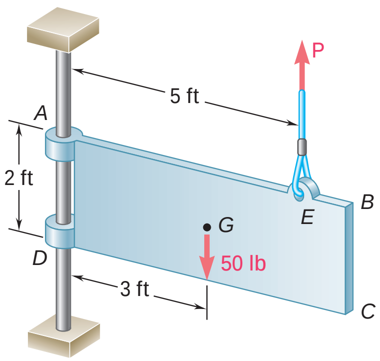


Fig. P8.29

8.30 In Prob. 8.29, determine the range of values of the magnitude *P* of the vertical force applied at *E* for which the plate will move downward.

8.31 A rod DE and a small cylinder are placed between two guides as shown. The rod is not to slip downward, however large the force **P** may be; i.e., the arrangement is said to be self-locking. Neglecting the weight of the cylinder, determine the minimum allowable coefficients of static friction at A, B, and C.

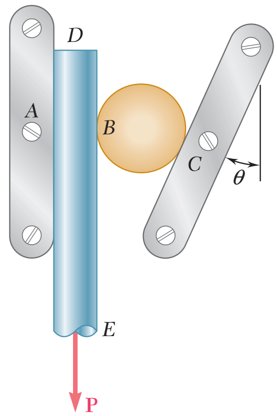


Fig. P8.31

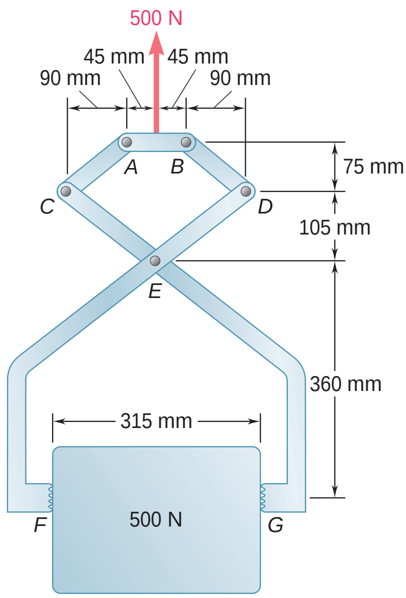


Fig. P8.32

8.32 A 500-N concrete block is to be lifted by the pair of tongs shown. Determine the smallest allowable value of the coefficient of static friction between the block and the tongs at F and G.

8.33 The 100-mm-radius cam shown is used to control the motion of the plate CD. Knowing that the coefficient of static friction between the cam and the plate is 0.45 and neglecting friction at the roller supports, determine (*a*) the force **P** required to maintain the motion of the plate, knowing that the plate is 20 mm thick, (*b*) the largest thickness of the plate for which the mechanism is self-locking (i.e., for which the plate cannot be moved however large the force **P** may be).

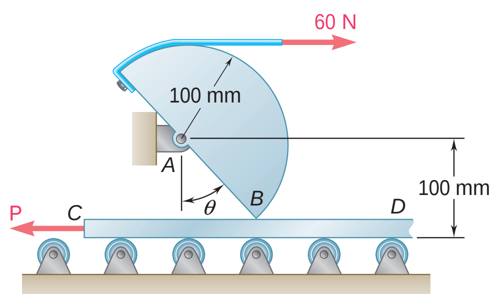


Fig. P8.33

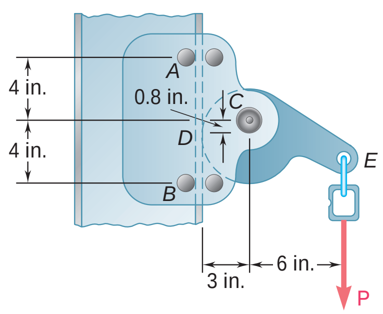


Fig. P8.34

8.34 A safety device used by workers climbing ladders fixed to high structures consists of a rail attached to the ladder and a sleeve that can slide on the flange of the rail. A chain connects the worker's belt to the end of an eccentric cam that can be rotated about an axle attached to the sleeve at C. Determine the smallest allowable common value of the coefficient of static friction between the flange of the rail, the pins at A and B, and the eccentric cam if the sleeve is not to slide down when the chain is pulled vertically downward.

8.35 To be of practical use, the safety sleeve described in Prob. 8.34 must be free to slide along the rail when pulled upward. Determine the largest allowable value of the coefficient of static friction between the flange of the rail and the pins at A and B if the sleeve is to be free to slide when pulled as shown in the figure, assuming (a) u = 60°, (b) u = 50°, (c) u = 40°.

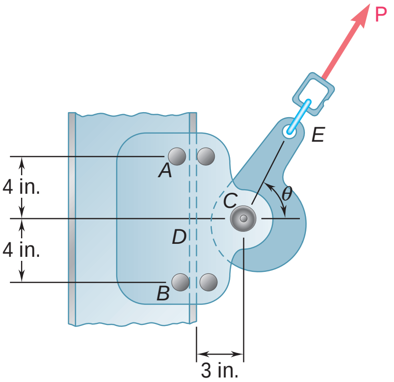


Fig. P8.35

8.36 Two 10-lb blocks A and B are connected by a slender rod of negligible weight. The coefficient of static friction is 0.30 between all surfaces of contact, and the rod forms an angle u = 30° with the vertical. (a) Show that the system is in equilibrium when $P = 0$. (b) Determine the largest value of P for which equilibrium is maintained.

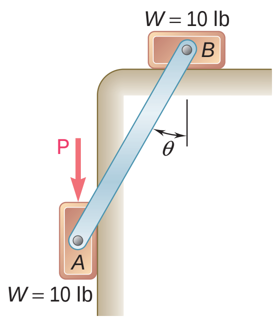


Fig. P8.36

8.37 Bar AB is attached to collars that can slide on the inclined rods shown. A force **P** is applied at point D located at a distance a from end A. Knowing that the coefficient of static friction m$_s$ between each collar and the rod upon which it slides is 0.30 and neglecting the weights of the bar and of the collars, determine the smallest value of the ratio a/L for which equilibrium is maintained.

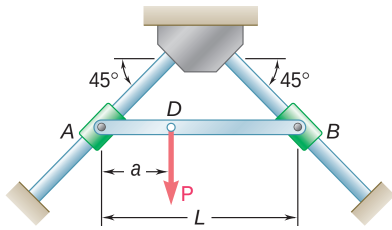


Fig. P8.37

8.38 Two identical uniform boards, each of weight 40 lb, are temporarily leaned against each other as shown. Knowing that the coefficient of static friction between all surfaces is 0.40, determine (a) the largest magnitude of the force **P** for which equilibrium will be maintained, (b) the surface at which motion will impend.

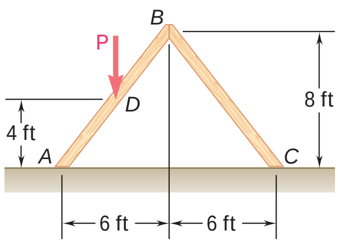


Fig. *P8.38*

8.39 Knowing that the coefficient of static friction between the collar and the rod is 0.35, determine the range of values of P for which equilibrium is maintained when u = 50° and $M = 20$ N · m.

8.40 Knowing that the coefficient of static friction between the collar and the rod is 0.40, determine the range of values of M for which equilibrium is maintained when u = 60° and $P = 200$ N.

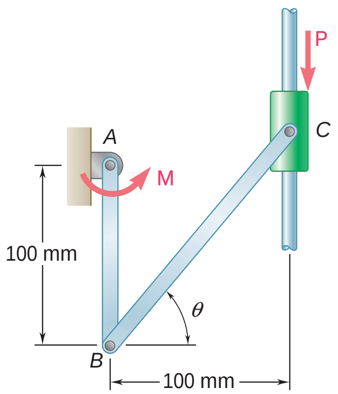


Fig. P8.39 and P8.40

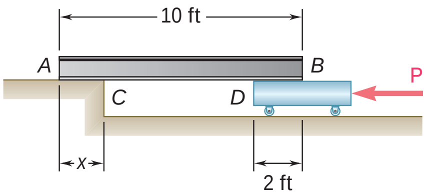


Fig. P8.41

8.41 A 10-ft beam, weighing 1200 lb, is to be moved to the left onto the platform. A horizontal force **P** is applied to the dolly, which is mounted on frictionless wheels. The coefficients of friction between all surfaces are $\mu_s = 0.30$ and $\mu_k = 0.25$, and initially $x = 2$ ft. Knowing that the top surface of the dolly is slightly higher than the platform, determine the force **P** required to start moving the beam. (*Hint:* The beam is supported at *A* and *D*.)

8.42 (*a*) Show that the beam of Prob. 8.41 *cannot* be moved if the top surface of the dolly is slightly *lower* than the platform. (*b*) Show that the beam *can* be moved if two 175-lb workers stand on the beam at *B* and determine how far to the left the beam can be moved.

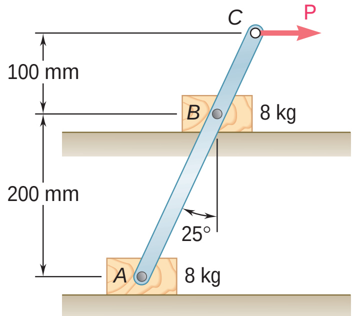


Fig. P8.43

8.43 Two 8-kg blocks *A* and *B* resting on shelves are connected by a rod of negligible mass. Knowing that the magnitude of a horizontal force **P** applied at *C* is slowly increased from zero, determine the value of *P* for which motion occurs, and what that motion is, when the coefficient of static friction between all surfaces is (*a*) $\mu_s = 0.40$, (*b*) $\mu_s = 0.50$.

8.44 A slender steel rod of length 225 mm is placed inside a pipe as shown. Knowing that the coefficient of static friction between the rod and the pipe is 0.20, determine the largest value of θ for which the rod will not fall into the pipe.

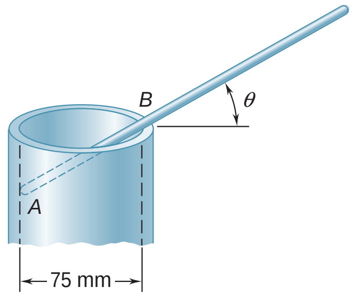


Fig. P8.44

8.45 In Prob. 8.44, determine the smallest value of θ for which the rod will not fall out of the pipe.

8.46 Two slender rods of negligible weight are pin-connected at *C* and attached to blocks *A* and *B*, each of weight *W*. Knowing that $\theta = 80°$ and that the coefficient of static friction between the blocks and the horizontal surface is 0.30, determine the largest value of *P* for which equilibrium is maintained.

8.47 Two slender rods of negligible weight are pin-connected at *C* and attached to blocks *A* and *B*, each of weight *W*. Knowing that $P = 1.260W$ and that the coefficient of static friction between the blocks and the horizontal surface is 0.30, determine the range of values of θ, between 0 and 180°, for which equilibrium is maintained.

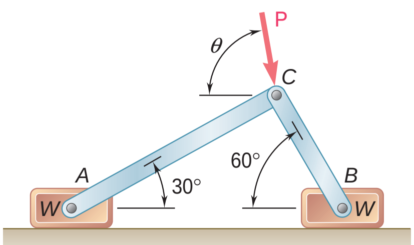


Fig. P8.46 and P8.47

8.5 WEDGES

Wedges are simple machines used to raise large stone blocks and other heavy loads. These loads can be raised by applying to the wedge a force usually considerably smaller than the weight of the load. In addition, because of the friction between the surfaces in contact, a properly shaped wedge will remain in place after being forced under the load. Wedges can thus be used advantageously to make small adjustments in the position of heavy pieces of machinery.

Consider the block *A* shown in Fig. 8.7*a*. This block rests against a vertical wall *B* and is to be raised slightly by forcing a wedge *C* between block *A* and a second wedge *D*. We want to find the minimum value of the force **P** which must be applied to the wedge *C* to move the block. It will be assumed that the weight **W** of the block is known, either given in pounds or determined in newtons from the mass of the block expressed in kilograms.

The free-body diagrams of block *A* and of wedge *C* have been drawn in Fig. 8.7*b* and *c*. The forces acting on the block include its weight and the normal and friction forces at the surfaces of contact with wall *B* and wedge *C*. The magnitudes of the friction forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal, respectively, to $\mu_s N_1$ and $\mu_s N_2$ since the motion of the block must be started. It is important to show the friction forces with their correct sense. Since the block will move upward, the force $\mathbf{F}_1$ exerted by the wall on the block must be directed downward. On the other hand, since the wedge *C* moves to the right, the relative motion of *A* with respect to *C* is to the left and the force $\mathbf{F}_2$ exerted by *C* on *A* must be directed to the right.

Considering now the free body *C* in Fig. 8.7*c*, we note that the forces acting on *C* include the applied force **P** and the normal and friction forces at the surfaces of contact with *A* and *D*. The weight of the wedge is small compared with the other forces involved and can be neglected. The forces exerted by *A* on *C* are equal and opposite to the forces $\mathbf{N}_2$ and $\mathbf{F}_2$ exerted by *C* on *A* and are denoted, respectively, by $-\mathbf{N}_2$ and $-\mathbf{F}_2$; the friction force $-\mathbf{F}_2$ must therefore be directed to the left. We check that the force $\mathbf{F}_3$ exerted by *D* is also directed to the left.

The total number of unknowns involved in the two free-body diagrams can be reduced to four if the friction forces are expressed in terms of the normal forces. Expressing that block *A* and wedge *C* are in equilibrium will provide four equations which can be solved to obtain the magnitude of **P**. It should be noted that in the example considered here, it will be more convenient to replace each pair of normal and friction forces by their resultant. Each free body is then subjected to only three forces, and the problem can be solved by drawing the corresponding force triangles (see Sample Prob. 8.4).

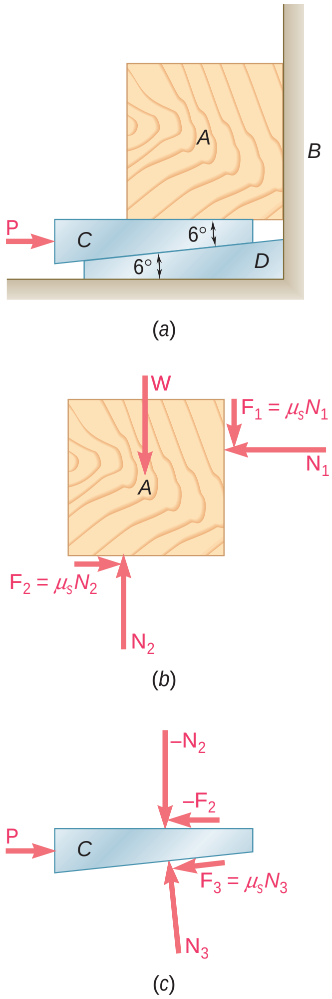


Fig. 8.7

8.6 SQUARE-THREADED SCREWS

Square-threaded screws are frequently used in jacks, presses, and other mechanisms. Their analysis is similar to the analysis of a block sliding along an inclined plane.

Consider the jack shown in Fig. 8.8. The screw carries a load **W** and is supported by the base of the jack. Contact between screw and base takes place along a portion of their threads. By applying a force **P** on the handle, the screw can be made to turn and to raise the load **W**.

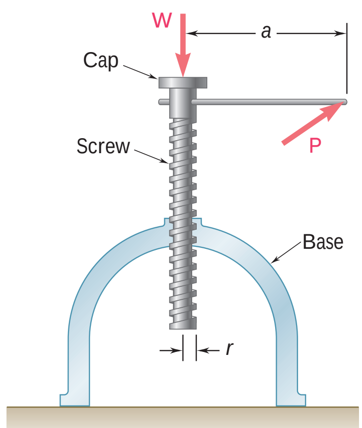
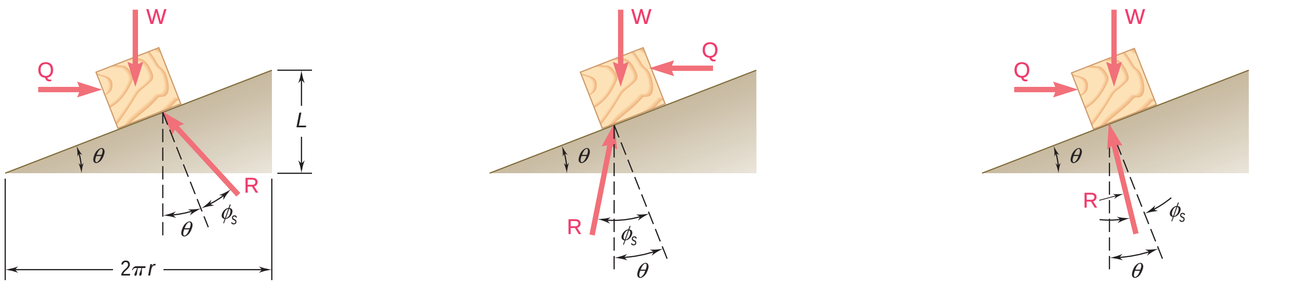


Fig. 8.8

Photo 8.2 Wedges are used as shown to split tree trunks because the normal forces exerted by the wedges on the wood are much larger than the forces required to insert the wedges.

The thread of the base has been unwrapped and shown as a straight line in Fig. 8.9*a*. The correct slope was obtained by plotting horizontally the product $2\pi r$, where r is the mean radius of the thread, and vertically the *lead* L of the screw, i.e., the distance through which the screw advances in one turn. The angle θ this line forms with the horizontal is the *lead angle*. Since the force of friction between two surfaces in contact does not depend upon the area of contact, a much smaller than actual area of contact between the two threads can be assumed, and the screw can be represented by the block shown in Fig. 8.9*a*. It should be noted, however, that in this analysis of the jack, the friction between cap and screw is neglected.

The free-body diagram of the block should include the load **W**, the reaction **R** of the base thread, and a horizontal force **Q** having the same effect as the force **P** exerted on the handle. The force **Q** should have the same moment as **P** about the axis of the screw and its magnitude should thus be $Q = Pa/r$. The force **Q**, and thus the force **P** required to raise the load **W**, can be obtained from the free-body diagram shown in Fig. 8.9*a*. The friction angle is taken equal to ϕ_s since the load will presumably be raised through a succession of short strokes. In mechanisms providing for the continuous rotation of a screw, it may be desirable to distinguish between the force required to start motion (using ϕ_s) and that required to maintain motion (using ϕ_k).

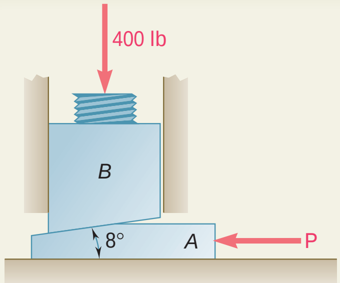


(*a*) Impending motion upward (*b*) Impending motion downward with $\phi_s > \theta$ (*c*) Impending motion downward with $\phi_s < \theta$

Fig. 8.9 Block-and-incline analysis of a screw.

If the friction angle ϕ_s is larger than the lead angle θ, the screw is said to be *self-locking;* it will remain in place under the load. To lower the load, we must then apply the force shown in Fig. 8.9*b*. If ϕ_s is smaller than θ, the screw will unwind under the load; it is then necessary to apply the force shown in Fig. 8.9*c* to maintain equilibrium.

The lead of a screw should not be confused with its *pitch*. The lead was defined as the distance through which the screw advances in one turn; the pitch is the distance measured between two consecutive threads. While lead and pitch are equal in the case of *single-threaded* screws, they are different in the case of *multiple-threaded* screws, i.e., screws having several independent threads. It is easily verified that for double-threaded screws, the lead is twice as large as the pitch; for triple-threaded screws, it is three times as large as the pitch; etc.

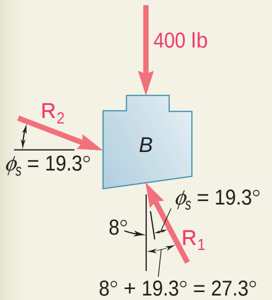


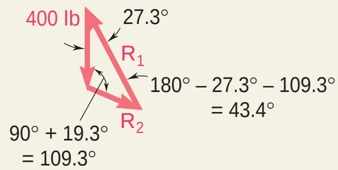
SAMPLE PROBLEM 8.4

The position of the machine block *B* is adjusted by moving the wedge *A*. Knowing that the coefficient of static friction is 0.35 between all surfaces of contact, determine the force **P** required (*a*) to raise block *B*, (*b*) to lower block *B*.

SOLUTION

For each part, the free-body diagrams of block *B* and wedge *A* are drawn, together with the corresponding force triangles, and the law of sines is used to find the desired forces. We note that since $m_s = 0.35$, the angle of friction is





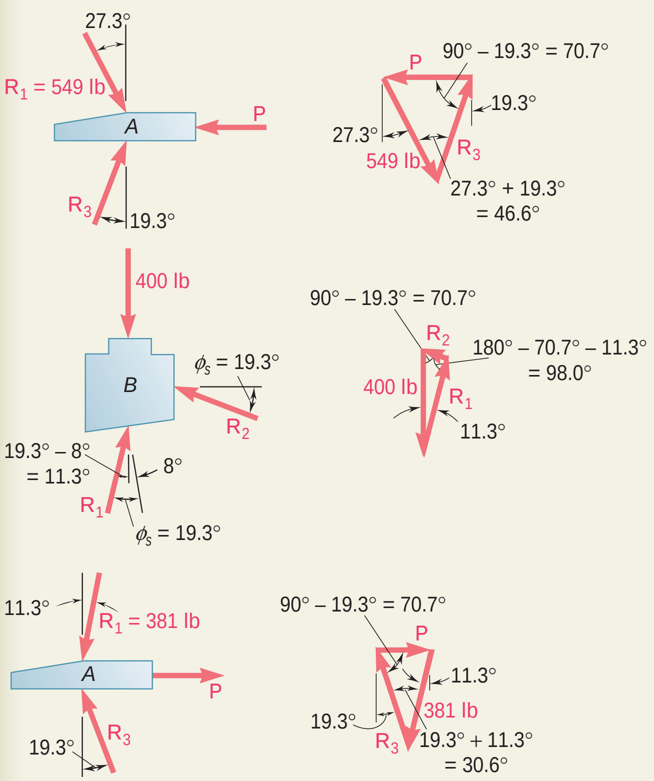
$$f_s = \tan^{-1} 0.35 = 19.3°$$

a. Force P to Raise Block

Free Body: Block B

$$\frac{R_1}{\sin 109.3°} = \frac{400 \text{ lb}}{\sin 43.4°}$$

$$R_1 = 549 \text{ lb}$$



Free Body: Wedge A

$$\frac{P}{\sin 46.6°} = \frac{549 \text{ lb}}{\sin 70.7°}$$

$P = 423$ lb **P** = 423 lb z ◀

b. Force P to Lower Block

Free Body: Block B

$$\frac{R_1}{\sin 70.7°} = \frac{400 \text{ lb}}{\sin 98.0°}$$

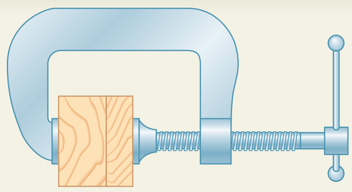
$$R_1 = 381 \text{ lb}$$

Free Body: Wedge A

$$\frac{P}{\sin 30.6°} = \frac{381 \text{ lb}}{\sin 70.7°}$$

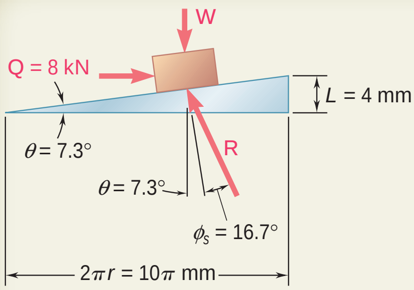
$P = 206$ lb **P** = 206 lb y ◀

SAMPLE PROBLEM 8.5



A clamp is used to hold two pieces of wood together as shown. The clamp has a double square thread of mean diameter equal to 10 mm with a pitch of 2 mm. The coefficient of friction between threads is $\mu_s = 0.30$. If a maximum couple of 40 N · m is applied in tightening the clamp, determine (*a*) the force exerted on the pieces of wood, (*b*) the couple required to loosen the clamp.

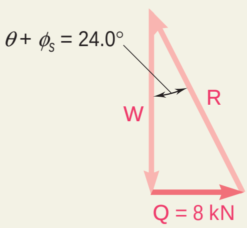
SOLUTION



a. Force Exerted by Clamp. The mean radius of the screw is $r = 5$ mm. Since the screw is double-threaded, the lead L is equal to twice the pitch: $L = 2(2 \text{ mm}) = 4$ mm. The lead angle θ and the friction angle ϕ_s are obtained by writing

$$\tan \theta = \frac{L}{2\pi r} = \frac{4 \text{ mm}}{10\pi \text{ mm}} = 0.1273 \qquad \theta = 7.3^\circ$$

$$\tan \phi_s = \mu_s = 0.30 \qquad \phi_s = 16.7^\circ$$



The force **Q** which should be applied to the block representing the screw is obtained by expressing that its moment Qr about the axis of the screw is equal to the applied couple.

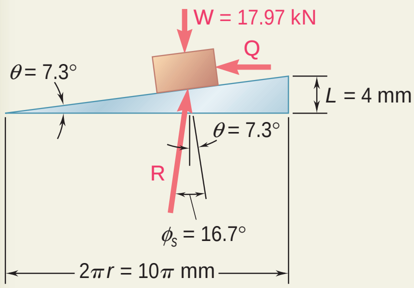
$$Q(5 \text{ mm}) = 40 \text{ N} \cdot \text{m}$$

$$Q = \frac{40 \text{ N} \cdot \text{m}}{5 \text{ mm}} = \frac{40 \text{ N} \cdot \text{m}}{5 \times 10^{-3} \text{ m}} = 8000 \text{ N} = 8 \text{ kN}$$

The free-body diagram and the corresponding force triangle can now be drawn for the block; the magnitude of the force **W** exerted on the pieces of wood is obtained by solving the triangle.

$$W = \frac{Q}{\tan(\theta + \phi_s)} = \frac{8 \text{ kN}}{\tan 24.0^\circ}$$

$$W = 17.97 \text{ kN} \quad ◀$$

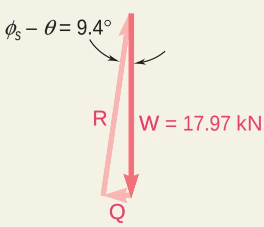


b. Couple Required to Loosen Clamp. The force **Q** required to loosen the clamp and the corresponding couple are obtained from the free-body diagram and force triangle shown.

$$Q = W \tan(\phi_s - \theta) = (17.97 \text{ kN}) \tan 9.4^\circ$$
$$= 2.975 \text{ kN}$$

$$\text{Couple} = Qr = (2.975 \text{ kN})(5 \text{ mm})$$
$$= (2.975 \times 10^3 \text{ N})(5 \times 10^{-3} \text{ m}) = 14.87 \text{ N} \cdot \text{m}$$

$$\text{Couple} = 14.87 \text{ N} \cdot \text{m} \quad ◀$$



SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to apply the laws of friction to the solution of problems involving *wedges* and *square-threaded screws*.

1. Wedges. Keep the following in mind when solving a problem involving a wedge:

a. First draw a free-body diagram of the wedge and of all the other bodies involved. Carefully note the sense of the relative motion of all surfaces of contact and show each friction force acting in *a direction opposite* to the direction of that relative motion.

b. Show the maximum static friction force $\mathbf{F}_m$ at each surface if the wedge is to be inserted or removed, *since motion will be impending in each of these cases*.

c. The reaction R and the angle of friction, rather than the normal force and the friction force, can be used in many applications. You can then draw one or more force triangles and determine the unknown quantities either graphically or by trigonometry [Sample Prob. 8.4].

2. Square-Threaded Screws. The analysis of a square-threaded screw is equivalent to the analysis of a block sliding on an incline. To draw the appropriate incline, you should unwrap the thread of the screw and represent it by a straight line [Sample Prob. 8.5]. When solving a problem involving a square-threaded screw, keep the following in mind:

a. Do not confuse the pitch of a screw with the lead of a screw. The *pitch* of a screw is the distance between two consecutive threads, while the *lead* of a screw is the distance the screw advances in one full turn. The lead and the pitch are equal only in single-threaded screws. In a double-threaded screw, the lead is twice the pitch.

b. The couple required to tighten a screw is different from the couple required to loosen it. Also, screws used in jacks and clamps are usually *self-locking;* that is, the screw will remain stationary as long as no couple is applied to it, and a couple must be applied to the screw to loosen it [Sample Prob. 8.5].

PROBLEMS

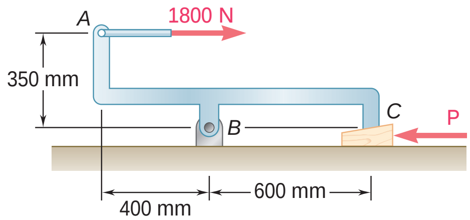


Fig. P8.48

8.48 The machine part ABC is supported by a frictionless hinge at B and a 10° wedge at C. Knowing that the coefficient of static friction at both surfaces of the wedge is 0.20, determine (*a*) the force **P** required to move the wedge, (*b*) the components of the corresponding reaction at B.

8.49 Solve Prob. 8.48 assuming that the force **P** is directed to the right.

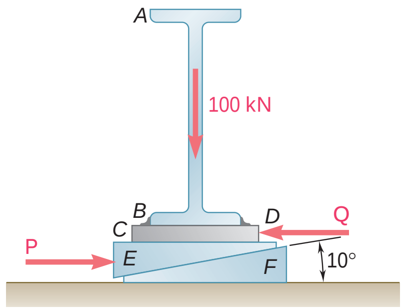


Fig. *P8.50*

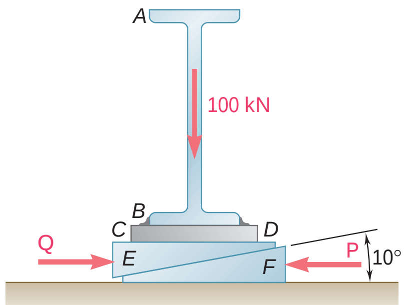


Fig. *P8.51*

8.50* and *8.51 The elevation of the end of the steel beam supported by a concrete floor is adjusted by means of the steel wedges E and F. The base plate CD has been welded to the lower flange of the beam, and the end reaction of the beam is known to be 100 kN. The coefficient of static friction is 0.30 between two steel surfaces and 0.60 between steel and concrete. If the horizontal motion of the beam is prevented by the force **Q**, determine (*a*) the force **P** required to raise the beam, (*b*) the corresponding force **Q**.

8.52 and 8.53 Two 10° wedges of negligible weight are used to move and position the 400-lb block. Knowing that the coefficient of static friction is 0.25 at all surfaces of contact, determine the smallest force **P** that should be applied as shown to one of the wedges.

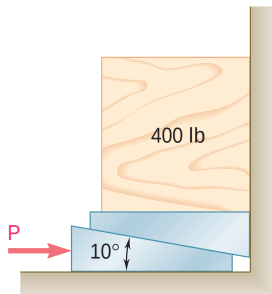


Fig. P8.52

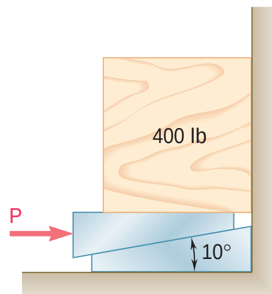


Fig. P8.53

8.54 Block A supports a pipe column and rests as shown on wedge B. Knowing that the coefficient of static friction at all surfaces of contact is 0.25 and that u = 45°, determine the smallest force **P** required to raise block A.

8.55 Block A supports a pipe column and rests as shown on wedge B. Knowing that the coefficient of static friction at all surfaces of contact is 0.25 and that u = 45°, determine the smallest force **P** for which equilibrium is maintained.

8.56 Block A supports a pipe column and rests as shown on wedge B. The coefficient of static friction at all surfaces of contact is 0.25. If **P** = 0, determine (*a*) the angle u for which sliding is impending, (*b*) the corresponding force exerted on the block by the vertical wall.

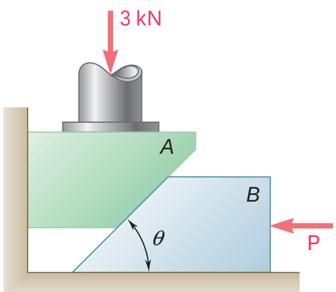


Fig. P8.54, P8.55, and P8.56

8.57 A wedge A of negligible weight is to be driven between two 100-lb plates B and C. The coefficient of static friction between all surfaces of contact is 0.35. Determine the magnitude of the force **P** required to start moving the wedge (*a*) if the plates are equally free to move, (*b*) if plate C is securely bolted to the surface.

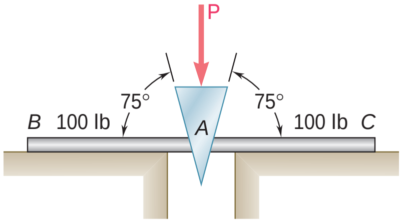


Fig. P8.57

8.58 A 10° wedge is used to split a section of a log. The coefficient of static friction between the wedge and the log is 0.35. Knowing that a force **P** of magnitude 600 lb was required to insert the wedge, determine the magnitude of the forces exerted on the wood by the wedge after insertion.

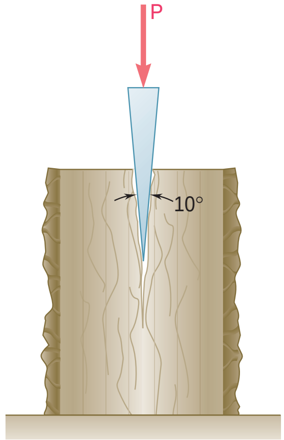


Fig. *P8.58*

8.59 A 10° wedge is to be forced under end B of the 5-kg rod AB. Knowing that the coefficient of static friction is 0.40 between the wedge and the rod and 0.20 between the wedge and the floor, determine the smallest force **P** required to raise end B of the rod.

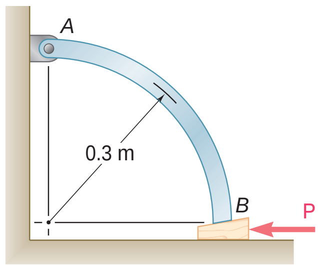


Fig. P8.59

8.60 The spring of the door latch has a constant of 1.8 lb/in. and in the position shown exerts a 0.6-lb force on the bolt. The coefficient of static friction between the bolt and the strike plate is 0.40; all other surfaces are well lubricated and may be assumed frictionless. Determine the magnitude of the force **P** required to start closing the door.

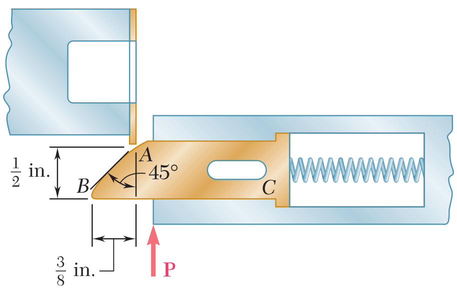


Fig. P8.60

8.61 In Prob. 8.60, determine the angle that the face of the bolt should form with the line BC if the force **P** required to close the door is to be the same for both the position shown and the position when B is almost at the strike plate.

8.62 A 5° wedge is to be forced under a 1400-lb machine base at A. Knowing that the coefficient of static friction at all surfaces is 0.20, (*a*) determine the force **P** required to move the wedge, (*b*) indicate whether the machine base will move.

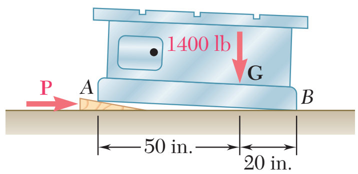


Fig. P8.62

8.63 Solve Prob. 8.62 assuming that the wedge is to be forced under the machine base at B instead of A.

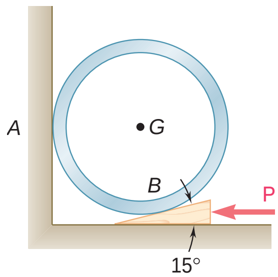


Fig. P8.64 and P8.65

8.64 A 15° wedge is forced under a 50-kg pipe as shown. The coefficient of static friction at all surfaces is 0.20. (*a*) Show that slipping will occur between the pipe and the vertical wall. (*b*) Determine the force **P** required to move the wedge.

8.65 A 15° wedge is forced under a 50-kg pipe as shown. Knowing that the coefficient of static friction at both surfaces of the wedge is 0.20, determine the largest coefficient of static friction between the pipe and the vertical wall for which slipping will occur at *A*.

***8.66** A 200-N block rests as shown on a wedge of negligible weight. The coefficient of static friction μ_s is the same at both surfaces of the wedge, and friction between the block and the vertical wall may be neglected. For $P = 100$ N, determine the value of μ_s for which motion is impending. (*Hint:* Solve the equation obtained by trial and error.)

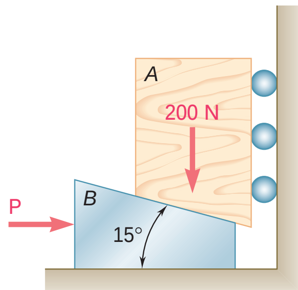


Fig. P8.66

***8.67** Solve Prob. 8.66 assuming that the rollers are removed and that μ_s is the coefficient of friction at all surfaces of contact.

8.68 Derive the following formulas relating the load **W** and the force **P** exerted on the handle of the jack discussed in Sec. 8.6. (*a*) $P = (Wr/a)\tan(\theta + \phi_s)$, to raise the load; (*b*) $P = (Wr/a)\tan(\phi_s - \theta)$, to lower the load if the screw is self-locking; (*c*) $P = (Wr/a)\tan(\theta - \phi_s)$, to hold the load if the screw is not self-locking.

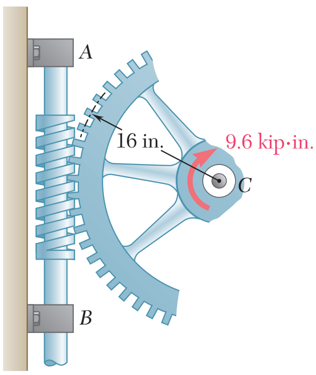


Fig. *P8.69*

8.69 The square-threaded worm gear shown has a mean radius of 2 in. and a lead of 0.5 in. The large gear is subjected to a constant clockwise couple of 9.6 kip · in. Knowing that the coefficient of static friction between the two gears is 0.12, determine the couple that must be applied to shaft *AB* in order to rotate the large gear counterclockwise. Neglect friction in the bearings at *A*, *B*, and *C*.

8.70 In Prob. 8.69, determine the couple that must be applied to shaft *AB* in order to rotate the large gear clockwise.

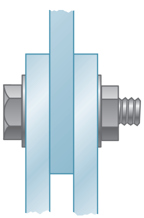


Fig. P8.71

8.71 High-strength bolts are used in the construction of many steel structures. For a 24-mm-nominal-diameter bolt, the required minimum bolt tension is 210 kN. Assuming the coefficient of friction to be 0.40, determine the required couple that should be applied to the bolt and nut. The mean diameter of the thread is 22.6 mm, and the lead is 3 mm. Neglect friction between the nut and washer, and assume the bolt to be square-threaded.

8.72 The position of the automobile jack shown is controlled by a screw ABC that is single-threaded at each end (right-handed thread at A, left-handed thread at C). Each thread has a pitch of 0.1 in. and a mean diameter of 0.375 in. If the coefficient of static friction is 0.15, determine the magnitude of the couple **M** that must be applied to raise the automobile.

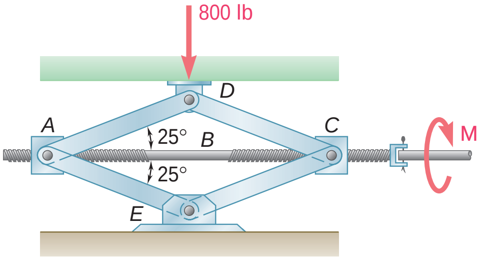


Fig. P8.72

8.73 For the jack of Prob. 8.72, determine the magnitude of the couple **M** that must be applied to lower the automobile.

8.74 In the gear-pulling assembly shown, the square-threaded screw AB has a mean radius of 15 mm and a lead of 4 mm. Knowing that the coefficient of static friction is 0.10, determine the couple that must be applied to the screw in order to produce a force of 3 kN on the gear. Neglect friction at end A of the screw.

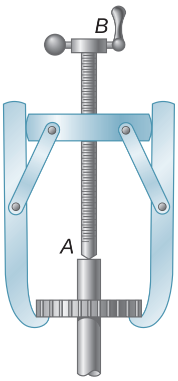


Fig. *P8.74*

8.75 The ends of two fixed rods A and B are each made in the form of a single-threaded screw of mean radius 6 mm and pitch 2 mm. Rod A has a right-handed thread and rod B has a left-handed thread. The coefficient of static friction between the rods and the threaded sleeve is 0.12. Determine the magnitude of the couple that must be applied to the sleeve in order to draw the rods closer together.

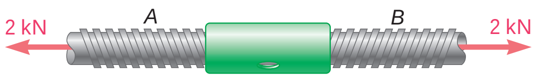


Fig. P8.75

8.76 Assuming that in Prob. 8.75 a right-handed thread is used on *both* rods A and B, determine the magnitude of the couple that must be applied to the sleeve in order to rotate it.

*8.7 JOURNAL BEARINGS. AXLE FRICTION

Journal bearings are used to provide lateral support to rotating shafts and axles. Thrust bearings, which will be studied in the next section, are used to provide axial support to shafts and axles. If the journal bearing is fully lubricated, the frictional resistance depends upon the speed of rotation, the clearance between axle and bearing, and the viscosity of the lubricant. As indicated in Sec. 8.1, such problems are studied in fluid mechanics. The methods of this chapter, however, can be applied to the study of axle friction when the bearing is not lubricated or only partially lubricated. It can then be assumed that the axle and the bearing are in direct contact along a single straight line.

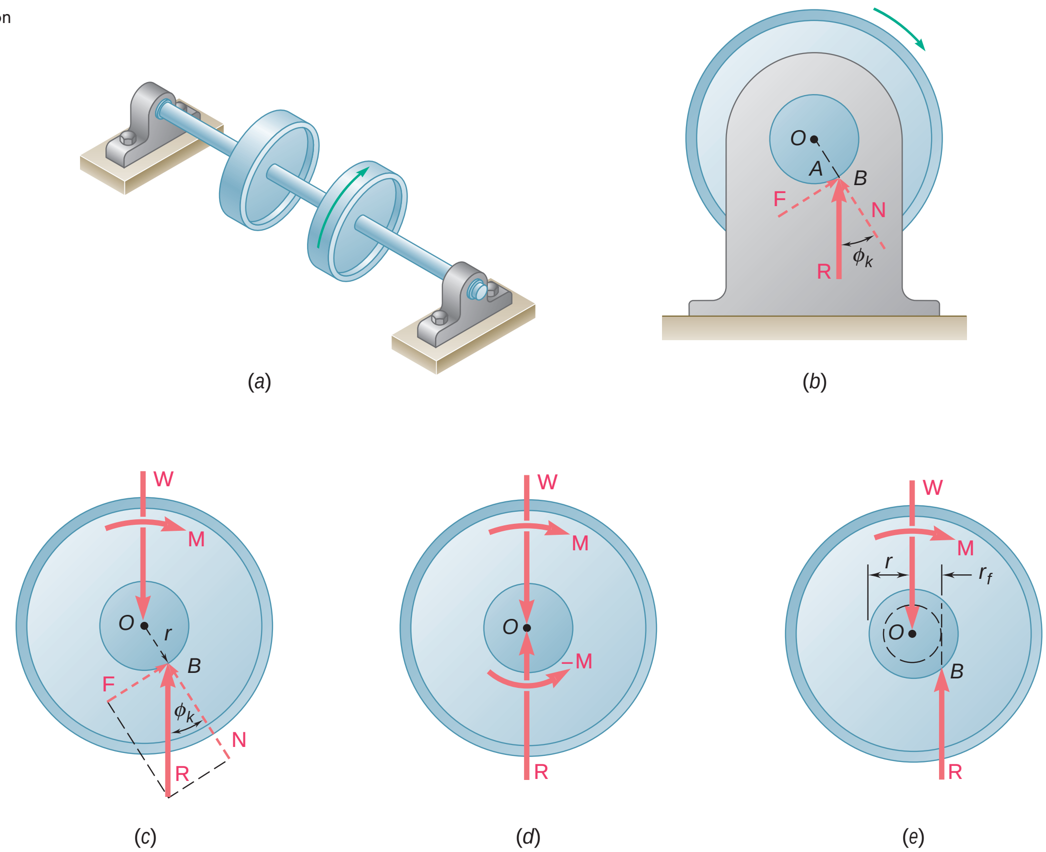


Fig. 8.10

Consider two wheels, each of weight **W**, rigidly mounted on an axle supported symmetrically by two journal bearings (Fig. 8.10*a*). If the wheels rotate, we find that to keep them rotating at constant speed, it is necessary to apply to each of them a couple **M**. The free-body diagram in Fig. 8.10*c* represents one of the wheels and the corresponding half axle in projection on a plane perpendicular to the axle. The forces acting on the free body include the weight **W** of the wheel, the couple **M** required to maintain its motion, and a force **R** representing the reaction of the bearing. This force is vertical, equal, and opposite to **W** but does not pass through the center O of the axle; **R** is located to the right of O at a distance such that its moment about O balances the moment **M** of the couple. Therefore, contact between the axle and bearing does not take place at the lowest point A when the axle rotates. It takes place at point B (Fig. 8.10*b*) or, rather, along a straight line intersecting the plane of the figure at B. Physically, this is explained by the fact that when the wheels are set in motion, the axle "climbs" in the bearings until slippage occurs. After sliding back slightly, the axle settles more or less in the position shown. This position is such that the angle between the reaction **R** and the normal to the surface of the bearing is equal to the angle of kinetic friction ϕ_k. The distance from O to the line of action of **R** is thus $r \sin \phi_k$, where r is the radius of the axle. Writing that $\Sigma M_O = 0$ for the forces acting on the free body considered, we obtain the magnitude of the couple **M** required to overcome the frictional resistance of one of the bearings:

$$M = Rr \sin \phi_k \tag{8.5}$$

Observing that, for small values of the angle of friction, sin ϕ_k can be replaced by tan ϕ_k, that is, by μ_k, we write the approximate formula

$$M \approx Rr\mu_k \tag{8.6}$$

In the solution of certain problems, it may be more convenient to let the line of action of **R** pass through O, as it does when the axle does not rotate. A couple $-\mathbf{M}$ of the same magnitude as the couple **M** but of opposite sense must then be added to the reaction **R** (Fig. 8.10*d*). This couple represents the frictional resistance of the bearing.

In case a graphical solution is preferred, the line of action of **R** can be readily drawn (Fig. 8.10*e*) if we note that it must be tangent to a circle centered at O and of radius

$$r_f = r \sin \phi_k \approx r\mu_k \tag{8.7}$$

This circle is called the *circle of friction* of the axle and bearing and is independent of the loading conditions of the axle.

*8.8 THRUST BEARINGS. DISK FRICTION

Two types of thrust bearings are used to provide axial support to rotating shafts and axles: (1) *end bearings* and (2) *collar bearings* (Fig. 8.11). In the case of collar bearings, friction forces develop between the two ring-shaped areas which are in contact. In the case of end bearings, friction takes place over full circular areas, or over ring-shaped areas when the end of the shaft is hollow. Friction between circular areas, called *disk friction,* also occurs in other mechanisms, such as *disk clutches*.

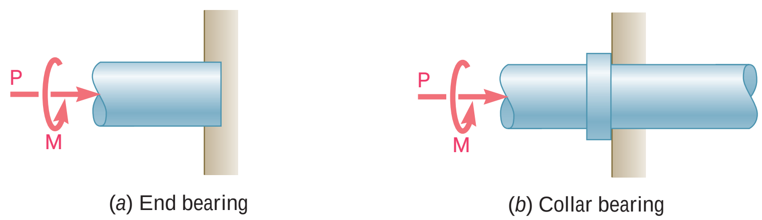


Fig. 8.11 Thrust bearings.

To obtain a formula which is valid in the most general case of disk friction, let us consider a rotating hollow shaft. A couple **M** keeps the shaft rotating at constant speed while a force **P** maintains it in contact with a fixed bearing (Fig. 8.12). Contact between the shaft and

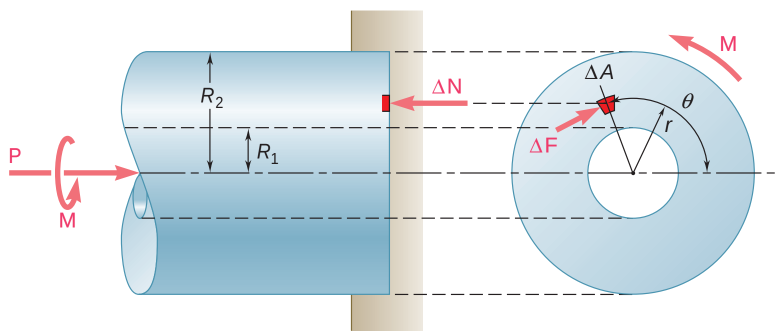


Fig. 8.12

the bearing takes place over a ring-shaped area of inner radius R_1 and outer radius R_2. Assuming that the pressure between the two surfaces in contact is uniform, we find that the magnitude of the normal force $\Delta\mathbf{N}$ exerted on an element of area ΔA is $\Delta N = P\,\Delta A/A$, where $A = \pi(R_2^2 - R_1^2)$, and that the magnitude of the friction force $\Delta\mathbf{F}$ acting on ΔA is $\Delta F = \mu_k\,\Delta N$. Denoting by r the distance from the axis of the shaft to the element of area ΔA, we express the magnitude ΔM of the moment of $\Delta\mathbf{F}$ about the axis of the shaft as follows:

$$\Delta M = r\,\Delta F = \frac{r\mu_k P\,\Delta A}{\pi(R_2^2 - R_1^2)}$$

The equilibrium of the shaft requires that the moment $\mathbf{M}$ of the couple applied to the shaft be equal in magnitude to the sum of the moments of the friction forces $\Delta\mathbf{F}$. Replacing ΔA by the infinitesimal element $dA = r\,d\theta\,dr$ used with polar coordinates, and integrating over the area of contact, we thus obtain the following expression for the magnitude of the couple $\mathbf{M}$ required to overcome the frictional resistance of the bearing:

$$M = \frac{\mu_k P}{\pi(R_2^2 - R_1^2)}\int_0^{2\pi}\int_{R_1}^{R_2} r^2\,dr\,d\theta$$

$$= \frac{\mu_k P}{\pi(R_2^2 - R_1^2)}\int_0^{2\pi} \tfrac{1}{3}(R_2^3 - R_1^3)\,d\theta$$

$$M = \tfrac{2}{3}\mu_k P\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2} \tag{8.8}$$

When contact takes place over a full circle of radius R, formula (8.8) reduces to

$$M = \tfrac{2}{3}\mu_k PR \tag{8.9}$$

The value of M is then the same as would be obtained if contact between shaft and bearing took place at a single point located at a distance $2R/3$ from the axis of the shaft.

The largest couple which can be transmitted by a disk clutch without causing slippage is given by a formula similar to (8.9), where μ_k has been replaced by the coefficient of static friction μ_s.

*8.9 WHEEL FRICTION. ROLLING RESISTANCE

The wheel is one of the most important inventions of our civilization. Its use makes it possible to move heavy loads with relatively little effort. Because the point of the wheel in contact with the ground at any given instant has no relative motion with respect to the ground, the wheel eliminates the large friction forces which would arise if the load were in direct contact with the ground. However, some resistance to the wheel's motion exists. This resistance has two distinct causes. It is due (1) to a combined effect of axle friction and friction at the rim and (2) to the fact that the wheel and the ground deform, with the result that contact between wheel and ground takes place over a certain area, rather than at a single point.

To understand better the first cause of resistance to the motion of a wheel, let us consider a railroad car supported by eight wheels mounted on axles and bearings. The car is assumed to be moving to the right at constant speed along a straight horizontal track. The free-body diagram of one of the wheels is shown in Fig. 8.13*a*. The forces acting on the free body include the load **W** supported by the wheel and the normal reaction **N** of the track. Since **W** is drawn through the center *O* of the axle, the frictional resistance of the bearing should be represented by a counterclockwise couple **M** (see Sec. 8.7). To keep the free body in equilibrium, we must add two equal and opposite forces **P** and **F**, forming a clockwise couple of moment $-\mathbf{M}$. The force **F** is the friction force exerted by the track on the wheel, and **P** represents the force which should be applied to the wheel to keep it rolling at constant speed. Note that the forces **P** and **F** would not exist if there were no friction between wheel and track. The couple **M** representing the axle friction would then be zero; the wheel would slide on the track without turning in its bearing.

The couple **M** and the forces **P** and **F** also reduce to zero when there is no axle friction. For example, a wheel which is not held in bearings and rolls freely and at constant speed on horizontal ground (Fig. 8.13*b*) will be subjected to only two forces: its own weight **W** and the normal reaction **N** of the ground. Regardless of the value of the coefficient of friction between wheel and ground no friction force will act on the wheel. A wheel rolling freely on horizontal ground should thus keep rolling indefinitely.

Experience, however, indicates that the wheel will slow down and eventually come to rest. This is due to the second type of resistance mentioned at the beginning of this section, known as the *rolling resistance*. Under the load **W**, both the wheel and the ground deform slightly, causing the contact between wheel and ground to take place over a certain area. Experimental evidence shows that the resultant of the forces exerted by the ground on the wheel over this area is a force **R** applied at a point *B*, which is not located directly under the center *O* of the wheel, but slightly in front of it (Fig. 8.13*c*). To balance the moment of **W** about *B* and to keep the wheel rolling at constant speed, it is necessary to apply a horizontal force **P** at the center of the wheel. Writing $\Sigma M_B = 0$, we obtain

$$Pr = Wb \tag{8.10}$$

where r = radius of wheel
b = horizontal distance between O and B

The distance b is commonly called the *coefficient of rolling resistance*. It should be noted that b is not a dimensionless coefficient since it represents a length; b is usually expressed in inches or in millimeters. The value of b depends upon several parameters in a manner which has not yet been clearly established. Values of the coefficient of rolling resistance vary from about 0.01 in. or 0.25 mm for a steel wheel on a steel rail to 5.0 in. or 125 mm for the same wheel on soft ground.

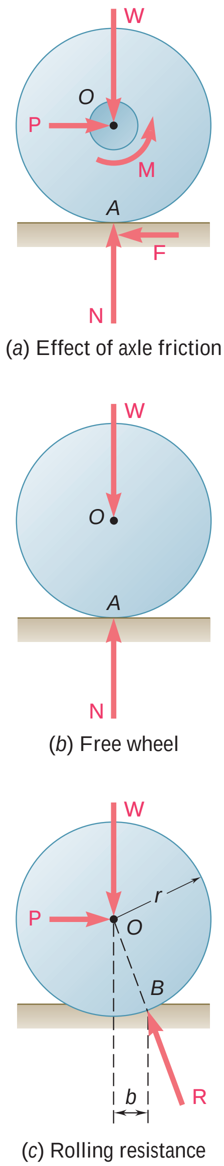
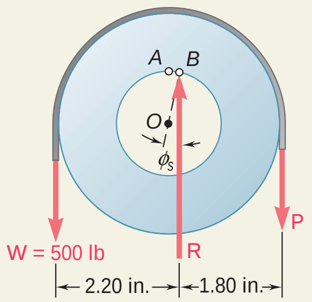


Fig. 8.13

SAMPLE PROBLEM 8.6

A pulley of diameter 4 in. can rotate about a fixed shaft of diameter 2 in. The coefficient of static friction between the pulley and shaft is 0.20. Determine (*a*) the smallest vertical force **P** required to start raising a 500-lb load, (*b*) the smallest vertical force **P** required to hold the load, (*c*) the smallest horizontal force **P** required to start raising the same load.

SOLUTION

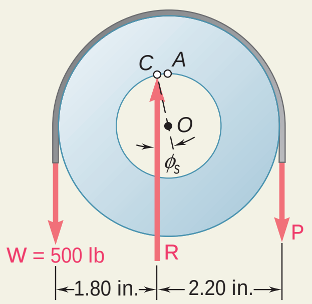


a. Vertical Force P Required to Start Raising the Load. When the forces in both parts of the rope are equal, contact between the pulley and shaft takes place at *A*. When **P** is increased, the pulley rolls around the shaft slightly and contact takes place at *B*. The free-body diagram of the pulley when motion is impending is drawn. The perpendicular distance from the center *O* of the pulley to the line of action of **R** is

$$r_f = r \sin \phi_s \approx r\mu_s \qquad r_f \approx (1 \text{ in.})0.20 = 0.20 \text{ in.}$$

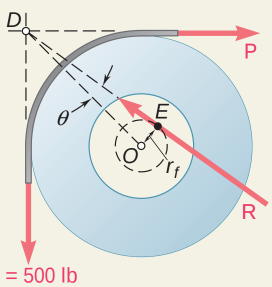
Summing moments about *B*, we write

$$+\circlearrowleft \Sigma M_B = 0: \quad (2.20 \text{ in.})(500 \text{ lb}) - (1.80 \text{ in.})P = 0$$
$$P = 611 \text{ lb} \qquad \mathbf{P} = 611 \text{ lb}\downarrow \blacktriangleleft$$



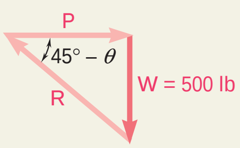
b. Vertical Force P to Hold the Load. As the force **P** is decreased, the pulley rolls around the shaft and contact takes place at *C*. Considering the pulley as a free body and summing moments about *C*, we write

$$+\circlearrowleft \Sigma M_C = 0: \quad (1.80 \text{ in.})(500 \text{ lb}) - (2.20 \text{ in.})P = 0$$
$$P = 409 \text{ lb} \qquad \mathbf{P} = 409 \text{ lb}\downarrow \blacktriangleleft$$



c. Horizontal Force P to Start Raising the Load. Since the three forces **W**, **P**, and **R** are not parallel, they must be concurrent. The direction of **R** is thus determined from the fact that its line of action must pass through the point of intersection *D* of **W** and **P**, and must be tangent to the circle of friction. Recalling that the radius of the circle of friction is $r_f = 0.20$ in., we write

$$\sin \theta = \frac{OE}{OD} = \frac{0.20 \text{ in.}}{(2 \text{ in.})\sqrt{2}} = 0.0707 \qquad \theta = 4.1°$$



From the force triangle, we obtain

$$P = W \cot (45° - \theta) = (500 \text{ lb}) \cot 40.9°$$
$$= 577 \text{ lb} \qquad \mathbf{P} = 577 \text{ lb} \rightarrow \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned about several additional engineering applications of the laws of friction.

1. Journal bearings and axle friction. In journal bearings, the *reaction does not pass through the center of the shaft or axle* which is being supported. The distance from the center of the shaft or axle to the line of action of the reaction (Fig. 8.10) is defined by the equation.

$$r_f = r \sin \phi_k \approx r\mu_k$$

if motion is actually taking place, and by the equation

$$r_f = r \sin \phi_s \approx r\mu_s$$

if the motion is impending.

Once you have determined the line of action of the reaction, you can draw a *free-body diagram* and use the corresponding equations of equilibrium to complete your solution [Sample Prob. 8.6]. In some problems, it is useful to observe that the line of action of the reaction must be tangent to a circle of radius $r_f \approx r\mu_k$, or $r_f \approx r\mu_s$, known as the *circle of friction* [Sample Prob. 8.6, part *c*].

2. Thrust bearings and disk friction. In a *thrust bearing* the magnitude of the couple required to overcome frictional resistance is equal to the sum of the moments of the *kinetic friction forces* exerted on the elements of the end of the shaft [Eqs. (8.8) and (8.9)].

An example of disk friction is the *disk clutch*. It is analyzed in the same way as a thrust bearing, except that to determine the largest couple that can be transmitted, you must compute the sum of the moments of the *maximum static friction forces* exerted on the disk.

3. Wheel friction and rolling resistance. You saw that the rolling resistance of a wheel is caused by deformations of both the wheel and the ground. The line of action of the reaction **R** of the ground on the wheel intersects the ground at a horizontal distance b from the center of the wheel. The distance b is known as the *coefficient of rolling resistance* and is expressed in inches or millimeters.

4. In problems involving both rolling resistance and axle friction, your free-body diagram should show that the line of action of the reaction **R** of the ground on the wheel is tangent to the friction circle of the axle and intersects the ground at a horizontal distance from the center of the wheel equal to the coefficient of rolling resistance.

PROBLEMS

8.77 A lever of negligible weight is loosely fitted onto a 30-mm-radius fixed shaft as shown. Knowing that a force **P** of magnitude 275 N will just start the lever rotating clockwise, determine (*a*) the coefficient of static friction between the shaft and the lever, (*b*) the smallest force **P** for which the lever does not start rotating counterclockwise.

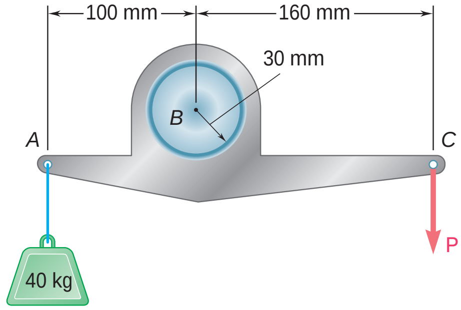


Fig. P8.77

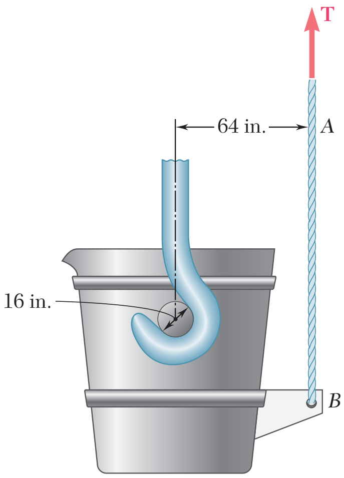


Fig. P8.78

8.78 A hot-metal ladle and its contents weigh 130 kips. Knowing that the coefficient of static friction between the hooks and the pinion is 0.30, determine the tension in cable *AB* required to start tipping the ladle.

8.79 and 8.80 The double pulley shown is attached to a 10-mm-radius shaft that fits loosely in a fixed bearing. Knowing that the coefficient of static friction between the shaft and the poorly lubricated bearing is 0.40, determine the magnitude of the force **P** required to start raising the load.

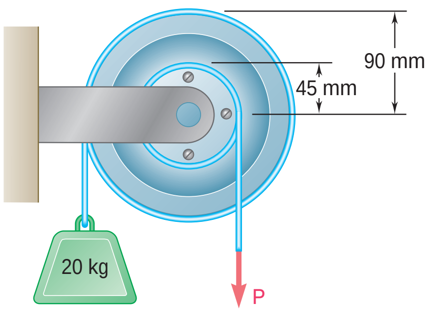


Fig. P8.79 and P8.81

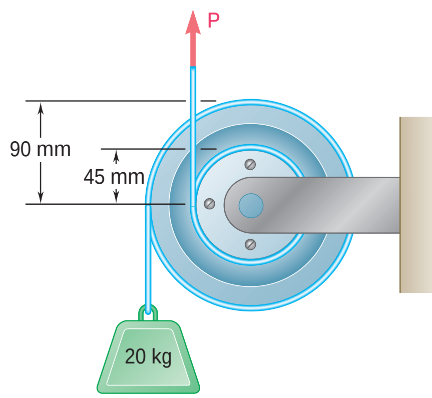


Fig. P8.80 and P8.82

8.81 and 8.82 The double pulley shown is attached to a 10-mm-radius shaft that fits loosely in a fixed bearing. Knowing that the coefficient of static friction between the shaft and the poorly lubricated bearing is 0.40, determine the magnitude of the smallest force **P** required to maintain equilibrium.

8.83 The block and tackle shown are used to raise a 150-lb load. Each of the 3-in.-diameter pulleys rotates on a 0.5-in.-diameter axle. Knowing that the coefficient of static friction is 0.20, determine the tension in each portion of the rope as the load is slowly raised.

8.84 The block and tackle shown are used to lower a 150-lb load. Each of the 3-in.-diameter pulleys rotates on a 0.5-in.-diameter axle. Knowing that the coefficient of static friction is 0.20, determine the tension in each portion of the rope as the load is slowly lowered.

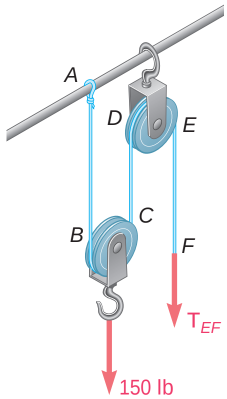


Fig. *P8.83* and P8.84

8.85 A scooter is to be designed to roll down a 2 percent slope at a constant speed. Assuming that the coefficient of kinetic friction between the 25-mm-diameter axles and the bearings is 0.10, determine the required diameter of the wheels. Neglect the rolling resistance between the wheels and the ground.

8.86 The link arrangement shown is frequently used in highway bridge construction to allow for expansion due to changes in temperature. At each of the 60-mm-diameter pins *A* and *B* the coefficient of static friction is 0.20. Knowing that the vertical component of the force exerted by *BC* on the link is 200 kN, determine (*a*) the horizontal force that should be exerted on beam *BC* to just move the link, (*b*) the angle that the resulting force exerted by beam *BC* on the link will form with the vertical.

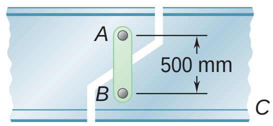


Fig. P8.86

8.87 and 8.88 A lever *AB* of negligible weight is loosely fitted onto a 2.5-in.-diameter fixed shaft. Knowing that the coefficient of static friction between the fixed shaft and the lever is 0.15, determine the force **P** required to start the lever rotating counterclockwise.

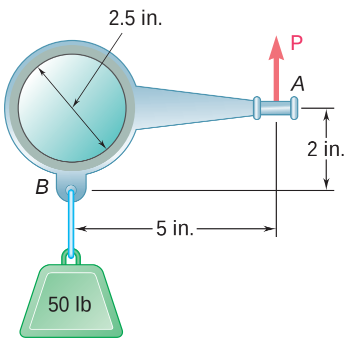


Fig. *P8.87* and P8.89

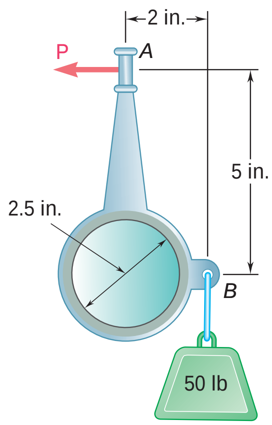


Fig. P8.88 and P8.90

8.89 and 8.90 A lever *AB* of negligible weight is loosely fitted onto a 2.5-in.-diameter fixed shaft. Knowing that the coefficient of static friction between the fixed shaft and the lever is 0.15, determine the force **P** required to start the lever rotating clockwise.

8.91 A loaded railroad car has a mass of 30 Mg and is supported by eight 800-mm-diameter wheels with 125-mm-diameter axles. Knowing that the coefficients of friction are $\mu_s = 0.020$ and $\mu_k = 0.015$, determine the horizontal force required (*a*) to start the car moving, (*b*) to keep the car moving at a constant speed. Neglect rolling resistance between the wheels and the rails.

8.92 Knowing that a couple of magnitude 30 N · m is required to start the vertical shaft rotating, determine the coefficient of static friction between the annular surfaces of contact.

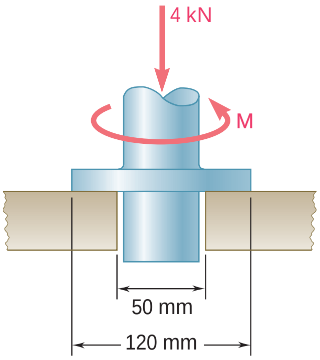


Fig. P8.92

8.93 A 50-lb electric floor polisher is operated on a surface for which the coefficient of kinetic friction is 0.25. Assuming that the normal force per unit area between the disk and the floor is uniformly distributed, determine the magnitude Q of the horizontal forces required to prevent motion of the machine.

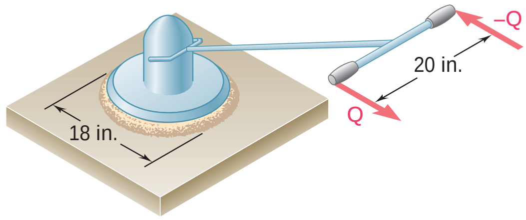


Fig. P8.93

***8.94** The frictional resistance of a thrust bearing decreases as the shaft and bearing surfaces wear out. It is generally assumed that the wear is directly proportional to the distance traveled by any given point of the shaft and thus to the distance r from the point to the axis of the shaft. Assuming, then, that the normal force per unit area is inversely proportional to r, show that the magnitude M of the couple required to overcome the frictional resistance of a worn-out end bearing (with contact over the full circular area) is equal to 75 percent of the value given by Eq. (8.9) for a new bearing.

***8.95** Assuming that bearings wear out as indicated in Prob. 8.94, show that the magnitude M of the couple required to overcome the frictional resistance of a worn-out collar bearing is

$$M = \tfrac{1}{2}\mu_k P(R_1 + R_2)$$

where P = magnitude of the total axial force
R_1, R_2 = inner and outer radii of the collar

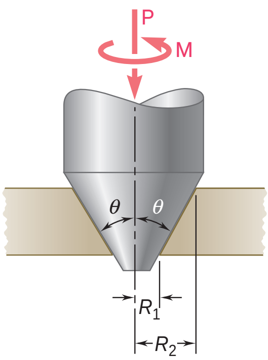


Fig. *P8.96*

***8.96** Assuming that the pressure between the surfaces of contact is uniform, show that the magnitude M of the couple required to overcome frictional resistance for the conical bearing shown is

$$M = \frac{2}{3}\frac{\mu_k P}{\sin\theta}\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}$$

8.97 Solve Prob. 8.93 assuming that the normal force per unit area between the disk and the floor varies linearly from a maximum at the center to zero at the circumference of the disk.

8.98 Determine the horizontal force required to move a 2500-lb automobile with 23-in.-diameter tires along a horizontal road at a constant speed. Neglect all forms of friction except rolling resistance, and assume the coefficient of rolling resistance to be 0.05 in.

8.99 Knowing that a 6-in.-diameter disk rolls at a constant velocity down a 2 percent incline, determine the coefficient of rolling resistance between the disk and the incline.

8.100 A 900-kg machine base is rolled along a concrete floor using a series of steel pipes with outside diameters of 100 mm. Knowing that the coefficient of rolling resistance is 0.5 mm between the pipes and the base and 1.25 mm between the pipes and the concrete floor, determine the magnitude of the force **P** required to slowly move the base along the floor.

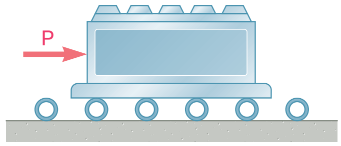


Fig. P8.100

8.101 Solve Prob. 8.85 including the effect of a coefficient of rolling resistance of 1.75 mm.

8.102 Solve Prob. 8.91 including the effect of a coefficient of rolling resistance of 0.5 mm.

8.10 BELT FRICTION

Consider a flat belt passing over a fixed cylindrical drum (Fig. 8.14*a*). We propose to determine the relation existing between the values T_1 and T_2 of the tension in the two parts of the belt when the belt is just about to slide toward the right.

Let us detach from the belt a small element PP' subtending an angle Δu. Denoting by T the tension at P and by $T + \Delta T$ the tension at P', we draw the free-body diagram of the element of the belt (Fig. 8.14*b*). Besides the two forces of tension, the forces acting on the free body are the normal component $\Delta\mathbf{N}$ of the reaction of the drum and the friction force $\Delta\mathbf{F}$. Since motion is assumed to be impending, we have $\Delta F = m_s \, \Delta N$. It should be noted that if Δu is made to approach zero, the magnitudes ΔN and ΔF, and the *difference* ΔT between the tension at P and the tension at P', will also approach zero; the value T of the tension at P, however, will remain unchanged. This observation helps in understanding our choice of notations.

Choosing the coordinate axes shown in Fig. 8.14*b*, we write the equations of equilibrium for the element PP':

$$\Sigma F_x = 0: \qquad (T + \Delta T)\cos\frac{\Delta u}{2} - T\cos\frac{\Delta u}{2} - m_s \Delta N = 0 \qquad (8.11)$$

$$\Sigma F_y = 0: \qquad \Delta N - (T + \Delta T)\sin\frac{\Delta u}{2} - T\sin\frac{\Delta u}{2} = 0 \qquad (8.12)$$

Solving Eq. (8.12) for ΔN and substituting into (8.11), we obtain after reductions

$$\Delta T \cos\frac{\Delta u}{2} - m_s(2T + \Delta T)\sin\frac{\Delta u}{2} = 0$$

Both terms are now divided by Δu. For the first term, this is done simply by dividing ΔT by Δu. The division of the second term is

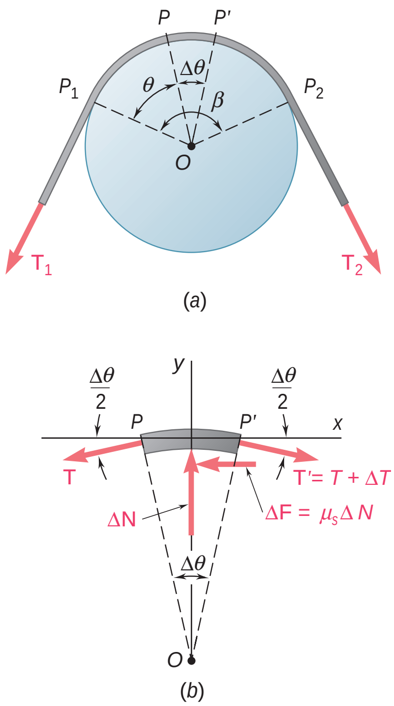


Fig. 8.14

carried out by dividing the terms in the parentheses by 2 and the sine by $\Delta\theta/2$. We write

$$\frac{\Delta T}{\Delta\theta}\cos\frac{\Delta\theta}{2} - \mu_s\left(T + \frac{\Delta T}{2}\right)\frac{\sin(\Delta\theta/2)}{\Delta\theta/2} = 0$$

If we now let $\Delta\theta$ approach 0, the cosine approaches 1 and $\Delta T/2$ approaches zero, as noted above. The quotient of sin $(\Delta\theta/2)$ over $\Delta\theta/2$ approaches 1, according to a lemma derived in all calculus textbooks. Since the limit of $\Delta T/\Delta\theta$ is by definition equal to the derivative $dT/d\theta$, we write

$$\frac{dT}{d\theta} - \mu_s T = 0 \qquad\qquad \frac{dT}{T} = \mu_s d\theta$$

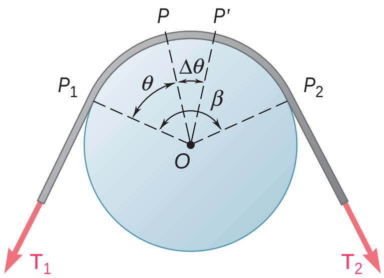


Fig. 8.14a (*repeated*)

Both members of the last equation (Fig. 8.14*a*) will now be integrated from P_1 to P_2. At P_1, we have $\theta = 0$ and $T = T_1$; at P_2, we have $\theta = \beta$ and $T = T_2$. Integrating between these limits, we write

$$\int_{T_1}^{T_2}\frac{dT}{T} = \int_0^{\beta}\mu_s\, d\theta$$
$$\ln T_2 - \ln T_1 = \mu_s\beta$$

or, noting that the left-hand member is equal to the natural logarithm of the quotient of T_2 and T_1,

$$\boxed{\ln\frac{T_2}{T_1} = \mu_s\beta} \qquad (8.13)$$

This relation can also be written in the form

$$\boxed{\frac{T_2}{T_1} = e^{\mu_s\beta}} \qquad (8.14)$$

The formulas we have derived apply equally well to problems involving flat belts passing over fixed cylindrical drums and to problems involving ropes wrapped around a post or capstan. They can also be used to solve problems involving band brakes. In such problems, it is the drum which is about to rotate, while the band remains fixed. The formulas can also be applied to problems involving belt drives. In these problems, both the pulley and the belt rotate; our concern is then to find whether the belt will slip, i.e., whether it will move *with respect* to the pulley.


Photo 8.3 By wrapping the rope around the bollard, the force exerted by the worker to control the rope is much smaller than the tension in the taut portion of the rope.

Formulas (8.13) and (8.14) should be used only if the belt, rope, or brake is *about to slip*. Formula (8.14) will be used if T_1 or T_2 is desired; formula (8.13) will be preferred if either μ_s or the angle of contact β is desired. We should note that T_2 is always larger than T_1; T_2 therefore represents the tension in that part of the belt or rope which *pulls*, while T_1 is the tension in the part which *resists*. We should also observe that the angle of contact β must be expressed in *radians*. The angle β may be larger than 2π; for example, if a rope is wrapped n times around a post, β is equal to $2\pi n$.

If the belt, rope, or brake is actually slipping, formulas similar to (8.13) and (8.14), but involving the coefficient of kinetic friction μ_k, should be used. If the belt, rope, or brake is not slipping and is not about to slip, none of these formulas can be used.

The belts used in belt drives are often V-shaped. In the V belt shown in Fig. 8.15*a* contact between belt and pulley takes place

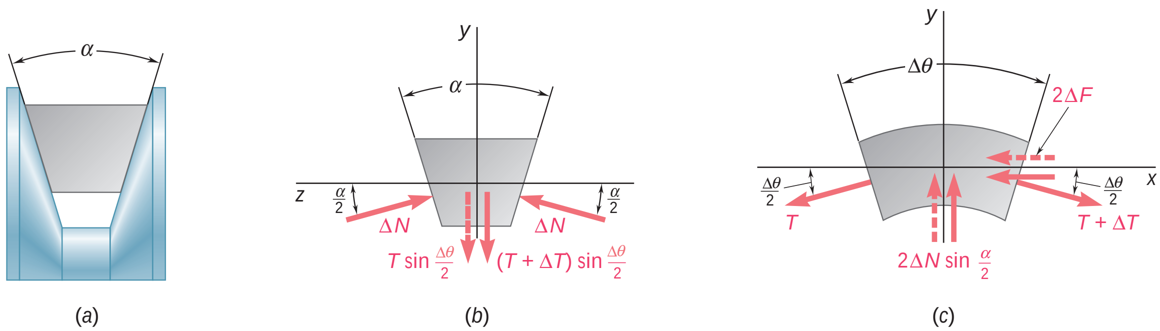


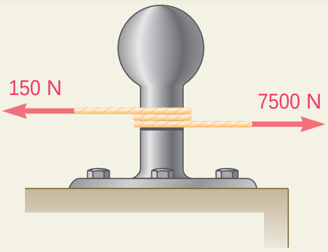
Fig. 8.15

along the sides of the groove. The relation existing between the values T_1 and T_2 of the tension in the two parts of the belt when the belt is just about to slip can again be obtained by drawing the free-body diagram of an element of belt (Fig. 8.15*b* and *c*). Equations similar to (8.11) and (8.12) are derived, but the magnitude of the total friction force acting on the element is now $2\ \Delta F$, and the sum of the y components of the normal forces is $2\ \Delta N \sin(\alpha/2)$. Proceeding as above, we obtain

$$\ln \frac{T_2}{T_1} = \frac{\mu_s \beta}{\sin(\alpha/2)} \tag{8.15}$$

or,

$$\frac{T_2}{T_1} = e^{\mu_s \beta / \sin(\alpha/2)} \tag{8.16}$$



SAMPLE PROBLEM 8.7

A hawser thrown from a ship to a pier is wrapped two full turns around a bollard. The tension in the hawser is 7500 N; by exerting a force of 150 N on its free end, a dockworker can just keep the hawser from slipping. (*a*) Determine the coefficient of friction between the hawser and the bollard. (*b*) Determine the tension in the hawser that could be resisted by the 150-N force if the hawser were wrapped three full turns around the bollard.

SOLUTION

a. Coefficient of Friction. Since slipping of the hawser is impending, we use Eq. (8.13):

$$\ln \frac{T_2}{T_1} = \mu_s \beta$$

Since the hawser is wrapped two full turns around the bollard, we have

$$\beta = 2(2\pi \text{ rad}) = 12.57 \text{ rad}$$
$$T_1 = 150 \text{ N} \qquad T_2 = 7500 \text{ N}$$

Therefore,

$$\mu_s \beta = \ln \frac{T_2}{T_1}$$
$$\mu_s(12.57 \text{ rad}) = \ln \frac{7500 \text{ N}}{150 \text{ N}} = \ln 50 = 3.91$$
$$\mu_s = 0.311 \qquad\qquad \mu_s = 0.311 \blacktriangleleft$$

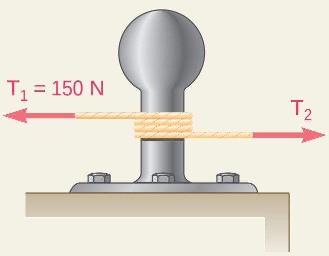
b. Hawser Wrapped Three Turns Around Bollard. Using the value of μ_s obtained in part *a*, we now have

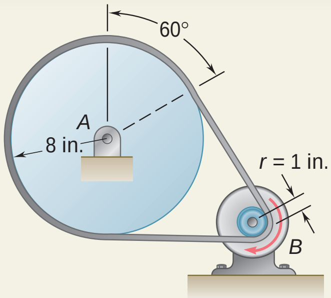
$$\beta = 3(2\pi \text{ rad}) = 18.85 \text{ rad}$$
$$T_1 = 150 \text{ N} \qquad \mu_s = 0.311$$

Substituting these values into Eq. (8.14), we obtain

$$\frac{T_2}{T_1} = e^{\mu_s \beta}$$
$$\frac{T_2}{150 \text{ N}} = e^{(0.311)(18.85)} = e^{5.862} = 351.5$$
$$T_2 = 52\,725 \text{ N}$$

$$T_2 = 52.7 \text{ kN} \blacktriangleleft$$

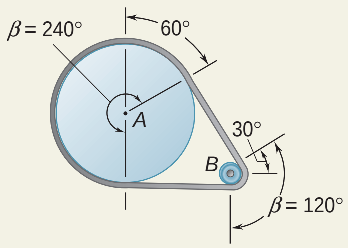




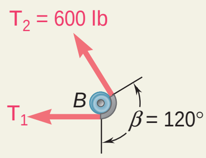
SAMPLE PROBLEM 8.8

A flat belt connects pulley A, which drives a machine tool, to pulley B, which is attached to the shaft of an electric motor. The coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$ between both pulleys and the belt. Knowing that the maximum allowable tension in the belt is 600 lb, determine the largest torque which can be exerted by the belt on pulley A.

SOLUTION



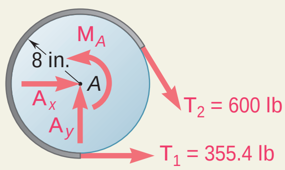
Since the resistance to slippage depends upon the angle of contact β between pulley and belt, as well as upon the coefficient of static friction μ_s, and since μ_s is the same for both pulleys, slippage will occur first on pulley B, for which β is smaller.



Pulley *B*. Using Eq. (8.14) with $T_2 = 600$ lb, $\mu_s = 0.25$, and $\beta = 120° = 2\pi/3$ rad, we write

$$\frac{T_2}{T_1} = e^{\mu_s \beta} \qquad \frac{600 \text{ lb}}{T_1} = e^{0.25(2\pi/3)} = 1.688$$

$$T_1 = \frac{600 \text{ lb}}{1.688} = 355.4 \text{ lb}$$



Pulley *A*. We draw the free-body diagram of pulley A. The couple $\mathbf{M}_A$ is applied to the pulley by the machine tool to which it is attached and is equal and opposite to the torque exerted by the belt. We write

$$+\circlearrowleft \Sigma M_A = 0: \qquad M_A - (600 \text{ lb})(8 \text{ in.}) + (355.4 \text{ lb})(8 \text{ in.}) = 0$$

$$M_A = 1957 \text{ lb} \cdot \text{in.} \qquad M_A = 163.1 \text{ lb} \cdot \text{ft} \quad \blacktriangleleft$$

Note. We may check that the belt does not slip on pulley A by computing the value of μ_s required to prevent slipping at A and verifying that it is smaller than the actual value of μ_s. From Eq. (8.13) we have

$$\mu_s \beta = \ln \frac{T_2}{T_1} = \ln \frac{600 \text{ lb}}{355.4 \text{ lb}} = 0.524$$

and, since $\beta = 240° = 4\pi/3$ rad,

$$\frac{4\pi}{3}\mu_s = 0.524 \qquad \mu_s = 0.125 < 0.25$$

SOLVING PROBLEMS ON YOUR OWN

In the preceding section you learned about *belt friction*. The problems you will solve include belts passing over fixed drums, band brakes in which the drum rotates while the band remains fixed, and belt drives.

1. Problems involving belt friction fall into one of the following two categories:

a. Problems in which slipping is impending. One of the following formulas, involving the *coefficient of static friction* μ_s, may then be used,

$$\ln\frac{T_2}{T_1} = \mu_s\beta \qquad (8.13)$$

or

$$\frac{T_2}{T_1} = e^{\mu_s\beta} \qquad (8.14)$$

b. Problems in which slipping is occurring. The formulas to be used can be obtained from Eqs. (8.13) and (8.14) by replacing μ_s with the *coefficient of kinetic friction* μ_k.

2. As you start solving a belt-friction problem, be sure to remember the following:

a. The angle β must be expressed in radians. In a belt-and-drum problem, this is the angle subtending the arc of the drum on which the belt is wrapped.

b. The larger tension is always denoted by T_2 and the *smaller tension is denoted by* T_1.

c. The larger tension occurs at the end of the belt which is in the direction of the motion, or impending motion, of the belt relative to the drum.

3. In each of the problems you will be asked to solve, three of the four quantities T_1, T_2, β, and μ_s (or μ_k) will either be given or readily found, and you will then solve the appropriate equation for the fourth quantity. Here are two kinds of problems that you will encounter:

a. Find μ_s between belt and drum, knowing that slipping is impending. From the given data, determine T_1, T_2, and β; substitute these values into Eq. (8.13) and solve for μ_s [Sample Prob. 8.7, part *a*]. Follow the same procedure to find the *smallest value* of μ_s for which slipping will not occur.

b. Find the magnitude of a force or couple applied to the belt or drum, knowing that slipping is impending. The given data should include μ_s and β. If it also includes T_1 or T_2, use Eq. (8.14) to find the other tension. If neither T_1 nor T_2 is known but some other data is given, use the free-body diagram of the belt-drum system to write an equilibrium equation that you will solve simultaneously with Eq. (8.14) for T_1 and T_2. You will then be able to find the magnitude of the specified force or couple from the free-body diagram of the system. Follow the same procedure to determine the *largest value* of a force or couple which can be applied to the belt or drum if no slipping is to occur [Sample Prob. 8.8].

PROBLEMS

8.103 A 300-lb block is supported by a rope that is wrapped $1\frac{1}{2}$ times around a horizontal rod. Knowing that the coefficient of static friction between the rope and the rod is 0.15, determine the range of values of P for which equilibrium is maintained.

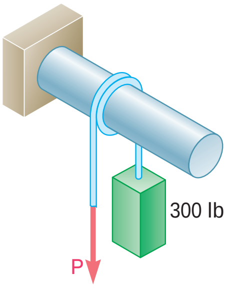


Fig. P8.103

8.104 A hawser is wrapped two full turns around a bollard. By exerting an 80-lb force on the free end of the hawser, a dockworker can resist a force of 5000 lb on the other end of the hawser. Determine (*a*) the coefficient of static friction between the hawser and the bollard, (*b*) the number of times the hawser should be wrapped around the bollard if a 20,000-lb force is to be resisted by the same 80-lb force.

8.105 A rope *ABCD* is looped over two pipes as shown. Knowing that the coefficient of static friction is 0.25, determine (*a*) the smallest value of the mass *m* for which equilibrium is possible, (*b*) the corresponding tension in portion *BC* of the rope.

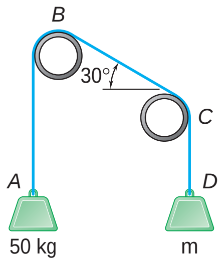


Fig. P8.105 and P8.106

8.106 A rope *ABCD* is looped over two pipes as shown. Knowing that the coefficient of static friction is 0.25, determine (*a*) the largest value of the mass *m* for which equilibrium is possible, (*b*) the corresponding tension in portion *BC* of the rope.

8.107 Knowing that the coefficient of static friction is 0.25 between the rope and the horizontal pipe and 0.20 between the rope and the vertical pipe, determine the range of values of P for which equilibrium is maintained.

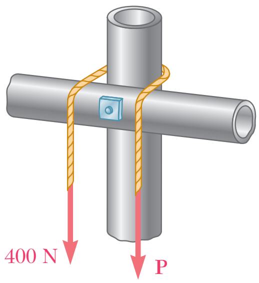


Fig. *P8.107* and *P8.108*

8.108 Knowing that the coefficient of static friction is 0.30 between the rope and the horizontal pipe and that the smallest value of P for which equilibrium is maintained is 80 N, determine (*a*) the largest value of P for which equilibrium is maintained, (*b*) the coefficient of static friction between the rope and the vertical pipe.

8.109 A band brake is used to control the speed of a flywheel as shown. The coefficients of friction are $\mu_s = 0.30$ and $\mu_k = 0.25$. Determine the magnitude of the couple being applied to the flywheel, knowing that $P = 45$ N and that the flywheel is rotating counterclockwise at a constant speed.

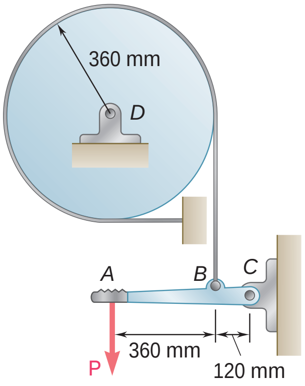


Fig. P8.109

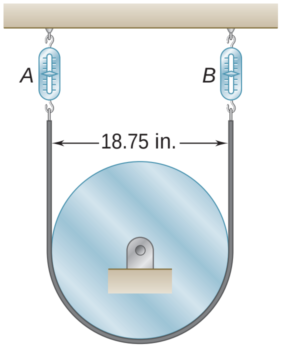


Fig. P8.110 and P8.111

8.110 The setup shown is used to measure the output of a small turbine. When the flywheel is at rest, the reading of each spring scale is 14 lb. If a 105-lb · in. couple must be applied to the flywheel to keep it rotating clockwise at a constant speed, determine (*a*) the reading of each scale at that time, (*b*) the coefficient of kinetic friction. Assume that the length of the belt does not change.

8.111 The setup shown is used to measure the output of a small turbine. The coefficient of kinetic friction is 0.20 and the reading of each spring scale is 16 lb when the flywheel is at rest. Determine (*a*) the reading of each scale when the flywheel is rotating clockwise at a constant speed, (*b*) the couple that must be applied to the flywheel. Assume that the length of the belt does not change.

8.112 A flat belt is used to transmit a couple from drum *B* to drum *A*. Knowing that the coefficient of static friction is 0.40 and that the allowable belt tension is 450 N, determine the largest couple that can be exerted on drum *A*.

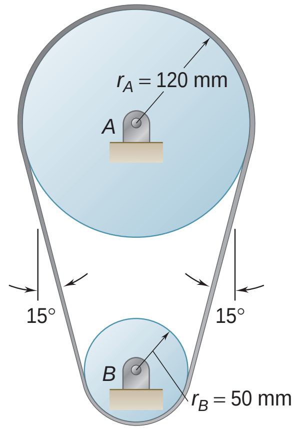


Fig. P8.112

8.113 A flat belt is used to transmit a couple from pulley *A* to pulley *B*. The radius of each pulley is 60 mm, and a force of magnitude $P = 900$ N is applied as shown to the axle of pulley *A*. Knowing that the coefficient of static friction is 0.35, determine (*a*) the largest couple that can be transmitted, (*b*) the corresponding maximum value of the tension in the belt.

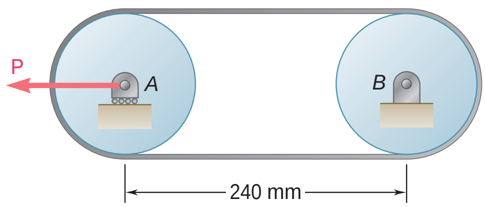


Fig. P8.113

8.114 Solve Prob. 8.113 assuming that the belt is looped around the pulleys in a figure eight.

8.115 The speed of the brake drum shown is controlled by a belt attached to the control bar AD. A force $\mathbf{P}$ of magnitude 25 lb is applied to the control bar at A. Determine the magnitude of the couple being applied to the drum, knowing that the coefficient of kinetic friction between the belt and the drum is 0.25, that $a = 4$ in., and that the drum is rotating at a constant speed (a) counterclockwise, (b) clockwise.

8.116 The speed of the brake drum shown is controlled by a belt attached to the control bar AD. Knowing that $a = 4$ in., determine the maximum value of the coefficient of static friction for which the brake is not self-locking when the drum rotates counterclockwise.

8.117 The speed of the brake drum shown is controlled by a belt attached to the control bar AD. Knowing that the coefficient of static friction is 0.30 and that the brake drum is rotating counterclockwise, determine the minimum value of a for which the brake is not self-locking.

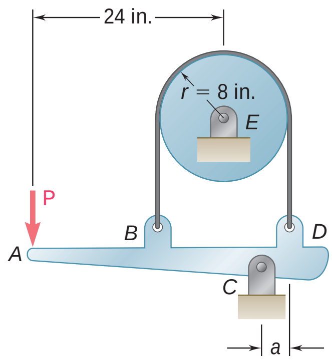


Fig. *P8.115, P8.116,* and P8.117

8.118 Bucket A and block C are connected by a cable that passes over drum B. Knowing that drum B rotates slowly counterclockwise and that the coefficients of friction at all surfaces are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine the smallest combined mass m of the bucket and its contents for which block C will (a) remain at rest, (b) start moving up the incline, (c) continue moving up the incline at a constant speed.

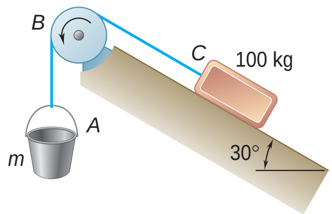


Fig. P8.118

8.119 Solve Prob. 8.118 assuming that drum B is frozen and cannot rotate.

8.120 and *8.122* A cable is placed around three parallel pipes. Knowing that the coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine (a) the smallest weight W for which equilibrium is maintained, (b) the largest weight W that can be raised if pipe B is slowly rotated counterclockwise while pipes A and C remain fixed.

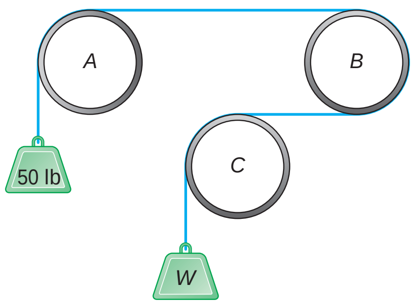


Fig. P8.120 and P8.121

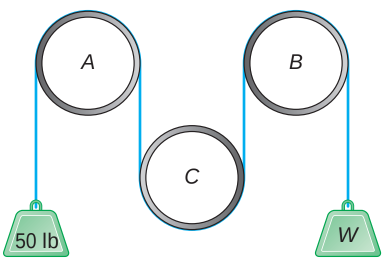


Fig. *P8.122* and *P8.123*

8.121 and *8.123* A cable is placed around three parallel pipes. Two of the pipes are fixed and do not rotate; the third pipe is slowly rotated. Knowing that the coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine the largest weight W that can be raised (a) if only pipe A is rotated counterclockwise, (b) if only pipe C is rotated clockwise.

8.124 A recording tape passes over the 20-mm-radius drive drum *B* and under the idler drum *C*. Knowing that the coefficients of friction between the tape and the drums are $\mu_s = 0.40$ and $\mu_k = 0.30$ and that drum *C* is free to rotate, determine the smallest allowable value of *P* if slipping of the tape on drum *B* is not to occur.

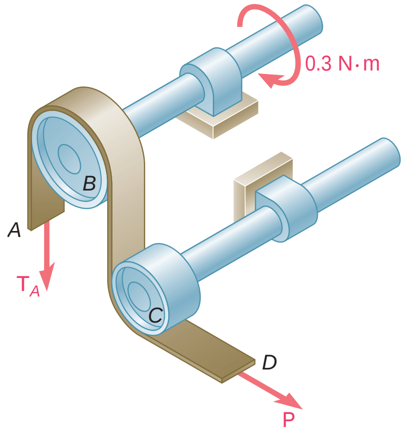


Fig. P8.124

8.125 Solve Prob. 8.124 assuming that the idler drum *C* is frozen and cannot rotate.

8.126 The strap wrench shown is used to grip the pipe firmly without marring the external surface of the pipe. Knowing that the coefficient of static friction is the same for all surfaces of contact, determine the smallest value of μ_s for which the wrench will be self-locking when $a = 200$ mm, $r = 30$ mm, and $\theta = 65°$.

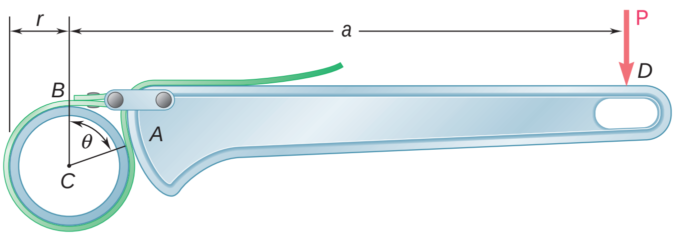


Fig. P8.126

8.127 Solve Prob. 8.126 assuming that $\theta = 75°$.

8.128 The 10-lb bar *AE* is suspended by a cable that passes over a 5-in.-radius drum. Vertical motion of end *E* of the bar is prevented by the two stops shown. Knowing that $\mu_s = 0.30$ between the cable and the drum, determine (*a*) the largest counterclockwise couple $\mathbf{M}_0$ that can be applied to the drum if slipping is not to occur, (*b*) the corresponding force exerted on end *E* of the bar.

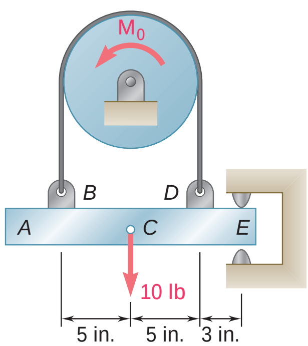


Fig. P8.128

8.129 Solve Prob. 8.128 assuming that a clockwise couple $\mathbf{M}_0$ is applied to the drum.

8.130 Prove that Eqs. (8.13) and (8.14) are valid for any shape of surface provided that the coefficient of friction is the same at all points of contact.

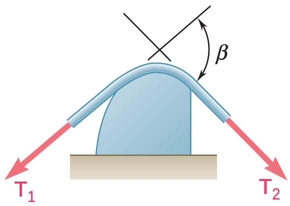


Fig. P8.130

8.131 Complete the derivation of Eq. (8.15), which relates the tension in both parts of a V belt.

8.132 Solve Prob. 8.112 assuming that the flat belt and drums are replaced by a V belt and V pulleys with $\alpha = 36°$. (The angle α is as shown in Fig. 8.15*a*.)

8.133 Solve Prob. 8.113 assuming that the flat belt and pulleys are replaced by a V belt and V pulleys with $\alpha = 36°$. (The angle α is as shown in Fig. 8.15*a*.)

REVIEW AND SUMMARY

This chapter was devoted to the study of *dry friction*, i.e., to problems involving rigid bodies which are in contact along *nonlubricated surfaces*.

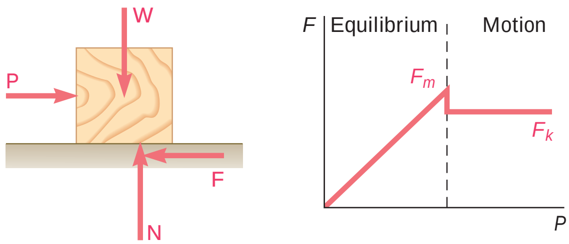


Fig. 8.16

Static and kinetic friction

Applying a horizontal force **P** to a block resting on a horizontal surface [Sec. 8.2], we note that the block at first does not move. This shows that a *friction force* **F** must have developed to balance **P** (Fig. 8.16). As the magnitude of **P** is increased, the magnitude of **F** also increases until it reaches a maximum value F_m. If **P** is further increased, the block starts sliding and the magnitude of **F** drops from F_m to a lower value F_k. Experimental evidence shows that F_m and F_k are proportional to the normal component N of the reaction of the surface. We have

$$F_m = \mu_s N \qquad F_k = \mu_k N \qquad (8.1, 8.2)$$

where μ_s and μ_k are called, respectively, the *coefficient of static friction* and the *coefficient of kinetic friction*. These coefficients depend on the nature and the condition of the surfaces in contact. Approximate values of the coefficients of static friction were given in Table 8.1.

Angles of friction

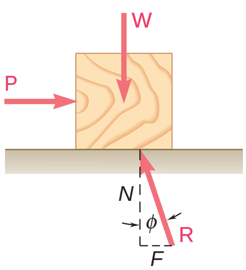


Fig. 8.17

It is sometimes convenient to replace the normal force **N** and the friction force **F** by their resultant **R** (Fig. 8.17). As the friction force increases and reaches its maximum value $F_m = \mu_s N$, the angle ϕ that **R** forms with the normal to the surface increases and reaches a maximum value ϕ_s, called the *angle of static friction*. If motion actually takes place, the magnitude of **F** drops to F_k; similarly the angle ϕ drops to a lower value ϕ_k, called the *angle of kinetic friction*. As shown in Sec. 8.3, we have

$$\tan \phi_s = \mu_s \qquad \tan \phi_k = \mu_k \qquad (8.3, 8.4)$$

Problems involving friction

When solving equilibrium problems involving friction, we should keep in mind that the magnitude F of the friction force is equal to $F_m = \mu_s N$ *only if the body is about to slide* [Sec. 8.4]. *If motion is not impending,* F and N should be considered as *independent unknowns* to be determined from the equilibrium equations (Fig. 8.18*a*). We should also check that the value of F required to maintain equilibrium is not larger than F_m; if it were, the body would move and the magnitude of the friction force would be $F_k = \mu_k N$ [Sample Prob. 8.1]. On the other hand, *if motion is known to be impending,* F has reached its maximum value $F_m = \mu_s N$ (Fig. 8.18*b*), and this expression may be substituted for F in the equilibrium equations [Sample Prob. 8.3]. When only three forces are involved in a free-body diagram, including the reaction **R** of the surface in contact with the body, it is usually more convenient to solve the problem by drawing a force triangle [Sample Prob. 8.2].

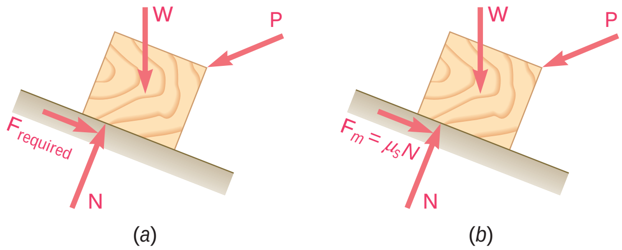


Fig. 8.18

When a problem involves the analysis of the forces exerted on each other by *two bodies A and B*, it is important to show the friction forces with their correct sense. The correct sense for the friction force exerted by B on A, for instance, is opposite to that of the *relative motion* (or impending motion) of A with respect to B [Fig. 8.6].

Wedges and screws

In the second part of the chapter we considered a number of specific engineering applications where dry friction plays an important role. In the case of *wedges,* which are simple machines used to raise heavy loads [Sec. 8.5], two or more free-body diagrams were drawn and care was taken to show each friction force with its correct sense [Sample Prob. 8.4]. The analysis of *square-threaded screws,* which are frequently used in jacks, presses, and other mechanisms, was reduced to the analysis of a block sliding on an incline by unwrapping the thread of the screw and showing it as a straight line [Sec. 8.6]. This is done again in Fig. 8.19, where r denotes the *mean radius* of the thread, L is the *lead* of the screw, i.e., the distance through which the screw advances in one turn, **W** is the load, and Qr is equal to the couple exerted on the screw. It was noted that in the case of multiple-threaded screws the lead L of the screw is *not* equal to its pitch, which is the distance measured between two consecutive threads.

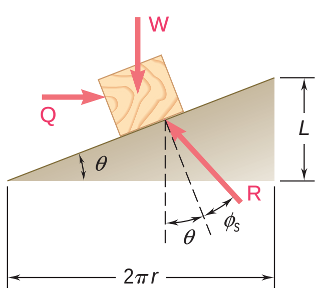


Fig. 8.19

Other engineering applications considered in this chapter were *journal bearings* and *axle friction* [Sec. 8.7], *thrust bearings* and *disk friction* [Sec. 8.8], *wheel friction* and *rolling resistance* [Sec. 8.9], and *belt friction* [Sec. 8.10].

Belt friction

In solving a problem involving a *flat belt* passing over a fixed cylinder, it is important to first determine the direction in which the belt slips or is about to slip. If the drum is rotating, the motion or impending motion of the belt should be determined *relative* to the rotating drum. For instance, if the belt shown in Fig. 8.20 is about to slip to

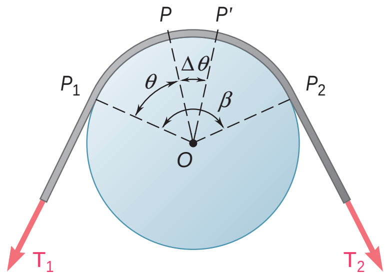


Fig. 8.20

the right relative to the drum, the friction forces exerted by the drum on the belt will be directed to the left and the tension will be larger in the right-hand portion of the belt than in the left-hand portion. Denoting the larger tension by T_2, the smaller tension by T_1, the coefficient of static friction by μ_s, and the angle (in radians) subtended by the belt by β, we derived in Sec. 8.10 the formulas

$$\ln \frac{T_2}{T_1} = \mu_s \beta \tag{8.13}$$

$$\frac{T_2}{T_1} = e^{\mu_s \beta} \tag{8.14}$$

which were used in solving Sample Probs. 8.7 and 8.8. If the belt actually slips on the drum, the coefficient of static friction μ_s should be replaced by the coefficient of kinetic friction μ_k in both of these formulas.

REVIEW PROBLEMS

8.134 Determine whether the block shown is in equilibrium and find the magnitude and direction of the friction force when $\theta = 35°$ and $P = 200$ N.

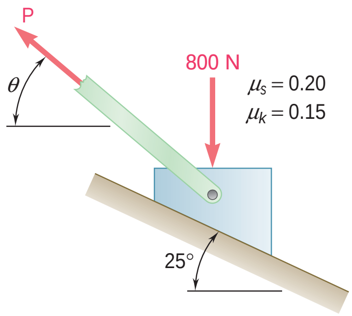


Fig. P8.134

8.135 Three 4-kg packages A, B, and C are placed on a conveyor belt that is at rest. Between the belt and both packages A and C the coefficients of friction are $\mu_s = 0.30$ and $\mu_k = 0.20$; between package B and the belt the coefficients are $\mu_s = 0.10$ and $\mu_k = 0.08$. The packages are placed on the belt so that they are in contact with each other and at rest. Determine which, if any, of the packages will move and the friction force acting on each package.

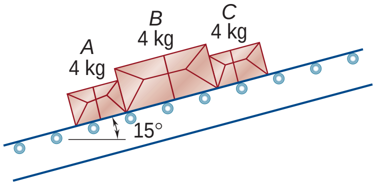


Fig. *P8.135*

8.136 The cylinder shown is of weight W and radius r. Express in terms W and r the magnitude of the largest couple **M** that can be applied to the cylinder if it is not to rotate, assuming the coefficient of static friction to be (*a*) zero at A and 0.30 at B, (*b*) 0.25 at A and 0.30 at B.

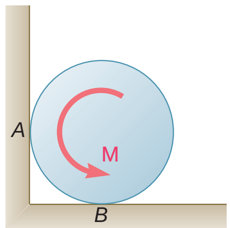


Fig. P8.136

8.137 End A of a slender, uniform rod of length L and weight W bears on a surface as shown, while end B is supported by a cord BC. Knowing that the coefficients of friction are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine (*a*) the largest value of θ for which motion is impending, (*b*) the corresponding value of the tension in the cord.

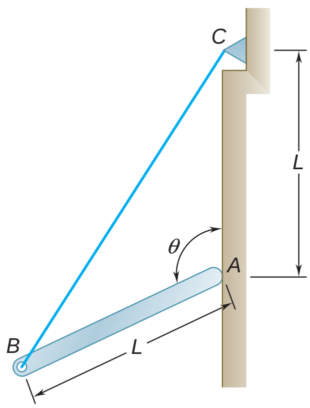


Fig. P8.137

8.138 A worker slowly moves a 50-kg crate to the left along a loading dock by applying a force **P** at corner B as shown. Knowing that the crate starts to tip about the edge E of the loading dock when $a = 200$ mm, determine (*a*) the coefficient of kinetic friction between the crate and the loading dock, (*b*) the corresponding magnitude P of the force.

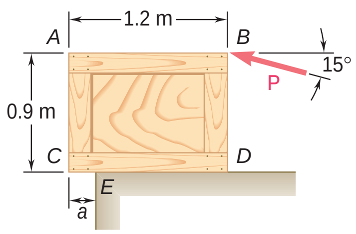


Fig. *P8.138*

8.139 A window sash weighing 10 lb is normally supported by two 5-lb sash weights. Knowing that the window remains open after one sash cord has broken, determine the smallest possible value of the coefficient of static friction. (Assume that the sash is slightly smaller than the frame and will bind only at points A and D.)

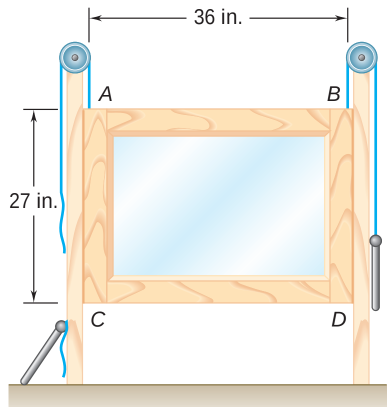


Fig. P8.139

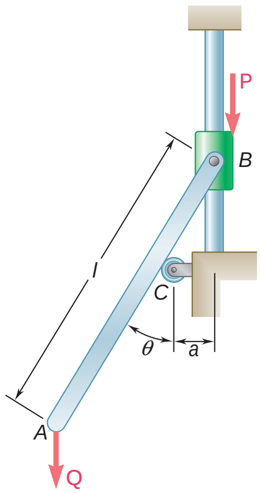


Fig. P8.140

8.140 The slender rod AB of length $l = 600$ mm is attached to a collar at B and rests on a small wheel located at a horizontal distance $a = 80$ mm from the vertical rod on which the collar slides. Knowing that the coefficient of static friction between the collar and the vertical rod is 0.25 and neglecting the radius of the wheel, determine the range of values of P for which equilibrium is maintained when $Q = 100$ N and u $= 30°$.

8.141 The machine part ABC is supported by a frictionless hinge at B and a 10° wedge at C. Knowing that the coefficient of static friction is 0.20 at both surfaces of the wedge, determine (*a*) the force **P** required to move the wedge to the left, (*b*) the components of the corresponding reaction at B.

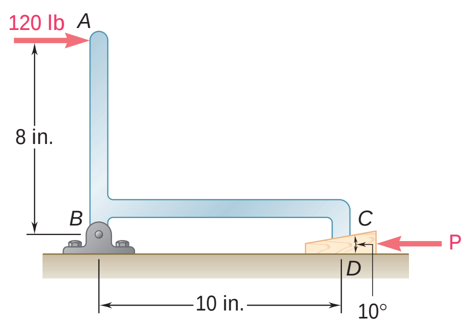


Fig. P8.141

8.142 A conical wedge is placed between two horizontal plates that are then slowly moved toward each other. Indicate what will happen to the wedge (*a*) if $\mu_s = 0.20$, (*b*) if $\mu_s = 0.30$.

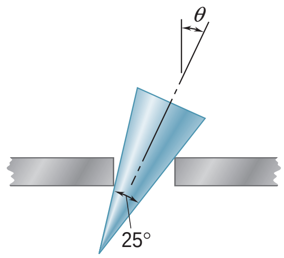


Fig. *P8.142*

8.143 In the machinist's vise shown, the movable jaw *D* is rigidly attached to the tongue *AB* that fits loosely into the fixed body of the vise. The screw is single-threaded into the fixed base and has a mean diameter of 0.75 in. and a pitch of 0.25 in. The coefficient of static friction is 0.25 between the threads and also between the tongue and the body. Neglecting bearing friction between the screw and the movable head, determine the couple that must be applied to the handle in order to produce a clamping force of 1 kip.

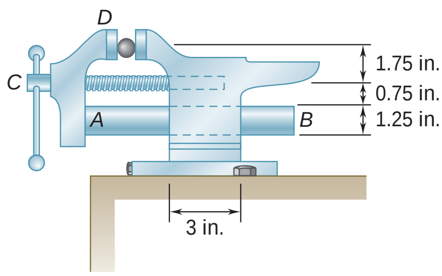


Fig. P8.143

8.144 A lever of negligible weight is loosely fitted onto a 75-mm-diameter fixed shaft. It is observed that the lever will just start rotating if a 3-kg mass is added at *C*. Determine the coefficient of static friction between the shaft and the lever.

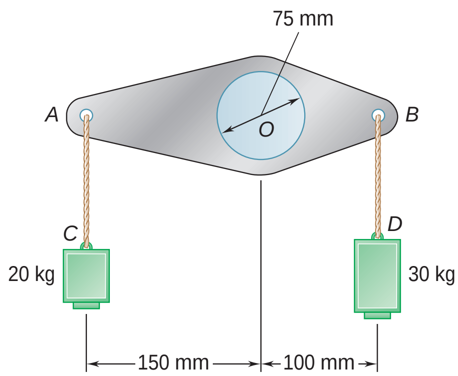


Fig. P8.144

8.145 In the pivoted motor mount shown, the weight **W** of the 175-lb motor is used to maintain tension in the drive belt. Knowing that the coefficient of static friction between the flat belt and drums *A* and *B* is 0.40, and neglecting the weight of platform *CD*, determine the largest couple that can be transmitted to drum *B* when the drive drum *A* is rotating clockwise.

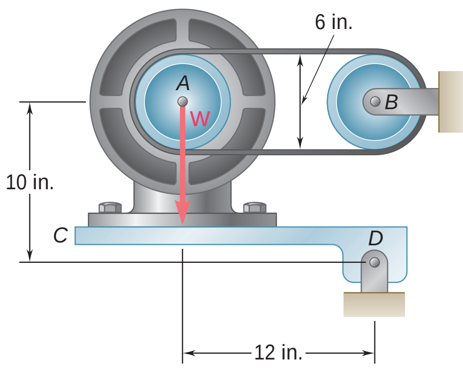


Fig. *P8.145*

COMPUTER PROBLEMS

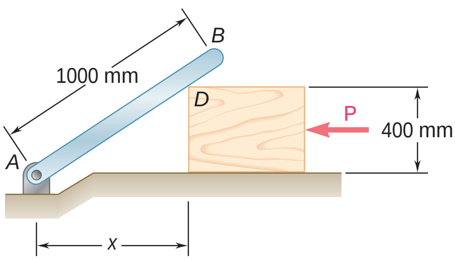


Fig. P8.C1

8.C1 The position of the 10-kg rod AB is controlled by the 2-kg block shown, which is slowly moved to the left by the force **P**. Knowing that the coefficient of kinetic friction between all surfaces of contact is 0.25, write a computer program and use it to calculate the magnitude P of the force for values of x from 900 to 100 mm, using 50-mm decrements. Using appropriate smaller decrements, determine the maximum value of P and the corresponding value of x.

8.C2 Blocks A and B are supported by an incline that is held in the position shown. Knowing that block A weighs 20 lb and that the coefficient of static friction between all surfaces of contact is 0.15, write a computer program and use it to calculate the value of θ for which motion is impending for weights of block B from 0 to 100 lb, using 10-lb increments.

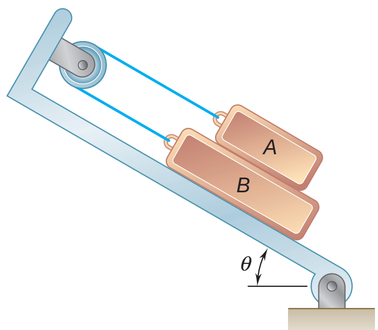


Fig. P8.C2

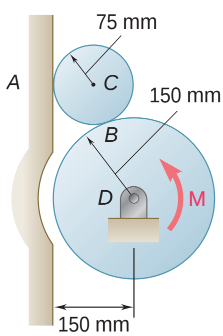


Fig. P8.C3

8.C3 A 300-g cylinder C rests on cylinder D as shown. Knowing that the coefficient of static friction μ_s is the same at A and B, write a computer program and use it to determine, for values of μ_s from 0 to 0.40 and using 0.05 increments, the largest counterclockwise couple **M** that can be applied to cylinder D if it is not to rotate.

8.C4 Two rods are connected by a slider block D and are held in equilibrium by the couple $\mathbf{M}_A$ as shown. Knowing that the coefficient of static friction between rod AC and the slider block is 0.40, write a computer program and use it to determine, for values of θ from 0 to 120° and using 10° increments, the range of values of M_A for which equilibrium is maintained.

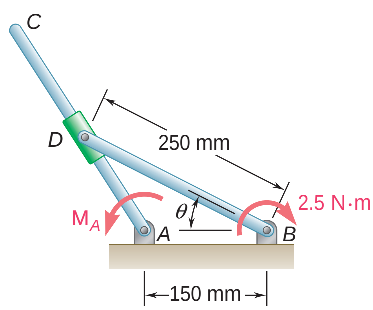


Fig. P8.C4

8.C5 The 10-lb block *A* is slowly moved up the circular cylindrical surface by a cable that passes over a small fixed cylindrical drum at *B*. The coefficient of kinetic friction is known to be 0.30 between the block and the surface and between the cable and the drum. Write a computer program and use it to calculate the force **P** required to maintain the motion for values of u from 0 to 90°, using 10° increments. For the same values of u calculate the magnitude of the reaction between the block and the surface. [Note that the angle of contact between the cable and the fixed drum is $b = p - (u/2)$.]

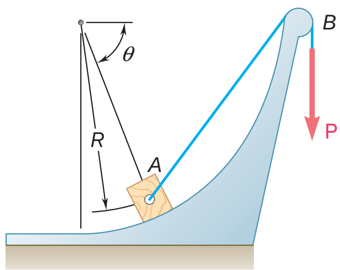


Fig. P8.C5

8.C6 A flat belt is used to transmit a couple from drum *A* to drum *B*. The radius of each drum is 80 mm, and the system is fitted with an idler wheel *C* that is used to increase the contact between the belt and the drums. The allowable belt tension is 200 N, and the coefficient of static friction between the belt and the drums is 0.30. Write a computer program and use it to calculate the largest couple that can be transmitted for values of u from 0 to 30°, using 5° increments.

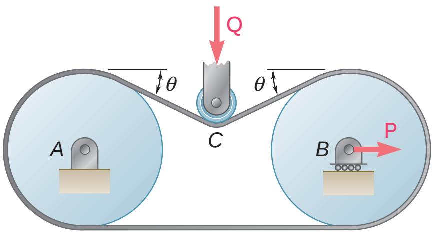


Fig. P8.C6

8.C7 Two collars *A* and *B* that slide on vertical rods with negligible friction are connected by a 30-in. cord that passes over a fixed shaft at *C*. The coefficient of static friction between the cord and the fixed shaft is 0.30. Knowing that the weight of collar *B* is 8 lb, write a computer program and use it to determine, for values of u from 0 to 60° and using 10° increments, the largest and smallest weight of collar *A* for which equilibrium is maintained.

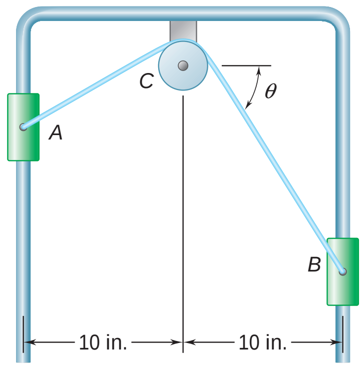


Fig. P8.C7

8.C8 The end *B* of a uniform beam of length *L* is being pulled by a stationary crane. Initially the beam lies on the ground with end *A* directly below pulley *C*. As the cable is slowly pulled in, the beam first slides to the left with $u = 0$ until it has moved through a distance x_0. In a second phase, end *B* is raised, while end *A* keeps sliding to the left until *x* reaches its maximum value x_m and u the corresponding value u_1. The beam then rotates about *A*′ while u keeps increasing. As u reaches the value u_2, end *A* starts sliding to the right and keeps sliding in an irregular manner until *B* reaches *C*. Knowing that the coefficients of friction between the beam and the ground are $m_s = 0.50$ and $m_k = 0.40$, (*a*) write a program to compute *x* for any value of u while the beam is sliding to the left and use this program to determine x_0, x_m, and u_1, (*b*) modify the program to compute for any u the value of *x* for which sliding would be impending to the right and use this new program to determine the value u_2 of u corresponding to $x = x_m$.

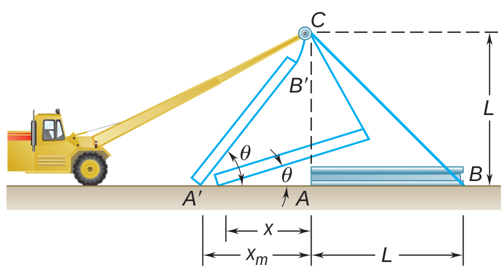
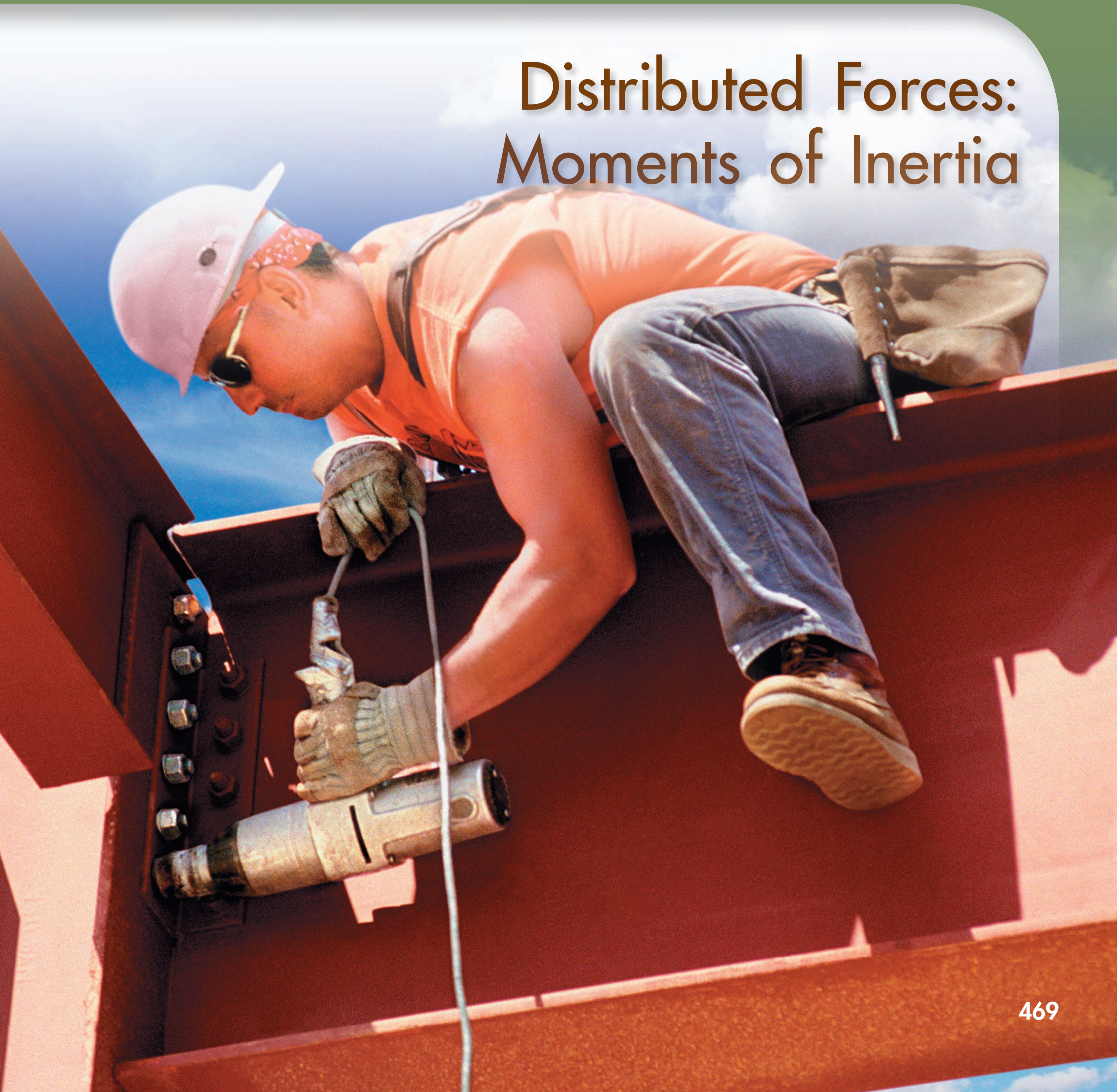


Fig. P8.C8

The strength of structural members used in the construction of buildings depends to a large extent on the properties of their cross sections. This includes the second moments of area, or moments of inertia, of these cross sections.

CHAPTER 9

Distributed Forces: Moments of Inertia



Chapter 9 Distributed Forces: Moments of Inertia

9.1 Introduction
9.2 Second Moment, or Moment of Inertia, of an Area
9.3 Determination of the Moment of Inertia of an Area by Integration
9.4 Polar Moment of Inertia
9.5 Radius of Gyration of an Area
9.6 Parallel-Axis Theorem
9.7 Moments of Inertia of Composite Areas
9.8 Product of Inertia
9.9 Principal Axes and Principal Moments of Inertia
9.10 Mohr's Circle for Moments and Products of Inertia
9.11 Moment of Inertia of a Mass
9.12 Parallel-Axis Theorem
9.13 Moments of Inertia of Thin Plates
9.14 Determination of the Moment of Inertia of a Three-Dimensional Body by Integration
9.15 Moments of Inertia of Composite Bodies
9.16 Moment of Inertia of a Body with Respect to an Arbitrary Axis Through *O*. Mass Products of Inertia
9.17 Ellipsoid of Inertia. Principal Axes of Inertia
9.18 Determination of the Principal Axes and Principal Moments of Inertia of a Body of Arbitrary Shape

9.1 INTRODUCTION

In Chap. 5, we analyzed various systems of forces distributed over an area or volume. The three main types of forces considered were (1) weights of homogeneous plates of uniform thickness (Secs. 5.3 through 5.6), (2) distributed loads on beams (Sec. 5.8) and hydrostatic forces (Sec. 5.9), and (3) weights of homogeneous three-dimensional bodies (Secs. 5.10 and 5.11). In the case of homogeneous plates, the magnitude ΔW of the weight of an element of a plate was proportional to the area ΔA of the element. For distributed loads on beams, the magnitude ΔW of each elemental weight was represented by an element of area $\Delta A = \Delta W$ under the load curve; in the case of hydrostatic forces on submerged rectangular surfaces, a similar procedure was followed. In the case of homogeneous three-dimensional bodies, the magnitude ΔW of the weight of an element of the body was proportional to the volume ΔV of the element. Thus, in all cases considered in Chap. 5, the distributed forces were proportional to the elemental areas or volumes associated with them. The resultant of these forces, therefore, could be obtained by summing the corresponding areas or volumes, and the moment of the resultant about any given axis could be determined by computing the first moments of the areas or volumes about that axis.

In the first part of this chapter, we consider distributed forces $\Delta\mathbf{F}$ whose magnitudes depend not only upon the elements of area ΔA on which these forces act but also upon the distance from ΔA to some given axis. More precisely, the magnitude of the force per unit area $\Delta F/\Delta A$ is assumed to vary linearly with the distance to the axis. As indicated in the next section, forces of this type are found in the study of the bending of beams and in problems involving submerged nonrectangular surfaces. Assuming that the elemental forces involved are distributed over an area A and vary linearly with the distance y to the x axis, it will be shown that while the magnitude of their resultant $\mathbf{R}$ depends upon the first moment $Q_x = \int y\, dA$ of the area A, the location of the point where $\mathbf{R}$ is applied depends upon the *second moment,* or *moment of inertia,* $I_x = \int y^2\, dA$ of the same area with respect to the x axis. You will learn to compute the moments of inertia of various areas with respect to given x and y axes. Also introduced in the first part of this chapter is the *polar moment of inertia* $J_O = \int r^2\, dA$ of an area, where r is the distance from the element of area dA to the point O. To facilitate your computations, a relation will be established between the moment of inertia I_x of an area A with respect to a given x axis and the moment of inertia $I_{x'}$ of the same area with respect to the parallel centroidal x' axis (parallel-axis theorem). You will also study the transformation of the moments of inertia of a given area when the coordinate axes are rotated (Secs. 9.9 and 9.10).

In the second part of the chapter, you will learn how to determine the moments of inertia of various *masses* with respect to a given axis. As you will see in Sec. 9.11, the moment of inertia of a given mass about an axis AA' is defined as $I = \int r^2\, dm$, where r is the distance from the axis AA' to the element of mass dm. Moments of inertia of masses are encountered in dynamics in problems involving the rotation of a rigid body about an axis. To facilitate the computation of mass moments of inertia, the parallel-axis theorem will be introduced (Sec. 9.12). Finally, you will learn to analyze the transformation of moments of inertia of masses when the coordinate axes are rotated (Secs. 9.16 through 9.18).

MOMENTS OF INERTIA OF AREAS

9.2 SECOND MOMENT, OR MOMENT OF INERTIA, OF AN AREA

In the first part of this chapter, we consider distributed forces $\Delta\mathbf{F}$ whose magnitudes ΔF are proportional to the elements of area ΔA on which the forces act and at the same time vary linearly with the distance from ΔA to a given axis.

Consider, for example, a beam of uniform cross section which is subjected to two equal and opposite couples applied at each end of the beam. Such a beam is said to be in *pure bending*, and it is shown in mechanics of materials that the internal forces in any section of the beam are distributed forces whose magnitudes $\Delta F = ky\,\Delta A$ vary linearly with the distance y between the element of area ΔA and an axis passing through the centroid of the section. This axis, represented by the x axis in Fig. 9.1, is known as the *neutral axis* of the section. The forces on one side of the neutral axis are forces of compression, while those on the other side are forces of tension; on the neutral axis itself the forces are zero.

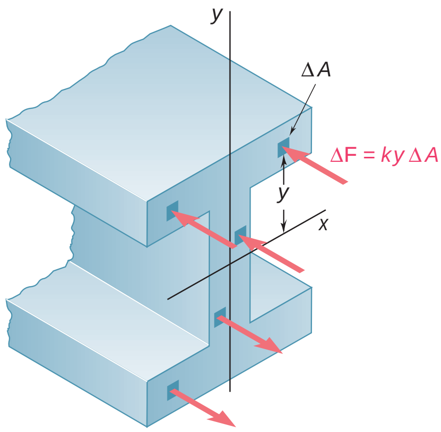


Fig. 9.1

The magnitude of the resultant $\mathbf{R}$ of the elemental forces $\Delta\mathbf{F}$ which act over the entire section is

$$R = \int ky\,dA = k\int y\,dA$$

The last integral obtained is recognized as the *first moment* Q_x of the section about the x axis; it is equal to $\bar{y}A$ and is thus equal to zero, since the centroid of the section is located on the x axis. The system of the forces $\Delta\mathbf{F}$ thus reduces to a couple. The magnitude M of this couple (bending moment) must be equal to the sum of the moments $\Delta M_x = y\,\Delta F = ky^2\,\Delta A$ of the elemental forces. Integrating over the entire section, we obtain

$$M = \int ky^2\,dA = k\int y^2\,dA$$

The last integral is known as the *second moment*, or *moment of inertia*,† of the beam section with respect to the x axis and is denoted by I_x. It is obtained by multiplying each element of area dA by the *square of its distance* from the x axis and integrating over the beam section. Since each product $y^2\,dA$ is positive, regardless of the sign of y, or zero (if y is zero), the integral I_x will always be positive.

Another example of a second moment, or moment of inertia, of an area is provided by the following problem from hydrostatics: A

†The term *second moment* is more proper than the term *moment of inertia*, since, logically, the latter should be used only to denote integrals of mass (see Sec. 9.11). In engineering practice, however, moment of inertia is used in connection with areas as well as masses.

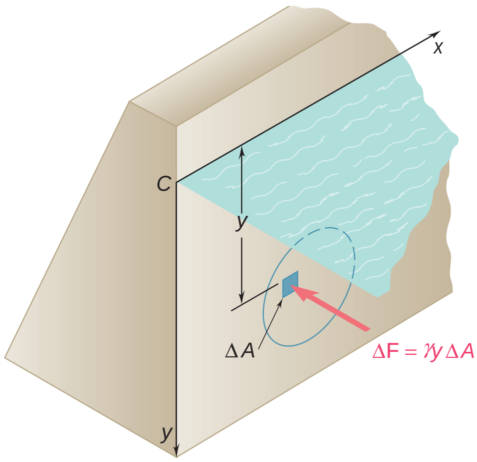


Fig. 9.2

vertical circular gate used to close the outlet of a large reservoir is submerged under water as shown in Fig. 9.2. What is the resultant of the forces exerted by the water on the gate, and what is the moment of the resultant about the line of intersection of the plane of the gate and the water surface (x axis)?

If the gate were rectangular, the resultant of the forces of pressure could be determined from the pressure curve, as was done in Sec. 5.9. Since the gate is circular, however, a more general method must be used. Denoting by y the depth of an element of area ΔA and by g the specific weight of water, the pressure at the element is $p = \mathrm{g}y$, and the magnitude of the elemental force exerted on ΔA is $\Delta F = p\,\Delta A = \mathrm{g}y\,\Delta A$. The magnitude of the resultant of the elemental forces is thus

$$R = \int \mathrm{g}y\,dA = \mathrm{g}\int y\,dA$$

and can be obtained by computing the first moment $Q_x = \int y\,dA$ of the area of the gate with respect to the x axis. The moment M_x of the resultant must be equal to the sum of the moments $\Delta M_x = y\,\Delta F = \mathrm{g}y^2\,\Delta A$ of the elemental forces. Integrating over the area of the gate, we have

$$M_x = \int \mathrm{g}y^2\,dA = \mathrm{g}\int y^2\,dA$$

Here again, the integral obtained represents the second moment, or moment of inertia, I_x of the area with respect to the x axis.

9.3 DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

We defined in the preceding section the second moment, or moment of inertia, of an area A with respect to the x axis. Defining in a similar way the moment of inertia I_y of the area A with respect to the y axis, we write (Fig. 9.3a)

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA \tag{9.1}$$

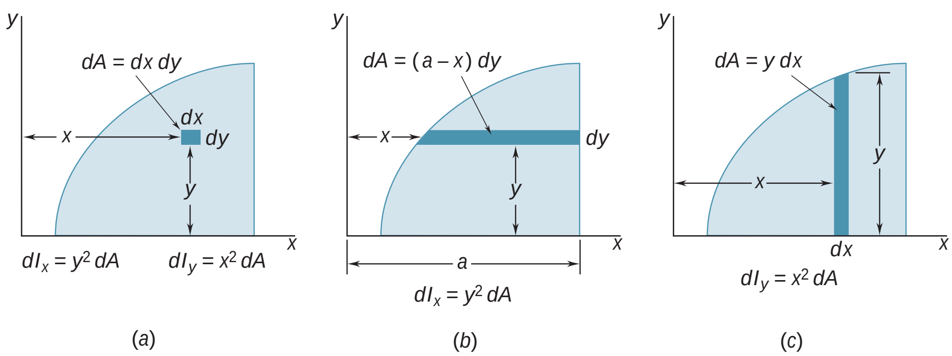


Fig. 9.3

These integrals, known as the *rectangular moments of inertia* of the area A, can be more easily evaluated if we choose dA to be a thin strip parallel to one of the coordinate axes. To compute I_x, the strip is chosen parallel to the x axis, so that all of the points of the strip are at the same distance y from the x axis (Fig. 9.3*b*); the moment of inertia dI_x of the strip is then obtained by multiplying the area dA of the strip by y^2. To compute I_y, the strip is chosen parallel to the y axis so that all of the points of the strip are at the same distance x from the y axis (Fig. 9.3*c*); the moment of inertia dI_y of the strip is $x^2\,dA$.

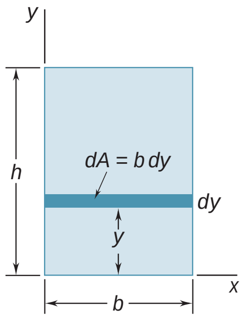


Fig. 9.4

Moment of Inertia of a Rectangular Area. As an example, let us determine the moment of inertia of a rectangle with respect to its base (Fig. 9.4). Dividing the rectangle into strips parallel to the x axis, we obtain

$$dA = b\,dy \qquad dI_x = y^2 b\,dy$$

$$I_x = \int_0^h by^2\,dy = \tfrac{1}{3}bh^3 \tag{9.2}$$

Computing I_x and I_y Using the Same Elemental Strips. The formula just derived can be used to determine the moment of inertia dI_x with respect to the x axis of a rectangular strip which is parallel to the y axis, such as the strip shown in Fig. 9.3*c*. Setting $b = dx$ and $h = y$ in formula (9.2), we write

$$dI_x = \tfrac{1}{3}y^3\,dx$$

On the other hand, we have

$$dI_y = x^2\,dA = x^2 y\,dx$$

The same element can thus be used to compute the moments of inertia I_x and I_y of a given area (Fig. 9.5).

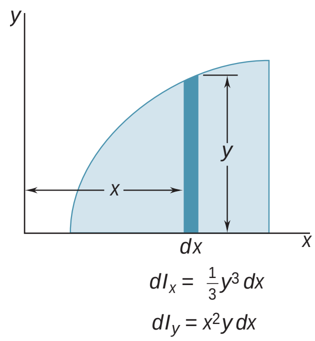


Fig. 9.5

9.4 POLAR MOMENT OF INERTIA

An integral of great importance in problems concerning the torsion of cylindrical shafts and in problems dealing with the rotation of slabs is

$$\boxed{J_O = \int r^2\,dA} \tag{9.3}$$

where r is the distance from O to the element of area dA (Fig. 9.6). This integral is the *polar moment of inertia* of the area A with respect to *the "pole"* O.

The polar moment of inertia of a given area can be computed from the rectangular moments of inertia I_x and I_y of the area if these quantities are already known. Indeed, noting that $r^2 = x^2 + y^2$, we write

$$J_O = \int r^2\,dA = \int (x^2 + y^2)\,dA = \int y^2\,dA + \int x^2\,dA$$

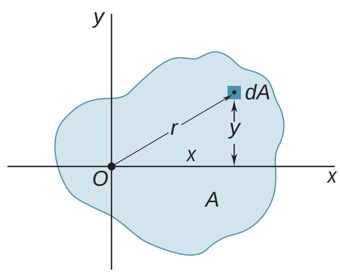


Fig. 9.6

that is,

$$J_O = I_x + I_y \tag{9.4}$$

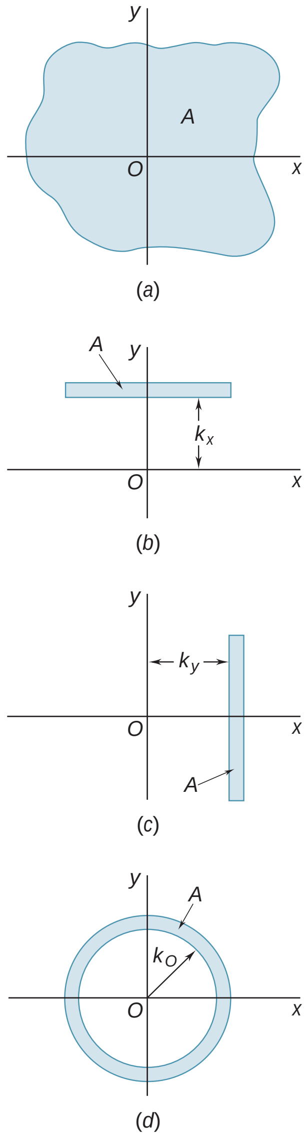


Fig. 9.7

9.5 RADIUS OF GYRATION OF AN AREA

Consider an area A which has a moment of inertia I_x with respect to the x axis (Fig. 9.7*a*). Let us imagine that we concentrate this area into a thin strip parallel to the x axis (Fig. 9.7*b*). If the area A, thus concentrated, is to have the same moment of inertia with respect to the x axis, the strip should be placed at a distance k_x from the x axis, where k_x is defined by the relation

$$I_x = k_x^2 A$$

Solving for k_x, we write

$$k_x = \sqrt{\frac{I_x}{A}} \tag{9.5}$$

The distance k_x is referred to as the *radius of gyration* of the area with respect to the x axis. In a similar way, we can define the radii of gyration k_y and k_O (Fig. 9.7*c* and *d*); we write

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}} \tag{9.6}$$

$$J_O = k_O^2 A \qquad k_O = \sqrt{\frac{J_O}{A}} \tag{9.7}$$

If we rewrite Eq. (9.4) in terms of the radii of gyration, we find that

$$k_O^2 = k_x^2 + k_y^2 \tag{9.8}$$

EXAMPLE For the rectangle shown in Fig. 9.8, let us compute the radius of gyration k_x with respect to its base. Using formulas (9.5) and (9.2), we write

$$k_x^2 = \frac{I_x}{A} = \frac{\frac{1}{3}bh^3}{bh} = \frac{h^2}{3} \qquad k_x = \frac{h}{\sqrt{3}}$$

The radius of gyration k_x of the rectangle is shown in Fig. 9.8. It should not be confused with the ordinate $\bar{y} = h/2$ of the centroid of the area. While k_x depends upon the *second moment*, or moment of inertia, of the area, the ordinate $\bar{y}$ is related to the *first moment* of the area. ■

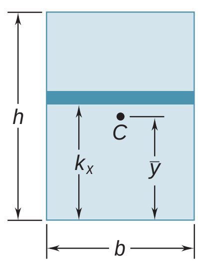
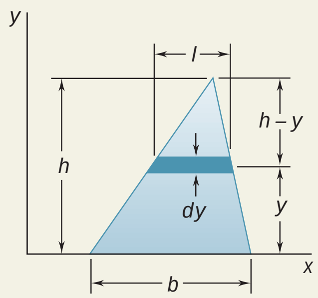


Fig. 9.8

SAMPLE PROBLEM 9.1

Determine the moment of inertia of a triangle with respect to its base.



SOLUTION

A triangle of base b and height h is drawn; the x axis is chosen to coincide with the base. A differential strip parallel to the x axis is chosen to be dA. Since all portions of the strip are at the same distance from the x axis, we write

$$dI_x = y^2\,dA \qquad dA = l\,dy$$

Using similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

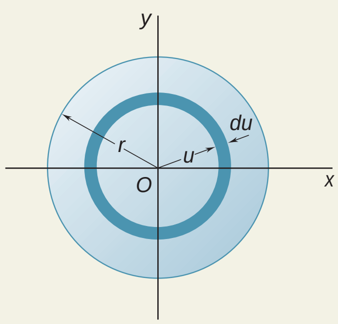
Integrating dI_x from $y = 0$ to $y = h$, we obtain

$$I_x = \int y^2\,dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h (hy^2 - y^3)\,dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h \qquad I_x = \frac{bh^3}{12} \blacktriangleleft$$

SAMPLE PROBLEM 9.2

(a) Determine the centroidal polar moment of inertia of a circular area by direct integration. (b) Using the result of part a, determine the moment of inertia of a circular area with respect to a diameter.



SOLUTION

a. Polar Moment of Inertia. An annular differential element of area is chosen to be dA. Since all portions of the differential area are at the same distance from the origin, we write

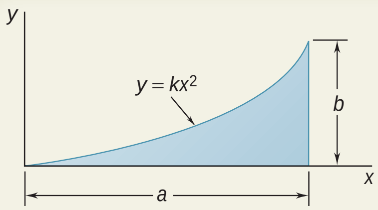
$$dJ_O = u^2\,dA \qquad dA = 2\pi u\,du$$

$$J_O = \int dJ_O = \int_0^r u^2(2\pi u\,du) = 2\pi\int_0^r u^3\,du$$

$$J_O = \frac{\pi}{2}r^4 \blacktriangleleft$$

b. Moment of Inertia with Respect to a Diameter. Because of the symmetry of the circular area, we have $I_x = I_y$. We then write

$$J_O = I_x + I_y = 2I_x \qquad \frac{\pi}{2}r^4 = 2I_x \qquad I_{\text{diameter}} = I_x = \frac{\pi}{4}r^4 \blacktriangleleft$$



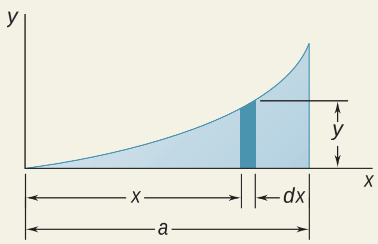
SAMPLE PROBLEM 9.3

(a) Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes. (Properties of this area were considered in Sample Prob. 5.4.) (b) Using the results of part a, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.

SOLUTION

Referring to Sample Prob. 5.4, we obtain the following expressions for the equation of the curve and the total area:

$$y = \frac{b}{a^2}x^2 \qquad A = \tfrac{1}{3}ab$$



Moment of Inertia I_x. A vertical differential element of area is chosen to be dA. Since all portions of this element are *not* at the same distance from the x axis, we must treat the element as a thin rectangle. The moment of inertia of the element with respect to the x axis is then

$$dI_x = \tfrac{1}{3}y^3\,dx = \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx = \frac{1}{3}\frac{b^3}{a^6}x^6\,dx$$

$$I_x = \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a^6}x^6\,dx = \left[\frac{1}{3}\frac{b^3}{a^6}\frac{x^7}{7}\right]_0^a$$

$$I_x = \frac{ab^3}{21} \quad ◀$$

Moment of Inertia I_y. The same vertical differential element of area is used. Since all portions of the element are at the same distance from the y axis, we write

$$dI_y = x^2\,dA = x^2(y\,dx) = x^2\left(\frac{b}{a^2}x^2\right)dx = \frac{b}{a^2}x^4\,dx$$

$$I_y = \int dI_y = \int_0^a \frac{b}{a^2}x^4\,dx = \left[\frac{b}{a^2}\frac{x^5}{5}\right]_0^a$$

$$I_y = \frac{a^3b}{5} \quad ◀$$

Radii of Gyration k_x and k_y. We have, by definition,

$$k_x^2 = \frac{I_x}{A} = \frac{ab^3/21}{ab/3} = \frac{b^2}{7} \qquad k_x = \sqrt{\tfrac{1}{7}}\,b \quad ◀$$

and

$$k_y^2 = \frac{I_y}{A} = \frac{a^3b/5}{ab/3} = \tfrac{3}{5}a^2 \qquad k_y = \sqrt{\tfrac{3}{5}}\,a \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

The purpose of this lesson was to introduce the *rectangular and polar moments of inertia of areas* and the corresponding *radii of gyration*. Although the problems you are about to solve may appear to be more appropriate for a calculus class than for one in mechanics, we hope that our introductory comments have convinced you of the relevance of the moments of inertia to your study of a variety of engineering topics.

1. Calculating the rectangular moments of inertia I_x and I_y. We defined these quantities as

$$I_x = \int y^2\, dA \qquad I_y = \int x^2\, dA \tag{9.1}$$

where dA is a differential element of area $dx\, dy$. The moments of inertia are *the second moments of the area;* it is for that reason that I_x, for example, depends on the perpendicular distance y to the area dA. As you study Sec. 9.3, you should recognize the importance of carefully defining the shape and the orientation of dA. Further, you should note the following points.

a. The moments of inertia of most areas can be obtained by means of a single integration. The expressions given in Figs. 9.3*b* and *c* and Fig. 9.5 can be used to calculate I_x and I_y. Regardless of whether you use a single or a double integration, be sure to show on your sketch the element dA that you have chosen.

b. The moment of inertia of an area is always positive, regardless of the location of the area with respect to the coordinate axes. This is because it is obtained by integrating the product of dA and the *square* of distance. (Note how this differs from the results for the first moment of the area.) Only when an area is *removed* (as in the case for a hole) will its moment of inertia be entered in your computations with a minus sign.

c. As a partial check of your work, observe that the moments of inertia are equal to an area times the square of a length. Thus, every term in an expression for a moment of inertia must be a *length to the fourth power.*

2. Computing the polar moment of inertia J_O. We defined J_O as

$$J_O = \int r^2\, dA \tag{9.3}$$

where $r^2 = x^2 + y^2$. If the given area has circular symmetry (as in Sample Prob. 9.2), it is possible to express dA as a function of r and to compute J_O with a single integration. When the area lacks circular symmetry, it is usually easier first to calculate I_x and I_y and then to determine J_O from

$$J_O = I_x + I_y \tag{9.4}$$

Lastly, if the equation of the curve that bounds the given area is expressed in polar coordinates, then $dA = r\, dr\, d\theta$ and a double integration is required to compute the integral for J_O [see Prob. 9.27].

3. Determining the radii of gyration k_x and k_y and the polar radius of gyration k_O. These quantities were defined in Sec. 9.5, and you should realize that they can be determined only after the area and the appropriate moments of inertia have been computed. It is important to remember that k_x is measured in the y direction, while k_y is measured in the x direction; you should carefully study Sec. 9.5 until you understand this point.

PROBLEMS

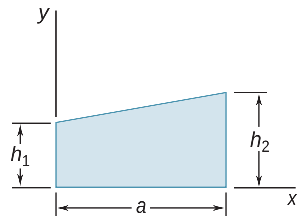


Fig. P9.1 and *P9.5*

9.1 through 9.4 Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

***9.5* through 9.8** Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

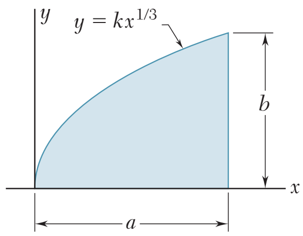


Fig. P9.2 and P9.6

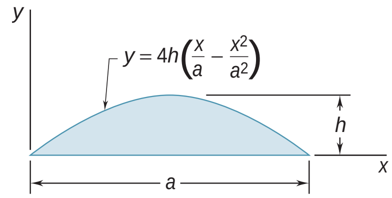


Fig. P9.3 and *P9.7*

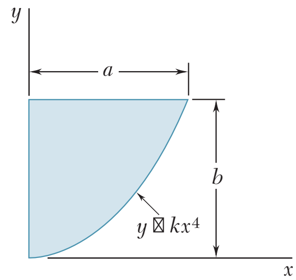


Fig. P9.4 and P9.8

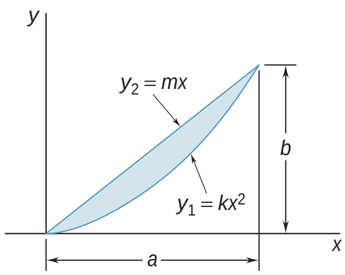


Fig. P9.9 and P9.12

9.9 through 9.11 Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

9.12 through *9.14* Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

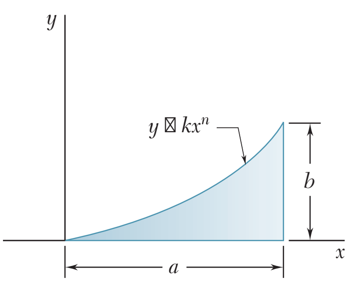


Fig. P9.10 and *P9.13*

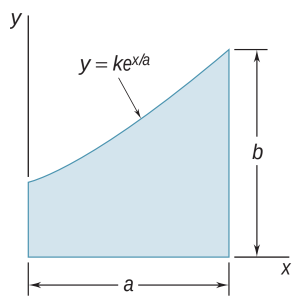


Fig. P9.11 and *P9.14*

9.15 and 9.16 Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

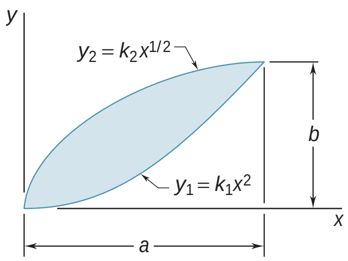


Fig. P9.15 and P9.17

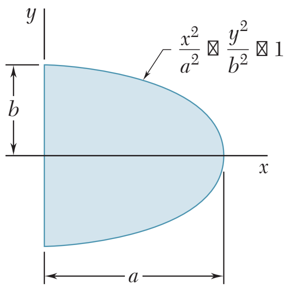


Fig. P9.16 and P9.18

9.17 and 9.18 Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

9.19 Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

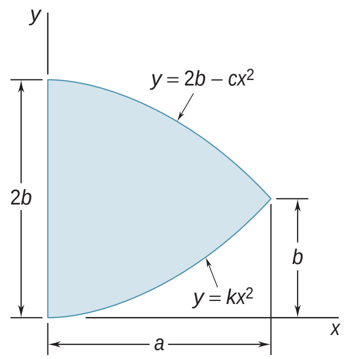


Fig. *P9.19* and *P9.20*

9.20 Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

9.21 and 9.22 Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to point P.

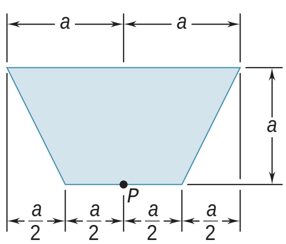


Fig. P9.21

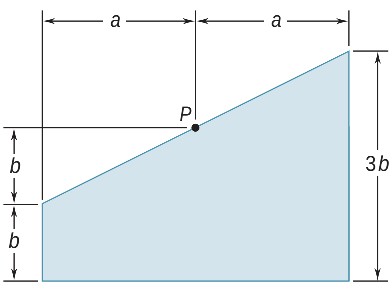


Fig. P9.22

9.23 and 9.24 Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to point *P*.

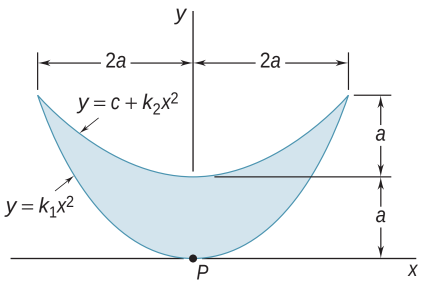


Fig. P9.23

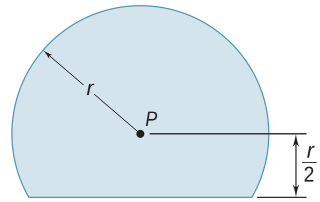


Fig. P9.24

9.25 (*a*) Determine by direct integration the polar moment of inertia of the semiannular area shown with respect to point *O*. (*b*) Using the result of part *a*, determine the moments of inertia of the given area with respect to the x and y axes.

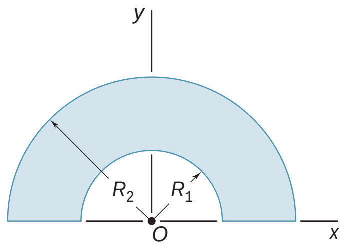


Fig. P9.25 and P9.26

9.26 (*a*) Show that the polar radius of gyration k_O of the semiannular area shown is approximately equal to the mean radius $R_m = (R_1 + R_2)/2$ for small values of the thickness $t = R_2 - R_1$. (*b*) Determine the percentage error introduced by using R_m in place of k_O for the following values of t/R_m: 1, $\frac{1}{2}$, and $\frac{1}{10}$.

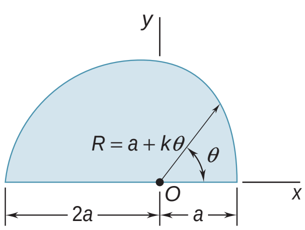


Fig. P9.27

9.27 Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to the point *O*.

9.28 Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to the point *O*.

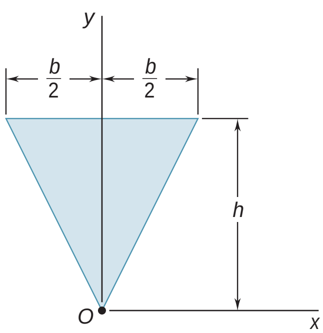


Fig. P9.28

***9.29** Using the polar moment of inertia of the isosceles triangle of Prob. 9.28, show that the centroidal polar moment of inertia of a circular area of radius r is $\pi r^4/2$. (*Hint:* As a circular area is divided into an increasing number of equal circular sectors, what is the approximate shape of each circular sector?)

***9.30** Prove that the centroidal polar moment of inertia of a given area A cannot be smaller than $A^2/2\pi$. (*Hint:* Compare the moment of inertia of the given area with the moment of inertia of a circle that has the same area and the same centroid.)

9.6 PARALLEL-AXIS THEOREM

Consider the moment of inertia I of an area A with respect to an axis AA' (Fig. 9.9). Denoting by y the distance from an element of area dA to AA', we write

$$I = \int y^2\, dA$$

Let us now draw through the centroid C of the area an axis BB' parallel to AA'; this axis is called a *centroidal axis*. Denoting by y'

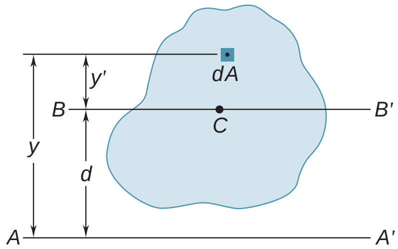


Fig. 9.9

the distance from the element dA to BB', we write $y = y' + d$, where d is the distance between the axes AA' and BB'. Substituting for y in the above integral, we write

$$I = \int y^2\, dA = \int (y' + d)^2\, dA$$

$$= \int y'^2\, dA + 2d \int y'\, dA + d^2 \int dA$$

The first integral represents the moment of inertia $\bar{I}$ of the area with respect to the centroidal axis BB'. The second integral represents the first moment of the area with respect to BB'; since the centroid C of the area is located on that axis, the second integral must be zero. Finally, we observe that the last integral is equal to the total area A. Therefore, we have

$$\boxed{I = \bar{I} + Ad^2} \tag{9.9}$$

This formula expresses that the moment of inertia I of an area with respect to any given axis AA' is equal to the moment of inertia $\bar{I}$ of the area with respect to a centroidal axis BB' parallel to AA' *plus* the product of the area A and the square of the distance d between the two axes. This theorem is known as the *parallel-axis theorem*. Substituting k^2A for I and $\bar{k}^2A$ for $\bar{I}$, the theorem can also be expressed as

$$k^2 = \bar{k}^2 + d^2 \tag{9.10}$$

A similar theorem can be used to relate the polar moment of inertia J_O of an area about a point O to the polar moment of inertia $\bar{J}_C$ of the same area about its centroid C. Denoting by d the distance between O and C, we write

$$J_O = \bar{J}_C + Ad^2 \qquad \text{or} \qquad k_O^2 = \bar{k}_C^2 + d^2 \tag{9.11}$$

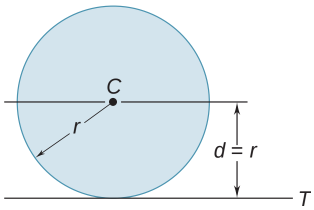


Fig. 9.10

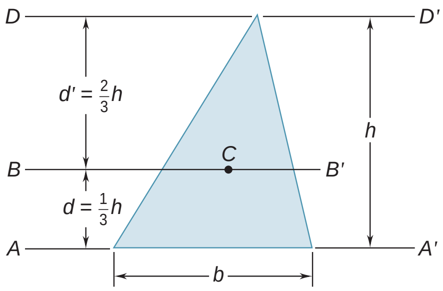


Fig. 9.11

Photo 9.1 Figure 9.13 tabulates data for a small sample of the rolled-steel shapes that are readily available. Shown above are two examples of wide-flange shapes that are commonly used in the construction of buildings.

EXAMPLE 1 As an application of the parallel-axis theorem, let us determine the moment of inertia I_T of a circular area with respect to a line tangent to the circle (Fig. 9.10). We found in Sample Prob. 9.2 that the moment of inertia of a circular area about a centroidal axis is $\bar{I} = \frac{1}{4}\pi r^4$. We can write, therefore,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + (\pi r^2)r^2 = \tfrac{5}{4}\pi r^4$$ ■

EXAMPLE 2 The parallel-axis theorem can also be used to determine the centroidal moment of inertia of an area when the moment of inertia of the area with respect to a parallel axis is known. Consider, for instance, a triangular area (Fig. 9.11). We found in Sample Prob. 9.1 that the moment of inertia of a triangle with respect to its base AA' is equal to $\frac{1}{12}bh^3$. Using the parallel-axis theorem, we write

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$\bar{I}_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh(\tfrac{1}{3}h)^2 = \tfrac{1}{36}bh^3$$

It should be observed that the product Ad^2 was *subtracted* from the given moment of inertia in order to obtain the centroidal moment of inertia of the triangle. Note that this product is *added* when transferring *from* a centroidal axis to a parallel axis, but it should be *subtracted* when transferring *to* a centroidal axis. In other words, the moment of inertia of an area is always smaller with respect to a centroidal axis than with respect to any parallel axis.

Returning to Fig. 9.11, we observe that the moment of inertia of the triangle with respect to the line DD' (which is drawn through a vertex) can be obtained by writing

$$I_{DD'} = \bar{I}_{BB'} + Ad'^2 = \tfrac{1}{36}bh^3 + \tfrac{1}{2}bh(\tfrac{2}{3}h)^2 = \tfrac{1}{4}bh^3$$

Note that $I_{DD'}$ could not have been obtained directly from $I_{AA'}$. The parallel-axis theorem can be applied only if one of the two parallel axes passes through the centroid of the area. ■

9.7 MOMENTS OF INERTIA OF COMPOSITE AREAS

Consider a composite area A made of several component areas A_1, A_2, A_3, . . . Since the integral representing the moment of inertia of A can be subdivided into integrals evaluated over A_1, A_2, A_3, . . . , the moment of inertia of A with respect to a given axis is obtained by adding the moments of inertia of the areas A_1, A_2, A_3, . . . , with respect to the same axis. The moment of inertia of an area consisting of several of the common shapes shown in Fig. 9.12 can thus be obtained by using the formulas given in that figure. Before adding the moments of inertia of the component areas, however, the parallel-axis theorem may have to be used to transfer each moment of inertia to the desired axis. This is shown in Sample Probs. 9.4 and 9.5.

The properties of the cross sections of various structural shapes are given in Fig. 9.13. As noted in Sec. 9.2, the moment of inertia of a beam section about its neutral axis is closely related to the computation of the bending moment in that section of the beam. The

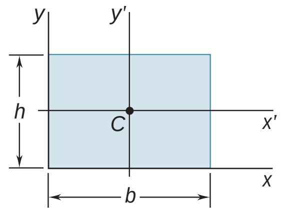
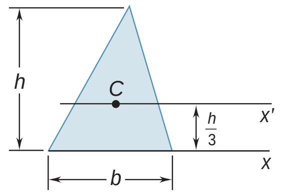
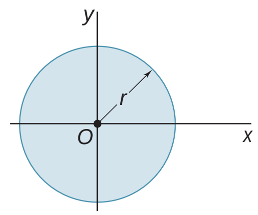
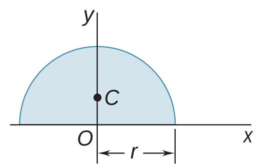
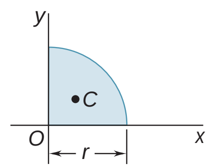
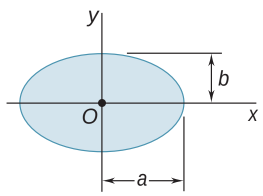
Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Fig. 9.12 Moments of inertia of common geometric shapes.

determination of moments of inertia is thus a prerequisite to the analysis and design of structural members.

It should be noted that the radius of gyration of a composite area is *not* equal to the sum of the radii of gyration of the component areas. In order to determine the radius of gyration of a composite area, it is first necessary to compute the moment of inertia of the area.

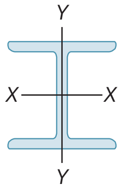
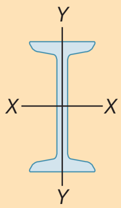
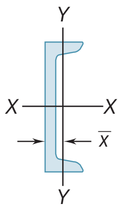
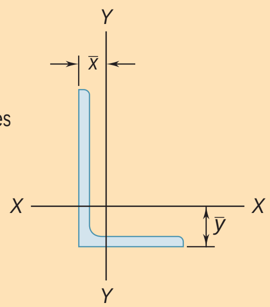
	Designation	Area in^2	Depth in.	Width in.	Axis X-X			Axis Y-Y		
					$\bar{I}_x$, in^4	$\bar{k}_x$, in.	$\bar{y}$, in.	$\bar{I}_y$, in^4	$\bar{k}_y$, in.	$\bar{x}$, in.
W Shapes (Wide-Flange Shapes) 	W18 × 76†	22.3	18.2	11.0	1330	7.73		152	2.61	
	W16 × 57	16.8	16.4	7.12	758	6.72		43.1	1.60	
	W14 × 38	11.2	14.1	6.77	385	5.87		26.7	1.55	
	W8 × 31	9.12	8.00	8.00	110	3.47		37.1	2.02	
S Shapes (American Standard Shapes) 	S18 × 54.7†	16.0	18.0	6.00	801	7.07		20.7	1.14	
	S12 × 31.8	9.31	12.0	5.00	217	4.83		9.33	1.00	
	S10 × 25.4	7.45	10.0	4.66	123	4.07		6.73	0.950	
	S6 × 12.5	3.66	6.00	3.33	22.0	2.45		1.80	0.702	
C Shapes (American Standard Channels) 	C12 × 20.7†	6.08	12.0	2.94	129	4.61		3.86	0.797	0.698
	C10 × 15.3	4.48	10.0	2.60	67.3	3.87		2.27	0.711	0.634
	C8 × 11.5	3.37	8.00	2.26	32.5	3.11		1.31	0.623	0.572
	C6 × 8.2	2.39	6.00	1.92	13.1	2.34		0.687	0.536	0.512
Angles 	L6 × 6 × 1‡	11.0			35.4	1.79	1.86	35.4	1.79	1.86
	L4 × 4 × $\frac{1}{2}$	3.75			5.52	1.21	1.18	5.52	1.21	1.18
	L3 × 3 × $\frac{1}{4}$	1.44			1.23	0.926	0.836	1.23	0.926	0.836
	L6 × 4 × $\frac{1}{2}$	4.75			17.3	1.91	1.98	6.22	1.14	0.981
	L5 × 3 × $\frac{1}{2}$	3.75			9.43	1.58	1.74	2.55	0.824	0.746
	L3 × 2 × $\frac{1}{4}$	1.19			1.09	0.953	0.980	0.390	0.569	0.487

Fig. 9.13A Properties of rolled-steel shapes (U.S. customary units).*

*Courtesy of the American Institute of Steel Construction, Chicago, Illinois

†Nominal depth in inches and weight in pounds per foot

‡Depth, width, and thickness in inches

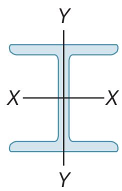
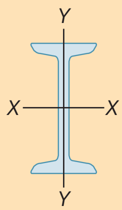
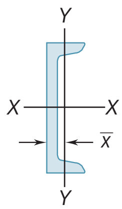
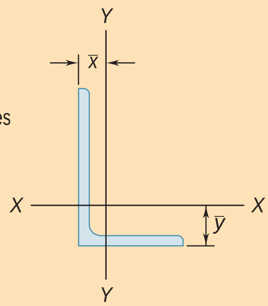
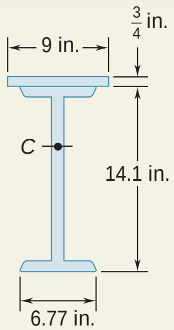
	Designation	Area mm²	Depth mm	Width mm	Axis X-X: $\bar{I}_x$ 10^6 mm⁴	Axis X-X: $\bar{k}_x$ mm	Axis X-X: $\bar{y}$ mm	Axis Y-Y: $\bar{I}_y$ 10^6 mm⁴	Axis Y-Y: $\bar{k}_y$ mm	Axis Y-Y: $\bar{x}$ mm
W Shapes (Wide-Flange Shapes) 	W460 × 113†	14400	462	279	554	196		63.3	66.3	
	W410 × 85	10800	417	181	316	171		17.9	40.6	
	W360 × 57.8	7230	358	172	160	149		11.1	39.4	
	W200 × 46.1	5880	203	203	45.8	88.1		15.4	51.3	
S Shapes (American Standard Shapes) 	S460 × 81.4†	10300	457	152	333	180		8.62	29.0	
	S310 × 47.3	6010	305	127	90.3	123		3.88	25.4	
	S250 × 37.8	4810	254	118	51.2	103		2.80	24.1	
	S150 × 18.6	2360	152	84.6	9.16	62.2		0.749	17.8	
C Shapes (American Standard Channels) 	C310 × 30.8†	3920	305	74.7	53.7	117		1.61	20.2	17.7
	C250 × 22.8	2890	254	66.0	28.0	98.3		0.945	18.1	16.1
	C200 × 17.1	2170	203	57.4	13.5	79.0		0.545	15.8	14.5
	C150 × 12.2	1540	152	48.8	5.45	59.4		0.286	13.6	13.0
Angles 	L152 × 152 × 25.4‡	7100			14.7	45.5	47.2	14.7	45.5	47.2
	L102 × 102 × 12.7	2420			2.30	30.7	30.0	2.30	30.7	30.0
	L76 × 76 × 6.4	929			0.512	23.5	21.2	0.512	23.5	21.2
	L152 × 102 × 12.7	3060			7.20	48.5	50.3	2.59	29.0	24.9
	L127 × 76 × 12.7	2420			3.93	40.1	44.2	1.06	20.9	18.9
	L76 × 51 × 6.4	768			0.454	24.2	24.9	0.162	14.5	12.4

Fig. 9.13B Properties of rolled-steel shapes (SI units).

†Nominal depth in millimeters and mass in kilograms per meter

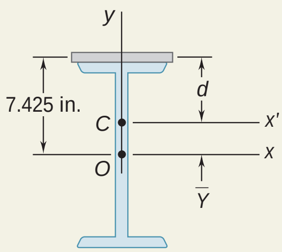
‡Depth, width, and thickness in millimeters

SAMPLE PROBLEM 9.4



The strength of a W14 × 38 rolled-steel beam is increased by attaching a $9 \times \frac{3}{4}$-in. plate to its upper flange as shown. Determine the moment of inertia and the radius of gyration of the composite section with respect to an axis which is parallel to the plate and passes through the centroid C of the section.

SOLUTION



The origin O of the coordinates is placed at the centroid of the wide-flange shape, and the distance $\bar{Y}$ to the centroid of the composite section is computed using the methods of Chap. 5. The area of the wide-flange shape is found by referring to Fig. 9.13A. The area and the y coordinate of the centroid of the plate are

$$A = (9 \text{ in.})(0.75 \text{ in.}) = 6.75 \text{ in}^2$$
$$\bar{y} = \tfrac{1}{2}(14.1 \text{ in.}) + \tfrac{1}{2}(0.75 \text{ in.}) = 7.425 \text{ in.}$$

Section	Area, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
Plate	6.75	7.425	50.12
Wide-flange shape	11.2	0	0
	$\Sigma A = 17.95$		$\Sigma \bar{y}A = 50.12$

$$\bar{Y}\Sigma A = \Sigma \bar{y}A \qquad \bar{Y}(17.95) = 50.12 \qquad \bar{Y} = 2.792 \text{ in.}$$

Moment of Inertia. The parallel-axis theorem is used to determine the moments of inertia of the wide-flange shape and the plate with respect to the x' axis. This axis is a centroidal axis for the composite section but *not* for either of the elements considered separately. The value of $\bar{I}_x$ for the wide-flange shape is obtained from Fig. 9.13A.

For the wide-flange shape,

$$I_{x'} = \bar{I}_x + A\bar{Y}^2 = 385 + (11.2)(2.792)^2 = 472.3 \text{ in}^4$$

For the plate,

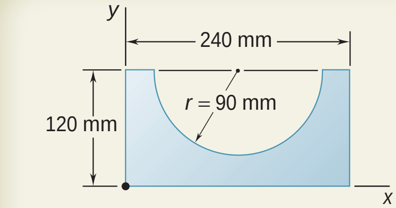
$$I_{x'} = \bar{I}_x + Ad^2 = (\tfrac{1}{12})(9)(\tfrac{3}{4})^3 + (6.75)(7.425 - 2.792)^2 = 145.2 \text{ in}^4$$

For the composite area,

$$I_{x'} = 472.3 + 145.2 = 617.5 \text{ in}^4 \qquad I_{x'} = 618 \text{ in}^4 \blacktriangleleft$$

Radius of Gyration. We have

$$k_{x'}^2 = \frac{I_{x'}}{A} = \frac{617.5 \text{ in}^4}{17.95 \text{ in}^2} \qquad k_{x'} = 5.87 \text{ in.} \blacktriangleleft$$

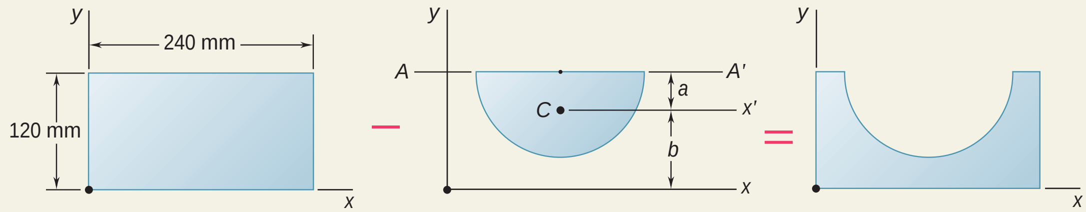


SAMPLE PROBLEM 9.5

Determine the moment of inertia of the shaded area with respect to the x axis.

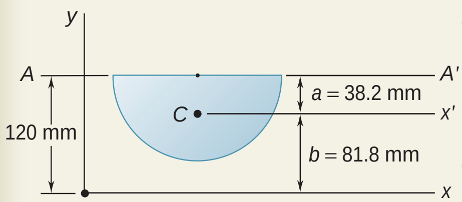
SOLUTION

The given area can be obtained by subtracting a half circle from a rectangle. The moments of inertia of the rectangle and the half circle will be computed separately.



Moment of Inertia of Rectangle. Referring to Fig. 9.12, we obtain

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240 \text{ mm})(120 \text{ mm})^3 = 138.2 \times 10^6 \text{ mm}^4$$



Moment of Inertia of Half Circle. Referring to Fig. 5.8, we determine the location of the centroid C of the half circle with respect to diameter AA'.

$$a = \frac{4r}{3\pi} = \frac{(4)(90 \text{ mm})}{3\pi} = 38.2 \text{ mm}$$

The distance b from the centroid C to the x axis is

$$b = 120 \text{ mm} - a = 120 \text{ mm} - 38.2 \text{ mm} = 81.8 \text{ mm}$$

Referring now to Fig. 9.12, we compute the moment of inertia of the half circle with respect to diameter AA'; we also compute the area of the half circle.

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90 \text{ mm})^4 = 25.76 \times 10^6 \text{ mm}^4$$
$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90 \text{ mm})^2 = 12.72 \times 10^3 \text{ mm}^2$$

Using the parallel-axis theorem, we obtain the value of $\bar{I}_{x'}$:

$$I_{AA'} = \bar{I}_{x'} + Aa^2$$
$$25.76 \times 10^6 \text{ mm}^4 = \bar{I}_{x'} + (12.72 \times 10^3 \text{ mm}^2)(38.2 \text{ mm})^2$$
$$\bar{I}_{x'} = 7.20 \times 10^6 \text{ mm}^4$$

Again using the parallel-axis theorem, we obtain the value of I_x:

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 \text{ mm}^4 + (12.72 \times 10^3 \text{ mm}^2)(81.8 \text{ mm})^2$$
$$= 92.3 \times 10^6 \text{mm}^4$$

Moment of Inertia of Given Area. Subtracting the moment of inertia of the half circle from that of the rectangle, we obtain

$$I_x = 138.2 \times 10^6 \text{ mm}^4 - 92.3 \times 10^6 \text{ mm}^4$$

$$I_x = 45.9 \times 10^6 \text{ mm}^4$$ ◀

SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced the *parallel-axis theorem* and illustrated how it can be used to simplify the computation of moments and polar moments of inertia of composite areas. The areas that you will consider in the following problems will consist of common shapes and rolled-steel shapes. You will also use the parallel-axis theorem to locate the point of application (the center of pressure) of the resultant of the hydrostatic forces acting on a submerged plane area.

1. Applying the parallel-axis theorem. In Sec. 9.6 we derived the parallel-axis theorem

$$I = \bar{I} + Ad^2 \tag{9.9}$$

which states that the moment of inertia I of an area A with respect to a given axis is equal to the sum of the moment of inertia $\bar{I}$ of that area with respect to a *parallel centroidal axis* and the product Ad^2, where d is the distance between the two axes. It is important that you remember the following points as you use the parallel-axis theorem.

a. The centroidal moment of inertia $\bar{I}$ of an area A can be obtained by subtracting the product Ad^2 from the moment of inertia I of the area with respect to a parallel axis. It follows that the moment of inertia $\bar{I}$ is *smaller* than the moment of inertia I of the same area with respect to any parallel axis.

b. The parallel-axis theorem can be applied only if one of the two axes involved is a centroidal axis. Therefore, as we noted in Example 2, to compute the moment of inertia of an area with respect to a *noncentroidal axis* when the moment of inertia of the area is known with respect to *another noncentroidal axis,* it is necessary to *first compute* the moment of inertia of the area with respect to a *centroidal axis parallel to the two given axes*.

2. Computing the moments and polar moments of inertia of composite areas. Sample Probs. 9.4 and 9.5 illustrate the steps you should follow to solve problems of this type. As with all composite-area problems, you should show on your sketch the common shapes or rolled-steel shapes that constitute the various elements of the given area, as well as the distances between the centroidal axes of the elements and the axes about which the moments of inertia are to be computed. In addition, it is important that the following points be noted.

a. The moment of inertia of an area is always positive, regardless of the location of the axis with respect to which it is computed. As pointed out in the comments for the preceding lesson, it is only when an area is *removed* (as in the case of a hole) that its moment of inertia should be entered in your computations with a minus sign.

b. The moments of inertia of a semiellipse and a quarter ellipse can be determined by dividing the moment of inertia of an ellipse by 2 and 4, respectively. It should be noted, however, that the moments of inertia obtained in this manner are *with respect to the axes of symmetry of the ellipse*. To obtain the *centroidal* moments of inertia of these shapes, the parallel-axis theorem should be used. Note that this remark also applies to a semicircle and to a quarter circle and that the expressions given for these shapes in Fig. 9.12 are *not* centroidal moments of inertia.

c. To calculate the polar moment of inertia of a composite area, you can use either the expressions given in Fig. 9.12 for J_O or the relationship

$$J_O = I_x + I_y \tag{9.4}$$

depending on the shape of the given area.

d. Before computing the centroidal moments of inertia of a given area, you may find it necessary to first locate the centroid of the area using the methods of Chap. 5.

3. Locating the point of application of the resultant of a system of hydrostatic forces. In Sec. 9.2 we found that

$$R = \gamma \int y\,dA = \gamma \bar{y} A$$

$$M_x = \gamma \int y^2\,dA = \gamma I_x$$

where $\bar{y}$ is the distance from the x axis to the centroid of the submerged plane area. Since $\mathbf{R}$ is equivalent to the system of elemental hydrostatic forces, it follows that

$$\Sigma M_x: \qquad y_P R = M_x$$

where y_P is the depth of the point of application of $\mathbf{R}$. Then

$$y_P(\gamma \bar{y} A) = \gamma I_x \qquad \text{or} \qquad y_P = \frac{I_x}{\bar{y} A}$$

In closing, we encourage you to carefully study the notation used in Fig. 9.13 for the rolled-steel shapes, as you will likely encounter it again in subsequent engineering courses.

PROBLEMS

9.31 and 9.32 Determine the moment of inertia and the radius of gyration of the shaded area with respect to the x axis.

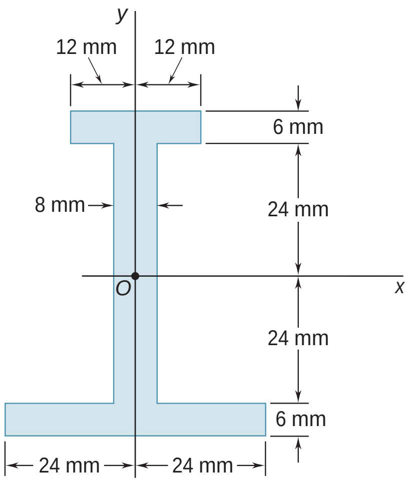


Fig. P9.31 and P9.33

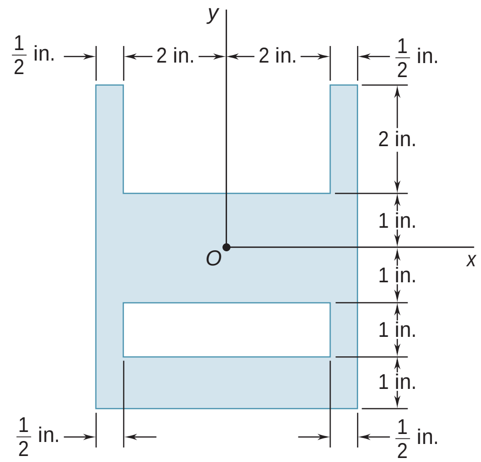


Fig. P9.32 and P9.34

9.33 and 9.34 Determine the moment of inertia and the radius of gyration of the shaded area with respect to the y axis.

9.35 and 9.36 Determine the moments of inertia of the shaded area shown with respect to the x and y axes when $a = 20$ mm.

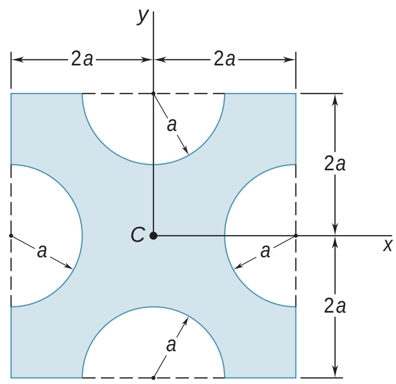


Fig. *P9.35*

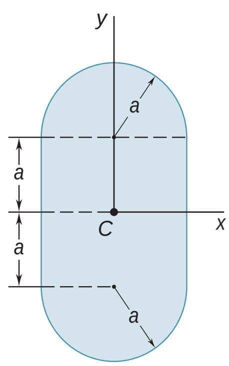


Fig. *P9.36*

9.37 The shaded area is equal to 50 in^2. Determine its centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$, knowing that $\bar{I}_y = 2\bar{I}_x$ and that the polar moment of inertia of the area about point A is $J_A = 2250$ in^4.

9.38 The polar moments of inertia of the shaded area with respect to points A, B, and D are, respectively, $J_A = 2880$ in^4, $J_B = 6720$ in^4, and $J_D = 4560$ in^4. Determine the shaded area, its centroidal moment of inertia $\bar{J}_C$, and the distance d from C to D.

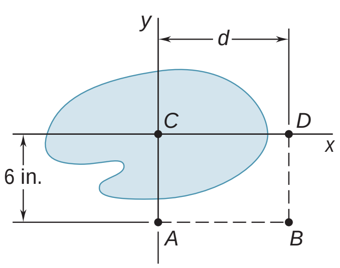


Fig. P9.37 and *P9.38*

9.39 Determine the shaded area and its moment of inertia with respect to the centroidal axis parallel to AA', knowing that $d_1 = 30$ mm and $d_2 = 10$ mm, and that the moments of inertia with respect to AA' and BB' are 4.1×10^6 mm^4 and 6.9×10^6 mm^4, respectively.

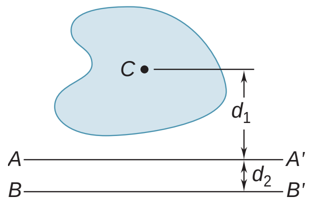


Fig. P9.39 and P9.40

9.40 Knowing that the shaded area is equal to 7500 mm^2 and that its moment of inertia with respect to AA' is 31×10^6 mm^4, determine its moment of inertia with respect to BB', for $d_1 = 60$ mm and $d_2 = 15$ mm.

9.41 through 9.44 Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

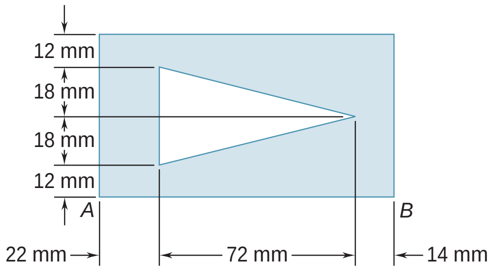


Fig. P9.41

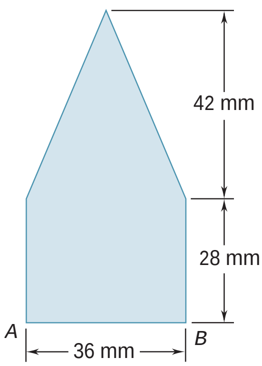


Fig. P9.42

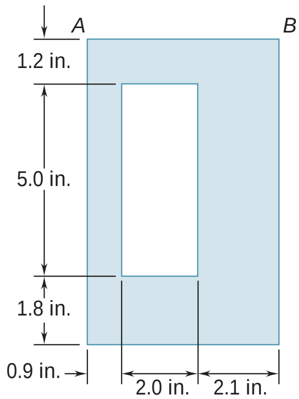


Fig. P9.43

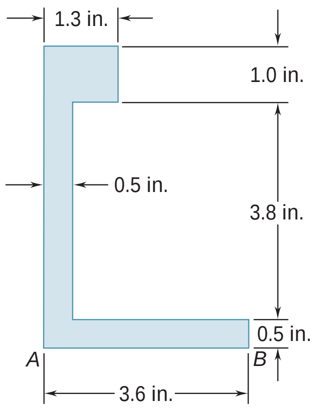


Fig. P9.44

9.45 and 9.46 Determine the polar moment of inertia of the area shown with respect to (*a*) point O, (*b*) the centroid of the area.

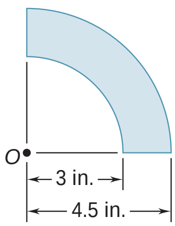


Fig. *P9.45*

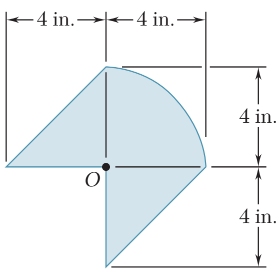


Fig. *P9.46*

9.47 and 9.48 Determine the polar moment of inertia of the area shown with respect to (*a*) point *O*, (*b*) the centroid of the area.

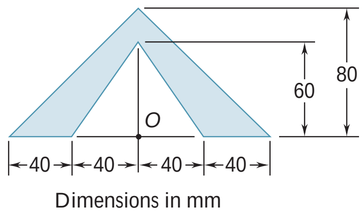


Fig. P9.47

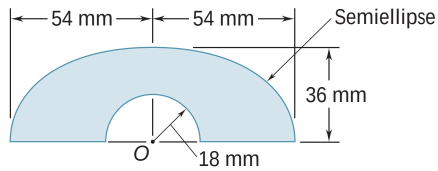


Fig. P9.48

9.49 Two channels and two plates are used to form the column section shown. For $b = 200$ mm, determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

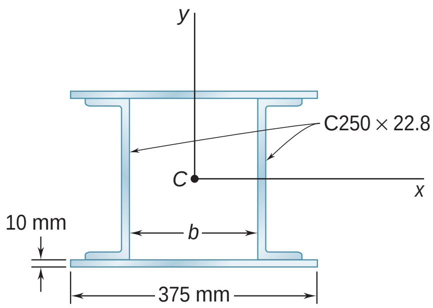


Fig. P9.49

9.50 Two L6 × 4 × $\frac{1}{2}$-in. angles are welded together to form the section shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

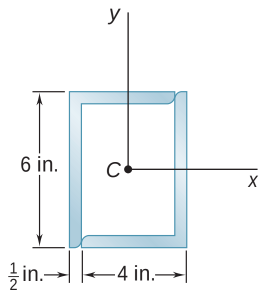


Fig. P9.50

9.51 Two channels are welded to a rolled W section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

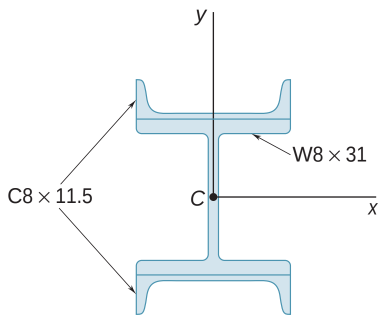


Fig. P9.51

9.52 Two 20-mm steel plates are welded to a rolled S section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

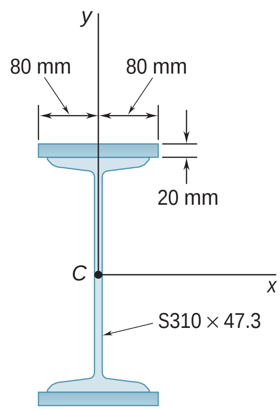


Fig. P9.52

9.53 A channel and a plate are welded together as shown to form a section that is symmetrical with respect to the y axis. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

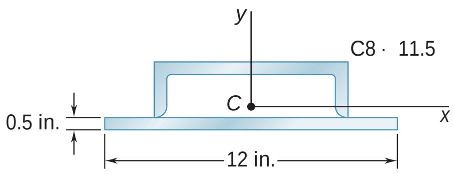


Fig. *P9.53*

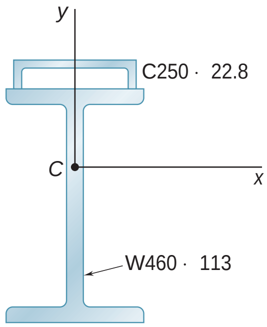


Fig. P9.54

9.54 The strength of the rolled W section shown is increased by welding a channel to its upper flange. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

9.55 Two L76 × 76 × 6.4-mm angles are welded to a C250 × 22.8 channel. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the web of the channel.

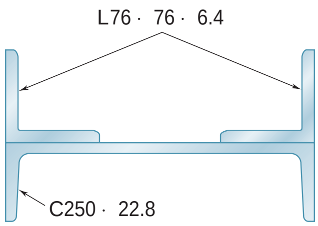


Fig. P9.55

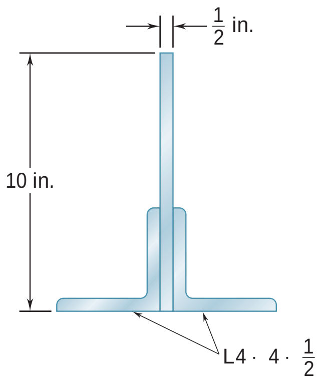


Fig. *P9.56*

9.56 Two L4 × 4 × $\frac{1}{2}$-in. angles are welded to a steel plate as shown. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the plate.

9.57 and 9.58 The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to Sec. 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

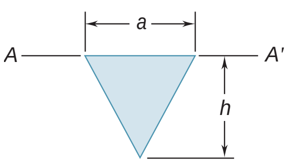


Fig. P9.57

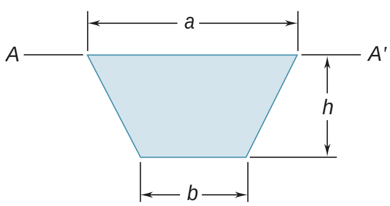


Fig. P9.58

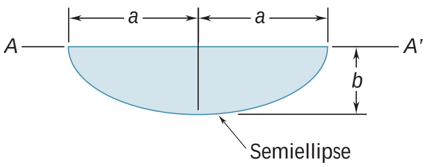


Fig. P9.59

9.59 and *9.60 The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to Sec. 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

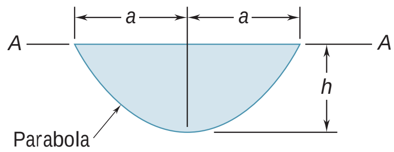


Fig. P9.60

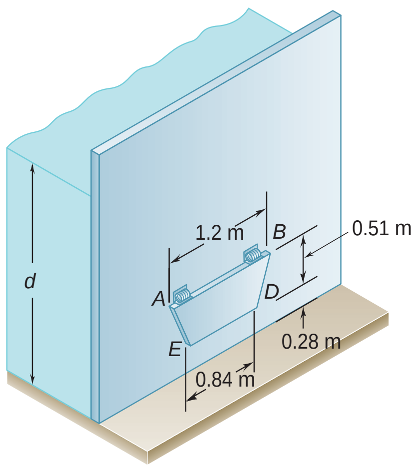


Fig. *P9.61*

9.61 A vertical trapezoidal gate that is used as an automatic valve is held shut by two springs attached to hinges located along edge AB. Knowing that each spring exerts a couple of magnitude 1470 N · m, determine the depth d of water for which the gate will open.

9.62 The cover for a 0.5-m-diameter access hole in a water storage tank is attached to the tank with four equally spaced bolts as shown. Determine the additional force on each bolt due to the water pressure when the center of the cover is located 1.4 m below the water surface.

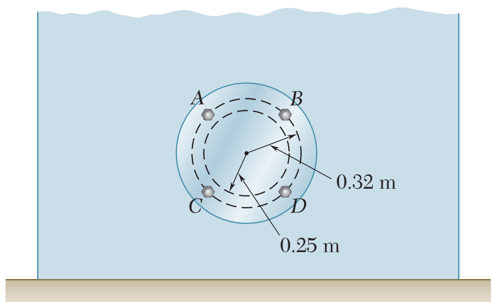


Fig. *P9.62*

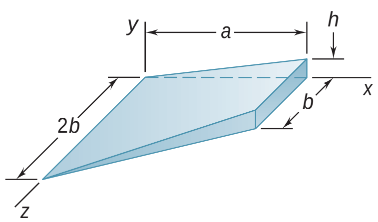


Fig. P9.63

***9.63** Determine the x coordinate of the centroid of the volume shown. (*Hint:* The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

***9.64** Determine the x coordinate of the centroid of the volume shown; this volume was obtained by intersecting an elliptic cylinder with an oblique plane. (See hint of Prob. 9.63.)

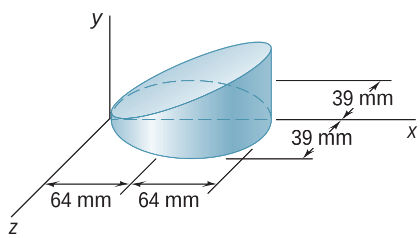


Fig. P9.64

***9.65** Show that the system of hydrostatic forces acting on a submerged plane area A can be reduced to a force $\mathbf{P}$ at the centroid C of the area and two couples. The force $\mathbf{P}$ is perpendicular to the area and is of magnitude $P = gA\bar{y} \sin u$, where g is the specific weight of the liquid, and the couples are $\mathbf{M}_{x'} = (g\bar{I}_{x'} \sin u)\mathbf{i}$ and $\mathbf{M}_{y'} = (g\bar{I}_{x'y'} \sin u)\mathbf{j}$, where $\bar{I}_{x'y'} = \int x'y'\, dA$ (see Sec. 9.8). Note that the couples are independent of the depth at which the area is submerged.

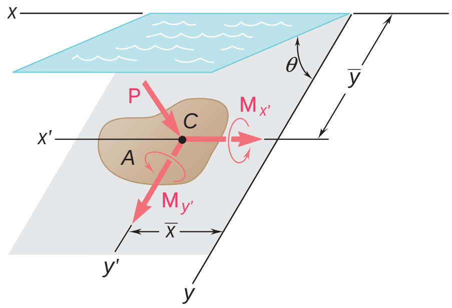


Fig. P9.65

***9.66** Show that the resultant of the hydrostatic forces acting on a submerged plane area A is a force $\mathbf{P}$ perpendicular to the area and of magnitude $P = gA\bar{y} \sin u = \bar{p}A$, where g is the specific weight of the liquid and $\bar{p}$ is the pressure at the centroid C of the area. Show that $\mathbf{P}$ is applied at a point C_P, called the center of pressure, whose coordinates are $x_p = I_{xy}/A\bar{y}$ and $y_p = I_x/A\bar{y}$, where $I_{xy} = \int xy\, dA$ (see Sec. 9.8). Show also that the difference of ordinates $y_p - \bar{y}$ is equal to $\bar{k}_{x'}^2/\bar{y}$ and thus depends upon the depth at which the area is submerged.

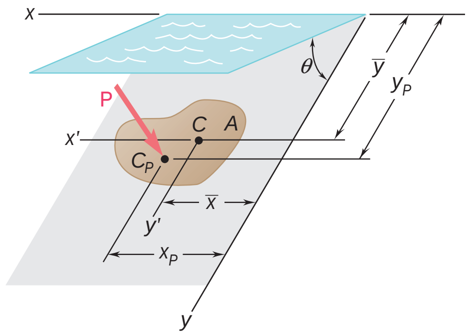


Fig. P9.66

*9.8 PRODUCT OF INERTIA

The integral

$$I_{xy} = \int xy\, dA \tag{9.12}$$

which is obtained by multiplying each element dA of an area A by its coordinates x and y and integrating over the area (Fig. 9.14), is known as the *product of inertia* of the area A with respect to the x and y axes. Unlike the moments of inertia I_x and I_y, the product of inertia I_{xy} can be positive, negative, or zero.

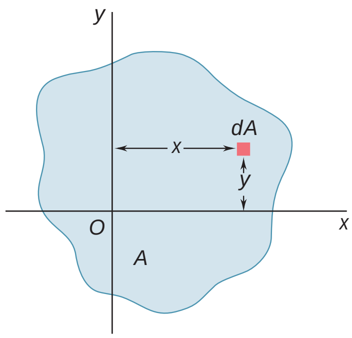


Fig. 9.14

When one or both of the x and y axes are axes of symmetry for the area A, the product of inertia I_{xy} is zero. Consider, for example, the channel section shown in Fig. 9.15. Since this section is symmetrical with respect to the x axis, we can associate with each element dA of coordinates x and y an element dA' of coordinates x and $-y$. Clearly, the contributions to I_{xy} of any pair of elements chosen in this way cancel out, and the integral (9.12) reduces to zero.

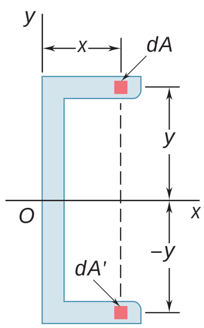


Fig. 9.15

A parallel-axis theorem similar to the one established in Sec. 9.6 for moments of inertia can be derived for products of inertia. Consider an area A and a system of rectangular coordinates x and y

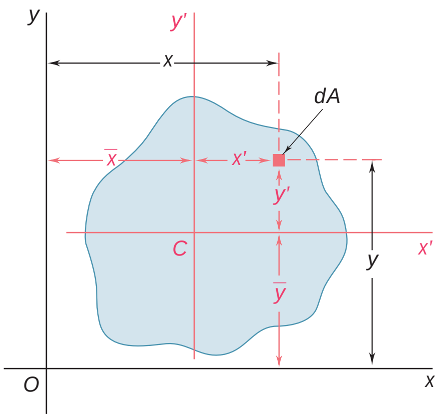


Fig. 9.16

(Fig. 9.16). Through the centroid C of the area, of coordinates $\bar{x}$ and $\bar{y}$, we draw two *centroidal axes* x' and y' which are parallel, respectively, to the x and y axes. Denoting by x and y the coordinates of an element of area dA with respect to the original axes, and by x' and y' the coordinates of the same element with respect to the centroidal axes, we write $x = x' + \bar{x}$ and $y = y' + \bar{y}$. Substituting into (9.12), we obtain the following expression for the product of inertia I_{xy}:

$$I_{xy} = \int xy\,dA = \int (x' + \bar{x})(y' + \bar{y})\,dA$$

$$= \int x'y'\,dA + \bar{y}\int x'\,dA + \bar{x}\int y'\,dA + \bar{x}\bar{y}\int dA$$

The first integral represents the product of inertia $\bar{I}_{x'y'}$ of the area A with respect to the centroidal axes x' and y'. The next two integrals represent first moments of the area with respect to the centroidal axes; they reduce to zero, since the centroid C is located on these axes. Finally, we observe that the last integral is equal to the total area A. Therefore, we have

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A \tag{9.13}$$

*9.9 PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA

Consider the area A and the coordinate axes x and y (Fig. 9.17). Assuming that the moments and product of inertia

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA \qquad I_{xy} = \int xy\,dA \tag{9.14}$$

of the area A are known, we propose to determine the moments and product of inertia $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ of A with respect to new axes x' and y' which are obtained by rotating the original axes about the origin through an angle u.

We first note the following relations between the coordinates x', y' and x, y of an element of area dA:

$$x' = x \cos u + y \sin u \qquad y' = y \cos u - x \sin u$$

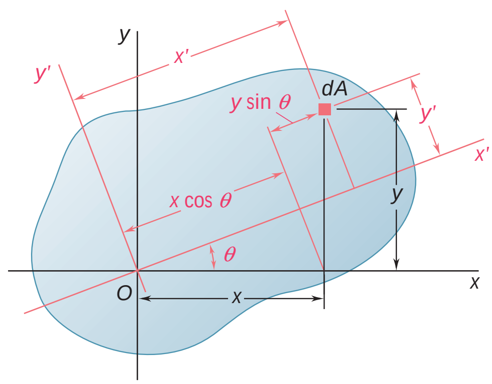


Fig. 9.17

Substituting for y' in the expression for $I_{x'}$, we write

$$I_{x'} = \int (y')^2 \, dA = \int (y \cos \theta - x \sin \theta)^2 \, dA$$
$$= \cos^2 \theta \int y^2 \, dA - 2 \sin \theta \cos \theta \int xy \, dA + \sin^2 \theta \int x^2 \, dA$$

Using the relations (9.14), we write

$$I_{x'} = I_x \cos^2 \theta - 2I_{xy} \sin \theta \cos \theta + I_y \sin^2 \theta \tag{9.15}$$

Similarly, we obtain for $I_{y'}$ and $I_{x'y'}$ the expressions

$$I_{y'} = I_x \sin^2 \theta + 2I_{xy} \sin \theta \cos \theta + I_y \cos^2 \theta \tag{9.16}$$
$$I_{x'y'} = (I_x - I_y) \sin \theta \cos \theta + I_{xy}(\cos^2 \theta - \sin^2 \theta) \tag{9.17}$$

Recalling the trigonometric relations

$$\sin 2\theta = 2 \sin \theta \cos \theta \qquad \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

and

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \qquad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

we can write (9.15), (9.16), and (9.17) as follows:

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \tag{9.18}$$
$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \tag{9.19}$$
$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \tag{9.20}$$

Adding (9.18) and (9.19) we observe that

$$I_{x'} + I_{y'} = I_x + I_y \tag{9.21}$$

This result could have been anticipated, since both members of (9.21) are equal to the polar moment of inertia J_O.

Equations (9.18) and (9.20) are the parametric equations of a circle. This means that if we choose a set of rectangular axes and plot a point M of abscissa $I_{x'}$ and ordinate $I_{x'y'}$ for any given value of the parameter θ, all of the points thus obtained will lie on a circle. To establish this property, we eliminate θ from Eqs. (9.18) and (9.20); this is done by transposing $(I_x + I_y)/2$ in Eq. (9.18), squaring both members of Eqs. (9.18) and (9.20), and adding. We write

$$\left(I_{x'} - \frac{I_x + I_y}{2}\right)^2 + I_{x'y'}^2 = \left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2 \tag{9.22}$$

Setting

$$I_{\text{ave}} = \frac{I_x + I_y}{2} \qquad \text{and} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \tag{9.23}$$

we write the identity (9.22) in the form

$$(I_{x'} - I_{\text{ave}})^2 + I_{x'y'}^2 = R^2 \tag{9.24}$$

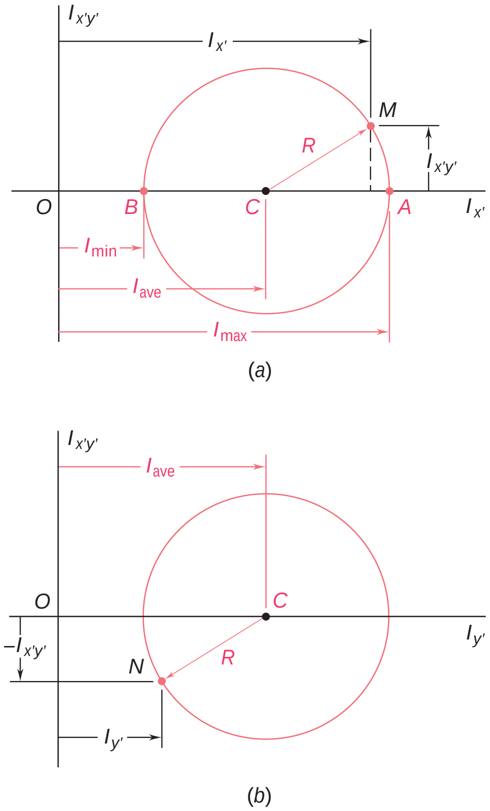


Fig. 9.18

which is the equation of a circle of radius R centered at the point C whose x and y coordinates are I_{ave} and 0, respectively (Fig. 9.18*a*). We observe that Eqs. (9.19) and (9.20) are the parametric equations of the same circle. Furthermore, because of the symmetry of the circle about the horizontal axis, the same result would have been obtained if instead of plotting M, we had plotted a point N of coordinates $I_{y'}$ and $-I_{x'y'}$ (Fig. 9.18*b*). This property will be used in Sec. 9.10.

The two points A and B where the above circle intersects the horizontal axis (Fig. 9.18*a*) are of special interest: Point A corresponds to the maximum value of the moment of inertia $I_{x'}$, while point B corresponds to its minimum value. In addition, both points correspond to a zero value of the product of inertia $I_{x'y'}$. Thus, the values θ_m of the parameter θ which correspond to the points A and B can be obtained by setting $I_{x'y'} = 0$ in Eq. (9.20). We obtain†

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} \tag{9.25}$$

This equation defines two values $2\theta_m$ which are 180° apart and thus two values θ_m which are 90° apart. One of these values corresponds to point A in Fig. 9.18*a* and to an axis through O in Fig. 9.17 with respect to which the moment of inertia of the given area is maximum; the other value corresponds to point B and to an axis through O with respect to which the moment of inertia of the area is minimum. The two axes thus defined, which are perpendicular to each other, are called the *principal axes of the area about* O, and the corresponding values I_{max} and I_{min} of the moment of inertia are called the *principal moments of inertia of the area about* O. Since the two values θ_m defined by Eq. (9.25) were obtained by setting $I_{x'y'} = 0$ in Eq. (9.20), it is clear that the product of inertia of the given area with respect to its principal axes is zero.

We observe from Fig. 9.18*a* that

$$I_{max} = I_{ave} + R \qquad I_{min} = I_{ave} - R \tag{9.26}$$

Using the values for I_{ave} and R from formulas (9.23), we write

$$I_{max,min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \tag{9.27}$$

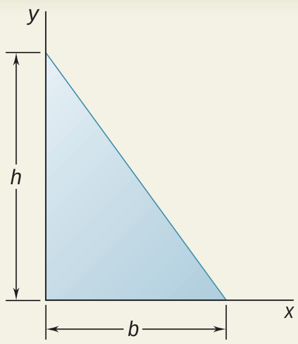
Unless it is possible to tell by inspection which of the two principal axes corresponds to I_{max} and which corresponds to I_{min}, it is necessary to substitute one of the values of θ_m into Eq. (9.18) in order to determine which of the two corresponds to the maximum value of the moment of inertia of the area about O.

Referring to Sec. 9.8, we note that if an area possesses an axis of symmetry through a point O, this axis must be a principal axis of the area about O. On the other hand, a principal axis does not need to be an axis of symmetry; whether or not an area possesses any axes of symmetry, it will have two principal axes of inertia about any point O.

The properties we have established hold for any point O located inside or outside the given area. If the point O is chosen to coincide with the centroid of the area, any axis through O is a centroidal axis; the two principal axes of the area about its centroid are referred to as the *principal centroidal axes of the area*.

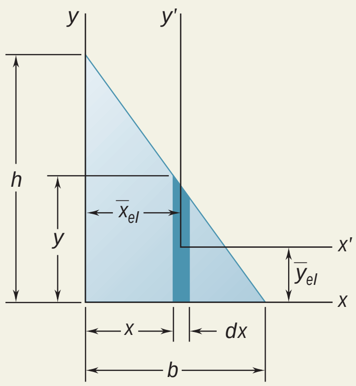
†This relation can also be obtained by differentiating $I_{x'}$ in Eq. (9.18) and setting $dI_{x'}/d\theta = 0$.

SAMPLE PROBLEM 9.6



Determine the product of inertia of the right triangle shown (*a*) with respect to the x and y axes and (*b*) with respect to centroidal axes parallel to the x and y axes.

SOLUTION



a. Product of Inertia I_{xy}. A vertical rectangular strip is chosen as the differential element of area. Using the parallel-axis theorem, we write

$$dI_{xy} = dI_{x'y'} + \bar{x}_{el}\bar{y}_{el}\, dA$$

Since the element is symmetrical with respect to the x' and y' axes, we note that $dI_{x'y'} = 0$. From the geometry of the triangle, we obtain

$$y = h\left(1 - \frac{x}{b}\right) \qquad dA = y\, dx = h\left(1 - \frac{x}{b}\right)dx$$

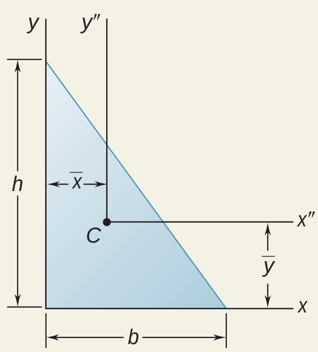
$$\bar{x}_{el} = x \qquad \bar{y}_{el} = \tfrac{1}{2}y = \tfrac{1}{2}h\left(1 - \frac{x}{b}\right)$$

Integrating dI_{xy} from $x = 0$ to $x = b$, we obtain

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el}\bar{y}_{el}\, dA = \int_0^b x(\tfrac{1}{2})h^2\left(1 - \frac{x}{b}\right)^2 dx$$

$$= h^2\int_0^b \left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2}\right)dx = h^2\left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2}\right]_0^b$$

$$I_{xy} = \tfrac{1}{24}b^2h^2 \quad ◀$$



b. Product of Inertia $\bar{I}_{x''y''}$. The coordinates of the centroid of the triangle relative to the x and y axes are

$$\bar{x} = \tfrac{1}{3}b \qquad \bar{y} = \tfrac{1}{3}h$$

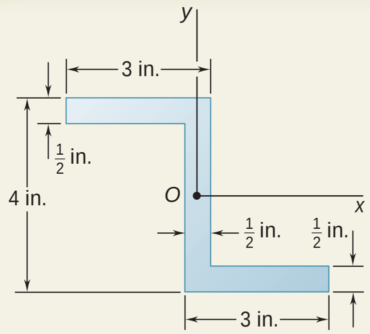
Using the expression for I_{xy} obtained in part *a*, we apply the parallel-axis theorem and write

$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\,\bar{y}A$$

$$\tfrac{1}{24}b^2h^2 = \bar{I}_{x''y''} + (\tfrac{1}{3}b)(\tfrac{1}{3}h)(\tfrac{1}{2}bh)$$

$$\bar{I}_{x''y''} = \tfrac{1}{24}b^2h^2 - \tfrac{1}{18}b^2h^2$$

$$\bar{I}_{x''y''} = -\tfrac{1}{72}b^2h^2 \quad ◀$$



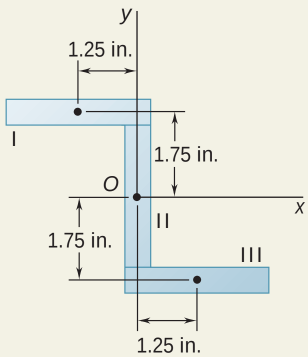
SAMPLE PROBLEM 9.7

For the section shown, the moments of inertia with respect to the x and y axes have been computed and are known to be

$$I_x = 10.38 \text{ in}^4 \qquad I_y = 6.97 \text{ in}^4$$

Determine (*a*) the orientation of the principal axes of the section about O, (*b*) the values of the principal moments of inertia of the section about O.

SOLUTION



We first compute the product of inertia with respect to the x and y axes. The area is divided into three rectangles as shown. We note that the product of inertia $\bar{I}_{x'y'}$ with respect to centroidal axes parallel to the x and y axes is zero for each rectangle. Using the parallel-axis theorem $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$, we find that I_{xy} reduces to $\bar{x}\,\bar{y}A$ for each rectangle.

Rectangle	Area, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\bar{y}A$, in^4
I	1.5	−1.25	+1.75	−3.28
II	1.5	0	0	0
III	1.5	+1.25	−1.75	−3.28
				$\Sigma\bar{x}\bar{y}A = -6.56$

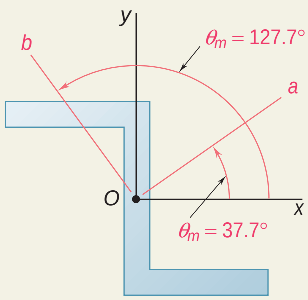
$$I_{xy} = \Sigma\bar{x}\,\bar{y}A = -6.56 \text{ in}^4$$

a. Principal Axes. Since the magnitudes of I_x, I_y, and I_{xy} are known, Eq. (9.25) is used to determine the values of θ_m:

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-6.56)}{10.38 - 6.97} = +3.85$$

$$2\theta_m = 75.4° \text{ and } 255.4°$$

$$\theta_m = 37.7° \qquad \text{and} \qquad \theta_m = 127.7° \quad ◀$$



b. Principal Moments of Inertia. Using Eq. (9.27), we write

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{10.38 + 6.97}{2} \pm \sqrt{\left(\frac{10.38 - 6.97}{2}\right)^2 + (-6.56)^2}$$

$$I_{\max} = 15.45 \text{ in}^4 \qquad I_{\min} = 1.897 \text{ in}^4 \quad ◀$$

Noting that the elements of the area of the section are more closely distributed about the b axis than about the a axis, we conclude that $I_a = I_{\max} = 15.45 \text{ in}^4$ and $I_b = I_{\min} = 1.897 \text{ in}^4$. This conclusion can be verified by substituting $\theta = 37.7°$ into Eqs. (9.18) and (9.19).

SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will continue your work with *moments of inertia* and will utilize various techniques for computing *products of inertia*. Although the problems are generally straightforward, several items are worth noting.

1. Calculating the product of inertia I_{xy} by integration. We defined this quantity as

$$I_{xy} = \int xy\, dA \tag{9.12}$$

and stated that its value can be positive, negative, or zero. The product of inertia can be computed directly from the above equation using double integration, or it can be determined using single integration as shown in Sample Prob. 9.6. When applying the latter technique and using the parallel-axis theorem, it is important to remember that $\bar{x}_{el}$ and $\bar{y}_{el}$ in the equation

$$dI_{xy} = dI_{x'y'} + \bar{x}_{el}\bar{y}_{el}\, dA$$

are the coordinates of the centroid of the element of area dA. Thus, if dA is not in the first quadrant, one or both of these coordinates will be negative.

2. Calculating the products of inertia of composite areas. They can easily be computed from the products of inertia of their component parts by using the parallel-axis theorem

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A \tag{9.13}$$

The proper technique to use for problems of this type is illustrated in Sample Probs. 9.6 and 9.7. In addition to the usual rules for composite-area problems, it is essential that you remember the following points.

a. If either of the centroidal axes of a component area is an axis of symmetry for that area, the product of inertia $\bar{I}_{x'y'}$ for that area is zero. Thus, $\bar{I}_{x'y'}$ is zero for component areas such as circles, semicircles, rectangles, and isosceles triangles which possess an axis of symmetry parallel to one of the coordinate axes.

b. Pay careful attention to the signs of the coordinates $\bar{x}$ and $\bar{y}$ of each component area when you use the parallel-axis theorem [Sample Prob. 9.7].

3. Determining the moments of inertia and the product of inertia for rotated coordinate axes. In Sec. 9.9 we derived Eqs. (9.18), (9.19), and (9.20), from which the moments of inertia and the product of inertia can be computed for coordinate axes which have been rotated about the origin O. To apply these equations, you must know a set of values I_x, I_y, and I_{xy} for a given orientation of the axes, and you must remember that u is positive for counterclockwise rotations of the axes and negative for clockwise rotations of the axes.

4. Computing the principal moments of inertia. We showed in Sec. 9.9 that there is a particular orientation of the coordinate axes for which the moments of inertia attain their maximum and minimum values, I_{max} and I_{min}, and for which the product of inertia is zero. Equation (9.27) can be used to compute these values, known as the *principal moments of inertia* of the area about O. The corresponding axes are referred to as the *principal axes* of the area about O, and their orientation is defined by Eq. (9.25). *To determine which of the principal axes corresponds to I_{max} and which corresponds to I_{min}*, you can either follow the procedure outlined in the text after Eq. (9.27) or observe about which of the two principal axes the area is more closely distributed; that axis corresponds to I_{min} [Sample Prob. 9.7].

PROBLEMS

9.67 through 9.70 Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

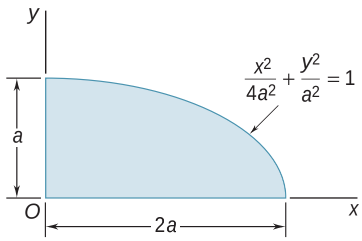


Fig. P9.67

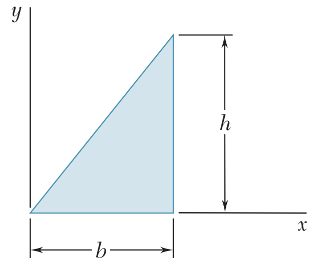


Fig. P9.68

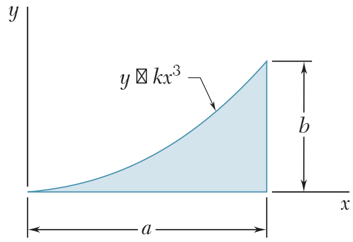


Fig. P9.69

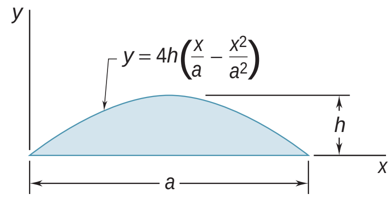


Fig. *P9.70*

9.71 through 9.74 Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

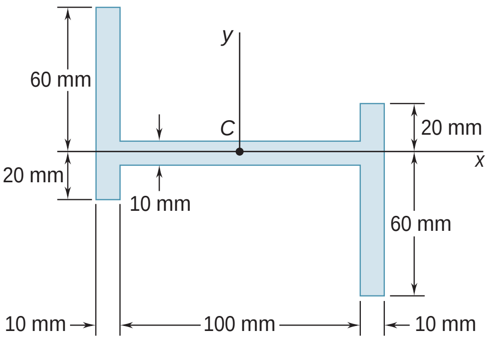


Fig. P9.71

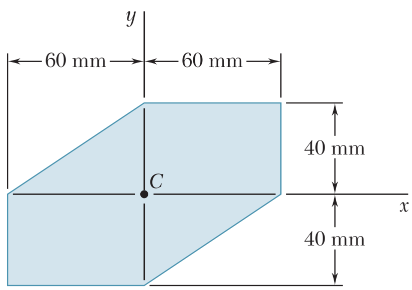


Fig. P9.72

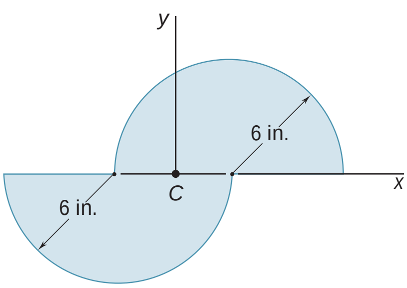


Fig. *P9.73*

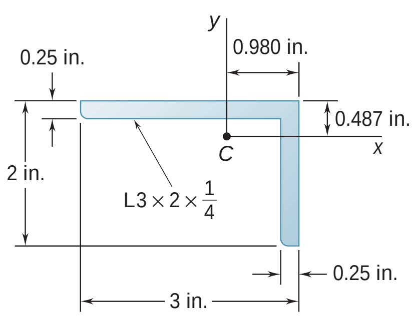


Fig. P9.74

9.75 through 9.78 Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

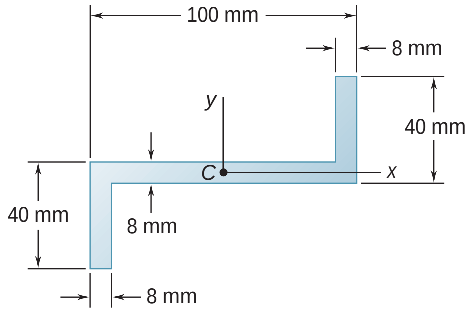


Fig. P9.75

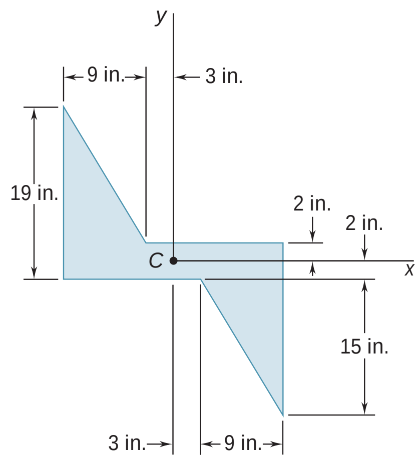


Fig. P9.76

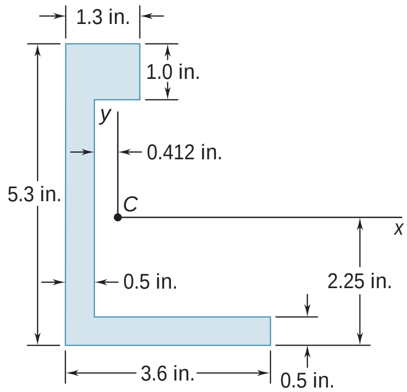


Fig. *P9.77*

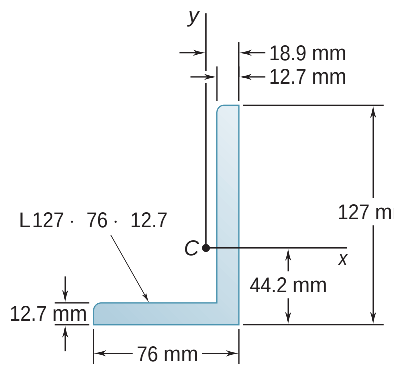


Fig. P9.78

9.79 Determine for the quarter ellipse of Prob. 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the x and y axes about O (*a*) through 45° counterclockwise, (*b*) through 30° clockwise.

9.80 Determine the moments of inertia and the product of inertia of the area of Prob. 9.72 with respect to new centroidal axes obtained by rotating the x and y axes 30° counterclockwise.

9.81 Determine the moments of inertia and the product of inertia of the area of Prob. 9.73 with respect to new centroidal axes obtained by rotating the x and y axes 60° counterclockwise.

9.82 Determine the moments of inertia and the product of inertia of the area of Prob. 9.75 with respect to new centroidal axes obtained by rotating the x and y axes 45° clockwise.

9.83 Determine the moments of inertia and the product of inertia of the L3 × 2 × $\frac{1}{4}$-in. angle cross section of Prob. 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

9.84 Determine the moments of inertia and the product of inertia of the L127 × 76 × 12.7-mm angle cross section of Prob. 9.78 with respect to new centroidal axes obtained by rotating the x and y axes 45° counterclockwise.

9.85 For the quarter ellipse of Prob. 9.67, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

9.86 through 9.88 For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

9.86 Area of Prob. 9.72
9.87 Area of Prob. 9.73
9.88 Area of Prob. 9.75

9.89 and 9.90 For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

9.89 The L3 × 2 × $\frac{1}{4}$-in. angle cross section of Prob. 9.74
9.90 The L127 × 76 × 12.7-mm angle cross section of Prob. 9.78

*9.10 MOHR'S CIRCLE FOR MOMENTS AND PRODUCTS OF INERTIA

The circle used in the preceding section to illustrate the relations existing between the moments and products of inertia of a given area with respect to axes passing through a fixed point O was first introduced by the German engineer Otto Mohr (1835–1918) and is known as *Mohr's circle*. It will be shown that if the moments and product of inertia of an area A are known with respect to two rectangular x and y axes which pass through a point O, Mohr's circle can be used to graphically determine (a) the principal axes and principal moments of inertia of the area about O and (b) the moments and product of inertia of the area with respect to any other pair of rectangular axes x' and y' through O.

Consider a given area A and two rectangular coordinate axes x and y (Fig. 9.19a). Assuming that the moments of inertia I_x and I_y and the product of inertia I_{xy} are known, we will represent them on a diagram by plotting a point X of coordinates I_x and I_{xy} and a point Y of coordinates I_y and $-I_{xy}$ (Fig. 9.19b). If I_{xy} is positive, as assumed in Fig. 9.19a, point X is located above the horizontal axis and point Y is located below, as shown in Fig. 9.19b. If I_{xy} is negative, X is located below the horizontal axis and Y is located above. Joining X and Y with a straight line, we denote by C the point of intersection of line XY with the horizontal axis and draw the circle of center C and diameter XY. Noting that the abscissa of C and the radius of the circle are respectively equal to the quantities I_{ave} and R defined by the formula (9.23), we conclude that the circle obtained is Mohr's circle for the given area about point O. Thus, the abscissas of the points A and B where the circle intersects the horizontal axis represent, respectively, the principal moments of inertia I_{max} and I_{min} of the area.

We also note that, since $\tan (XCA) = 2I_{xy}/(I_x - I_y)$, the angle XCA is equal in magnitude to one of the angles $2\theta_m$ which satisfy Eq. (9.25);

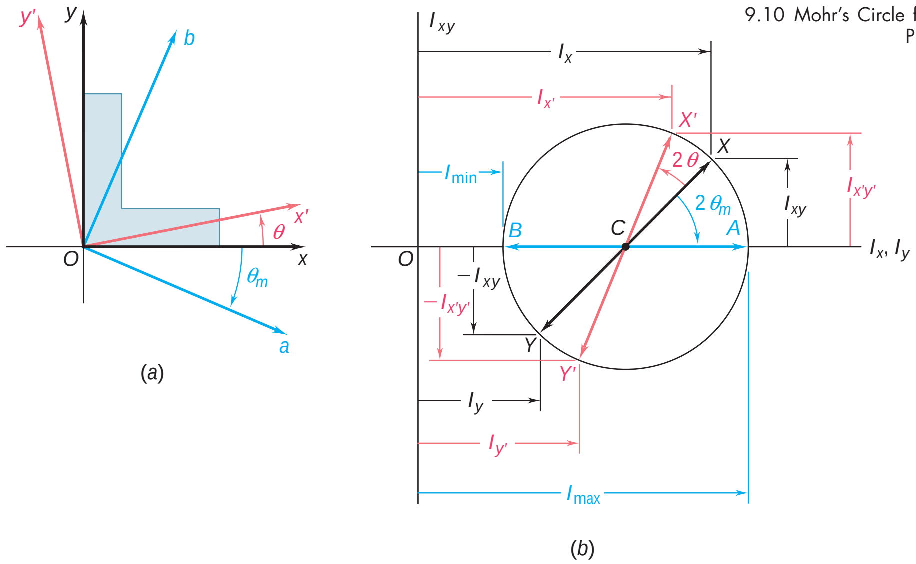
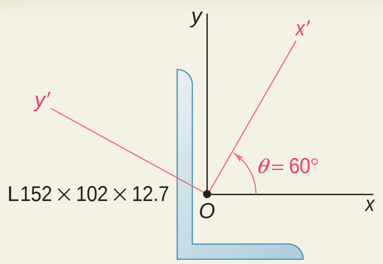


Fig. 9.19

thus, the angle θ_m, which defines in Fig. 9.19*a* the principal axis *Oa* corresponding to point *A* in Fig. 9.19*b*, is equal to half of the angle *XCA* of Mohr's circle. We further observe that if $I_x > I_y$ and $I_{xy} > 0$, as in the case considered here, the rotation which brings *CX* into *CA* is clockwise. Also, under these conditions, the angle θ_m obtained from Eq. (9.25), which defines the principal axis *Oa* in Fig. 9.19*a*, is negative; thus, the rotation which brings *Ox* into *Oa* is also clockwise. We conclude that the senses of rotation in both parts of Fig. 9.19 are the same. If a clockwise rotation through $2\theta_m$ is required to bring *CX* into *CA* on Mohr's circle, a clockwise rotation through θ_m will bring *Ox* into the corresponding principal axis *Oa* in Fig. 9.19*a*.

Since Mohr's circle is uniquely defined, the same circle can be obtained by considering the moments and product of inertia of the area *A* with respect to the rectangular axes x' and y' (Fig. 9.19*a*). The point X' of coordinates $I_{x'}$ and $I_{x'y'}$ and the point Y' of coordinates $I_{y'}$ and $-I_{x'y'}$ are thus located on Mohr's circle, and the angle $X'CA$ in Fig. 9.19*b* must be equal to twice the angle $x'Oa$ in Fig. 9.19*a*. Since, as noted above, the angle *XCA* is twice the angle *xOa*, it follows that the angle XCX' in Fig. 9.19*b* is twice the angle xOx' in Fig. 9.19*a*. The diameter $X'Y'$, which defines the moments and product of inertia $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ of the given area with respect to rectangular axes x' and y' forming an angle θ with the x and y axes can be obtained by rotating through an angle 2θ the diameter *XY* which corresponds to the moments and product of inertia I_x, I_y, and I_{xy}. We note that the rotation which brings the diameter *XY* into the diameter $X'Y'$ in Fig. 9.19*b* has the same sense as the rotation which brings the x and y axes into the x' and y' axes in Fig. 9.19*a*.

It should be noted that the use of Mohr's circle is not limited to graphical solutions, i.e., to solutions based on the careful drawing and measuring of the various parameters involved. By merely sketching Mohr's circle and using trigonometry, one can easily derive the various relations required for a numerical solution of a given problem (see Sample Prob. 9.8).

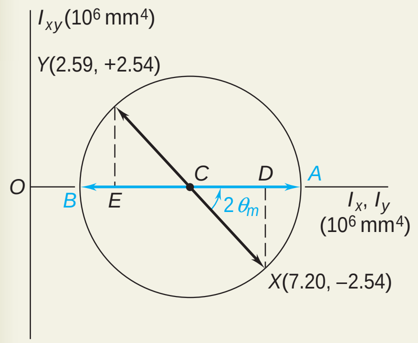


SAMPLE PROBLEM 9.8

For the section shown, the moments and product of inertia with respect to the x and y axes are known to be

$$I_x = 7.20 \times 10^6 \text{ mm}^4 \qquad I_y = 2.59 \times 10^6 \text{ mm}^4 \qquad I_{xy} = -2.54 \times 10^6 \text{ mm}^4$$

Using Mohr's circle, determine (*a*) the principal axes of the section about O, (*b*) the values of the principal moments of inertia of the section about O, (*c*) the moments and product of inertia of the section with respect to the x' and y' axes which form an angle of 60° with the x and y axes.



SOLUTION

Drawing Mohr's Circle. We first plot point X of coordinates $I_x = 7.20$, $I_{xy} = -2.54$, and point Y of coordinates $I_y = 2.59$, $-I_{xy} = +2.54$. Joining X and Y with a straight line, we define the center C of Mohr's circle. The abscissa of C, which represents I_{ave}, and the radius R of the circle can be measured directly or calculated as follows:

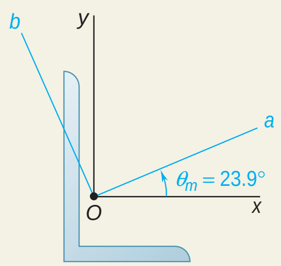
$$I_{ave} = OC = \tfrac{1}{2}(I_x + I_y) = \tfrac{1}{2}(7.20 \times 10^6 + 2.59 \times 10^6) = 4.895 \times 10^6 \text{ mm}^4$$

$$CD = \tfrac{1}{2}(I_x - I_y) = \tfrac{1}{2}(7.20 \times 10^6 - 2.59 \times 10^6) = 2.305 \times 10^6 \text{ mm}^4$$

$$R = \sqrt{(CD)^2 + (DX)^2} = \sqrt{(2.305 \times 10^6)^2 + (2.54 \times 10^6)^2}$$
$$= 3.430 \times 10^6 \text{ mm}^4$$

a. Principal Axes. The principal axes of the section correspond to points A and B on Mohr's circle, and the angle through which we should rotate CX to bring it into CA defines $2u_m$. We have

$$\tan 2u_m = \frac{DX}{CD} = \frac{2.54}{2.305} = 1.102 \qquad 2u_m = 47.8° \text{ l} \qquad u_m = 23.9° \text{ l} \quad ◀$$

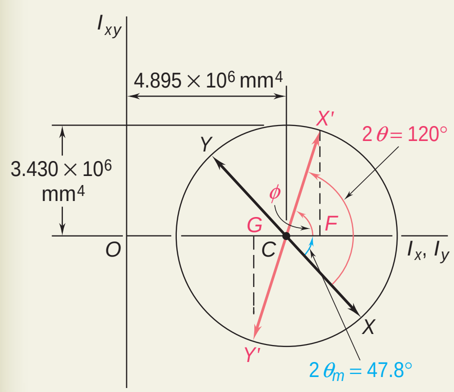


Thus, the principal axis Oa corresponding to the maximum value of the moment of inertia is obtained by rotating the x axis through 23.9° counterclockwise; the principal axis Ob corresponding to the minimum value of the moment of inertia can be obtained by rotating the y axis through the same angle.

b. Principal Moments of Inertia. The principal moments of inertia are represented by the abscissas of A and B. We have

$$I_{max} = OA = OC + CA = I_{ave} + R = (4.895 + 3.430)10^6 \text{ mm}^4$$
$$I_{max} = 8.33 \times 10^6 \text{ mm}^4 \quad ◀$$

$$I_{min} = OB = OC - BC = I_{ave} - R = (4.895 - 3.430)10^6 \text{ mm}^4$$
$$I_{min} = 1.47 \times 10^6 \text{ mm}^4 \quad ◀$$



c. Moments and Product of Inertia with Respect to the x' and y' Axes. On Mohr's circle, the points X' and Y', which correspond to the x' and y' axes, are obtained by rotating CX and CY through an angle $2u = 2(60°) = 120°$ counterclockwise. The coordinates of X' and Y' yield the desired moments and product of inertia. Noting that the angle that CX' forms with the horizontal axis is $f = 120° - 47.8° = 72.2°$, we write

$$I_{x'} = OF = OC + CF = 4.895 \times 10^6 \text{ mm}^4 + (3.430 \times 10^6 \text{ mm}^4) \cos 72.2°$$
$$I_{x'} = 5.94 \times 10^6 \text{ mm}^4 \quad ◀$$

$$I_{y'} = OG = OC - GC = 4.895 \times 10^6 \text{ mm}^4 - (3.430 \times 10^6 \text{ mm}^4) \cos 72.2°$$
$$I_{y'} = 3.85 \times 10^6 \text{ mm}^4 \quad ◀$$

$$I_{x'y'} = FX' = (3.430 \times 10^6 \text{ mm}^4) \sin 72.2°$$
$$I_{x'y'} = 3.27 \times 10^6 \text{ mm}^4 \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will use *Mohr's circle* to determine the moments and products of inertia of a given area for different orientations of the coordinate axes. Although in some cases using Mohr's circle may not be as direct as substituting into the appropriate equations [Eqs. (9.18) through (9.20)], this method of solution has the advantage of providing a visual representation of the relationships among the various variables. Further, Mohr's circle shows all of the values of the moments and products of inertia which are possible for a given problem.

Using Mohr's circle. The underlying theory was presented in Sec. 9.9, and we discussed the application of this method in Sec. 9.10 and in Sample Prob. 9.8. In the same problem, we presented the steps you should follow to determine the *principal axes*, the *principal moments of inertia*, and the *moments and product of inertia with respect to a specified orientation of the coordinates axes*. When you use Mohr's circle to solve problems, it is important that you remember the following points.

a. Mohr's circle is completely defined by the quantities R and I_{ave}, which represent, respectively, the radius of the circle and the distance from the origin O to the center C of the circle. These quantities can be obtained from Eqs. (9.23) if the moments and product of inertia are known for a given orientation of the axes. However, Mohr's circle can be defined by other combinations of known values [Probs. 9.103, 9.106, and 9.107]. For these cases, it may be necessary to first make one or more assumptions, such as choosing an arbitrary location for the center when I_{ave} is unknown, assigning relative magnitudes to the moments of inertia (for example, $I_x > I_y$), or selecting the sign of the product of inertia.

b. Point X of coordinates (I_x, I_{xy}) and point Y of coordinates $(I_y, -I_{xy})$ are both located on Mohr's circle and are diametrically opposite.

c. Since moments of inertia must be positive, the entire Mohr's circle must lie to the right of the I_{xy} axis; it follows that $I_{ave} > R$ for all cases.

d. As the coordinate axes are rotated through an angle θ, the associated rotation of the diameter of Mohr's circle is equal to 2θ and is in the same sense (clockwise or counterclockwise). We strongly suggest that the known points on the circumference of the circle be labeled with the appropriate capital letter, as was done in Fig. 9.19*b* and for the Mohr circles of Sample Prob. 9.8. This will enable you to determine, for each value of θ, the sign of the corresponding product of inertia and to determine which moment of inertia is associated with each of the coordinate axes [Sample Prob. 9.8, parts *a* and *c*].

Although we have introduced Mohr's circle within the specific context of the study of moments and products of inertia, the Mohr circle technique is also applicable to the solution of analogous but physically different problems in mechanics of materials. This multiple use of a specific technique is not unique, and as you pursue your engineering studies, you will encounter several methods of solution which can be applied to a variety of problems.

PROBLEMS

9.91 Using Mohr's circle, determine for the quarter ellipse of Prob. 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the x and y axes about O (*a*) through 45° counterclockwise, (*b*) through 30° clockwise.

9.92 Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Prob. 9.72 with respect to new centroidal axes obtained by rotating the x and y axes 30° counterclockwise.

9.93 Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Prob. 9.73 with respect to new centroidal axes obtained by rotating the x and y axes 60° counterclockwise.

9.94 Using Mohr's circle, determine the moments of inertia and the product of inertia of the area of Prob. 9.75 with respect to new centroidal axes obtained by rotating the x and y axes 45° clockwise.

9.95 Using Mohr's circle, determine the moments of inertia and the product of inertia of the L3 × 2 × $\frac{1}{4}$-in. angle cross section of Prob. 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

9.96 Using Mohr's circle, determine the moments of inertia and the product of inertia of the L127 × 76 × 12.7-mm angle cross section of Prob. 9.78 with respect to new centroidal axes obtained by rotating the x and y axes 45° counterclockwise.

9.97 For the quarter ellipse of Prob. 9.67, use Mohr's circle to determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

9.98 through *9.102* Using Mohr's circle, determine for the area indicated the orientation of the principal centroidal axes and the corresponding values of the moments of inertia.

- **9.98** Area of Prob. 9.72
- **9.99** Area of Prob. 9.76
- **9.100** Area of Prob. 9.73
- ***9.101*** Area of Prob. 9.74
- ***9.102*** Area of Prob. 9.77

(The moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area of Prob. 9.102 were determined in Prob. 9.44.)

9.103 The moments and product of inertia of an L4 × 3 × $\frac{1}{4}$-in. angle cross section with respect to two rectangular axes x and y through C are, respectively, $\bar{I}_x = 1.33 \text{ in}^4$, $\bar{I}_y = 2.75 \text{ in}^4$, and $\bar{I}_{xy} < 0$, with the minimum value of the moment of inertia of the area with respect to any axis through C being $\bar{I}_{\min} = 0.692 \text{ in}^4$. Using Mohr's circle, determine (*a*) the product of inertia $\bar{I}_{xy}$ of the area, (*b*) the orientation of the principal axes, (*c*) the value of $\bar{I}_{\max}$.

9.104 and 9.105 Using Mohr's circle, determine for the cross section of the rolled-steel angle shown the orientation of the principal centroidal axes and the corresponding values of the moments of inertia. (Properties of the cross sections are given in Fig. 9.13.)

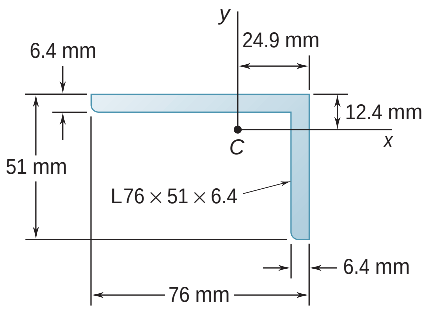


Fig. P9.104

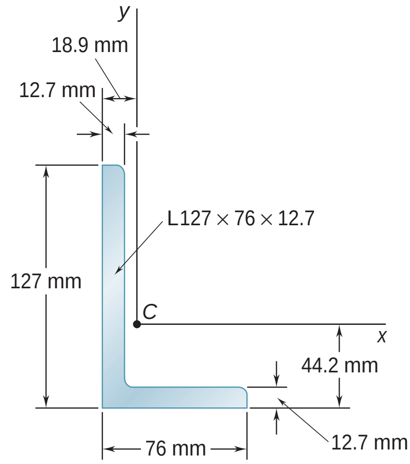


Fig. P9.105

***9.106** For a given area the moments of inertia with respect to two rectangular centroidal x and y axes are $\bar{I}_x = 1200\ \text{in}^4$ and $\bar{I}_y = 300\ \text{in}^4$, respectively. Knowing that after rotating the x and y axes about the centroid 30° counterclockwise, the moment of inertia relative to the rotated x axis is $1450\ \text{in}^4$, use Mohr's circle to determine (*a*) the orientation of the principal axes, (*b*) the principal centroidal moments of inertia.

9.107 It is known that for a given area $\bar{I}_y = 48 \times 10^6\ \text{mm}^4$ and $\bar{I}_{xy} = -20 \times 10^6\ \text{mm}^4$, where the x and y axes are rectangular centroidal axes. If the axis corresponding to the maximum product of inertia is obtained by rotating the x axis 67.5° counterclockwise about C, use Mohr's circle to determine (*a*) the moment of inertia $\bar{I}_x$ of the area, (*b*) the principal centroidal moments of inertia.

9.108 Using Mohr's circle, show that for any regular polygon (such as a pentagon) (*a*) the moment of inertia with respect to every axis through the centroid is the same, (*b*) the product of inertia with respect to every pair of rectangular axes through the centroid is zero.

9.109 Using Mohr's circle, prove that the expression $I_{x'}I_{y'} - I_{x'y'}^2$ is independent of the orientation of the x' and y' axes, where $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ represent the moments and product of inertia, respectively, of a given area with respect to a pair of rectangular axes x' and y' through a given point O. Also show that the given expression is equal to the square of the length of the tangent drawn from the origin of the coordinate system to Mohr's circle.

9.110 Using the invariance property established in the preceding problem, express the product of inertia I_{xy} of an area A with respect to a pair of rectangular axes through O in terms of the moments of inertia I_x and I_y of A and the principal moments of inertia $I_{\min}$ and $I_{\max}$ of A about O. Use the formula obtained to calculate the product of inertia I_{xy} of the L3 × 2 × $\frac{1}{4}$-in. angle cross section shown in Fig. 9.13A, knowing that its maximum moment of inertia is $1.257\ \text{in}^4$.

MOMENTS OF INERTIA OF MASSES

9.11 MOMENT OF INERTIA OF A MASS

Consider a small mass Δm mounted on a rod of negligible mass which can rotate freely about an axis AA' (Fig. 9.20*a*). If a couple is applied to the system, the rod and mass, assumed to be initially at rest, will start rotating about AA'. The details of this motion will be studied later in dynamics. At present, we wish only to indicate that the time required for the system to reach a given speed of rotation is proportional to the mass Δm and to the square of the distance r. The product $r^2 \Delta m$ provides, therefore, a measure of the *inertia* of the system, i.e., a measure of the resistance the system offers when we try to set it in motion. For this reason, the product $r^2 \Delta m$ is called the *moment of inertia* of the mass Δm with respect to the axis AA'.

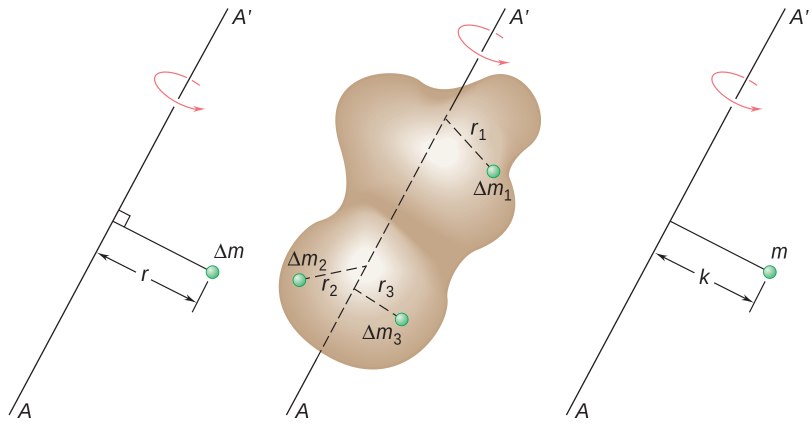


Fig. 9.20

Consider now a body of mass m which is to be rotated about an axis AA' (Fig. 9.20*b*). Dividing the body into elements of mass Δm_1, Δm_2, etc., we find that the body's resistance to being rotated is measured by the sum $r_1^2 \Delta m_1 + r_2^2 \Delta m_2 + \cdots$. This sum defines, therefore, the moment of inertia of the body with respect to the axis AA'. Increasing the number of elements, we find that the moment of inertia is equal, in the limit, to the integral

$$I = \int r^2 \, dm \tag{9.28}$$

The *radius of gyration* k of the body with respect to the axis AA' is defined by the relation

$$I = k^2 m \qquad \text{or} \qquad k = \sqrt{\frac{I}{m}} \tag{9.29}$$

The radius of gyration k represents, therefore, the distance at which the entire mass of the body should be concentrated if its moment of inertia with respect to AA' is to remain unchanged (Fig. 9.20*c*). Whether it is kept in its original shape (Fig. 9.20*b*) or whether it is concentrated as shown in Fig. 9.20*c*, the mass m will react in the same way to a rotation, or *gyration,* about AA'.

If SI units are used, the radius of gyration k is expressed in meters and the mass m in kilograms, and thus the unit used for the moment of inertia of a mass is $\text{kg} \cdot \text{m}^2$. If U.S. customary units are used, the radius of gyration is expressed in feet and the mass in slugs (i.e., in $\text{lb} \cdot \text{s}^2/\text{ft}$), and thus the derived unit used for the moment of inertia of a mass is $\text{lb} \cdot \text{ft} \cdot \text{s}^2$.†

The moment of inertia of a body with respect to a coordinate axis can easily be expressed in terms of the coordinates x, y, z of the element of mass dm (Fig. 9.21). Noting, for example, that the square of the distance r from the element dm to the y axis is $z^2 + x^2$, we express the moment of inertia of the body with respect to the y axis as

$$I_y = \int r^2\, dm = \int (z^2 + x^2)\, dm$$

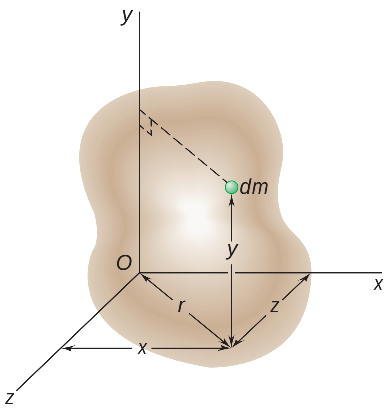


Fig. 9.21

Similar expressions can be obtained for the moments of inertia with respect to the x and z axes. We write

$$\boxed{\begin{aligned} I_x &= \int (y^2 + z^2)\, dm \\ I_y &= \int (z^2 + x^2)\, dm \\ I_z &= \int (x^2 + y^2)\, dm \end{aligned}} \tag{9.30}$$

Photo 9.2 As you will discuss in your dynamics course, the rotational behavior of the camshaft shown is dependent upon the mass moment of inertia of the camshaft with respect to its axis of rotation.

†It should be kept in mind when converting the moment of inertia of a mass from U.S. customary units to SI units that the base unit *pound* used in the derived unit $\text{lb} \cdot \text{ft} \cdot \text{s}^2$ is a unit of force (*not* of mass) and should therefore be converted into newtons. We have

$$1\ \text{lb} \cdot \text{ft} \cdot \text{s}^2 = (4.45\ \text{N})(0.3048\ \text{m})(1\ \text{s})^2 = 1.356\ \text{N} \cdot \text{m} \cdot \text{s}^2$$

or, since $1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$,

$$1\ \text{lb} \cdot \text{ft} \cdot \text{s}^2 = 1.356\ \text{kg} \cdot \text{m}^2$$

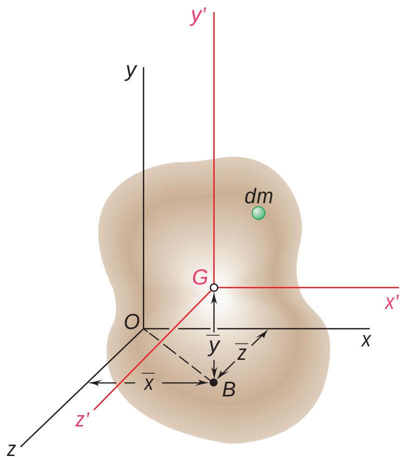


Fig. 9.22

9.12 PARALLEL-AXIS THEOREM

Consider a body of mass m. Let $Oxyz$ be a system of rectangular coordinates whose origin is at the arbitrary point O, and $Gx'y'z'$ a system of parallel *centroidal axes*, i.e., a system whose origin is at the center of gravity G of the body† and whose axes x', y', z' are parallel to the x, y, and z axes, respectively (Fig. 9.22). Denoting by $\bar{x}$, $\bar{y}$, $\bar{z}$ the coordinates of G with respect to $Oxyz$, we write the following relations between the coordinates x, y, z of the element dm with respect to $Oxyz$ and its coordinates x', y', z' with respect to the centroidal axes $Gx'y'z'$:

$$x = x' + \bar{x} \qquad y = y' + \bar{y} \qquad z = z' + \bar{z} \tag{9.31}$$

Referring to Eqs. (9.30), we can express the moment of inertia of the body with respect to the x axis as follows:

$$\begin{aligned} I_x &= \int (y^2 + z^2)\, dm = \int [(y' + \bar{y})^2 + (z' + \bar{z})^2]\, dm \\ &= \int (y'^2 + z'^2)\, dm + 2\bar{y}\int y'\, dm + 2\bar{z}\int z'\, dm + (\bar{y}^2 + \bar{z}^2)\int dm \end{aligned}$$

The first integral in this expression represents the moment of inertia $\bar{I}_{x'}$ of the body with respect to the centroidal axis x'; the second and third integrals represent the first moment of the body with respect to the $z'x'$ and $x'y'$ planes, respectively, and, since both planes contain G, the two integrals are zero; the last integral is equal to the total mass m of the body. We write, therefore,

$$\boxed{I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2)} \tag{9.32}$$

and, similarly,

$$\boxed{I_y = \bar{I}_{y'} + m(\bar{z}^2 + \bar{x}^2) \qquad I_z = \bar{I}_{z'} + m(\bar{x}^2 + \bar{y}^2)} \tag{9.32$'$}$$

We easily verify from Fig. 9.22 that the sum $\bar{z}^2 + \bar{x}^2$ represents the square of the distance OB between the y and y' axes. Similarly, $\bar{y}^2 + \bar{z}^2$ and $\bar{x}^2 + \bar{y}^2$ represent the squares of the distance between the x and x' axes and the z and z' axes, respectively. Denoting by d the distance between an arbitrary axis AA' and a parallel centroidal axis BB' (Fig. 9.23), we can, therefore, write the following general relation between the moment of inertia I of the body with respect to AA' and its moment of inertia $\bar{I}$ with respect to BB':

$$\boxed{I = \bar{I} + md^2} \tag{9.33}$$

Expressing the moments of inertia in terms of the corresponding radii of gyration, we can also write

$$k^2 = \bar{k}^2 + d^2 \tag{9.34}$$

where k and $\bar{k}$ represent the radii of gyration of the body about AA' and BB', respectively.

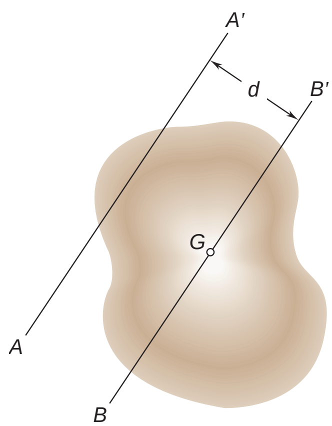


Fig. 9.23

†Note that the term *centroidal* is used here to define an axis passing through the center of gravity G of the body, whether or not G coincides with the centroid of the volume of the body.

9.13 MOMENTS OF INERTIA OF THIN PLATES

Consider a thin plate of uniform thickness t, which is made of a homogeneous material of density ρ (density = mass per unit volume). The mass moment of inertia of the plate with respect to an axis AA' *contained in the plane* of the plate (Fig. 9.24*a*) is

$$I_{AA', \text{mass}} = \int r^2 \, dm$$

Since $dm = \rho t \, dA$, we write

$$I_{AA', \text{mass}} = \rho t \int r^2 \, dA$$

But r represents the distance of the element of area dA to the axis

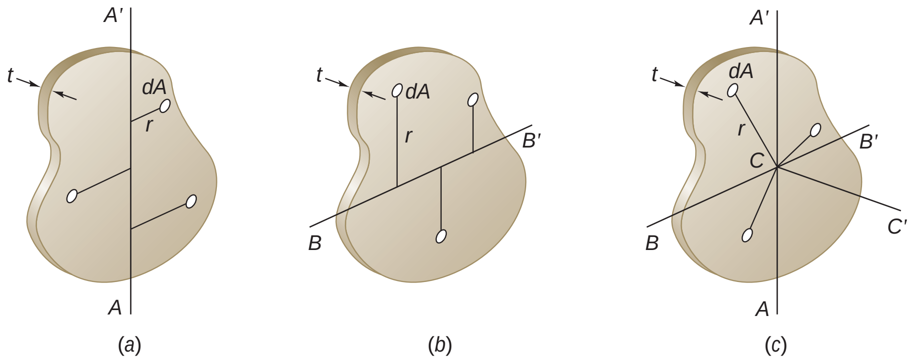


Fig. 9.24

AA'; the integral is therefore equal to the moment of inertia of the area of the plate with respect to AA'. We have

$$I_{AA', \text{mass}} = \rho t I_{AA', \text{area}} \tag{9.35}$$

Similarly, for an axis BB' which is contained in the plane of the plate and is perpendicular to AA' (Fig. 9.24*b*), we have

$$I_{BB', \text{mass}} = \rho t I_{BB', \text{area}} \tag{9.36}$$

Considering now the axis CC' which is *perpendicular* to the plate and passes through the point of intersection C of AA' and BB' (Fig. 9.24*c*), we write

$$I_{CC', \text{mass}} = \rho t J_{C, \text{area}} \tag{9.37}$$

where J_C is the *polar* moment of inertia of the area of the plate with respect to point C.

Recalling the relation $J_C = I_{AA'} + I_{BB'}$ which exists between polar and rectangular moments of inertia of an area, we write the following relation between the mass moments of inertia of a thin plate:

$$I_{CC'} = I_{AA'} + I_{BB'} \tag{9.38}$$

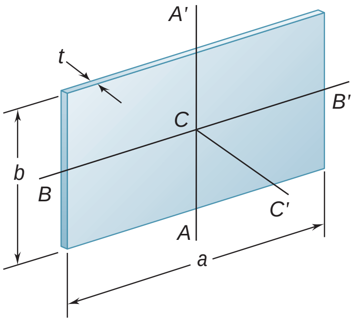


Fig. 9.25

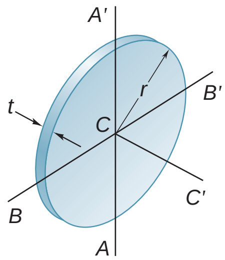
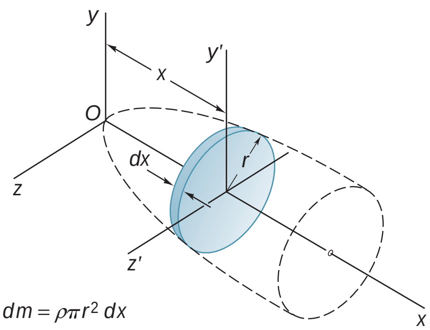


Fig. 9.26



$$dm = \rho\pi r^2\, dx$$

$$dI_x = \tfrac{1}{2} r^2\, dm$$

$$dI_y = dI_{y'} + x^2\, dm = \left(\tfrac{1}{4} r^2 + x^2\right) dm$$

$$dI_z = dI_{z'} + x^2\, dm = \left(\tfrac{1}{4} r^2 + x^2\right) dm$$

Fig. 9.27 Determination of the moment of inertia of a body of revolution.

Rectangular Plate. In the case of a rectangular plate of sides a and b (Fig. 9.25), we obtain the following mass moments of inertia with respect to axes through the center of gravity of the plate:

$$I_{AA', \text{mass}} = \rho t I_{AA', \text{area}} = \rho t(\tfrac{1}{12} a^3 b)$$
$$I_{BB', \text{mass}} = \rho t I_{BB', \text{area}} = \rho t(\tfrac{1}{12} a b^3)$$

Observing that the product ρabt is equal to the mass m of the plate, we write the mass moments of inertia of a thin rectangular plate as follows:

$$I_{AA'} = \tfrac{1}{12} m a^2 \qquad I_{BB'} = \tfrac{1}{12} m b^2 \tag{9.39}$$
$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{12} m(a^2 + b^2) \tag{9.40}$$

Circular Plate. In the case of a circular plate, or disk, of radius r (Fig. 9.26), we write

$$I_{AA', \text{mass}} = \rho t I_{AA', \text{area}} = \rho t(\tfrac{1}{4} \pi r^4)$$

Observing that the product $\rho \pi r^2 t$ is equal to the mass m of the plate and that $I_{AA'} = I_{BB'}$, we write the mass moments of inertia of a circular plate as follows:

$$I_{AA'} = I_{BB'} = \tfrac{1}{4} m r^2 \tag{9.41}$$
$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{2} m r^2 \tag{9.42}$$

9.14 DETERMINATION OF THE MOMENT OF INERTIA OF A THREE-DIMENSIONAL BODY BY INTEGRATION

The moment of inertia of a three-dimensional body is obtained by evaluating the integral $I = \int r^2\, dm$. If the body is made of a homogeneous material of density ρ, the element of mass dm is equal to $\rho\, dV$ and we can write $I = \rho \int r^2\, dV$. This integral depends only upon the shape of the body. Thus, in order to compute the moment of inertia of a three-dimensional body, it will generally be necessary to perform a triple, or at least a double, integration.

However, if the body possesses two planes of symmetry, it is usually possible to determine the body's moment of inertia with a single integration by choosing as the element of mass dm a thin slab which is perpendicular to the planes of symmetry. In the case of bodies of revolution, for example, the element of mass would be a thin disk (Fig. 9.27). Using formula (9.42), the moment of inertia of the disk with respect to the axis of revolution can be expressed as indicated in Fig. 9.27. Its moment of inertia with respect to each of the other two coordinate axes is obtained by using formula (9.41) and the parallel-axis theorem. Integration of the expression obtained yields the desired moment of inertia of the body.

9.15 MOMENTS OF INERTIA OF COMPOSITE BODIES

The moments of inertia of a few common shapes are shown in Fig. 9.28. For a body consisting of several of these simple shapes, the moment of inertia of the body with respect to a given axis can be obtained by first computing the moments of inertia of its component parts about the desired axis and then adding them together. As was the case for areas, the radius of gyration of a composite body *cannot* be obtained by adding the radii of gyration of its component parts.

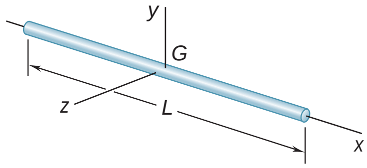
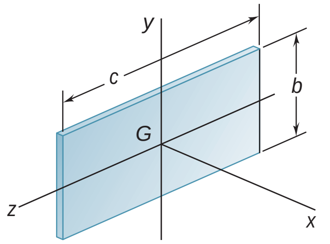
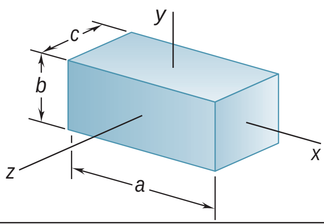
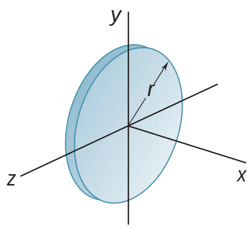
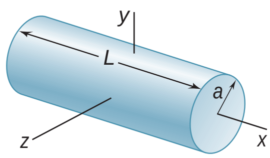
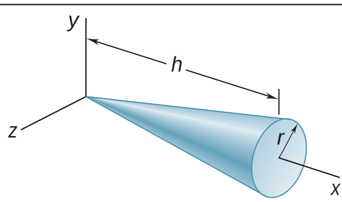
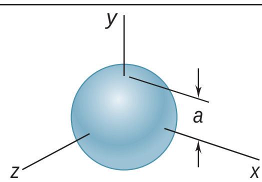
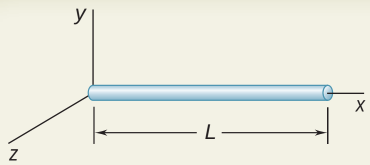
Slender rod		$I_y = I_z = \frac{1}{12}mL^2$
Thin rectangular plate		$I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}mc^2$ $I_z = \frac{1}{12}mb^2$
Rectangular prism		$I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}m(c^2 + a^2)$ $I_z = \frac{1}{12}m(a^2 + b^2)$
Thin disk		$I_x = \frac{1}{2}mr^2$ $I_y = I_z = \frac{1}{4}mr^2$
Circular cylinder		$I_x = \frac{1}{2}ma^2$ $I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$
Circular cone		$I_x = \frac{3}{10}ma^2$ $I_y = I_z = \frac{3}{5}m(\frac{1}{4}a^2 + h^2)$
Sphere		$I_x = I_y = I_z = \frac{2}{5}ma^2$

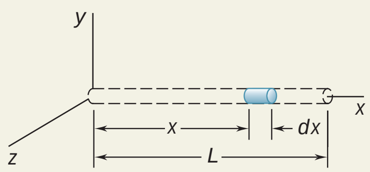
Fig. 9.28 Mass moments of inertia of common geometric shapes.

SAMPLE PROBLEM 9.9



Determine the moment of inertia of a slender rod of length L and mass m with respect to an axis which is perpendicular to the rod and passes through one end of the rod.

SOLUTION

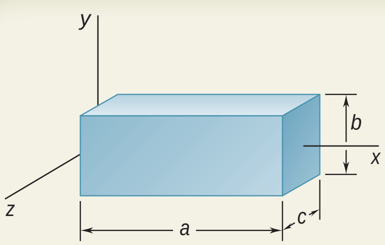


Choosing the differential element of mass shown, we write

$$dm = \frac{m}{L}dx$$

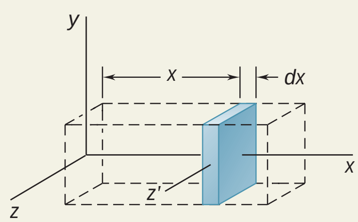
$$I_y = \int x^2\, dm = \int_0^L x^2 \frac{m}{L} dx = \left[\frac{m}{L}\frac{x^3}{3}\right]_0^L \qquad I_y = \tfrac{1}{3}mL^2 \blacktriangleleft$$

SAMPLE PROBLEM 9.10



For the homogeneous rectangular prism shown, determine the moment of inertia with respect to the z axis.

SOLUTION



We choose as the differential element of mass the thin slab shown; thus

$$dm = \rho bc\, dx$$

Referring to Sec. 9.13, we find that the moment of inertia of the element with respect to the z' axis is

$$dI_{z'} = \tfrac{1}{12}b^2\, dm$$

Applying the parallel-axis theorem, we obtain the mass moment of inertia of the slab with respect to the z axis.

$$dI_z = dI_{z'} + x^2\, dm = \tfrac{1}{12}b^2\, dm + x^2\, dm = (\tfrac{1}{12}b^2 + x^2)\rho bc\, dx$$

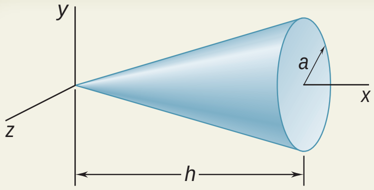
Integrating from $x = 0$ to $x = a$, we obtain

$$I_z = \int dI_z = \int_0^a (\tfrac{1}{12}b^2 + x^2)\rho bc\, dx = \rho abc(\tfrac{1}{12}b^2 + \tfrac{1}{3}a^2)$$

Since the total mass of the prism is $m = \rho abc$, we can write

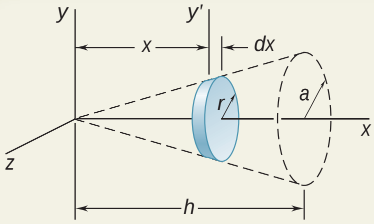
$$I_z = m(\tfrac{1}{12}b^2 + \tfrac{1}{3}a^2) \qquad I_z = \tfrac{1}{12}m(4a^2 + b^2) \blacktriangleleft$$

We note that if the prism is thin, b is small compared to a, and the expression for I_z reduces to $\frac{1}{3}ma^2$, which is the result obtained in Sample Prob. 9.9 when $L = a$.



SAMPLE PROBLEM 9.11

Determine the moment of inertia of a right circular cone with respect to (*a*) its longitudinal axis, (*b*) an axis through the apex of the cone and perpendicular to its longitudinal axis, (*c*) an axis through the centroid of the cone and perpendicular to its longitudinal axis.



SOLUTION

We choose the differential element of mass shown.

$$r = a\frac{x}{h} \qquad dm = \rho\pi r^2\,dx = \rho\pi\frac{a^2}{h^2}x^2\,dx$$

a. Moment of Inertia I_x. Using the expression derived in Sec. 9.13 for a thin disk, we compute the mass moment of inertia of the differential element with respect to the x axis.

$$dI_x = \tfrac{1}{2}r^2\,dm = \tfrac{1}{2}\left(a\frac{x}{h}\right)^2\left(\rho\pi\frac{a^2}{h^2}x^2\,dx\right) = \tfrac{1}{2}\rho\pi\frac{a^4}{h^4}x^4\,dx$$

Integrating from $x = 0$ to $x = h$, we obtain

$$I_x = \int dI_x = \int_0^h \tfrac{1}{2}\rho\pi\frac{a^4}{h^4}x^4\,dx = \tfrac{1}{2}\rho\pi\frac{a^4}{h^4}\frac{h^5}{5} = \tfrac{1}{10}\rho\pi a^4 h$$

Since the total mass of the cone is $m = \tfrac{1}{3}\rho\pi a^2 h$, we can write

$$I_x = \tfrac{1}{10}\rho\pi a^4 h = \tfrac{3}{10}a^2(\tfrac{1}{3}\rho\pi a^2 h) = \tfrac{3}{10}ma^2 \qquad I_x = \tfrac{3}{10}ma^2 \;\blacktriangleleft$$

b. Moment of Inertia I_y. The same differential element is used. Applying the parallel-axis theorem and using the expression derived in Sec. 9.13 for a thin disk, we write

$$dI_y = dI_{y'} + x^2\,dm = \tfrac{1}{4}r^2\,dm + x^2\,dm = (\tfrac{1}{4}r^2 + x^2)\,dm$$

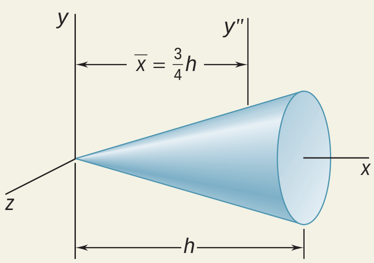
Substituting the expressions for r and dm into the equation, we obtain

$$dI_y = \left(\frac{1}{4}\frac{a^2}{h^2}x^2 + x^2\right)\left(\rho\pi\frac{a^2}{h^2}x^2\,dx\right) = \rho\pi\frac{a^2}{h^2}\left(\frac{a^2}{4h^2} + 1\right)x^4\,dx$$

$$I_y = \int dI_y = \int_0^h \rho\pi\frac{a^2}{h^2}\left(\frac{a^2}{4h^2} + 1\right)x^4\,dx = \rho\pi\frac{a^2}{h^2}\left(\frac{a^2}{4h^2} + 1\right)\frac{h^5}{5}$$

Introducing the total mass of the cone m, we rewrite I_y as follows:

$$I_y = \tfrac{3}{5}(\tfrac{1}{4}a^2 + h^2)\tfrac{1}{3}\rho\pi a^2 h \qquad I_y = \tfrac{3}{5}m(\tfrac{1}{4}a^2 + h^2) \;\blacktriangleleft$$



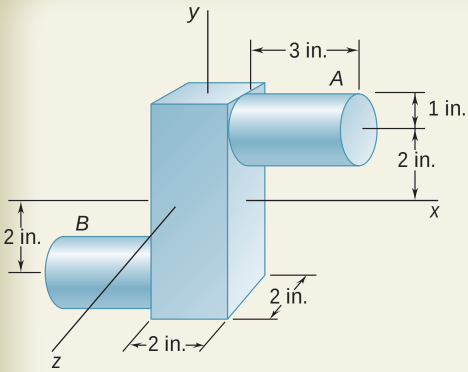
c. Moment of Inertia $\bar{I}_{y''}$. We apply the parallel-axis theorem and write

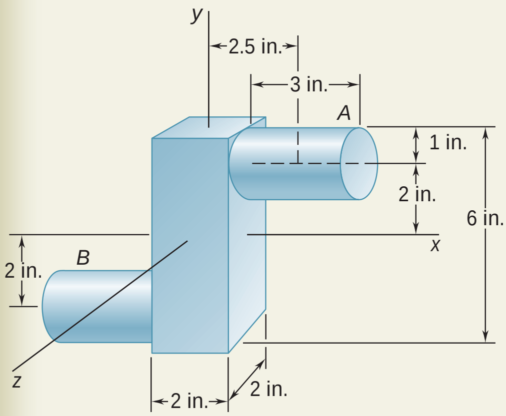
$$I_y = \bar{I}_{y''} + m\bar{x}^2$$

Solving for $\bar{I}_{y''}$ and recalling that $\bar{x} = \tfrac{3}{4}h$, we have

$$\bar{I}_{y''} = I_y - m\bar{x}^2 = \tfrac{3}{5}m(\tfrac{1}{4}a^2 + h^2) - m(\tfrac{3}{4}h)^2$$

$$\bar{I}_{y''} = \tfrac{3}{20}m(a^2 + \tfrac{1}{4}h^2) \;\blacktriangleleft$$





SAMPLE PROBLEM 9.12

A steel forging consists of a $6 \times 2 \times 2$-in. rectangular prism and two cylinders of diameter 2 in. and length 3 in. as shown. Determine the moments of inertia of the forging with respect to the coordinate axes, knowing that the specific weight of steel is 490 lb/ft^3.

SOLUTION

Computation of Masses

Prism

$$V = (2 \text{ in.})(2 \text{ in.})(6 \text{ in.}) = 24 \text{ in}^3$$

$$W = \frac{(24 \text{ in}^3)(490 \text{ lb/ft}^3)}{1728 \text{ in}^3/\text{ft}^3} = 6.81 \text{ lb}$$

$$m = \frac{6.81 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.211 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Each Cylinder

$$V = \text{p}(1 \text{ in.})^2(3 \text{ in.}) = 9.42 \text{ in}^3$$

$$W = \frac{(9.42 \text{ in}^3)(490 \text{ lb/ft}^3)}{1728 \text{ in}^3/\text{ft}^3} = 2.67 \text{ lb}$$

$$m = \frac{2.67 \text{ lb}}{32.2 \text{ ft/s}^2} = 0.0829 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Moments of Inertia. The moments of inertia of each component are computed from Fig. 9.28, using the parallel-axis theorem when necessary. Note that all lengths should be expressed in feet.

Prism

$$I_x = I_z = \tfrac{1}{12}(0.211 \text{ lb} \cdot \text{s}^2/\text{ft})[(\tfrac{6}{12} \text{ ft})^2 + (\tfrac{2}{12} \text{ ft})^2] = 4.88 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_y = \tfrac{1}{12}(0.211 \text{ lb} \cdot \text{s}^2/\text{ft})[(\tfrac{2}{12} \text{ ft})^2 + (\tfrac{2}{12} \text{ ft})^2] = 0.977 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Each Cylinder

$$I_x = \tfrac{1}{2}ma^2 + m\bar{y}^2 = \tfrac{1}{2}(0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})(\tfrac{1}{12} \text{ ft})^2 + (0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})(\tfrac{2}{12} \text{ ft})^2 = 2.59 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_y = \tfrac{1}{12}m(3a^2 + L^2) = m\bar{x}^2 = \tfrac{1}{12}(0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})[3(\tfrac{1}{12} \text{ ft})^2 + (\tfrac{3}{12} \text{ ft})^2] + (0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})(\tfrac{2.5}{12} \text{ ft})^2 = 4.17 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

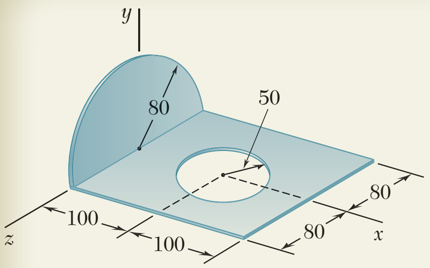
$$I_z = \tfrac{1}{12}m(3a^2 + L^2) + m(\bar{x}^2 + \bar{y}^2) = \tfrac{1}{12}(0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})[3(\tfrac{1}{12} \text{ ft})^2 + (\tfrac{3}{12} \text{ ft})^2] + (0.0829 \text{ lb} \cdot \text{s}^2/\text{ft})[(\tfrac{2.5}{12} \text{ ft})^2 + (\tfrac{2}{12} \text{ ft})^2] = 6.48 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Entire Body. Adding the values obtained,

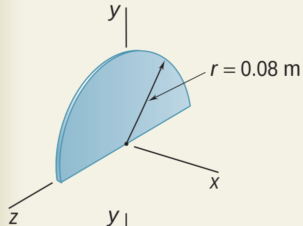
$$I_x = 4.88 \times 10^{-3} + 2(2.59 \times 10^{-3}) \qquad I_x = 10.06 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \blacktriangleleft$$

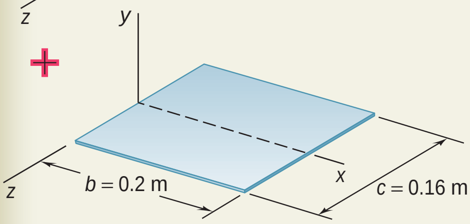
$$I_y = 0.977 \times 10^{-3} + 2(4.17 \times 10^{-3}) \qquad I_y = 9.32 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \blacktriangleleft$$

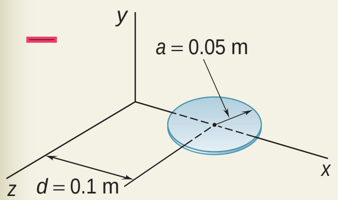
$$I_z = 4.88 \times 10^{-3} + 2(6.48 \times 10^{-3}) \qquad I_z = 17.84 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \blacktriangleleft$$



Dimensions in mm







SAMPLE PROBLEM 9.13

A thin steel plate which is 4 mm thick is cut and bent to form the machine part shown. Knowing that the density of steel is 7850 kg/m^3, determine the moments of inertia of the machine part with respect to the coordinate axes.

SOLUTION

We observe that the machine part consists of a semicircular plate and a rectangular plate from which a circular plate has been removed.

Computation of Masses. ***Semicircular Plate***

$$V_1 = \tfrac{1}{2}\pi r^2 t = \tfrac{1}{2}\pi(0.08\text{ m})^2(0.004\text{ m}) = 40.21 \times 10^{-6}\text{ m}^3$$
$$m_1 = \rho V_1 = (7.85 \times 10^3\text{ kg/m}^3)(40.21 \times 10^{-6}\text{ m}^3) = 0.3156\text{ kg}$$

Rectangular Plate

$$V_2 = (0.200\text{ m})(0.160\text{ m})(0.004\text{ m}) = 128 \times 10^{-6}\text{ m}^3$$
$$m_2 = \rho V_2 = (7.85 \times 10^3\text{ kg/m}^3)(128 \times 10^{-6}\text{ m}^3) = 1.005\text{ kg}$$

Circular Plate

$$V_3 = \pi a^2 t = \pi(0.050\text{ m})^2(0.004\text{ m}) = 31.42 \times 10^{-6}\text{ m}^3$$
$$m_3 = \rho V_3 = (7.85 \times 10^3\text{ kg/m}^3)(31.42 \times 10^{-6}\text{ m}^3) = 0.2466\text{ kg}$$

Moments of Inertia. Using the method presented in Sec. 9.13, we compute the moments of inertia of each component.

Semicircular Plate. From Fig. 9.28, we observe that for a circular plate of mass m and radius r

$$I_x = \tfrac{1}{2}mr^2 \qquad I_y = I_z = \tfrac{1}{4}mr^2$$

Because of symmetry, we note that for a semicircular plate

$$I_x = \tfrac{1}{2}(\tfrac{1}{2}mr^2) \qquad I_y = I_z = \tfrac{1}{2}(\tfrac{1}{4}mr^2)$$

Since the mass of the semicircular plate is $m_1 = \tfrac{1}{2}m$, we have

$$I_x = \tfrac{1}{2}m_1r^2 = \tfrac{1}{2}(0.3156\text{ kg})(0.08\text{ m})^2 = 1.010 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_y = I_z = \tfrac{1}{4}(\tfrac{1}{2}mr^2) = \tfrac{1}{4}m_1r^2 = \tfrac{1}{4}(0.3156\text{ kg})(0.08\text{ m})^2 = 0.505 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

Rectangular Plate

$$I_x = \tfrac{1}{12}m_2c^2 = \tfrac{1}{12}(1.005\text{ kg})(0.16\text{ m})^2 = 2.144 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_z = \tfrac{1}{3}m_2b^2 = \tfrac{1}{3}(1.005\text{ kg})(0.2\text{ m})^2 = 13.400 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_y = I_x + I_z = (2.144 + 13.400)(10^{-3}) = 15.544 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

Circular Plate

$$I_x = \tfrac{1}{4}m_3a^2 = \tfrac{1}{4}(0.2466\text{ kg})(0.05\text{ m})^2 = 0.154 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_y = \tfrac{1}{2}m_3a^2 + m_3d^2$$
$$= \tfrac{1}{2}(0.2466\text{ kg})(0.05\text{ m})^2 + (0.2466\text{ kg})(0.1\text{ m})^2 = 2.774 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$
$$I_z = \tfrac{1}{4}m_3a^2 + m_3d^2 = \tfrac{1}{4}(0.2466\text{ kg})(0.05\text{ m})^2 + (0.2466\text{ kg})(0.1\text{ m})^2$$
$$= 2.620 \times 10^{-3}\text{ kg}\cdot\text{m}^2$$

Entire Machine Part

$$I_x = (1.010 + 2.144 - 0.154)(10^{-3})\text{ kg}\cdot\text{m}^2 \qquad I_x = 3.00 \times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$
$$I_y = (0.505 + 15.544 - 2.774)(10^{-3})\text{ kg}\cdot\text{m}^2 \qquad I_y = 13.28 \times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$
$$I_z = (0.505 + 13.400 - 2.620)(10^{-3})\text{ kg}\cdot\text{m}^2 \qquad I_z = 11.29 \times 10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced the *mass moment of inertia* and the *radius of gyration* of a three-dimensional body with respect to a given axis [Eqs. (9.28) and (9.29)]. We also derived a *parallel-axis theorem* for use with mass moments of inertia and discussed the computation of the mass moments of inertia of thin plates and three-dimensional bodies.

1. Computing mass moments of inertia. The mass moment of inertia I of a body with respect to a given axis can be calculated directly from the definition given in Eq. (9.28) for simple shapes [Sample Prob. 9.9]. In most cases, however, it is necessary to divide the body into thin slabs, compute the moment of inertia of a typical slab with respect to the given axis—using the parallel-axis theorem if necessary—and integrate the expression obtained.

2. Applying the parallel-axis theorem. In Sec. 9.12 we derived the parallel-axis theorem for mass moments of inertia

$$I = \bar{I} + md^2 \tag{9.33}$$

which states that the moment of inertia I of a body of mass m with respect to a given axis is equal to the sum of the moment of inertia $\bar{I}$ of that body with respect to a *parallel centroidal axis* and the product md^2, where d is the distance between the two axes. When the moment of inertia of a three-dimensional body is calculated with respect to one of the coordinate axes, d^2 can be replaced by the sum of the squares of distances measured along the other two coordinate axes [Eqs. (9.32) and (9.32′)].

3. Avoiding unit-related errors. To avoid errors, it is essential that you be consistent in your use of units. Thus, all lengths should be expressed in meters or feet, as appropriate, and for problems using U.S. customary units, masses should be given in $\text{lb} \cdot \text{s}^2/\text{ft}$. In addition, we strongly recommend that you include units as you perform your calculations [Sample Probs. 9.12 and 9.13].

4. Calculating the mass moment of inertia of thin plates. We showed in Sec. 9.13 that the mass moment of inertia of a thin plate with respect to a given axis can be obtained by multiplying the corresponding moment of inertia of the area of the plate by the density ρ and the thickness t of the plate [Eqs. (9.35) through (9.37)]. Note that since the axis CC' in Fig. 9.24*c* is *perpendicular to the plate*, $I_{CC',\text{ mass}}$ is associated with the *polar* moment of inertia $J_{C,\text{ area}}$.

Instead of calculating directly the moment of inertia of a thin plate with respect to a specified axis, you may sometimes find it convenient to first compute its moment of inertia with respect to an axis parallel to the specified axis and then apply the parallel-axis theorem. Further, to determine the moment of inertia of a thin plate with respect to an axis perpendicular to the plate, you may wish to first determine its moments of inertia with respect to two perpendicular in-plane axes and then use Eq. (9.38). Finally, remember that the mass of a plate of area A, thickness t, and density ρ is $m = \rho t A$.

5. Determining the moment of inertia of a body by direct single integration. We discussed in Sec. 9.14 and illustrated in Sample Probs. 9.10 and 9.11 how single integration can be used to compute the moment of inertia of a body that can be divided into a series of thin, parallel slabs. For such cases, you will often need to express the mass of the body in terms of the body's density and dimensions. Assuming that the body has been divided, as in the sample problems, into thin slabs perpendicular to the x axis, you will need to express the dimensions of each slab as functions of the variable x.

a. In the special case of a body of revolution, the elemental slab is a thin disk, and the equations given in Fig. 9.27 should be used to determine the moments of inertia of the body [Sample Prob. 9.11].

b. In the general case, when the body is not of revolution, the differential element is not a disk, but a thin slab of a different shape, and the equations of Fig. 9.27 cannot be used. See, for example, Sample Prob. 9.10, where the element was a thin, rectangular slab. For more complex configurations, you may want to use one or more of the following equations, which are based on Eqs. (9.32) and (9.32′) of Sec. 9.12.

$$dI_x = dI_{x'} + (\bar{y}_{el}^2 + \bar{z}_{el}^2)\,dm$$
$$dI_y = dI_{y'} + (\bar{z}_{el}^2 + \bar{x}_{el}^2)\,dm$$
$$dI_z = dI_{z'} + (\bar{x}_{el}^2 + \bar{y}_{el}^2)\,dm$$

where the primes denote the centroidal axes of each elemental slab, and where $\bar{x}_{el}$, $\bar{y}_{el}$, and $\bar{z}_{el}$ represent the coordinates of its centroid. The centroidal moments of inertia of the slab are determined in the manner described earlier for a thin plate: Referring to Fig. 9.12 on page 483, calculate the corresponding moments of inertia of the area of the slab and multiply the result by the density ρ and the thickness t of the slab. Also, assuming that the body has been divided into thin slabs perpendicular to the x axis, remember that you can obtain $dI_{x'}$ by adding $dI_{y'}$ and $dI_{z'}$ instead of computing it directly. Finally, using the geometry of the body, express the result obtained in terms of the single variable x and integrate in x.

6. Computing the moment of inertia of a composite body. As stated in Sec. 9.15, the moment of inertia of a composite body with respect to a specified axis is equal to the sum of the moments of its components with respect to that axis. Sample Probs. 9.12 and 9.13 illustrate the appropriate method of solution. You must also remember that the moment of inertia of a component will be negative only if the component is *removed* (as in the case of a hole).

Although the composite-body problems in this lesson are relatively straightforward, you will have to work carefully to avoid computational errors. In addition, if some of the moments of inertia that you need are not given in Fig. 9.28, you will have to derive your own formulas, using the techniques of this lesson.

PROBLEMS

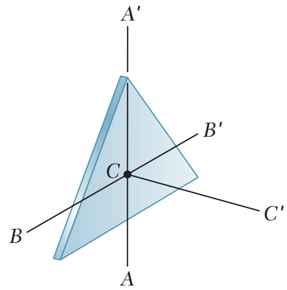


Fig. P9.111

9.111 A thin plate of mass m is cut in the shape of an equilateral triangle of side a. Determine the mass moment of inertia of the plate with respect to (*a*) the centroidal axes AA' and BB', (*b*) the centroidal axis CC' that is perpendicular to the plate.

9.112 The elliptical ring shown was cut from a thin, uniform plate. Denoting the mass of the ring by m, determine its mass moment of inertia with respect to (*a*) the centroidal axis BB', (*b*) the centroidal axis CC' that is perpendicular to the plane of the ring.

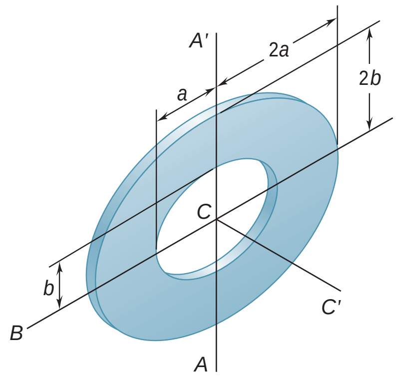


Fig. P9.112

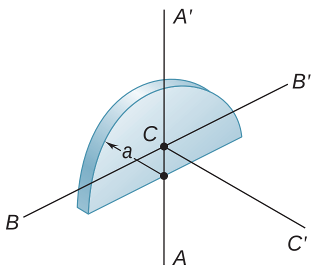


Fig. P9.113

9.113 A thin semicircular plate has a radius a and a mass m. Determine the mass moment of inertia of the plate with respect to (*a*) the centroidal axis BB', (*b*) the centroidal axis CC' that is perpendicular to the plate.

9.114 The quarter ring shown has a mass m and was cut from a thin, uniform plate. Knowing that $r_1 = \frac{3}{4}r_2$, determine the mass moment of inertia of the quarter ring with respect to (*a*) the axis AA', (*b*) the centroidal axis CC' that is perpendicular to the plane of the quarter ring.

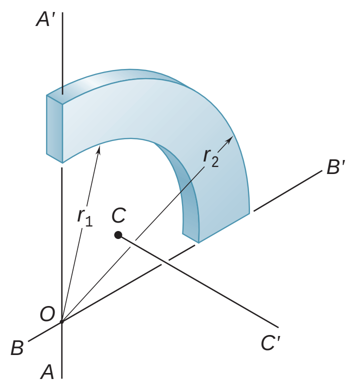


Fig. P9.114

9.115 A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m, determine its mass moment of inertia with respect to (*a*) the x axis, (*b*) the y axis.

9.116 A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m, determine its mass moment of inertia with respect to (*a*) the axis AA', (*b*) the axis BB', where the AA' and BB' axes are parallel to the x axis and lie in a plane parallel to and at a distance a above the xz plane.

9.117 A thin plate of mass m was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (*a*) the x axis, (*b*) the axis BB', which is perpendicular to the plate.

9.118 A thin plate of mass m was cut in the shape of a parallelogram as shown. Determine the mass moment of inertia of the plate with respect to (*a*) the y axis, (*b*) the axis AA', which is perpendicular to the plate.

9.119 Determine by direct integration the mass moment of inertia with respect to the z axis of the right circular cylinder shown, assuming that it has a uniform density and a mass m.

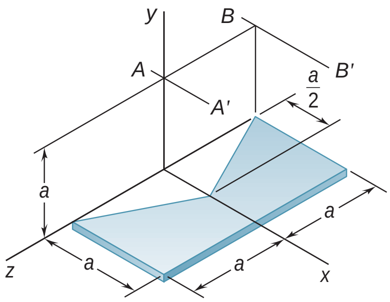


Fig. *P9.115* and *P9.116*

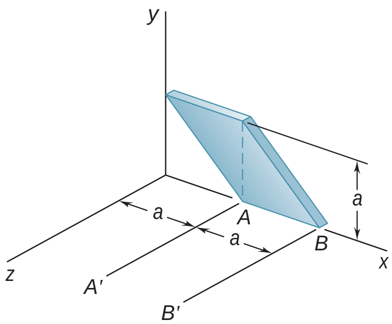


Fig. P9.117 and P9.118

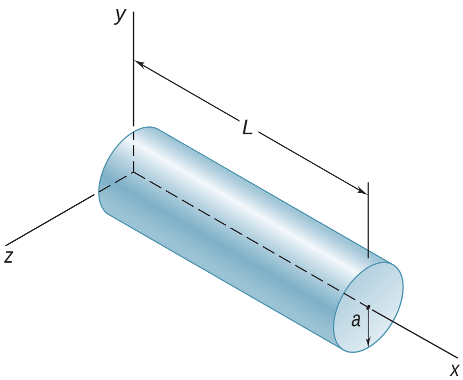


Fig. P9.119

9.120 The area shown is revolved about the x axis to form a homogeneous solid of revolution of mass m. Using direct integration, express the mass moment of inertia of the solid with respect to the x axis in terms of m and h.

9.121 The area shown is revolved about the x axis to form a homogeneous solid of revolution of mass m. Determine by direct integration the mass moment of inertia of the solid with respect to (*a*) the x axis, (*b*) the y axis. Express your answers in terms of m and the dimensions of the solid.

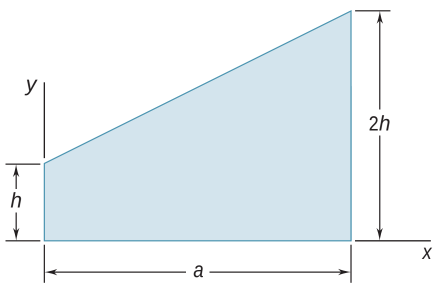


Fig. P9.120

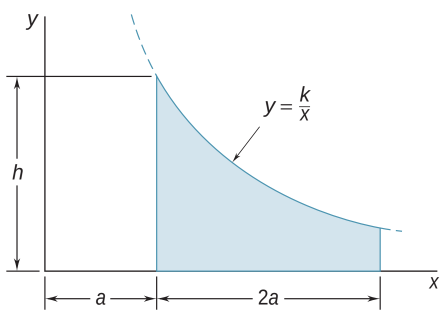


Fig. P9.121

9.122 Determine by direct integration the mass moment of inertia with respect to the x axis of the pyramid shown, assuming that it has a uniform density and a mass m.

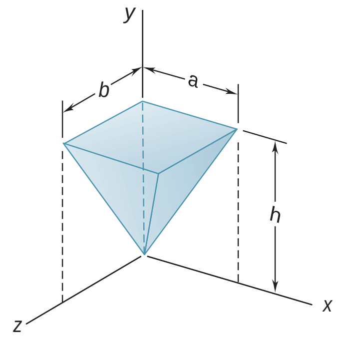


Fig. P9.122 and *P9.123*

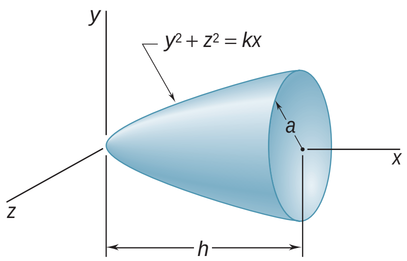


Fig. P9.124

9.123 Determine by direct integration the mass moment of inertia with respect to the y axis of the pyramid shown, assuming that it has a uniform density and a mass m.

9.124 Determine by direct integration the mass moment of inertia with respect to the y axis of the paraboloid shown, assuming that it has a uniform density and a mass m.

9.125 A thin rectangular plate of mass m is welded to a vertical shaft AB as shown. Knowing that the plate forms an angle u with the y axis, determine by direct integration the mass moment of inertia of the plate with respect to (*a*) the y axis, (*b*) the z axis.

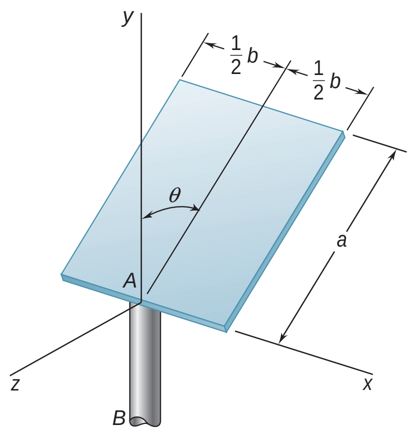


Fig. *P9.125*

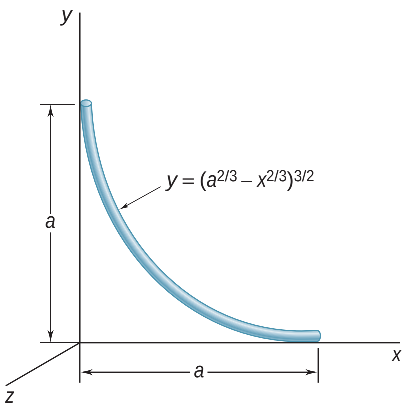


Fig. P9.126

***9.126** A thin steel wire is bent into the shape shown. Denoting the mass per unit length of the wire by m', determine by direct integration the mass moment of inertia of the wire with respect to each of the coordinate axes.

9.127 Shown is the cross section of an idler roller. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA'. (The specific weight of bronze is 0.310 lb/in^3; of aluminum, 0.100 lb/in^3; and of neoprene, 0.0452 lb/in^3.)

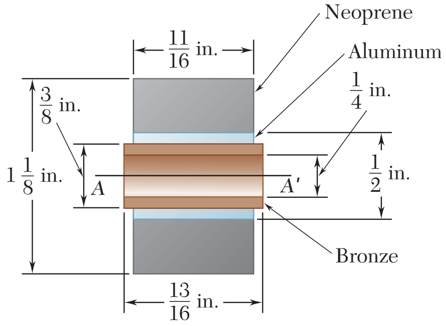


Fig. P9.127

9.128 Shown is the cross section of a molded flat-belt pulley. Determine its mass moment of inertia and its radius of gyration with respect to the axis AA'. (The density of brass is 8650 kg/m^3 and the density of the fiber-reinforced polycarbonate used is 1250 kg/m^3.)

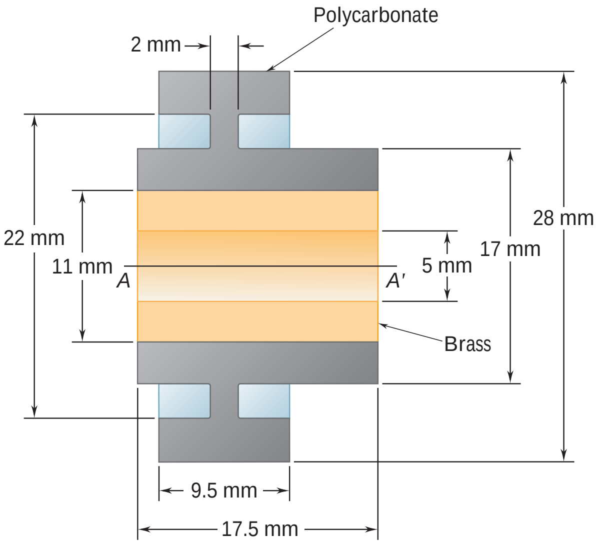


Fig. P9.128

9.129 The machine part shown is formed by machining a conical surface into a circular cylinder. For $b = \frac{1}{2}h$, determine the mass moment of inertia and the radius of gyration of the machine part with respect to the y axis.

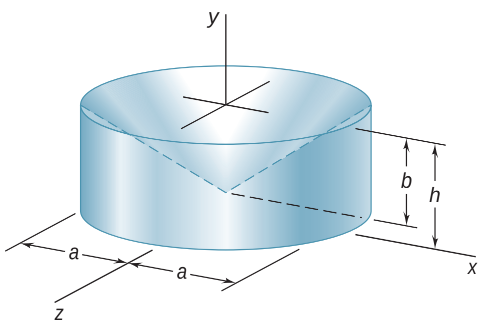


Fig. *P9.129*

9.130 Given the dimensions and the mass m of the thin conical shell shown, determine the mass moment of inertia and the radius of gyration of the shell with respect to the x axis. (*Hint:* Assume that the shell was formed by removing a cone with a circular base of radius a from a cone with a circular base of radius $a + t$, where t is the thickness of the wall. In the resulting expressions, neglect terms containing t^2, t^3, etc. Do not forget to account for the difference in the heights of the two cones.)

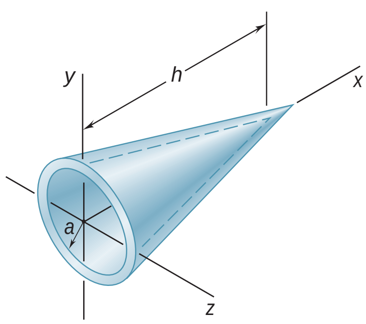


Fig. P9.130

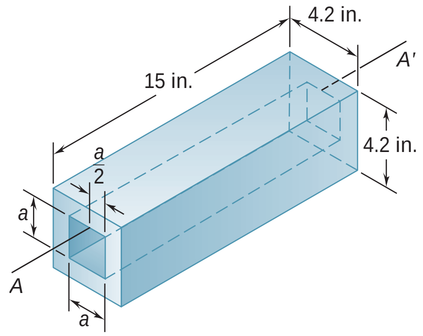


Fig. P9.131

9.131 A square hole is centered in and extends through the aluminum machine component shown. Determine (*a*) the value of *a* for which the mass moment of inertia of the component with respect to the axis AA', which bisects the top surface of the hole, is maximum, (*b*) the corresponding values of the mass moment of inertia and the radius of gyration with respect to the axis AA'. (The specific weight of aluminum is 0.100 lb/in^3.)

9.132 The cups and the arms of an anemometer are fabricated from a material of density r. Knowing that the mass moment of inertia of a thin, hemispherical shell of mass *m* and thickness *t* with respect to its centroidal axis GG' is $5ma^2/12$, determine (*a*) the mass moment of inertia of the anemometer with respect to the axis AA', (*b*) the ratio of *a* to *l* for which the centroidal moment of inertia of the cups is equal to 1 percent of the moment of inertia of the cups with respect to the axis AA'.

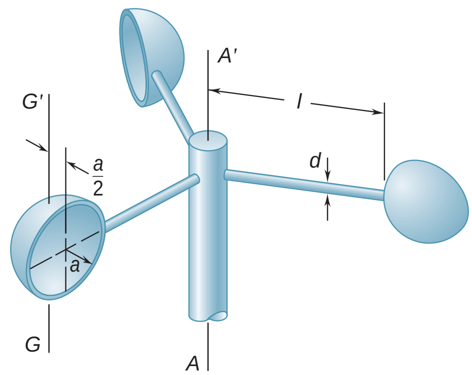


Fig. P9.132

9.133 After a period of use, one of the blades of a shredder has been worn to the shape shown and is of mass 0.18 kg. Knowing that the mass moments of inertia of the blade with respect to the AA' and BB' axes are $0.320 \text{ g} \cdot \text{m}^2$ and $0.680 \text{ g} \cdot \text{m}^2$, respectively, determine (*a*) the location of the centroidal axis GG', (*b*) the radius of gyration with respect to axis GG'.

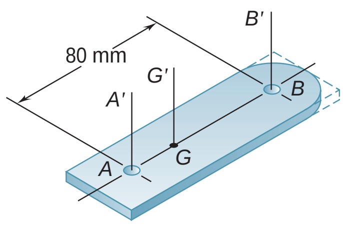


Fig. P9.133

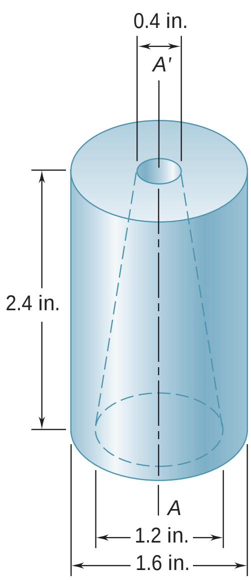


Fig. P9.134

9.134 Determine the mass moment of inertia of the 0.9-lb machine component shown with respect to the axis AA'.

9.135 and 9.136 A 2-mm-thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

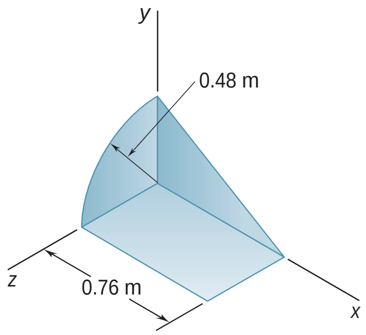


Fig. P9.135

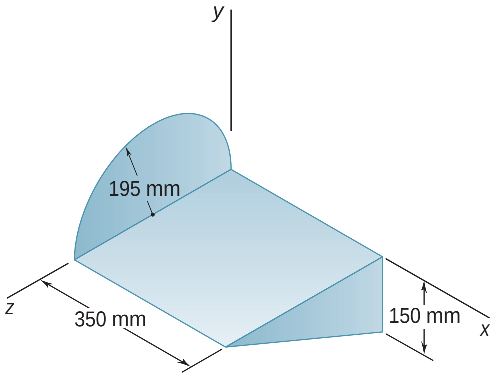


Fig. P9.136

9.137 A subassembly for a model airplane is fabricated from three pieces of 1.5-mm plywood. Neglecting the mass of the adhesive used to assemble the three pieces, determine the mass moment of inertia of the subassembly with respect to each of the coordinate axes. (The density of the plywood is 780 kg/m^3.)

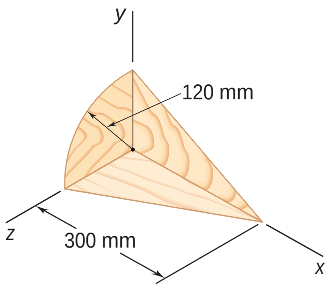


Fig. *P9.137*

9.138 The cover for an electronic device is formed from sheet aluminum that is 0.05 in. thick. Determine the mass moment of inertia of the cover with respect to each of the coordinate axes. (The specific weight of aluminum is 0.100 lb/in^3.)

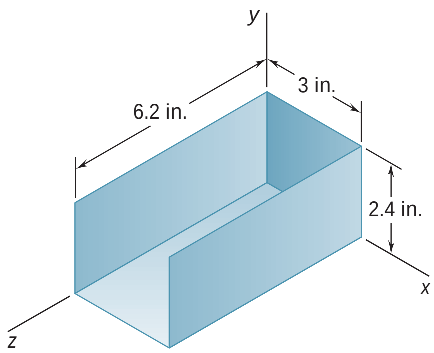


Fig. P9.138

9.139 A framing anchor is formed of 0.05-in.-thick galvanized steel. Determine the mass moment of inertia of the anchor with respect to each of the coordinate axes. (The specific weight of galvanized steel is 470 lb/ft^3.)

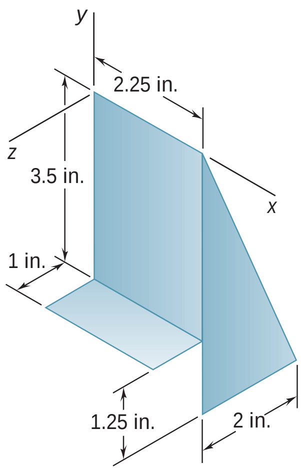


Fig. P9.139

***9.140** A farmer constructs a trough by welding a rectangular piece of 2-mm-thick sheet steel to half of a steel drum. Knowing that the density of steel is 7850 kg/m^3 and that the thickness of the walls of the drum is 1.8 mm, determine the mass moment of inertia of the trough with respect to each of the coordinate axes. Neglect the mass of the welds.

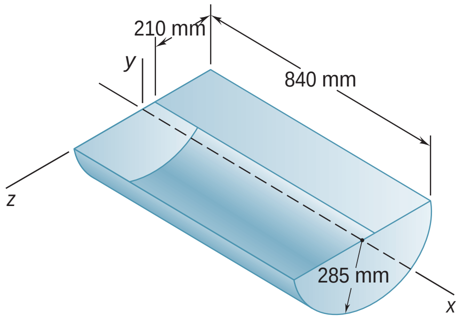


Fig. *P9.140*

9.141 The machine element shown is fabricated from steel. Determine the mass moment of inertia of the assembly with respect to (*a*) the x axis, (*b*) the y axis, (*c*) the z axis. (The density of steel is 7850 kg/m^3.)

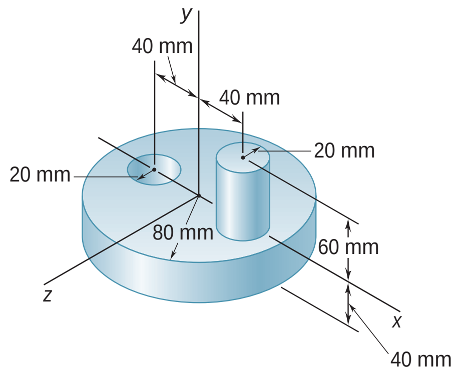


Fig. P9.141

9.142 Determine the mass moments of inertia and the radii of gyration of the steel machine element shown with respect to the x and y axes. (The density of steel is 7850 kg/m^3.)

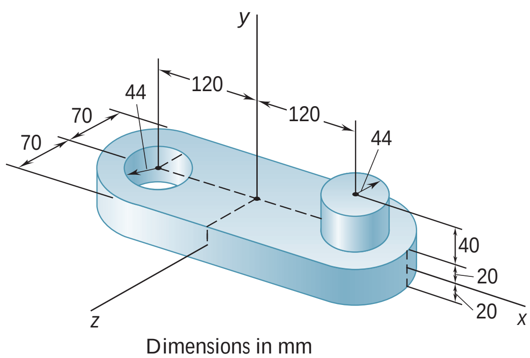


Fig. P9.142

9.143 Determine the mass moment of inertia of the steel machine element shown with respect to the y axis. (The specific weight of steel is 490 lb/ft^3.)

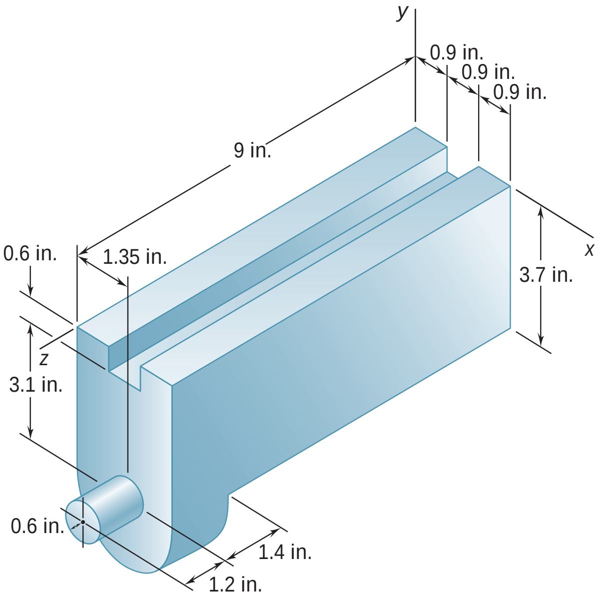


Fig. P9.143 and *P9.144*

9.144 Determine the mass moment of inertia of the steel machine element shown with respect to the z axis. (The specific weight of steel is 490 lb/ft^3.)

9.145 Determine the mass moment of inertia of the steel fixture shown with respect to (*a*) the x axis, (*b*) the y axis, (*c*) the z axis. (The density of steel is 7850 kg/m^3.)

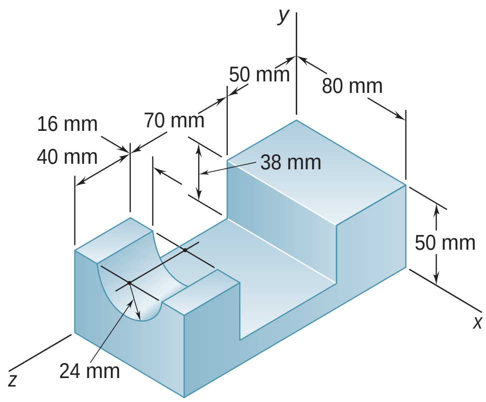


Fig. P9.145

9.146 Aluminum wire with a weight per unit length of 0.033 lb/ft is used to form the circle and the straight members of the figure shown. Determine the mass moment of inertia of the assembly with respect to each of the coordinate axes.

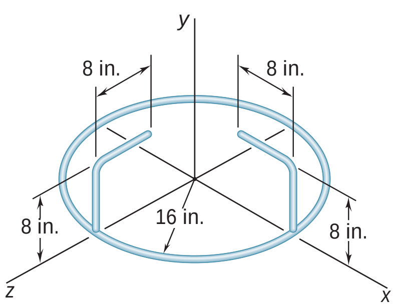


Fig. *P9.146*

9.147 The figure shown is formed of $\frac{1}{8}$-in.-diameter steel wire. Knowing that the specific weight of the steel is 490 lb/ft^3, determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

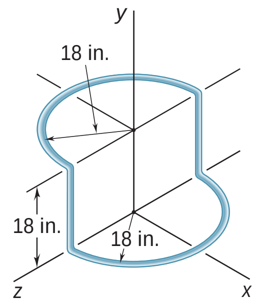


Fig. P9.147

9.148 A homogeneous wire with a mass per unit length of 0.056 kg/m is used to form the figure shown. Determine the mass moment of inertia of the wire with respect to each of the coordinate axes.

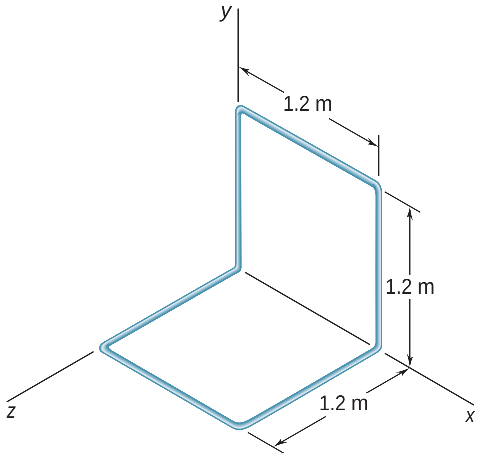


Fig. P9.148

*9.16 MOMENT OF INERTIA OF A BODY WITH RESPECT TO AN ARBITRARY AXIS THROUGH *O*. MASS PRODUCTS OF INERTIA

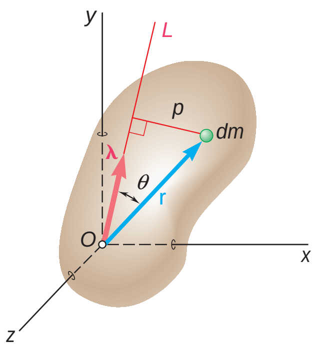


Fig. 9.29

In this section you will see how the moment of inertia of a body can be determined with respect to an arbitrary axis OL through the origin (Fig. 9.29) if its moments of inertia with respect to the three coordinate axes, as well as certain other quantities to be defined below, have already been determined.

The moment of inertia I_{OL} of the body with respect to OL is equal to $\int p^2 dm$, where p denotes the perpendicular distance from the element of mass dm to the axis OL. If we denote by $\boldsymbol{\lambda}$ the unit vector along OL and by $\mathbf{r}$ the position vector of the element dm, we observe that the perpendicular distance p is equal to $r \sin \theta$, which is the magnitude of the vector product $\boldsymbol{\lambda} \times \mathbf{r}$. We therefore write

$$I_{OL} = \int p^2 \, \mathrm{dm} = \int |\boldsymbol{\lambda} \times \mathbf{r}|^2 \, dm \tag{9.43}$$

Expressing $|\boldsymbol{\lambda} \times \mathbf{r}|^2$ in terms of the rectangular components of the vector product, we have

$$I_{OL} = \int [(\lambda_x y - \lambda_y x)^2 + (\lambda_y z - \lambda_z y)^2 + (\lambda_z x - \lambda_x z)^2] \, dm$$

where the components λ_x, λ_y, λ_z of the unit vector $\boldsymbol{\lambda}$ represent the direction cosines of the axis OL and the components x, y, z of $\mathbf{r}$ represent the coordinates of the element of mass dm. Expanding the squares and rearranging the terms, we write

$$\begin{aligned} I_{OL} = {} & \lambda_x^2 \int (y^2 + z^2)\, dm + \lambda_y^2 \int (z^2 + x^2)\, dm + \lambda_z^2 \int (x^2 + y^2)\, dm \\ & - 2\lambda_x \lambda_y \int xy \, dm - 2\lambda_y \lambda_z \int yz \, dm - 2\lambda_z \lambda_x \int zx \, dm \end{aligned} \tag{9.44}$$

Referring to Eqs. (9.30), we note that the first three integrals in (9.44) represent, respectively, the moments of inertia I_x, I_y, and I_z of the body with respect to the coordinate axes. The last three integrals in (9.44), which involve products of coordinates, are called the *products of inertia* of the body with respect to the x and y axes, the y and z axes, and the z and x axes, respectively. We write

$$I_{xy} = \int xy\,dm \qquad I_{yz} = \int yz\,dm \qquad I_{zx} = \int zx\,dm \tag{9.45}$$

Rewriting Eq. (9.44) in terms of the integrals defined in Eqs. (9.30) and (9.45), we have

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x \tag{9.46}$$

We note that the definition of the products of inertia of a mass given in Eqs. (9.45) is an extension of the definition of the product of inertia of an area (Sec. 9.8). Mass products of inertia reduce to zero under the same conditions of symmetry as do products of inertia of areas, and the parallel-axis theorem for mass products of inertia is expressed by relations similar to the formula derived for the product of inertia of an area. Substituting the expressions for x, y, and z given in Eqs. (9.31) into Eqs. (9.45), we find that

$$\begin{aligned} I_{xy} &= \bar{I}_{x'y'} + m\bar{x}\,\bar{y} \\ I_{yz} &= \bar{I}_{y'z'} + m\bar{y}\,\bar{z} \\ I_{zx} &= \bar{I}_{z'x'} + m\bar{z}\,\bar{x} \end{aligned} \tag{9.47}$$

where $\bar{x}, \bar{y}, \bar{z}$ are the coordinates of the center of gravity G of the body and $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, $\bar{I}_{z'x'}$ denote the products of inertia of the body with respect to the centroidal axes x', y', z' (See Fig. 9.22).

*9.17 ELLIPSOID OF INERTIA. PRINCIPAL AXES OF INERTIA

Let us assume that the moment of inertia of the body considered in the preceding section has been determined with respect to a large number of axes OL through the fixed point O and that a point Q has been plotted on each axis OL at a distance $OQ = 1/\sqrt{I_{OL}}$ from O. The locus of the points Q thus obtained forms a surface (Fig. 9.30). The equation of that surface can be obtained by substituting $1/(OQ)^2$ for I_{OL} in (9.46) and then multiplying both sides of the equation by $(OQ)^2$. Observing that

$$(OQ)\lambda_x = x \qquad (OQ)\lambda_y = y \qquad (OQ)\lambda_z = z$$

where x, y, z denote the rectangular coordinates of Q, we write

$$I_x x^2 + I_y y^2 + I_z z^2 - 2I_{xy}xy - 2I_{yz}yz - 2I_{zx}zx = 1 \tag{9.48}$$

The equation obtained is the equation of a *quadric surface*. Since the moment of inertia I_{OL} is different from zero for every axis OL, no point Q can be at an infinite distance from O. Thus, the quadric surface obtained is an *ellipsoid*. This ellipsoid, which defines the

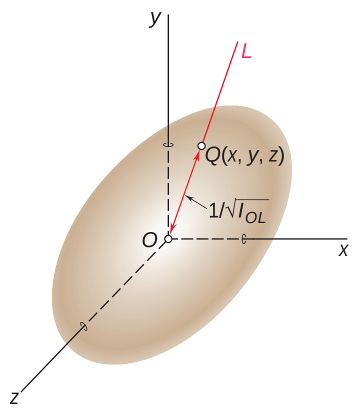


Fig. 9.30

moment of inertia of the body with respect to any axis through O, is known as the *ellipsoid of inertia* of the body at O.

We observe that if the axes in Fig. 9.30 are rotated, the coefficients of the equation defining the ellipsoid change, since they are equal to the moments and products of inertia of the body with respect to the rotated coordinate axes. However, the *ellipsoid itself remains unaffected*, since its shape depends only upon the distribution of mass in the given body. Suppose that we choose as coordinate axes the principal axes x', y', z' of the ellipsoid of inertia (Fig. 9.31). The equation of the ellipsoid with respect to these coordinate axes is known to be of the form

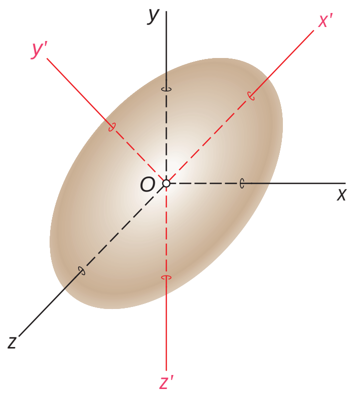


Fig. 9.31

$$I_{x'}x'^2 + I_{y'}y'^2 + I_{z'}z'^2 = 1 \tag{9.49}$$

which does not contain any products of the coordinates. Comparing Eqs. (9.48) and (9.49), we observe that the products of inertia of the body with respect to the x', y', z' axes must be zero. The x', y', z' axes are known as the *principal axes of inertia* of the body at O, and the coefficients $I_{x'}$, $I_{y'}$, $I_{z'}$ are referred to as the *principal moments of inertia* of the body at O. Note that, given a body of arbitrary shape and a point O, it is always possible to find axes which are the principal axes of inertia of the body at O, that is, axes with respect to which the products of inertia of the body are zero. Indeed, whatever the shape of the body, the moments and products of inertia of the body with respect to x, y, and z axes through O will define an ellipsoid, and this ellipsoid will have principal axes which, by definition, are the principal axes of inertia of the body at O.

If the principal axes of inertia x', y', z' are used as coordinate axes, the expression obtained in Eq. (9.46) for the moment of inertia of a body with respect to an arbitrary axis through O reduces to

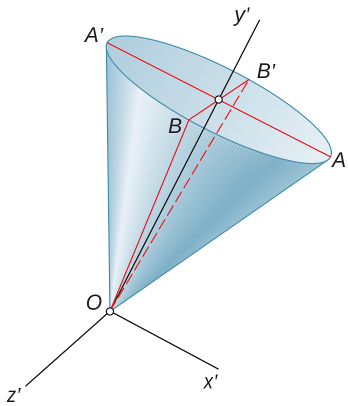


Fig. 9.32

$$\boxed{I_{OL} = I_{x'}\lambda_{x'}^2 + I_{y'}\lambda_{y'}^2 + I_{z'}\lambda_{z'}^2} \tag{9.50}$$

The determination of the principal axes of inertia of a body of arbitrary shape is somewhat involved and will be discussed in the next section. There are many cases, however, where these axes can be spotted immediately. Consider, for instance, the homogeneous cone of elliptical base shown in Fig. 9.32; this cone possesses two mutually perpendicular planes of symmetry OAA' and OBB'. From the definition (9.45), we observe that if the $x'y'$ and $y'z'$ planes are chosen to coincide with the two planes of symmetry, all of the products of inertia are zero. The x', y', and z' axes thus selected are therefore the principal axes of inertia of the cone at O. In the case of the homogeneous regular tetrahedron $OABC$ shown in Fig. 9.33, the line joining the corner O to the center D of the opposite face is a principal axis of inertia at O, and any line through O perpendicular to OD is also a principal axis of inertia at O. This property is apparent if we observe that rotating the tetrahedron through 120° about OD leaves its shape and mass distribution unchanged. It follows that the ellipsoid of inertia at O also remains unchanged under this rotation. The ellipsoid, therefore, is a body of revolution whose axis of revolution is OD, and the line OD, as well as any perpendicular line through O, must be a principal axis of the ellipsoid.

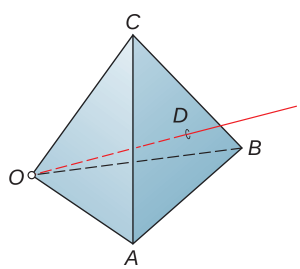


Fig. 9.33

*9.18 DETERMINATION OF THE PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA OF A BODY OF ARBITRARY SHAPE

The method of analysis described in this section should be used when the body under consideration has no obvious property of symmetry.

Consider the ellipsoid of inertia of the body at a given point O (Fig. 9.34); let $\mathbf{r}$ be the radius vector of a point P on the surface of the ellipsoid and let $\mathbf{n}$ be the unit vector along the normal to that surface at P. We observe that the only points where $\mathbf{r}$ and $\mathbf{n}$ are collinear are the points P_1, P_2, and P_3, where the principal axes intersect the visible portion of the surface of the ellipsoid, and the corresponding points on the other side of the ellipsoid.

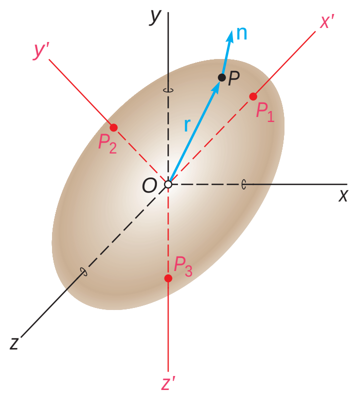


Fig. 9.34

We now recall from calculus that the direction of the normal to a surface of equation $f(x, y, z) = 0$ at a point $P(x, y, z)$ is defined by the gradient ∇f of the function f at that point. To obtain the points where the principal axes intersect the surface of the ellipsoid of inertia, we must therefore write that $\mathbf{r}$ and ∇f are collinear,

$$\nabla f = (2K)\mathbf{r} \tag{9.51}$$

where K is a constant, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial x}\mathbf{k}$$

Recalling Eq. (9.48), we note that the function $f(x, y, z)$ corresponding to the ellipsoid of inertia is

$$f(x, y, z) = I_x x^2 + I_y y^2 + I_z z^2 - 2I_{xy}xy - 2I_{yz}yz - 2I_{zx}zx - 1$$

Substituting for $\mathbf{r}$ and ∇f into Eq. (9.51) and equating the coefficients of the unit vectors, we write

$$\begin{aligned} I_x x \quad - I_{xy} y - I_{zx} z &= Kx \\ -I_{xy} x + I_y y \quad - I_{yz} z &= Ky \\ -I_{zx} x - I_{yz} y + I_z z \quad &= Kz \end{aligned} \tag{9.52}$$

Dividing each term by the distance r from O to P, we obtain similar equations involving the direction cosines λ_x, λ_y, and λ_z:

$$\begin{aligned} I_x\lambda_x - I_{xy}\lambda_y - I_{zx}\lambda_z &= K\lambda_x \\ -I_{xy}\lambda_x + I_y\lambda_y - I_{yz}\lambda_z &= K\lambda_y \\ -I_{zx}\lambda_x - I_{yz}\lambda_y + I_z\lambda_z &= K\lambda_z \end{aligned} \tag{9.53}$$

Transposing the right-hand members leads to the following homogeneous linear equations:

$$\begin{aligned} (I_x - K)\lambda_x - I_{xy}\lambda_y - I_{zx}\lambda_z &= 0 \\ -I_{xy}\lambda_x + (I_y - K)\lambda_y - I_{yz}\lambda_z &= 0 \\ -I_{zx}\lambda_x - I_{yz}\lambda_y + (I_z - K)\lambda_z &= 0 \end{aligned} \tag{9.54}$$

For this system of equations to have a solution different from $\lambda_x = \lambda_y = \lambda_z = 0$, its discriminant must be zero:

$$\begin{vmatrix} I_x - K & -I_{xy} & -I_{zx} \\ -I_{xy} & I_y - K & -I_{yz} \\ -I_{zx} & -I_{yz} & I_z - K \end{vmatrix} = 0 \tag{9.55}$$

Expanding this determinant and changing signs, we write

$$K^3 - (I_x + I_y + I_z)K^2 + (I_xI_y + I_yI_z + I_zI_x - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)K \\ - (I_xI_yI_z - I_xI_{yz}^2 - I_yI_{zx}^2 - I_zI_{xy}^2 - 2I_{xy}I_{yz}I_{zx}) = 0 \tag{9.56}$$

This is a cubic equation in K, which yields three real, positive roots K_1, K_2, and K_3.

To obtain the direction cosines of the principal axis corresponding to the root K_1 we substitute K_1 for K in Eqs. (9.54). Since these equations are now linearly dependent, only two of them may be used to determine λ_x, λ_y, and λ_z. An additional equation may be obtained, however, by recalling from Sec. 2.12 that the direction cosines must satisfy the relation

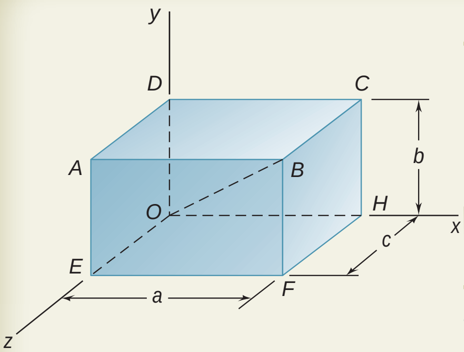
$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \tag{9.57}$$

Repeating this procedure with K_2 and K_3, we obtain the direction cosines of the other two principal axes.

We will now show that *the roots K_1, K_2, and K_3 of Eq. (9.56) are the principal moments of inertia of the given body*. Let us substitute for K in Eqs. (9.53) the root K_1, and for λ_x, λ_y, and λ_z the corresponding values $(\lambda_x)_1$, $(\lambda_y)_1$, and $(\lambda_z)_1$ of the direction cosines; the three equations will be satisfied. We now multiply by $(\lambda_x)_1$, $(\lambda_y)_1$, and $(\lambda_z)_1$, respectively, each term in the first, second, and third equation and add the equations obtained in this way. We write

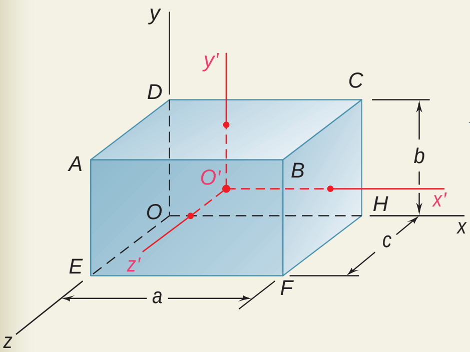
$$I_x^2(\lambda_x)_1^2 + I_y^2(\lambda_y)_1^2 + I_z^2(\lambda_z)_1^2 - 2I_{xy}(\lambda_x)_1(\lambda_y)_1 \\ - 2I_{yz}(\lambda_y)_1(\lambda_z)_1 - 2I_{zx}(\lambda_z)_1(\lambda_x)_1 = K_1[(\lambda_x)_1^2 + (\lambda_y)_1^2 + (\lambda_z)_1^2]$$

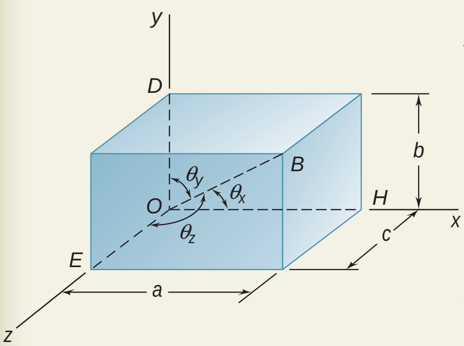
Recalling Eq. (9.46), we observe that the left-hand member of this equation represents the moment of inertia of the body with respect to the principal axis corresponding to K_1; it is thus the principal moment of inertia corresponding to that root. On the other hand, recalling Eq. (9.57), we note that the right-hand member reduces to K_1. Thus K_1 itself is the principal moment of inertia. We can show in the same fashion that K_2 and K_3 are the other two principal moments of inertia of the body.

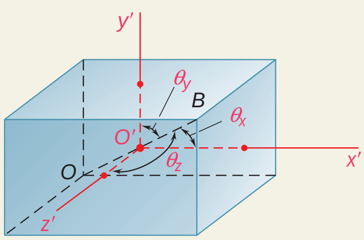


SAMPLE PROBLEM 9.14

Consider a rectangular prism of mass m and sides a, b, c. Determine (a) the moments and products of inertia of the prism with respect to the coordinate axes shown, (b) its moment of inertia with respect to the diagonal OB.







SOLUTION

a. Moments and Products of Inertia with Respect to the Coordinate Axes. ***Moments of Inertia.*** Introducing the centroidal axes x', y', z', with respect to which the moments of inertia are given in Fig. 9.28, we apply the parallel-axis theorem:

$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2) = \tfrac{1}{12}m(b^2 + c^2) + m(\tfrac{1}{4}b^2 + \tfrac{1}{4}c^2)$$

$$I_x = \tfrac{1}{3}m(b^2 + c^2) \quad ◀$$

Similarly, $$I_y = \tfrac{1}{3}m(c^2 + a^2) \qquad I_z = \tfrac{1}{3}m(a^2 + b^2) \quad ◀$$

Products of Inertia. Because of symmetry, the products of inertia with respect to the centroidal axes x', y', z' are zero, and these axes are principal axes of inertia. Using the parallel-axis theorem, we have

$$I_{xy} = \bar{I}_{x'y'} + m\bar{x}\bar{y} = 0 + m(\tfrac{1}{2}a)(\tfrac{1}{2}b) \qquad I_{xy} = \tfrac{1}{4}mab \quad ◀$$

Similarly, $$I_{yz} = \tfrac{1}{4}mbc \qquad I_{zx} = \tfrac{1}{4}mca \quad ◀$$

b. Moment of Inertia with Respect to *OB*. We recall Eq. (9.46):

$$I_{OB} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

where the direction cosines of OB are

$$\lambda_x = \cos\theta_x = \frac{OH}{OB} = \frac{a}{(a^2 + b^2 + c^2)^{1/2}}$$

$$\lambda_y = \frac{b}{(a^2 + b^2 + c^2)^{1/2}} \qquad \lambda_z = \frac{c}{(a^2 + b^2 + c^2)^{1/2}}$$

Substituting the values obtained for the moments and products of inertia and for the direction cosines into the equation for I_{OB}, we have

$$I_{OB} = \frac{1}{a^2 + b^2 + c^2}[\tfrac{1}{3}m(b^2 + c^2)a^2 + \tfrac{1}{3}m(c^2 + a^2)b^2 + \tfrac{1}{3}m(a^2 + b^2)c^2$$

$$-\tfrac{1}{2}ma^2b^2 - \tfrac{1}{2}mb^2c^2 - \tfrac{1}{2}mc^2a^2]$$

$$I_{OB} = \frac{m}{6}\frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2} \quad ◀$$

Alternative Solution. The moment of inertia I_{OB} can be obtained directly from the principal moments of inertia $\bar{I}_{x'}$, $\bar{I}_{y'}$, $\bar{I}_{z'}$, since the line OB passes through the centroid O'. Since the x', y', z' axes are principal axes of inertia, we use Eq. (9.50) to write

$$I_{OB} = \bar{I}_{x'}\lambda_x^2 + \bar{I}_{y'}\lambda_y^2 + \bar{I}_{z'}\lambda_z^2$$

$$= \frac{1}{a^2 + b^2 + c^2}\left[\frac{m}{12}(b^2 + c^2)a^2 + \frac{m}{12}(c^2 + a^2)b^2 + \frac{m}{12}(a^2 + b^2)c^2\right]$$

$$I_{OB} = \frac{m}{6}\frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2} \quad ◀$$

SAMPLE PROBLEM 9.15

If $a = 3c$ and $b = 2c$ for the rectangular prism of Sample Prob. 9.14, determine (*a*) the principal moments of inertia at the origin O, (*b*) the principal axes of inertia at O.

SOLUTION

a. Principal Moments of Inertia at the Origin *O*. Substituting $a = 3c$ and $b = 2c$ into the solution to Sample Prob. 9.14, we have

$$I_x = \tfrac{5}{3}mc^2 \qquad I_y = \tfrac{10}{3}mc^2 \qquad I_z = \tfrac{13}{3}mc^2$$
$$I_{xy} = \tfrac{3}{2}mc^2 \qquad I_{yz} = \tfrac{1}{2}mc^2 \qquad I_{zx} = \tfrac{3}{4}mc^2$$

Substituting the values of the moments and products of inertia into Eq. (9.56) and collecting terms yields

$$K^3 - (\tfrac{28}{3}mc^2)K^2 + (\tfrac{3479}{144}m^2c^4)K - \tfrac{589}{54}m^3c^6 = 0$$

We then solve for the roots of this equation; from the discussion in Sec. 9.18, it follows that these roots are the principal moments of inertia of the body at the origin.

$$K_1 = 0.568867mc^2 \qquad K_2 = 4.20885mc^2 \qquad K_3 = 4.55562mc^2$$
$$K_1 = 0.569mc^2 \qquad K_2 = 4.21mc^2 \qquad K_3 = 4.56mc^2$$ ◀

b. Principal Axes of Inertia at *O*. To determine the direction of a principal axis of inertia, we first substitute the corresponding value of K into two of the equations (9.54); the resulting equations together with Eq. (9.57) constitute a system of three equations from which the direction cosines of the corresponding principal axis can be determined. Thus, we have for the first principal moment of inertia K_1:

$$(\tfrac{5}{3} - 0.568867)mc^2(\lambda_x)_1 - \tfrac{3}{2}mc^2(\lambda_y)_1 - \tfrac{3}{4}mc^2(\lambda_z)_1 = 0$$
$$-\tfrac{3}{2}mc^2(\lambda_x)_1 + (\tfrac{10}{3} - 0.568867)mc^2(\lambda_y)_1 - \tfrac{1}{2}mc^2(\lambda_z)_1 = 0$$
$$(\lambda_x)_1^2 + (\lambda_y)_1^2 + (\lambda_z)_1^2 = 1$$

Solving yields

$$(\lambda_x)_1 = 0.836600 \qquad (\lambda_y)_1 = 0.496001 \qquad (\lambda_z)_1 = 0.232557$$

The angles that the first principal axis of inertia forms with the coordinate axes are then

$$(\theta_x)_1 = 33.2° \qquad (\theta_y)_1 = 60.3° \qquad (\theta_z)_1 = 76.6°$$ ◀

Using the same set of equations successively with K_2 and K_3, we find that the angles associated with the second and third principal moments of inertia at the origin are, respectively,

$$(\theta_x)_2 = 57.8° \qquad (\theta_y)_2 = 146.6° \qquad (\theta_z)_2 = 98.0°$$ ◀

and

$$(\theta_x)_3 = 82.8° \qquad (\theta_y)_3 = 76.1° \qquad (\theta_z)_3 = 164.3°$$ ◀

SOLVING PROBLEMS ON YOUR OWN

In this lesson we defined the *mass products of inertia* I_{xy}, I_{yz}, and I_{zx} of a body and showed you how to determine the moments of inertia of that body with respect to an arbitrary axis passing through the origin O. You also learned how to determine at the origin O the *principal axes of inertia* of a body and the corresponding *principal moments of inertia*.

1. Determining the mass products of inertia of a composite body. The mass products of inertia of a composite body with respect to the coordinate axes can be expressed as the sums of the products of inertia of its component parts with respect to those axes. For each component part, we can use the parallel-axis theorem and write Eqs. (9.47)

$$I_{xy} = \bar{I}_{x'y'} + m\bar{x}\,\bar{y} \qquad I_{yz} = \bar{I}_{y'z'} + m\bar{y}\,\bar{z} \qquad I_{zx} = \bar{I}_{z'x'} + m\bar{z}\,\bar{x}$$

where the primes denote the centroidal axes of each component part and where $\bar{x}$, $\bar{y}$, and $\bar{z}$ represent the coordinates of its center of gravity. Keep in mind that the mass products of inertia can be positive, negative, or zero, and be sure to take into account the signs of $\bar{x}$, $\bar{y}$, and $\bar{z}$.

a. From the properties of symmetry of a component part, you can deduce that two or all three of its centroidal mass products of inertia are zero. For instance, you can verify that for a thin plate parallel to the xy plane; a wire lying in a plane parallel to the xy plane; a body with a plane of symmetry parallel to the xy plane; and a body with an axis of symmetry parallel to the z axis, *the products of inertia* $\bar{I}_{y'z'}$ *and* $\bar{I}_{z'x'}$ *are zero*.

For rectangular, circular, or semicircular plates with axes of symmetry parallel to the coordinate axes; straight wires parallel to a coordinate axis; circular and semicircular wires with axes of symmetry parallel to the coordinate axes; and rectangular prisms with axes of symmetry parallel to the coordinate axes, *the products of inertia* $\bar{I}_{x'y'}$, $\bar{I}_{y'z'}$, *and* $\bar{I}_{z'x'}$ *are all zero*.

b. Mass products of inertia which are different from zero can be computed from Eqs. (9.45). Although, in general, a triple integration is required to determine a mass product of inertia, a single integration can be used if the given body can be divided into a series of thin, parallel slabs. The computations are then similar to those discussed in the previous lesson for moments of inertia.

(*continued*)

2. Computing the moment of inertia of a body with respect to an arbitrary axis *OL*. An expression for the moment of inertia I_{OL} was derived in Sec. 9.16 and is given in Eq. (9.46). Before computing I_{OL}, you must first determine the mass moments and products of inertia of the body with respect to the given coordinate axes as well as the direction cosines of the unit vector $\mathbf{L}$ along OL.

3. Calculating the principal moments of inertia of a body and determining its principal axes of inertia. You saw in Sec. 9.17 that it is always possible to find an orientation of the coordinate axes for which the mass products of inertia are zero. These axes are referred to as the *principal axes of inertia* and the corresponding moments of inertia are known as the *principal moments of inertia* of the body. In many cases, the principal axes of inertia of a body can be determined from its properties of symmetry. The procedure required to determine the principal moments and principal axes of inertia of a body with no obvious property of symmetry was discussed in Sec. 9.18 and was illustrated in Sample Prob. 9.15. It consists of the following steps.

a. Expand the determinant in Eq. (9.55) and solve the resulting cubic equation. The solution can be obtained by trial and error or, preferably, with an advanced scientific calculator or with the appropriate computer software. The roots K_1, K_2, and K_3 of this equation are the principal moments of inertia of the body.

b. To determine the direction of the principal axis corresponding to K_1, substitute this value for K in two of the equations (9.54) and solve these equations together with Eq. (9.57) for the direction cosines of the principal axis corresponding to K_1.

c. Repeat this procedure with K_2 and K_3 to determine the directions of the other two principal axes. As a check of your computations, you may wish to verify that the scalar product of any two of the unit vectors along the three axes you have obtained is zero and, thus, that these axes are perpendicular to each other.

PROBLEMS

9.149 Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the steel fixture shown. (The density of steel is 7850 kg/m^3.)

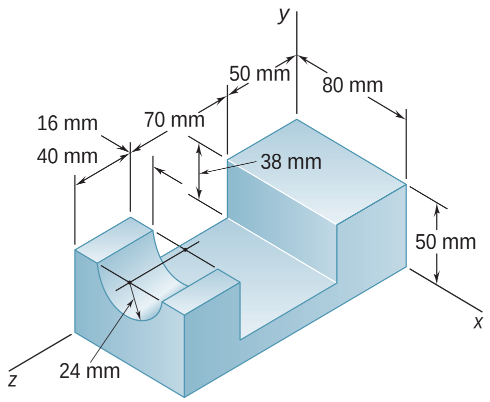


Fig. P9.149

9.150 Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the steel machine element shown. (The density of steel is 7850 kg/m^3.)

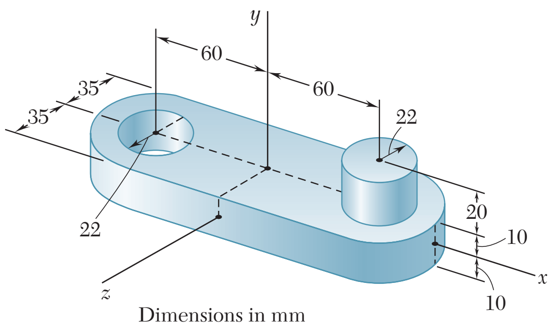


Fig. P9.150

9.151 and 9.152 Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the cast aluminum machine component shown. (The specific weight of aluminum is 0.100 lb/in^3.)

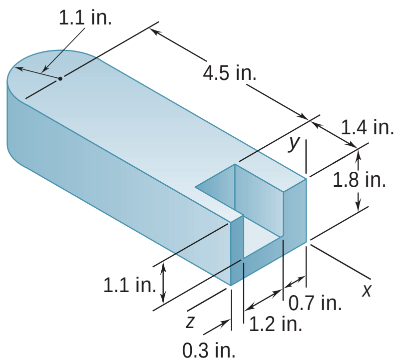


Fig. P9.151

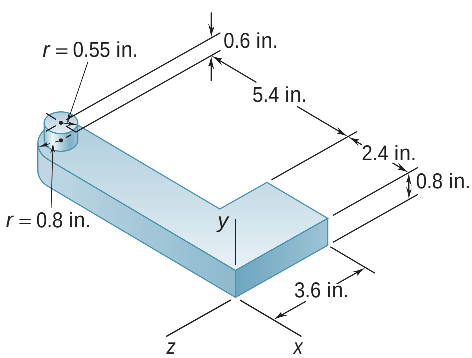


Fig. P9.152

9.153 through 9.156 A section of sheet steel 2 mm thick is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the component.

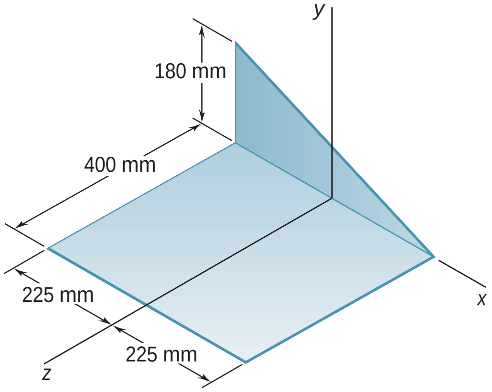


Fig. P9.153

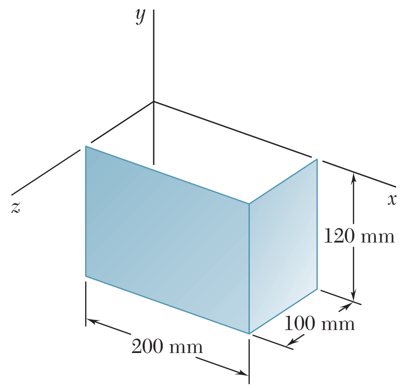


Fig. P9.154

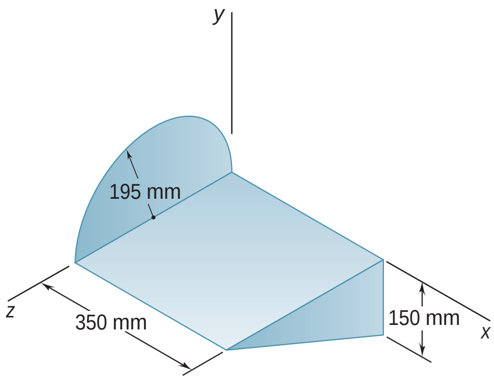


Fig. P9.155

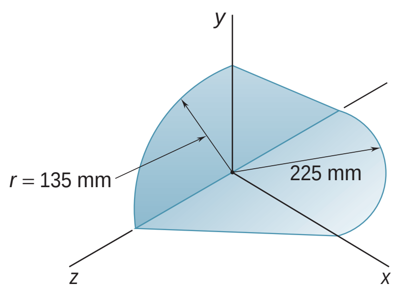


Fig. P9.156

9.157 The figure shown is formed of 1.5-mm-diameter aluminum wire. Knowing that the density of aluminum is 2800 kg/m^3, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

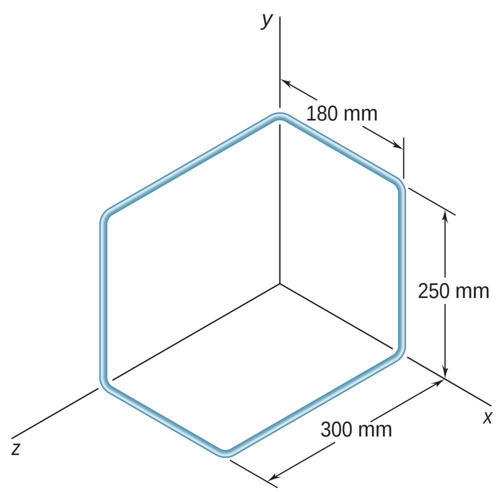


Fig. P9.157

9.158 Thin aluminum wire of uniform diameter is used to form the figure shown. Denoting by m' the mass per unit length of the wire, determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

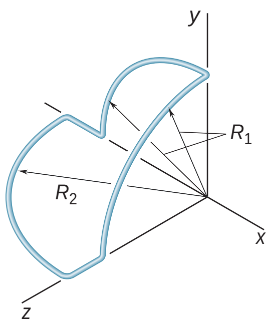


Fig. P9.158

9.159 and 9.160 Brass wire with a weight per unit length w is used to form the figure shown. Determine the mass products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

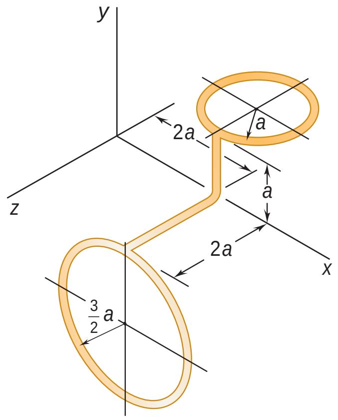


Fig. P9.159

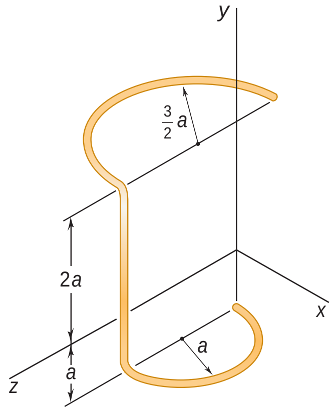


Fig. P9.160

9.161 Complete the derivation of Eqs. (9.47), which express the parallel-axis theorem for mass products of inertia.

9.162 For the homogeneous tetrahedron of mass m shown, (*a*) determine by direct integration the mass product of inertia I_{zx}, (*b*) deduce I_{yz} and I_{xy} from the result obtained in part *a*.

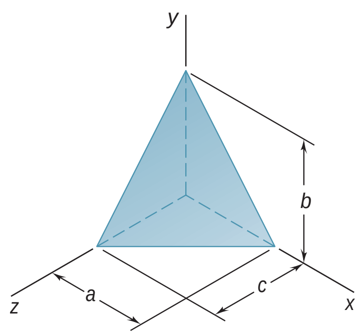


Fig. P9.162

9.163 The homogeneous circular cone shown has a mass m. Determine the mass moment of inertia of the cone with respect to the line joining the origin O and point A.

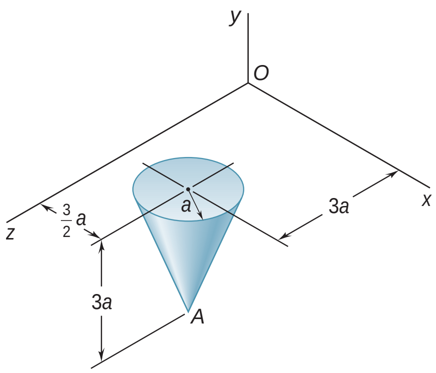


Fig. *P9.163*

9.164 The homogeneous circular cylinder shown has a mass m. Determine the mass moment of inertia of the cylinder with respect to the line joining the origin O and point A that is located on the perimeter of the top surface of the cylinder.

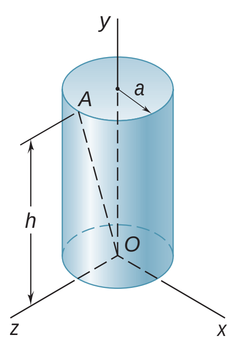


Fig. *P9.164*

9.165 Shown is the machine element of Prob. 9.141. Determine its mass moment of inertia with respect to the line joining the origin O and point A.

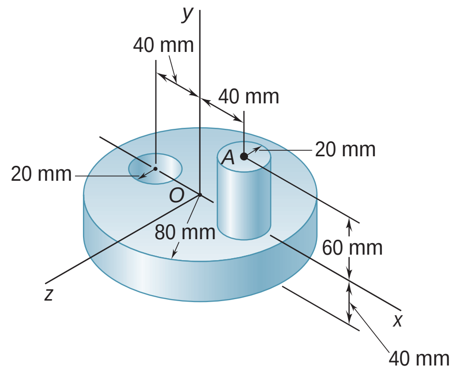


Fig. P9.165

9.166 Determine the mass moment of inertia of the steel fixture of Probs. 9.145 and 9.149 with respect to the axis through the origin that forms equal angles with the x, y, and z axes.

9.167 The thin bent plate shown is of uniform density and weight W. Determine its mass moment of inertia with respect to the line joining the origin O and point A.

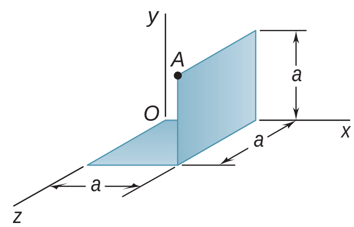


Fig. P9.167

9.168 A piece of sheet steel of thickness t and specific weight γ is cut and bent into the machine component shown. Determine the mass moment of inertia of the component with respect to the line joining the origin O and point A.

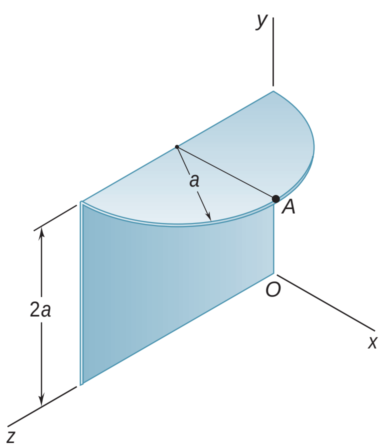


Fig. P9.168

9.169 Determine the mass moment of inertia of the machine component of Probs. 9.136 and 9.155 with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-4\mathbf{i} + 8\mathbf{j} + \mathbf{k})/9$.

9.170 through *9.172* For the wire figure of the problem indicated, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

9.170 Prob. 9.148
9.171 Prob. 9.147
9.172 Prob. 9.146

9.173 For the homogeneous circular cylinder shown, of radius a and length L, determine the value of the ratio a/L for which the ellipsoid of inertia of the cylinder is a sphere when computed (*a*) at the centroid of the cylinder, (*b*) at point A.

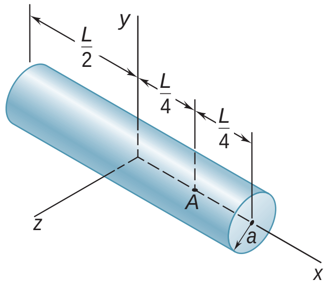


Fig. P9.173

9.174 For the rectangular prism shown, determine the values of the ratios b/a and c/a so that the ellipsoid of inertia of the prism is a sphere when computed (*a*) at point A, (*b*) at point B.

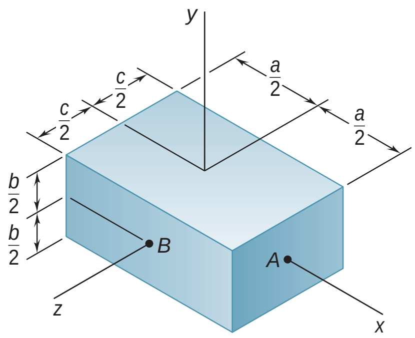


Fig. P9.174

9.175 For the right circular cone of Sample Prob. 9.11, determine the value of the ratio a/h for which the ellipsoid of inertia of the cone is a sphere when computed (*a*) at the apex of the cone, (*b*) at the center of the base of the cone.

9.176 Given an arbitrary body and three rectangular axes x, y, and z, prove that the mass moment of inertia of the body with respect to any one of the three axes cannot be larger than the sum of the mass moments of inertia of the body with respect to the other two axes. That is, prove that the inequality $I_x \leq I_y + I_z$ and the two similar inequalities are satisfied. Further, prove that $I_y \geq \frac{1}{2} I_x$ if the body is a homogeneous solid of revolution, where x is the axis of revolution and y is a transverse axis.

9.177 Consider a cube of mass m and side a. (*a*) Show that the ellipsoid of inertia at the center of the cube is a sphere, and use this property to determine the moment of inertia of the cube with respect to one of its diagonals. (*b*) Show that the ellipsoid of inertia at one of the corners of the cube is an ellipsoid of revolution, and determine the principal moments of inertia of the cube at that point.

9.178 Given a homogeneous body of mass m and of arbitrary shape and three rectangular axes x, y, and z with origin at O, prove that the sum $I_x + I_y + I_z$ of the mass moments of inertia of the body cannot be smaller than the similar sum computed for a sphere of the same mass and the same material centered at O. Further, using the result of Prob. 9.176, prove that if the body is a solid of revolution, where x is the axis of revolution, its mass moment of inertia I_y about a transverse axis y cannot be smaller than $3ma^2/10$, where a is the radius of the sphere of the same mass and the same material.

***9.179** The homogeneous circular cylinder shown has a mass m, and the diameter OB of its top surface forms 45° angles with the x and z axes. (*a*) Determine the principal mass moments of inertia of the cylinder at the origin O. (*b*) Compute the angles that the principal axes of inertia at O form with the coordinate axes. (*c*) Sketch the cylinder, and show the orientation of the principal axes of inertia relative to the x, y, and z axes.

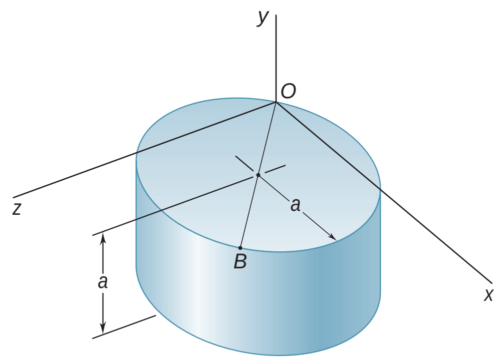


Fig. P9.179

9.180 through *9.184* For the component described in the problem indicated, determine (*a*) the principal mass moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the x, y, and z axes.

- ***9.180** Prob. 9.165
- ****9.181*** Probs. 9.145 and 9.149
- ***9.182** Prob. 9.167
- ***9.183** Prob. 9.168
- ****9.184*** Probs. 9.148 and 9.170

REVIEW AND SUMMARY

In the first half of this chapter, we discussed the determination of the resultant $\mathbf{R}$ of forces $\Delta\mathbf{F}$ distributed over a plane area A when the magnitudes of these forces are proportional to both the areas ΔA of the elements on which they act and the distances y from these elements to a given x axis; we thus had $\Delta F = ky\ \Delta A$. We found that the magnitude of the resultant $\mathbf{R}$ is proportional to the first moment $Q_x = \int y\, dA$ of the area A, while the moment of $\mathbf{R}$ about the x axis is proportional to the *second moment*, or *moment of inertia*, $I_x = \int y^2\, dA$ of A with respect to the same axis [Sec. 9.2].

Rectangular moments of inertia

The *rectangular moments of inertia I_x and I_y of an area* [Sec. 9.3] were obtained by evaluating the integrals

$$I_x = \int y^2\, dA \qquad I_y = \int x^2\, dA \tag{9.1}$$

These computations can be reduced to single integrations by choosing dA to be a thin strip parallel to one of the coordinate axes. We also recall that it is possible to compute I_x and I_y from the same elemental strip (Fig. 9.35) using the formula for the moment of inertia of a rectangular area [Sample Prob. 9.3].

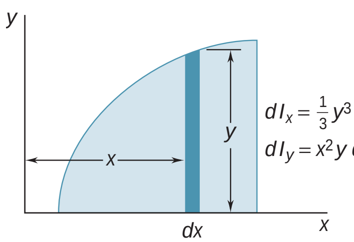


Fig. 9.35

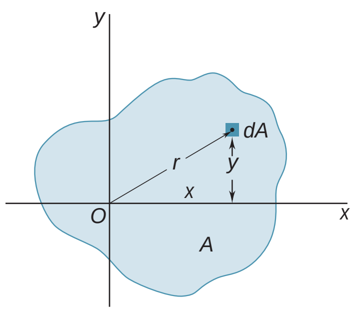


Fig. 9.36

Polar moment of inertia

The *polar moment of inertia of an area* A with respect to the pole O [Sec. 9.4] was defined as

$$J_O = \int r^2\, dA \tag{9.3}$$

where r is the distance from O to the element of area dA (Fig. 9.36). Observing that $r^2 = x^2 + y^2$, we established the relation

$$J_O = I_x + I_y \tag{9.4}$$

Radius of gyration

The *radius of gyration of an area A* with respect to the x axis [Sec. 9.5] was defined as the distance k_x, where $I_x = K_x^2 A$. With similar definitions for the radii of gyration of A with respect to the y axis and with respect to O, we had

$$k_x = \sqrt{\frac{I_x}{A}} \qquad k_y = \sqrt{\frac{I_y}{A}} \qquad k_O = \sqrt{\frac{J_O}{A}} \tag{9.5–9.7}$$

Parallel-axis theorem

The parallel-axis theorem was presented in Sec. 9.6. It states that the moment of inertia I of an area with respect to any given axis AA' (Fig. 9.37) is equal to the moment of inertia $\bar{I}$ of the area with respect to the centroidal axis BB' that is parallel to AA' *plus* the product of the area A and the square of the distance d between the two axes:

$$I = \bar{I} + Ad^2 \tag{9.9}$$

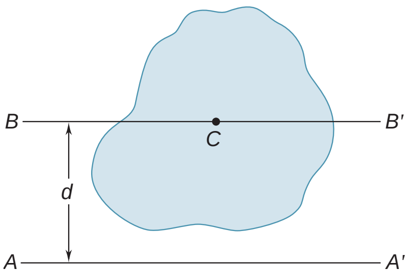


Fig. 9.37

This formula can also be used to determine the moment of inertia $\bar{I}$ of an area with respect to a centroidal axis BB' when its moment of inertia I with respect to a parallel axis AA' is known. In this case, however, the product Ad^2 should be *subtracted* from the known moment of inertia I.

A similar relation holds between the polar moment of inertia J_O of an area about a point O and the polar moment of inertia $\bar{J}_C$ of the same area about its centroid C. Letting d be the distance between O and C, we have

$$J_O = \bar{J}_C + Ad^2 \tag{9.11}$$

Composite areas

The parallel-axis theorem can be used very effectively to compute the *moment of inertia of a composite area* with respect to a given axis [Sec. 9.7]. Considering each component area separately, we first compute the moment of inertia of each area with respect to its centroidal axis, using the data provided in Figs. 9.12 and 9.13 whenever possible. The parallel-axis theorem is then applied to determine the moment of inertia of each component area with respect to the desired axis, and the various values obtained are added [Sample Probs. 9.4 and 9.5].

Product of inertia

Sections 9.8 through 9.10 were devoted to the transformation of the moments of inertia of an area *under a rotation of the coordinate axes*. First, we defined the *product of inertia of an area A* as

$$I_{xy} = \int xy\, dA \tag{9.12}$$

and showed that $I_{xy} = 0$ if the area A is symmetrical with respect to either or both of the coordinate axes. We also derived the *parallel-axis theorem for products of inertia*. We had

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A \tag{9.13}$$

where $\bar{I}_{x'y'}$ is the product of inertia of the area with respect to the centroidal axes x' and y' which are parallel to the x and y axis and $\bar{x}$ and $\bar{y}$ are the coordinates of the centroid of the area [Sec. 9.8].

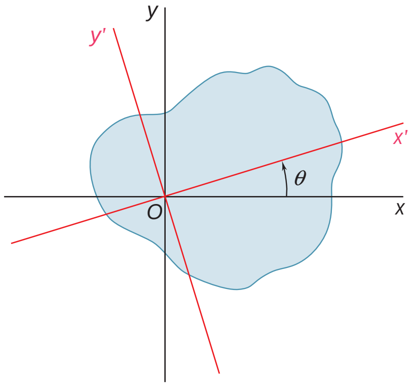


Fig. 9.38

Rotation of axes

In Sec. 9.9 we determined the moments and product of inertia $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ of an area with respect to x' and y' axes obtained by rotating the original x and y coordinate axes through an angle θ counterclockwise (Fig. 9.38). We expressed $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ in terms of the moments and product of inertia I_x, I_y, and I_{xy} computed with respect to the original x and y axes. We had

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \qquad (9.18)$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \qquad (9.19)$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \qquad (9.20)$$

Principal axes

The *principal axes of the area about O* were defined as the two axes perpendicular to each other, with respect to which the moments of inertia of the area are maximum and minimum. The corresponding values of θ, denoted by θ_m, were obtained from the formula

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} \qquad (9.25)$$

Principal moments of inertia

The corresponding maximum and minimum values of I are called the *principal moments of inertia* of the area about O; we had

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \qquad (9.27)$$

We also noted that the corresponding value of the product of inertia is zero.

Mohr's circle

The transformation of the moments and product of inertia of an area under a rotation of axes can be represented graphically by drawing *Mohr's circle* [Sec. 9.10]. Given the moments and product of inertia I_x, I_y, and I_{xy} of the area with respect to the x and y coordinate axes, we

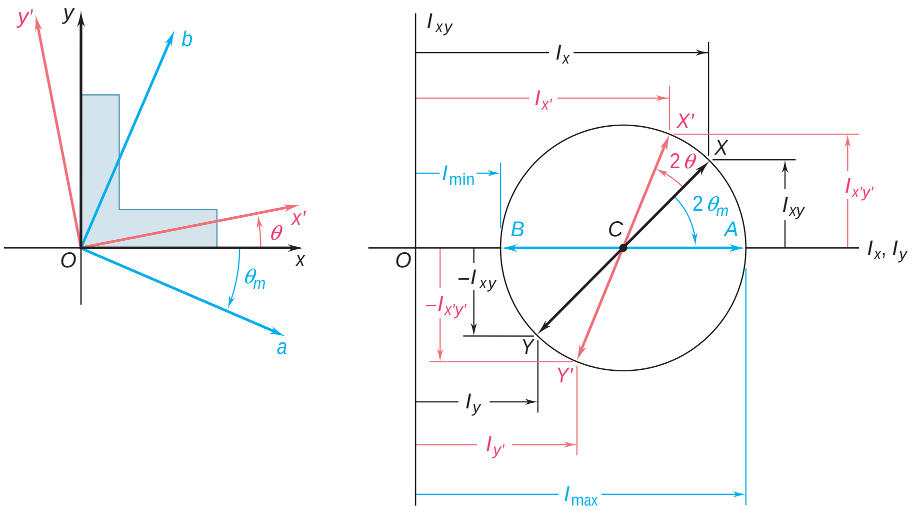


Fig. 9.39

plot points X (I_x, I_{xy}) and Y $(I_y, -I_{xy})$ and draw the line joining these two points (Fig. 9.39). This line is a diameter of Mohr's circle and thus defines this circle. As the coordinate axes are rotated through θ, the diameter rotates through *twice that angle,* and the coordinates of X' and Y' yield the new values $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ of the moments and product of inertia of the area. Also, the angle θ_m and the coordinates of points A and B define the principal axes a and b and the principal moments of inertia of the area [Sample Prob. 9.8].

Moments of inertia of masses

The second half of the chapter was devoted to the determination of *moments of inertia of masses,* which are encountered in dynamics in problems involving the rotation of a rigid body about an axis. The mass moment of inertia of a body with respect to an axis AA' (Fig. 9.40) was defined as

$$I = \int r^2\, dm \tag{9.28}$$

where r is the distance from AA' to the element of mass [Sec. 9.11]. The *radius of gyration* of the body was defined as

$$k = \sqrt{\frac{I}{m}} \tag{9.29}$$

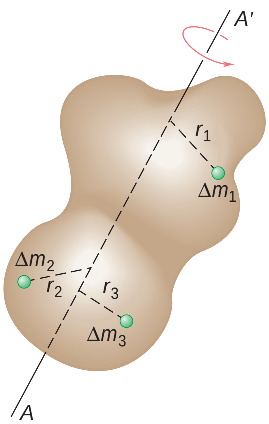


Fig. 9.40

The moments of inertia of a body with respect to the coordinates axes were expressed as

$$\begin{aligned} I_x &= \int (y^2 + z^2)\, dm \\ I_y &= \int (z^2 + x^2)\, dm \\ I_z &= \int (x^2 + y^2)\, dm \end{aligned} \tag{9.30}$$

We saw that the *parallel-axis theorem* also applies to mass moments of inertia [Sec. 9.12]. Thus, the moment of inertia I of a body with respect to an arbitrary axis AA' (Fig. 9.41) can be expressed as

Parallel-axis theorem

$$I = \bar{I} + md^2 \tag{9.33}$$

where $\bar{I}$ is the moment of inertia of the body with respect to the centroidal axis BB' which is parallel to the axis AA', m is the mass of the body, and d is the distance between the two axes.

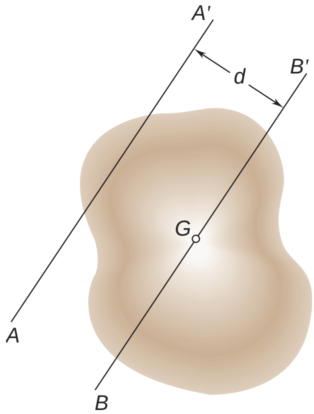


Fig. 9.41

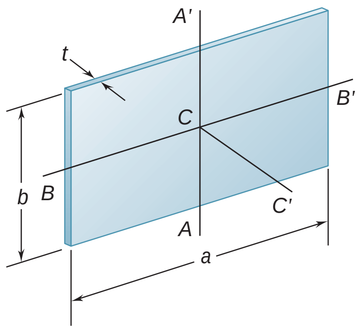


Fig. 9.42

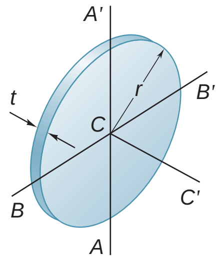


Fig. 9.43

Moments of inertia of thin plates

The moments of inertia of *thin plates* can be readily obtained from the moments of inertia of their areas [Sec. 9.13]. We found that for a *rectangular plate* the moments of inertia with respect to the axes shown (Fig. 9.42) are

$$I_{AA'} = \tfrac{1}{12}ma^2 \qquad I_{BB'} = \tfrac{1}{12}mb^2 \tag{9.39}$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{12}m(a^2 + b^2) \tag{9.40}$$

while for a *circular plate* (Fig. 9.43) they are

$$I_{AA'} = I_{BB'} = \tfrac{1}{4}mr^2 \tag{9.41}$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{2}mr^2 \tag{9.42}$$

Composite bodies

When a body possesses *two planes of symmetry,* it is usually possible to use a single integration to determine its moment of inertia with respect to a given axis by selecting the element of mass dm to be a thin plate [Sample Probs. 9.10 and 9.11]. On the other hand, when a body consists of *several common geometric shapes,* its moment of inertia with respect to a given axis can be obtained by using the formulas given in Fig. 9.28 together with the parallel-axis theorem [Sample Probs. 9.12 and 9.13].

Moment of inertia with respect to an arbitrary axis

In the last portion of the chapter, we learned to determine the moment of inertia of a body *with respect to an arbitrary axis OL* which is drawn through the origin O [Sec. 9.16]. Denoting by I_x, I_y,

λ_z the components of the unit vector $\boldsymbol{\lambda}$ along OL (Fig. 9.44) and introducing the *products of inertia*

$$I_{xy} = \int xy\,dm \qquad I_{yz} = \int yz\,dm \qquad I_{zx} = \int zx\,dm \tag{9.45}$$

we found that the moment of inertia of the body with respect to OL could be expressed as

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x \tag{9.46}$$

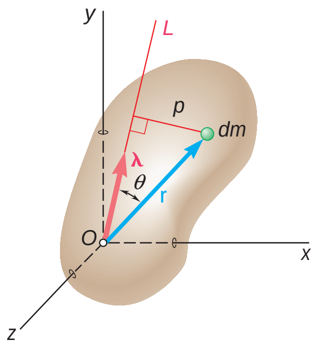


Fig. 9.44

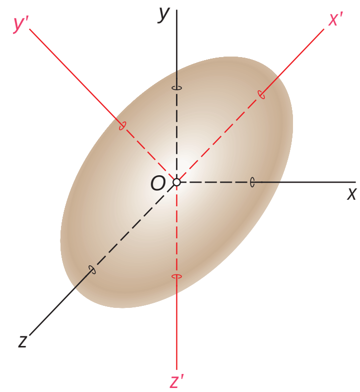


Fig. 9.45

Ellipsoid of inertia

By plotting a point Q along each axis OL at a distance $OQ = 1/\sqrt{I_{OL}}$ from O [Sec. 9.17], we obtained the surface of an ellipsoid, known as the *ellipsoid of inertia* of the body at point O. The principal axes x', y', z' of this ellipsoid (Fig. 9.45) are the *principal axes of inertia* of the body; that is, the products of inertia $I_{x'y'}$, $I_{y'z'}$, $I_{z'x'}$ of the body with respect to these axes are all zero. There are many situations when the principal axes of inertia of a body can be deduced from properties of symmetry of the body. Choosing these axes to be the coordinate axes, we can then express I_{OL} as

Principal axes of inertia
Principal moments of inertia

$$I_{OL} = I_{x'}\lambda_{x'}^2 + I_{y'}\lambda_{y'}^2 + I_{z'}\lambda_{z'}^2 \tag{9.50}$$

where $I_{x'}$, $I_{y'}$, $I_{z'}$ are the *principal moments of inertia* of the body at O.

When the principal axes of inertia cannot be obtained by observation [Sec. 9.17], it is necessary to solve the cubic equation

$$\begin{aligned} K^3 - (I_x + I_y + I_z)K^2 + (I_xI_y + I_yI_z + I_zI_x - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)K \\ - (I_xI_yI_z - I_xI_{yz}^2 - I_yI_{zx}^2 - I_zI_{xy}^2 - 2I_{xy}I_{yz}I_{zx}) = 0 \end{aligned} \tag{9.56}$$

We found [Sec. 9.18] that the roots K_1, K_2, and K_3 of this equation are the principal moments of inertia of the given body. The direction cosines $(\lambda_x)_1$, $(\lambda_y)_1$, and $(\lambda_z)_1$ of the principal axis corresponding to the principal moment of inertia K_1 are then determined by substituting K_1 into Eqs. (9.54) and solving two of these equations and Eq. (9.57) simultaneously. The same procedure is then repeated using K_2 and K_3 to determine the direction cosines of the other two principal axes [Sample Prob. 9.15].

REVIEW PROBLEMS

9.185 Determine by direct integration the moments of inertia of the shaded area with respect to the x and y axes.

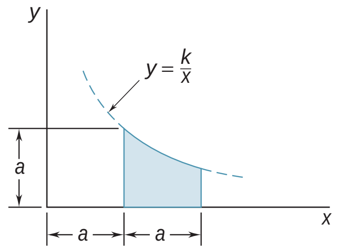


Fig. P9.185

9.186 Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

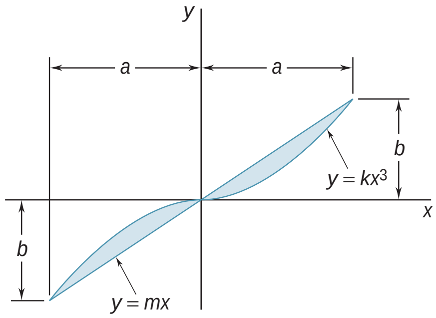


Fig. P9.186

9.187 Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

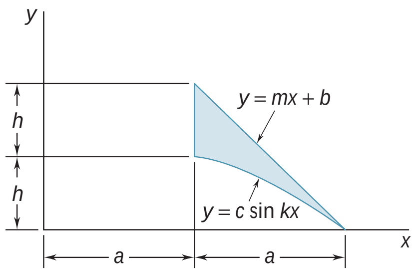


Fig. *P9.187*

9.188 Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

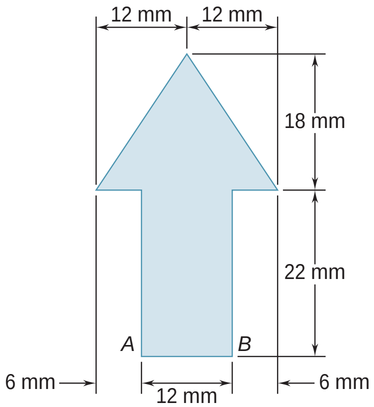


Fig. P9.188

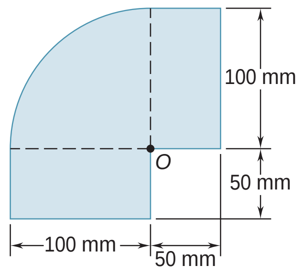


Fig. P9.189

9.189 Determine the polar moment of inertia of the area shown with respect to (*a*) point *O*, (*b*) the centroid of the area.

9.190 Two L5 × 3 × $\frac{1}{2}$-in. angles are welded to a $\frac{1}{2}$-in. steel plate. Determine the distance b and the centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the combined section, knowing that $\bar{I}_y = 4\bar{I}_x$.

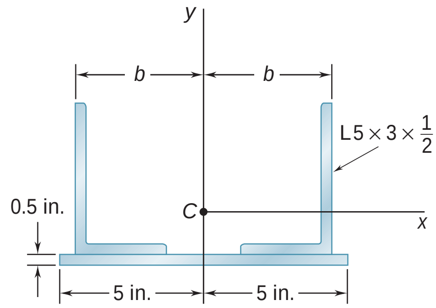


Fig. *P9.190*

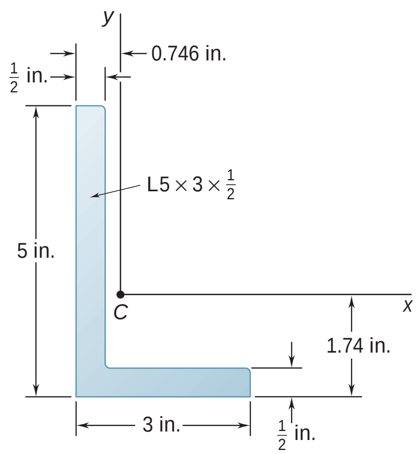


Fig. P9.191 and *P9.192*

9.191 Using the parallel-axis theorem, determine the product of inertia of the L5 × 3 × $\frac{1}{2}$-in. angle cross section shown with respect to the centroidal x and y axes.

9.192 For the L5 × 3 × $\frac{1}{2}$-in. angle cross section shown, use Mohr's circle to determine (*a*) the moments of inertia and the product of inertia with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise, (*b*) the orientation of the principal axes through the centroid and the corresponding values of the moments of inertia.

9.193 A thin plate of mass m has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (*a*) the x axis, (*b*) the y axis.

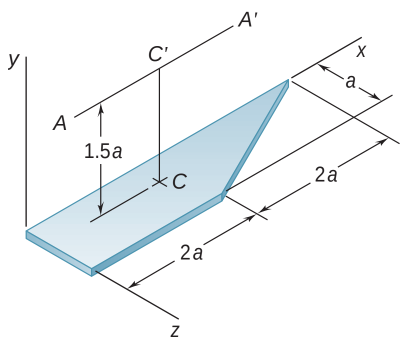


Fig. P9.193 and *P9.194*

9.194 A thin plate of mass m has the trapezoidal shape shown. Determine the mass moment of inertia of the plate with respect to (*a*) the centroidal axis CC' that is perpendicular to the plate, (*b*) the axis AA' that is parallel to the x axis and is located at a distance $1.5a$ from the plate.

9.195 A 2-mm-thick piece of sheet steel is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to each of the coordinate axes.

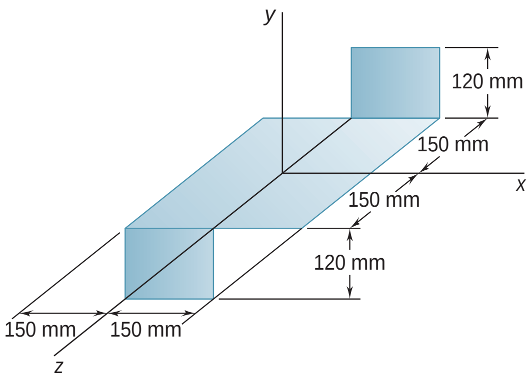


Fig. P9.195

9.196 Determine the mass moment of inertia and the radius of gyration of the steel machine element shown with respect to the x axis. (The density of steel is 7850 kg/m^3.)

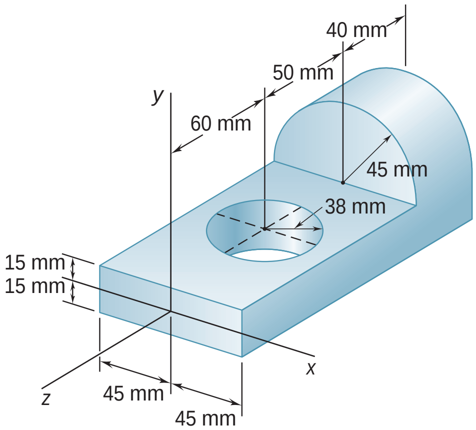


Fig. P9.196

COMPUTER PROBLEMS

9.C1 Write a computer program that, for an area with known moments and product of inertia I_x, I_y, and I_{xy}, can be used to calculate the moments and product of inertia $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ of the area with respect to axes x' and y' obtained by rotating the original axes counterclockwise through an angle u. Use this program to compute $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ for the section of Sample Prob. 9.7 for values of u from 0 to 90° using 5° increments.

9.C2 Write a computer program that, for an area with known moments and product of inertia I_x, I_y, and I_{xy}, can be used to calculate the orientation of the principal axes of the area and the corresponding values of the principal moments of inertia. Use this program to solve (*a*) Prob. 9.89, (*b*) Sample Prob. 9.7.

9.C3 Many cross sections can be approximated by a series of rectangles as shown. Write a computer program that can be used to calculate the moments of inertia and the radii of gyration of cross sections of this type with respect to horizontal and vertical centroidal axes. Apply this program to the cross sections shown in (*a*) Figs. P9.31 and P9.33, (*b*) Figs. P9.32 and P9.34, (*c*) Fig. P9.43, (*d*) Fig. P9.44.

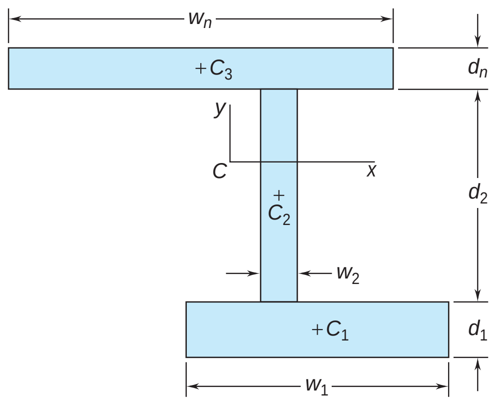


Fig. P9.C3 and P9.C4

9.C4 Many cross sections can be approximated by a series of rectangles as shown. Write a computer program that can be used to calculate the products of inertia of cross sections of this type with respect to horizontal and vertical centroidal axes. Use this program to solve (*a*) Prob. 9.71, (*b*) Prob. 9.75, (*c*) Prob. 9.77.

9.C5 The area shown is revolved about the x axis to form a homogeneous solid of mass m. Approximate the area using a series of 400 rectangles of the form $bcc'b'$, each of width Δl, and then write a computer program that can be used to determine the mass moment of inertia of the solid with respect to the x axis. Use this program to solve part a of (*a*) Sample Prob. 9.11, (*b*) Prob. 9.121, assuming that in these problems $m = 2$ kg, $a = 100$ mm, and $h = 400$ mm.

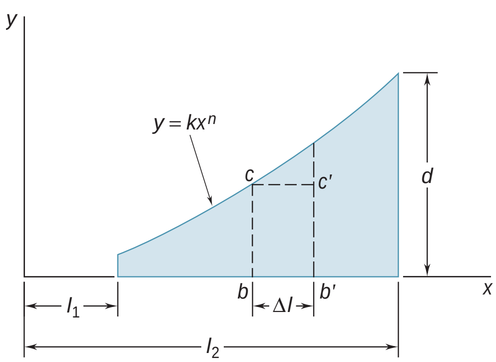


Fig. P9.C5

9.C6 A homogeneous wire with a weight per unit length of 0.04 lb/ft is used to form the figure shown. Approximate the figure using 10 straight line segments, and then write a computer program that can be used to determine the mass moment of inertia I_x of the wire with respect to the x axis. Use this program to determine I_x when (*a*) $a = 1$ in., $L = 11$ in., $h = 4$ in., (*b*) $a = 2$ in., $L = 17$ in., $h = 10$ in., (*c*) $a = 5$ in., $L = 25$ in., $h = 6$ in.

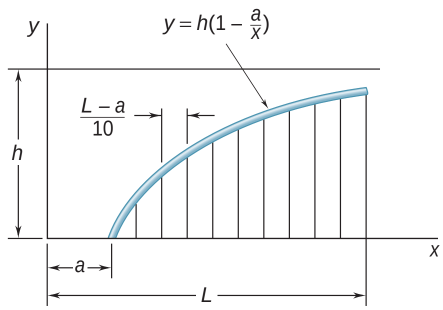


Fig. P9.C6

***9.C7** Write a computer program that, for a body with known mass moments and products of inertia I_x, I_y, I_z, I_{xy}, I_{yz}, and I_{zx}, can be used to calculate the principal mass moments of inertia K_1, K_2, and K_3 of the body at the origin. Use this program to solve part *a* of (*a*) Prob. 9.180, (*b*) Prob. 9.181, (*c*) Prob. 9.184.

***9.C8** Extend the computer program of Prob. 9.C7 to include the computation of the angles that the principal axes of inertia at the origin form with the coordinate axes. Use this program to solve (*a*) Prob. 9.180, (*b*) Prob. 9.181, (*c*) Prob. 9.184.

The method of virtual work is particularly effective when a simple relation can be found among the displacements of the points of application of the various forces involved. This is the case for the scissor lift platform being used by workers to gain access to a highway bridge under construction.

CHAPTER 10

Method of Virtual Work

Chapter 10 Method of Virtual Work

10.1 Introduction
10.2 Work of a Force
10.3 Principle of Virtual Work
10.4 Applications of the Principle of Virtual Work
10.5 Real Machines. Mechanical Efficiency
10.6 Work of a Force During a Finite Displacement
10.7 Potential Energy
10.8 Potential Energy and Equilibrium
10.9 Stability of Equilibrium

*10.1 INTRODUCTION

In the preceding chapters, problems involving the equilibrium of rigid bodies were solved by expressing that the external forces acting on the bodies were balanced. The equations of equilibrium $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_A = 0$ were written and solved for the desired unknowns. A different method, which will prove more effective for solving certain types of equilibrium problems, will now be considered. This method is based on the *principle of virtual work* and was first formally used by the Swiss mathematician Jean Bernoulli in the eighteenth century.

As you will see in Sec. 10.3, the principle of virtual work states that if a particle or rigid body, or, more generally, a system of connected rigid bodies, which is in equilibrium under various external forces, is given an arbitrary displacement from that position of equilibrium, the total work done by the external forces during the displacement is zero. This principle is particularly effective when applied to the solution of problems involving the equilibrium of machines or mechanisms consisting of several connected members.

In the second part of the chapter, the method of virtual work will be applied in an alternative form based on the concept of *potential energy.* It will be shown in Sec. 10.8 that if a particle, rigid body, or system of rigid bodies is in equilibrium, then the derivative of its potential energy with respect to a variable defining its position must be zero.

In this chapter, you will also learn to evaluate the mechanical efficiency of a machine (Sec. 10.5) and to determine whether a given position of equilibrium is stable, unstable, or neutral (Sec. 10.9).

*10.2 WORK OF A FORCE

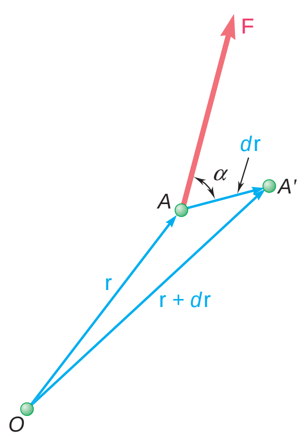


Fig. 10.1

Let us first define the terms *displacement* and *work* as they are used in mechanics. Consider a particle which moves from a point A to a neighboring point A' (Fig. 10.1). If $\mathbf{r}$ denotes the position vector corresponding to point A, the small vector joining A and A' may be denoted by the differential $d\mathbf{r}$; the vector $d\mathbf{r}$ is called the *displacement* of the particle. Now let us assume that a force $\mathbf{F}$ is acting on the particle. The *work of the force* $\mathbf{F}$ *corresponding to the displacement* $d\mathbf{r}$ is defined as the quantity

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{10.1}$$

obtained by forming the scalar product of the force $\mathbf{F}$ and the displacement $d\mathbf{r}$. Denoting respectively by F and ds the magnitudes of the force and of the displacement, and by α the angle formed by $\mathbf{F}$ and $d\mathbf{r}$, and recalling the definition of the scalar product of two vectors (Sec. 3.9), we write

$$dU = F\, ds \cos \alpha \tag{10.1$'$}$$

Being a *scalar quantity,* work has a magnitude and a sign, but no direction. We also note that work should be expressed in units obtained by multiplying units of length by units of force. Thus, if U.S. customary units are used, work should be expressed in ft · lb or in · lb. If SI units are used, work should be expressed in N · m. The unit of work N · m is called a *joule* (J).†

It follows from (10.1′) that the work dU is positive if the angle α is acute and negative if α is obtuse. Three particular cases are of special interest. If the force **F** has the same direction as $d\mathbf{r}$, the work dU reduces to $F\,ds$. If **F** has a direction opposite to that of $d\mathbf{r}$, the work is $dU = -F\,ds$. Finally, if **F** is perpendicular to $d\mathbf{r}$, the work dU is zero.

The work dU of a force **F** during a displacement $d\mathbf{r}$ can also be considered as the product of F and the component $ds \cos \alpha$ of the displacement $d\mathbf{r}$ along **F** (Fig. 10.2*a*). This view is particularly

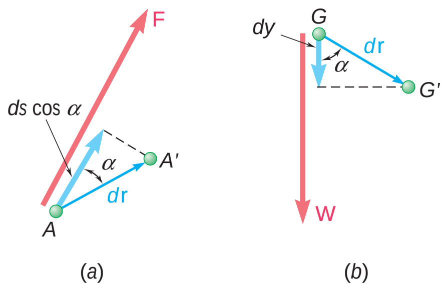


Fig. 10.2

Photo 10.1 The forces exerted by the hydraulic cylinders to position the bucket lift shown can be effectively determined using the method of virtual work since a simple relation exists among the displacements of the points of application of the forces acting on the members of the lift.

useful in the computation of the work done by the weight **W** of a body (Fig. 10.2*b*). The work of **W** is equal to the product of W and the vertical displacement dy of the center of gravity G of the body. If the displacement is downward, the work is positive; if it is upward, the work is negative.

A number of forces frequently encountered in statics *do no work:* forces applied to fixed points ($ds = 0$) or acting in a direction perpendicular to the displacement ($\cos \alpha = 0$). Among these forces are the reaction at a frictionless pin when the body supported rotates about the pin; the reaction at a frictionless surface when the body in contact moves along the surface; the reaction at a roller moving along its track; the weight of a body when its center of gravity moves horizontally; and the friction force acting on a wheel rolling without slipping (since at any instant the point of contact does not move). Examples of forces which *do work* are the weight of a body (except in the case considered above), the friction force acting on a body sliding on a rough surface, and most forces applied on a moving body.

†The joule is the SI unit of *energy,* whether in mechanical form (work, potential energy, kinetic energy) or in chemical, electrical, or thermal form. We should note that even though N · m = J, the moment of a force must be expressed in N · m, and not in joules, since the moment of a force is not a form of energy.

In certain cases, the sum of the work done by several forces is zero. Consider, for example, two rigid bodies AC and BC connected at C by a *frictionless pin* (Fig. 10.3*a*). Among the forces acting on AC is the force $\mathbf{F}$ exerted at C by BC. In general, the work of this

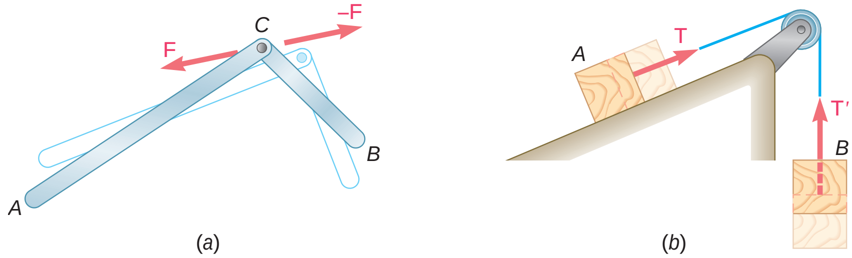


Fig. 10.3

force will not be zero, but it will be equal in magnitude and opposite in sign to the work of the force $-\mathbf{F}$ exerted by AC on BC, since these forces are equal and opposite and are applied to the same particle. Thus, when the total work done by all the forces acting on AB and BC is considered, the work of the two internal forces at C cancels out. A similar result is obtained if we consider a system consisting of two blocks connected by an *inextensible cord AB* (Fig. 10.3*b*). The work of the tension force $\mathbf{T}$ at A is equal in magnitude to the work of the tension force $\mathbf{T}'$ at B, since these forces have the same magnitude and the points A and B move through the same distance; but in one case the work is positive, and in the other it is negative. Thus, the work of the internal forces again cancels out.

It can be shown that the total work of the internal forces holding together the particles of a rigid body is zero. Consider two particles A and B of a rigid body and the two equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ they exert on each other (Fig. 10.4). While, in general,

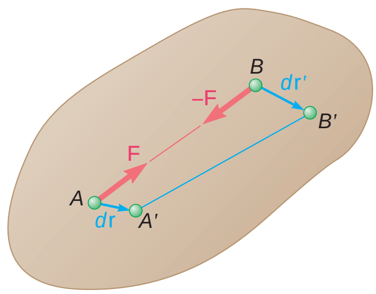


Fig. 10.4

small displacements $d\mathbf{r}$ and $d\mathbf{r}'$ of the two particles are different, the components of these displacements along AB must be equal; otherwise, the particles would not remain at the same distance from each other, and the body would not be rigid. Therefore, the work of $\mathbf{F}$ is equal in magnitude and opposite in sign to the work of $-\mathbf{F}$, and their sum is zero.

In computing the work of the external forces acting on a rigid body, it is often convenient to determine the work of a couple without considering separately the work of each of the two forces forming the couple. Consider the two forces $\mathbf{F}$ and $-\mathbf{F}$ forming a couple of moment **M** and acting on a rigid body (Fig. 10.5). Any small displacement of the rigid body bringing A and B, respectively, into A' and B'' can be divided into two parts, one in which points A and B undergo equal displacements $d\mathbf{r}_1$, the other in which A' remains fixed while B' moves into B'' through a displacement $d\mathbf{r}_2$ of magnitude $ds_2 = r\,d$u. In the first part of the motion, the work of **F** is equal in magnitude and opposite in sign to the work of $-\mathbf{F}$, and their sum is zero. In the second part of the motion, only force **F** works, and its work is $dU = F\,ds_2 = Fr\,d$u. But the product Fr is equal to the magnitude M of the moment of the couple. Thus, the work of a couple of moment **M** acting on a rigid body is

$$dU = M\,d\text{u} \tag{10.2}$$

where du is the small angle expressed in radians through which the body rotates. We again note that work should be expressed in units obtained by multiplying units of force by units of length.

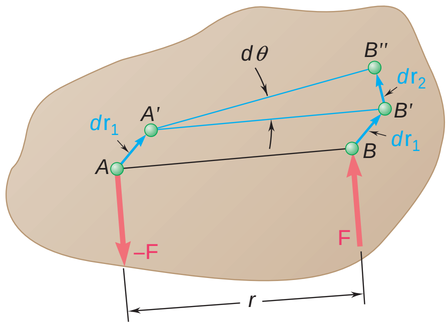


Fig. 10.5

*10.3 PRINCIPLE OF VIRTUAL WORK

Consider a particle acted upon by several forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ (Fig. 10.6). We can imagine that the particle undergoes a small displacement from A to A'. This displacement is possible, but it will not necessarily take place. The forces may be balanced and the particle at rest, or the particle may move under the action of the given forces in a direction different from that of AA'. Since the displacement considered does not actually occur, it is called a *virtual displacement* and is denoted by dr. The symbol dr represents a differential of the first order; it is used to distinguish the virtual displacement from the displacement $d\mathbf{r}$ which would take place under actual motion. As you will see, virtual displacements can be used to determine whether the conditions of equilibrium of a particle are satisfied.

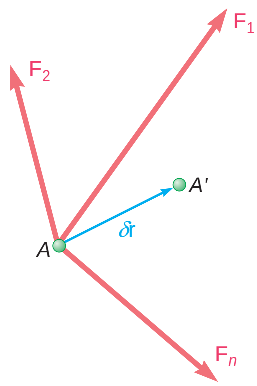


Fig. 10.6

The work of each of the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ during the virtual displacement dr is called *virtual work*. The virtual work of all the forces acting on the particle of Fig. 10.6 is

$$\begin{aligned} dU &= \mathbf{F}_1 \cdot d\mathbf{r} + \mathbf{F}_2 \cdot d\mathbf{r} + \cdots + \mathbf{F}_n \cdot d\mathbf{r} \\ &= (\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n) \cdot d\mathbf{r} \end{aligned}$$

or

$$dU = \mathbf{R} \cdot d\mathbf{r} \tag{10.3}$$

where **R** is the resultant of the given forces. Thus, the total virtual work of the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ is equal to the virtual work of their resultant **R**.

The principle of virtual work for a particle states that *if a particle is in equilibrium, the total virtual work of the forces acting on the particle is zero for any virtual displacement of the particle.* This condition is necessary: if the particle is in equilibrium, the resultant **R** of the forces is zero, and it follows from (10.3) that the total virtual work dU is zero. The condition is also sufficient: if the total virtual work dU is zero for any virtual displacement, the scalar product $\mathbf{R} \cdot d\mathbf{r}$ is zero for any dr, and the resultant **R** must be zero.

In the case of a rigid body, the principle of virtual work states that *if a rigid body is in equilibrium, the total virtual work of the external forces acting on the rigid body is zero for any virtual displacement of the body.* The condition is necessary: if the body is in equilibrium, all the particles forming the body are in equilibrium and the total virtual work of the forces acting on all the particles must be zero; but we have seen in the preceding section that the total work of the internal forces is zero; the total work of the external forces must therefore also be zero. The condition can also be proved to be sufficient.

The principle of virtual work can be extended to the case of a *system of connected rigid bodies.* If the system remains connected during the virtual displacement, *only the work of the forces external to the system need be considered,* since the total work of the internal forces at the various connections is zero.

*10.4 APPLICATIONS OF THE PRINCIPLE OF VIRTUAL WORK

The principle of virtual work is particularly effective when applied to the solution of problems involving machines or mechanisms consisting of several connected rigid bodies. Consider, for instance, the toggle vise ACB of Fig. 10.7a, used to compress a wooden block. We

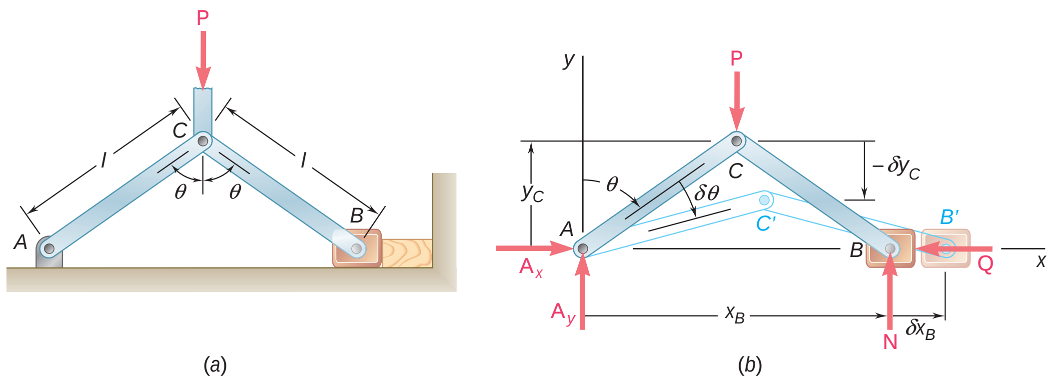


Fig. 10.7

wish to determine the force exerted by the vise on the block when a given force **P** is applied at C, assuming that there is no friction. Denoting by **Q** the reaction of the block on the vise, we draw the free-body diagram of the vise and consider the virtual displacement obtained by giving a positive increment du to the angle u (Fig. 10.7b). Choosing a system of coordinate axes with origin at A, we note that x_B increases while y_C decreases. This is indicated in the figure, where a positive increment dx_B and a negative increment −dy_C are shown. The reactions $\mathbf{A}_x$, $\mathbf{A}_y$, and **N** will do no work during the virtual displacement considered, and we need only compute the work of **P** and **Q**. Since **Q** and dx_B have opposite senses, the virtual work of **Q** is d$U_Q = -Q$ dx_B. Since **P** and the increment shown (−dy_C) have the same sense, the virtual work of **P** is d$U_P = +P($−d$y_C) = -P$ dy_C. The minus signs obtained could have been predicted by simply noting that the forces **Q** and **P** are directed opposite to the positive

x and y axes, respectively. Expressing the coordinates x_B and y_C in terms of the angle θ and differentiating, we obtain

$$\begin{aligned} x_B &= 2l \sin\theta & y_C &= l\cos\theta \\ \delta x_B &= 2l\cos\theta\,\delta\theta & \delta y_C &= -l\sin\theta\,\delta\theta \end{aligned} \qquad (10.4)$$

The total virtual work of the forces $\mathbf{Q}$ and $\mathbf{P}$ is thus

$$\begin{aligned} \delta U &= \delta U_Q + \delta U_P = -Q\,\delta x_B - P\,\delta y_C \\ &= -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta \end{aligned}$$

Making $\delta U = 0$, we obtain

$$2Ql\cos\theta\,\delta\theta = Pl\sin\theta\,\delta\theta \qquad (10.5)$$

$$Q = \tfrac{1}{2}P\tan\theta \qquad (10.6)$$

The superiority of the method of virtual work over the conventional equilibrium equations in the problem considered here is clear: by using the method of virtual work, we were able to eliminate all unknown reactions, while the equation $\Sigma M_A = 0$ would have eliminated only two of the unknown reactions. This property of the method of virtual work can be used in solving many problems involving machines and mechanisms. *If the virtual displacement considered is consistent with the constraints imposed by the supports and connections, all reactions and internal forces are eliminated and only the work of the loads, applied forces, and friction forces need be considered.*

The method of virtual work can also be used to solve problems involving completely constrained structures, although the virtual displacements considered will never actually take place. Consider, for example, the frame ACB shown in Fig. 10.8*a*. If point A is kept fixed, while B is given a horizontal virtual displacement (Fig. 10.8*b*), we need consider only the work of $\mathbf{P}$ and $\mathbf{B}_x$. We can thus determine

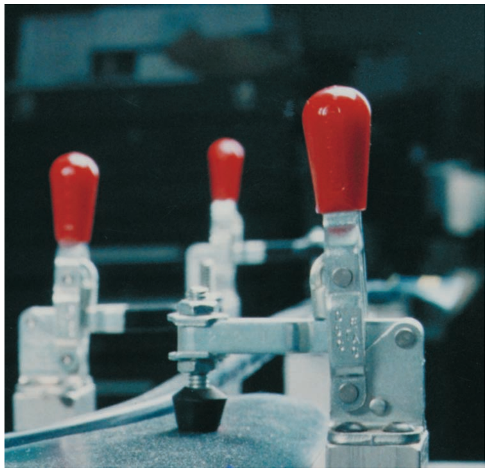


Photo 10.2 The clamping force of the toggle clamp shown can be expressed as a function of the force applied to the handle by first establishing the geometric relations among the members of the clamp and then applying the method of virtual work.

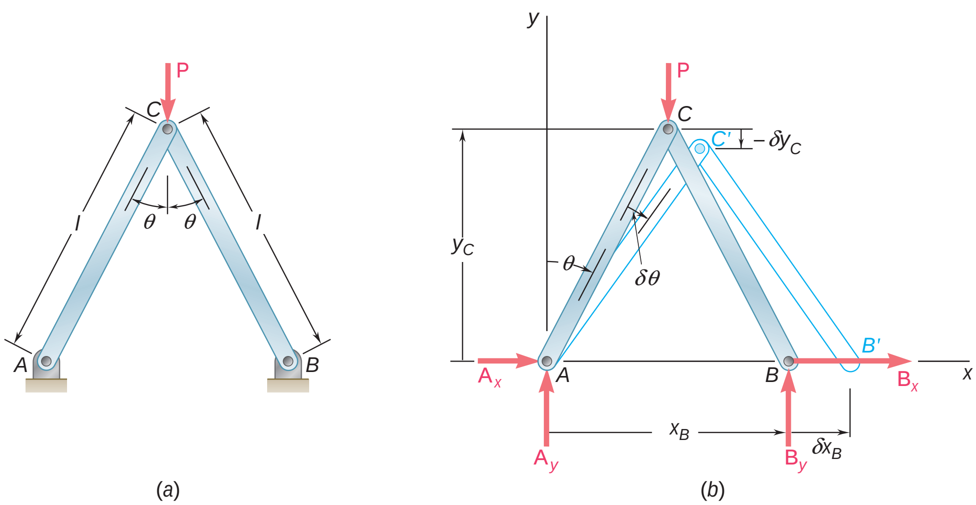


Fig. 10.8

the reaction component $\mathbf{B}_x$ in the same way as the force $\mathbf{Q}$ of the preceding example (Fig. 10.7*b*); we have

$$B_x = -\tfrac{1}{2}P \tan \theta$$

Keeping B fixed and giving to A a horizontal virtual displacement, we can similarly determine the reaction component $\mathbf{A}_x$. The components $\mathbf{A}_y$ and $\mathbf{B}_y$ can be determined by rotating the frame ACB as a rigid body about B and A, respectively.

The method of virtual work can also be used to determine the configuration of a system in equilibrium under given forces. For example, the value of the angle θ for which the linkage of Fig. 10.7 is in equilibrium under two given forces $\mathbf{P}$ and $\mathbf{Q}$ can be obtained by solving Eq. (10.6) for $\tan \theta$.

It should be noted, however, that the attractiveness of the method of virtual work depends to a large extent upon the existence of simple geometric relations between the various virtual displacements involved in the solution of a given problem. When no such simple relations exist, it is usually advisable to revert to the conventional method of Chap. 6.

*10.5 REAL MACHINES. MECHANICAL EFFICIENCY

In analyzing the toggle vise in the preceding section, we assumed that no friction forces were involved. Thus, the virtual work consisted only of the work of the applied force $\mathbf{P}$ and of the reaction $\mathbf{Q}$. But the work of the reaction $\mathbf{Q}$ is equal in magnitude and opposite in sign to the work of the force exerted by the vise on the block. Equation (10.5), therefore, expresses that the *output work* $2Ql \cos \theta\, \delta\theta$ is equal to the *input work* $Pl \sin \theta\, \delta\theta$. A machine in which input and output work are equal is said to be an "ideal" machine. In a "real" machine, friction forces will always do some work, and the output work will be smaller than the input work.

Consider, for example, the toggle vise of Fig. 10.7*a*, and assume now that a friction force $\mathbf{F}$ develops between the sliding block B and the horizontal plane (Fig. 10.9). Using the conventional methods of statics and summing moments about A, we find $N = P/2$. Denoting by μ the coefficient of friction between block B and the horizontal

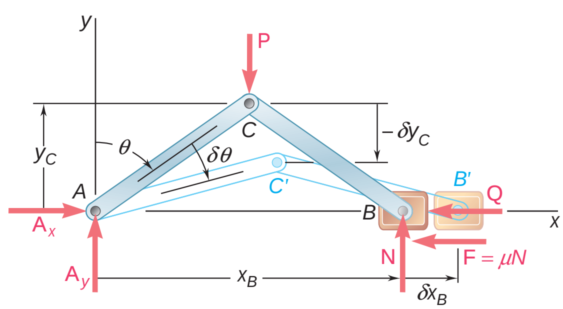


Fig. 10.9

plane, we have $F = \mathrm{m}N = \mathrm{m}P/2$. Recalling formulas (10.4), we find that the total virtual work of the forces **Q**, **P**, and **F** during the virtual displacement shown in Fig. 10.9 is

$$\begin{aligned} \mathrm{d}U &= -Q\,\mathrm{d}x_B - P\,\mathrm{d}y_C - F\,\mathrm{d}x_B \\ &= -2Ql \cos \mathrm{u}\,\mathrm{du} + Pl \sin \mathrm{u}\,\mathrm{du} - \mathrm{m}Pl \cos \mathrm{u}\,\mathrm{du} \end{aligned}$$

Making $\mathrm{d}U = 0$, we obtain

$$2Ql \cos \mathrm{u}\,\mathrm{du} = Pl \sin \mathrm{u}\,\mathrm{du} - \mathrm{m}Pl \cos \mathrm{u}\,\mathrm{du} \tag{10.7}$$

which expresses that the output work is equal to the input work minus the work of the friction force. Solving for Q, we have

$$Q = \tfrac{1}{2}P(\tan \mathrm{u} - \mathrm{m}) \tag{10.8}$$

We note that $Q = 0$ when $\tan \mathrm{u} = \mathrm{m}$, that is, when u is equal to the angle of friction f, and that $Q < 0$ when $\mathrm{u} < \mathrm{f}$. The toggle vise may thus be used only for values of u larger than the angle of friction.

The *mechanical efficiency* of a machine is defined as the ratio

$$\mathrm{h} = \frac{\text{output work}}{\text{input work}} \tag{10.9}$$

Clearly, the mechanical efficiency of an ideal machine is $\mathrm{h} = 1$, since input and output work are then equal, while the mechanical efficiency of a real machine will always be less than 1.

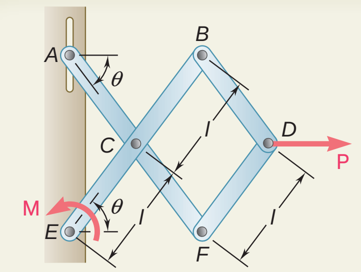
In the case of the toggle vise we have just analyzed, we write

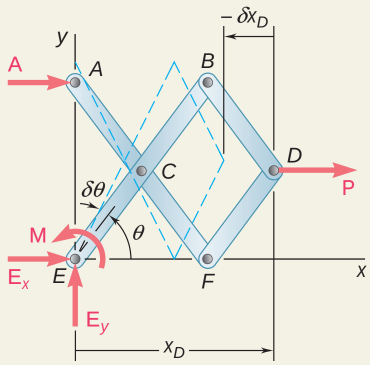
$$\mathrm{h} = \frac{\text{output work}}{\text{input work}} = \frac{2Ql \cos \mathrm{u}\,\mathrm{du}}{Pl \sin \mathrm{u}\,\mathrm{du}}$$

Substituting from (10.8) for Q, we obtain

$$\mathrm{h} = \frac{P(\tan \mathrm{u} - \mathrm{m})l \cos \mathrm{u}\,\mathrm{du}}{Pl \sin \mathrm{u}\,\mathrm{du}} = 1 - \mathrm{m} \cot \mathrm{u} \tag{10.10}$$

We check that in the absence of friction forces, we would have $\mathrm{m} = 0$ and $\mathrm{h} = 1$. In the general case, when m is different from zero, the efficiency h becomes zero for $\mathrm{m} \cot \mathrm{u} = 1$, that is, for $\tan \mathrm{u} = \mathrm{m}$, or $\mathrm{u} = \tan^{-1} \mathrm{m} = \mathrm{f}$. We note again that the toggle vise can be used only for values of u larger than the angle of friction f.





SAMPLE PROBLEM 10.1

Using the method of virtual work, determine the magnitude of the couple **M** required to maintain the equilibrium of the mechanism shown.

SOLUTION

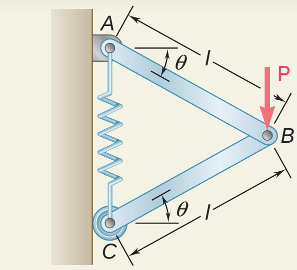
Choosing a coordinate system with origin at E, we write

$$x_D = 3l \cos u \qquad \text{d}x_D = -3l \sin u \, \text{du}$$

Principle of Virtual Work. Since the reactions $\mathbf{A}$, $\mathbf{E}_x$, and $\mathbf{E}_y$ will do no work during the virtual displacement, the total virtual work done by **M** and **P** must be zero. Noting that **P** acts in the positive x direction and **M** acts in the positive u direction, we write

$$\text{d}U = 0: \qquad +M \, \text{du} + P \, \text{d}x_D = 0$$
$$+M \, \text{du} + P(-3l \sin u \, \text{du}) = 0$$

$$M = 3Pl \sin u$$



SAMPLE PROBLEM 10.2

Determine the expressions for u and for the tension in the spring which correspond to the equilibrium position of the mechanism. The unstretched length of the spring is h, and the constant of the spring is k. Neglect the weight of the mechanism.

SOLUTION

With the coordinate system shown

$$y_B = l \sin u \qquad y_C = 2l \sin u$$
$$\text{d}y_B = l \cos u \, \text{du} \qquad \text{d}y_C = 2l \cos u \, \text{du}$$

The elongation of the spring is $\quad s = y_C - h = 2l \sin u - h$

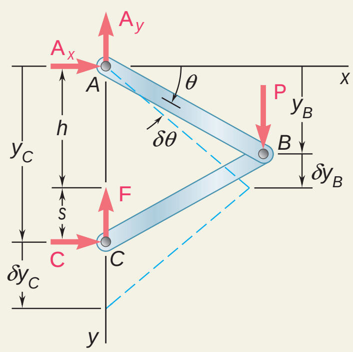
The magnitude of the force exerted at C by the spring is

$$F = ks = k(2l \sin u - h) \qquad (1)$$

Principle of Virtual Work. Since the reactions $\mathbf{A}_x$, $\mathbf{A}_y$, and **C** do no work, the total virtual work done by **P** and **F** must be zero.

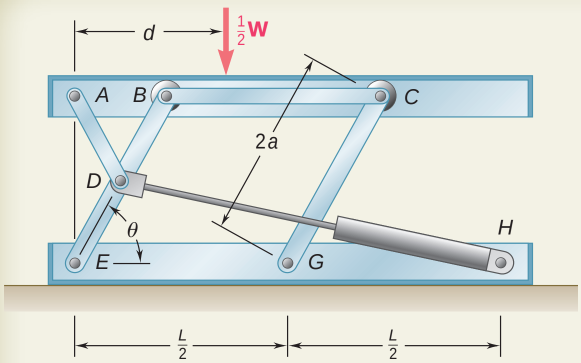
$$\text{d}U = 0: \qquad P \, \text{d}y_B - F \, \text{d}y_C = 0$$
$$P(l \cos u \, \text{du}) - k(2l \sin u - h)(2l \cos u \, \text{du}) = 0$$

$$\sin u = \frac{P + 2kh}{4kl}$$



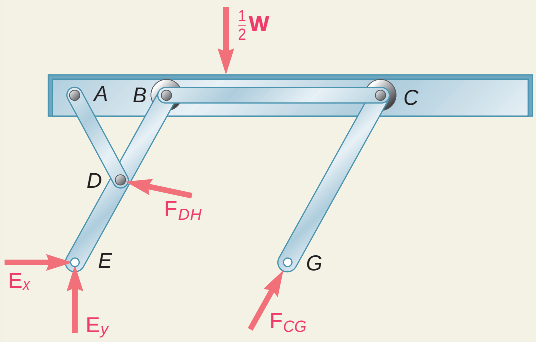
Substituting this expression into (1), we obtain $\qquad F = \frac{1}{2}P$

SAMPLE PROBLEM 10.3



A hydraulic-lift table is used to raise a 1000-kg crate. It consists of a platform and of two identical linkages on which hydraulic cylinders exert equal forces. (Only one linkage and one cylinder are shown.) Members EDB and CG are each of length $2a$, and member AD is pinned to the midpoint of EDB. If the crate is placed on the table, so that half of its weight is supported by the system shown, determine the force exerted by each cylinder in raising the crate for $\theta = 60°$, $a = 0.70$ m, and $L = 3.20$ m. This mechanism has been previously considered in Sample Prob. 6.7.

SOLUTION



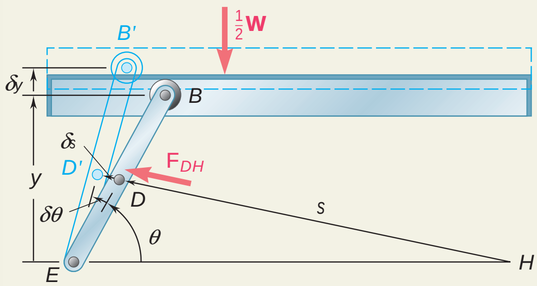
The machine considered consists of the platform and of the linkage, with an input force $\mathbf{F}_{DH}$ exerted by the cylinder and an output force equal and opposite to $\frac{1}{2}\mathbf{W}$.

Principle of Virtual Work. We first observe that the reactions at E and G do no work. Denoting by y the elevation of the platform above the base, and by s the length DH of the cylinder-and-piston assembly, we write

$$\delta U = 0: \qquad -\tfrac{1}{2}W\,\delta y + F_{DH}\,\delta s = 0 \tag{1}$$

The vertical displacement δy of the platform is expressed in terms of the angular displacement $\delta\theta$ of EDB as follows:

$$y = (EB)\sin\theta = 2a\sin\theta$$
$$\delta y = 2a\cos\theta\,\delta\theta$$



To express δs similarly in terms of $\delta\theta$, we first note that by the law of cosines,

$$s^2 = a^2 + L^2 - 2aL\cos\theta$$

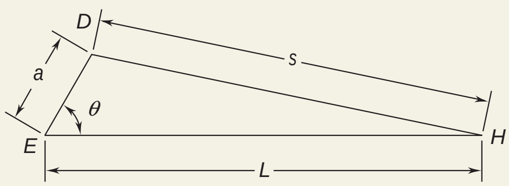
Differentiating,

$$2s\,\delta s = -2aL(-\sin\theta)\,\delta\theta$$
$$\delta s = \frac{aL\sin\theta}{s}\,\delta\theta$$

Substituting for δy and δs into (1), we write

$$(-\tfrac{1}{2}W)2a\cos\theta\,\delta\theta + F_{DH}\frac{aL\sin\theta}{s}\,\delta\theta = 0$$

$$F_{DH} = W\frac{s}{L}\cot\theta$$



With the given numerical data, we have

$$W = mg = (1000\text{ kg})(9.81\text{ m/s}^2) = 9810\text{ N} = 9.81\text{ kN}$$
$$s^2 = a^2 + L^2 - 2aL\cos\theta$$
$$= (0.70)^2 + (3.20)^2 - 2(0.70)(3.20)\cos 60° = 8.49$$
$$s = 2.91\text{ m}$$

$$F_{DH} = W\frac{s}{L}\cot\theta = (9.81\text{ kN})\frac{2.91\text{ m}}{3.20\text{ m}}\cot 60°$$

$$F_{DH} = 5.15\text{ kN} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to use the *method of virtual work,* which is a different way of solving problems involving the equilibrium of rigid bodies.

The work done by a force during a displacement of its point of application or by a couple during a rotation is found by using Eqs. (10.1) and (10.2), respectively:

$$dU = F\, ds \cos \alpha \tag{10.1}$$
$$dU = M\, d\theta \tag{10.2}$$

Principle of virtual work. In its more general and more useful form, this principle can be stated as follows: *If a system of connected rigid bodies is in equilibrium, the total virtual work of the external forces applied to the system is zero for any virtual displacement of the system.*

As you apply the principle of virtual work, keep in mind the following:

1. Virtual displacement. A machine or mechanism in equilibrium has no tendency to move. However, *we can cause, or imagine, a small displacement.* Since it does not actually occur, such a displacement is called a *virtual displacement.*

2. Virtual work. The work done by a force or couple during a virtual displacement is called *virtual work.*

3. You need consider only the forces which do work during the virtual displacement.

4. Forces which do no work during a virtual displacement that is consistent with the constraints imposed on the system are:

- **a.** Reactions at supports
- **b.** Internal forces at connections
- **c.** Forces exerted by inextensible cords and cables

None of these forces need be considered when you use the method of virtual work.

5. Be sure to express the various virtual displacements involved in your computations in terms of a *single virtual displacement.* This is done in each of the three preceding sample problems, where the virtual displacements are all expressed in terms of $d\theta$.

6. Remember that the method of virtual work is effective only in those cases where the geometry of the system makes it relatively easy to relate the displacements involved.

PROBLEMS

10.1 and 10.2 Determine the vertical force **P** that must be applied at G to maintain the equilibrium of the linkage.

10.3 and 10.4 Determine the couple **M** that must be applied to member $DEFG$ to maintain the equilibrium of the linkage.

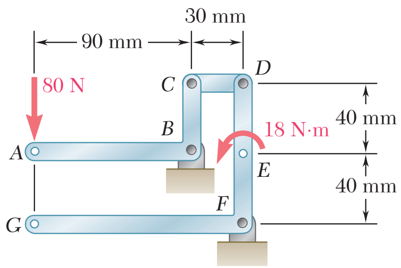


Fig. P10.1 and P10.3

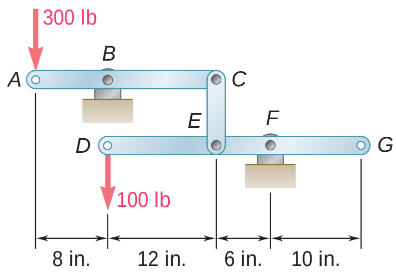


Fig. P10.2 and P10.4

10.5 Determine the force **P** required to maintain the equilibrium of the linkage shown. All members are of the same length and the wheels at A and B roll freely on the horizontal rod.

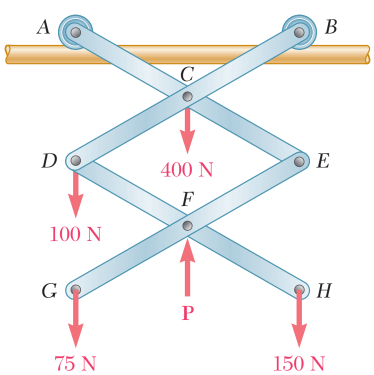


Fig. P10.5

10.6 Solve Prob. 10.5 assuming that the vertical force **P** is applied at point E.

10.7 The two-bar linkage shown is supported by a pin and bracket at B and a collar at D that slides freely on a vertical rod. Determine the force **P** required to maintain the equilibrium of the linkage.

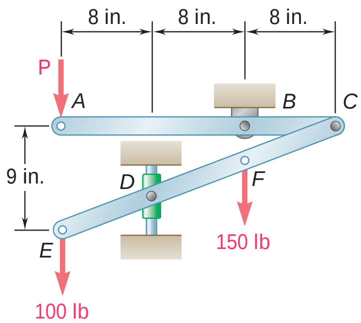


Fig. *P10.7*

10.8 Knowing that the maximum friction force exerted by the bottle on the cork is 60 lb, determine (*a*) the force **P** that must be applied to the corkscrew to open the bottle, (*b*) the maximum force exerted by the base of the corkscrew on the top of the bottle.

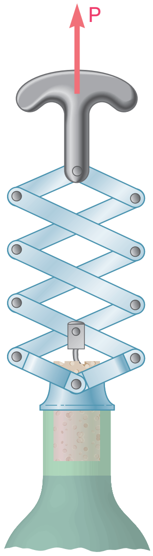


Fig. *P10.8*

10.9 Rod *AD* is acted upon by a vertical force **P** at end *A* and by two equal and opposite horizontal forces of magnitude *Q* at points *B* and *C*. Derive an expression for the magnitude *Q* of the horizontal forces required for equilibrium.

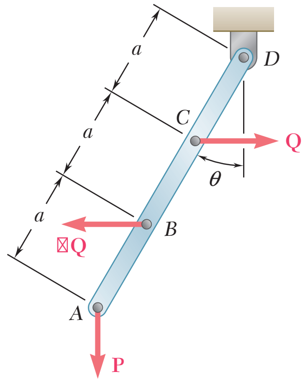


Fig. P10.9

10.10 and *10.11* The slender rod *AB* is attached to a collar *A* and rests on a small wheel at *C*. Neglecting the radius of the wheel and the effect of friction, derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the rod.

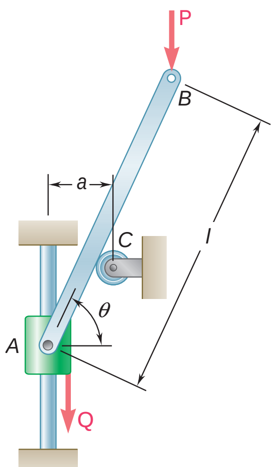


Fig. P10.10

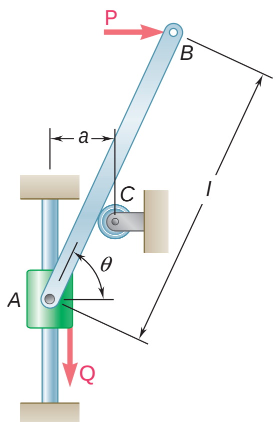


Fig. *P10.11*

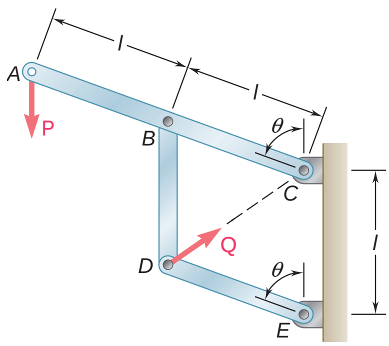


Fig. P10.12

10.12 Knowing that the line of action of the force **Q** passes through point *C*, derive an expression for the magnitude of **Q** required to maintain equilibrium.

10.13 Solve Prob. 10.12 assuming that the force **P** applied at point *A* acts horizontally to the left.

10.14 The mechanism shown is acted upon by the force **P**; derive an expression for the magnitude of the force **Q** required to maintain equilibrium.

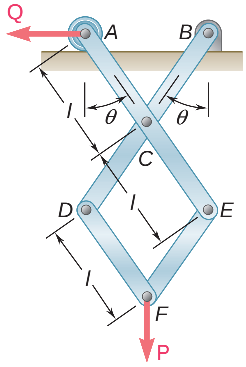


Fig. P10.14

10.15 and 10.16 Derive an expression for the magnitude of the couple **M** required to maintain the equilibrium of the linkage shown.

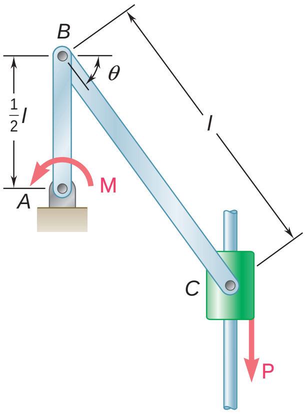


Fig. P10.15

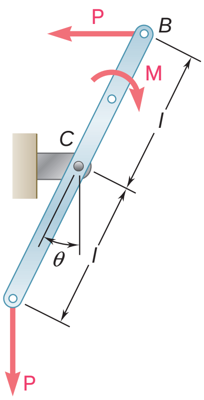


Fig. P10.16

10.17 A uniform rod AB of length l and weight W is suspended from two cords AC and BC of equal length. Derive an expression for the magnitude of the couple **M** required to maintain equilibrium of the rod in the position shown.

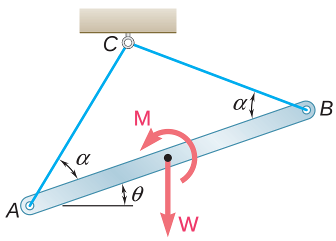


Fig. P10.17

10.18 Collar B can slide along rod AC and is attached by a pin to a block that can slide in the vertical slot shown. Derive an expression for the magnitude of the couple **M** required to maintain equilibrium.

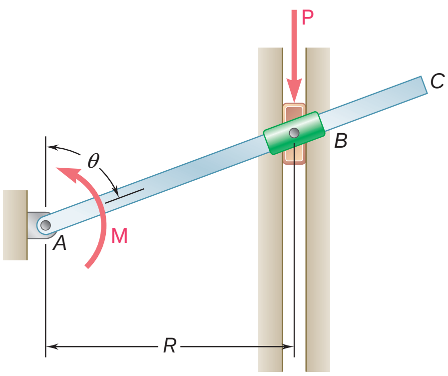


Fig. P10.18

10.19 For the linkage shown, determine the couple **M** required for equilibrium when l = 1.8 ft, Q = 40 lb, and u = 65°.

10.20 For the linkage shown, determine the force **Q** required for equilibrium when l = 18 in., M = 600 lb · in., and u = 70°.

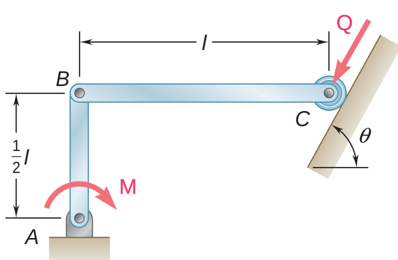


Fig. P10.19 and P10.20

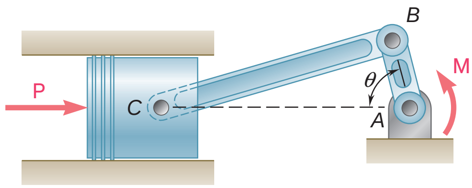


Fig. *P10.21* and *P10.22*

10.21 A 4-kN force **P** is applied as shown to the piston of the engine system. Knowing that $AB = 50$ mm and $BC = 200$ mm, determine the couple **M** required to maintain the equilibrium of the system when (*a*) u = 30°, (*b*) u = 150°.

10.22 A couple **M** of magnitude 100 N · m is applied as shown to the crank of the engine system. Knowing that $AB = 50$ mm and $BC = 200$ mm, determine the force **P** required to maintain the equilibrium of the system when (*a*) u = 60°, (*b*) u = 120°.

10.23 A slender rod of length l is attached to a collar at B and rests on a portion of a circular cylinder of radius r. Neglecting the effect of friction, determine the value of u corresponding to the equilibrium position of the mechanism when $l = 200$ mm, $r = 60$ mm, $P = 40$ N, and $Q = 80$ N.

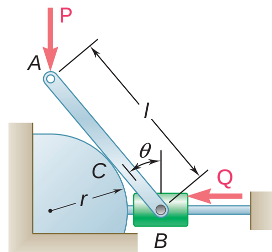


Fig. P10.23 and *P10.24*

10.24 A slender rod of length l is attached to a collar at B and rests on a portion of a circular cylinder of radius r. Neglecting the effect of friction, determine the value of u corresponding to the equilibrium position of the mechanism when $l = 14$ in., $r = 5$ in., $P = 75$ lb, and $Q = 150$ lb.

10.25 Determine the value of u corresponding to the equilibrium position of the rod of Prob. 10.10 when $l = 30$ in., $a = 5$ in., $P = 25$ lb, and $Q = 40$ lb.

10.26 Determine the values of u corresponding to the equilibrium position of the rod of Prob. 10.11 when $l = 600$ mm, $a = 100$ mm, $P = 50$ N, and $Q = 90$ N.

10.27 Determine the value of u corresponding to the equilibrium position of the mechanism of Prob. 10.12 when $P = 80$ N and $Q = 100$ N.

10.28 Determine the value of u corresponding to the equilibrium position of the mechanism of Prob. 10.14 when $P = 270$ N and $Q = 960$ N.

10.29 A load **W** of magnitude 600 N is applied to the linkage at B. The constant of the spring is $k = 2.5$ kN/m, and the spring is unstretched when AB and BC are horizontal. Neglecting the weight of the linkage and knowing that $l = 300$ mm, determine the value of u corresponding to equilibrium.

10.30 A vertical load **W** is applied to the linkage at B. The constant of the spring is k, and the spring is unstretched when AB and BC are horizontal. Neglecting the weight of the linkage, derive an equation in u, W, l, and k that must be satisfied when the linkage is in equilibrium.

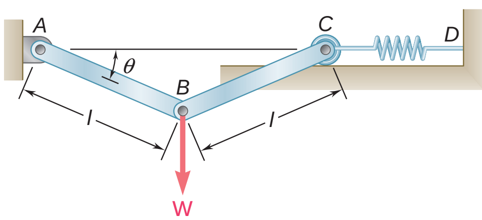


Fig. P10.29 and *P10.30*

10.31 Two bars AD and DG are connected by a pin at D and by a spring AG. Knowing that the spring is 12 in. long when unstretched and that the constant of the spring is 125 lb/in., determine the value of x corresponding to equilibrium when a 900-lb load is applied at E as shown.

10.32 Solve Prob. 10.31 assuming that the 900-lb vertical force is applied at C instead of E.

10.33 Two 5-kg bars AB and BC are connected by a pin at B and by a spring DE. Knowing that the spring is 150 mm long when unstretched and that the constant of the spring is 1 kN/m, determine the value of x corresponding to equilibrium.

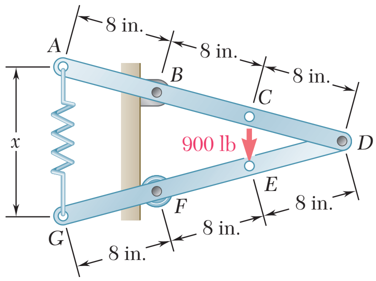


Fig. P10.31

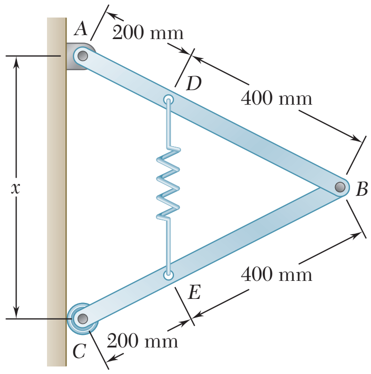


Fig. *P10.33*

10.34 Rod ABC is attached to blocks A and B that can move freely in the guides shown. The constant of the spring attached at A is $k = 3$ kN/m, and the spring is unstretched when the rod is vertical. For the loading shown, determine the value of θ corresponding to equilibrium.

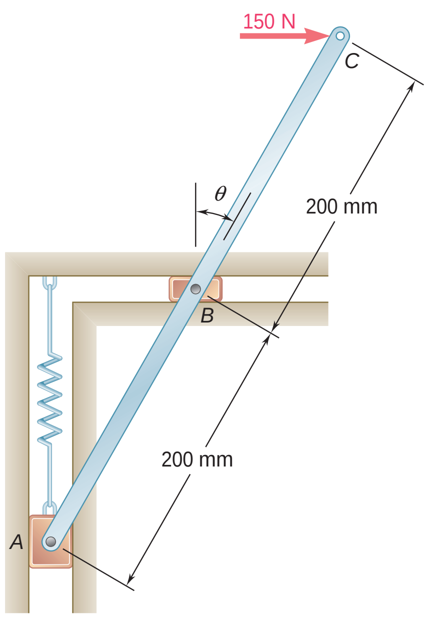


Fig. P10.34

10.35 A vertical force **P** of magnitude 150 N is applied to end E of cable CDE, which passes over a small pulley D and is attached to the mechanism at C. The constant of the spring is $k = 4$ kN/m, and the spring is unstretched when $\theta = 0$. Neglecting the weight of the mechanism and the radius of the pulley, determine the value of θ corresponding to equilibrium.

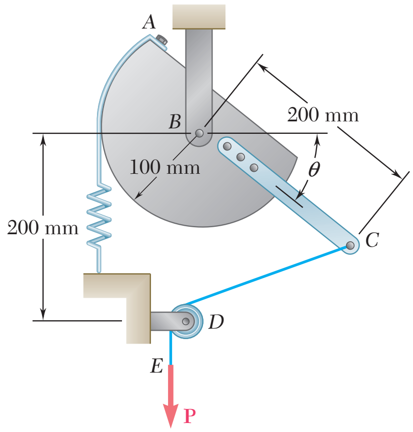


Fig. P10.35

10.36 A horizontal force **P** of magnitude 40 lb is applied to the mechanism at C. The constant of the spring is $k = 9$ lb/in., and the spring is unstretched when $\theta = 0$. Neglecting the weight of the mechanism, determine the value of θ corresponding to equilibrium.

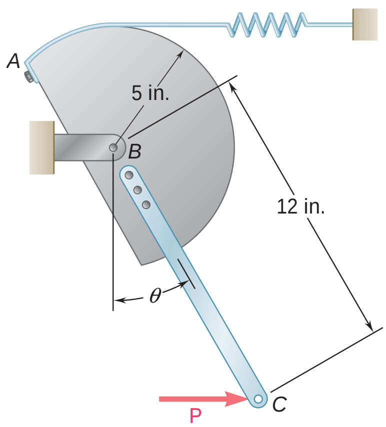


Fig. P10.36

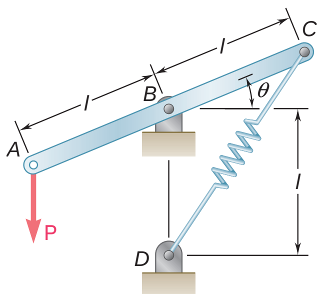


Fig. P10.37 and P10.38

10.37 and 10.38 Knowing that the constant of spring CD is k and that the spring is unstretched when rod ABC is horizontal, determine the value of θ corresponding to equilibrium for the data indicated.

10.37 $P = 300$ N, $l = 400$ mm, $k = 5$ kN/m
10.38 $P = 75$ lb, $l = 15$ in., $k = 20$ lb/in.

10.39 The lever *AB* is attached to the horizontal shaft *BC* that passes through a bearing and is welded to a fixed support at *C*. The torsional spring constant of the shaft *BC* is *K*; that is, a couple of magnitude *K* is required to rotate end *B* through 1 rad. Knowing that the shaft is untwisted when *AB* is horizontal, determine the value of u corresponding to the position of equilibrium when $P = 100$ N, $l = 250$ mm, and $K = 12.5$ N · m/rad.

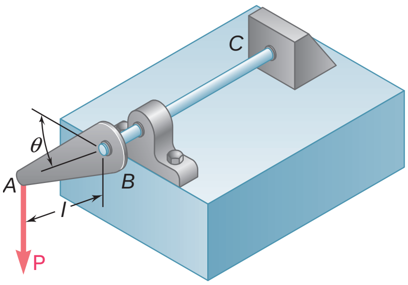


Fig. P10.39

10.40 Solve Prob. 10.39 assuming that $P = 350$ N, $l = 250$ mm, and $K = 12.5$ N · m/rad. Obtain answers in each of the following quadrants: 0 < u < 90°, 270° < u < 360°, 360° < u < 450°.

10.41 The position of boom *ABC* is controlled by the hydraulic cylinder *BD*. For the loading shown, determine the force exerted by the hydraulic cylinder on pin *B* when u = 65°.

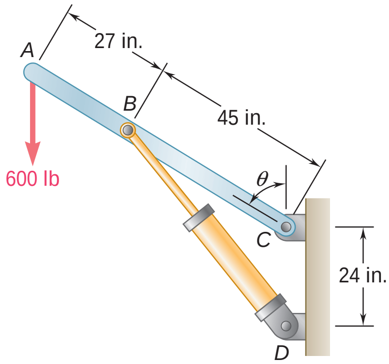


Fig. *P10.41* and *P10.42*

10.42 The position of boom *ABC* is controlled by the hydraulic cylinder *BD*. For the loading shown, (*a*) express the force exerted by the hydraulic cylinder on pin *B* as a function of the length *BD*, (*b*) determine the smallest possible value of the angle u if the maximum force that the cylinder can exert on pin *B* is 2.5 kips.

10.43 The position of member *ABC* is controlled by the hydraulic cylinder *CD*. For the loading shown, determine the force exerted by the hydraulic cylinder on pin *C* when u = 55°.

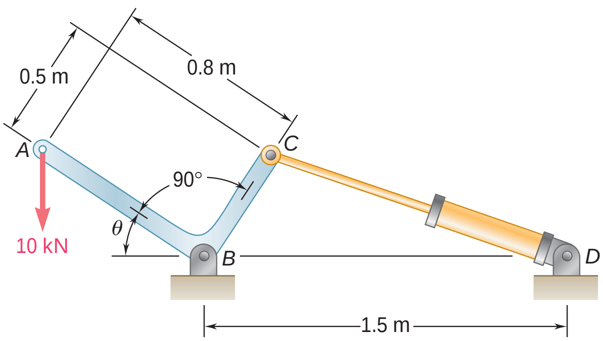


Fig. P10.43 and P10.44

10.44 The position of member *ABC* is controlled by the hydraulic cylinder *CD*. Determine the angle u knowing that the hydraulic cylinder exerts a 15-kN force on pin *C*.

10.45 The telescoping arm ABC is used to provide an elevated platform for construction workers. The workers and the platform together weigh 500 lb and their combined center of gravity is located directly above C. For the position when $\theta = 20°$, determine the force exerted on pin B by the single hydraulic cylinder BD.

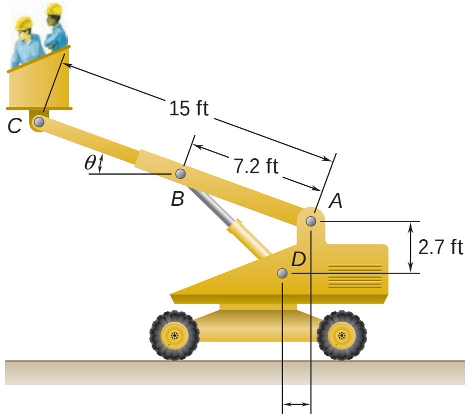


Fig. P10.45

10.46 Solve Prob. 10.45 assuming that the workers are lowered to a point near the ground so that $\theta = -20°$.

10.47 Denoting by μ_s the coefficient of static friction between collar C and the vertical rod, derive an expression for the magnitude of the largest couple **M** for which equilibrium is maintained in the position shown. Explain what happens if $\mu_s \geq \tan \theta$.

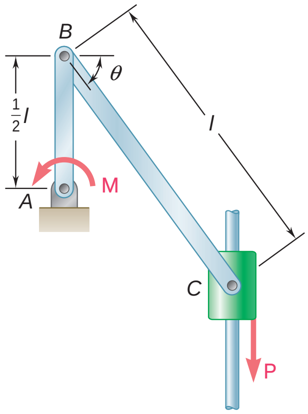


Fig. *P10.47* and P10.48

10.48 Knowing that the coefficient of static friction between collar C and the vertical rod is 0.40, determine the magnitude of the largest and smallest couple **M** for which equilibrium is maintained in the position shown, when $\theta = 35°$, $l = 600$ mm, and $P = 300$ N.

10.49 A block of weight W is pulled up a plane forming an angle α with the horizontal by a force **P** directed along the plane. If μ is the coefficient of friction between the block and the plane, derive an expression for the mechanical efficiency of the system. Show that the mechanical efficiency cannot exceed $\frac{1}{2}$ if the block is to remain in place when the force **P** is removed.

10.50 Derive an expression for the mechanical efficiency of the jack discussed in Sec. 8.6. Show that if the jack is to be self-locking, the mechanical efficiency cannot exceed $\frac{1}{2}$.

10.51 Denoting by m_s the coefficient of static friction between the block attached to rod *ACE* and the horizontal surface, derive expressions in terms of *P*, m_s, and u for the largest and smallest magnitude of the force **Q** for which equilibrium is maintained.

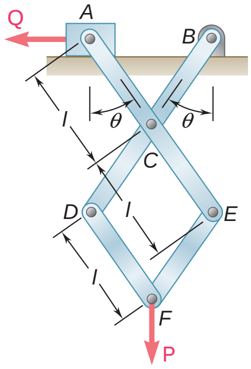


Fig. *P10.51* and P10.52

10.52 Knowing that the coefficient of static friction between the block attached to rod *ACE* and the horizontal surface is 0.15, determine the magnitude of the largest and smallest force **Q** for which equilibrium is maintained when u = 30°, l = 0.2 m, and *P* = 40 N.

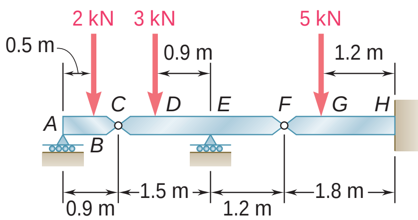


Fig. P10.53 and P10.54

10.53 Using the method of virtual work, determine the reaction at *E*.

10.54 Using the method of virtual work, determine separately the force and couple representing the reaction at *H*.

10.55 Referring to Prob. 10.43 and using the value found for the force exerted by the hydraulic cylinder *CD*, determine the change in the length of *CD* required to raise the 10-kN load by 15 mm.

10.56 Referring to Prob. 10.45 and using the value found for the force exerted by the hydraulic cylinder *BD*, determine the change in the length of *BD* required to raise the platform attached at *C* by 2.5 in.

10.57 Determine the vertical movement of joint *D* if the length of member *BF* is increased by 1.5 in. (*Hint:* Apply a vertical load at joint *D*, and, using the methods of Chap. 6, compute the force exerted by member *BF* on joints *B* and *F*. Then apply the method of virtual work for a virtual displacement resulting in the specified increase in length of member *BF*. This method should be used only for small changes in the lengths of members.)

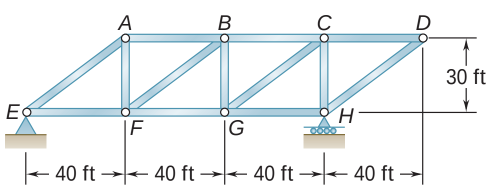


Fig. P10.57 and P10.58

10.58 Determine the horizontal movement of joint *D* if the length of member *BF* is increased by 1.5 in. (See the hint for Prob. 10.57.)

*10.6 WORK OF A FORCE DURING A FINITE DISPLACEMENT

Consider a force **F** acting on a particle. The work of **F** corresponding to an infinitesimal displacement $d\mathbf{r}$ of the particle was defined in Sec. 10.2 as

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{10.1}$$

The work of **F** corresponding to a finite displacement of the particle from A_1 to A_2 (Fig. 10.10*a*) is denoted by $U_{1\rightarrow 2}$ and is obtained by integrating (10.1) along the curve described by the particle:

$$U_{1\rightarrow 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \tag{10.11}$$

Using the alternative expression

$$dU = F\,ds \cos \alpha \tag{10.1'}$$

given in Sec. 10.2 for the elementary work dU, we can also express the work $U_{1\rightarrow 2}$ as

$$U_{1\rightarrow 2} = \int_{s_1}^{s_2} (F \cos \alpha)\,ds \tag{10.11'}$$

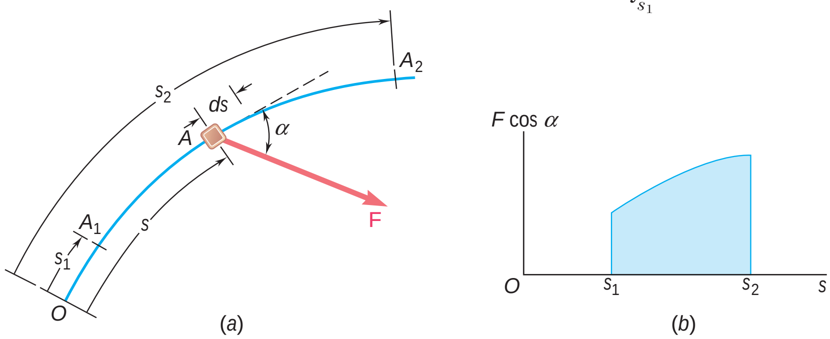


Fig. 10.10

where the variable of integration s measures the distance along the path traveled by the particle. The work $U_{1\rightarrow 2}$ is represented by the area under the curve obtained by plotting $F \cos \alpha$ against s (Fig. 10.10*b*). In the case of a force **F** of constant magnitude acting in the direction of motion, formula (10.11′) yields $U_{1\rightarrow 2} = F(s_2 - s_1)$.

Recalling from Sec. 10.2 that the work of a couple of moment **M** during an infinitesimal rotation $d\theta$ of a rigid body is

$$dU = M\,d\theta \tag{10.2}$$

we express as follows the work of the couple during a finite rotation of the body:

$$U_{1\rightarrow 2} = \int_{\theta_1}^{\theta_2} M\,d\theta \tag{10.12}$$

In the case of a constant couple, formula (10.12) yields

$$U_{1\rightarrow 2} = M(\theta_2 - \theta_1)$$

Work of a Weight. It was stated in Sec. 10.2 that the work of the weight **W** of a body during an infinitesimal displacement of the body is equal to the product of W and the vertical displacement of the center of gravity of the body. With the y axis pointing upward, the work of **W** during a finite displacement of the body (Fig. 10.11) is obtained by writing

$$dU = -W\,dy$$

Integrating from A_1 to A_2, we have

$$U_{1 y 2} = -\int_{y_1}^{y_2} W\,dy = Wy_1 - Wy_2 \tag{10.13}$$

or

$$U_{1 y 2} = -W(y_2 - y_1) = -W\,\Delta y \tag{10.13$'$}$$

where Δy is the vertical displacement from A_1 to A_2. The work of the weight **W** is thus equal to *the product of* W *and the vertical displacement of the center of gravity of the body.* The work is *positive* when $\Delta y < 0$, that is, *when the body moves down.*

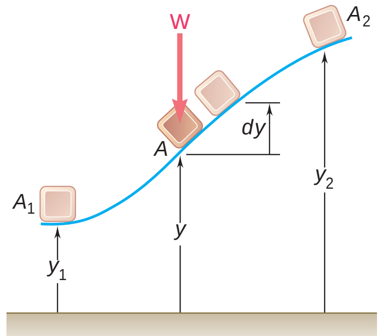


Fig. 10.11

Work of the Force Exerted by a Spring. Consider a body A attached to a fixed point B by a spring; it is assumed that the spring is undeformed when the body is at A_0 (Fig. 10.12*a*). Experimental evidence shows that the magnitude of the force **F** exerted by the spring on a body A is proportional to the deflection x of the spring measured from the position A_0. We have

$$F = kx \tag{10.14}$$

where k is the *spring constant,* expressed in N/m if SI units are used and expressed in lb/ft or lb/in. if U.S. customary units are used. The work of the force **F** exerted by the spring during a finite displacement of the body from $A_1(x = x_1)$ to $A_2(x = x_2)$ is obtained by writing

$$dU = -F\,dx = -kx\,dx$$

$$U_{1 y 2} = -\int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \tag{10.15}$$

Care should be taken to express k and x in consistent units. For example, if U.S. customary units are used, k should be expressed in lb/ft and x expressed in feet, or k in lb/in. and x in inches; in the first case, the work is obtained in ft · lb; in the second case, in in · lb. We note that the work of the force **F** exerted by the spring on the body is *positive* when $x_2 < x_1$, that is, *when the spring is returning to its undeformed position.*

Since Eq. (10.14) is the equation of a straight line of slope k passing through the origin, the work $U_{1 y 2}$ of **F** during the displacement from A_1 to A_2 can be obtained by evaluating the area of the trapezoid shown in Fig. 10.12*b*. This is done by computing the values F_1 and F_2 and multiplying the base Δx of the trapezoid by its mean height $\frac{1}{2}(F_1 + F_2)$. Since the work of the force **F** exerted by the spring is positive for a negative value of Δx, we write

$$U_{1 y 2} = -\tfrac{1}{2}(F_1 + F_2)\,\Delta x \tag{10.16}$$

Formula (10.16) is usually more convenient to use than (10.15) and affords fewer chances of confusing the units involved.

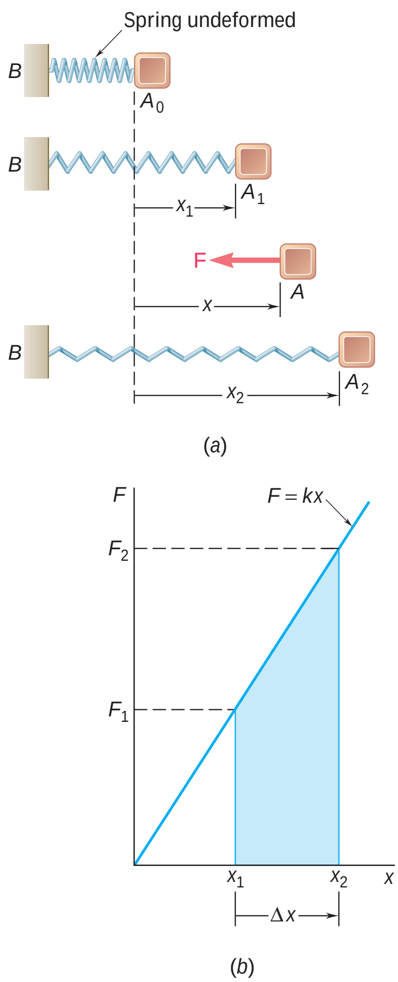


Fig. 10.12

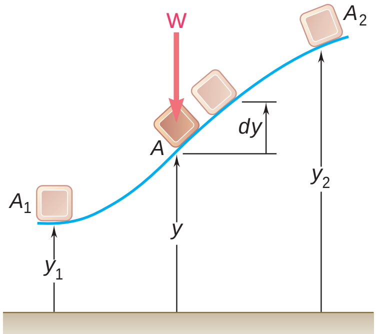


Fig. 10.11 (*repeated*)

*10.7 POTENTIAL ENERGY

Considering again the body of Fig. 10.11, we note from Eq. (10.13) that the work of the weight **W** during a finite displacement is obtained by subtracting the value of the function Wy corresponding to the second position of the body from its value corresponding to the first position. The work of **W** is thus independent of the actual path followed; it depends only upon the initial and final values of the function Wy. This function is called the *potential energy* of the body with respect to the *force of gravity* **W** and is denoted by V_g. We write

$$U_{1 y 2} = (V_g)_1 - (V_g)_2 \qquad \text{with } V_g = Wy \tag{10.17}$$

We note that if $(V_g)_2 > (V_g)_1$, that is, *if the potential energy increases* during the displacement (as in the case considered here), *the work* $U_{1 y 2}$ *is negative.* If, on the other hand, the work of **W** is positive, the potential energy decreases. Therefore, the potential energy V_g of the body provides a measure of *the work which can be done* by its weight **W**. Since only the *change* in potential energy, and not the actual value of V_g, is involved in formula (10.17), an arbitrary constant can be added to the expression obtained for V_g. In other words, the level from which the elevation y is measured can be chosen arbitrarily. Note that potential energy is expressed in the same units as work, i.e., in joules (J) if SI units are used† and in ft · lb or in · lb if U.S. customary units are used.

Considering now the body of Fig. 10.12*a*, we note from Eq. (10.15) that the work of the elastic force **F** is obtained by subtracting the value of the function $\frac{1}{2}kx^2$ corresponding to the second position of the body from its value corresponding to the first position. This function is denoted by V_e and is called the *potential energy* of the body with respect to the *elastic force* **F**. We write

$$U_{1 y 2} = (V_e)_1 - (V_e)_2 \qquad \text{with } V_e = \tfrac{1}{2}kx^2 \tag{10.18}$$

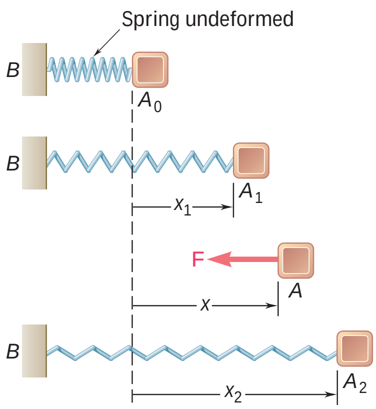


Fig. 10.12a (*repeated*)

and observe that during the displacement considered, the work of the force **F** exerted by the spring on the body is negative and the potential energy V_e increases. We should note that the expression obtained for V_e is valid only if the deflection of the spring is measured from its undeformed position.

The concept of potential energy can be used when forces other than gravity forces and elastic forces are involved. It remains valid as long as the elementary work dU of the force considered is an *exact differential.* It is then possible to find a function V, called potential energy, such that

$$\boxed{dU = -dV} \tag{10.19}$$

Integrating (10.19) over a finite displacement, we obtain the general formula

$$\boxed{U_{1 y 2} = V_1 - V_2} \tag{10.20}$$

which expresses that *the work of the force is independent of the path followed and is equal to minus the change in potential energy.* A force which satisfies Eq. (10.20) is said to be a *conservative force.*‡

†See footnote, page 559.

‡A detailed discussion of conservative forces is given in Sec. 13.7 of *Dynamics.*

*10.8 POTENTIAL ENERGY AND EQUILIBRIUM

The application of the principle of virtual work is considerably simplified when the potential energy of a system is known. In the case of a virtual displacement, formula (10.19) becomes $\delta U = -\delta V$. Moreover, if the position of the system is defined by a single independent variable θ, we can write $\delta V = (dV/d\theta)\,\delta\theta$. Since $\delta\theta$ must be different from zero, the condition $\delta U = 0$ for the equilibrium of the system becomes

$$\boxed{\frac{dV}{d\theta} = 0} \tag{10.21}$$

In terms of potential energy, therefore, the principle of virtual work states that *if a system is in equilibrium, the derivative of its total potential energy is zero.* If the position of the system depends upon several independent variables (the system is then said to possess *several degrees of freedom*), the partial derivatives of V with respect to each of the independent variables must be zero.

Consider, for example, a structure made of two members AC and CB and carrying a load W at C. The structure is supported by a pin at A and a roller at B, and a spring BD connects B to a fixed point D (Fig. 10.13*a*). The constant of the spring is k, and it is assumed that the natural length of the spring is equal to AD and thus that the spring is undeformed when B coincides with A. Neglecting the friction forces and the weight of the members, we find that the only forces which work during a displacement of the structure are the weight $\mathbf{W}$ and the force $\mathbf{F}$ exerted by the spring at point B (Fig. 10.13*b*). The total potential energy of the system will thus be obtained by adding the potential energy V_g corresponding to the gravity force $\mathbf{W}$ and the potential energy V_e corresponding to the elastic force $\mathbf{F}$.

Choosing a coordinate system with origin at A and noting that the deflection of the spring, measured from its undeformed position, is $AB = x_B$, we write

$$V_e = \tfrac{1}{2}kx_B^2 \qquad V_g = Wy_C$$

Expressing the coordinates x_B and y_C in terms of the angle θ, we have

$$\begin{aligned} x_B &= 2l\sin\theta & y_C &= l\cos\theta \\ V_e &= \tfrac{1}{2}k(2l\sin\theta)^2 & V_g &= W(l\cos\theta) \end{aligned}$$
$$V = V_e + V_g = 2kl^2\sin^2\theta + Wl\cos\theta \tag{10.22}$$

The positions of equilibrium of the system are obtained by equating to zero the derivative of the potential energy V. We write

$$\frac{dV}{d\theta} = 4kl^2\sin\theta\cos\theta - Wl\sin\theta = 0$$

or, factoring $l\sin\theta$,

$$\frac{dV}{d\theta} = l\sin\theta(4kl\cos\theta - W) = 0$$

There are therefore two positions of equilibrium, corresponding to the values $\theta = 0$ and $\theta = \cos^{-1}(W/4kl)$, respectively.†

†The second position does not exist if $W > 4kl$.

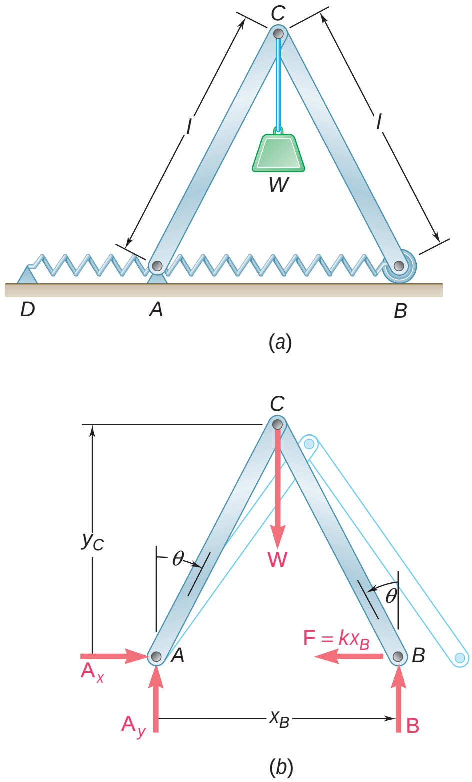
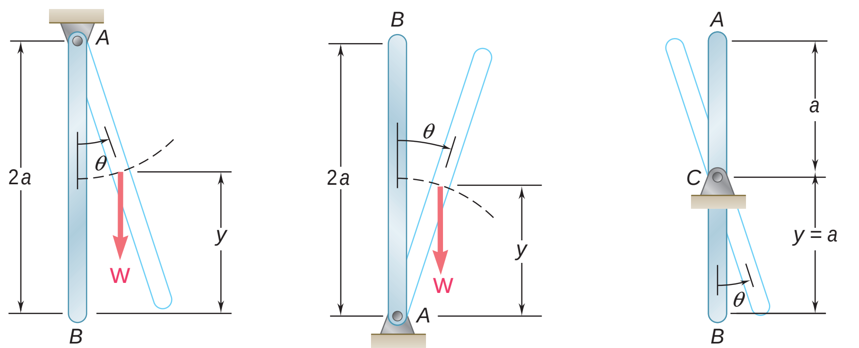


Fig. 10.13

*10.9 STABILITY OF EQUILIBRIUM

Consider the three uniform rods of length $2a$ and weight **W** shown in Fig. 10.14. While each rod is in equilibrium, there is an important difference between the three cases considered. Suppose that each rod is slightly disturbed from its position of equilibrium and then released: rod *a* will move back toward its original position, rod *b* will keep moving away from its original position, and rod *c* will remain in its new position. In case *a*, the equilibrium of the rod is said to be *stable;* in case *b*, it is said to be *unstable;* and, in case *c*, it is said to be *neutral*.



(*a*) Stable equilibrium (*b*) Unstable equilibrium (*c*) Neutral equilibrium

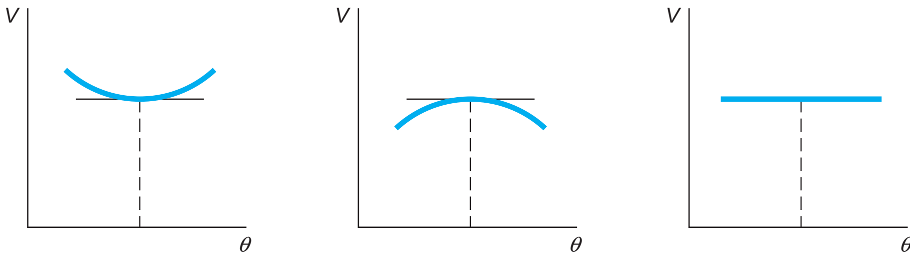
Fig. 10.14

Recalling from Sec. 10.7 that the potential energy V_g with respect to gravity is equal to Wy, where y is the elevation of the point of application of **W** measured from an arbitrary level, we observe that the potential energy of rod *a* is minimum in the position of equilibrium considered, that the potential energy of rod *b* is maximum, and that the potential energy of rod *c* is constant. Equilibrium is thus *stable, unstable,* or *neutral* according to whether the potential energy is *minimum*, *maximum,* or *constant* (Fig. 10.15).

That the result obtained is quite general can be seen as follows: We first observe that a force always tends to do positive work and thus to decrease the potential energy of the system on which it is applied. Therefore, when a system is disturbed from its position of equilibrium, the forces acting on the system will tend to bring it back to its original position if V is minimum (Fig. 10.15*a*) and to move it farther away if V is maximum (Fig. 10.15*b*). If V is constant (Fig. 10.15*c*), the forces will not tend to move the system either way.

Recalling from calculus that a function is minimum or maximum according to whether its second derivative is positive or negative, we can summarize the conditions for the equilibrium of a system with one degree of freedom (i.e., a system the position of which is defined by a single independent variable u) as follows:

$$\frac{dV}{d\mathrm{u}} = 0 \qquad \frac{d^2V}{d\mathrm{u}^2} > 0\text{: stable equilibrium}$$
$$\frac{dV}{d\mathrm{u}} = 0 \qquad \frac{d^2V}{d\mathrm{u}^2} < 0\text{: unstable equilibrium} \tag{10.23}$$



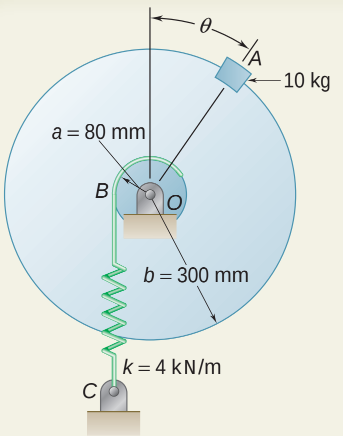
(a) Stable equilibrium (b) Unstable equilibrium (c) Neutral equilibrium

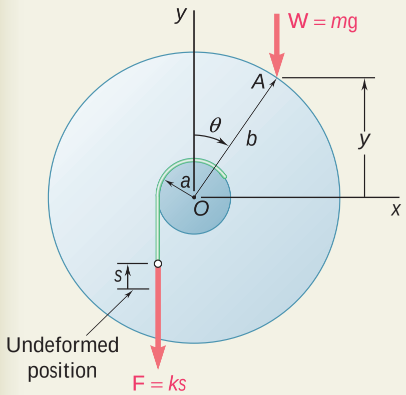
Fig. 10.15

If both the first and the second derivatives of V are zero, it is necessary to examine derivatives of a higher order to determine whether the equilibrium is stable, unstable, or neutral. The equilibrium will be neutral if all derivatives are zero, since the potential energy V is then a constant. The equilibrium will be stable if the first derivative found to be different from zero is of even order and positive. In all other cases the equilibrium will be unstable.

If the system considered possesses *several degrees of freedom,* the potential energy V depends upon several variables, and it is thus necessary to apply the theory of functions of several variables to determine whether V is minimum. It can be verified that a system with 2 degrees of freedom will be stable, and the corresponding potential energy $V(\mathrm{u}_1, \mathrm{u}_2)$ will be minimum, if the following relations are satisfied simultaneously:

$$\frac{\partial V}{\partial \mathrm{u}_1} = \frac{\partial V}{\partial \mathrm{u}_2} = 0$$
$$\left(\frac{\partial^2 V}{\partial \mathrm{u}_1\,\partial \mathrm{u}_2}\right)^2 - \frac{\partial^2 V}{\partial \mathrm{u}_1^2}\frac{\partial^2 V}{\partial \mathrm{u}_2^2} < 0 \tag{10.24}$$
$$\frac{\partial^2 V}{\partial \mathrm{u}_1^2} > 0 \qquad \text{or} \qquad \frac{\partial^2 V}{\partial \mathrm{u}_2^2} > 0$$





SAMPLE PROBLEM 10.4

A 10-kg block is attached to the rim of a 300-mm-radius disk as shown. Knowing that spring BC is unstretched when $\theta = 0$, determine the position or positions of equilibrium, and state in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION

Potential Energy. Denoting by s the deflection of the spring from its undeformed position and placing the origin of coordinates at O, we obtain

$$V_e = \tfrac{1}{2}ks^2 \qquad V_g = Wy = mgy$$

Measuring θ in radians, we have

$$s = a\theta \qquad y = b\cos\theta$$

Substituting for s and y in the expressions for V_e and V_g, we write

$$V_e = \tfrac{1}{2}ka^2\theta^2 \qquad V_g = mgb\cos\theta$$
$$V = V_e + V_g = \tfrac{1}{2}ka^2\theta^2 + mgb\cos\theta$$

Positions of Equilibrium. Setting $dV/d\theta = 0$, we write

$$\frac{dV}{d\theta} = ka^2\theta - mgb\sin\theta = 0$$

$$\sin\theta = \frac{ka^2}{mgb}\theta$$

Substituting $a = 0.08$ m, $b = 0.3$ m, $k = 4$ kN/m, and $m = 10$ kg, we obtain

$$\sin\theta = \frac{(4\text{ kN/m})(0.08\text{ m})^2}{(10\text{ kg})(9.81\text{ m/s}^2)(0.3\text{ m})}\theta$$
$$\sin\theta = 0.8699\,\theta$$

where θ is expressed in radians. Solving by trial and error for θ, we find

$$\theta = 0 \qquad \text{and} \qquad \theta = 0.902\text{ rad}$$
$$\theta = 0 \qquad \text{and} \qquad \theta = 51.7° \quad ◀$$

Stability of Equilibrium. The second derivative of the potential energy V with respect to θ is

$$\begin{aligned}\frac{d^2V}{d\theta^2} &= ka^2 - mgb\cos\theta \\ &= (4\text{ kN/m})(0.08\text{ m})^2 - (10\text{ kg})(9.81\text{ m/s}^2)(0.3\text{ m})\cos\theta \\ &= 25.6 - 29.43\cos\theta\end{aligned}$$

For $\theta = 0$: $\quad \dfrac{d^2V}{d\theta^2} = 25.6 - 29.43\cos 0° = -3.83 < 0$

The equilibrium is unstable for $\theta = 0$ ◀

For $\theta = 51.7°$: $\quad \dfrac{d^2V}{d\theta^2} = 25.6 - 29.43\cos 51.7° = +7.36 > 0$

The equilibrium is stable for $\theta = 51.7°$ ◀

SOLVING PROBLEMS ON YOUR OWN

In this lesson we defined the *work of a force during a finite displacement* and the *potential energy* of a rigid body or a system of rigid bodies. You learned to use the concept of potential energy to determine the *equilibrium position* of a rigid body or a system of rigid bodies.

1. The potential energy *V* of a system is the sum of the potential energies associated with the various forces acting on the system that *do work* as the system moves. In the problems of this lesson you will determine the following:

a. Potential energy of a weight. This is the potential energy due to *gravity*, $V_g = Wy$, where y is the elevation of the weight W measured from some arbitrary reference level. Note that the potential energy V_g may be used with any vertical force $\mathbf{P}$ of constant magnitude directed downward; we write $V_g = Py$.

b. Potential energy of a spring. This is the potential energy due to the *elastic* force exerted by a spring, $V_e = \frac{1}{2}kx^2$, where k is the constant of the spring and x is the deformation of the spring *measured from its unstretched position.*

Reactions at fixed supports, internal forces at connections, forces exerted by inextensible cords and cables, and other forces which do no work do not contribute to the potential energy of the system.

2. Express all distances and angles in terms of a single variable, such as an angle u, when computing the potential energy V of a system. This is necessary, since the determination of the equilibrium position of the system requires the computation of the derivative dV/du.

3. When a system is in equilibrium, the first derivative of its potential energy is zero. Therefore:

a. To determine a position of equilibrium of a system, once its potential energy V has been expressed in terms of the single variable u, compute its derivative and solve the equation $dV/du = 0$ for u.

b. To determine the force or couple required to maintain a system in a given position of equilibrium, substitute the known value of u in the equation $dV/du = 0$ and solve this equation for the desired force or couple.

4. Stability of equilibrium. The following rules generally apply:

a. Stable equilibrium occurs when the potential energy of the system is *minimum,* that is, when $dV/du = 0$ and $d^2V/du^2 > 0$ (Figs. 10.14*a* and 10.15*a*).

b. Unstable equilibrium occurs when the potential energy of the system is *maximum,* that is, when $dV/du = 0$ and $d^2V/du^2 < 0$ (Figs. 10.14*b* and 10.15*b*).

c. Neutral equilibrium occurs when the potential energy of the system is *constant;* dV/du, d^2V/du^2, and all the successive derivatives of V are then equal to zero (Figs. 10.14*c* and 10.15*c*).

See page 583 for a discussion of the case when dV/du, d^2V/du^2 but *not all* of the successive derivatives of V are equal to zero.

PROBLEMS

10.59 Using the method of Sec. 10.8, solve Prob. 10.29.

10.60 Using the method of Sec. 10.8, solve Prob. 10.30.

10.61 Using the method of Sec. 10.8, solve Prob. 10.31.

10.62 Using the method of Sec. 10.8, solve Prob. 10.32.

10.63 Using the method of Sec. 10.8, solve Prob. 10.33.

10.64 Using the method of Sec. 10.8, solve Prob. 10.35.

10.65 Using the method of Sec. 10.8, solve Prob. 10.37.

10.66 Using the method of Sec. 10.8, solve Prob. 10.38.

10.67 Show that equilibrium is neutral in Prob. 10.1.

10.68 Show that equilibrium is neutral in Prob. 10.7.

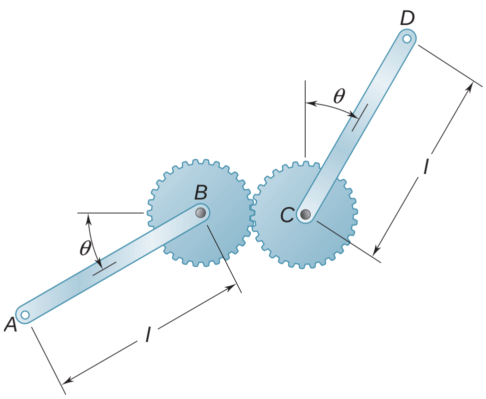


Fig. P10.69 and P10.70

10.69 Two uniform rods, each of mass m, are attached to gears of equal radii as shown. Determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

10.70 Two uniform rods, AB and CD, are attached to gears of equal radii as shown. Knowing that $W_{AB} = 8$ lb and $W_{CD} = 4$ lb, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

10.71 Two uniform rods, each of mass m and length l, are attached to gears as shown. For the range $0 \leq$ u $\leq 180°$, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

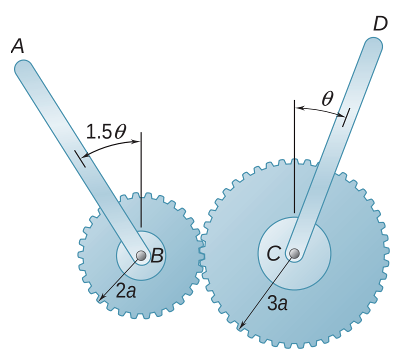


Fig. P10.71

10.72 Two uniform rods, each of mass m and length l, are attached to drums that are connected by a belt as shown. Assuming that no slipping occurs between the belt and the drums, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

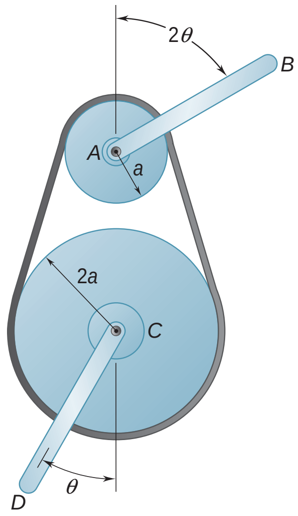


Fig. P10.72

10.73 Using the method of Sec. 10.8, solve Prob. 10.39. Determine whether the equilibrium is stable, unstable, or neutral. (*Hint:* The potential energy corresponding to the couple exerted by a torsion spring is $\frac{1}{2}K$u^2, where K is the torsional spring constant and u is the angle of twist.)

10.74 In Prob. 10.40, determine whether each of the positions of equilibrium is stable, unstable, or neutral. (See hint for Prob. 10.73.)

10.75 A load **W** of magnitude 100 lb is applied to the mechanism at C. Knowing that the spring is unstretched when u = 15°, determine that value of u corresponding to equilibrium and check that the equilibrium is stable.

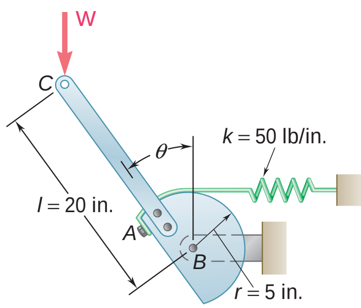


Fig. *P10.75* and *P10.76*

10.76 A load **W** of magnitude 100 lb is applied to the mechanism at C. Knowing that the spring is unstretched when u = 30°, determine that value of u corresponding to equilibrium and check that the equilibrium is stable.

10.77 A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. Knowing that the spring is unstretched when $y = 0$, determine the value of y corresponding to equilibrium when $W = 80$ N, $l = 500$ mm, and $k = 600$ N/m.

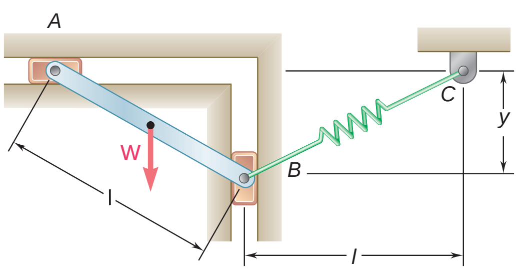


Fig. P10.77

10.78 A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. Knowing that both springs are unstretched when $y = 0$, determine the value of y corresponding to equilibrium when $W = 80$ N, $l = 500$ mm, and $k = 600$ N/m.

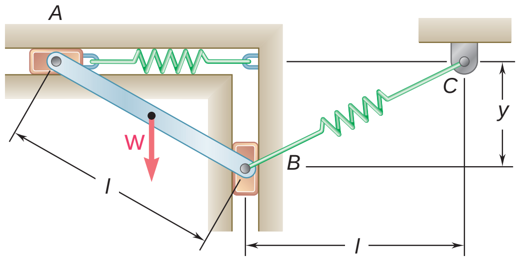


Fig. P10.78

10.79 A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. The constant of the spring is k, and the spring is unstretched when AB is horizontal. Neglecting the weight of the blocks, derive an equation in θ, W, l, and k that must be satisfied when the rod is in equilibrium.

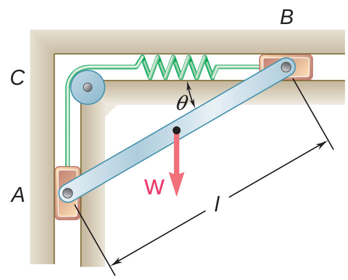


Fig. *P10.79* and P10.80

10.80 A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. Knowing that the spring is unstretched when AB is horizontal, determine three values of θ corresponding to equilibrium when $W = 300$ lb, $l = 16$ in., and $k = 75$ lb/in. State in each case whether the equilibrium is stable, unstable, or neutral.

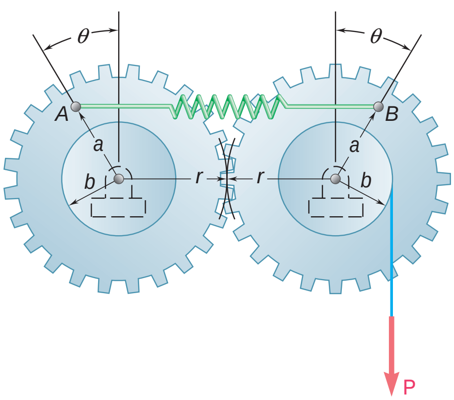


Fig. P10.81 and *P10.82*

10.81 A spring AB of constant k is attached to two identical gears as shown. Knowing that the spring is undeformed when $\theta = 0$, determine two values of the angle θ corresponding to equilibrium when $P = 30$ lb, $a = 4$ in., $b = 3$ in., $r = 6$ in., and $k = 5$ lb/in. State in each case whether the equilibrium is stable, unstable, or neutral.

10.82 A spring AB of constant k is attached to two identical gears as shown. Knowing that the spring is undeformed when $\theta = 0$, and given that $a = 60$ mm, $b = 45$ mm, $r = 90$ mm, and $k = 6$ kN/m, determine (*a*) the range of values of P for which a position of equilibrium exists, (*b*) two values of θ corresponding to equilibrium if the value of P is equal to half the upper limit of the range found in part *a*.

10.83 A slender rod AB is attached to two collars A and B that can move freely along the guide rods shown. Knowing that b = 30° and $P = Q = 400$ N, determine the value of the angle u corresponding to equilibrium.

10.84 A slender rod AB is attached to two collars A and B that can move freely along the guide rods shown. Knowing that b = 30°, $P =$ 100 N, and $Q = 25$ N, determine the value of the angle u corresponding to equilibrium.

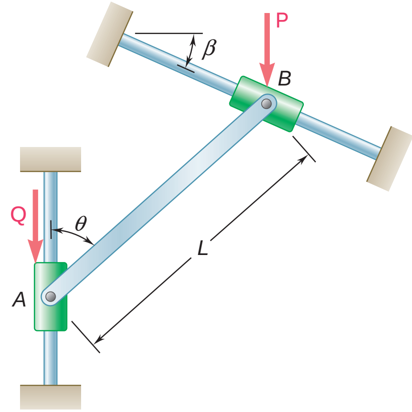


Fig. P10.83 and *P10.84*

***10.85* and 10.86** Cart B, which weighs 75 kN, rolls along a sloping track that forms an angle b with the horizontal. The spring constant is 5 kN/m, and the spring is unstretched when $x = 0$. Determine the distance x corresponding to equilibrium for the angle b indicated.

10.85 Angle b = 30°
10.86 Angle b = 60°

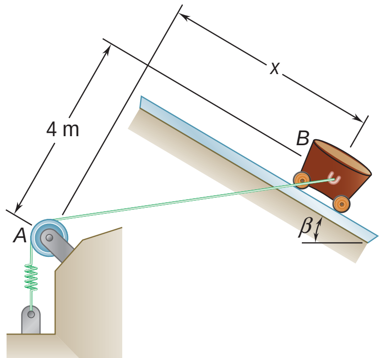


Fig. *P10.85* and P10.86

10.87 and 10.88 Collar A can slide freely on the semicircular rod shown. Knowing that the constant of the spring is k and that the unstretched length of the spring is equal to the radius r, determine the value of u corresponding to equilibrium when $W = 50$ lb, $r =$ 9 in., and $k = 15$ lb/in.

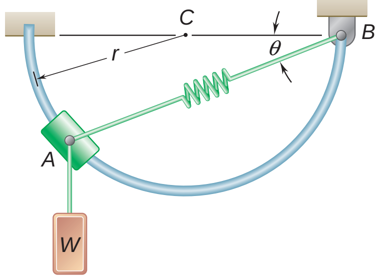


Fig. P10.87

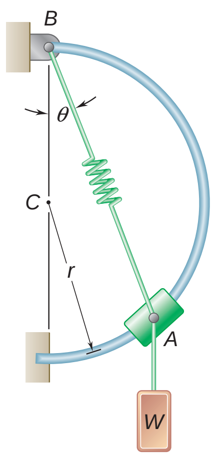


Fig. P10.88

10.89 Two bars AB and BC of negligible weight are attached to a single spring of constant k that is unstretched when the bars are horizontal. Determine the range of values of the magnitude P of two equal and opposite forces $\mathbf{P}$ and $-\mathbf{P}$ for which the equilibrium of the system is stable in the position shown.

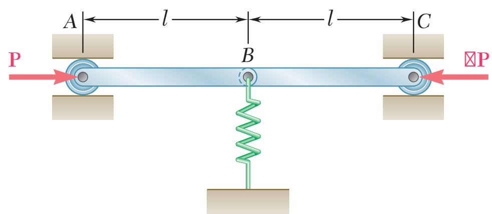


Fig. P10.89

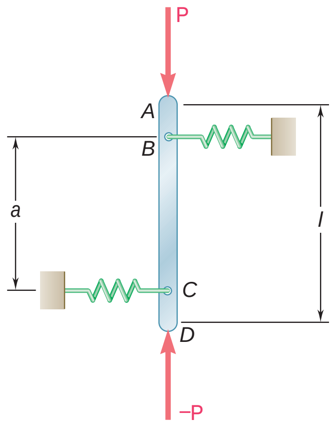


Fig. *P10.90*

10.90 A vertical bar AD is attached to two springs of constant k and is in equilibrium in the position shown. Determine the range of values of the magnitude P of two equal and opposite vertical forces $\mathbf{P}$ and $-\mathbf{P}$ for which the equilibrium position is stable if (*a*) $AB = CD$, (*b*) $AB = 2CD$.

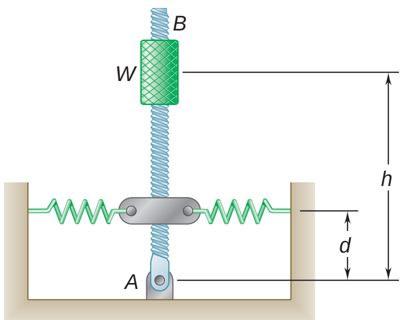


Fig. P10.91 and P10.92

10.91 Rod AB is attached to a hinge at A and to two springs, each of constant k. If $h = 25$ in., $d = 12$ in., and $W = 80$ lb, determine the range of values of k for which the equilibrium of the rod is stable in the position shown. Each spring can act in either tension or compression.

10.92 Rod AB is attached to a hinge at A and to two springs, each of constant k. If $h = 45$ in., $k = 6$ lb/in., and $W = 60$ lb, determine the smallest distance d for which the equilibrium of the rod is stable in the position shown. Each spring can act in either tension or compression.

10.93 and 10.94 Two bars are attached to a single spring of constant k that is unstretched when the bars are vertical. Determine the range of values of P for which the equilibrium of the system is stable in the position shown.

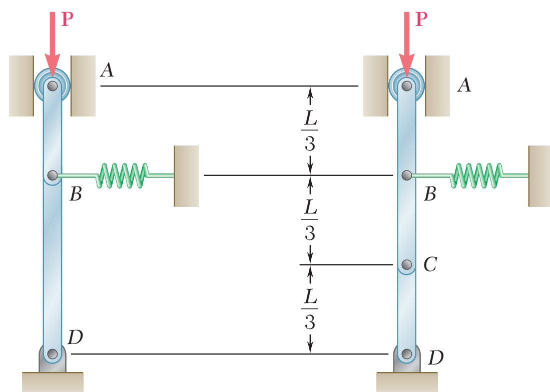


Fig. P10.93 **Fig. P10.94**

10.95 The horizontal bar *BEH* is connected to three vertical bars. The collar at *E* can slide freely on bar *DF*. Determine the range of values of *Q* for which the equilibrium of the system is stable in the position shown when $a = 24$ in., $b = 20$ in., and $P = 150$ lb.

10.96 The horizontal bar *BEH* is connected to three vertical bars. The collar at *E* can slide freely on bar *DF*. Determine the range of values of *P* for which the equilibrium of the system is stable in the position shown when $a = 150$ mm, $b = 200$ mm, and $Q = 45$ N.

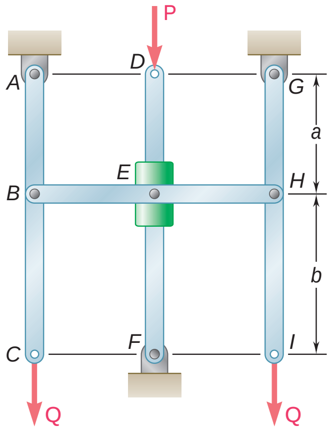


Fig. *P10.95* and P10.96

***10.97** Bars *AB* and *BC*, each of length l and of negligible weight, are attached to two springs, each of constant k. The springs are undeformed and the system is in equilibrium when $u_1 = u_2 = 0$. Determine the range of values of *P* for which the equilibrium position is stable.

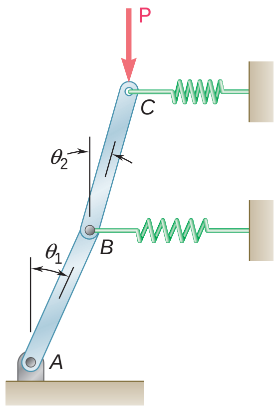


Fig. *P10.97*

***10.98** Solve Prob. 10.97 knowing that $l = 800$ mm and $k = 2.5$ kN/m.

***10.99** Two rods of negligible weight are attached to drums of radius r that are connected by a belt and spring of constant k. Knowing that the spring is undeformed when the rods are vertical, determine the range of values of *P* for which the equilibrium position $u_1 = u_2 = 0$ is stable.

***10.100** Solve Prob. 10.99 knowing that $k = 20$ lb/in., $r = 3$ in., $l = 6$ in., and (*a*) $W = 15$ lb, (*b*) $W = 60$ lb.

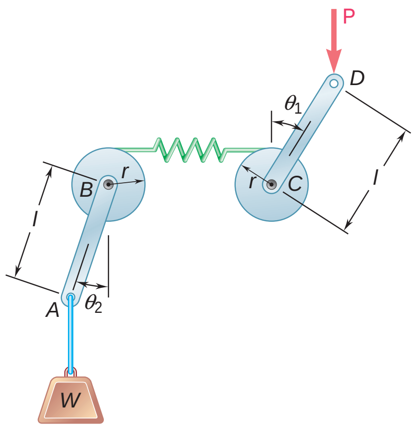


Fig. *P10.99*

REVIEW AND SUMMARY

Work of a force

The first part of this chapter was devoted to the *principle of virtual work* and to its direct application to the solution of equilibrium problems. We first defined the *work of a force* **F** *corresponding to the small displacement* $d\mathbf{r}$ [Sec. 10.2] as the quantity

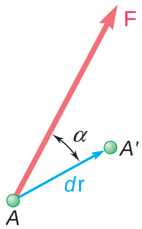


Fig. 10.16

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{10.1}$$

obtained by forming the scalar product of the force **F** and the displacement $d\mathbf{r}$ (Fig. 10.16). Denoting respectively by F and ds the magnitudes of the force and of the displacement, and by α the angle formed by **F** and $d\mathbf{r}$, we wrote

$$dU = F\,ds\cos\alpha \tag{10.1′}$$

The work dU is positive if $\alpha < 90°$, zero if $\alpha = 90°$, and negative if $\alpha > 90°$. We also found that the *work of a couple of moment* **M** acting on a rigid body is

$$dU = M\,d\theta \tag{10.2}$$

where $d\theta$ is the small angle expressed in radians through which the body rotates.

Virtual displacement

Considering a particle located at A and acted upon by several forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ [Sec. 10.3], we imagined that the particle moved to a new position A' (Fig. 10.17). Since this displacement did not actually take place, it was referred to as a *virtual displacement* and denoted by $\delta\mathbf{r}$, while the corresponding work of the forces was called *virtual work* and denoted by δU. We had

$$\delta U = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \cdots + \mathbf{F}_n \cdot \delta\mathbf{r}$$

Principle of virtual work

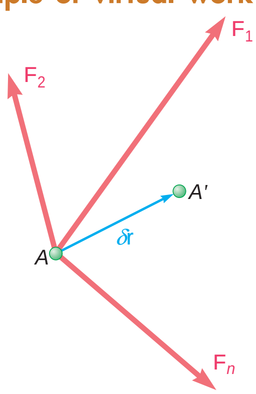


Fig. 10.17

The *principle of virtual work* states that *if a particle is in equilibrium, the total virtual work* δU *of the forces acting on the particle is zero for any virtual displacement of the particle.*

The principle of virtual work can be extended to the case of rigid bodies and systems of rigid bodies. Since it involves *only forces which do work,* its application provides a useful alternative to the use of the equilibrium equations in the solution of many engineering problems. It is particularly effective in the case of machines and mechanisms consisting of connected rigid bodies, since the work of the reactions at the supports is zero and the work of the internal forces at the pin connections cancels out [Sec. 10.4; Sample Probs. 10.1, 10.2, and 10.3].

In the case of *real machines*, however [Sec. 10.5], the work of the friction forces should be taken into account, with the result that the *output work will be less than the input work.* Defining the *mechanical efficiency* of a machine as the ratio

Mechanical efficiency

$$\eta = \frac{\text{output work}}{\text{input work}} \tag{10.9}$$

we also noted that for an ideal machine (no friction) $\eta = 1$, while for a real machine $\eta < 1$.

Work of a force over a finite displacement

In the second part of the chapter we considered the *work of forces corresponding to finite displacements* of their points of application. The work $U_{1\to 2}$ of the force $\mathbf{F}$ corresponding to a displacement of the particle A from A_1 to A_2 (Fig. 10.18) was obtained by integrating the right-hand member of Eq. (10.1) or (10.1′) along the curve described by the particle [Sec. 10.6]:

$$U_{1\to 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \tag{10.11}$$

or

$$U_{1\to 2} = \int_{s_1}^{s_2} (F \cos \alpha)\, ds \tag{10.11′}$$

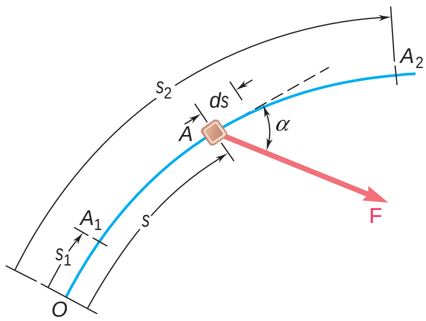


Fig. 10.18

Similarly, the work of a couple of moment $\mathbf{M}$ corresponding to a finite rotation from θ_1 to θ_2 of a rigid body was expressed as

$$U_{1\to 2} = \int_{\theta_1}^{\theta_2} M\, d\theta \tag{10.12}$$

Work of a weight

The *work of the weight* $\mathbf{W}$ *of a body* as its center of gravity moves from the elevation y_1 to y_2 (Fig. 10.19) can be obtained by making $F = W$ and $\alpha = 180°$ in Eq. (10.11′):

$$U_{1\to 2} = -\int_{y_1}^{y_2} W\, dy = Wy_1 - Wy_2 \tag{10.13}$$

The work of $\mathbf{W}$ is therefore positive *when the elevation y decreases.*

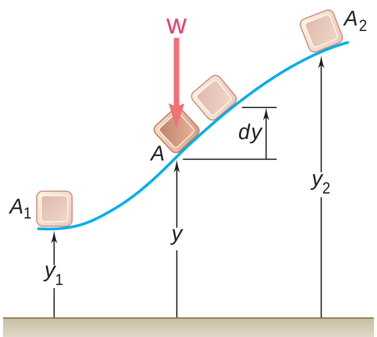


Fig. 10.19

Work of the force exerted by a spring

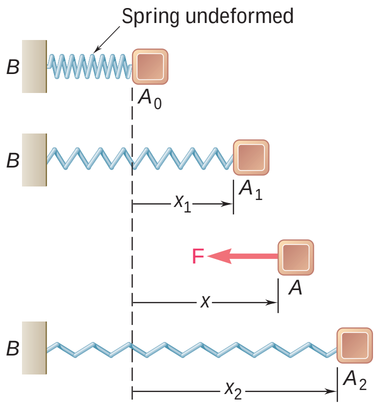


Fig. 10.20

The *work of the force* **F** *exerted by a spring* on a body A as the spring is stretched from x_1 to x_2 (Fig. 10.20) can be obtained by making $F = kx$, where k is the constant of the spring, and a $= 180°$ in Eq. (10.11′):

$$U_{1 \, y \, 2} = -\int_{x_1}^{x_2} kx \, dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \qquad (10.15)$$

The work of **F** is therefore positive *when the spring is returning to its undeformed position.*

Potential energy

When the work of a force **F** is independent of the path actually followed between A_1 and A_2, the force is said to be a *conservative force,* and its work can be expressed as

$$U_{1 \, y \, 2} = V_1 - V_2 \qquad (10.20)$$

where V is the *potential energy* associated with **F**, and V_1 and V_2 represent the values of V at A_1 and A_2, respectively [Sec. 10.7]. The potential energies associated, respectively, with the *force of gravity* **W** and the *elastic force* **F** exerted by a spring were found to be

$$V_g = Wy \qquad \text{and} \qquad V_e = \tfrac{1}{2}kx^2 \qquad (10.17, 10.18)$$

Alternative expression for the principle of virtual work

When the position of a mechanical system depends upon a single independent variable u, the potential energy of the system is a function V(u) of that variable, and it follows from Eq. (10.20) that dU = −dV = −(dV/du) du. The condition dU = 0 required by the principle of virtual work for the equilibrium of the system can thus be replaced by the condition

$$\frac{dV}{d\text{u}} = 0 \qquad (10.21)$$

When all the forces involved are conservative, it may be preferable to use Eq. (10.21) rather than apply the principle of virtual work directly [Sec. 10.8; Sample Prob. 10.4].

Stability of equilibrium

This approach presents another advantage, since it is possible to determine from the sign of the second derivative of V whether the equilibrium of the system is *stable, unstable,* or *neutral* [Sec. 10.9]. If $d^2V/d\text{u}^2 > 0$, V is *minimum* and the equilibrium is *stable;* if $d^2V/d\text{u}^2 < 0$, V is *maximum* and the equilibrium is *unstable;* if $d^2V/d\text{u}^2 = 0$, it is necessary to examine derivatives of a higher order.

REVIEW PROBLEMS

10.101 Determine the horizontal force **P** that must be applied at *A* to maintain the equilibrium of the linkage.

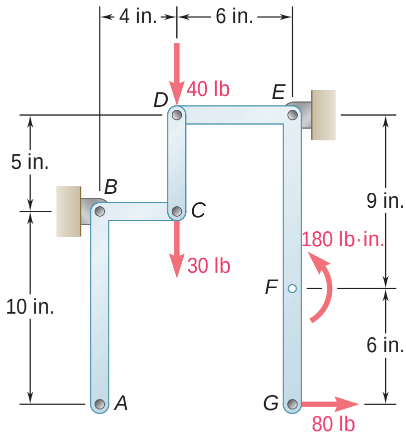


Fig. P10.101 and P10.102

10.102 Determine the couple **M** that must be applied to member *ABC* to maintain the equilibrium of the linkage.

10.103 A spring of constant 15 kN/m connects points *C* and *F* of the linkage shown. Neglecting the weight of the spring and linkage, determine the force in the spring and the vertical motion of point *G* when a vertical downward 120-N force is applied (*a*) at point *C*, (*b*) at points *C* and *H*.

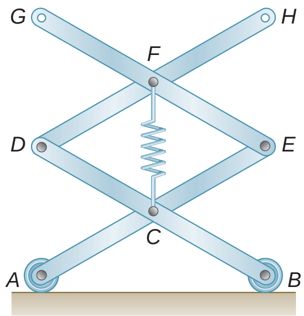


Fig. P10.103

10.104 Derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the mechanism shown.

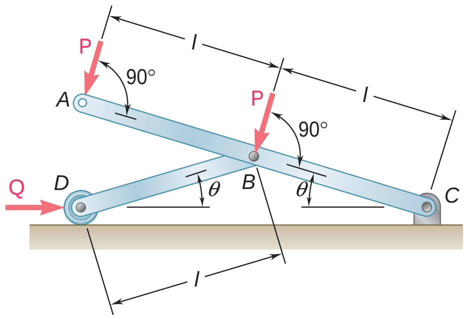


Fig. *P10.104*

10.105 Derive an expression for the magnitude of the couple **M** required to maintain the equilibrium of the linkage shown.

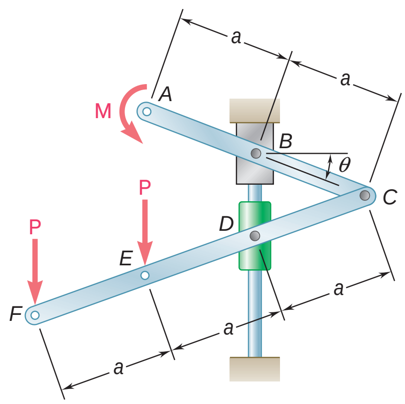


Fig. P10.105

10.106 Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is *k*, and the spring is unstretched when u = 30°. For the loading shown, derive an equation in *P*, u, *l*, and *k* that must be satisfied when the system is in equilibrium.

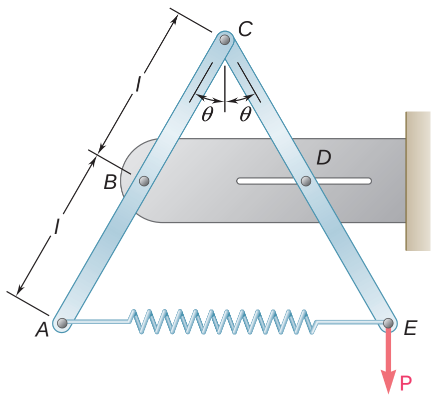


Fig. *P10.106*

10.107 A force **P** of magnitude 240 N is applied to end *E* of cable *CDE*, which passes under pulley *D* and is attached to the mechanism at *C*. Neglecting the weight of the mechanism and the radius of the pulley, determine the value of u corresponding to equilibrium. The constant of the spring is $k = 4$ kN/m, and the spring is unstretched when u = 90°.

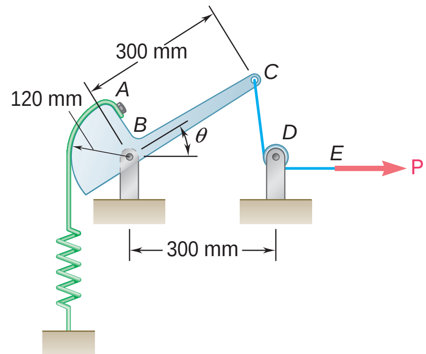


Fig. P10.107

10.108 Two identical rods *ABC* and *DBE* are connected by a pin at *B* and by a spring *CE*. Knowing that the spring is 4 in. long when unstretched and that the constant of the spring is 8 lb/in., determine the distance *x* corresponding to equilibrium when a 24-lb load is applied at *E* as shown.

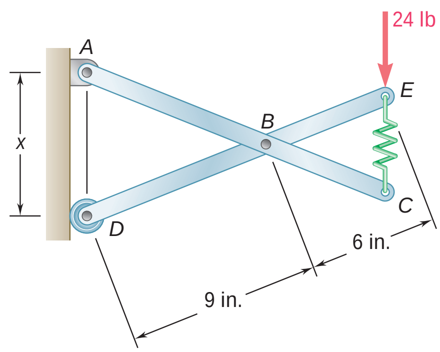


Fig. P10.108

10.109 Solve Prob. 10.108 assuming that the 24-lb load is applied at *C* instead of *E*.

10.110 Two bars *AB* and *BC* are attached to a single spring of constant *k* that is unstretched when the bars are vertical. Determine the range of values of *P* for which the equilibrium of the system is stable in the position shown.

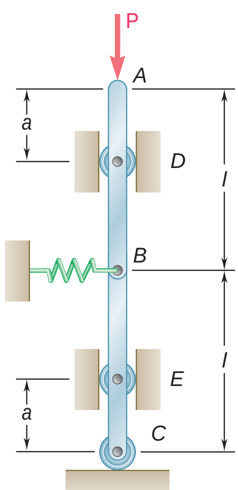


Fig. P10.110

10.111 A homogeneous hemisphere of radius r is placed on an incline as shown. Assuming that friction is sufficient to prevent slipping between the hemisphere and the incline, determine the angle u corresponding to equilibrium when b = 10°.

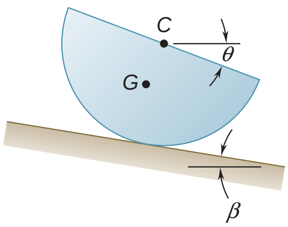


Fig. *P10.111* and P10.112

10.112 A homogeneous hemisphere of radius r is placed on an incline as shown. Assuming that friction is sufficient to prevent slipping between the hemisphere and the incline, determine (*a*) the largest angle b for which a position of equilibrium exists, (*b*) the angle u corresponding to equilibrium when the angle b is equal to half the value found in part *a*.

COMPUTER PROBLEMS

10.C1 A couple **M** is applied to crank AB in order to maintain the equilibrium of the engine system shown when a force **P** is applied to the piston. Knowing that $b = 2.4$ in. and $l = 7.5$ in., write a computer program that can be used to calculate the ratio M/P for values of u from 0 to 180° using 10° increments. Using appropriate smaller increments, determine the value of u for which the ratio M/P is maximum, and the corresponding value of M/P.

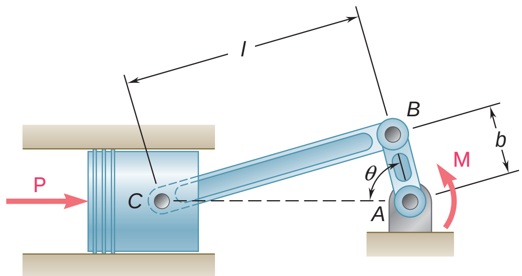


Fig. P10.C1

10.C2 Knowing that $a = 500$ mm, $b = 150$ mm, $L = 500$ mm, and $P = 100$ N, write a computer program that can be used to calculate the force in member BD for values of u from 30° to 150° using 10° increments. Using appropriate smaller increments, determine the range of values of u for which the absolute value of the force in member BD is less than 400 N.

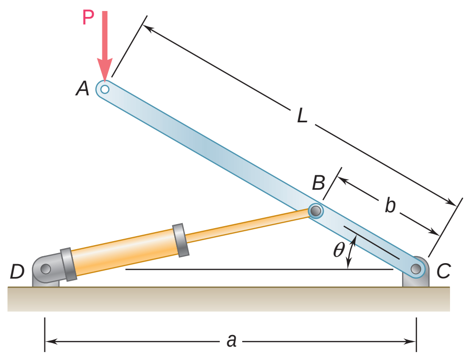


Fig. P10.C2

10.C3 Solve Prob. 10.C2 assuming that the force **P** applied at A is directed horizontally to the right.

10.C4 The constant of spring AB is k, and the spring is unstretched when u = 0. (*a*) Neglecting the weight of the member BCD, write a computer program that can be used to calculate the potential energy of the system and its derivative dV/du. (*b*) For $W = 150$ lb, $a = 10$ in., and $k = 75$ lb/in., calculate and plot the potential energy versus u for values of u from 0 to 165° using 15° increments. (*c*) Using appropriate smaller increments, determine the values of u for which the system is in equilibrium and state in each case whether the equilibrium is stable, unstable, or neutral.

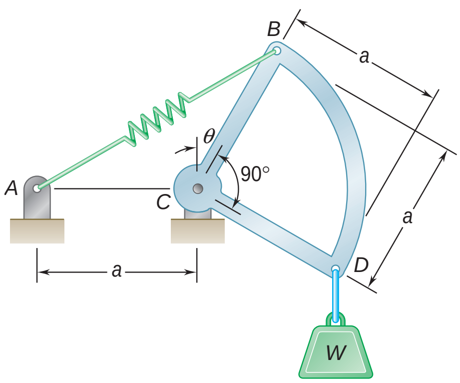


Fig. P10.C4

10.C5 Two rods, AC and DE, each of length L, are connected by a collar that is attached to rod AC at its midpoint B. (a) Write a computer program that can be used to calculate the potential energy V of the system and its derivative dV/du. (b) For $W = 75$ N, $P = 200$ N, and $L = 500$ mm, calculate V and dV/du for values of u from 0 to 70° using 5° increments. (c) Using appropriate smaller increments, determine the values of u for which the system is in equilibrium and state in each case whether the equilibrium is stable, unstable, or neutral.

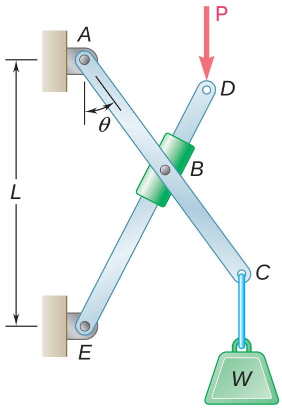


Fig. P10.C5

10.C6 A slender rod ABC is attached to blocks A and B that can move freely in the guides shown. The constant of the spring is k, and the spring is unstretched when the rod is vertical. (a) Neglecting the weights of the rod and of the blocks, write a computer program that can be used to calculate the potential energy V of the system and its derivative dV/du. (b) For $P = 150$ N, $l = 200$ mm, and $k = 3$ kN/m, calculate and plot the potential energy versus u for values of u from 0 to 75° using 5° increments. (c) Using appropriate smaller increments, determine any positions of equilibrium in the range $0 \leq$ u $\leq 75°$ and state in each case whether the equilibrium is stable, unstable, or neutral.

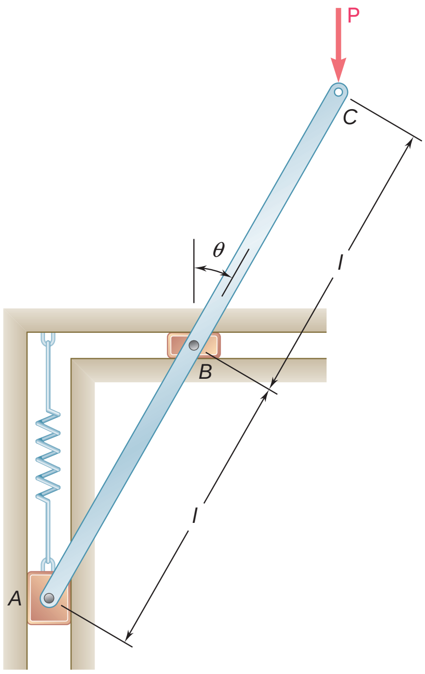


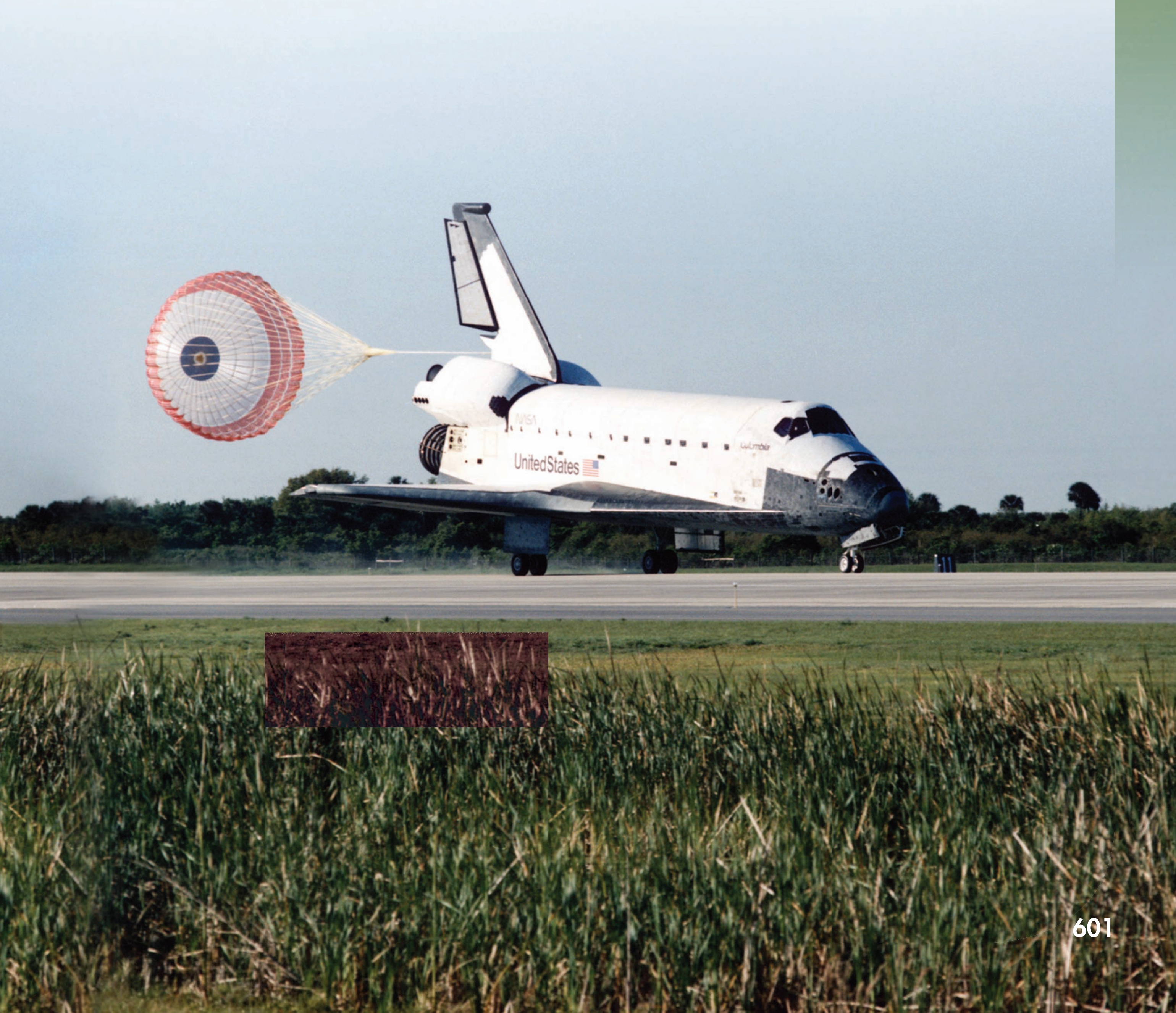
Fig. P10.C6

10.C7 Solve Prob. 10.C6 assuming that the force **P** applied at C is directed horizontally to the right.

The motion of the space shuttle can be described in terms of its *position, velocity,* and *acceleration.* When landing, the pilot of the shuttle needs to consider the wind velocity and the *relative motion* of the shuttle with respect to the wind. The study of motion is known as *kinematics* and is the subject of this chapter.

CHAPTER 11

Kinematics of Particles



Chapter 11 Kinematics of Particles

11.1 Introduction to Dynamics
11.2 Position, Velocity, and Acceleration
11.3 Determination of the Motion of a Particle
11.4 Uniform Rectilinear Motion
11.5 Uniformly Accelerated Rectilinear Motion
11.6 Motion of Several Particles
11.7 Graphical Solution of Rectilinear-Motion Problems
11.8 Other Graphical Methods
11.9 Position Vector, Velocity, and Acceleration
11.10 Derivatives of Vector Functions
11.11 Rectangular Components of Velocity and Acceleration
11.12 Motion Relative to a Frame in Translation
11.13 Tangential and Normal Components
11.14 Radial and Transverse Components

11.1 INTRODUCTION TO DYNAMICS

Chapters 1 to 10 were devoted to *statics,* i.e., to the analysis of bodies at rest. We now begin the study of *dynamics,* the part of mechanics that deals with the analysis of bodies in motion.

While the study of statics goes back to the time of the Greek philosophers, the first significant contribution to dynamics was made by Galileo (1564–1642). Galileo's experiments on uniformly accelerated bodies led Newton (1642–1727) to formulate his fundamental laws of motion.

Dynamics includes:

1. *Kinematics,* which is the study of the geometry of motion. Kinematics is used to relate displacement, velocity, acceleration, and time, without reference to the cause of the motion.
2. *Kinetics,* which is the study of the relation existing between the forces acting on a body, the mass of the body, and the motion of the body. Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

Chapters 11 to 14 are devoted to the *dynamics of particles;* in Chap. 11 the *kinematics of particles* will be considered. The use of the word *particles* does not mean that our study will be restricted to small corpuscles; rather, it indicates that in these first chapters the motion of bodies—possibly as large as cars, rockets, or airplanes—will be considered without regard to their size. By saying that the bodies are analyzed as particles, we mean that only their motion as an entire unit will be considered; any rotation about their own mass center will be neglected. There are cases, however, when such a rotation is not negligible; the bodies cannot then be considered as particles. Such motions will be analyzed in later chapters, dealing with the *dynamics of rigid bodies.*

In the first part of Chap. 11, the rectilinear motion of a particle will be analyzed; that is, the position, velocity, and acceleration of a particle will be determined at every instant as it moves along a straight line. First, general methods of analysis will be used to study the motion of a particle; then two important particular cases will be considered, namely, the uniform motion and the uniformly accelerated motion of a particle (Secs. 11.4 and 11.5). In Sec. 11.6 the simultaneous motion of several particles will be considered, and the concept of the relative motion of one particle with respect to another will be introduced. The first part of this chapter concludes with a study of graphical methods of analysis and their application to the solution of various problems involving the rectilinear motion of particles (Secs. 11.7 and 11.8).

In the second part of this chapter, the motion of a particle as it moves along a curved path will be analyzed. Since the position, velocity, and acceleration of a particle will be defined as vector quantities, the concept of the derivative of a vector function will be introduced in Sec. 11.10 and added to our mathematical tools. Applications in which the motion of a particle is defined by the rectangular components of its velocity and acceleration will then be considered; at this point, the motion of a projectile will be analyzed (Sec. 11.11). In Sec. 11.12, the motion of a particle relative to a reference frame in translation will be considered. Finally, the curvilinear motion of a particle will be analyzed in terms of components other than rectangular. The tangential and normal components of a particular velocity and an acceleration will be introduced in Sec. 11.13 and the radial and transverse components of its velocity and acceleration in Sec. 11.14.

RECTILINEAR MOTION OF PARTICLES

11.2 POSITION, VELOCITY, AND ACCELERATION

A particle moving along a straight line is said to be in *rectilinear motion*. At any given instant t, the particle will occupy a certain position on the straight line. To define the position P of the particle, we choose a fixed origin O on the straight line and a positive direction along the line. We measure the distance x from O to P and record it with a plus or minus sign, according to whether P is reached from O by moving along the line in the positive or the negative direction. The distance x, with the appropriate sign, completely defines the position of the particle; it is called the *position coordinate* of the particle considered. For example, the position coordinate corresponding to P in Fig. 11.1*a* is $x = +5$ m; the coordinate corresponding to P' in Fig. 11.1*b* is $x' = -2$ m.

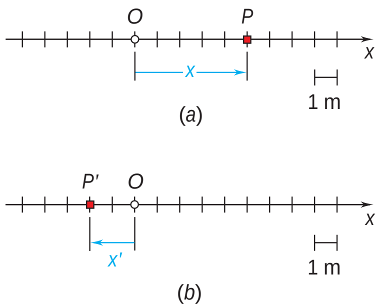


Fig. 11.1

When the position coordinate x of a particle is known for every value of time t, we say that the motion of the particle is known. The "timetable" of the motion can be given in the form of an equation in x and t, such as $x = 6t^2 - t^3$, or in the form of a graph of x versus t as shown in Fig. 11.6. The units most often used to measure the position coordinate x are the meter (m) in the SI system of units† and the foot (ft) in the U.S. customary system of units. Time t is usually measured in seconds (s).

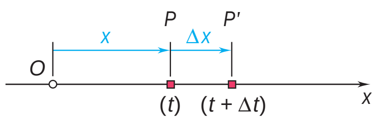


Fig. 11.2

Consider the position P occupied by the particle at time t and the corresponding coordinate x (Fig. 11.2). Consider also the position P' occupied by the particle at a later time $t + \Delta t$; the position coordinate of P' can be obtained by adding to the coordinate x of P the small displacement Δx, which will be positive or negative according to whether P' is to the right or to the left of P. The *average velocity* of the particle over the time interval Δt is defined as the quotient of the displacement Δx and the time interval Δt:

$$\text{Average velocity} = \frac{\Delta x}{\Delta t}$$

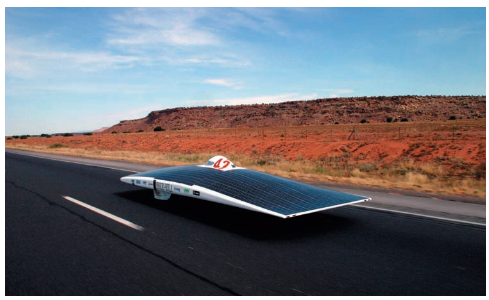


Photo 11.1 The motion of this solar car can be described by its position, velocity, and acceleration.

†Cf. Sec. 1.3.

If SI units are used, Δx is expressed in meters and Δt in seconds; the average velocity will thus be expressed in meters per second (m/s). If U.S. customary units are used, Δx is expressed in feet and Δt in seconds; the average velocity will then be expressed in feet per second (ft/s).

The *instantaneous velocity* v of the particle at the instant t is obtained from the average velocity by choosing shorter and shorter time intervals Δt and displacements Δx:

$$\text{Instantaneous velocity} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

The instantaneous velocity will also be expressed in m/s or ft/s. Observing that the limit of the quotient is equal, by definition, to the derivative of x with respect to t, we write

$$\boxed{v = \frac{dx}{dt}} \tag{11.1}$$

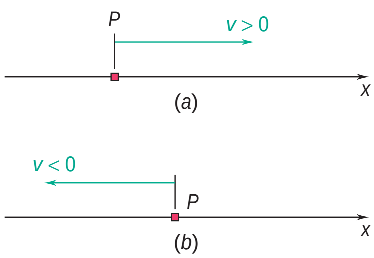


Fig. 11.3

The velocity v is represented by an algebraic number which can be positive or negative.† A positive value of v indicates that x increases, i.e., that the particle moves in the positive direction (Fig. 11.3*a*); a negative value of v indicates that x decreases, i.e., that the particle moves in the negative direction (Fig. 11.3*b*). The magnitude of v is known as the *speed* of the particle.

Consider the velocity v of the particle at time t and also its velocity $v + \Delta v$ at a later time $t + \Delta t$ (Fig. 11.4). The *average acceleration* of the particle over the time interval Δt is defined as the quotient of Δv and Δt:

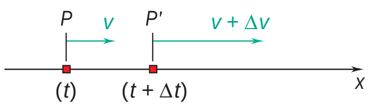


Fig. 11.4

$$\text{Average acceleration} = \frac{\Delta v}{\Delta t}$$

If SI units are used, Δv is expressed in m/s and Δt in seconds; the average acceleration will thus be expressed in m/s^2. If U.S. customary units are used, Δv is expressed in ft/s and Δt in seconds; the average acceleration will then be expressed in ft/s^2.

The *instantaneous acceleration* a of the particle at the instant t is obtained from the average acceleration by choosing smaller and smaller values for Δt and Δv:

$$\text{Instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

The instantaneous acceleration will also be expressed in m/s^2 or ft/s^2. The limit of the quotient, which is by definition the derivative of v

†As you will see in Sec. 11.9, the velocity is actually a vector quantity. However, since we are considering here the rectilinear motion of a particle, where the velocity of the particle has a known and fixed direction, we need only specify the sense and magnitude of the velocity; this can be conveniently done by using a scalar quantity with a plus or minus sign. The same is true of the acceleration of a particle in rectilinear motion.

with respect to t, measures the rate of change of the velocity. We write

$$a = \frac{dv}{dt} \tag{11.2}$$

or, substituting for v from (11.1),

$$a = \frac{d^2x}{dt^2} \tag{11.3}$$

The acceleration a is represented by an algebraic number which can be positive or negative.† A positive value of a indicates that the velocity (i.e., the algebraic number v) increases. This may mean that the particle is moving faster in the positive direction (Fig. 11.5*a*) or that it is moving more slowly in the negative direction (Fig. 11.5*b*); in both cases, Δv is positive. A negative value of a indicates that the velocity decreases; either the particle is moving more slowly in the positive direction (Fig. 11.5*c*) or it is moving faster in the negative direction (Fig. 11.5*d*).

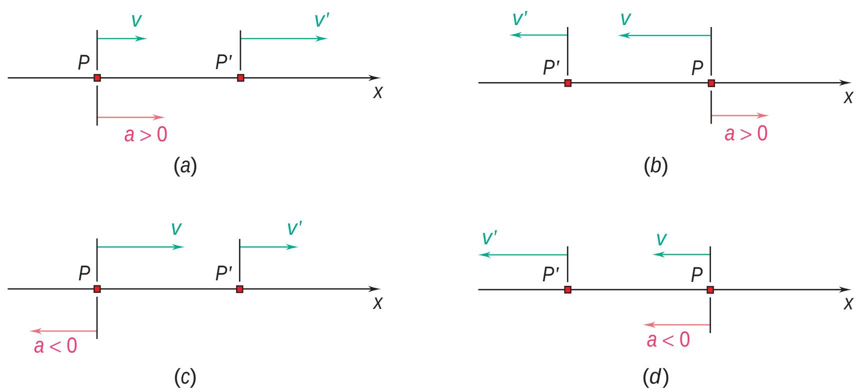


Fig. 11.5

The term *deceleration* is sometimes used to refer to a when the speed of the particle (i.e., the magnitude of v) decreases; the particle is then moving more slowly. For example, the particle of Fig. 11.5 is decelerated in parts b and c; it is truly accelerated (i.e., moves faster) in parts a and d.

Another expression for the acceleration can be obtained by eliminating the differential dt in Eqs. (11.1) and (11.2). Solving (11.1) for dt, we obtain $dt = dx/v$; substituting into (11.2), we write

$$a = v\frac{dv}{dx} \tag{11.4}$$

†See footnote, page 604.

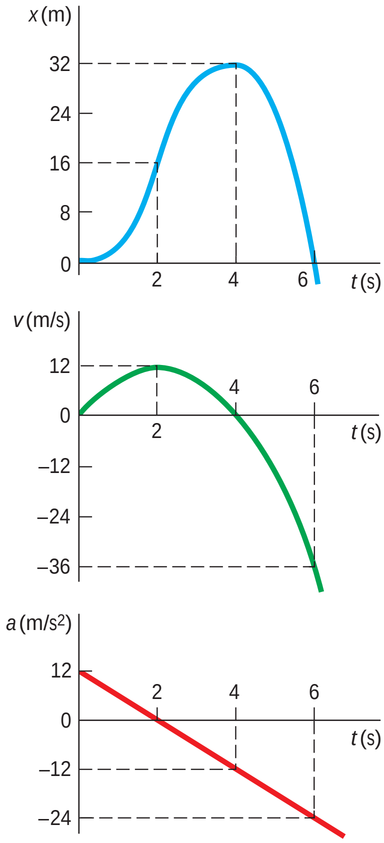


Fig. 11.6

EXAMPLE Consider a particle moving in a straight line, and assume that its position is defined by the equation

$$x = 6t^2 - t^3$$

where t is expressed in seconds and x in meters. The velocity v at any time t is obtained by differentiating x with respect to t:

$$v = \frac{dx}{dt} = 12t - 3t^2$$

The acceleration a is obtained by differentiating again with respect to t:

$$a = \frac{dv}{dt} = 12 - 6t$$

The position coordinate, the velocity, and the acceleration have been plotted against t in Fig. 11.6. The curves obtained are known as *motion curves*. Keep in mind, however, that the particle does not move along any of these curves; the particle moves in a straight line. Since the derivative of a function measures the slope of the corresponding curve, the slope of the x–t curve at any given time is equal to the value of v at that time and the slope of the v–t curve is equal to the value of a. Since $a = 0$ at $t = 2$ s, the slope of the v–t curve must be zero at $t = 2$ s; the velocity reaches a maximum at this instant. Also, since $v = 0$ at $t = 0$ and at $t = 4$ s, the tangent to the x–t curve must be horizontal for both of these values of t.

A study of the three motion curves of Fig. 11.6 shows that the motion of the particle from $t = 0$ to $t = \infty$ can be divided into four phases:

1. The particle starts from the origin, $x = 0$, with no velocity but with a positive acceleration. Under this acceleration, the particle gains a positive velocity and moves in the positive direction. From $t = 0$ to $t = 2$ s, x, v, and a are all positive.
2. At $t = 2$ s, the acceleration is zero; the velocity has reached its maximum value. From $t = 2$ s to $t = 4$ s, v is positive, but a is negative; the particle still moves in the positive direction but more and more slowly; the particle is decelerating.
3. At $t = 4$ s, the velocity is zero; the position coordinate x has reached its maximum value. From then on, both v and a are negative; the particle is accelerating and moves in the negative direction with increasing speed.
4. At $t = 6$ s, the particle passes through the origin; its coordinate x is then zero, while the total distance traveled since the beginning of the motion is 64 m. For values of t larger than 6 s, x, v, and a will all be negative. The particle keeps moving in the negative direction, away from O, faster and faster. ■

11.3 DETERMINATION OF THE MOTION OF A PARTICLE

We saw in the preceding section that the motion of a particle is said to be known if the position of the particle is known for every value of the time t. In practice, however, a motion is seldom defined by a relation between x and t. More often, the conditions of the motion will be specified by the type of acceleration that the particle possesses. For example, a freely falling body will have a constant acceleration, directed downward and equal to 9.81 m/s^2, or 32.2 ft/s^2; a mass attached to a spring which has been stretched will have an acceleration proportional to the instantaneous elongation of the spring measured from the equilibrium position, etc. In general, the acceleration of the particle can be expressed as a function of one or more of the variables x, v, and t. In order to determine the position coordinate x in terms of t, it will thus be necessary to perform two successive integrations.

Let us consider three common classes of motion:

1. $a = f(t)$. *The Acceleration Is a Given Function of* t. Solving (11.2) for dv and substituting $f(t)$ for a, we write

$$dv = a\,dt$$
$$dv = f(t)\,dt$$

Integrating both members, we obtain the equation

$$\int dv = \int f(t)\,dt$$

which defines v in terms of t. It should be noted, however, that an arbitrary constant will be introduced as a result of the integration. This is due to the fact that there are many motions which correspond to the given acceleration $a = f(t)$. In order to uniquely define the motion of the particle, it is necessary to specify the *initial conditions* of the motion, i.e., the value v_0 of the velocity and the value x_0 of the position coordinate at $t = 0$. Replacing the indefinite integrals by *definite integrals* with lower limits corresponding to the initial conditions $t = 0$ and $v = v_0$ and upper limits corresponding to $t = t$ and $v = v$, we write

$$\int_{v_0}^{v} dv = \int_0^t f(t)\,dt$$

$$v - v_0 = \int_0^t f(t)\,dt$$

which yields v in terms of t.

Equation (11.1) can now be solved for dx,

$$dx = v\,dt$$

and the expression just obtained substituted for v. Both members are then integrated, the left-hand member with respect to x from $x = x_0$ to $x = x$, and the right-hand member with respect to t from $t = 0$ to $t = t$. The position coordinate x is thus obtained in terms of t; the motion is completely determined.

Two important particular cases will be studied in greater detail in Secs. 11.4 and 11.5: the case when $a = 0$, corresponding to a *uniform motion*, and the case when $a =$ constant, corresponding to a *uniformly accelerated motion*.

2. $a = f(x)$. *The Acceleration Is a Given Function of* x. Rearranging Eq. (11.4) and substituting $f(x)$ for a, we write

$$v\,dv = a\,dx$$
$$v\,dv = f(x)\,dx$$

Since each member contains only one variable, we can integrate the equation. Denoting again by v_0 and x_0, respectively, the initial values of the velocity and of the position coordinate, we obtain

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} f(x)\,dx$$
$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \int_{x_0}^{x} f(x)\,dx$$

which yields v in terms of x. We now solve (11.1) for dt,

$$dt = \frac{dx}{v}$$

and substitute for v the expression just obtained. Both members can then be integrated to obtain the desired relation between x and t. However, in most cases this last integration cannot be performed analytically and one must resort to a numerical method of integration.

3. $a = f(v)$. *The Acceleration Is a Given Function of* v. We can now substitute $f(v)$ for a in either (11.2) or (11.4) to obtain either of the following relations:

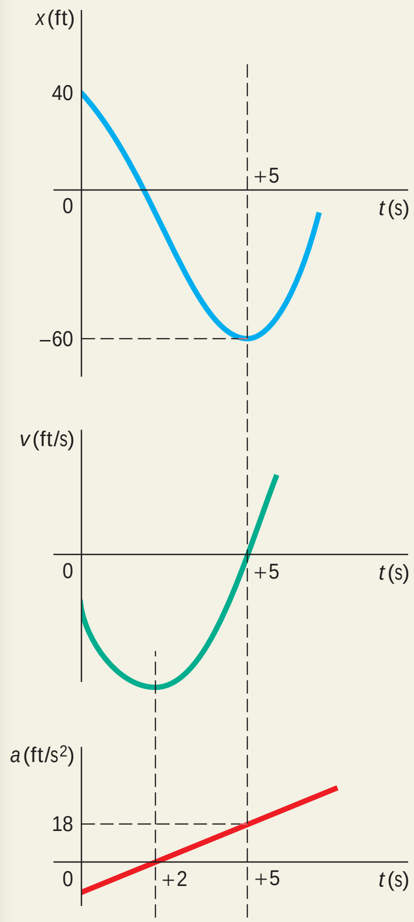
$$f(v) = \frac{dv}{dt} \qquad f(v) = v\frac{dv}{dx}$$
$$dt = \frac{dv}{f(v)} \qquad dx = \frac{v\,dv}{f(v)}$$

Integration of the first equation will yield a relation between v and t; integration of the second equation will yield a relation between v and x. Either of these relations can be used in conjunction with Eq. (11.1) to obtain the relation between x and t which characterizes the motion of the particle.

SAMPLE PROBLEM 11.1

The position of a particle which moves along a straight line is defined by the relation $x = t^3 - 6t^2 - 15t + 40$, where x is expressed in feet and t in seconds. Determine (*a*) the time at which the velocity will be zero, (*b*) the position and distance traveled by the particle at that time, (*c*) the acceleration of the particle at that time, (*d*) the distance traveled by the particle from $t = 4$ s to $t = 6$ s.

SOLUTION



The equations of motion are

$$x = t^3 - 6t^2 - 15t + 40 \tag{1}$$

$$v = \frac{dx}{dt} = 3t^2 - 12t - 15 \tag{2}$$

$$a = \frac{dv}{dt} = 6t - 12 \tag{3}$$

a. Time at Which $v = 0$. We set $v = 0$ in (2):

$$3t^2 - 12t - 15 = 0 \qquad t = -1 \text{ s} \qquad \text{and} \qquad t = +5 \text{ s} \quad ◀$$

Only the root $t = +5$ s corresponds to a time after the motion has begun: for $t < 5$ s, $v < 0$, the particle moves in the negative direction; for $t > 5$ s, $v > 0$, the particle moves in the positive direction.

b. Position and Distance Traveled When $v = 0$. Carrying $t = +5$ s into (1), we have

$$x_5 = (5)^3 - 6(5)^2 - 15(5) + 40 \qquad x_5 = -60 \text{ ft} \quad ◀$$

The initial position at $t = 0$ was $x_0 = +40$ ft. Since $v \neq 0$ during the interval $t = 0$ to $t = 5$ s, we have

$$\text{Distance traveled} = x_5 - x_0 = -60 \text{ ft} - 40 \text{ ft} = -100 \text{ ft}$$

Distance traveled = 100 ft in the negative direction ◀

c. Acceleration When $v = 0$. We substitute $t = +5$ s into (3):

$$a_5 = 6(5) - 12 \qquad a_5 = +18 \text{ ft/s}^2 \quad ◀$$

d. Distance Traveled from $t = 4$ s to $t = 6$ s. The particle moves in the negative direction from $t = 4$ s to $t = 5$ s and in the positive direction from $t = 5$ s to $t = 6$ s; therefore, the distance traveled during each of these time intervals will be computed separately.

From $t = 4$ s to $t = 5$ s: $\quad x_5 = -60$ ft

$$x_4 = (4)^3 - 6(4)^2 - 15(4) + 40 = -52 \text{ ft}$$

$$\begin{aligned}\text{Distance traveled} &= x_5 - x_4 = -60 \text{ ft} - (-52 \text{ ft}) = -8 \text{ ft}\\ &= 8 \text{ ft in the negative direction}\end{aligned}$$

From $t = 5$ s to $t = 6$ s: $\quad x_5 = -60$ ft

$$x_6 = (6)^3 - 6(6)^2 - 15(6) + 40 = -50 \text{ ft}$$

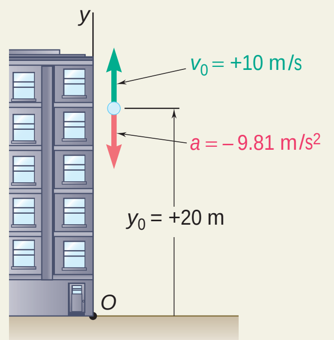
$$\begin{aligned}\text{Distance traveled} &= x_6 - x_5 = -50 \text{ ft} - (-60 \text{ ft}) = +10 \text{ ft}\\ &= 10 \text{ ft in the positive direction}\end{aligned}$$

Total distance traveled from $t = 4$ s to $t = 6$ s is 8 ft + 10 ft $= 18$ ft ◀

SAMPLE PROBLEM 11.2

A ball is tossed with a velocity of 10 m/s directed vertically upward from a window located 20 m above the ground. Knowing that the acceleration of the ball is constant and equal to 9.81 m/s^2 downward, determine (*a*) the velocity v and elevation y of the ball above the ground at any time t, (*b*) the highest elevation reached by the ball and the corresponding value of t, (*c*) the time when the ball will hit the ground and the corresponding velocity. Draw the v–t and y–t curves.

SOLUTION



a. Velocity and Elevation. The y axis measuring the position coordinate (or elevation) is chosen with its origin O on the ground and its positive sense upward. The value of the acceleration and the initial values of v and y are as indicated. Substituting for a in $a = dv/dt$ and noting that at $t = 0$, $v_0 = +10$ m/s, we have

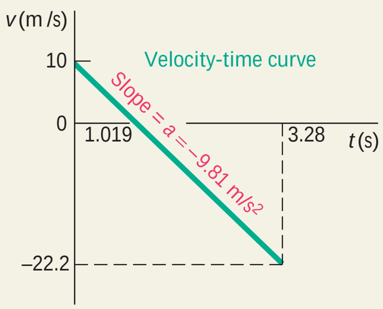
$$\frac{dv}{dt} = a = -9.81\,\text{m/s}^2$$

$$\int_{v_0=10}^{v} dv = -\int_0^t 9.81\,dt$$

$$[v]_{10}^{v} = -[9.81t]_0^t$$

$$v - 10 = -9.81t$$

$$v = 10 - 9.81t \quad (1) \quad ◀$$



Substituting for v in $v = dy/dt$ and noting that at $t = 0$, $y_0 = 20$ m, we have

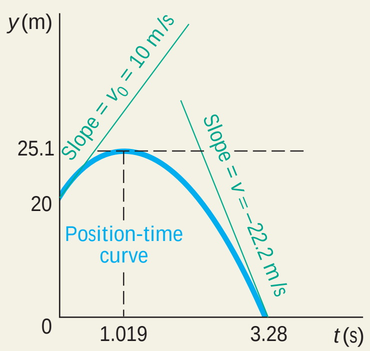
$$\frac{dy}{dt} = v = 10 - 9.81t$$

$$\int_{y_0=20}^{y} dy = \int_0^t (10 - 9.81t)\,dt$$

$$[y]_{20}^{y} = [10t - 4.905t^2]_0^t$$

$$y - 20 = 10t - 4.905t^2$$

$$y = 20 + 10t - 4.905t^2 \quad (2) \quad ◀$$



b. Highest Elevation. When the ball reaches its highest elevation, we have $v = 0$. Substituting into (1), we obtain

$$10 - 9.81t = 0 \qquad t = 1.019\text{ s} \quad ◀$$

Carrying $t = 1.019$ s into (2), we have

$$y = 20 + 10(1.019) - 4.905(1.019)^2 \qquad y = 25.1\text{ m} \quad ◀$$

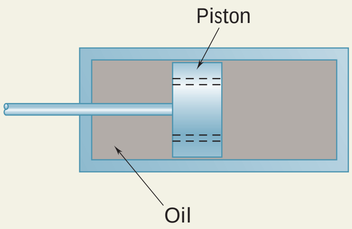
c. Ball Hits the Ground. When the ball hits the ground, we have $y = 0$. Substituting into (2), we obtain

$$20 + 10t - 4.905t^2 = 0 \qquad t = -1.243\text{ s} \quad \text{and} \quad t = +3.28\text{ s} \quad ◀$$

Only the root $t = +3.28$ s corresponds to a time after the motion has begun. Carrying this value of t into (1), we have

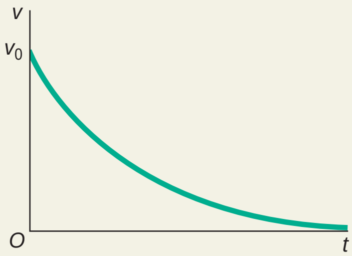
$$v = 10 - 9.81(3.28) = -22.2\text{ m/s} \qquad v = 22.2\text{ m/s w} \quad ◀$$

SAMPLE PROBLEM 11.3



The brake mechanism used to reduce recoil in certain types of guns consists essentially of a piston attached to the barrel and moving in a fixed cylinder filled with oil. As the barrel recoils with an initial velocity v_0, the piston moves and oil is forced through orifices in the piston, causing the piston and the barrel to decelerate at a rate proportional to their velocity; that is, $a = -kv$. Express (*a*) v in terms of t, (*b*) x in terms of t, (*c*) v in terms of x. Draw the corresponding motion curves.

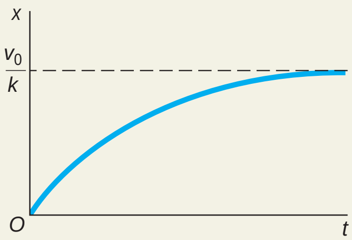
SOLUTION



a. *v* in Terms of *t*. Substituting $-kv$ for a in the fundamental formula defining acceleration, $a = dv/dt$, we write

$$-kv = \frac{dv}{dt} \qquad \frac{dv}{v} = -k\,dt \qquad \int_{v_0}^{v} \frac{dv}{v} = -k\int_0^t dt$$

$$\ln\frac{v}{v_0} = -kt \qquad\qquad v = v_0 e^{-kt} \quad ◀$$



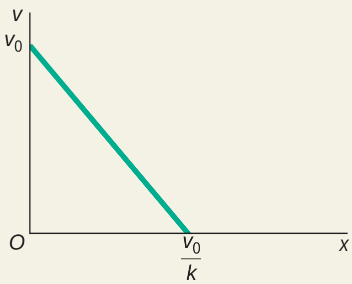
b. *x* in Terms of *t*. Substituting the expression just obtained for v into $v = dx/dt$, we write

$$v_0 e^{-kt} = \frac{dx}{dt}$$

$$\int_0^x dx = v_0\int_0^t e^{-kt}\,dt$$

$$x = -\frac{v_0}{k}[e^{-kt}]_0^t = -\frac{v_0}{k}(e^{-kt} - 1)$$

$$x = \frac{v_0}{k}(1 - e^{-kt}) \quad ◀$$



c. *v* in Terms of *x*. Substituting $-kv$ for a in $a = v\,dv/dx$, we write

$$-kv = v\frac{dv}{dx}$$

$$dv = -k\,dx$$

$$\int_{v_0}^{v} dv = -k\int_0^x dx$$

$$v - v_0 = -kx \qquad\qquad v = v_0 - kx \quad ◀$$

Check. Part *c* could have been solved by eliminating t from the answers obtained for parts *a* and *b*. This alternative method can be used as a check. From part *a* we obtain $e^{-kt} = v/v_0$; substituting into the answer of part *b*, we obtain

$$x = \frac{v_0}{k}(1 - e^{-kt}) = \frac{v_0}{k}\left(1 - \frac{v}{v_0}\right) \qquad v = v_0 - kx \qquad \text{(checks)}$$

SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will be asked to determine the *position,* the *velocity,* or the *acceleration* of a particle in *rectilinear motion*. As you read each problem, it is important that you identify both the independent variable (typically t or x) and what is required (for example, the need to express v as a function of x). You may find it helpful to start each problem by writing down both the given information and a simple statement of what is to be determined.

1. Determining $v(t)$ and $a(t)$ for a given $x(t)$. As explained in Sec. 11.2, the first and the second derivatives of x with respect to t are respectively equal to the velocity and the acceleration of the particle [Eqs. (11.1) and (11.2)]. If the velocity and the acceleration have opposite signs, the particle can come to rest and then move in the opposite direction [Sample Prob. 11.1]. Thus, when computing the total distance traveled by a particle, you should first determine if the particle will come to rest during the specified interval of time. Constructing a diagram similar to that of Sample Prob. 11.1 that shows the position and the velocity of the particle at each critical instant ($v = v_{max}$, $v = 0$, etc.) will help you to visualize the motion.

2. Determining $v(t)$ and $x(t)$ for a given $a(t)$. The solution of problems of this type was discussed in the first part of Sec. 11.3. We used the initial conditions, $t = 0$ and $v = v_0$, for the lower limits of the integrals in t and v, but any other known state (for example, $t = t_1$, $v = v_1$) could have been used instead. Also, if the given function $a(t)$ contains an unknown constant (for example, the constant k if $a = kt$), you will first have to determine that constant by substituting a set of known values of t and a in the equation defining $a(t)$.

3. Determining $v(x)$ and $x(t)$ for a given $a(x)$. This is the second case considered in Sec. 11.3. We again note that the lower limits of integration can be any known state (for example, $x = x_1$, $v = v_1$). In addition, since $v = v_{max}$ when $a = 0$, the positions where the maximum values of the velocity occur are easily determined by writing $a(x) = 0$ and solving for x.

4. Determining $v(x)$, $v(t)$, and $x(t)$ for a given $a(v)$. This is the last case treated in Sec. 11.3; the appropriate solution techniques for problems of this type are illustrated in Sample Prob. 11.3. All of the general comments for the preceding cases once again apply. Note that Sample Prob. 11.3 provides a summary of how and when to use the equations $v = dx/dt$, $a = dv/dt$, and $a = v\, dv/dx$.

PROBLEMS†

CONCEPT QUESTIONS

11.CQ1 A bus travels the 100 miles between A and B at 50 mi/h and then another 100 miles between B and C at 70 mi/h. The average speed of the bus for the entire 200-mile trip is:

a. More than 60 mi/h.
b. Equal to 60 mi/h.
c. Less than 60 mi/h.

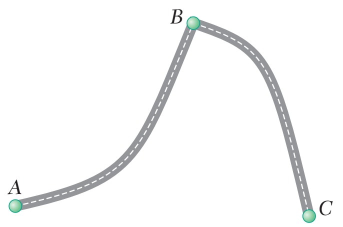


Fig. P11.CQ1

11.CQ2 Two cars A and B race each other down a straight road. The position of each car as a function of time is shown. Which of the following statements are true (more than one answer can be correct)?

a. At time t_2 both cars have traveled the same distance.
b. At time t_1 both cars have the same speed.
c. Both cars have the same speed at some time $t < t_1$.
d. Both cars have the same acceleration at some time $t < t_1$.
e. Both cars have the same acceleration at some time $t_1 < t < t_2$.

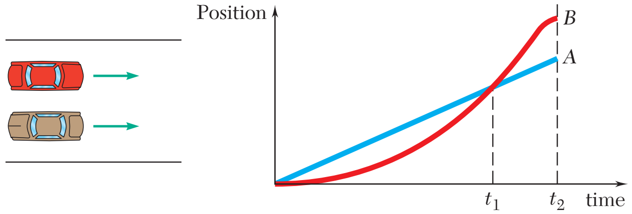


Fig. P11.CQ2

END-OF-SECTION PROBLEMS

11.1 The motion of a particle is defined by the relation $x = t^4 - 10t^2 + 8t + 12$, where x and t are expressed in inches and seconds, respectively. Determine the position, the velocity, and the acceleration of the particle when $t = 1$ s.

11.2 The motion of a particle is defined by the relation $x = 2t^3 - 9t^2 + 12t + 10$, where x and t are expressed in feet and seconds, respectively. Determine the time, the position, and the acceleration of the particle when $v = 0$.

†Answers to all problems set in straight type (such as **11.1**) are given at the end of the book. Answers to problems with a number set in italic type (such as ***11.7***) are not given.

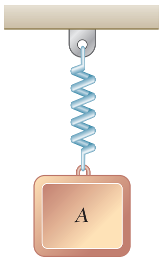


Fig. P11.3

11.3 The vertical motion of mass A is defined by the relation $x = 10 \sin 2t + 15 \cos 2t + 100$, where x and t are expressed in millimeters and seconds, respectively. Determine (*a*) the position, velocity, and acceleration of A when $t = 1$ s, (*b*) the maximum velocity and acceleration of A.

11.4 A loaded railroad car is rolling at a constant velocity when it couples with a spring and dashpot bumper system. After the coupling, the motion of the car is defined by the relation $x = 60e^{-4.8t} \sin 16t$, where x and t are expressed in millimeters and seconds, respectively. Determine the position, the velocity, and the acceleration of the railroad car when (*a*) $t = 0$, (*b*) $t = 0.3$ s.

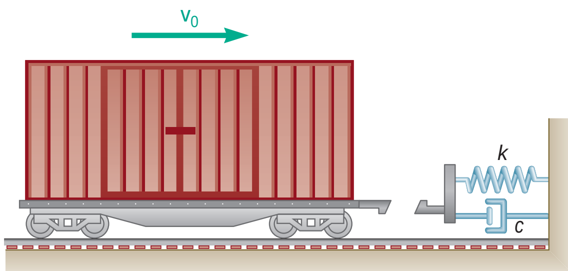


Fig. P11.4

11.5 The motion of a particle is defined by the relation $x = 6t^4 - 2t^3 - 12t^2 + 3t + 3$, where x and t are expressed in meters and seconds, respectively. Determine the time, the position, and the velocity when $a = 0$.

11.6 The motion of a particle is defined by the relation $x = t^3 - 9t^2 + 24t - 8$, where x and t are expressed in inches and seconds, respectively. Determine (*a*) when the velocity is zero, (*b*) the position and the total distance traveled when the acceleration is zero.

11.7 The motion of a particle is defined by the relation $x = 2t^3 - 15t^2 + 24t + 4$, where x is expressed in meters and t in seconds. Determine (*a*) when the velocity is zero, (*b*) the position and the total distance traveled when the acceleration is zero.

11.8 The motion of a particle is defined by the relation $x = t^3 - 6t^2 - 36t - 40$, where x and t are expressed in feet and seconds, respectively. Determine (*a*) when the velocity is zero, (*b*) the velocity, the acceleration, and the total distance traveled when $x = 0$.

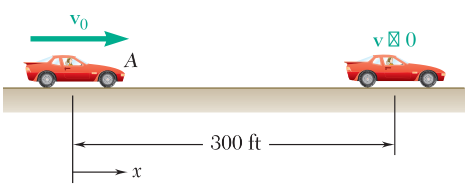


Fig. P11.9

11.9 The brakes of a car are applied, causing it to slow down at a rate of 10 ft/s^2. Knowing that the car stops in 300 ft, determine (*a*) how fast the car was traveling immediately before the brakes were applied, (*b*) the time required for the car to stop.

11.10 The acceleration of a particle is directly proportional to the time t. At $t = 0$, the velocity of the particle is $v = 16$ in./s. Knowing that $v = 15$ in./s and that $x = 20$ in. when $t = 1$ s, determine the velocity, the position, and the total distance traveled when $t = 7$ s.

11.11 The acceleration of a particle is directly proportional to the square of the time t. When $t = 0$, the particle is at $x = 24$ m. Knowing that at $t = 6$ s, $x = 96$ m and $v = 18$ m/s, express x and v in terms of t.

11.12 The acceleration of a particle is defined by the relation $a = kt^2$. (a) Knowing that $v = -8$ m/s when $t = 0$ and that $v = +8$ m/s when $t = 2$ s, determine the constant k. (b) Write the equations of motion, knowing also that $x = 0$ when $t = 2$ s.

11.13 The acceleration of point A is defined by the relation $a = -1.8 \sin kt$, where a and t are expressed in m/s^2 and seconds, respectively, and $k = 3$ rad/s. Knowing that $x = 0$ and $v = 0.6$ m/s when $t = 0$, determine the velocity and position of point A when $t = 0.5$ s.

11.14 The acceleration of point A is defined by the relation $a = -1.08 \sin kt - 1.44 \cos kt$, where a and t are expressed in m/s^2 and seconds, respectively, and $k = 3$ rad/s. Knowing that $x = 0.16$ m and $v = 0.36$ m/s when $t = 0$, determine the velocity and position of point A when $t = 0.5$ s.

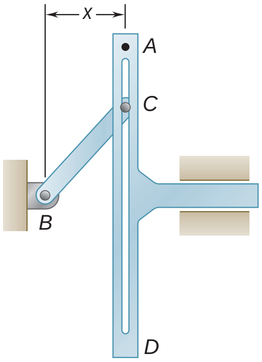


Fig. *P11.13* and *P11.14*

11.15 A piece of electronic equipment that is surrounded by packing material is dropped so that it hits the ground with a speed of 4 m/s. After contact the equipment experiences an acceleration of $a = -kx$, where k is a constant and x is the compression of the packing material. If the packing material experiences a maximum compression of 20 mm, determine the maximum acceleration of the equipment.

Fig. P11.15

11.16 A projectile enters a resisting medium at $x = 0$ with an initial velocity $\mathbf{v}_0 = 900$ ft/s and travels 4 in. before coming to rest. Assuming that the velocity of the projectile is defined by the relation $v = v_0 - kx$, where v is expressed in ft/s and x is in feet, determine (a) the initial acceleration of the projectile, (b) the time required for the projectile to penetrate 3.9 in. into the resisting medium.

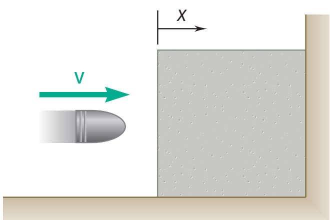


Fig. P11.16

11.17 The acceleration of a particle is defined by the relation $a = -k/x$. It has been experimentally determined that $v = 15$ ft/s when $x = 0.6$ ft and that $v = 9$ ft/s when $x = 1.2$ ft. Determine (a) the velocity of the particle when $x = 1.5$ ft, (b) the position of the particle at which its velocity is zero.

11.18 A brass (nonmagnetic) block A and a steel magnet B are in equilibrium in a brass tube under the magnetic repelling force of another steel magnet C located at a distance $x = 0.004$ m from B. The force is inversely proportional to the square of the distance between B and C. If block A is suddenly removed, the acceleration of block B is $a = -9.81 + k/x^2$, where a and x are expressed in m/s^2 and meters, respectively, and $k = 4 \times 10^{-4}$ m^3/s^2. Determine the maximum velocity and acceleration of B.

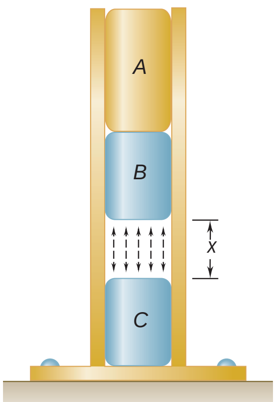


Fig. P11.18

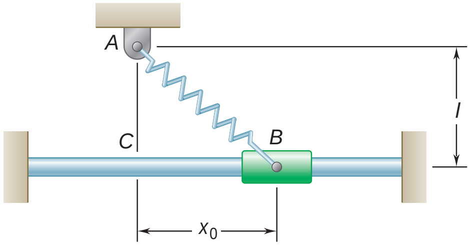


Fig. *P11.20*

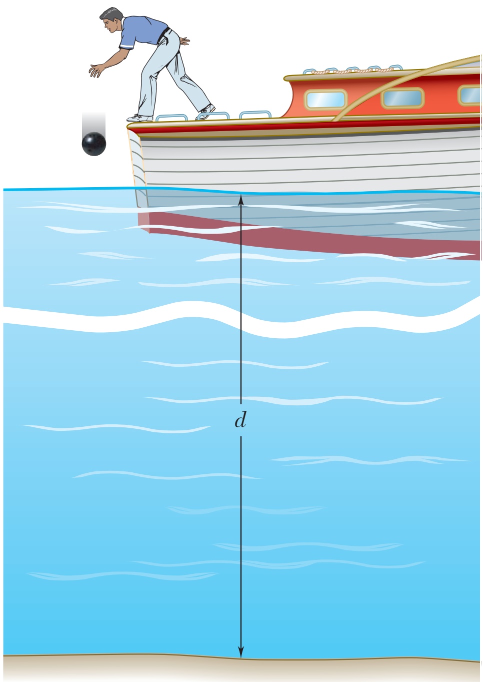


Fig. P11.23

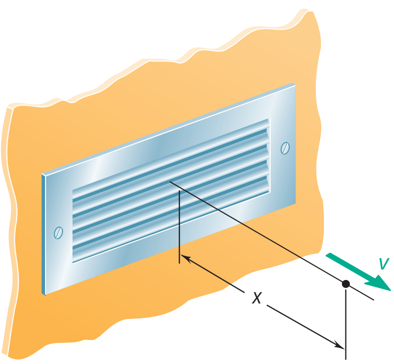


Fig. P11.27

11.19 Based on experimental observations, the acceleration of a particle is defined by the relation $a = -(0.1 + \sin x/b)$, where a and x are expressed in m/s^2 and meters, respectively. Knowing that $b = 0.8$ m and that $v = 1$ m/s when $x = 0$, determine (*a*) the velocity of the particle when $x = -1$ m, (*b*) the position where the velocity is maximum, (*c*) the maximum velocity.

11.20 A spring AB is attached to a support at A and to a collar. The unstretched length of the spring is l. Knowing that the collar is released from rest at $x = x_0$ and has an acceleration defined by the relation $a = -100(x - lx/\sqrt{l^2 + x^2})$, determine the velocity of the collar as it passes through point C.

11.21 The acceleration of a particle is defined by the relation $a = -0.8v$, where a is expressed in m/s^2 and v in m/s. Knowing that at $t = 0$ the velocity is 1 m/s, determine (*a*) the distance the particle will travel before coming to rest, (*b*) the time required for the particle's velocity to be reduced by 50 percent of its initial value.

11.22 Starting from $x = 0$ with no initial velocity, a particle is given an acceleration $a = 0.1\sqrt{v^2 + 16}$, where a and v are expressed in ft/s^2 and ft/s, respectively. Determine (*a*) the position of the particle when $v = 3$ ft/s, (*b*) the speed and acceleration of the particle when $x = 4$ ft.

11.23 A ball is dropped from a boat so that it strikes the surface of a lake with a speed of 16.5 ft/s. While in the water the ball experiences an acceleration of $a = 10 - 0.8v$, where a and v are expressed in ft/s^2 and ft/s, respectively. Knowing the ball takes 3 s to reach the bottom of the lake, determine (*a*) the depth of the lake, (*b*) the speed of the ball when it hits the bottom of the lake.

11.24 The acceleration of a particle is defined by the relation $a = -k\sqrt{v}$, where k is a constant. Knowing that $x = 0$ and $v = 81$ m/s at $t = 0$ and that $v = 36$ m/s when $x = 18$ m, determine (*a*) the velocity of the particle when $x = 20$ m, (*b*) the time required for the particle to come to rest.

11.25 A particle is projected to the right from the position $x = 0$ with an initial velocity of 9 m/s. If the acceleration of the particle is defined by the relation $a = -0.6v^{3/2}$, where a and v are expressed in m/s^2 and m/s, respectively, determine (*a*) the distance the particle will have traveled when its velocity is 4 m/s, (*b*) the time when $v = 1$ m/s, (*c*) the time required for the particle to travel 6 m.

11.26 The acceleration of a particle is defined by the relation $a = 0.4(1 - kv)$, where k is a constant. Knowing that at $t = 0$ the particle starts from rest at $x = 4$ m and that when $t = 15$ s, $v = 4$ m/s, determine (*a*) the constant k, (*b*) the position of the particle when $v = 6$ m/s, (*c*) the maximum velocity of the particle.

11.27 Experimental data indicate that in a region downstream of a given louvered supply vent the velocity of the emitted air is defined by $v = 0.18v_0/x$, where v and x are expressed in m/s and meters, respectively, and v_0 is the initial discharge velocity of the air. For $v_0 = 3.6$ m/s, determine (*a*) the acceleration of the air at $x = 2$ m, (*b*) the time required for the air to flow from $x = 1$ to $x = 3$ m.

11.28 Based on observations, the speed of a jogger can be approximated by the relation $v = 7.5(1 - 0.04x)^{0.3}$, where v and x are expressed in mi/h and miles, respectively. Knowing that $x = 0$ at $t = 0$, determine (*a*) the distance the jogger has run when $t = 1$ h, (*b*) the jogger's acceleration in ft/s^2 at $t = 0$, (*c*) the time required for the jogger to run 6 mi.

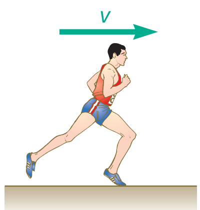


Fig. P11.28

11.29 The acceleration due to gravity at an altitude y above the surface of the earth can be expressed as

$$a = \frac{-32.2}{[1 + (y/20.9 \times 10^6)]^2}$$

where a and y are expressed in ft/s^2 and feet, respectively. Using this expression, compute the height reached by a projectile fired vertically upward from the surface of the earth if its initial velocity is (*a*) 1800 ft/s, (*b*) 3000 ft/s, (*c*) 36,700 ft/s.

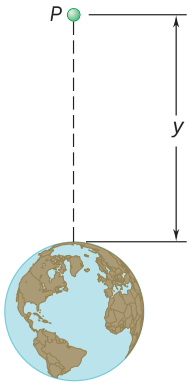


Fig. *P11.29*

11.30 The acceleration due to gravity of a particle falling toward the earth is $a = -gR^2/r^2$, where r is the distance from the *center* of the earth to the particle, R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth. If $R = 3960$ mi, calculate the *escape velocity,* that is, the minimum velocity with which a particle must be projected vertically upward from the surface of the earth if it is not to return to the earth. (*Hint:* $v = 0$ for $r = \infty$.)

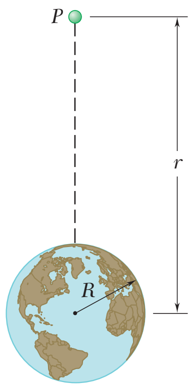


Fig. *P11.30*

11.31 The velocity of a particle is $v = v_0[1 - \sin(\mathrm{p}t/T)]$. Knowing that the particle starts from the origin with an initial velocity v_0, determine (*a*) its position and its acceleration at $t = 3T$, (*b*) its average velocity during the interval $t = 0$ to $t = T$.

11.32 The velocity of a slider is defined by the relation $v = v'\sin(\mathrm{v}_n t + \mathrm{f})$. Denoting the velocity and the position of the slider at $t = 0$ by v_0 and x_0, respectively, and knowing that the maximum displacement of the slider is $2x_0$, show that (*a*) $v' = (v_0^2 + x_0^2\mathrm{v}_n^2)/2x_0\mathrm{v}_n$, (*b*) the maximum value of the velocity occurs when $x = x_0[3 - (v_0/x_0\mathrm{v}_n)^2]/2$.

11.4 UNIFORM RECTILINEAR MOTION

Uniform rectilinear motion is a type of straight-line motion which is frequently encountered in practical applications. In this motion, the acceleration a of the particle is zero for every value of t. The velocity v is therefore constant, and Eq. (11.1) becomes

$$\frac{dx}{dt} = v = \text{constant}$$

The position coordinate x is obtained by integrating this equation. Denoting by x_0 the initial value of x, we write

$$\int_{x_0}^{x} dv = v \int_0^t dt$$

$$x - x_0 = vt$$

$$\boxed{x = x_0 + vt} \tag{11.5}$$

This equation can be used *only if the velocity of the particle is known to be constant*.

11.5 UNIFORMLY ACCELERATED RECTILINEAR MOTION

Uniformly accelerated rectilinear motion is another common type of motion. In this motion, the acceleration a of the particle is constant, and Eq. (11.2) becomes

$$\frac{dv}{dt} = a = \text{constant}$$

The velocity v of the particle is obtained by integrating this equation:

$$\int_{v_0}^{v} dv = a \int_0^t dt$$

$$v - v_0 = at$$

$$\boxed{v = v_0 + at} \tag{11.6}$$

where v_0 is the initial velocity. Substituting for v in (11.1), we write

$$\frac{dx}{dt} = v_0 + at$$

Denoting by x_0 the initial value of x and integrating, we have

$$\int_{x_0}^{x} dx = \int_0^t (v_0 + at)\, dt$$

$$x - x_0 = v_0 t + \tfrac{1}{2}at^2$$

$$\boxed{x = x_0 + v_0 t + \tfrac{1}{2}at^2} \tag{11.7}$$

We can also use Eq. (11.4) and write

$$v\frac{dv}{dx} = a = \text{constant}$$

$$v\,dv = a\,dx$$

Integrating both sides, we obtain

$$\int_{v_0}^{v} v\,dv = a\int_{x_0}^{x} dx$$

$$\tfrac{1}{2}(v^2 - v_0^2) = a(x - x_0)$$

$$\boxed{v^2 = v_0^2 + 2a(x - x_0)} \tag{11.8}$$

The three equations we have derived provide useful relations among position coordinate, velocity, and time in the case of a uniformly accelerated motion, as soon as appropriate values have been substituted for a, v_0, and x_0. The origin O of the x axis should first be defined and a positive direction chosen along the axis; this direction will be used to determine the signs of a, v_0, and x_0. Equation (11.6) relates v and t and should be used when the value of v corresponding to a given value of t is desired, or inversely. Equation (11.7) relates x and t; Eq. (11.8) relates v and x. An important application of uniformly accelerated motion is the motion of a *freely falling body*. The acceleration of a freely falling body (usually denoted by g) is equal to 9.81 m/s^2 or 32.2 ft/s^2.

It is important to keep in mind that the three equations can be used *only when the acceleration of the particle is known to be constant*. If the acceleration of the particle is variable, its motion should be determined from the fundamental equations (11.1) to (11.4) according to the methods outlined in Sec. 11.3.

11.6 MOTION OF SEVERAL PARTICLES

When several particles move independently along the same line, independent equations of motion can be written for each particle. Whenever possible, time should be recorded from the same initial instant for all particles, and displacements should be measured from the same origin and in the same direction. In other words, a single clock and a single measuring tape should be used.

Relative Motion of Two Particles. Consider two particles A and B moving along the same straight line (Fig. 11.7). If the position coordinates x_A and x_B are measured from the same origin, the difference $x_B - x_A$ defines the *relative position coordinate of B with respect to A* and is denoted by $x_{B/A}$. We write

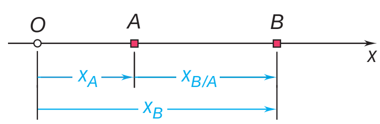


Fig. 11.7

$$x_{B/A} = x_B - x_A \qquad \text{or} \qquad \boxed{x_B = x_A + x_{B/A}} \tag{11.9}$$

Regardless of the positions of A and B with respect to the origin, a positive sign for $x_{B/A}$ means that B is to the right of A, and a negative sign means that B is to the left of A.

Photo 11.2 Multiple cables and pulleys are used by this shipyard crane.

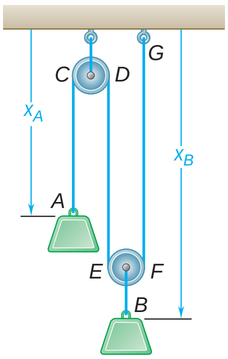


Fig. 11.8

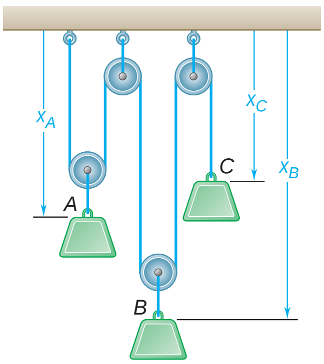


Fig. 11.9

The rate of change of $x_{B/A}$ is known as the *relative velocity of B with respect to A* and is denoted by $v_{B/A}$. Differentiating (11.9), we write

$$v_{B/A} = v_B - v_A \qquad \text{or} \qquad v_B = v_A + v_{B/A} \tag{11.10}$$

A positive sign for $v_{B/A}$ means that B is *observed from A* to move in the positive direction; a negative sign means that it is observed to move in the negative direction.

The rate of change of $v_{B/A}$ is known as the *relative acceleration of B with respect to A* and is denoted by $a_{B/A}$. Differentiating (11.10), we obtain†

$$a_{B/A} = a_B - a_A \qquad \text{or} \qquad a_B = a_A + a_{B/A} \tag{11.11}$$

Dependent Motions. Sometimes, the position of a particle will depend upon the position of another particle or of several other particles. The motions are then said to be *dependent*. For example, the position of block B in Fig. 11.8 depends upon the position of block A. Since the rope $ACDEFG$ is of constant length, and since the lengths of the portions of rope CD and EF wrapped around the pulleys remain constant, it follows that the sum of the lengths of the segments AC, DE, and FG is constant. Observing that the length of the segment AC differs from x_A only by a constant and that, similarly, the lengths of the segments DE and FG differ from x_B only by a constant, we write

$$x_A + 2x_B = \text{constant}$$

Since only one of the two coordinates x_A and x_B can be chosen arbitrarily, we say that the system shown in Fig. 11.8 has *one degree of freedom*. From the relation between the position coordinates x_A and x_B, it follows that if x_A is given an increment Δx_A, that is, if block A is lowered by an amount Δx_A, the coordinate x_B will receive an increment $\Delta x_B = -\frac{1}{2}\Delta x_A$. In other words, block B will rise by half the same amount; this can easily be checked directly from Fig. 11.8.

In the case of the three blocks of Fig. 11.9, we can again observe that the length of the rope which passes over the pulleys is constant, and thus the following relation must be satisfied by the position coordinates of the three blocks:

$$2x_A + 2x_B + x_C = \text{constant}$$

Since two of the coordinates can be chosen arbitrarily, we say that the system shown in Fig. 11.9 has *two degrees of freedom*.

When the relation existing between the position coordinates of several particles is *linear*, a similar relation holds between the velocities and between the accelerations of the particles. In the case of the blocks of Fig. 11.9, for instance, we differentiate twice the equation obtained and write

$$2\frac{dx_A}{dt} + 2\frac{dx_B}{dt} + \frac{dx_C}{dt} = 0 \qquad \text{or} \qquad 2v_A + 2v_B + v_C = 0$$

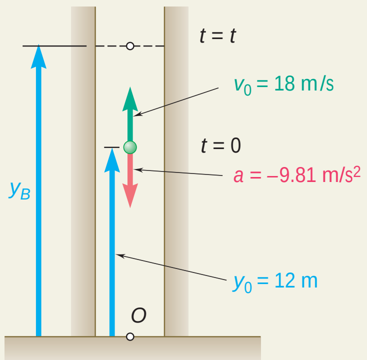
$$2\frac{dv_A}{dt} + 2\frac{dv_B}{dt} + \frac{dv_C}{dt} = 0 \qquad \text{or} \qquad 2a_A + 2a_B + a_C = 0$$

†Note that the product of the subscripts A and B/A used in the right-hand member of Eqs. (11.9), (11.10), and (11.11) is equal to the subscript B used in their left-hand member.

SAMPLE PROBLEM 11.4

A ball is thrown vertically upward from the 12-m level in an elevator shaft with an initial velocity of 18 m/s. At the same instant an open-platform elevator passes the 5-m level, moving upward with a constant velocity of 2 m/s. Determine (*a*) when and where the ball will hit the elevator, (*b*) the relative velocity of the ball with respect to the elevator when the ball hits the elevator.

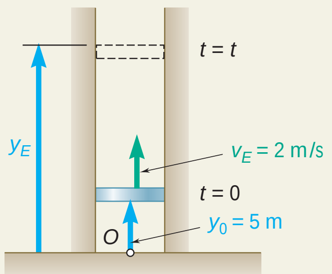
SOLUTION



Motion of Ball. Since the ball has a constant acceleration, its motion is *uniformly accelerated*. Placing the origin O of the y axis at ground level and choosing its positive direction upward, we find that the initial position is $y_0 = +12$ m, the initial velocity is $v_0 = +18$ m/s, and the acceleration is $a = -9.81$ m/s^2. Substituting these values in the equations for uniformly accelerated motion, we write

$$v_B = v_0 + at \qquad v_B = 18 - 9.81t \tag{1}$$

$$y_B = y_0 + v_0 t + \tfrac{1}{2}at^2 \qquad y_B = 12 + 18t - 4.905t^2 \tag{2}$$



Motion of Elevator. Since the elevator has a constant velocity, its motion is *uniform*. Again placing the origin O at the ground level and choosing the positive direction upward, we note that $y_0 = +5$ m and write

$$v_E = +2 \text{ m/s} \tag{3}$$

$$y_E = y_0 + v_E t \qquad y_E = 5 + 2t \tag{4}$$

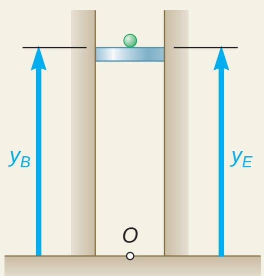
Ball Hits Elevator. We first note that the same time t and the same origin O were used in writing the equations of motion of both the ball and the elevator. We see from the figure that when the ball hits the elevator,

$$y_E = y_B \tag{5}$$

Substituting for y_E and y_B from (2) and (4) into (5), we have

$$5 + 2t = 12 + 18t - 4.905t^2$$

$$t = -0.39 \text{ s} \qquad \text{and} \qquad t = 3.65 \text{ s} \blacktriangleleft$$



Only the root $t = 3.65$ s corresponds to a time after the motion has begun. Substituting this value into (4), we have

$$y_E = 5 + 2(3.65) = 12.30 \text{ m}$$

$$\text{Elevation from ground} = 12.30 \text{ m} \blacktriangleleft$$

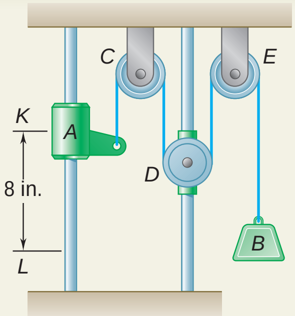
The relative velocity of the ball with respect to the elevator is

$$v_{B/E} = v_B - v_E = (18 - 9.81t) - 2 = 16 - 9.81t$$

When the ball hits the elevator at time $t = 3.65$ s, we have

$$v_{B/E} = 16 - 9.81(3.65) \qquad v_{B/E} = -19.81 \text{ m/s} \blacktriangleleft$$

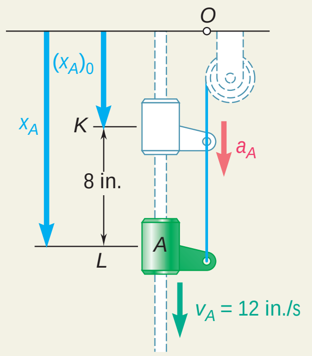
The negative sign means that the ball is observed from the elevator to be moving in the negative sense (downward).



SAMPLE PROBLEM 11.5

Collar A and block B are connected by a cable passing over three pulleys C, D, and E as shown. Pulleys C and E are fixed, while D is attached to a collar which is pulled downward with a constant velocity of 3 in./s. At $t = 0$, collar A starts moving downward from position K with a constant acceleration and no initial velocity. Knowing that the velocity of collar A is 12 in./s as it passes through point L, determine the change in elevation, the velocity, and the acceleration of block B when collar A passes through L.

SOLUTION

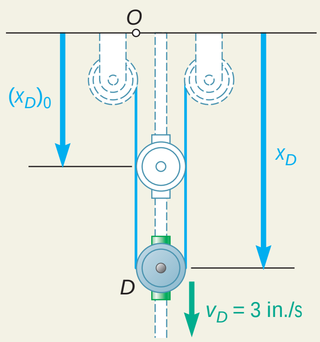


Motion of Collar *A*. We place the origin O at the upper horizontal surface and choose the positive direction downward. We observe that when $t = 0$, collar A is at the position K and $(v_A)_0 = 0$. Since $v_A = 12$ in./s and $x_A - (x_A)_0 = 8$ in. when the collar passes through L, we write

$$v_A^2 = (v_A)_0^2 + 2a_A[x_A - (x_A)_0] \qquad (12)^2 = 0 + 2a_A(8)$$
$$a_A = 9 \text{ in./s}^2$$

The time at which collar A reaches point L is obtained by writing

$$v_A = (v_A)_0 + a_A t \qquad 12 = 0 + 9t \qquad t = 1.333 \text{ s}$$



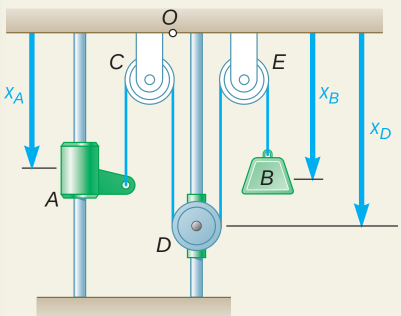
Motion of Pulley *D*. Recalling that the positive direction is downward, we write

$$a_D = 0 \qquad v_D = 3 \text{ in./s} \qquad x_D = (x_D)_0 + v_D t = (x_D)_0 + 3t$$

When collar A reaches L, at $t = 1.333$ s, we have

$$x_D = (x_D)_0 + 3(1.333) = (x_D)_0 + 4$$

Thus,
$$x_D - (x_D)_0 = 4 \text{ in.}$$



Motion of Block *B*. We note that the total length of cable $ACDEB$ differs from the quantity $(x_A + 2x_D + x_B)$ only by a constant. Since the cable length is constant during the motion, this quantity must also remain constant. Thus, considering the times $t = 0$ and $t = 1.333$ s, we write

$$x_A + 2x_D + x_B = (x_A)_0 + 2(x_D)_0 + (x_B)_0 \qquad (1)$$
$$[x_A - (x_A)_0] + 2[x_D - (x_D)_0] + [x_B - (x_B)_0] = 0 \qquad (2)$$

But we know that $x_A - (x_A)_0 = 8$ in. and $x_D - (x_D)_0 = 4$ in.; substituting these values in (2), we find

$$8 + 2(4) + [x_B - (x_B)_0] = 0 \qquad x_B - (x_B)_0 = -16 \text{ in.}$$

Thus: Change in elevation of B = 16 in. x ◀

Differentiating (1) twice, we obtain equations relating the velocities and the accelerations of A, B, and D. Substituting for the velocities and accelerations of A and D at $t = 1.333$ s, we have

$$v_A + 2v_D + v_B = 0: \qquad 12 + 2(3) + v_B = 0$$
$$v_B = -18 \text{ in./s} \qquad v_B = 18 \text{ in./s x} \;◀$$
$$a_A + 2a_D + a_B = 0: \qquad 9 + 2(0) + a_B = 0$$
$$a_B = -9 \text{ in./s}^2 \qquad a_B = 9 \text{ in./s}^2 \text{ x} \;◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we derived the equations that describe *uniform rectilinear motion* (constant velocity) and *uniformly accelerated rectilinear motion* (constant acceleration). We also introduced the concept of *relative motion*. The equations for relative motion [Eqs. (11.9) to (11.11)] can be applied to the independent or dependent motions of any two particles moving along the same straight line.

A. Independent motion of one or more particles. The solution of problems of this type should be organized as follows:

1. Begin your solution by listing the given information, sketching the system, and selecting the origin and the positive direction of the coordinate axis [Sample Prob. 11.4]. It is always advantageous to have a visual representation of problems of this type.

2. Write the equations that describe the motions of the various particles as well as those that describe how these motions are related [Eq. (5) of Sample Prob. 11.4].

3. Define the initial conditions, i.e., specify the state of the system corresponding to $t = 0$. This is especially important if the motions of the particles begin at different times. In such cases, either of two approaches can be used.

a. Let $t = 0$ be the time when the last particle begins to move. You must then determine the initial position x_0 and the initial velocity v_0 of each of the other particles.

b. Let $t = 0$ be the time when the first particle begins to move. You must then, in each of the equations describing the motion of another particle, replace t with $t - t_0$, where t_0 is the time at which that specific particle begins to move. It is important to recognize that the equations obtained in this way are valid only for $t \geq t_0$.

B. Dependent motion of two or more particles. In problems of this type the particles of the system are connected to each other, typically by ropes or by cables. The method of solution of these problems is similar to that of the preceding group of problems, except that it will now be necessary to describe the *physical connections* between the particles. In the following problems, the connection is provided by one or more cables. For each cable, you will have to write equations similar to the last three equations of Sec. 11.6. We suggest that you use the following procedure:

1. Draw a sketch of the system and select a coordinate system, indicating clearly a positive sense for each of the coordinate axes. For example, in Sample Prob. 11.5 lengths are measured downward from the upper horizontal support. It thus follows that those displacements, velocities, and accelerations which have positive values are directed downward.

2. Write the equation describing the constraint imposed by each cable on the motion of the particles involved. Differentiating this equation twice, you will obtain the corresponding relations among velocities and accelerations.

3. If several directions of motion are involved, you must select a coordinate axis and a positive sense for each of these directions. You should also try to locate the origins of your coordinate axes so that the equations of constraints will be as simple as possible. For example, in Sample Prob. 11.5 it is easier to define the various coordinates by measuring them downward from the upper support than by measuring them upward from the bottom support.

Finally, keep in mind that the method of analysis described in this lesson and the corresponding equations can be used only for particles moving with *uniform* or *uniformly accelerated rectilinear motion*.

PROBLEMS

11.33 A stone is thrown vertically upward from a point on a bridge located 40 m above the water. Knowing that it strikes the water 4 s after release, determine (*a*) the speed with which the stone was thrown upward, (*b*) the speed with which the stone strikes the water.

11.34 A motorist is traveling at 54 km/h when she observes that a traffic light 240 m ahead of her turns red. The traffic light is timed to stay red for 24 s. If the motorist wishes to pass the light without stopping just as it turns green again, determine (*a*) the required uniform deceleration of the car, (*b*) the speed of the car as it passes the light.

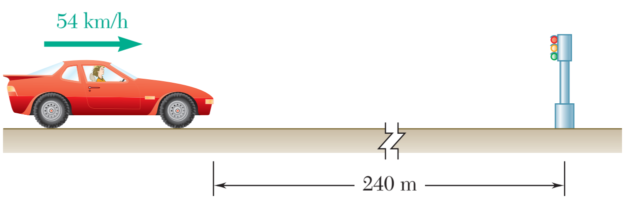


Fig. P11.34

11.35 A motorist enters a freeway at 30 mi/h and accelerates uniformly to 60 mi/h. From the odometer in the car, the motorist knows that she traveled 550 ft while accelerating. Determine (*a*) the acceleration of the car, (*b*) the time required to reach 60 mi/h.

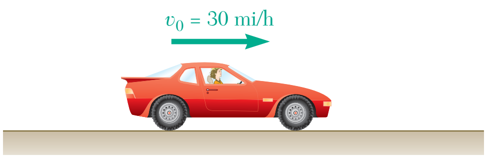


Fig. P11.35

11.36 A group of students launches a model rocket in the vertical direction. Based on tracking data, they determine that the altitude of the rocket was 89.6 ft at the end of the powered portion of the flight and that the rocket landed 16 s later. Knowing that the descent parachute failed to deploy so that the rocket fell freely to the ground after reaching its maximum altitude and assuming that $g = 32.2\ \text{ft/s}^2$, determine (*a*) the speed v_1 of the rocket at the end of powered flight, (*b*) the maximum altitude reached by the rocket.

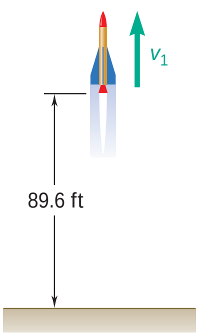


Fig. P11.36

11.37 A small package is released from rest at A and moves along the skate wheel conveyor $ABCD$. The package has a uniform acceleration of 4.8 m/s^2 as it moves down sections AB and CD, and its velocity is constant between B and C. If the velocity of the package at D is 7.2 m/s, determine (a) the distance d between C and D, (b) the time required for the package to reach D.

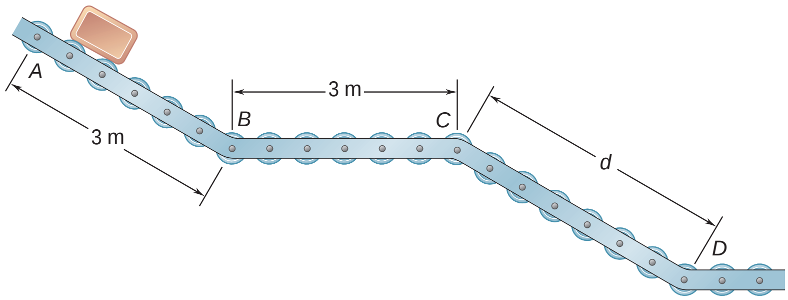


Fig. *P11.37*

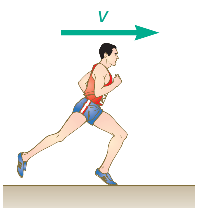


Fig. *P11.38*

11.38 A sprinter in a 100-m race accelerates uniformly for the first 35 m and then runs with constant velocity. If the sprinter's time for the first 35 m is 5.4 s, determine (a) his acceleration, (b) his final velocity, (c) his time for the race.

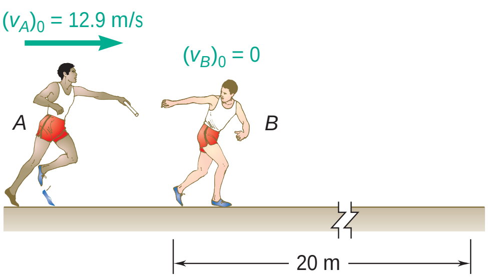


Fig. P11.39

11.39 As relay runner A enters the 20-m-long exchange zone with a speed of 12.9 m/s, he begins to slow down. He hands the baton to runner B 1.82 s later as they leave the exchange zone with the same velocity. Determine (a) the uniform acceleration of each of the runners, (b) when runner B should begin to run.

11.40 In a boat race, boat A is leading boat B by 50 m and both boats are traveling at a constant speed of 180 km/h. At t = 0, the boats accelerate at constant rates. Knowing that when B passes A, t = 8 s and v_A = 225 km/h, determine (a) the acceleration of A, (b) the acceleration of B.

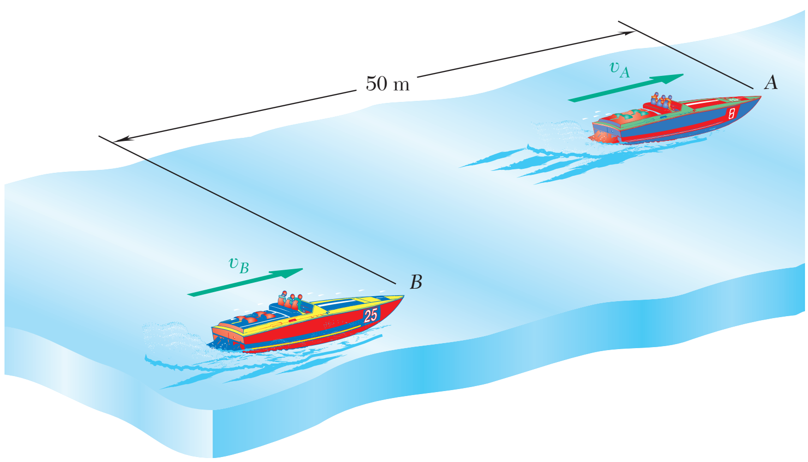


Fig. P11.40

11.41 A police officer in a patrol car parked in a 45 mi/h speed zone observes a passing automobile traveling at a slow, constant speed. Believing that the driver of the automobile might be intoxicated, the officer starts his car, accelerates uniformly to 60 mi/h in 8 s, and, maintaining a constant velocity of 60 mi/h, overtakes the motorist 42 s after the automobile passed him. Knowing that 18 s elapsed before the officer began pursuing the motorist, determine (*a*) the distance the officer traveled before overtaking the motorist, (*b*) the motorist's speed.

11.42 Automobiles *A* and *B* are traveling in adjacent highway lanes and at $t = 0$ have the positions and speeds shown. Knowing that automobile *A* has a constant acceleration of 1.8 ft/s^2 and that *B* has a constant deceleration of 1.2 ft/s^2, determine (*a*) when and where *A* will overtake *B*, (*b*) the speed of each automobile at that time.

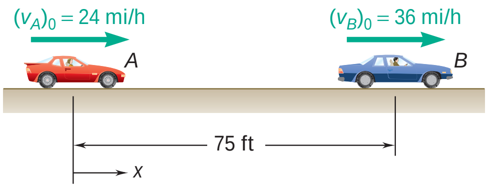


Fig. P11.42

11.43 Two automobiles *A* and *B* are approaching each other in adjacent highway lanes. At $t = 0$, *A* and *B* are 3200 ft apart, their speeds are $v_A = 65$ mi/h and $v_B = 40$ mi/h, and they are at points *P* and *Q*, respectively. Knowing that *A* passes point *Q* 40 s after *B* was there and that *B* passes point *P* 42 s after *A* was there, determine (*a*) the uniform accelerations of *A* and *B*, (*b*) when the vehicles pass each other, (*c*) the speed of *B* at that time.

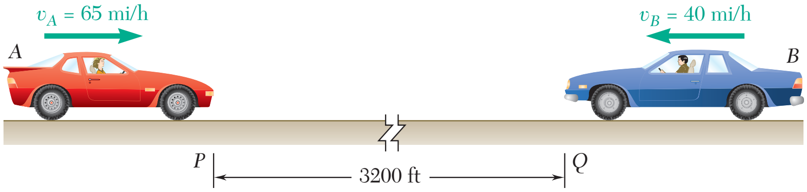


Fig. P11.43

11.44 An elevator is moving upward at a constant speed of 4 m/s. A man standing 10 m above the top of the elevator throws a ball upward with a speed of 3 m/s. Determine (*a*) when the ball will hit the elevator, (*b*) where the ball will hit the elevator with respect to the location of the man.

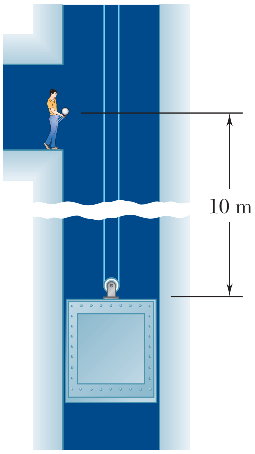


Fig. P11.44

11.45 Two rockets are launched at a fireworks display. Rocket A is launched with an initial velocity $v_0 = 100$ m/s and rocket B is launched t_1 s later with the same initial velocity. The two rockets are timed to explode simultaneously at a height of 300 m as A is falling and B is rising. Assuming a constant acceleration $g = 9.81$ m/s^2, determine (a) the time t_1, (b) the velocity of B relative to A at the time of the explosion.

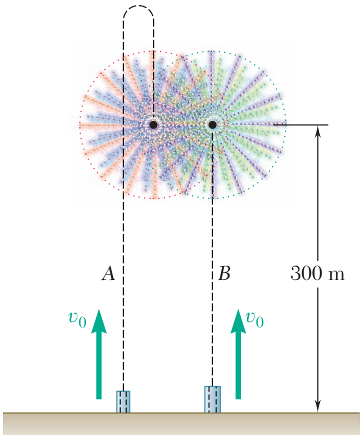


Fig. *P11.45*

11.46 Car A is parked along the northbound lane of a highway, and car B is traveling in the southbound lane at a constant speed of 60 mi/h. At $t = 0$, A starts and accelerates at a constant rate a_A, while at $t = 5$ s, B begins to slow down with a constant deceleration of magnitude $a_A/6$. Knowing that when the cars pass each other $x = 294$ ft and $v_A = v_B$, determine (a) the acceleration a_A, (b) when the vehicles pass each other, (c) the distance d between the vehicles at $t = 0$.

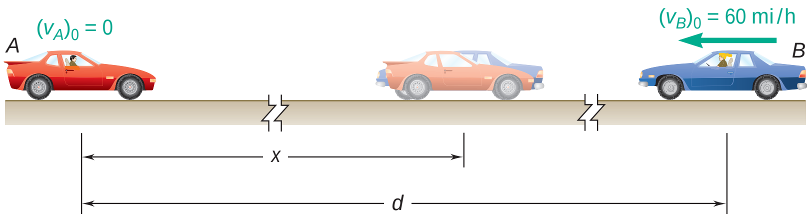


Fig. *P11.46*

11.47 The elevator shown in the figure moves downward with a constant velocity of 4 m/s. Determine (a) the velocity of the cable C, (b) the velocity of the counterweight W, (c) the relative velocity of the cable C with respect to the elevator, (d) the relative velocity of the counterweight W with respect to the elevator.

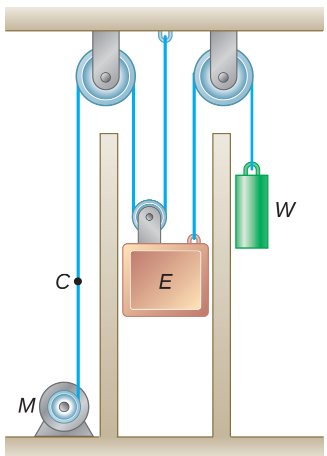


Fig. P11.47 and P11.48

11.48 The elevator shown starts from rest and moves upward with a constant acceleration. If the counterweight W moves through 30 ft in 5 s, determine (a) the acceleration of the elevator and the cable C, (b) the velocity of the elevator after 5 s.

11.49 Slider block A moves to the left with a constant velocity of 6 m/s. Determine (a) the velocity of block B, (b) the velocity of portion D of the cable, (c) the relative velocity of portion C of the cable with respect to portion D.

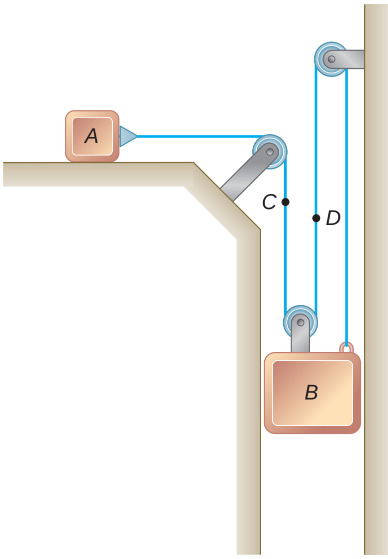


Fig. P11.49 and P11.50

11.50 Block B starts from rest and moves downward with a constant acceleration. Knowing that after slider block A has moved 9 in. its velocity is 6 ft/s, determine (a) the accelerations of A and B, (b) the velocity and the change in position of B after 2 s.

11.51 Slider block B moves to the right with a constant velocity of 300 mm/s. Determine (a) the velocity of slider block A, (b) the velocity of portion C of the cable, (c) the velocity of portion D of the cable, (d) the relative velocity of portion C of the cable with respect to slider block A.

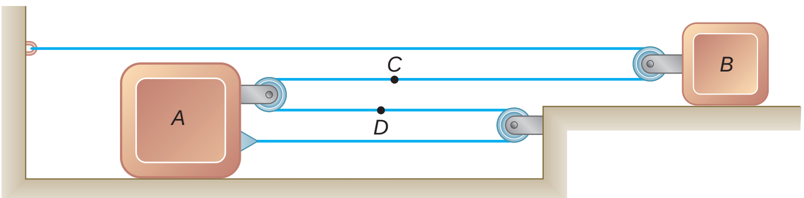


Fig. P11.51 and P11.52

11.52 At the instant shown, slider block B is moving with a constant acceleration, and its speed is 150 mm/s. Knowing that after slider block A has moved 240 mm to the right its velocity is 60 mm/s, determine (a) the accelerations of A and B, (b) the acceleration of portion D of the cable, (c) the velocity and the change in position of slider block B after 4 s.

11.53 Collar A starts from rest and moves upward with a constant acceleration. Knowing that after 8 s the relative velocity of collar B with respect to collar A is 24 in./s, determine (a) the accelerations of A and B, (b) the velocity and the change in position of B after 6 s.

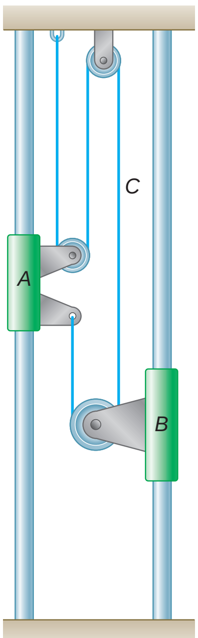


Fig. *P11.53*

11.54 The motor M reels in the cable at a constant rate of 100 mm/s. Determine (a) the velocity of load L, (b) the velocity of pulley B with respect to load L.

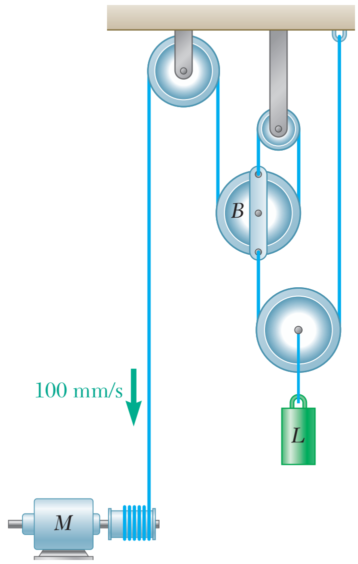


Fig. *P11.54*

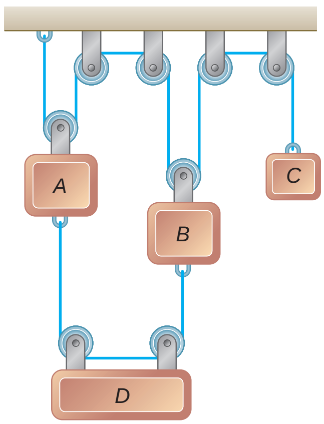


Fig. P11.55

11.55 Block C starts from rest at $t = 0$ and moves downward with a constant acceleration of 4 in./s^2. Knowing that block B has a constant velocity of 3 in./s upward, determine (a) the time when the velocity of block A is zero, (b) the time when the velocity of block A is equal to the velocity of block D, (c) the change in position of block A after 5 s.

11.56 Block A starts from rest at $t = 0$ and moves downward with a constant acceleration of 6 in./s^2. Knowing that block B moves up with a constant velocity of 3 in./s, determine (a) the time when the velocity of block C is zero, (b) the corresponding position of block C.

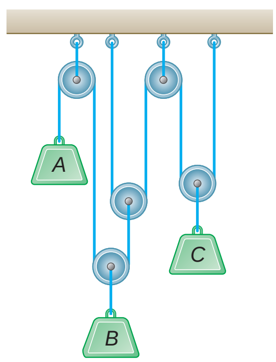


Fig. P11.56

11.57 Block B starts from rest, block A moves with a constant acceleration, and slider block C moves to the right with a constant acceleration of 75 mm/s^2. Knowing that at $t = 2$ s the velocities of B and C are 480 mm/s downward and 280 mm/s to the right, respectively, determine (*a*) the accelerations of A and B, (*b*) the initial velocities of A and C, (*c*) the change in position of slider block C after 3 s.

11.58 Block B moves downward with a constant velocity of 20 mm/s. At $t = 0$, block A is moving upward with a constant acceleration, and its velocity is 30 mm/s. Knowing that at $t = 3$ s slider block C has moved 57 mm to the right, determine (*a*) the velocity of slider block C at $t = 0$, (*b*) the accelerations of A and C, (*c*) the change in position of block A after 5 s.

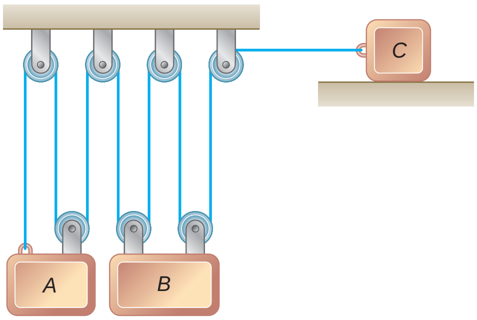


Fig. P11.57 and P11.58

11.59 The system shown starts from rest, and each component moves with a constant acceleration. If the relative acceleration of block C with respect to collar B is 60 mm/s^2 upward and the relative acceleration of block D with respect to block A is 110 mm/s^2 downward, determine (*a*) the velocity of block C after 3 s, (*b*) the change in position of block D after 5 s.

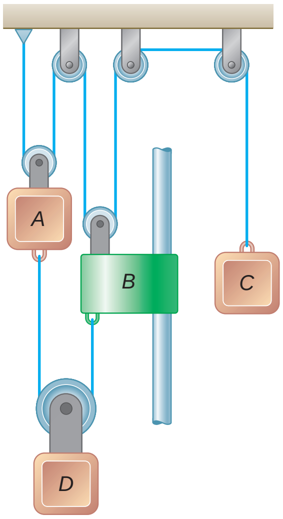


Fig. *P11.59* and *P11.60*

11.60 The system shown starts from rest, and the length of the upper cord is adjusted so that A, B, and C are initially at the same level. Each component moves with a constant acceleration, and after 2 s the relative change in position of block C with respect to block A is 280 mm upward. Knowing that when the relative velocity of collar B with respect to block A is 80 mm/s downward, the displacements of A and B are 160 mm downward and 320 mm downward, respectively, determine (*a*) the accelerations of A and B if $a_B > 10$ mm/s^2, (*b*) the change in position of block D when the velocity of block C is 600 mm/s upward.

*11.7 GRAPHICAL SOLUTION OF RECTILINEAR-MOTION PROBLEMS

It was observed in Sec. 11.2 that the fundamental formulas

$$v = \frac{dx}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}$$

have a geometrical significance. The first formula expresses that the velocity at any instant is equal to the slope of the x–t curve at the same instant (Fig. 11.10). The second formula expresses that the acceleration is equal to the slope of the v–t curve. These two properties can be used to determine graphically the v–t and a–t curves of a motion when the x–t curve is known.

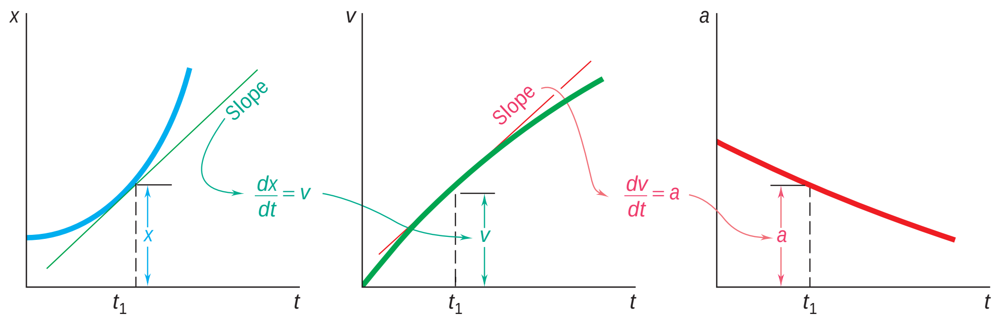


Fig. 11.10

Integrating the two fundamental formulas from a time t_1 to a time t_2, we write

$$x_2 - x_1 = \int_{t_1}^{t_2} v\, dt \qquad \text{and} \qquad v_2 - v_1 = \int_{t_1}^{t_2} a\, dt \tag{11.12}$$

The first formula expresses that the area measured under the v–t curve from t_1 to t_2 is equal to the change in x during that time interval (Fig. 11.11). Similarly, the second formula expresses that the area measured under the a–t curve from t_1 to t_2 is equal to the change in v during that time interval. These two properties can be used to determine graphically the x–t curve of a motion when its v–t curve or its a–t curve is known (see Sample Prob. 11.6).

Graphical solutions are particularly useful when the motion considered is defined from experimental data and when x, v, and a are not analytical functions of t. They can also be used to advantage when the motion consists of distinct parts and when its analysis requires writing a different equation for each of its parts. When using a graphical solution, however, one should be careful to note that (1) the area under the v–t curve measures the *change in* x, not x itself, and similarly, that the area under the a–t curve measures the change in v; (2) an area above the t axis corresponds to an *increase* in x or v, while an area located below the t axis measures a *decrease* in x or v.

It will be useful to remember in drawing motion curves that if the velocity is constant, it will be represented by a horizontal straight line; the position coordinate x will then be a linear function of t and will be represented by an oblique straight line. If the acceleration is

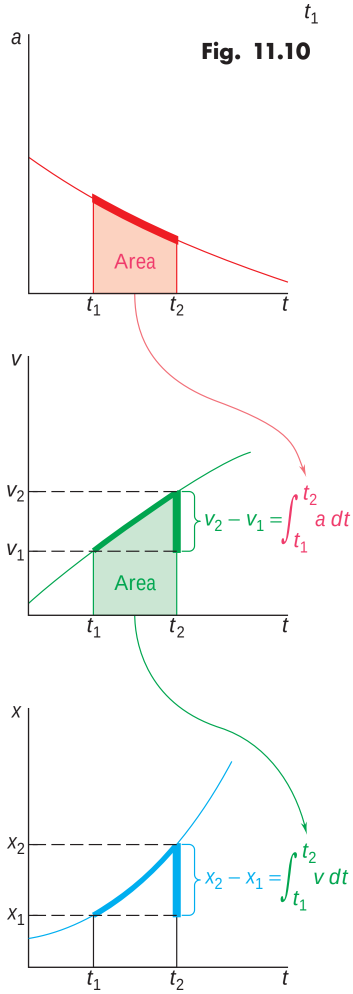


Fig. 11.11

constant and different from zero, it will be represented by a horizontal straight line; v will then be a linear function of t, represented by an oblique straight line, and x will be expressed as a second-degree polynomial in t, represented by a parabola. If the acceleration is a linear function of t, the velocity and the position coordinate will be equal, respectively, to second-degree and third-degree polynomials; a will then be represented by an oblique straight line, v by a parabola, and x by a cubic. In general, if the acceleration is a polynomial of degree n in t, the velocity will be a polynomial of degree $n + 1$ and the position coordinate a polynomial of degree $n + 2$; these polynomials are represented by motion curves of a corresponding degree.

*11.8 OTHER GRAPHICAL METHODS

An alternative graphical solution can be used to determine the position of a particle at a given instant directly from the a–t curve. Denoting the values of x and v at $t = 0$ as x_0 and v_0 and their values at $t = t_1$ as x_1 and v_1, and observing that the area under the v–t curve can be divided into a rectangle of area $v_0 t_1$ and horizontal differential elements of area $(t_1 - t)\, dv$ (Fig. 11.12*a*), we write

$$x_1 - x_0 = \text{area under } v\text{–}t \text{ curve} = v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t)\, dv$$

Substituting $dv = a\, dt$ in the integral, we obtain

$$x_1 - x_0 = v_0 t_1 + \int_0^{t_1} (t_1 - t)\, a\, dt$$

Referring to Fig. 11.12*b*, we note that the integral represents the first moment of the area under the a–t curve with respect to the line $t = t_1$ bounding the area on the right. This method of solution is known, therefore, as the *moment-area method*. If the abscissa $\bar{t}$ of the centroid C of the area is known, the position coordinate x_1 can be obtained by writing

$$x_1 = x_0 + v_0 t_1 + (\text{area under } a\text{–}t \text{ curve})(t_1 - \bar{t}) \qquad (11.13)$$

If the area under the a–t curve is a composite area, the last term in (11.13) can be obtained by multiplying each component area by the distance from its centroid to the line $t = t_1$. Areas above the t axis should be considered as positive and areas below the t axis as negative.

Another type of motion curve, the v–x curve, is sometimes used. If such a curve has been plotted (Fig. 11.13), the acceleration a can be obtained at any time by drawing the normal AC to the curve and *measuring the subnormal BC*. Indeed, observing that the angle between AC and AB is equal to the angle u between the horizontal and the tangent at A (the slope of which is tan u $= dv/dx$), we write

$$BC = AB \tan \text{u} = v \frac{dv}{dx}$$

and thus, recalling formula (11.4),

$$BC = a$$

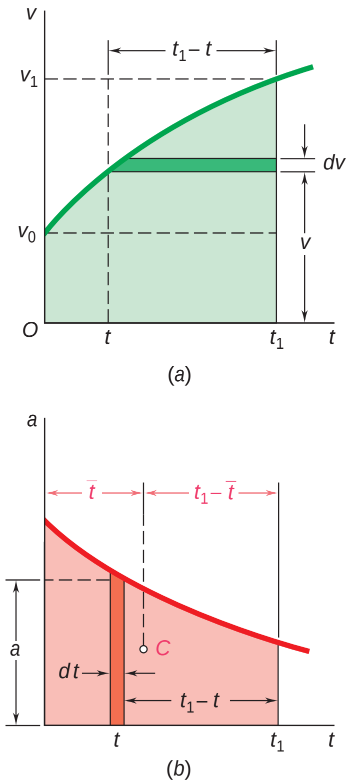


Fig. 11.12

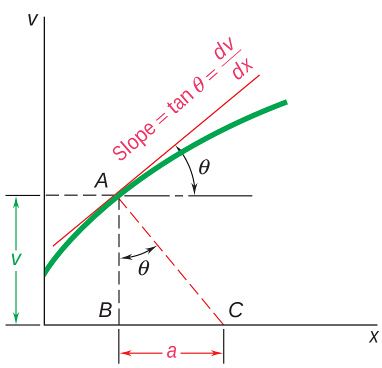
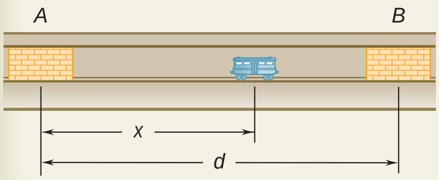


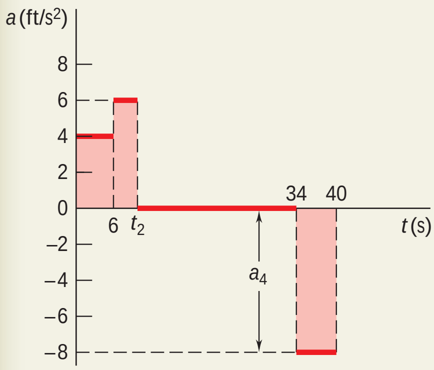
Fig. 11.13

SAMPLE PROBLEM 11.6



A subway car leaves station A; it gains speed at the rate of 4 ft/s^2 for 6 s and then at the rate of 6 ft/s^2 until it has reached the speed of 48 ft/s. The car maintains the same speed until it approaches station B; brakes are then applied, giving the car a constant deceleration and bringing it to a stop in 6 s. The total running time from A to B is 40 s. Draw the a–t, v–t, and x–t curves, and determine the distance between stations A and B.

SOLUTION



Acceleration-Time Curve. Since the acceleration is either constant or zero, the a–t curve is made of horizontal straight-line segments. The values of t_2 and a_4 are determined as follows:

$0 < t < 6$: Change in v = area under a–t curve

$$v_6 - 0 = (6 \text{ s})(4 \text{ ft/s}^2) = 24 \text{ ft/s}$$

$6 < t < t_2$: Since the velocity increases from 24 to 48 ft/s,

Change in v = area under a–t curve

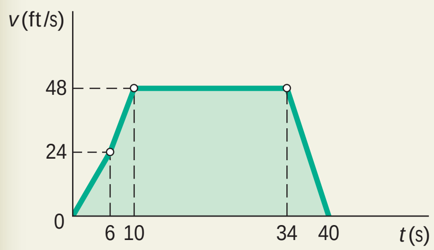
$$48 \text{ ft/s} - 24 \text{ ft/s} = (t_2 - 6)(6 \text{ ft/s}^2) \qquad t_2 = 10 \text{ s}$$

$t_2 < t < 34$: Since the velocity is constant, the acceleration is zero.

$34 < t < 40$: Change in v = area under a–t curve

$$0 - 48 \text{ ft/s} = (6 \text{ s})a_4 \qquad a_4 = -8 \text{ ft/s}^2$$

The acceleration being negative, the corresponding area is below the t axis; this area represents a decrease in velocity.



Velocity-Time Curve. Since the acceleration is either constant or zero, the v–t curve is made of straight-line segments connecting the points determined above.

Change in x = area under v–t curve

$0 < t < 6$: $\quad x_6 - 0 = \frac{1}{2}(6)(24) = 72$ ft

$6 < t < 10$: $\quad x_{10} - x_6 = \frac{1}{2}(4)(24 + 48) = 144$ ft

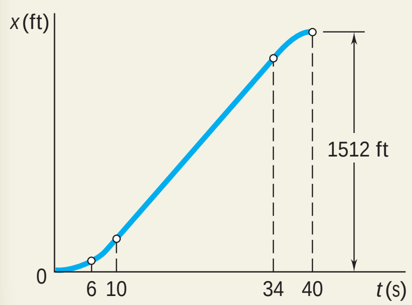
$10 < t < 34$: $\quad x_{34} - x_{10} = (24)(48) = 1152$ ft

$34 < t < 40$: $\quad x_{40} - x_{34} = \frac{1}{2}(6)(48) = 144$ ft

Adding the changes in x, we obtain the distance from A to B:

$$d = x_{40} - 0 = 1512 \text{ ft}$$

$d = 1512$ ft ◀



Position-Time Curve. The points determined above should be joined by three arcs of parabola and one straight-line segment. In constructing the x–t curve, keep in mind that for any value of t the slope of the tangent to the x–t curve is equal to the value of v at that instant.

SOLVING PROBLEMS ON YOUR OWN

In this lesson (Secs. 11.7 and 11.8), we reviewed and developed several *graphical techniques for the solution of problems involving rectilinear motion*. These techniques can be used to solve problems directly or to complement analytical methods of solution by providing a visual description, and thus a better understanding, of the motion of a given body. We suggest that you sketch one or more motion curves for several of the problems in this lesson, even if these problems are not part of your homework assignment.

1. Drawing *x–t*, *v–t*, and *a–t* curves and applying graphical methods. The following properties were indicated in Sec. 11.7 and should be kept in mind as you use a graphical method of solution.

a. The slopes of the *x–t* and *v–t* curves at a time t_1 are respectively equal to the *velocity* and the *acceleration* at time t_1.

b. The areas under the *a–t* and *v–t* curves between the times t_1 and t_2 are respectively equal to the change Δv in the velocity and to the change Δx in the position coordinate during that time interval.

c. If one of the motion curves is known, the fundamental properties we have summarized in paragraphs *a* and *b* will enable you to construct the other two curves. However, when using the properties of paragraph *b*, the velocity and the position coordinate at time t_1 must be known in order to determine the velocity and the position coordinate at time t_2. Thus, in Sample Prob. 11.6, knowing that the initial value of the velocity was zero allowed us to find the velocity at $t = 6$ s: $v_6 = v_0 + \Delta v = 0 + 24 \text{ ft/s} = 24 \text{ ft/s}$.

If you have previously studied the shear and bending-moment diagrams for a beam, you should recognize the analogy that exists between the three motion curves and the three diagrams representing respectively the distributed load, the shear, and the bending moment in the beam. Thus, any techniques that you learned regarding the construction of these diagrams can be applied when drawing the motion curves.

2. Using approximate methods. When the *a–t* and *v–t* curves are not represented by analytical functions or when they are based on experimental data, it is often necessary to use approximate methods to calculate the areas under these curves. In those cases, the given area is approximated by a series of rectangles of width Δt. The smaller the value of Δt, the smaller the error introduced by the approximation. The velocity and the position coordinate are obtained by writing

$$v = v_0 + \Sigma a_{\text{ave}}\,\Delta t \qquad x = x_0 + \Sigma v_{\text{ave}}\,\Delta t$$

where a_{ave} and v_{ave} are the heights of an acceleration rectangle and a velocity rectangle, respectively.

(continued)

3. Applying the moment-area method. This graphical technique is used when the $a\text{–}t$ curve is given and the change in the position coordinate is to be determined. We found in Sec. 11.8 that the position coordinate x_1 can be expressed as

$$x_1 = x_0 + v_0 t_1 + (\text{area under } a\text{–}t \text{ curve})(t_1 - \bar{t}) \tag{11.13}$$

Keep in mind that when the area under the $a\text{–}t$ curve is a composite area, the same value of t_1 should be used for computing the contribution of each of the component areas.

4. Determining the acceleration from a *v–x* curve. You saw in Sec. 11.8 that it is possible to determine the acceleration from a $v\text{–}x$ curve by direct measurement. It is important to note, however, that this method is applicable only if the same linear scale is used for the v and x axes (for example, 1 in. = 10 ft and 1 in. = 10 ft/s). When this condition is not satisfied, the acceleration can still be determined from the equation

$$a = v\frac{dv}{dx}$$

where the slope dv/dx is obtained as follows: First, draw the tangent to the curve at the point of interest. Next, using appropriate scales, measure along that tangent corresponding increments Δx and Δv. The desired slope is equal to the ratio $\Delta v/\Delta x$.

PROBLEMS

11.61 A particle moves in a straight line with the acceleration shown in the figure. Knowing that it starts from the origin with $v_0 = -14$ ft/s, plot the v–t and x–t curves for $0 < t < 15$ s and determine (*a*) the maximum value of the velocity of the particle, (*b*) the maximum value of its position coordinate.

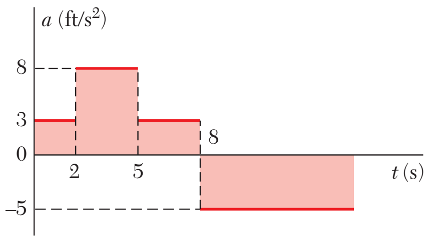


Fig. P11.61 and P11.62

11.62 For the particle and motion of Prob. 11.61, plot the v–t and x–t curves for $0 < t < 15$ s and determine the velocity of the particle, its position, and the total distance traveled after 10 s.

11.63 A particle moves in a straight line with the velocity shown in the figure. Knowing that $x = -540$ m at $t = 0$, (*a*) construct the a–t and x–t curves for $0 < t < 50$ s, and determine (*b*) the total distance traveled by the particle when $t = 50$ s, (*c*) the two times at which $x = 0$.

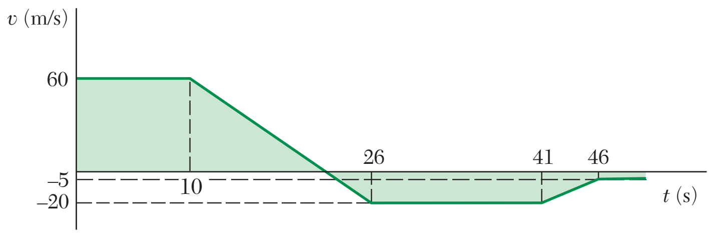


Fig. P11.63 and P11.64

11.64 A particle moves in a straight line with the velocity shown in the figure. Knowing that $x = -540$ m at $t = 0$, (*a*) construct the a–t and x–t curves for $0 < t < 50$ s, and determine (*b*) the maximum value of the position coordinate of the particle, (*c*) the values of t for which the particle is at $x = 100$ m.

11.65 During a finishing operation the bed of an industrial planer moves alternately 30 in. to the right and 30 in. to the left. The velocity of the bed is limited to a maximum value of 6 in./s to the right and 12 in./s to the left; the acceleration is successively equal to 6 in./s^2 to the right, zero, 6 in./s^2 to the left, zero, etc. Determine the time required for the bed to complete a full cycle, and draw the v–t and x–t curves.

11.66 A parachutist is in free fall at a rate of 200 km/h when he opens his parachute at an altitude of 600 m. Following a rapid and constant deceleration, he then descends at a constant rate of 50 km/h from 586 m to 30 m, where he maneuvers the parachute into the wind to further slow his descent. Knowing that the parachutist lands with a negligible downward velocity, determine (*a*) the time required for the parachutist to land after opening his parachute, (*b*) the initial deceleration.

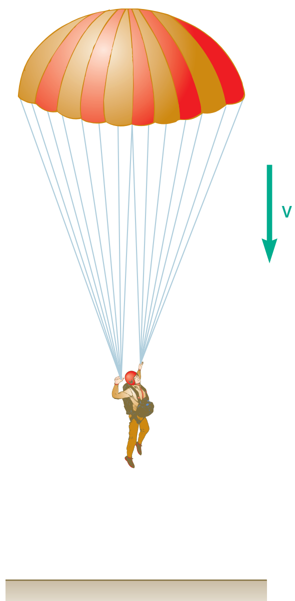


Fig. P11.66

11.67 A commuter train traveling at 40 mi/h is 3 mi from a station. The train then decelerates so that its speed is 20 mi/h when it is 0.5 mi from the station. Knowing that the train arrives at the station 7.5 min after beginning to decelerate and assuming constant decelerations, determine (*a*) the time required for the train to travel the first 2.5 mi, (*b*) the speed of the train as it arrives at the station, (*c*) the final constant deceleration of the train.

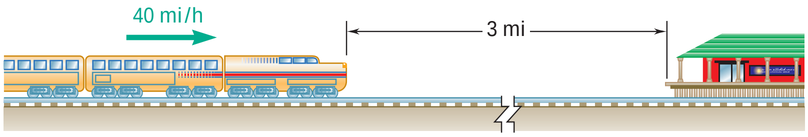


Fig. *P11.67*

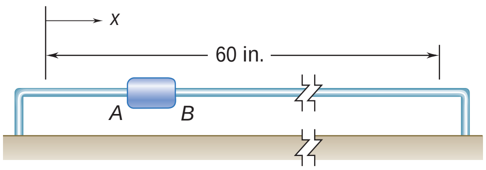


Fig. *P11.68*

11.68 A temperature sensor is attached to slider *AB* which moves back and forth through 60 in. The maximum velocities of the slider are 12 in./s to the right and 30 in./s to the left. When the slider is moving to the right, it accelerates and decelerates at a constant rate of 6 in./s^2; when moving to the left, the slider accelerates and decelerates at a constant rate of 20 in./s^2. Determine the time required for the slider to complete a full cycle, and construct the v–t and x–t curves of its motion.

11.69 In a water-tank test involving the launching of a small model boat, the model's initial horizontal velocity is 6 m/s and its horizontal acceleration varies linearly from -12 m/s^2 at $t = 0$ to -2 m/s^2 at $t = t_1$ and then remains equal to -2 m/s^2 until $t = 1.4$ s. Knowing that $v = 1.8$ m/s when $t = t_1$, determine (*a*) the value of t_1, (*b*) the velocity and the position of the model at $t = 1.4$ s.

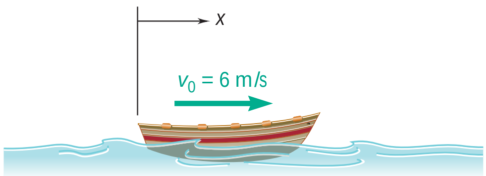


Fig. P11.69

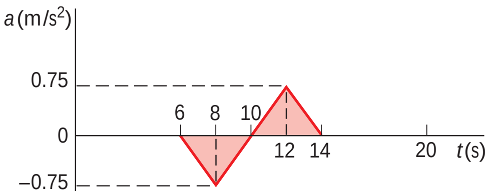


Fig. P11.70

11.70 The acceleration record shown was obtained for a small airplane traveling along a straight course. Knowing that $x = 0$ and $v = 60$ m/s when $t = 0$, determine (*a*) the velocity and position of the plane at $t = 20$ s, (*b*) its average velocity during the interval $6\text{ s} < t < 14\text{ s}$.

11.71 In a 400-m race, runner *A* reaches her maximum velocity v_A in 4 s with constant acceleration and maintains that velocity until she reaches the halfway point with a split time of 25 s. Runner *B* reaches her maximum velocity v_B in 5 s with constant acceleration and maintains that velocity until she reaches the halfway point with a split time of 25.2 s. Both runners then run the second half of the race with the same constant deceleration of 0.1 m/s^2. Determine (*a*) the race times for both runners, (*b*) the position of the winner relative to the loser when the winner reaches the finish line.

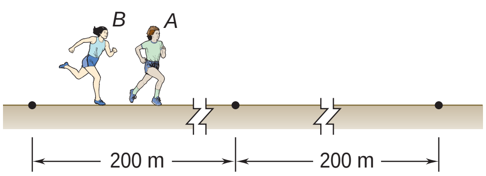


Fig. P11.71

11.72 A car and a truck are both traveling at the constant speed of 35 mi/h; the car is 40 ft behind the truck. The driver of the car wants to pass the truck, i.e., he wishes to place his car at *B*, 40 ft in front of the truck, and then resume the speed of 35 mi/h. The maximum acceleration of the car is 5 ft/s^2 and the maximum deceleration obtained by applying the brakes is 20 ft/s^2. What is the shortest time in which the driver of the car can complete the passing operation if he does not at any time exceed a speed of 50 mi/h? Draw the v–t curve.

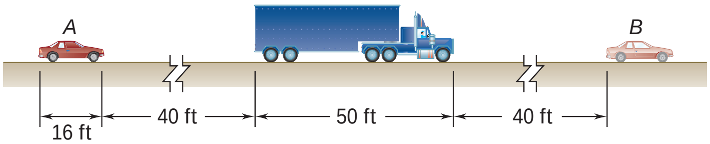


Fig. P11.72

11.73 Solve Prob. 11.72, assuming that the driver of the car does not pay any attention to the speed limit while passing and concentrates on reaching position *B* and resuming a speed of 35 mi/h in the shortest possible time. What is the maximum speed reached? Draw the v–t curve.

11.74 Car *A* is traveling on a highway at a constant speed $(v_A)_0 = 60$ mi/h and is 380 ft from the entrance of an access ramp when car *B* enters the acceleration lane at that point at a speed $(v_B)_0 = 15$ mi/h. Car *B* accelerates uniformly and enters the main traffic lane after traveling 200 ft in 5 s. It then continues to accelerate at the same rate until it reaches a speed of 60 mi/h, which it then maintains. Determine the final distance between the two cars.

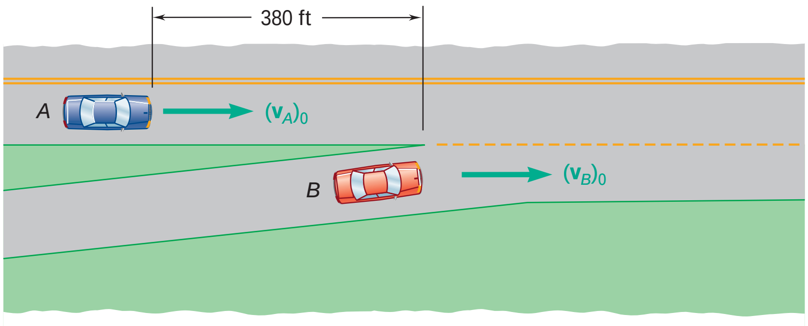


Fig. P11.74

11.75 An elevator starts from rest and moves upward, accelerating at a rate of 1.2 m/s^2 until it reaches a speed of 7.8 m/s, which it then maintains. Two seconds after the elevator begins to move, a man standing 12 m above the initial position of the top of the elevator throws a ball upward with an initial velocity of 20 m/s. Determine when the ball will hit the elevator.

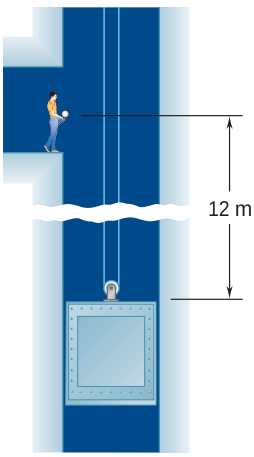


Fig. P11.75

11.76 Car A is traveling at 40 mi/h when it enters a 30 mi/h speed zone. The driver of car A decelerates at a rate of 16 ft/s^2 until reaching a speed of 30 mi/h, which she then maintains. When car B, which was initially 60 ft behind car A and traveling at a constant speed of 45 mi/h, enters the speed zone, its driver decelerates at a rate of 20 ft/s^2 until reaching a speed of 28 mi/h. Knowing that the driver of car B maintains a speed of 28 mi/h, determine (*a*) the closest that car B comes to car A, (*b*) the time at which car A is 70 ft in front of car B.

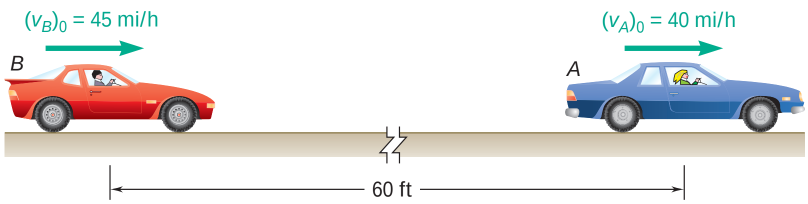


Fig. *P11.76*

11.77 An accelerometer record for the motion of a given part of a mechanism is approximated by an arc of a parabola for 0.2 s and a straight line for the next 0.2 s as shown in the figure. Knowing that $v = 0$ when $t = 0$ and $x = 0.8$ ft when $t = 0.4$ s, (*a*) construct the v–t curve for $0 \leq t \leq 0.4$ s, (*b*) determine the position of the part at $t = 0.3$ s and $t = 0.2$ s.

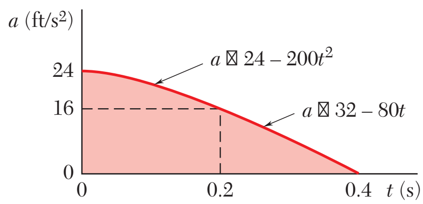


Fig. *P11.77*

11.78 A car is traveling at a constant speed of 54 km/h when its driver sees a child run into the road. The driver applies her brakes until the child returns to the sidewalk and then accelerates to resume her original speed of 54 km/h; the acceleration record of the car is shown in the figure. Assuming $x = 0$ when $t = 0$, determine (*a*) the time t_1 at which the velocity is again 54 km/h, (*b*) the position of the car at that time, (*c*) the average velocity of the car during the interval $1 \text{ s} \leq t \leq t_1$.

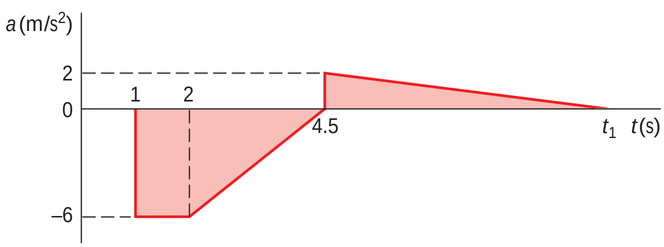


Fig. P11.78

11.79 An airport shuttle train travels between two terminals that are 1.6 mi apart. To maintain passenger comfort, the acceleration of the train is limited to $\pm 4 \text{ ft/s}^2$, and the jerk, or rate of change of acceleration, is limited to $\pm 0.8 \text{ ft/s}^2$ per second. If the shuttle has a maximum speed of 20 mi/h, determine (*a*) the shortest time for the shuttle to travel between the two terminals, (*b*) the corresponding average velocity of the shuttle.

11.80 During a manufacturing process, a conveyor belt starts from rest and travels a total of 1.2 ft before temporarily coming to rest. Knowing that the jerk, or rate of change of acceleration, is limited to $\pm 4.8 \text{ ft/s}^2$ per second, determine (*a*) the shortest time required for the belt to move 1.2 ft, (*b*) the maximum and average values of the velocity of the belt during that time.

11.81 Two seconds are required to bring the piston rod of an air cylinder to rest; the acceleration record of the piston rod during the 2 s is as shown. Determine by approximate means (*a*) the initial velocity of the piston rod, (*b*) the distance traveled by the piston rod as it is brought to rest.

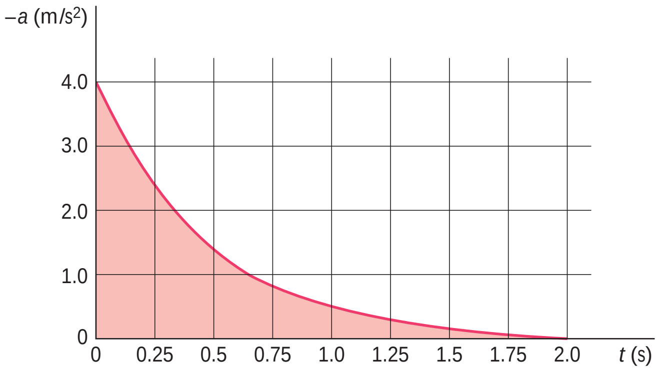


Fig. *P11.81*

11.82 The acceleration record shown was obtained during the speed trials of a sports car. Knowing that the car starts from rest, determine by approximate means (*a*) the velocity of the car at $t = 8$ s, (*b*) the distance the car has traveled at $t = 20$ s.

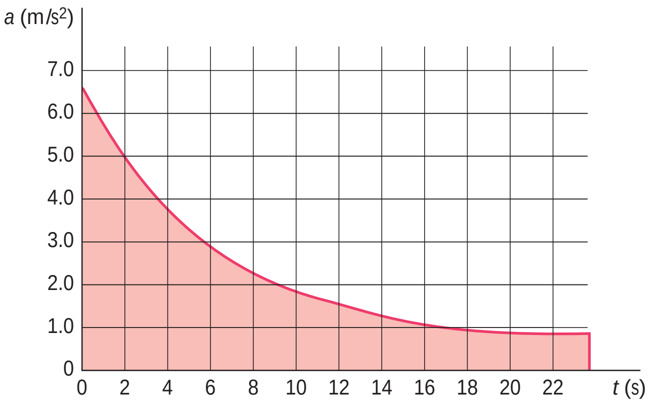


Fig. *P11.82*

11.83 A training airplane has a velocity of 126 ft/s when it lands on an aircraft carrier. As the arresting gear of the carrier brings the airplane to rest, the velocity and the acceleration of the airplane are recorded; the results are shown (solid curve) in the figure. Determine by approximate means (*a*) the time required for the airplane to come to rest, (*b*) the distance traveled in that time.

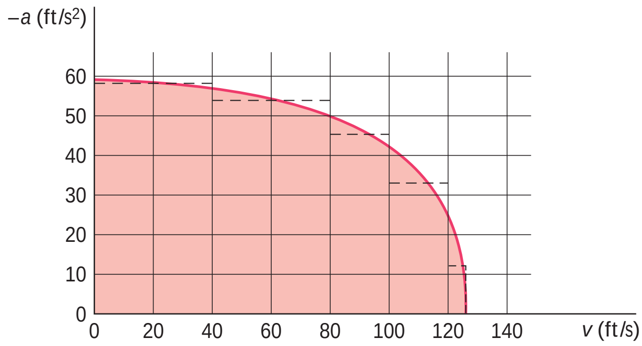


Fig. P11.83

11.84 Shown in the figure is a portion of the experimentally determined v–x curve for a shuttle cart. Determine by approximate means the acceleration of the cart when (*a*) $x = 10$ in., (*b*) $v = 80$ in./s.

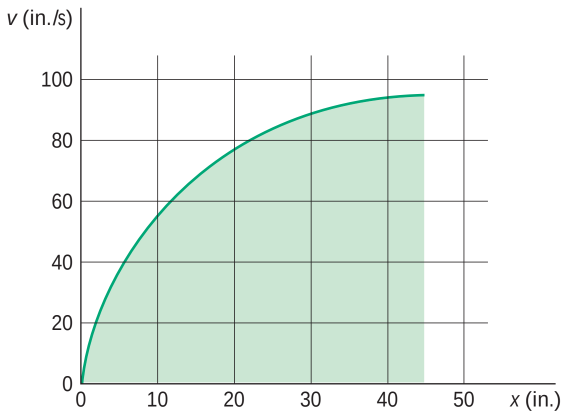


Fig. P11.84

11.85 Using the method of Sec. 11.8, derive the formula $x = x_0 + v_0 t + \frac{1}{2}at^2$ for the position coordinate of a particle in uniformly accelerated rectilinear motion.

11.86 Using the method of Sec. 11.8, determine the position of the particle of Prob. 11.61 when $t = 8$ s.

11.87 The acceleration of an object subjected to the pressure wave of a large explosion is defined approximately by the curve shown. The object is initially at rest and is again at rest at time t_1. Using the method of Sec. 11.8, determine (*a*) the time t_1, (*b*) the distance through which the object is moved by the pressure wave.

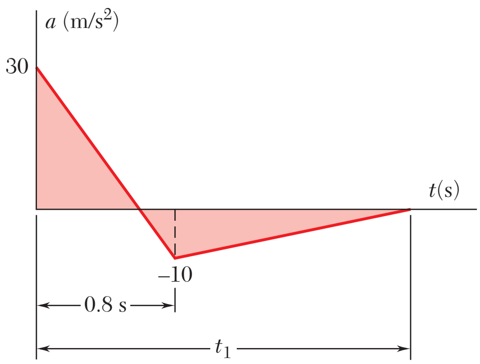


Fig. *P11.87*

11.88 For the particle of Prob. 11.63, draw the a–t curve and determine, using the method of Sec. 11.8, (*a*) the position of the particle when $t = 52$ s, (*b*) the maximum value of its position coordinate.

CURVILINEAR MOTION OF PARTICLES

11.9 POSITION VECTOR, VELOCITY, AND ACCELERATION

When a particle moves along a curve other than a straight line, we say that the particle is in *curvilinear motion*. To define the position P occupied by the particle at a given time t, we select a fixed reference system, such as the x, y, z axes shown in Fig. 11.14*a*, and draw the vector $\mathbf{r}$ joining the origin O and point P. Since the vector $\mathbf{r}$ is characterized by its magnitude r and its direction with respect to the reference axes, it completely defines the position of the particle with respect to those axes; the vector $\mathbf{r}$ is referred to as the *position vector* of the particle at time t.

Consider now the vector $\mathbf{r}'$ defining the position P' occupied by the same particle at a later time $t + \Delta t$. The vector $\Delta\mathbf{r}$ joining P and P' represents the change in the position vector during the time interval Δt since, as we can easily check from Fig. 11.14*a*, the vector $\mathbf{r}'$ is obtained by adding the vectors $\mathbf{r}$ and $\Delta\mathbf{r}$ according to the triangle rule. We note that $\Delta\mathbf{r}$ represents a change in *direction* as well as a change in *magnitude* of the position vector $\mathbf{r}$. The *average velocity* of the particle over the time interval Δt is defined as the quotient of $\Delta\mathbf{r}$ and Δt. Since $\Delta\mathbf{r}$ is a vector and Δt is a scalar, the quotient $\Delta\mathbf{r}/\Delta t$ is a vector attached at P, of the same direction as $\Delta\mathbf{r}$ and of magnitude equal to the magnitude of $\Delta\mathbf{r}$ divided by Δt (Fig. 11.14*b*).

The *instantaneous velocity* of the particle at time t is obtained by choosing shorter and shorter time intervals Δt and, correspondingly, shorter and shorter vector increments $\Delta\mathbf{r}$. The instantaneous velocity is thus represented by the vector

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} \tag{11.14}$$

As Δt and $\Delta\mathbf{r}$ become shorter, the points P and P' get closer; the vector $\mathbf{v}$ obtained in the limit must therefore be tangent to the path of the particle (Fig. 11.14*c*).

Since the position vector $\mathbf{r}$ depends upon the time t, we can refer to it as a *vector function* of the scalar variable t and denote it by $\mathbf{r}(t)$. Extending the concept of derivative of a scalar function introduced in elementary calculus, we will refer to the limit of the quotient $\Delta\mathbf{r}/\Delta t$ as the *derivative* of the vector function $\mathbf{r}(t)$. We write

$$\boxed{\mathbf{v} = \frac{d\mathbf{r}}{dt}} \tag{11.15}$$

The magnitude v of the vector $\mathbf{v}$ is called the *speed* of the particle. It can be obtained by substituting for the vector $\Delta\mathbf{r}$ in formula (11.14) the magnitude of this vector represented by the straight-line segment PP'. But the length of the segment PP' approaches the length Δs of the arc PP' as Δt decreases (Fig. 11.14*a*), and we can write

$$v = \lim_{\Delta t \to 0} \frac{PP'}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \qquad \boxed{v = \frac{ds}{dt}} \tag{11.16}$$

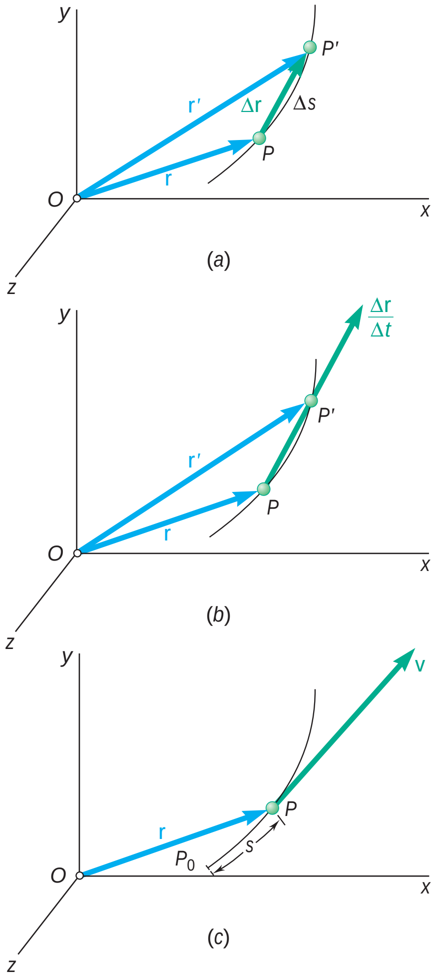
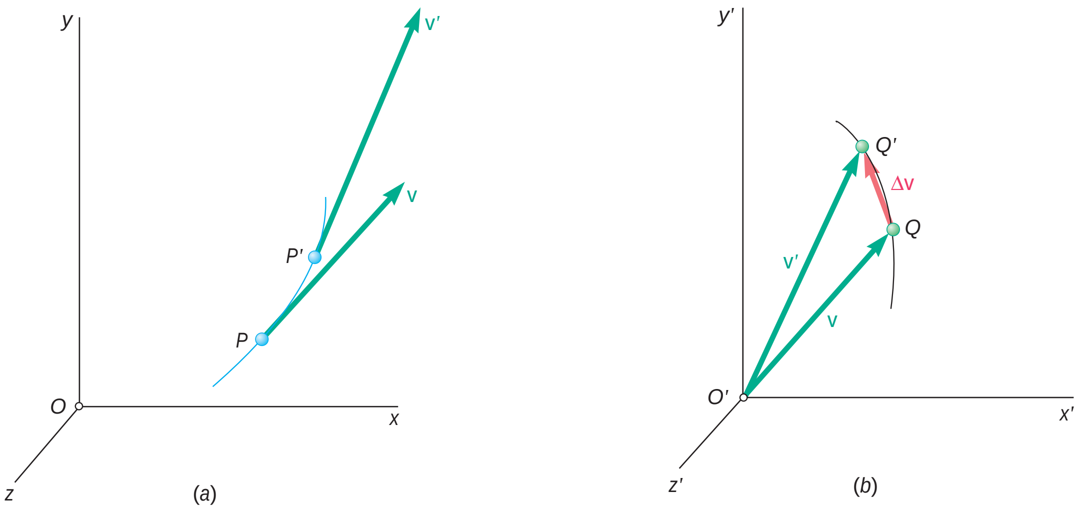


Fig. 11.14



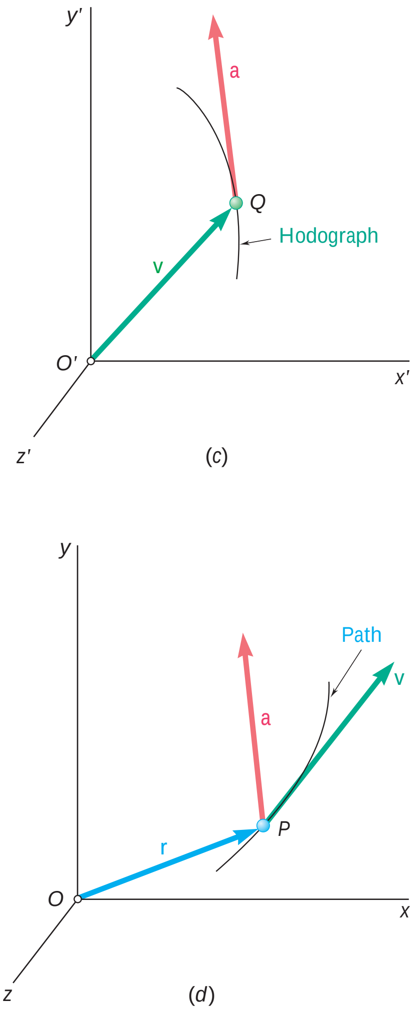


Fig. 11.15

The speed v can thus be obtained by differentiating with respect to t the length s of the arc described by the particle.

Consider the velocity $\mathbf{v}$ of the particle at time t and its velocity $\mathbf{v}'$ at a later time $t + \Delta t$ (Fig. 11.15*a*). Let us draw both vectors $\mathbf{v}$ and $\mathbf{v}'$ from the same origin O' (Fig. 11.15*b*). The vector $\Delta\mathbf{v}$ joining Q and Q' represents the change in the velocity of the particle during the time interval Δt, since the vector $\mathbf{v}'$ can be obtained by adding the vectors $\mathbf{v}$ and $\Delta\mathbf{v}$. We should note that $\Delta\mathbf{v}$ represents a change in the *direction* of the velocity as well as a change in *speed*. The *average acceleration* of the particle over the time interval Δt is defined as the quotient of $\Delta\mathbf{v}$ and Δt. Since $\Delta\mathbf{v}$ is a vector and Δt a scalar, the quotient $\Delta\mathbf{v}/\Delta t$ is a vector of the same direction as $\Delta\mathbf{v}$.

The *instantaneous acceleration* of the particle at time t is obtained by choosing smaller and smaller values for Δt and $\Delta\mathbf{v}$. The instantaneous acceleration is thus represented by the vector

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} \tag{11.17}$$

Noting that the velocity $\mathbf{v}$ is a vector function $\mathbf{v}(t)$ of the time t, we can refer to the limit of the quotient $\Delta\mathbf{v}/\Delta t$ as the derivative of $\mathbf{v}$ with respect to t. We write

$$\boxed{\mathbf{a} = \frac{d\mathbf{v}}{dt}} \tag{11.18}$$

We observe that the acceleration $\mathbf{a}$ is tangent to the curve described by the tip Q of the vector $\mathbf{v}$ when the latter is drawn from a fixed origin O' (Fig. 11.15*c*) and that, in general, the acceleration is *not* tangent to the path of the particle (Fig. 11.15*d*). The curve described by the tip of $\mathbf{v}$ and shown in Fig. 11.15*c* is called the *hodograph* of the motion.

11.10 DERIVATIVES OF VECTOR FUNCTIONS

We saw in the preceding section that the velocity $\mathbf{v}$ of a particle in curvilinear motion can be represented by the derivative of the vector function $\mathbf{r}(t)$ characterizing the position of the particle. Similarly, the acceleration $\mathbf{a}$ of the particle can be represented by the derivative of the vector function $\mathbf{v}(t)$. In this section, we will give a formal definition of the derivative of a vector function and establish a few rules governing the differentiation of sums and products of vector functions.

Let $\mathbf{P}(u)$ be a vector function of the scalar variable u. By that we mean that the scalar u completely defines the magnitude and direction of the vector $\mathbf{P}$. If the vector $\mathbf{P}$ is drawn from a fixed origin O and the scalar u is allowed to vary, the tip of $\mathbf{P}$ will describe a given curve in space. Consider the vectors $\mathbf{P}$ corresponding, respectively, to the values u and $u + \Delta u$ of the scalar variable (Fig. 11.16*a*). Let $\Delta\mathbf{P}$ be the vector joining the tips of the two given vectors; we write

$$\Delta\mathbf{P} = \mathbf{P}(u + \Delta u) - \mathbf{P}(u)$$

Dividing through by Δu and letting Δu approach zero, *we define the derivative of the vector function* $\mathbf{P}(u)$:

$$\frac{d\mathbf{P}}{du} = \lim_{\Delta u \to 0} \frac{\Delta\mathbf{P}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\mathbf{P}(u + \Delta u) - \mathbf{P}(u)}{\Delta u} \tag{11.19}$$

As Δu approaches zero, the line of action of $\Delta\mathbf{P}$ becomes tangent to the curve of Fig. 11.16*a*. Thus, the derivative $d\mathbf{P}/du$ of the vector function $\mathbf{P}(u)$ *is tangent to the curve described by the tip of* $\mathbf{P}(u)$ (Fig. 11.16*b*).

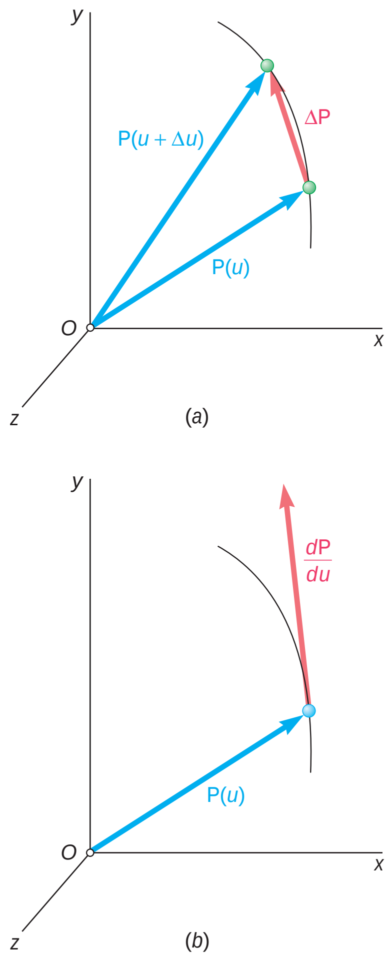


Fig. 11.16

The standard rules for the differentiation of the sums and products of scalar functions can be extended to vector functions. Consider first the *sum of two vector functions* $\mathbf{P}(u)$ and $\mathbf{Q}(u)$ of the same scalar variable u. According to the definition given in (11.19), the derivative of the vector $\mathbf{P} + \mathbf{Q}$ is

$$\frac{d(\mathbf{P} + \mathbf{Q})}{du} = \lim_{\Delta u \to 0} \frac{\Delta(\mathbf{P} + \mathbf{Q})}{\Delta u} = \lim_{\Delta u \to 0} \left(\frac{\Delta\mathbf{P}}{\Delta u} + \frac{\Delta\mathbf{Q}}{\Delta u}\right)$$

or since the limit of a sum is equal to the sum of the limits of its terms,

$$\frac{d(\mathbf{P} + \mathbf{Q})}{du} = \lim_{\Delta u \to 0} \frac{\Delta\mathbf{P}}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta\mathbf{Q}}{\Delta u}$$

$$\frac{d(\mathbf{P} + \mathbf{Q})}{du} = \frac{d\mathbf{P}}{du} + \frac{d\mathbf{Q}}{du} \tag{11.20}$$

The *product of a scalar function f(u) and a vector function* $\mathbf{P}(u)$ of the same scalar variable u will now be considered. The derivative of the vector $f\mathbf{P}$ is

$$\frac{d(f\mathbf{P})}{du} = \lim_{\Delta u \to 0} \frac{(f + \Delta f)(\mathbf{P} + \Delta\mathbf{P}) - f\mathbf{P}}{\Delta u} = \lim_{\Delta u \to 0} \left(\frac{\Delta f}{\Delta u}\mathbf{P} + f\frac{\Delta\mathbf{P}}{\Delta u}\right)$$

or recalling the properties of the limits of sums and products,

$$\frac{d(f\mathbf{P})}{du} = \frac{df}{du}\mathbf{P} + f\frac{d\mathbf{P}}{du} \tag{11.21}$$

The derivatives of the *scalar product* and the *vector product* of two vector functions $\mathbf{P}(u)$ and $\mathbf{Q}(u)$ can be obtained in a similar way. We have

$$\frac{d(\mathbf{P}\cdot\mathbf{Q})}{du} = \frac{d\mathbf{P}}{du}\cdot\mathbf{Q} + \mathbf{P}\cdot\frac{d\mathbf{Q}}{du} \tag{11.22}$$

$$\frac{d(\mathbf{P}\times\mathbf{Q})}{du} = \frac{d\mathbf{P}}{du}\times\mathbf{Q} + \mathbf{P}\times\frac{d\mathbf{Q}}{du} \tag{11.23}$$†

The properties established above can be used to determine the *rectangular components of the derivative of a vector function* $\mathbf{P}(u)$. Resolving $\mathbf{P}$ into components along fixed rectangular axes x, y, z, we write

$$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k} \tag{11.24}$$

where P_x, P_y, P_z are the rectangular scalar components of the vector $\mathbf{P}$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors corresponding, respectively, to the x, y, and z axes (Sec. 2.12). By (11.20), the derivative of $\mathbf{P}$ is equal to the sum of the derivatives of the terms in the right-hand member. Since each of these terms is the product of a scalar and a vector function, we should use (11.21). But the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ have a constant magnitude (equal to 1) and fixed directions. Their derivatives are therefore zero, and we write

$$\frac{d\mathbf{P}}{du} = \frac{dP_x}{du}\mathbf{i} + \frac{dP_y}{du}\mathbf{j} + \frac{dP_z}{du}\mathbf{k} \tag{11.25}$$

Noting that the coefficients of the unit vectors are, by definition, the scalar components of the vector $d\mathbf{P}/du$, we conclude that *the rectangular scalar components of the derivative $d\mathbf{P}/du$ of the vector function $\mathbf{P}(u)$* are obtained by differentiating the corresponding scalar components of $\mathbf{P}$.

Rate of Change of a Vector. When the vector $\mathbf{P}$ is a function of the time t, its derivative $d\mathbf{P}/dt$ represents the *rate of change* of $\mathbf{P}$ with respect to the frame $Oxyz$. Resolving $\mathbf{P}$ into rectangular components, we have, by (11.25),

$$\frac{d\mathbf{P}}{dt} = \frac{dP_x}{dt}\mathbf{i} + \frac{dP_y}{dt}\mathbf{j} + \frac{dP_z}{dt}\mathbf{k}$$

or, using dots to indicate differentiation with respect to t,

$$\dot{\mathbf{P}} = \dot{P}_x\mathbf{i} + \dot{P}_y\mathbf{j} + \dot{P}_z\mathbf{k} \tag{11.25$'$}$$

†Since the vector product is not commutative (Sec. 3.4), the order of the factors in Eq. (11.23) must be maintained.

As you will see in Sec. 15.10, the rate of change of a vector as observed from a *moving frame of reference* is, in general, different from its rate of change as observed from a fixed frame of reference. However, if the moving frame $O'x'y'z'$ is in *translation*, i.e., if its axes remain parallel to the corresponding axes of the fixed frame $Oxyz$ (Fig. 11.17), the same unit vectors **i**, **j**, **k** are used in both frames, and at any given instant the vector **P** has the same components P_x, P_y, P_z in both frames. It follows from (11.25′) that the rate of change $\dot{\mathbf{P}}$ is the same with respect to the frames $Oxyz$ and $O'x'y'z'$. We state, therefore: *The rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation.* This property will greatly simplify our work, since we will be concerned mainly with frames in translation.

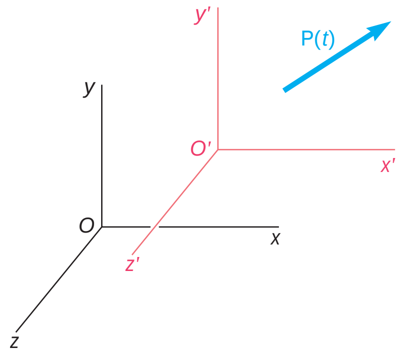


Fig. 11.17

11.11 RECTANGULAR COMPONENTS OF VELOCITY AND ACCELERATION

When the position of a particle P is defined at any instant by its rectangular coordinates x, y, and z, it is convenient to resolve the velocity **v** and the acceleration **a** of the particle into rectangular components (Fig. 11.18).

Resolving the position vector **r** of the particle into rectangular components, we write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{11.26}$$

where the coordinates x, y, z are functions of t. Differentiating twice, we obtain

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{11.27}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \tag{11.28}$$

where $\dot{x}, \dot{y}, \dot{z}$ and $\ddot{x}, \ddot{y}, \ddot{z}$ represent, respectively, the first and second derivatives of x, y, and z with respect to t. It follows from (11.27) and (11.28) that the scalar components of the velocity and acceleration are

$$v_x = \dot{x} \qquad v_y = \dot{y} \qquad v_z = \dot{z} \tag{11.29}$$

$$a_x = \ddot{x} \qquad a_y = \ddot{y} \qquad a_z = \ddot{z} \tag{11.30}$$

A positive value for v_x indicates that the vector component $\mathbf{v}_x$ is directed to the right, and a negative value indicates that it is directed to the left. The sense of each of the other vector components can be determined in a similar way from the sign of the corresponding scalar component. If desired, the magnitudes and directions of the velocity and acceleration can be obtained from their scalar components by the methods of Secs. 2.7 and 2.12.

The use of rectangular components to describe the position, the velocity, and the acceleration of a particle is particularly effective when the component a_x of the acceleration depends only upon t, x, and/or v_x, and when, similarly, a_y depends only upon t, y, and/or v_y,

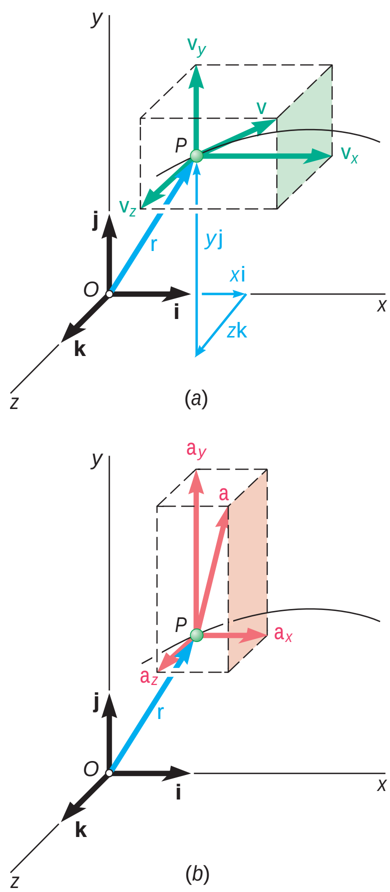
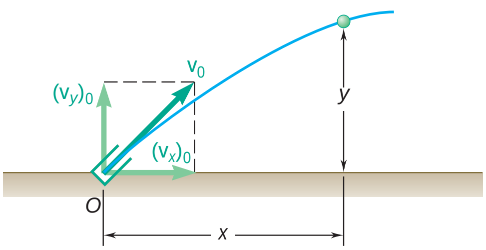
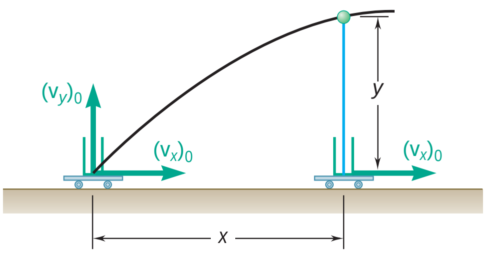


Fig. 11.18

Photo 11.3 The motion of this snowboarder in the air will be a parabola assuming we can neglect air resistance.



(*a*) Motion of a projectile



(*b*) Equivalent rectilinear motions

Fig. 11.19

and a_z upon t, z, and/or v_z. Equations (11.30) can then be integrated independently, and so can Eqs. (11.29). In other words, the motion of the particle in the x direction, its motion in the y direction, and its motion in the z direction can be considered separately.

In the case of the *motion of a projectile*, for example, it can be shown (see Sec. 12.5) that the components of the acceleration are

$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0$$

if the resistance of the air is neglected. Denoting by x_0, y_0, and z_0 the coordinates of a gun, and by $(v_x)_0$, $(v_y)_0$, and $(v_z)_0$ the components of the initial velocity $\mathbf{v}_0$ of the projectile (a bullet), we integrate twice in t and obtain

$$\begin{array}{lll} v_x = \dot{x} = (v_x)_0 & v_y = \dot{y} = (v_y)_0 - gt & v_z = \dot{z} = (v_z)_0 \\ x = x_0 + (v_x)_0 t & y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2 & z = z_0 + (v_z)_0 t \end{array}$$

If the projectile is fired in the xy plane from the origin O, we have $x_0 = y_0 = z_0 = 0$ and $(v_z)_0 = 0$, and the equations of motion reduce to

$$\begin{array}{lll} v_x = (v_x)_0 & v_y = (v_y)_0 - gt & v_z = 0 \\ x = (v_x)_0 t & y = (v_y)_0 t - \frac{1}{2}gt^2 & z = 0 \end{array}$$

These equations show that the projectile remains in the xy plane, that its motion in the horizontal direction is uniform, and that its motion in the vertical direction is uniformly accelerated. The motion of a projectile can thus be replaced by two independent rectilinear motions, which are easily visualized if we assume that the projectile is fired vertically with an initial velocity $(\mathbf{v}_y)_0$ from a platform moving with a constant horizontal velocity $(\mathbf{v}_x)_0$ (Fig. 11.19). The coordinate x of the projectile is equal at any instant to the distance traveled by the platform, and its coordinate y can be computed as if the projectile were moving along a vertical line.

It can be observed that the equations defining the coordinates x and y of a projectile at any instant are the parametric equations of a parabola. Thus, the trajectory of a projectile is *parabolic*. This result, however, ceases to be valid when the resistance of the air or the variation with altitude of the acceleration of gravity is taken into account.

11.12 MOTION RELATIVE TO A FRAME IN TRANSLATION

In the preceding section, a single frame of reference was used to describe the motion of a particle. In most cases this frame was attached to the earth and was considered as fixed. Situations in which it is convenient to use several frames of reference simultaneously will now be analyzed. If one of the frames is attached to the earth, it will be called a *fixed frame of reference*, and the other frames will be referred to as *moving frames of reference*. It should be understood, however, that the selection of a fixed frame of reference is purely arbitrary. Any frame can be designated as "fixed"; all other frames not rigidly attached to this frame will then be described as "moving."

Consider two particles A and B moving in space (Fig. 11.20); the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ define their positions at any given instant with respect to the fixed frame of reference $Oxyz$. Consider now a system of axes x', y', z' centered at A and parallel to the x, y, z axes. While the origin of these axes moves, their orientation remains the same; the frame of reference $Ax'y'z'$ is in *translation* with respect to $Oxyz$. The vector $\mathbf{r}_{B/A}$ joining A and B defines *the position of B relative to the moving frame* $Ax'y'z'$ (or, for short, *the position of B relative to* A).

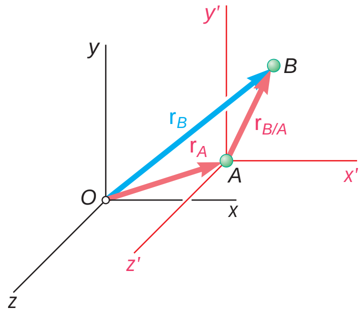


Fig. 11.20

We note from Fig. 11.20 that the position vector $\mathbf{r}_B$ of particle B is the sum of the position vector $\mathbf{r}_A$ of particle A and of the position vector $\mathbf{r}_{B/A}$ of B relative to A; we write

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \tag{11.31}$$

Differentiating (11.31) with respect to t within the fixed frame of reference, and using dots to indicate time derivatives, we have

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{B/A} \tag{11.32}$$

The derivatives $\dot{\mathbf{r}}_A$ and $\dot{\mathbf{r}}_B$ represent, respectively, the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of the particles A and B. Since $Ax'y'z'$ is in translation, the derivative $\dot{\mathbf{r}}_{B/A}$ represents the rate of change of $\mathbf{r}_{B/A}$ with respect to the frame $Ax'y'z'$ as well as with respect to the fixed frame (Sec. 11.10). This derivative, therefore, defines *the velocity* $\mathbf{v}_{B/A}$ *of B relative to the frame* $Ax'y'z'$ (or, for short, *the velocity* $\mathbf{v}_{B/A}$ *of B relative to* A). We write

Photo 11.4 The pilot of a helicopter must take into account the relative motion of the ship when landing.

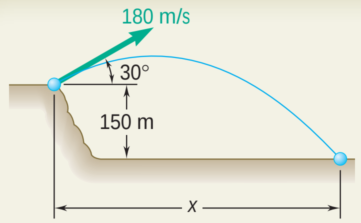
$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{11.33}$$

Differentiating Eq. (11.33) with respect to t, and using the derivative $\dot{\mathbf{v}}_{B/A}$ to define *the acceleration* $\mathbf{a}_{B/A}$ *of B relative to the frame* $Ax'y'z'$ (or, for short, *the acceleration* $\mathbf{a}_{B/A}$ *of B relative to* A), we write

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{11.34}$$

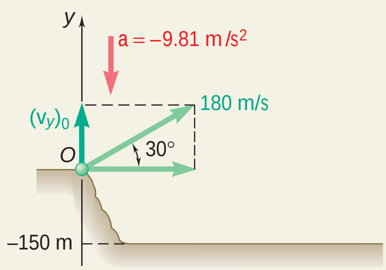
The motion of B with respect to the fixed frame $Oxyz$ is referred to as the *absolute motion of B*. The equations derived in this section show that *the absolute motion of B can be obtained by combining the motion of A and the relative motion of B with respect to the moving frame attached to A*. Equation (11.33), for example, expresses that the absolute velocity $\mathbf{v}_B$ of particle B can be obtained by adding vectorially the velocity of A and the velocity of B relative to the frame $Ax'y'z'$. Equation (11.34) expresses a similar property in terms of the accelerations.† We should keep in mind, however, that *the frame* $Ax'y'z'$ *is in translation;* that is, while it moves with A, it maintains the same orientation. As you will see later (Sec. 15.14), different relations must be used in the case of a rotating frame of reference.

†Note that the product of the subscripts A and B/A used in the right-hand member of Eqs. (11.31) through (11.34) is equal to the subscript B used in their left-hand member.



SAMPLE PROBLEM 11.7

A projectile is fired from the edge of a 150-m cliff with an initial velocity of 180 m/s at an angle of 30° with the horizontal. Neglecting air resistance, find (*a*) the horizontal distance from the gun to the point where the projectile strikes the ground, (*b*) the greatest elevation above the ground reached by the projectile.



SOLUTION

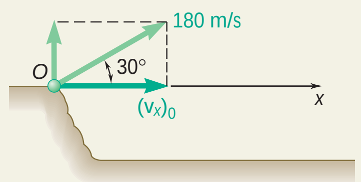
The vertical and the horizontal motion will be considered separately.

Vertical Motion. ***Uniformly Accelerated Motion.*** Choosing the positive sense of the y axis upward and placing the origin O at the gun, we have

$$(v_y)_0 = (180 \text{ m/s}) \sin 30° = +90 \text{ m/s}$$
$$a = -9.81 \text{ m/s}^2$$

Substituting into the equations of uniformly accelerated motion, we have

$$v_y = (v_y)_0 + at \qquad v_y = 90 - 9.81t \qquad (1)$$
$$y = (v_y)_0 t + \tfrac{1}{2}at^2 \qquad y = 90t - 4.90t^2 \qquad (2)$$
$$v_y^2 = (v_y)_0^2 + 2ay \qquad v_y^2 = 8100 - 19.62y \qquad (3)$$



Horizontal Motion. ***Uniform Motion.*** Choosing the positive sense of the x axis to the right, we have

$$(v_x)_0 = (180 \text{ m/s}) \cos 30° = +155.9 \text{ m/s}$$

Substituting into the equation of uniform motion, we obtain

$$x = (v_x)_0 t \qquad x = 155.9t \qquad (4)$$

a. Horizontal Distance. When the projectile strikes the ground, we have

$$y = -150 \text{ m}$$

Carrying this value into Eq. (2) for the vertical motion, we write

$$-150 = 90t - 4.90t^2 \qquad t^2 - 18.37t - 30.6 = 0 \qquad t = 19.91 \text{ s}$$

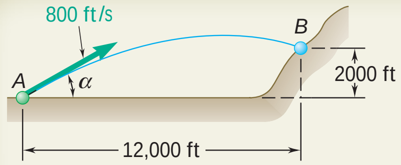
Carrying $t = 19.91$ s into Eq. (4) for the horizontal motion, we obtain

$$x = 155.9(19.91) \qquad x = 3100 \text{ m} \quad ◀$$

b. Greatest Elevation. When the projectile reaches its greatest elevation, we have $v_y = 0$; carrying this value into Eq. (3) for the vertical motion, we write

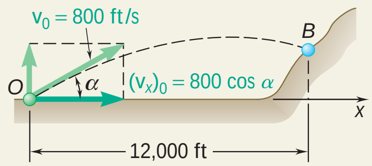
$$0 = 8100 - 19.62y \qquad y = 413 \text{ m}$$

Greatest elevation above ground $= 150 \text{ m} + 413 \text{ m} = 563 \text{ m}$ ◀



SAMPLE PROBLEM 11.8

A projectile is fired with an initial velocity of 800 ft/s at a target B located 2000 ft above the gun A and at a horizontal distance of 12,000 ft. Neglecting air resistance, determine the value of the firing angle α.



SOLUTION

The horizontal and the vertical motion will be considered separately.

Horizontal Motion. Placing the origin of the coordinate axes at the gun, we have

$$(v_x)_0 = 800 \cos \alpha$$

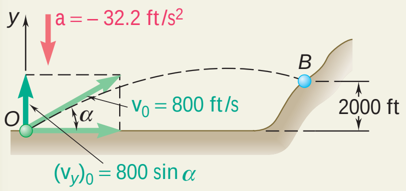
Substituting into the equation of uniform horizontal motion, we obtain

$$x = (v_x)_0 t \qquad x = (800 \cos \alpha)t$$

The time required for the projectile to move through a horizontal distance of 12,000 ft is obtained by setting x equal to 12,000 ft.

$$12{,}000 = (800 \cos \alpha)t$$

$$t = \frac{12{,}000}{800 \cos \alpha} = \frac{15}{\cos \alpha}$$



Vertical Motion

$$(v_y)_0 = 800 \sin \alpha \qquad a = -32.2 \text{ ft/s}^2$$

Substituting into the equation of uniformly accelerated vertical motion, we obtain

$$y = (v_y)_0 t + \tfrac{1}{2}at^2 \qquad y = (800 \sin \alpha)t - 16.1t^2$$

Projectile Hits Target. When $x = 12{,}000$ ft, we must have $y = 2000$ ft. Substituting for y and setting t equal to the value found above, we write

$$2000 = 800 \sin \alpha \frac{15}{\cos \alpha} - 16.1\left(\frac{15}{\cos \alpha}\right)^2$$

Since $1/\cos^2 \alpha = \sec^2 \alpha = 1 + \tan^2 \alpha$, we have

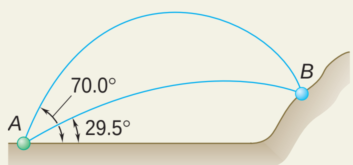
$$2000 = 800(15) \tan \alpha - 16.1(15^2)(1 + \tan^2 \alpha)$$

$$3622 \tan^2 \alpha - 12{,}000 \tan \alpha + 5622 = 0$$

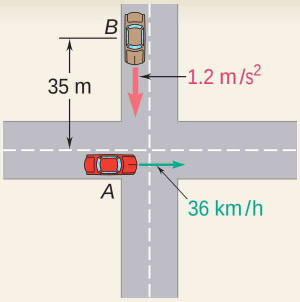
Solving this quadratic equation for $\tan \alpha$, we have

$$\tan \alpha = 0.565 \qquad \text{and} \qquad \tan \alpha = 2.75$$

$$\alpha = 29.5° \qquad \text{and} \qquad \alpha = 70.0° \quad ◀$$



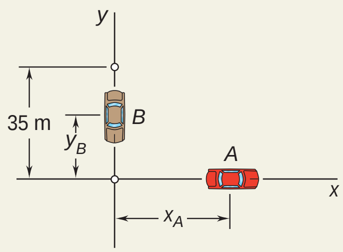
The target will be hit if either of these two firing angles is used (see figure).



SAMPLE PROBLEM 11.9

Automobile A is traveling east at the constant speed of 36 km/h. As automobile A crosses the intersection shown, automobile B starts from rest 35 m north of the intersection and moves south with a constant acceleration of 1.2 m/s^2. Determine the position, velocity, and acceleration of B relative to A 5 s after A crosses the intersection.

SOLUTION



We choose x and y axes with origin at the intersection of the two streets and with positive senses directed respectively east and north.

Motion of Automobile *A*. First the speed is expressed in m/s:

$$v_A = \left(36\frac{\text{km}}{\text{h}}\right)\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 10 \text{ m/s}$$

Noting that the motion of A is uniform, we write, for any time t,

$$a_A = 0$$
$$v_A = +10 \text{ m/s}$$
$$x_A = (x_A)_0 + v_A t = 0 + 10t$$

For $t = 5$ s, we have

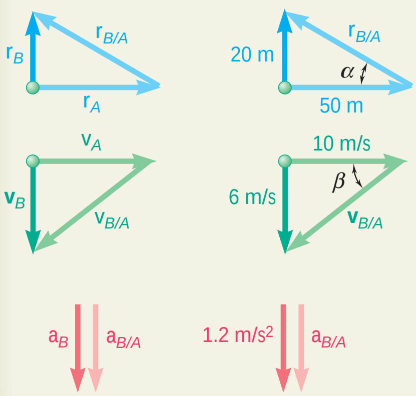
$$a_A = 0 \qquad \mathbf{a}_A = 0$$
$$v_A = +10 \text{ m/s} \qquad \mathbf{v}_A = 10 \text{ m/s} \rightarrow$$
$$x_A = +(10 \text{ m/s})(5 \text{ s}) = +50 \text{ m} \qquad \mathbf{r}_A = 50 \text{ m} \rightarrow$$

Motion of Automobile *B*. We note that the motion of B is uniformly accelerated and write

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = (v_B)_0 + at = 0 - 1.2\,t$$
$$y_B = (y_B)_0 + (v_B)_0 t + \tfrac{1}{2}a_B t^2 = 35 + 0 - \tfrac{1}{2}(1.2)t^2$$

For $t = 5$ s, we have

$$a_B = -1.2 \text{ m/s}^2 \qquad \mathbf{a}_B = 1.2 \text{ m/s}^2 \downarrow$$
$$v_B = -(1.2 \text{ m/s}^2)(5 \text{ s}) = -6 \text{ m/s} \qquad \mathbf{v}_B = 6 \text{ m/s} \downarrow$$
$$y_B = 35 - \tfrac{1}{2}(1.2 \text{ m/s}^2)(5 \text{ s})^2 = +20 \text{ m} \qquad \mathbf{r}_B = 20 \text{ m} \uparrow$$



Motion of *B* Relative to *A*. We draw the triangle corresponding to the vector equation $\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$ and obtain the magnitude and direction of the position vector of B relative to A.

$$r_{B/A} = 53.9 \text{ m} \qquad \alpha = 21.8° \qquad \mathbf{r}_{B/A} = 53.9 \text{ m} \measuredangle\ 21.8° \blacktriangleleft$$

Proceeding in a similar fashion, we find the velocity and acceleration of B relative to A.

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$
$$v_{B/A} = 11.66 \text{ m/s} \qquad \beta = 31.0° \qquad \mathbf{v}_{B/A} = 11.66 \text{ m/s} \measuredangle\ 31.0° \blacktriangleleft$$
$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \qquad \mathbf{a}_{B/A} = 1.2 \text{ m/s}^2 \downarrow \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will analyze the *two- and three-dimensional motion* of a particle. While the physical interpretations of the velocity and acceleration are the same as in the first lessons of the chapter, you should remember that these quantities are vectors. In addition, you should understand from your experience with vectors in statics that it will often be advantageous to express position vectors, velocities, and accelerations in terms of their rectangular scalar components [Eqs. (11.27) and (11.28)]. Furthermore, given two vectors $\mathbf{A}$ and $\mathbf{B}$, recall that $\mathbf{A} \cdot \mathbf{B} = 0$ if $\mathbf{A}$ and $\mathbf{B}$ are perpendicular to each other, while $\mathbf{A} \times \mathbf{B} = 0$ if $\mathbf{A}$ and $\mathbf{B}$ are parallel.

A. Analyzing the motion of a projectile. Many of the following problems deal with the two-dimensional motion of a projectile, where the resistance of the air can be neglected. In Sec. 11.11, we developed the equations which describe this type of motion, and we observed that the horizontal component of the velocity remained constant (uniform motion) while the vertical component of the acceleration was constant (uniformly accelerated motion). We were able to consider separately the horizontal and the vertical motions of the particle. Assuming that the projectile is fired from the origin, we can write the two equations

$$x = (v_x)_0 t \qquad y = (v_y)_0 t - \tfrac{1}{2}gt^2$$

1. If the initial velocity and firing angle are known, the value of y corresponding to any given value of x (or the value of x for any value of y) can be obtained by solving one of the above equations for t and substituting for t into the other, equation [Sample Prob. 11.7].

2. If the initial velocity and the coordinates of a point of the trajectory are known, and you wish to *determine the firing angle* α, begin your solution by expressing the components $(v_x)_0$ and $(v_y)_0$ of the initial velocity as functions of the angle α. These expressions and the known values of x and y are then substituted into the above equations. Finally, solve the first equation for t and substitute that value of t into the second equation to obtain a trigonometric equation in α, which you can solve for that unknown [Sample Prob. 11.8].

(continued)

B. Solving translational two-dimensional relative-motion problems. You saw in Sec. 11.12 that the absolute motion of a particle B can be obtained by combining the motion of a particle A and the *relative motion* of B with respect to a frame attached to A which is in *translation*. The velocity and acceleration of B can then be expressed as shown in Eqs. (11.33) and (11.34), respectively.

1. To visualize the relative motion of *B* with respect to *A*, imagine that you are attached to particle A as you observe the motion of particle B. For example, to a passenger in automobile A of Sample Prob. 11.9, automobile B appears to be heading in a southwesterly direction (*south* should be obvious; and *west* is due to the fact that automobile A is moving to the east—automobile B then appears to travel to the west). Note that this conclusion is consistent with the direction of $\mathbf{v}_{B/A}$.

2. To solve a relative-motion problem, first write the vector equations (11.31), (11.33), and (11.34), which relate the motions of particles A and B. You may then use either of the following methods:

a. Construct the corresponding vector triangles and solve them for the desired position vector, velocity, and acceleration [Sample Prob. 11.9].

b. Express all vectors in terms of their rectangular components and solve the two independent sets of scalar equations obtained in that way. If you choose this approach, be sure to select the same positive direction for the displacement, velocity, and acceleration of each particle.

PROBLEMS

CONCEPT QUESTIONS

11.CQ3 Two model rockets are fired simultaneously from a ledge and follow the trajectories shown. Neglecting air resistance, which of the rockets will hit the ground first?

a. *A*.
b. *B*.
c. They hit at the same time.
d. The answer depends on *h*.

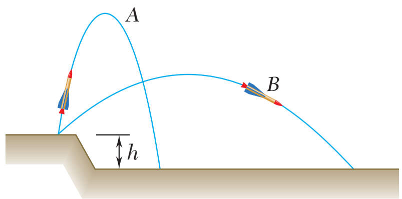


Fig. P11.CQ3

11.CQ4 Ball *A* is thrown straight up. Which of the following statements about the ball are true at the highest point in its path?

a. The velocity and acceleration are both zero.
b. The velocity is zero, but the acceleration is not zero.
c. The velocity is not zero, but the acceleration is zero.
d. Neither the velocity nor the acceleration is zero.

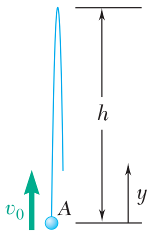


Fig. P11.CQ4

11.CQ5 Ball *A* is thrown straight up with an initial speed v_0 and reaches a maximum elevation *h* before falling back down. When *A* reaches its maximum elevation, a second ball is thrown straight upward with the same initial speed v_0. At what height, *y*, will the balls cross paths?

a. $y = h$
b. $y > h/2$
c. $y = h/2$
d. $y < h/2$
e. $y = 0$

11.CQ6 Two cars are approaching an intersection at constant speeds as shown. What velocity will car *B* appear to have to an observer in car *A*?

a. → **b.** ↘ **c.** ↖ **d.** ↗ **e.** ↙

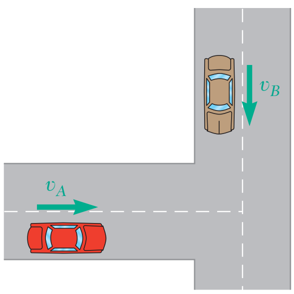
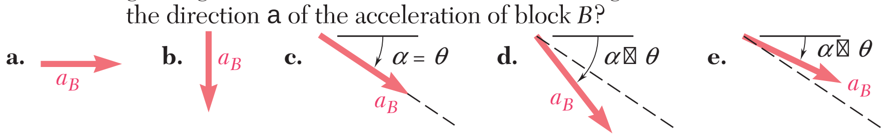


Fig. P11.CQ6

11.CQ7 Blocks *A* and *B* are released from rest in the positions shown. Neglecting friction between all surfaces, which figure best indicates the direction a of the acceleration of block *B*?



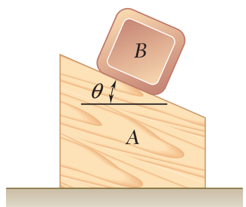


Fig. P11.CQ7

END-OF-SECTION PROBLEMS

11.89 A ball is thrown so that the motion is defined by the equations $x = 5t$ and $y = 2 + 6t - 4.9t^2$, where x and y are expressed in meters and t is expressed in seconds. Determine (*a*) the velocity at $t = 1$ s, (*b*) the horizontal distance the ball travels before hitting the ground.

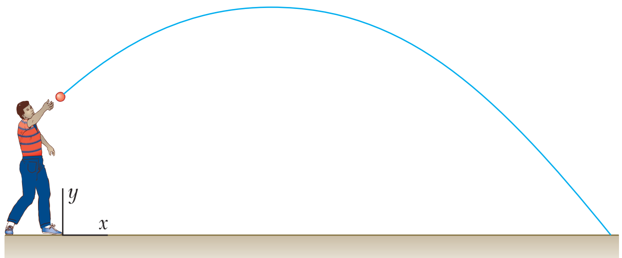


Fig. P11.89

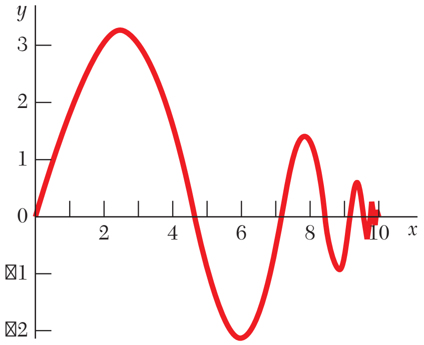


Fig. P11.90

11.90 The motion of a vibrating particle is defined by the position vector $\mathbf{r} = 10(1 - e^{-3t})\mathbf{i} + (4e^{-2t} \sin 15t)\mathbf{j}$, where $\mathbf{r}$ and t are expressed in millimeters and seconds, respectively. Determine the velocity and acceleration when (*a*) $t = 0$, (*b*) $t = 0.5$ s.

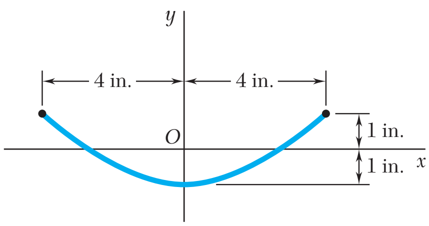


Fig. P11.91

11.91 The motion of a vibrating particle is defined by the position vector $\mathbf{r} = (4 \sin pt)\mathbf{i} - (\cos 2pt)\mathbf{j}$, where r is expressed in inches and t in seconds. (*a*) Determine the velocity and acceleration when $t = 1$ s. (*b*) Show that the path of the particle is parabolic.

11.92 The motion of a particle is defined by the equations $x = 10t - 5 \sin t$ and $y = 10 - 5 \cos t$, where x and y are expressed in feet and t is expressed in seconds. Sketch the path of the particle for the time interval $0 \le t \le 2p$, and determine (*a*) the magnitudes of the smallest and largest velocities reached by the particle, (*b*) the corresponding times, positions, and directions of the velocities.

11.93 The damped motion of a vibrating particle is defined by the position vector $\boldsymbol{r} = x_1[1 - 1/(t + 1)]\mathbf{i} + (y_1 e^{-pt/2} \cos 2pt)\mathbf{j}$, where t is expressed in seconds. For $x_1 = 30$ mm and $y_1 = 20$ mm, determine the position, the velocity, and the acceleration of the particle when (*a*) $t = 0$, (*b*) $t = 1.5$ s.

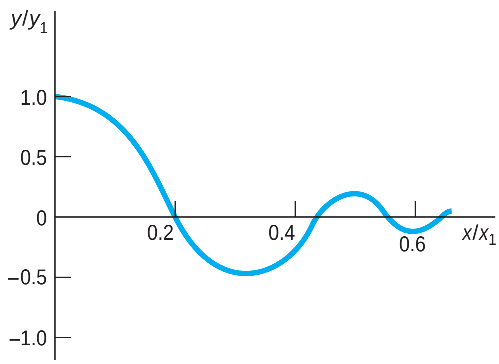


Fig. *P11.93*

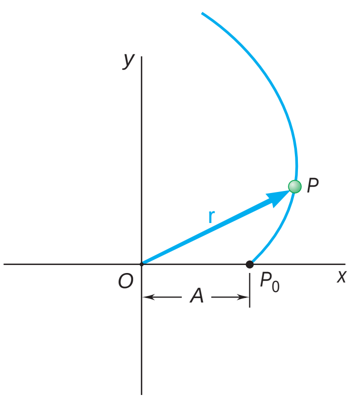


Fig. *P11.94*

11.94 The motion of a particle is defined by the position vector $\boldsymbol{r} = A(\cos t + t \sin t)\mathbf{i} + A(\sin t - t \cos t)\mathbf{j}$, where t is expressed in seconds. Determine the values of t for which the position vector and the acceleration are (*a*) perpendicular, (*b*) parallel.

11.95 The three-dimensional motion of a particle is defined by the position vector $\boldsymbol{r} = (Rt \cos \omega_n t)\mathbf{i} + ct\mathbf{j} + (Rt \sin \omega_n t)\mathbf{k}$. Determine the magnitudes of the velocity and acceleration of the particle. (The space curve described by the particle is a conic helix.)

***11.96** The three-dimensional motion of a particle is defined by the position vector $\boldsymbol{r} = (At \cos t)\mathbf{i} + (A\sqrt{t^2 + 1})\mathbf{j} + (Bt \sin t)\mathbf{k}$, where r and t are expressed in feet and seconds, respectively. Show that the curve described by the particle lies on the hyperboloid $(y/A)^2 - (x/A)^2 - (z/B)^2 = 1$. For $A = 3$ and $B = 1$, determine (*a*) the magnitudes of the velocity and acceleration when $t = 0$, (*b*) the smallest nonzero value of t for which the position vector and the velocity are perpendicular to each other.

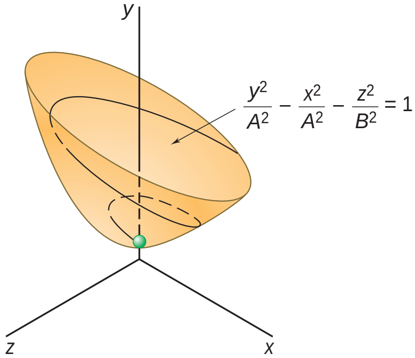


Fig. *P11.96*

11.97 An airplane used to drop water on brushfires is flying horizontally in a straight line at 180 mi/h at an altitude of 300 ft. Determine the distance d at which the pilot should release the water so that it will hit the fire at B.

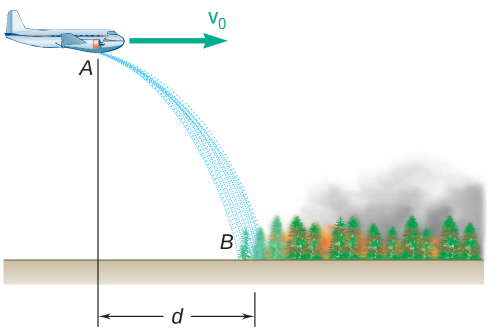


Fig. P11.97

11.98 A helicopter is flying with a constant horizontal velocity of 180 km/h and is directly above point A when a loose part begins to fall. The part lands 6.5 s later at point B on an inclined surface. Determine (*a*) the distance d between points A and B. (*b*) the initial height h.

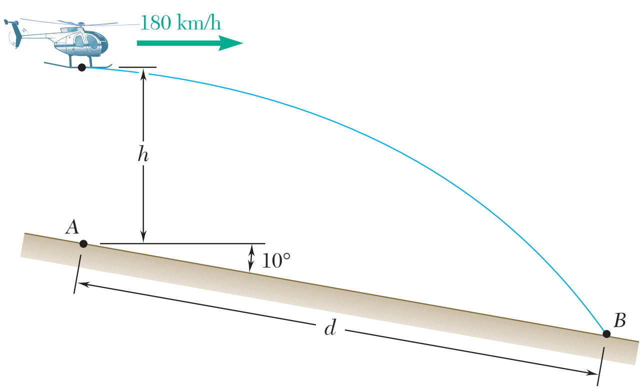


Fig. P11.98

11.99 A baseball pitching machine "throws" baseballs with a horizontal velocity $\mathbf{v}_0$. Knowing that height h varies between 788 mm and 1068 mm, determine (*a*) the range of values of v_0, (*b*) the values of α corresponding to $h = 788$ mm and $h = 1068$ mm.

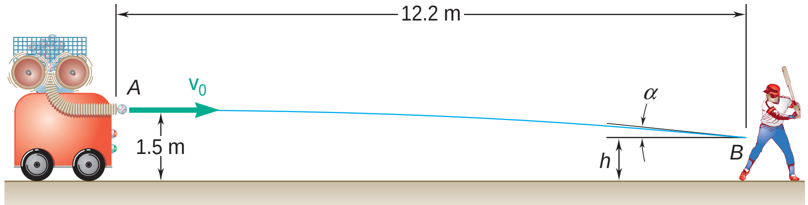


Fig. P11.99

11.100 While delivering newspapers, a girl throws a newspaper with a horizontal velocity $\mathbf{v}_0$. Determine the range of values of v_0 if the newspaper is to land between points B and C.

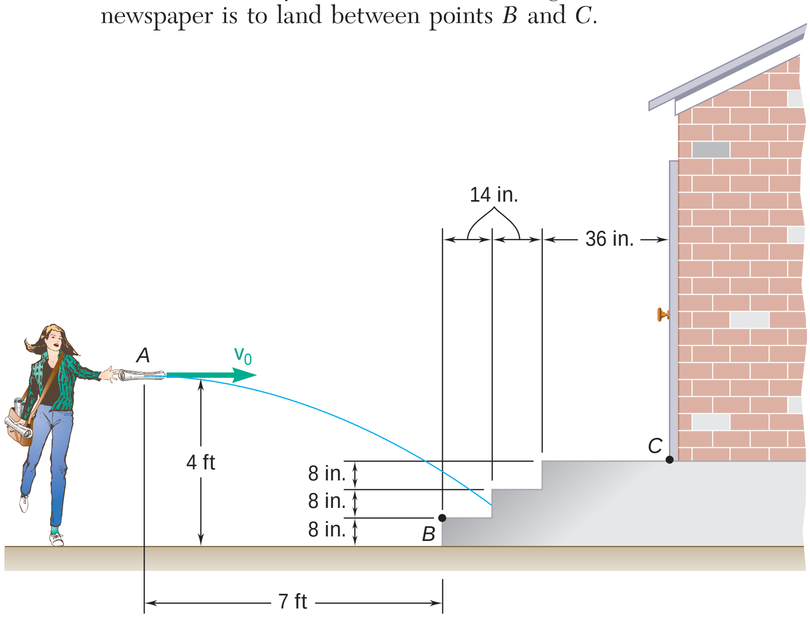


Fig. P11.100

11.101 Water flows from a drain spout with an initial velocity of 2.5 ft/s at an angle of 15° with the horizontal. Determine the range of values of the distance d for which the water will enter the trough BC.

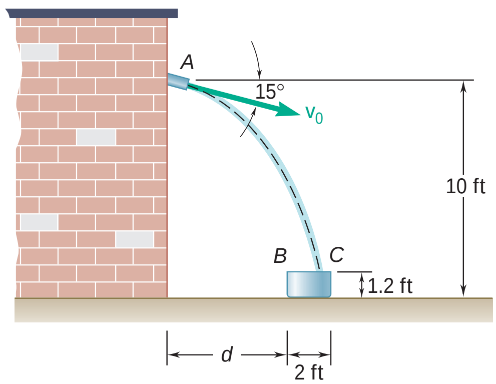


Fig. *P11.101*

11.102 Milk is poured into a glass of height 140 mm and inside diameter 66 mm. If the initial velocity of the milk is 1.2 m/s at an angle of 40° with the horizontal, determine the range of values of the height h for which the milk will enter the glass.

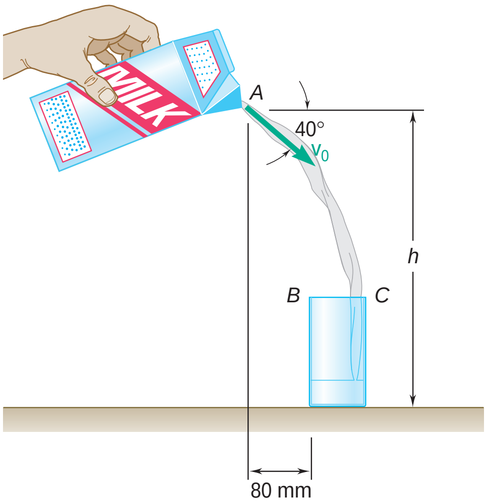


Fig. P11.102

11.103 A volleyball player serves the ball with an initial velocity $\mathbf{v}_0$ of magnitude 13.40 m/s at an angle of 20° with the horizontal. Determine (*a*) if the ball will clear the top of the net, (*b*) how far from the net the ball will land.

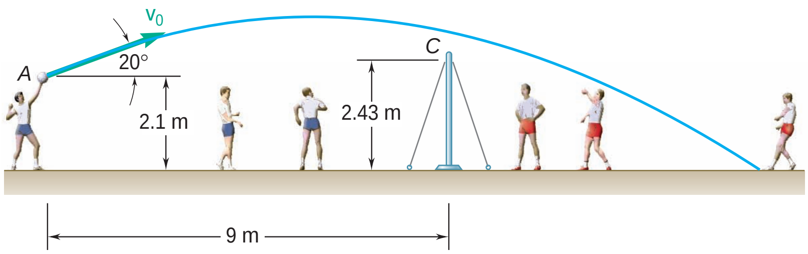


Fig. P11.103

11.104 A golfer hits a golf ball with an initial velocity of 160 ft/s at an angle of 25° with the horizontal. Knowing that the fairway slopes downward at an average angle of 5°, determine the distance d between the golfer and point B where the ball first lands.

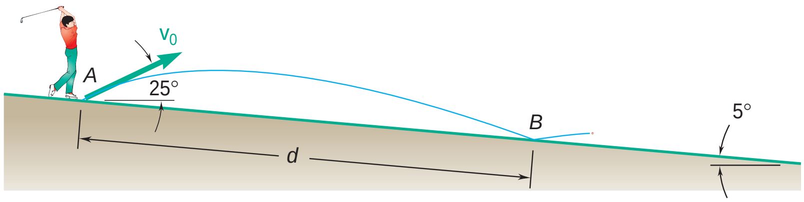


Fig. *P11.104*

11.105 A homeowner uses a snowblower to clear his driveway. Knowing that the snow is discharged at an average angle of 40° with the horizontal, determine the initial velocity v_0 of the snow.

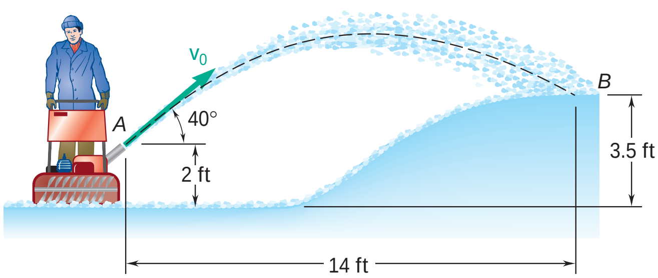


Fig. P11.105

11.106 At halftime of a football game souvenir balls are thrown to the spectators with a velocity $\mathbf{v}_0$. Determine the range of values of v_0 if the balls are to land between points B and C.

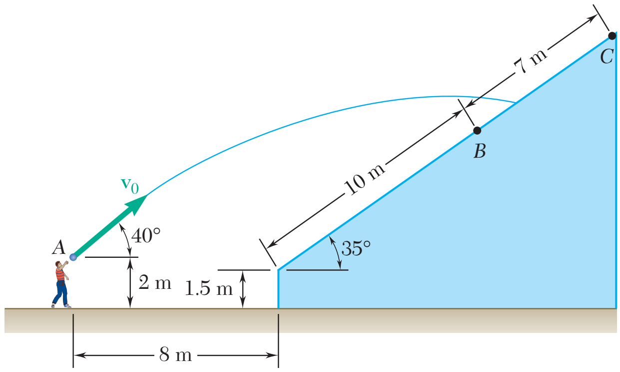


Fig. P11.106

11.107 A basketball player shoots when she is 16 ft from the backboard. Knowing that the ball has an initial velocity $\mathbf{v}_0$ at an angle of 30° with the horizontal, determine the value of v_0 when d is equal to (*a*) 9 in., (*b*) 17 in.

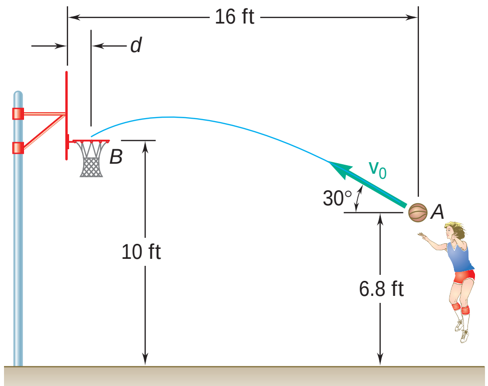


Fig. P11.107

11.108 A tennis player serves the ball at a height $h = 2.5$ m with an initial velocity of $\mathbf{v}_0$ at an angle of 5° with the horizontal. Determine the range of v_0 for which the ball will land in the service area that extends to 6.4 m beyond the net.

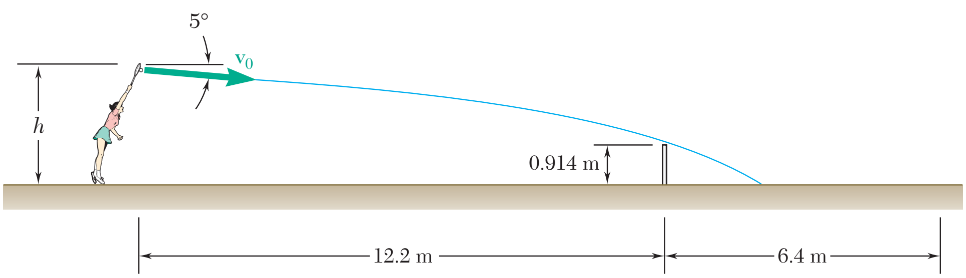


Fig. P11.108

11.109 The nozzle at A discharges cooling water with an initial velocity $\mathbf{v}_0$ at an angle of 6° with the horizontal onto a grinding wheel 350 mm in diameter. Determine the range of values of the initial velocity for which the water will land on the grinding wheel between points B and C.

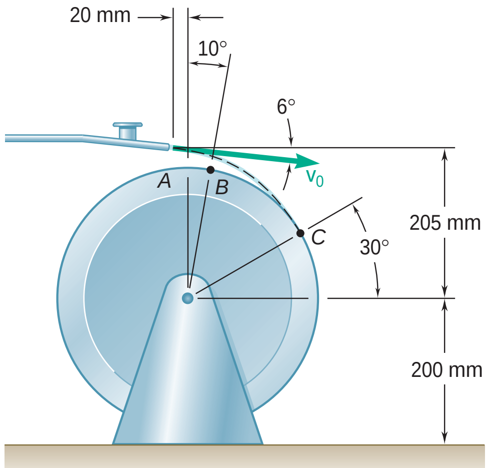


Fig. P11.109

11.110 While holding one of its ends, a worker lobs a coil of rope over the lowest limb of a tree. If he throws the rope with an initial velocity $\mathbf{v}_0$ at an angle of 65° with the horizontal, determine the range of values of v_0 for which the rope will go over only the lowest limb.

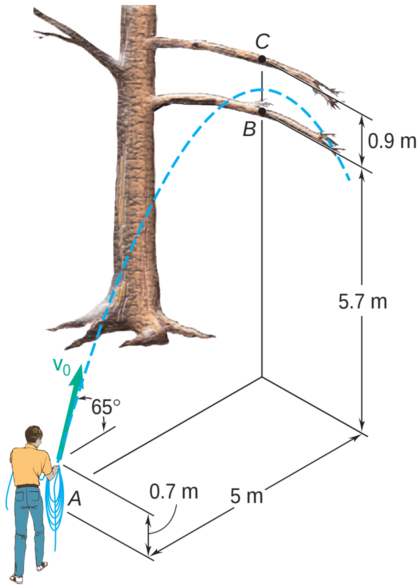


Fig. P11.110

11.111 The pitcher in a softball game throws a ball with an initial velocity $\mathbf{v}_0$ of 72 km/h at an angle a with the horizontal. If the height of the ball at point B is 0.68 m, determine (*a*) the angle a, (*b*) the angle u that the velocity of the ball at point B forms with the horizontal.

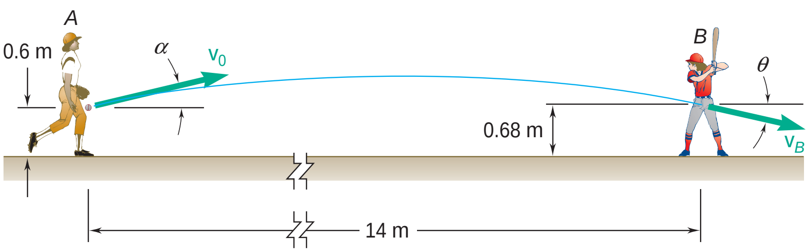


Fig. P11.111

11.112 A model rocket is launched from point A with an initial velocity $\mathbf{v}_0$ of 75 m/s. If the rocket's descent parachute does not deploy and the rocket lands a distance d = 100 m from A, determine (*a*) the angle a that $\mathbf{v}_0$ forms with the vertical, (*b*) the maximum height above point A reached by the rocket, (*c*) the duration of the flight.

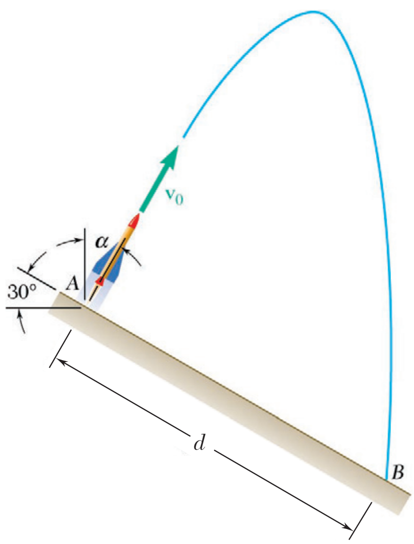


Fig. P11.112

11.113 The initial velocity $\mathbf{v}_0$ of a hockey puck is 105 mi/h. Determine (*a*) the largest value (less than 45°) of the angle α for which the puck will enter the net, (*b*) the corresponding time required for the puck to reach the net.

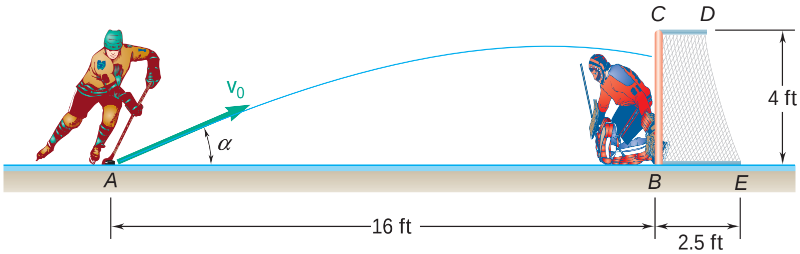


Fig. P11.113

11.114 A worker uses high-pressure water to clean the inside of a long drainpipe. If the water is discharged with an initial velocity $\mathbf{v}_0$ of 11.5 m/s, determine (*a*) the distance *d* to the farthest point *B* on the top of the pipe that the worker can wash from his position at *A*, (*b*) the corresponding angle α.

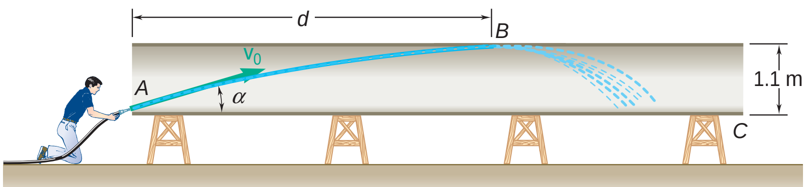


Fig. P11.114

11.115 An oscillating garden sprinkler which discharges water with an initial velocity $\mathbf{v}_0$ of 8 m/s is used to water a vegetable garden. Determine the distance *d* to the farthest point *B* that will be watered and the corresponding angle α when (*a*) the vegetables are just beginning to grow, (*b*) the height *h* of the corn is 1.8 m.

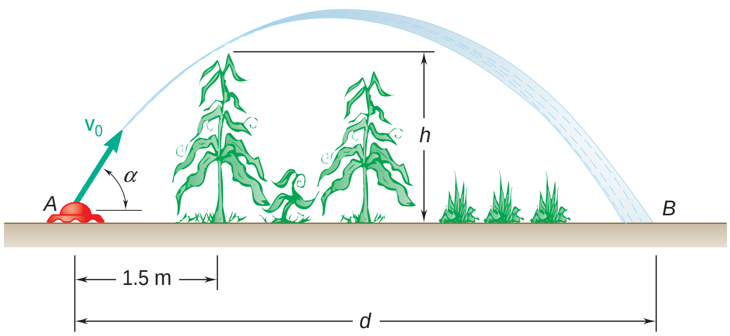


Fig. P11.115

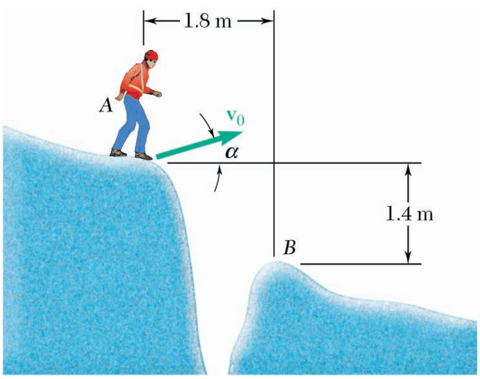


Fig. P11.116

***11.116** A mountain climber plans to jump from *A* to *B* over a crevasse. Determine the smallest value of the climber's initial velocity $\mathbf{v}_0$ and the corresponding value of angle α so that he lands at *B*.

11.117 The velocities of skiers A and B are as shown. Determine the velocity of A with respect to B.

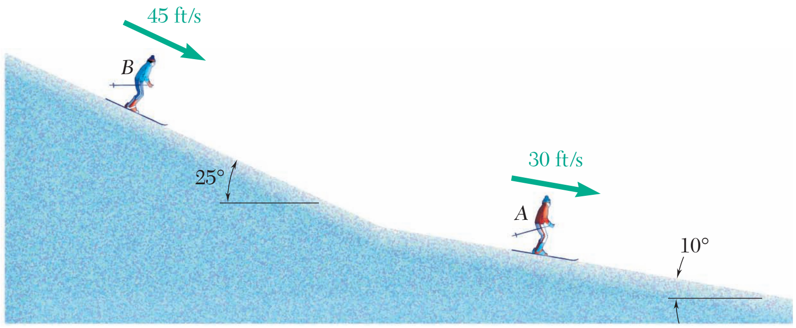


Fig. P11.117

11.118 The three blocks shown move with constant velocities. Find the velocity of each block, knowing that the relative velocity of A with respect to C is 300 mm/s upward and that the relative velocity of B with respect to A is 200 mm/s downward.

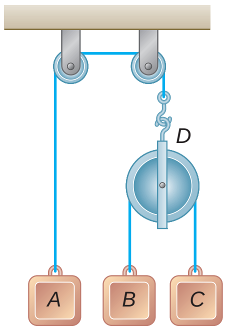


Fig. P11.118

11.119 Three seconds after automobile B passes through the intersection shown, automobile A passes through the same intersection. Knowing that the speed of each automobile is constant, determine (*a*) the relative velocity of B with respect to A, (*b*) the change in position of B with respect to A during a 4-s interval, (*c*) the distance between the two automobiles 2 s after A has passed through the intersection.

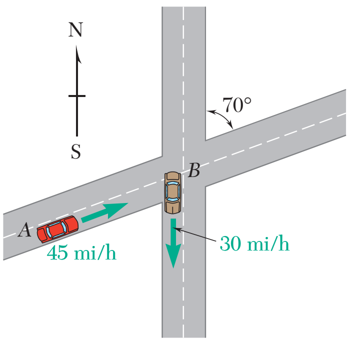


Fig. P11.119

11.120 Shore-based radar indicates that a ferry leaves its slip with a velocity $\mathbf{v}$ = 18 km/h d 70°, while instruments aboard the ferry indicate a speed of 18.4 km/h and a heading of 30° west of south relative to the river. Determine the velocity of the river.

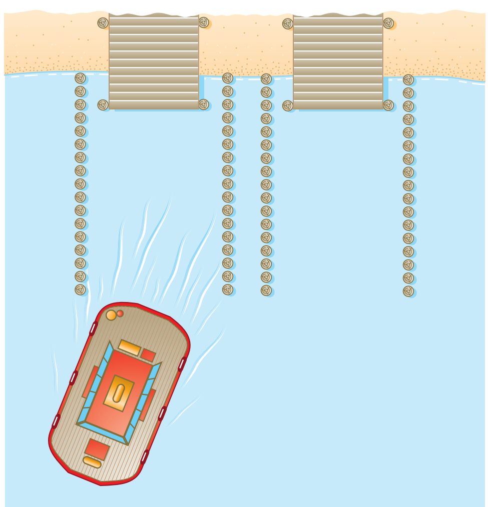


Fig. P11.120

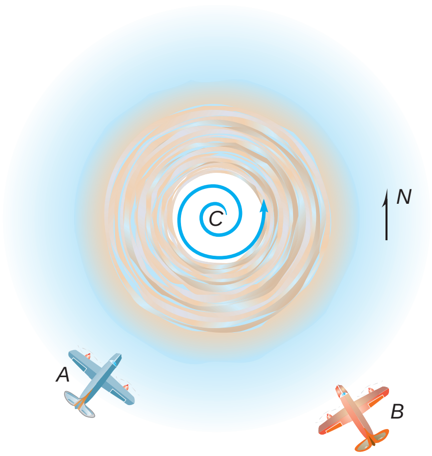


Fig. *P11.121*

11.121 Airplanes A and B are flying at the same altitude and are tracking the eye of hurricane C. The relative velocity of C with respect to A is $\mathbf{v}_{C/A}$ = 350 km/h d 75°, and the relative velocity of C with respect to B is $\mathbf{v}_{C/B}$ = 400 km/h c 40°. Determine (*a*) the relative velocity of B with respect to A, (*b*) the velocity of A if ground-based radar indicates that the hurricane is moving at a speed of 30 km/h due north, (*c*) the change in position of C with respect to B during a 15-min interval.

11.122 Pin P moves at a constant speed of 150 mm/s in a counterclockwise sense along a circular slot which has been milled in the slider block A shown. Knowing that the block moves downward at a constant speed of 100 mm/s, determine the velocity of pin P when (*a*) u = 30°, (*b*) u = 120°.

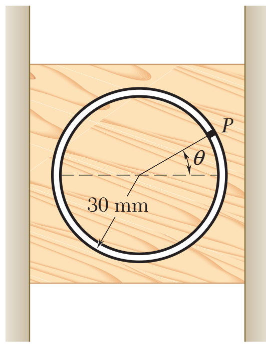


Fig. *P11.122*

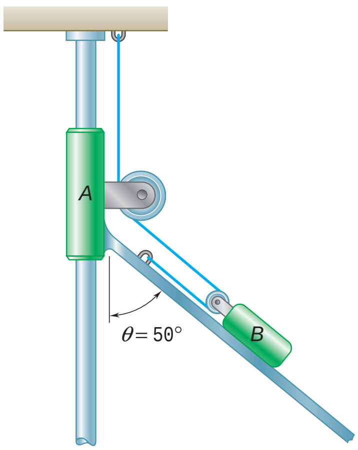


Fig. P11.123

11.123 Knowing that at the instant shown assembly A has a velocity of 9 in./s and an acceleration of 15 in./s^2 both directed downward, determine (*a*) the velocity of block B, (*b*) the acceleration of block B.

11.124 Knowing that at the instant shown block A has a velocity of 8 in./s and an acceleration of 6 in./s^2 both directed down the incline, determine (*a*) the velocity of block B, (*b*) the acceleration of block B.

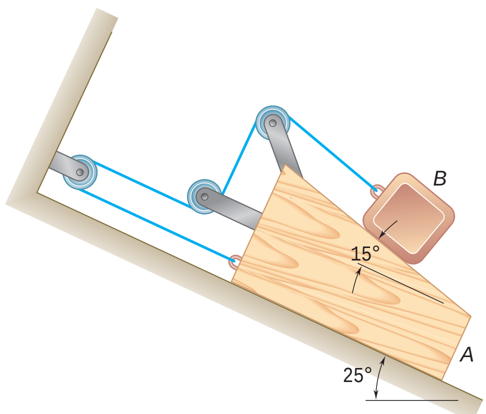


Fig. P11.124

11.125 A boat is moving to the right with a constant deceleration of 0.3 m/s^2 when a boy standing on the deck D throws a ball with an initial velocity relative to the deck which is vertical. The ball rises to a maximum height of 8 m above the release point and the boy must step forward a distance d to catch it at the same height as the release point. Determine (*a*) the distance d, (*b*) the relative velocity of the ball with respect to the deck when the ball is caught.

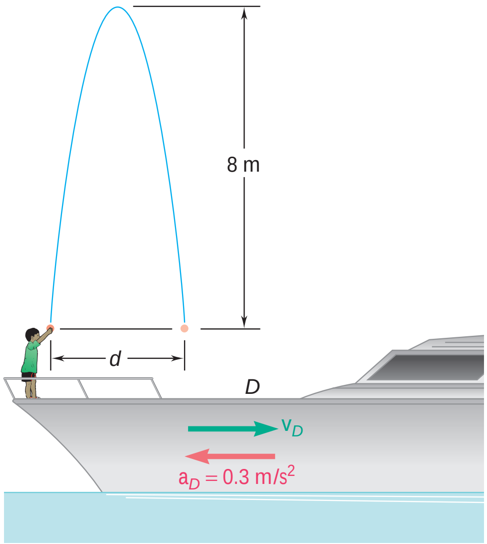


Fig. P11.125

11.126 The assembly of rod A and wedge B starts from rest and moves to the right with a constant acceleration of 2 mm/s^2. Determine (*a*) the acceleration of wedge C, (*b*) the velocity of wedge C when $t = 10$ s.

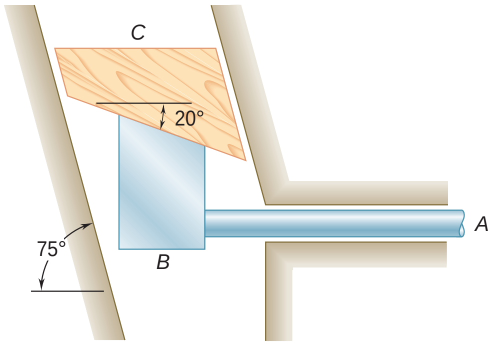


Fig. P11.126

11.127 Determine the required velocity of the belt B if the relative velocity with which the sand hits belt B is to be (*a*) vertical, (*b*) as small as possible.

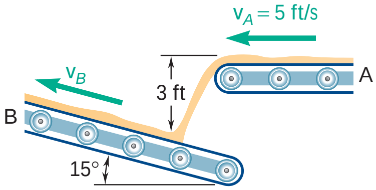


Fig. P11.127

11.128 Conveyor belt A, which forms a 20° angle with the horizontal, moves at a constant speed of 4 ft/s and is used to load an airplane. Knowing that a worker tosses duffel bag B with an initial velocity of 2.5 ft/s at an angle of 30° with the horizontal, determine the velocity of the bag relative to the belt as it lands on the belt.

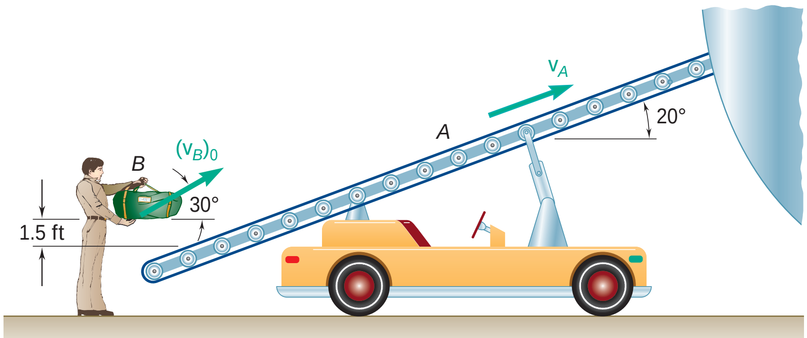


Fig. P11.128

11.129 During a rainstorm the paths of the raindrops appear to form an angle of 30° with the vertical and to be directed to the left when observed from a side window of a train moving at a speed of 15 km/h. A short time later, after the speed of the train has increased to 24 km/h, the angle between the vertical and the paths of the drops appears to be 45°. If the train were stopped, at what angle and with what velocity would the drops be observed to fall?

11.130 As observed from a ship moving due east at 9 km/h, the wind appears to blow from the south. After the ship has changed course and speed, and as it is moving north at 6 km/h, the wind appears to blow from the southwest. Assuming that the wind velocity is constant during the period of observation, determine the magnitude and direction of the true wind velocity.

11.131 When a small boat travels north at 5 km/h, a flag mounted on its stern forms an angle $\theta = 50°$ with the centerline of the boat as shown. A short time later, when the boat travels east at 20 km/h, angle θ is again 50°. Determine the speed and the direction of the wind.

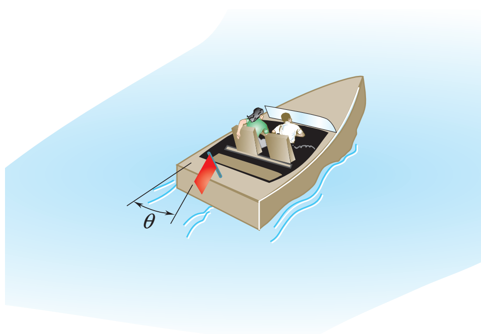


Fig. P11.131

11.132 As part of a department store display, a model train D runs on a slight incline between the store's up and down escalators. When the train and shoppers pass point A, the train appears to a shopper on the up escalator B to move downward at an angle of 22° with the horizontal, and to a shopper on the down escalator C to move upward at an angle of 23° with the horizontal and to travel to the left. Knowing that the speed of the escalators is 3 ft/s, determine the speed and the direction of the train.

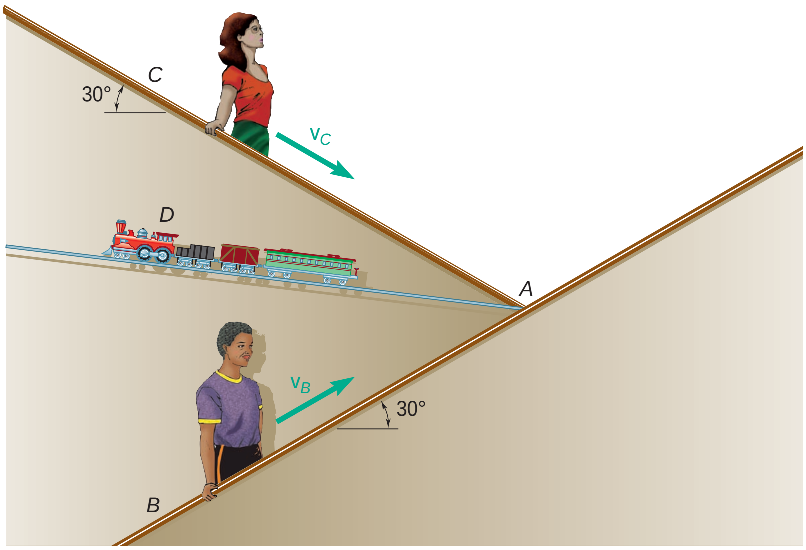


Fig. *P11.132*

11.13 TANGENTIAL AND NORMAL COMPONENTS

We saw in Sec. 11.9 that the velocity of a particle is a vector tangent to the path of the particle but that, in general, the acceleration is not tangent to the path. It is sometimes convenient to resolve the acceleration into components directed, respectively, along the tangent and the normal to the path of the particle.

Plane Motion of a Particle. First, let us consider a particle which moves along a curve contained in the plane of the figure. Let P be the position of the particle at a given instant. We attach at P a unit vector $\mathbf{e}_t$ tangent to the path of the particle and pointing in the direction of motion (Fig. 11.21*a*). Let $\mathbf{e}'_t$ be the unit vector corresponding to the position P' of the particle at a later instant. Drawing both vectors from the same origin O', we define the vector $\Delta\mathbf{e}_t = \mathbf{e}'_t - \mathbf{e}_t$ (Fig. 11.21*b*). Since $\mathbf{e}_t$ and $\mathbf{e}'_t$ are of unit length, their tips lie on a circle of radius 1. Denoting by Δu the angle formed by $\mathbf{e}_t$ and $\mathbf{e}'_t$, we find that the magnitude of $\Delta\mathbf{e}_t$ is $2 \sin(\Delta u/2)$. Considering now the vector $\Delta\mathbf{e}_t/\Delta u$, we note that as Δu approaches zero, this vector becomes tangent to the unit circle of Fig. 11.21*b*, i.e., perpendicular to $\mathbf{e}_t$, and that its magnitude approaches

$$\lim_{\Delta u\, y\, 0} \frac{2\sin(\Delta u/2)}{\Delta u} = \lim_{\Delta u\, y\, 0} \frac{\sin(\Delta u/2)}{\Delta u/2} = 1$$

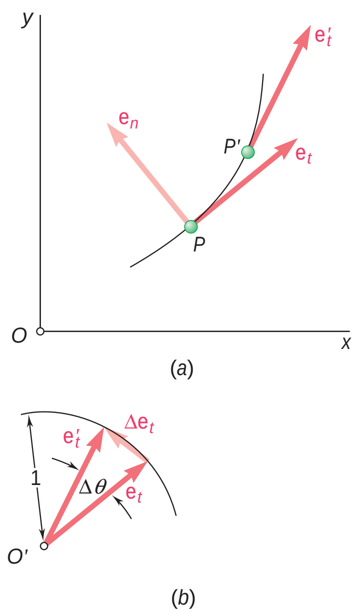


Fig. 11.21

Thus, the vector obtained in the limit is a unit vector along the normal to the path of the particle, in the direction toward which $\mathbf{e}_t$ turns. Denoting this vector by $\mathbf{e}_n$, we write

$$\mathbf{e}_n = \lim_{\Delta\theta \to 0} \frac{\Delta \mathbf{e}_t}{\Delta\theta}$$

$$\boxed{\mathbf{e}_n = \frac{d\mathbf{e}_t}{d\theta}} \tag{11.35}$$

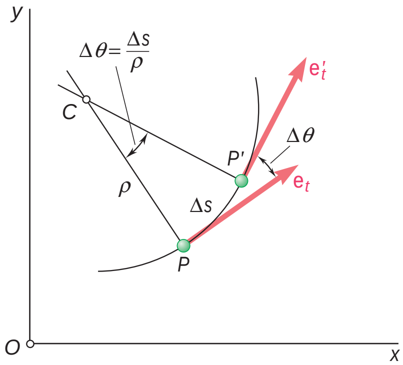


Fig. 11.22

Since the velocity $\mathbf{v}$ of the particle is tangent to the path, it can be expressed as the product of the scalar v and the unit vector $\mathbf{e}_t$. We have

$$\mathbf{v} = v\mathbf{e}_t \tag{11.36}$$

To obtain the acceleration of the particle, (11.36) will be differentiated with respect to t. Applying the rule for the differentiation of the product of a scalar and a vector function (Sec. 11.10), we write

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\mathbf{e}_t}{dt} \tag{11.37}$$

But

$$\frac{d\mathbf{e}_t}{dt} = \frac{d\mathbf{e}_t}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

Photo 11.5 The passengers in a train traveling around a curve will experience a normal acceleration toward the center of curvature of the path.

Recalling from (11.16) that $ds/dt = v$, from (11.35) that $d\mathbf{e}_t/d\theta = \mathbf{e}_n$, and from elementary calculus that $d\theta/ds$ is equal to $1/\rho$, where ρ is the radius of curvature of the path at P (Fig. 11.22), we have

$$\frac{d\mathbf{e}_t}{dt} = \frac{v}{\rho}\mathbf{e}_n \tag{11.38}$$

Substituting into (11.37), we obtain

$$\boxed{\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n} \tag{11.39}$$

Thus, the scalar components of the acceleration are

$$a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho} \tag{11.40}$$

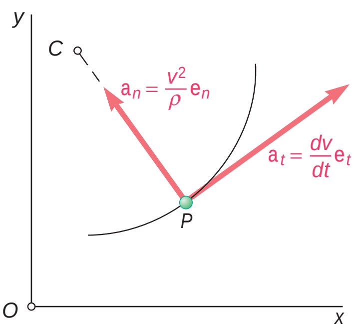


Fig. 11.23

The relations obtained express that the *tangential component* of the acceleration is equal to the *rate of change of the speed of the particle,* while the *normal component* is equal to the *square of the speed divided by the radius of curvature of the path at P*. If the speed of the particle increases, a_t is positive and the vector component $\mathbf{a}_t$ points in the direction of motion. If the speed of the particle decreases, a_t is negative and $\mathbf{a}_t$ points against the direction of motion. The vector component $\mathbf{a}_n$, on the other hand, *is always directed toward the center of curvature C of the path* (Fig. 11.23).

We conclude from the above that the tangential component of the acceleration reflects a change in the speed of the particle, while its normal component reflects a change in the direction of motion of the particle. The acceleration of a particle will be zero only if both its components are zero. Thus, the acceleration of a particle moving with constant speed along a curve will not be zero unless the particle happens to pass through a point of inflection of the curve (where the radius of curvature is infinite) or unless the curve is a straight line.

The fact that the normal component of the acceleration depends upon the radius of curvature of the path followed by the particle is taken into account in the design of structures or mechanisms as widely different as airplane wings, railroad tracks, and cams. In order to avoid sudden changes in the acceleration of the air particles flowing past a wing, wing profiles are designed without any sudden change in curvature. Similar care is taken in designing railroad curves, to avoid sudden changes in the acceleration of the cars (which would be hard on the equipment and unpleasant for the passengers). A straight section of track, for instance, is never directly followed by a circular section. Special transition sections are used to help pass smoothly from the infinite radius of curvature of the straight section to the finite radius of the circular track. Likewise, in the design of high-speed cams, abrupt changes in acceleration are avoided by using transition curves which produce a continuous change in acceleration.

Motion of a Particle in Space. The relations (11.39) and (11.40) still hold in the case of a particle moving along a space curve. However, since there are an infinite number of straight lines which are perpendicular to the tangent at a given point P of a space curve, it is necessary to define more precisely the direction of the unit vector $\mathbf{e}_n$.

Let us consider again the unit vectors $\mathbf{e}_t$ and $\mathbf{e}'_t$ tangent to the path of the particle at two neighboring points P and P' (Fig. 11.24*a*) and the vector $\Delta\mathbf{e}_t$ representing the difference between $\mathbf{e}_t$ and $\mathbf{e}'_t$ (Fig. 11.24*b*). Let us now imagine a plane through P (Fig. 11.24*a*) parallel to the plane defined by the vectors $\mathbf{e}_t$, $\mathbf{e}'_t$, and $\Delta\mathbf{e}_t$ (Fig. 11.24*b*). This plane contains the tangent to the curve at P and is parallel to the tangent at P'. If we let P' approach P, we obtain in the limit the plane which fits the curve most closely in the neighborhood of P. This plane is called the *osculating plane* at P.† It follows from this

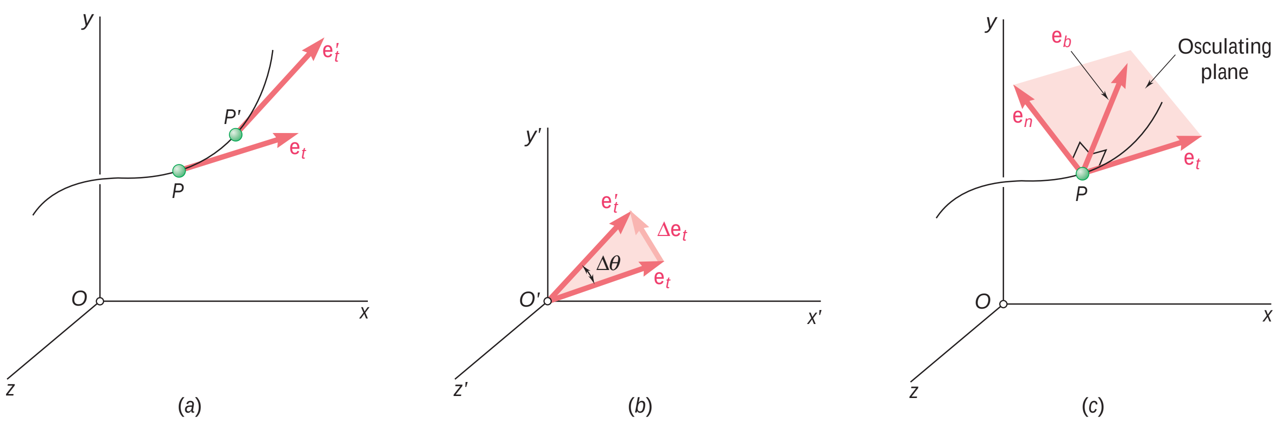


Fig. 11.24

†From the Latin *osculari*, to kiss.

definition that the osculating plane contains the unit vector $\mathbf{e}_n$, since this vector represents the limit of the vector $\Delta\mathbf{e}_t/\Delta\theta$. The normal defined by $\mathbf{e}_n$ is thus contained in the osculating plane; it is called the *principal normal* at P. The unit vector $\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n$ which completes the right-handed triad $\mathbf{e}_t$, $\mathbf{e}_n$, $\mathbf{e}_b$ (Fig. 11.24*c*) defines the *binormal* at P. The binormal is thus perpendicular to the osculating plane. We conclude that the acceleration of the particle at P can be resolved into two components, one along the tangent, the other along the principal normal at P, as indicated in Eq. (11.39). Note that the acceleration has no component along the binormal.

11.14 RADIAL AND TRANSVERSE COMPONENTS

In certain problems of plane motion, the position of the particle P is defined by its polar coordinates r and θ (Fig. 11.25*a*). It is then convenient to resolve the velocity and acceleration of the particle into components parallel and perpendicular, respectively, to the line OP. These components are called *radial* and *transverse components*.

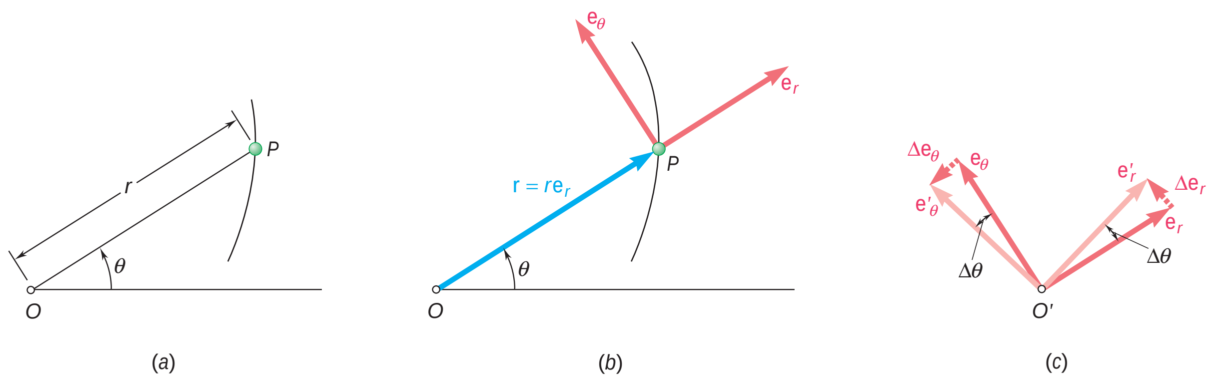


Fig. 11.25

We attach at P two unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (Fig. 11.25*b*). The vector $\mathbf{e}_r$ is directed along OP and the vector $\mathbf{e}_\theta$ is obtained by rotating $\mathbf{e}_r$ through 90° counterclockwise. The unit vector $\mathbf{e}_r$ defines the *radial* direction, i.e., the direction in which P would move if r were increased and θ were kept constant; the unit vector $\mathbf{e}_\theta$ defines the *transverse* direction, i.e., the direction in which P would move if θ were increased and r were kept constant. A derivation similar to the one we used in Sec. 11.13 to determine the derivative of the unit vector $\mathbf{e}_t$ leads to the relations

$$\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta \qquad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r \tag{11.41}$$

Photo 11.6 The footpads on an elliptical trainer undergo curvilinear motion.

where $-\mathbf{e}_r$ denotes a unit vector of sense opposite to that of $\mathbf{e}_r$ (Fig. 11.25*c*). Using the chain rule of differentiation, we express the time derivatives of the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as follows:

$$\frac{d\mathbf{e}_r}{dt} = \frac{d\mathbf{e}_r}{d\theta}\frac{d\theta}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt} \qquad \frac{d\mathbf{e}_\theta}{dt} = \frac{d\mathbf{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\mathbf{e}_r\frac{d\theta}{dt}$$

or, using dots to indicate differentiation with respect to t,

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r \tag{11.42}$$

To obtain the velocity $\mathbf{v}$ of the particle P, we express the position vector $\mathbf{r}$ of P as the product of the scalar r and the unit vector $\mathbf{e}_r$ and differentiate with respect to t:

$$\mathbf{v} = \frac{d}{dt}(r\mathbf{e}_r) = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$$

or, recalling the first of the relations (11.42),

$$\boxed{\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta} \tag{11.43}$$

Differentiating again with respect to t to obtain the acceleration, we write

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta$$

or, substituting for $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$ from (11.42) and factoring $\mathbf{e}_r$ and $\mathbf{e}_\theta$,

$$\boxed{\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta} \tag{11.44}$$

The scalar components of the velocity and the acceleration in the radial and transverse directions are, therefore,

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta} \tag{11.45}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{11.46}$$

It is important to note that a_r is *not* equal to the time derivative of v_r and that a_θ is *not* equal to the time derivative of v_θ.

In the case of a particle moving along a circle of center O, we have r = constant and $\dot{r} = \ddot{r} = 0$, and the formulas (11.43) and (11.44) reduce, respectively, to

$$\mathbf{v} = r\dot{\theta}\mathbf{e}_\theta \qquad \mathbf{a} = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta \tag{11.47}$$

Extension to the Motion of a Particle in Space: Cylindrical Coordinates. The position of a particle P in space is sometimes defined by its cylindrical coordinates R, θ, and z (Fig. 11.26*a*). It is then convenient to use the unit vectors $\mathbf{e}_R$, $\mathbf{e}_\theta$, and $\mathbf{k}$ shown in Fig. 11.26*b*. Resolving the position vector $\mathbf{r}$ of the particle P into components along the unit vectors, we write

$$\mathbf{r} = R\mathbf{e}_R + z\mathbf{k} \tag{11.48}$$

Observing that $\mathbf{e}_R$ and $\mathbf{e}_\theta$ define, respectively, the radial and transverse directions in the horizontal xy plane, and that the vector $\mathbf{k}$, which defines the *axial* direction, is constant in direction as well as in magnitude, we easily verify that

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{R}\mathbf{e}_R + R\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{k} \tag{11.49}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\ddot{R} - R\dot{\theta}^2)\mathbf{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\mathbf{e}_\theta + \ddot{z}\mathbf{k} \tag{11.50}$$

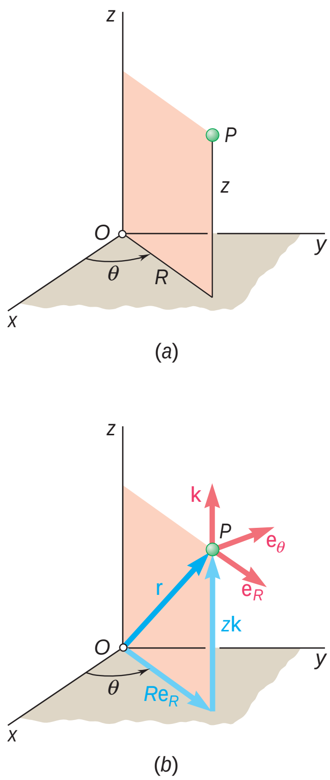
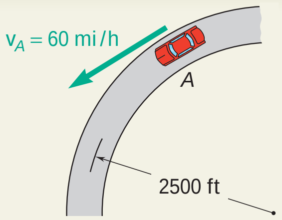


Fig. 11.26



SAMPLE PROBLEM 11.10

A motorist is traveling on a curved section of highway of radius 2500 ft at the speed of 60 mi/h. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 s the speed has been reduced to 45 mi/h, determine the acceleration of the automobile immediately after the brakes have been applied.

SOLUTION

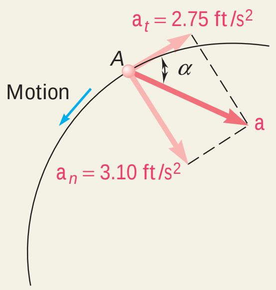
Tangential Component of Acceleration. First the speeds are expressed in ft/s.

$$60 \text{ mi/h} = \left(60 \frac{\text{mi}}{\text{h}}\right)\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 88 \text{ ft/s}$$
$$45 \text{ mi/h} = 66 \text{ ft/s}$$

Since the automobile slows down at a constant rate, we have

$$a_t = \text{average } a_t = \frac{\Delta v}{\Delta t} = \frac{66 \text{ ft/s} - 88 \text{ ft/s}}{8 \text{ s}} = -2.75 \text{ ft/s}^2$$

Normal Component of Acceleration. Immediately after the brakes have been applied, the speed is still 88 ft/s, and we have



$$a_n = \frac{v^2}{\rho} = \frac{(88 \text{ ft/s})^2}{2500 \text{ ft}} = 3.10 \text{ ft/s}^2$$

Magnitude and Direction of Acceleration. The magnitude and direction of the resultant **a** of the components $\mathbf{a}_n$ and $\mathbf{a}_t$ are

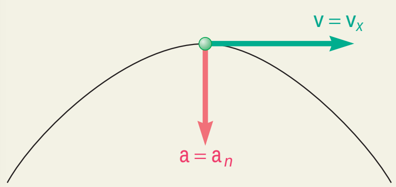
$$\tan \alpha = \frac{a_n}{a_t} = \frac{3.10 \text{ ft/s}^2}{2.75 \text{ ft/s}^2} \qquad \alpha = 48.4° \quad \blacktriangleleft$$

$$a = \frac{a_n}{\sin \alpha} = \frac{3.10 \text{ ft/s}^2}{\sin 48.4°} \qquad \mathbf{a} = 4.14 \text{ ft/s}^2 \quad \blacktriangleleft$$

SAMPLE PROBLEM 11.11

Determine the minimum radius of curvature of the trajectory described by the projectile considered in Sample Prob. 11.7.

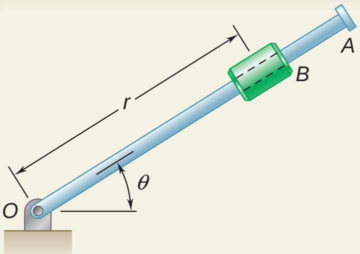
SOLUTION



Since $a_n = v^2/\rho$, we have $\rho = v^2/a_n$. The radius will be small when v is small or when a_n is large. The speed v is minimum at the top of the trajectory since $v_y = 0$ at that point; a_n is maximum at that same point, since the direction of the vertical coincides with the direction of the normal. Therefore, the minimum radius of curvature occurs at the top of the trajectory. At this point, we have

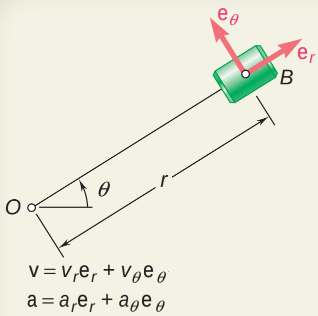
$$v = v_x = 155.9 \text{ m/s} \qquad a_n = a = 9.81 \text{ m/s}^2$$

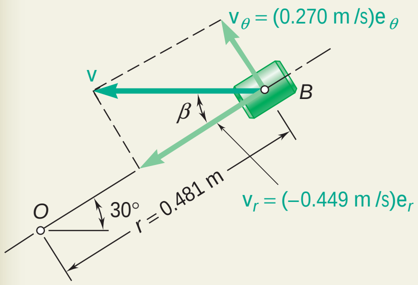
$$\rho = \frac{v^2}{a_n} = \frac{(155.9 \text{ m/s})^2}{9.81 \text{ m/s}^2} \qquad \rho = 2480 \text{ m} \quad \blacktriangleleft$$

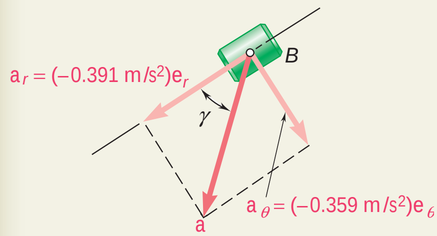


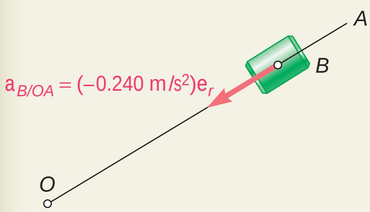
SAMPLE PROBLEM 11.12

The rotation of the 0.9-m arm OA about O is defined by the relation u $= 0.15t^2$, where u is expressed in radians and t in seconds. Collar B slides along the arm in such a way that its distance from O is $r = 0.9 - 0.12t^2$, where r is expressed in meters and t in seconds. After the arm OA has rotated through 30°, determine (*a*) the total velocity of the collar, (*b*) the total acceleration of the collar, (*c*) the relative acceleration of the collar with respect to the arm.









SOLUTION

Time t at which U = 30°. Substituting u = 30° = 0.524 rad into the expression for u, we obtain

$$u = 0.15t^2 \qquad 0.524 = 0.15t^2 \qquad t = 1.869 \text{ s}$$

Equations of Motion. Substituting $t = 1.869$ s in the expressions for r, u, and their first and second derivatives, we have

$$r = 0.9 - 0.12t^2 = 0.481 \text{ m} \qquad u = 0.15t^2 = 0.524 \text{ rad}$$
$$\dot{r} = -0.24t = -0.449 \text{ m/s} \qquad \dot{u} = 0.30t = 0.561 \text{ rad/s}$$
$$\ddot{r} = -0.24 = -0.240 \text{ m/s}^2 \qquad \ddot{u} = 0.30 = 0.300 \text{ rad/s}^2$$

a. Velocity of *B*. Using Eqs. (11.45), we obtain the values of v_r and v_u when $t = 1.869$ s.

$$v_r = \dot{r} = -0.449 \text{ m/s}$$
$$v_u = r\dot{u} = 0.481(0.561) = 0.270 \text{ m/s}$$

Solving the right triangle shown, we obtain the magnitude and direction of the velocity,

$$v = 0.524 \text{ m/s} \qquad b = 31.0° \quad ◀$$

b. Acceleration of *B*. Using Eqs. (11.46), we obtain

$$a_r = \ddot{r} - r\dot{u}^2$$
$$= -0.240 - 0.481(0.561)^2 = -0.391 \text{ m/s}^2$$
$$a_u = r\ddot{u} + 2\dot{r}\dot{u}$$
$$= 0.481(0.300) + 2(-0.449)(0.561) = -0.359 \text{ m/s}^2$$
$$a = 0.531 \text{ m/s}^2 \qquad g = 42.6° \quad ◀$$

c. Acceleration of *B* with Respect to Arm *OA*. We note that the motion of the collar with respect to the arm is rectilinear and defined by the coordinate r. We write

$$a_{B/OA} = \ddot{r} = -0.240 \text{ m/s}^2$$
$$a_{B/OA} = 0.240 \text{ m/s}^2 \text{ toward } O. \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

You will be asked in the following problems to express the velocity and the acceleration of particles in terms of either their *tangential and normal components* or their *radial and transverse components*. Although those components may not be as familiar to you as the rectangular components, you will find that they can simplify the solution of many problems and that certain types of motion are more easily described when they are used.

1. Using tangential and normal components. These components are most often used when the particle of interest travels along a circular path or when the radius of curvature of the path is to be determined. Remember that the unit vector $\mathbf{e}_t$ is tangent to the path of the particle (and thus aligned with the velocity) while the unit vector $\mathbf{e}_n$ is directed along the normal to the path and always points toward its center of curvature. It follows that, as the particle moves, the directions of the two unit vectors are constantly changing.

2. Expressing the acceleration in terms of its tangential and normal components. We derived in Sec. 11.13 the following equation, applicable to both the two-dimensional and the three-dimensional motion of a particle:

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n \tag{11.39}$$

The following observations may help you in solving the problems of this lesson.

a. The tangential component of the acceleration measures the rate of change of the speed: $a_t = dv/dt$. It follows that when a_t is constant, the equations for uniformly accelerated motion can be used with the acceleration equal to a_t. Furthermore, when a particle moves at a constant speed, we have $a_t = 0$ and the acceleration of the particle reduces to its normal component.

b. The normal component of the acceleration is always directed toward the center of curvature of the path of the particle, and its magnitude is $a_n = v^2/\rho$. Thus, the normal component can be easily determined if the speed of the particle and the radius of curvature ρ of the path are known. Conversely, when the speed and normal acceleration of the particle are known, the radius of curvature of the path can be obtained by solving this equation for ρ [Sample Prob. 11.11].

c. In three-dimensional motion, a third unit vector is used, $\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n$, which defines the direction of the *binormal*. Since this vector is perpendicular to both the velocity and the acceleration, it can be obtained by writing

$$\mathbf{e}_b = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$$

3. Using radial and transverse components. These components are used to analyze the plane motion of a particle P, when the position of P is defined by its polar coordinates r and θ. As shown in Fig. 11.25, the unit vector $\mathbf{e}_r$, which defines the *radial* direction, is attached to P and points away from the fixed point O, while the unit vector $\mathbf{e}_\theta$, which defines the *transverse* direction, is obtained by rotating $\mathbf{e}_r$ *counterclockwise* through 90°. The velocity and the acceleration of a particle were expressed in terms of their radial and transverse components in Eqs. (11.43) and (11.44), respectively. You will note that the expressions obtained contain the first and second derivatives with respect to t of both coordinates r and θ.

In the problems of this lesson, you will encounter the following types of problems involving radial and transverse components:

a. Both r and θ are known functions of t. In this case, you will compute the first and second derivatives of r and θ and substitute the expressions obtained into Eqs. (11.43) and (11.44).

b. A certain relationship exists between r and θ. First, you should determine this relationship from the geometry of the given system and use it to express r as a function of θ. Once the function $r = f(\theta)$ is known, you can apply the chain rule to determine $\dot{r}$ in terms of θ and $\dot{\theta}$, and $\ddot{r}$ in terms of θ, $\dot{\theta}$, $\ddot{\theta}$:

$$\dot{r} = f'(\theta)\dot{\theta}$$

$$\ddot{r} = f''(\theta)\dot{\theta}^2 + f'(\theta)\ddot{\theta}$$

The expressions obtained can then be substituted into Eqs. (11.43) and (11.44).

c. The three-dimensional motion of a particle, as indicated at the end of Sec. 11.14, can often be effectively described in terms of the *cylindrical coordinates* R, θ, and z (Fig. 11.26). The unit vectors then should consist of $\mathbf{e}_R$, $\mathbf{e}_\theta$, and $\mathbf{k}$. The corresponding components of the velocity and the acceleration are given in Eqs. (11.49) and (11.50). Please note that the radial distance R is always measured in a plane parallel to the xy plane, and be careful not to confuse the position vector $\mathbf{r}$ with its radial component $R\mathbf{e}_R$.

PROBLEMS

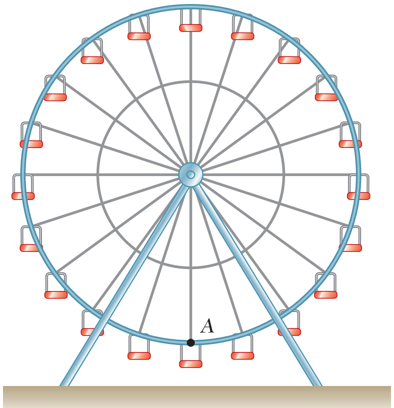


Fig. P11.CQ8

CONCEPT QUESTIONS

11.CQ8 The Ferris wheel is rotating with a constant angular velocity ω. What is the direction of the acceleration of point A?

a. $\rightarrow$ **b.** $\uparrow$ **c.** $\downarrow$ **d.** $\leftarrow$ **e.** The acceleration is zero.

11.CQ9 A race car travels around the track shown at a constant speed. At which point will the race car have the largest acceleration?

a. A. **b.** B. **c.** C. **d.** D. **e.** The acceleration will be zero at all the points.

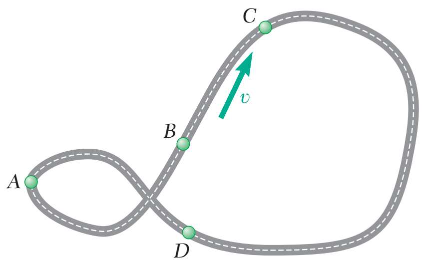


Fig. P11.CQ9

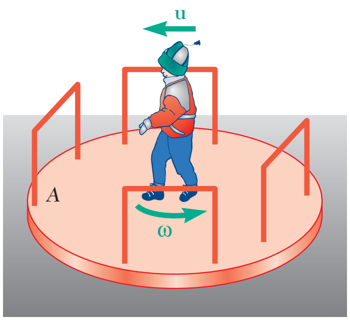


Fig. P11.CQ10

11.CQ10 A child walks across merry-go-round A with a constant speed u relative to A. The merry-go-round undergoes fixed-axis rotation about its center with a constant angular velocity ω counterclockwise. When the child is at the center of A, as shown, what is the direction of his acceleration when viewed from above?

a. $\rightarrow$ **b.** $\leftarrow$ **c.** $\uparrow$ **d.** $\downarrow$ **e.** The acceleration is zero.

END-OF-SECTION PROBLEMS

11.133 Determine the smallest radius that should be used for a highway if the normal component of the acceleration of a car traveling at 72 km/h is not to exceed 0.8 m/s^2.

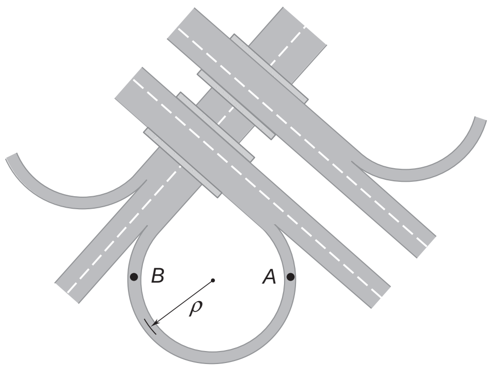


Fig. P11.133

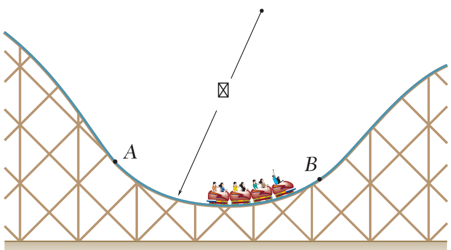


Fig. P11.134

11.134 Determine the maximum speed that the cars of the roller-coaster can reach along the circular portion AB of the track if $\rho = 25$ m and the normal component of their acceleration cannot exceed 3g.

11.135 A bull-roarer is a piece of wood that produces a roaring sound when attached to the end of a string and whirled around in a circle. Determine the magnitude of the normal acceleration of a bull-roarer when it is spun in a circle of radius 0.9 m at a speed of 20 m/s.

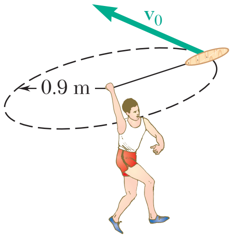


Fig. P11.135

11.136 To test its performance, an automobile is driven around a circular test track of diameter d. Determine (*a*) the value of d if when the speed of the automobile is 45 mi/h, the normal component of the acceleration is 11 ft/s^2, (*b*) the speed of the automobile if d = 600 ft and the normal component of the acceleration is measured to be 0.6g.

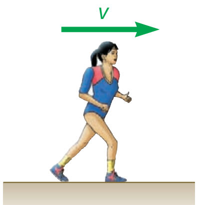


Fig. P11.137

11.137 An outdoor track is 420 ft in diameter. A runner increases her speed at a constant rate from 14 to 24 ft/s over a distance of 95 ft. Determine the magnitude of the total acceleration of the runner 2 s after she begins to increase her speed.

11.138 A robot arm moves so that P travels in a circle about point B, which is not moving. Knowing that P starts from rest, and its speed increases at a constant rate of 10 mm/s^2, determine (*a*) the magnitude of the acceleration when t = 4 s, (*b*) the time for the magnitude of the acceleration to be 80 mm/s^2.

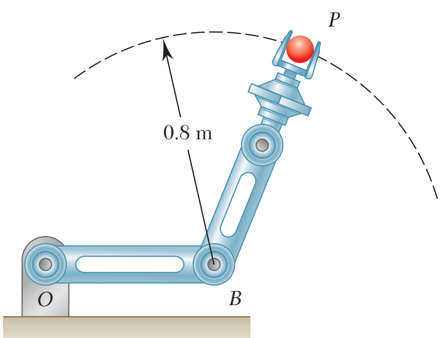


Fig. P11.138

11.139 A monorail train starts from rest on a curve of radius 400 m and accelerates at the constant rate a_t. If the maximum total acceleration of the train must not exceed 1.5 m/s^2, determine (*a*) the shortest distance in which the train can reach a speed of 72 km/h, (*b*) the corresponding constant rate of acceleration a_t.

11.140 A motorist starts from rest at point A on a circular entrance ramp when t = 0, increases the speed of her automobile at a constant rate and enters the highway at point B. Knowing that her speed continues to increase at the same rate until it reaches 100 km/h at point C, determine (*a*) the speed at point B, (*b*) the magnitude of the total acceleration when t = 20 s.

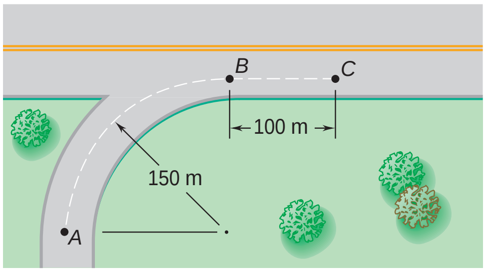


Fig. *P11.140*

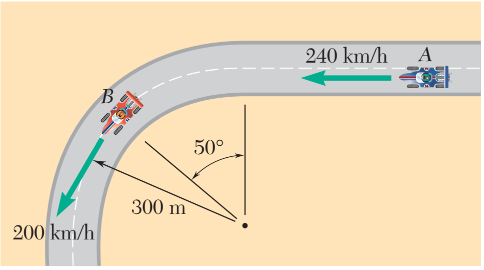


Fig. P11.141

11.141 Race car A is traveling on a straight portion of the track while race car B is traveling on a circular portion of the track. At the instant shown, the speed of A is increasing at the rate of 10 m/s^2, and the speed of B is decreasing at the rate of 6 m/s^2. For the position shown, determine (*a*) the velocity of B relative to A, (*b*) the acceleration of B relative to A.

11.142 At a given instant in an airplane race, airplane A is flying horizontally in a straight line, and its speed is being increased at the rate of 8 m/s^2. Airplane B is flying at the same altitude as airplane A and, as it rounds a pylon, is following a circular path of 300-m radius. Knowing that at the given instant the speed of B is being decreased at the rate of 3 m/s^2, determine, for the positions shown, (*a*) the velocity of B relative to A, (*b*) the acceleration of B relative to A.

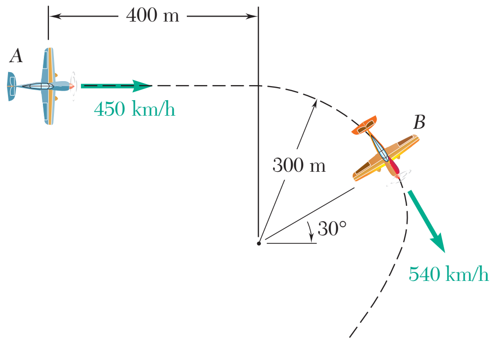


Fig. *P11.142*

11.143 From a photograph of a homeowner using a snowblower, it is determined that the radius of curvature of the trajectory of the snow was 30 ft as the snow left the discharge chute at A. Determine (*a*) the discharge velocity $\mathbf{v}_A$ of the snow, (*b*) the radius of curvature of the trajectory at its maximum height.

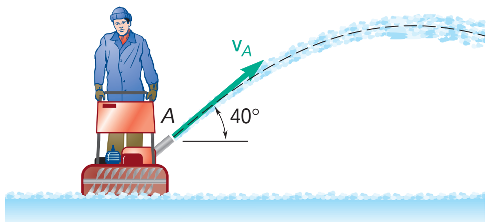


Fig. P11.143

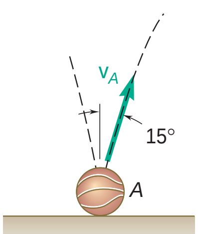


Fig. P11.144

11.144 A basketball is bounced on the ground at point A and rebounds with a velocity $\mathbf{v}_A$ of magnitude 7.5 ft/s as shown. Determine the radius of curvature of the trajectory described by the ball (*a*) at point A, (*b*) at the highest point of the trajectory.

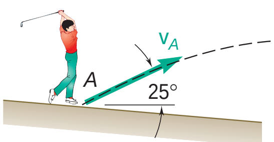


Fig. P11.145

11.145 A golfer hits a golf ball from point A with an initial velocity of 50 m/s at an angle of 25° with the horizontal. Determine the radius of curvature of the trajectory described by the ball (*a*) at point A, (*b*) at the highest point of the trajectory.

11.146 Three children are throwing snowballs at each other. Child A throws a snowball with a horizontal velocity $\mathbf{v}_0$. If the snowball just passes over the head of child B and hits child C, determine the radius of curvature of the trajectory described by the snowball (*a*) at point B, (*b*) at point C.

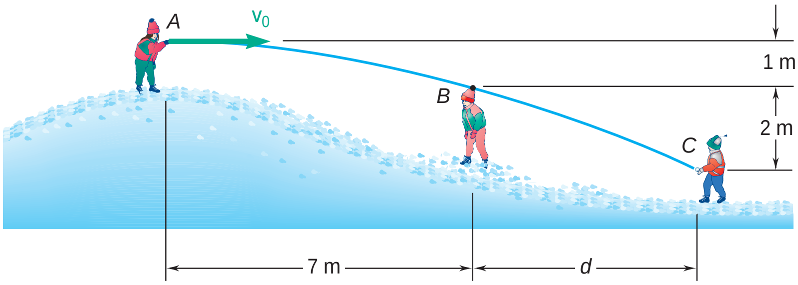


Fig. P11.146

11.147 Coal is discharged from the tailgate A of a dump truck with an initial velocity $\mathbf{v}_A$ = 2 m/s d 50°. Determine the radius of curvature of the trajectory described by the coal (*a*) at point A, (*b*) at the point of the trajectory 1 m below point A.

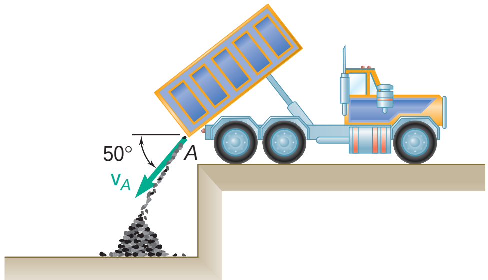


Fig. P11.147

11.148 From measurements of a photograph, it has been found that as the stream of water shown left the nozzle at A, it had a radius of curvature of 25 m. Determine (*a*) the initial velocity $\mathbf{v}_A$ of the stream, (*b*) the radius of curvature of the stream as it reaches its maximum height at B.

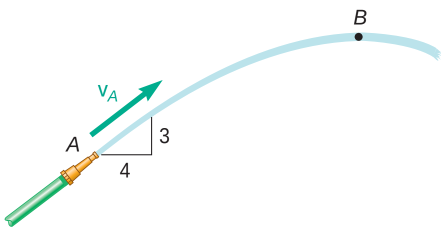


Fig. *P11.148*

11.149 A child throws a ball from point A with an initial velocity $\mathbf{v}_A$ of 20 m/s at an angle of 25° with the horizontal. Determine the velocity of the ball at the points of the trajectory described by the ball where the radius of curvature is equal to three-quarters of its value at A.

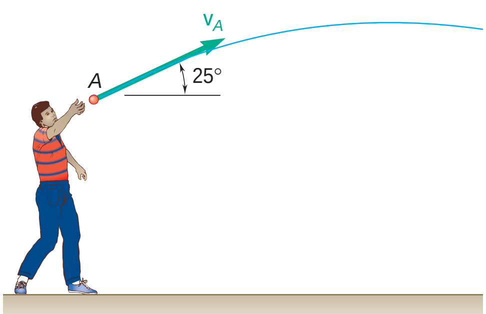


Fig. P11.149

11.150 A projectile is fired from point A with an initial velocity $\mathbf{v}_0$. (*a*) Show that the radius of curvature of the trajectory of the projectile reaches its minimum value at the highest point B of the trajectory. (*b*) Denoting by θ the angle formed by the trajectory and the horizontal at a given point C, show that the radius of curvature of the trajectory at C is $\rho = \rho_{min}/\cos^3\theta$.

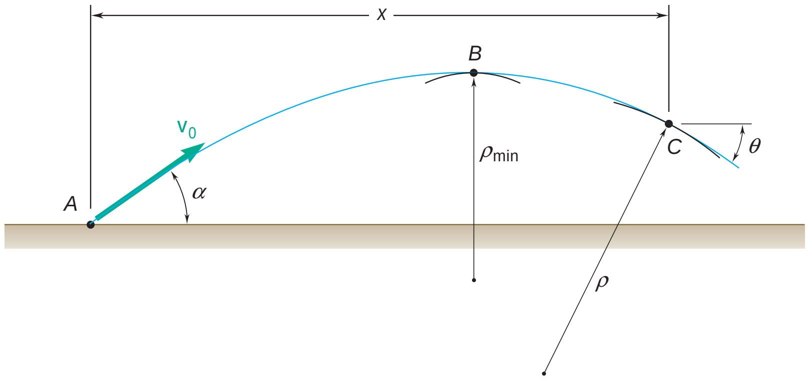


Fig. P11.150

***11.151** Determine the radius of curvature of the path described by the particle of Prob. 11.95 when $t = 0$.

***11.152** Determine the radius of curvature of the path described by the particle of Prob. 11.96 when $t = 0$, $A = 3$, and $B = 1$.

11.153 and 11.154 A satellite will travel indefinitely in a circular orbit around a planet if the normal component of the acceleration of the satellite is equal to $g(R/r)^2$, where g is the acceleration of gravity at the surface of the planet, R is the radius of the planet, and r is the distance from the center of the planet to the satellite. Knowing that the diameter of the sun is 1.39 Gm and that the acceleration of gravity at its surface is 274 m/s^2, determine the radius of the orbit of the indicated planet around the sun assuming that the orbit is circular.

11.153 Earth: $(v_{mean})_{orbit} = 107$ Mm/h.
11.154 Saturn: $(v_{mean})_{orbit} = 34.7$ Mm/h.

11.155 through 11.157 Determine the speed of a satellite relative to the indicated planet if the satellite is to travel indefinitely in a circular orbit 100 mi above the surface of the planet. (See information given in Probs. 11.153–11.154.)

11.155 Venus: $g = 29.20$ ft/s^2, $R = 3761$ mi.
11.156 Mars: $g = 12.17$ ft/s^2, $R = 2102$ mi.
11.157 Jupiter: $g = 75.35$ ft/s^2, $R = 44{,}432$ mi.

11.158 A satellite is traveling in a circular orbit around Mars at an altitude of 300 km. After the altitude of the satellite is adjusted, it is found that the time of one orbit has increased by 10 percent. Knowing that the radius of Mars is 3382 km, determine the new altitude of the satellite. (See information given in Probs. 11.153–11.154).

11.159 Knowing that the radius of the earth is 6370 km, determine the time of one orbit of the Hubble Space Telescope knowing that the telescope travels in a circular orbit 590 km above the surface of the earth. (See information given in Probs. 11.153–11.154.)

11.160 Satellites A and B are traveling in the same plane in circular orbits around the earth at altitudes of 120 and 200 mi, respectively. If at $t = 0$ the satellites are aligned as shown and knowing that the radius of the earth is $R = 3960$ mi, determine when the satellites will next be radially aligned. (See information given in Probs. 11.153–11.154.)

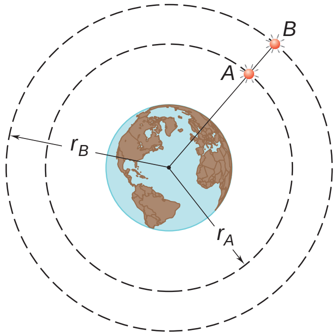


Fig. *P11.160*

11.161 The oscillation of rod OA about O is defined by the relation $\theta = (3/\pi)(\sin \pi t)$, where θ and t are expressed in radians and seconds, respectively. Collar B slides along the rod so that its distance from O is $r = 6(1 - e^{-2t})$ where r and t are expressed in inches and seconds, respectively. When $t = 1$ s, determine (*a*) the velocity of the collar, (*b*) the acceleration of the collar, (*c*) the acceleration of the collar relative to the rod.

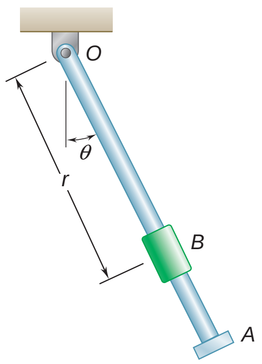


Fig. P11.161 and P11.162

11.162 The rotation of rod OA about O is defined by the relation $\theta = t^3 - 4t$, where θ and t are expressed in radians and seconds, respectively. Collar B slides along the rod so that its distance from O is $r = 2.5t^3 - 5t^2$, where r and t are expressed in inches and seconds, respectively. When $t = 1$ s, determine (*a*) the velocity of the collar, (*b*) the acceleration of the collar, (*c*) the radius of curvature of the path of the collar.

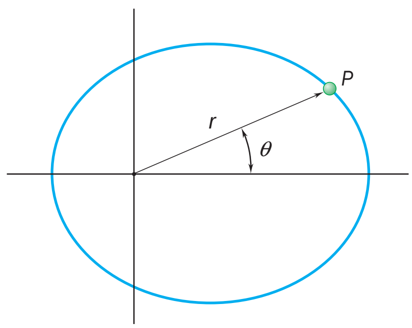


Fig. P11.163

11.163 The path of particle P is the ellipse defined by the relations $r = 2/(2 - \cos \pi t)$ and $\theta = \pi t$, where r is expressed in meters, t is in seconds, and θ is in radians. Determine the velocity and the acceleration of the particle when (*a*) $t = 0$, (*b*) $t = 0.5$ s.

11.164 The two-dimensional motion of a particle is defined by the relations $r = 2a \cos \theta$ and $\theta = bt^2/2$, where a and b are constants. Determine (*a*) the magnitudes of the velocity and acceleration at any instant, (*b*) the radius of curvature of the path. What conclusion can you draw regarding the path of the particle?

11.165 As rod OA rotates, pin P moves along the parabola BCD. Knowing that the equation of this parabola is $r = 2b/(1 + \cos\theta)$ and that $\theta = kt$, determine the velocity and acceleration of P when (*a*) $\theta = 0$, (*b*) $\theta = 90°$.

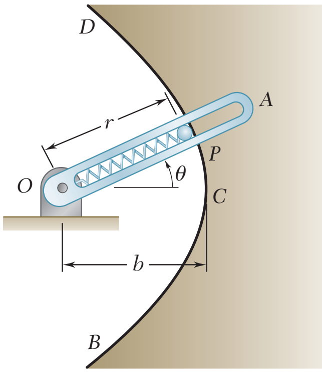


Fig. P11.165

11.166 The pin at B is free to slide along the circular slot DE and along the rotating rod OC. Assuming that the rod OC rotates at a constant rate $\dot{\theta}$, (*a*) show that the acceleration of pin B is of constant magnitude, (*b*) determine the direction of the acceleration of pin B.

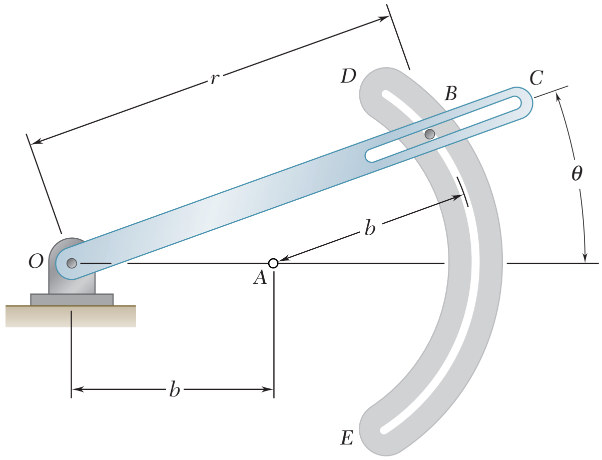


Fig. P11.166

11.167 To study the performance of a race car, a high-speed camera is positioned at point A. The camera is mounted on a mechanism which permits it to record the motion of the car as the car travels on straightaway BC. Determine (*a*) the speed of the car in terms of b, θ, and $\dot{\theta}$, (*b*) the magnitude of the acceleration in terms of b, θ, $\dot{\theta}$, and $\ddot{\theta}$.

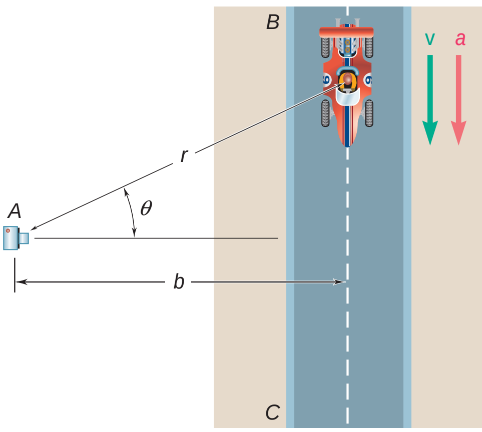


Fig. *P11.167*

11.168 After taking off, a helicopter climbs in a straight line at a constant angle β. Its flight is tracked by radar from point A. Determine the speed of the helicopter in terms of d, β, θ, and $\dot{\theta}$.

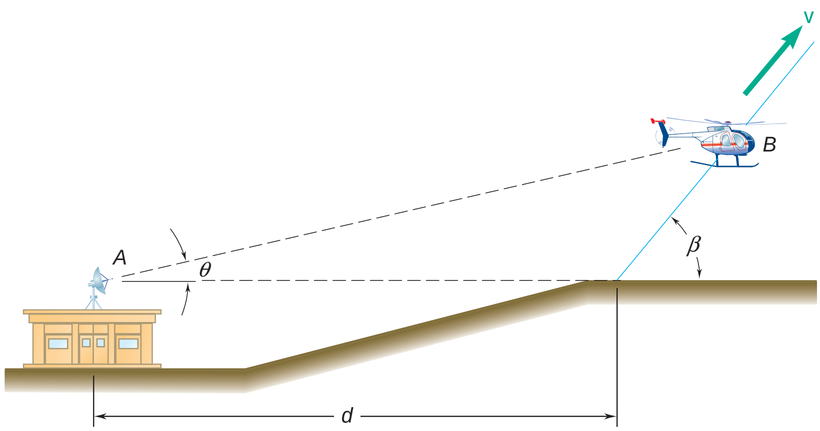


Fig. *P11.168*

11.169 At the bottom of a loop in the vertical plane an airplane has a horizontal velocity of 315 mi/h and is speeding up at a rate of 10 ft/s^2. The radius of curvature of the loop is 1 mi. The plane is being tracked by radar at O. What are the recorded values of $\dot{r}$, $\ddot{r}$, $\dot{u}$, and $\ddot{u}$ for this instant?

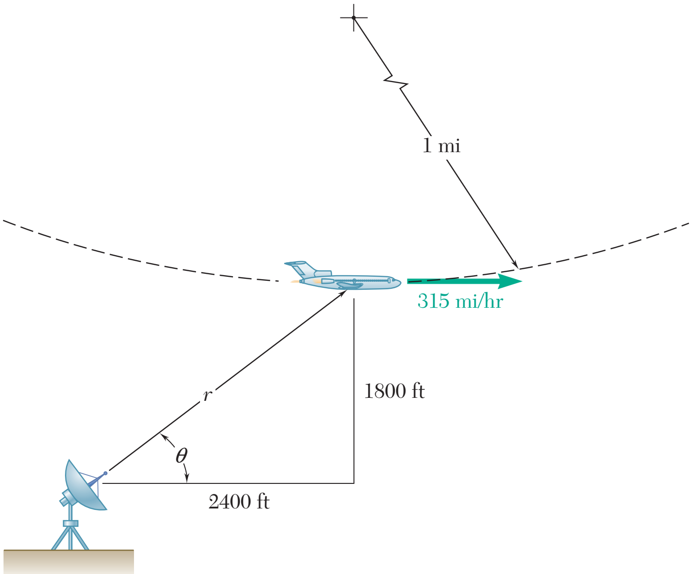


Fig. P11.169

11.170 Pin C is attached to rod BC and slides freely in the slot of rod OA which rotates at the constant rate v. At the instant when b = 60°, determine (*a*) $\dot{r}$ and $\dot{u}$, (*b*) $\ddot{r}$ and $\ddot{u}$. Express your answers in terms of d and v.

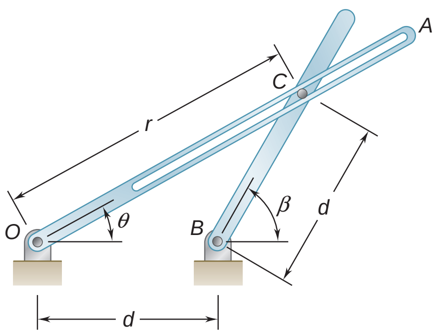


Fig. P11.170

11.171 For the race car of Prob. 11.167, it was found that it took 0.5 s for the car to travel from the position u = 60° to the position u = 35°. Knowing that b = 25 m, determine the average speed of the car during the 0.5-s interval.

11.172 For the helicopter of Prob. 11.168, it was found that when the helicopter was at B, the distance and the angle of elevation of the helicopter were r = 3000 ft and u = 20°, respectively. Four seconds later, the radar station sighted the helicopter at r = 3320 ft and u = 23.1°. Determine the average speed and the angle of climb b of the helicopter during the 4-s interval.

11.173 and 11.174 A particle moves along the spiral shown; determine the magnitude of the velocity of the particle in terms of b, u, and $\dot{\text{u}}$.

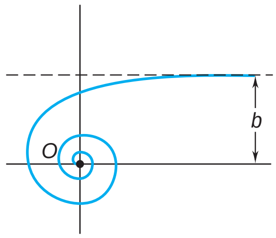


Fig. P11.173 and P11.175

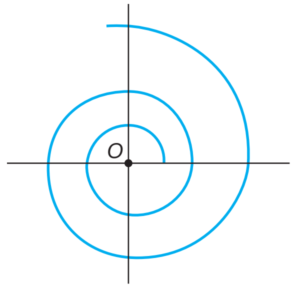


Fig. P11.174 and P11.176

11.175 and 11.176 A particle moves along the spiral shown. Knowing that $\dot{\text{u}}$ is constant and denoting this constant by v, determine the magnitude of the acceleration of the particle in terms of b, u, and v.

11.177 The motion of a particle on the surface of a right circular cylinder is defined by the relations $R = A$, u $= 2$pt, and $z = B \sin 2$pnt, where A and B are constants and n is an integer. Determine the magnitudes of the velocity and acceleration of the particle at any time t.

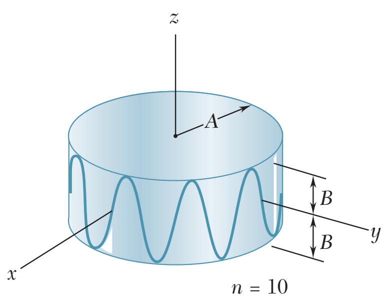


Fig. P11.177

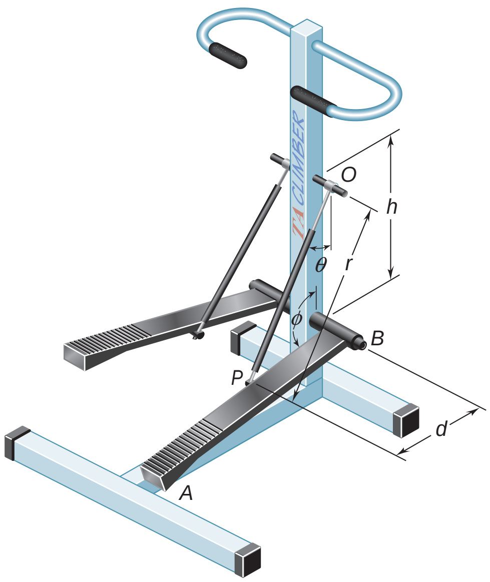


Fig. P11.178

11.178 Show that $\dot{r} = h\dot{\text{f}} \sin$ u knowing that at the instant shown, step AB of the step exerciser is rotating counterclockwise at a constant rate $\dot{\text{f}}$.

11.179 The three-dimensional motion of a particle is defined by the relations $R = A(1 - e^{-t})$, u $= 2$pt, and $z = B(1 - e^{-t})$. Determine the magnitudes of the velocity and acceleration when (*a*) $t = 0$, (*b*) $t = \infty$.

***11.180** For the conic helix of Prob. 11.95, determine the angle that the osculating plane forms with the y axis.

***11.181** Determine the direction of the binormal of the path described by the particle of Prob. 11.96 when (*a*) $t = 0$, (*b*) $t =$ p$/2$ s.

REVIEW AND SUMMARY

Position coordinate of a particle in rectilinear motion

In the first half of the chapter, we analyzed the *rectilinear motion of a particle,* i.e., the motion of a particle along a straight line. To define the position P of the particle on that line, we chose a fixed origin O and a positive direction (Fig. 11.27). The distance x from O to P, with the appropriate sign, completely defines the position of the particle on the line and is called the *position coordinate* of the particle [Sec. 11.2].

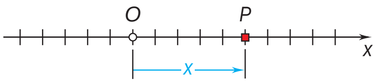


Fig. 11.27

Velocity and acceleration in rectilinear motion

The *velocity* v of the particle was shown to be equal to the time derivative of the position coordinate x,

$$v = \frac{dx}{dt} \tag{11.1}$$

and the *acceleration* a was obtained by differentiating v with respect to t,

$$a = \frac{dv}{dt} \tag{11.2}$$

or

$$a = \frac{d^2x}{dt^2} \tag{11.3}$$

We also noted that a could be expressed as

$$a = v\frac{dv}{dx} \tag{11.4}$$

We observed that the velocity v and the acceleration a were represented by algebraic numbers which can be positive or negative. A positive value for v indicates that the particle moves in the positive direction, and a negative value that it moves in the negative direction. A positive value for a, however, may mean that the particle is truly accelerated (i.e., moves faster) in the positive direction, or that it is decelerated (i.e., moves more slowly) in the negative direction. A negative value for a is subject to a similar interpretation [Sample Prob. 11.1].

Determination of the velocity and acceleration by integration

In most problems, the conditions of motion of a particle are defined by the type of acceleration that the particle possesses and by the initial conditions [Sec. 11.3]. The velocity and position of the particle can then be obtained by integrating two of the equations (11.1) to (11.4). Which of these equations should be selected depends upon the type of acceleration involved [Sample Probs. 11.2 and 11.3].

Uniform rectilinear motion

Two types of motion are frequently encountered: the *uniform rectilinear motion* [Sec. 11.4], in which the velocity v of the particle is constant and

$$x = x_0 + vt \tag{11.5}$$

Uniformly accelerated rectilinear motion

and the *uniformly accelerated rectilinear motion* [Sec. 11.5], in which the acceleration a of the particle is constant and we have

$$v = v_0 + at \tag{11.6}$$
$$x = x_0 + v_0 t + \tfrac{1}{2}at^2 \tag{11.7}$$
$$v^2 = v_0^2 + 2a(x - x_0) \tag{11.8}$$

Relative motion of two particles

When two particles A and B move along the same straight line, we may wish to consider the *relative motion* of B with respect to A

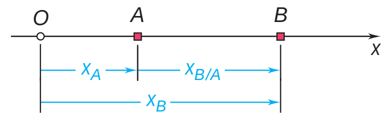


Fig. 11.28

[Sec. 11.6]. Denoting by $x_{B/A}$ the *relative position coordinate* of B with respect to A (Fig. 11.28), we had

$$x_B = x_A + x_{B/A} \tag{11.9}$$

Differentiating Eq. (11.9) twice with respect to t, we obtained successively

$$v_B = v_A + v_{B/A} \tag{11.10}$$
$$a_B = a_A + a_{B/A} \tag{11.11}$$

where $v_{B/A}$ and $a_{B/A}$ represent, respectively, the *relative velocity* and the *relative acceleration* of B with respect to A.

Blocks connected by inextensible cords

When several blocks are *connected by inextensible cords,* it is possible to write a *linear relation* between their position coordinates. Similar relations can then be written between their velocities and between their accelerations and can be used to analyze their motion [Sample Prob. 11.5].

Graphical solutions

It is sometimes convenient to use a *graphical solution* for problems involving the rectilinear motion of a particle [Secs. 11.7 and 11.8]. The graphical solution most commonly used involves the x–t, v–t, and a–t curves [Sec. 11.7; Sample Prob. 11.6]. It was shown that, at any given time t,

$$v = \text{slope of } x\text{–}t \text{ curve}$$
$$a = \text{slope of } v\text{–}t \text{ curve}$$

while, over any given time interval from t_1 to t_2,

$$v_2 - v_1 = \text{area under } a\text{–}t \text{ curve}$$
$$x_2 - x_1 = \text{area under } v\text{–}t \text{ curve}$$

Position vector and velocity in curvilinear motion

In the second half of the chapter, we analyzed the *curvilinear motion of a particle,* i.e., the motion of a particle along a curved path. The position P of the particle at a given time [Sec. 11.9] was defined by the *position vector* **r** joining the O of the coordinates and point P (Fig. 11.29). The *velocity* **v** of the particle was defined by the relation

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{11.15}$$

and was found to be a *vector tangent to the path of the particle* and of magnitude v (called the *speed* of the particle) equal to the time derivative of the length s of the arc described by the particle:

$$v = \frac{ds}{dt} \tag{11.16}$$

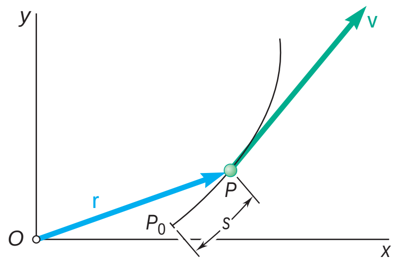


Fig. 11.29

Acceleration in curvilinear motion

The *acceleration* **a** of the particle was defined by the relation

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{11.18}$$

and we noted that, in general, *the acceleration is not tangent to the path of the particle.*

Derivative of a vector function

Before proceeding to the consideration of the components of velocity and acceleration, we reviewed the formal definition of the derivative of a vector function and established a few rules governing the differentiation of sums and products of vector functions. We then showed that the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation [Sec. 11.10].

Rectangular components of velocity and acceleration

Denoting by x, y, and z the rectangular coordinates of a particle P, we found that the rectangular components of the velocity and acceleration of P equal, respectively, the first and second derivatives with respect to t of the corresponding coordinates:

$$v_x = \dot{x} \qquad v_y = \dot{y} \qquad v_z = \dot{z} \tag{11.29}$$
$$a_x = \ddot{x} \qquad a_y = \ddot{y} \qquad a_z = \ddot{z} \tag{11.30}$$

Component motions

When the component a_x of the acceleration depends only upon t, x, and/or v_x, and when similarly a_y depends only upon t, y, and/or v_y, and a_z upon t, z, and/or v_z, Eq. (11.30) can be integrated independently. The analysis of the given curvilinear motion can thus be reduced to the analysis of three independent rectilinear component motions [Sec. 11.11]. This approach is particularly effective in the study of the motion of projectiles [Sample Probs. 11.7 and 11.8].

Relative motion of two particles

For two particles A and B moving in space (Fig. 11.30), we considered the relative motion of B with respect to A, or more precisely, with respect to a moving frame attached to A and in translation with A [Sec. 11.12]. Denoting by $\mathbf{r}_{B/A}$ the *relative position vector* of B with respect to A (Fig. 11.30), we had

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \tag{11.31}$$

Denoting by $\mathbf{v}_{B/A}$ and $\mathbf{a}_{B/A}$, respectively, the *relative velocity* and the *relative acceleration* of B with respect to A, we also showed that

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{11.33}$$

and

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{11.34}$$

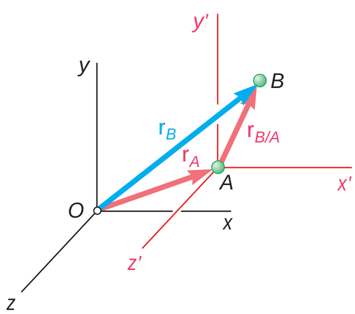


Fig. 11.30

Tangential and normal components

It is sometimes convenient to resolve the velocity and acceleration of a particle P into components other than the rectangular x, y, and z components. For a particle P moving along a path contained in a plane, we attached to P unit vectors $\mathbf{e}_t$ tangent to the path and $\mathbf{e}_n$ normal to the path and directed toward the center of curvature of the path [Sec. 11.13]. We then expressed the velocity and acceleration of the particle in terms of tangential and normal components. We wrote

$$\mathbf{v} = v\mathbf{e}_t \qquad (11.36)$$

and

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n \qquad (11.39)$$

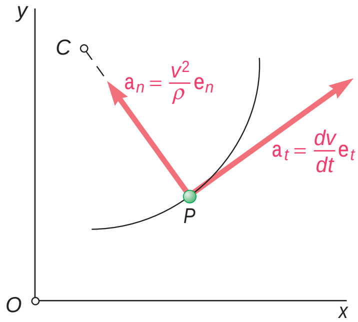


Fig. 11.31

where v is the speed of the particle and ρ the radius of curvature of its path [Sample Probs. 11.10 and 11.11]. We observed that while the velocity $\mathbf{v}$ is directed along the tangent to the path, the acceleration $\mathbf{a}$ consists of a component $\mathbf{a}_t$ directed along the tangent to the path and a component $\mathbf{a}_n$ directed toward the center of curvature of the path (Fig. 11.31).

Motion along a space curve

For a particle P moving along a space curve, we defined the plane which most closely fits the curve in the neighborhood of P as the *osculating plane*. This plane contains the unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$ which define, respectively, the tangent and principal normal to the curve. The unit vector $\mathbf{e}_b$ which is perpendicular to the osculating plane defines the *binormal*.

Radial and transverse components

When the position of a particle P moving in a plane is defined by its polar coordinates r and θ, it is convenient to use radial and transverse components directed, respectively, along the position vector $\mathbf{r}$ of the particle and in the direction obtained by rotating $\mathbf{r}$ through 90° counterclockwise [Sec. 11.14]. We attached to P unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ directed, respectively, in the radial and transverse directions (Fig. 11.32). We then expressed the velocity and acceleration of the particle in terms of radial and transverse components

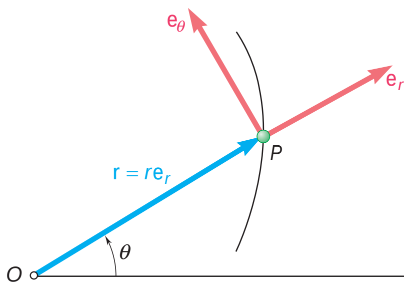


Fig. 11.32

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \qquad (11.43)$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \qquad (11.44)$$

where dots are used to indicate differentiation with respect to time. The scalar components of the velocity and acceleration in the radial and transverse directions are therefore

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta} \qquad (11.45)$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad (11.46)$$

It is important to note that a_r is *not* equal to the time derivative of v_r, and that a_θ is *not* equal to the time derivative of v_θ [Sample Prob. 11.12].

The chapter ended with a discussion of the use of cylindrical coordinates to define the position and motion of a particle in space.

REVIEW PROBLEMS

11.182 The motion of a particle is defined by the relation $x = 2t^3 - 15t^2 + 24t + 4$, where x and t are expressed in meters and seconds, respectively. Determine (*a*) when the velocity is zero, (*b*) the position and the total distance traveled when the acceleration is zero.

11.183 A particle starting from rest at $x = 1$ m is accelerated so that its velocity doubles in magnitude between $x = 2$ m and $x = 8$ m. Knowing that the acceleration of the particle is defined by the relation $a = k[x - (A/x)]$, determine the values of the constants A and k if the particle has a velocity of 29 m/s when $x = 16$ m.

11.184 A particle moves in a straight line with the acceleration shown in the figure. Knowing that the particle starts from the origin with $v_0 = -2$ m/s, (*a*) construct the v–t and x–t curves for $0 < t < 18$ s, (*b*) determine the position and the velocity of the particle and the total distance traveled when $t = 18$ s.

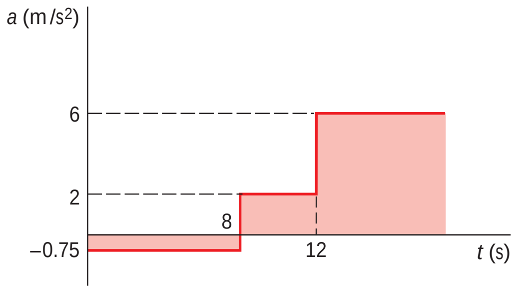


Fig. *P11.184*

11.185 The velocities of commuter trains A and B are as shown. Knowing that the speed of each train is constant and that B reaches the crossing 10 min after A passed through the same crossing, determine (*a*) the relative velocity of B with respect to A, (*b*) the distance between the fronts of the engines 3 min after A passed through the crossing.

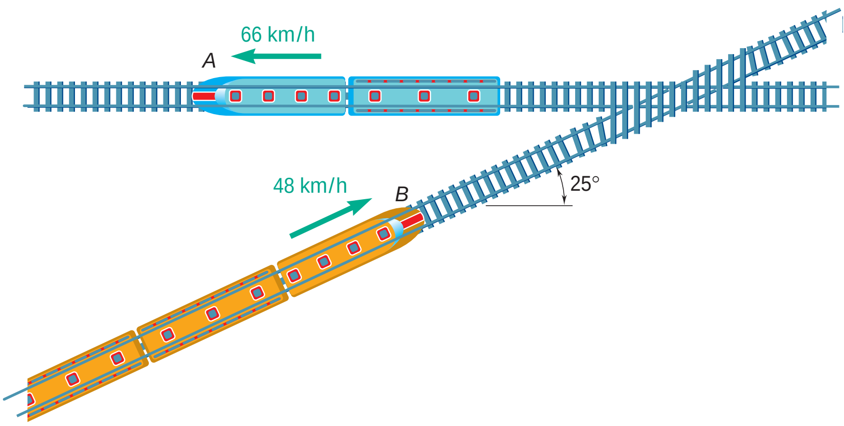


Fig. P11.185

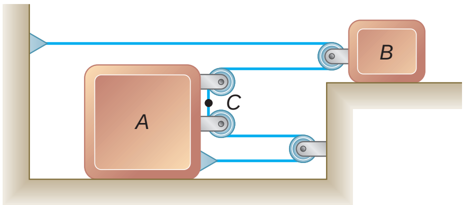


Fig. *P11.186*

11.186 Slider block B starts from rest and moves to the right with a constant acceleration of 1 ft/s^2. Determine (*a*) the relative acceleration of portion C of the cable with respect to slider block A, (*b*) the velocity of portion C of the cable after 2 s.

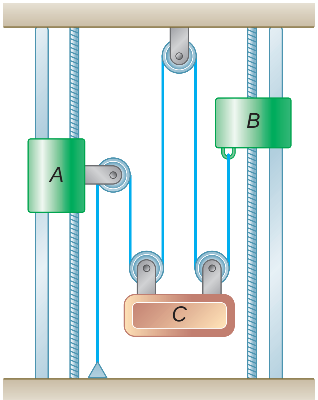


Fig. P11.187

11.187 Collar A starts from rest at $t = 0$ and moves downward with a constant acceleration of 7 in./s^2. Collar B moves upward with a constant acceleration, and its initial velocity is 8 in./s. Knowing that collar B moves through 20 in. between $t = 0$ and $t = 2$ s, determine (*a*) the accelerations of collar B and block C, (*b*) the time at which the velocity of block C is zero, (*c*) the distance through which block C will have moved at that time.

11.188 A golfer hits a ball with an initial velocity of magnitude v_0 at an angle α with the horizontal. Knowing that the ball must clear the tops of two trees and land as close as possible to the flag, determine v_0 and the distance d when the golfer uses (*a*) a six-iron with $\alpha = 31°$, (*b*) a five-iron with $\alpha = 27°$.

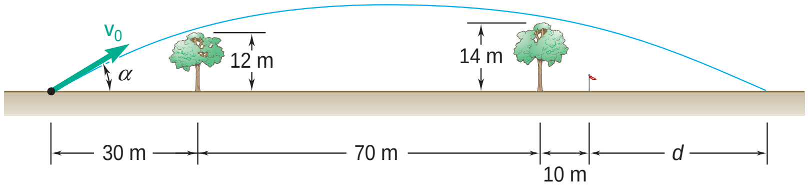


Fig. P11.188

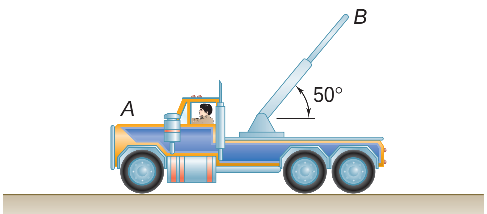


Fig. P11.189

11.189 As the truck shown begins to back up with a constant acceleration of 4 ft/s^2, the outer section B of its boom starts to retract with a constant acceleration of 1.6 ft/s^2 relative to the truck. Determine (*a*) the acceleration of section B, (*b*) the velocity of section B when $t = 2$ s.

11.190 A motorist traveling along a straight portion of a highway is decreasing the speed of his automobile at a constant rate before exiting from the highway onto a circular exit ramp with a radius of 560 ft. He continues to decelerate at the same constant rate so that 10 s after entering the ramp, his speed has decreased to 20 mi/h, a speed which he then maintains. Knowing that at this constant speed the total acceleration of the automobile is equal to one-quarter of its value prior to entering the ramp, determine the maximum value of the total acceleration of the automobile.

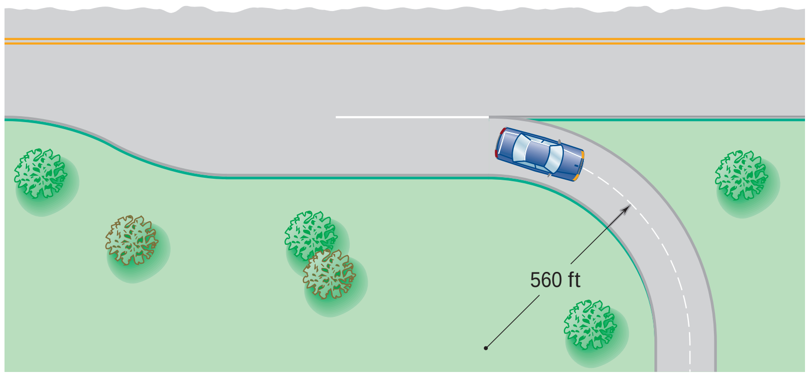


Fig. P11.190

11.191 Sand is discharged at A from a conveyor belt and falls onto the top of a stockpile at B. Knowing that the conveyor belt forms an angle $\alpha = 25°$ with the horizontal, determine (*a*) the speed v_0 of the belt, (*b*) the radius of curvature of the trajectory described by the sand at point B.

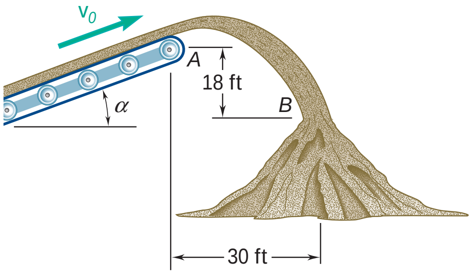


Fig. P11.191

11.192 The end point B of a boom is originally 5 m from fixed point A when the driver starts to retract the boom with a constant radial acceleration of $\ddot{r} = -1.0\ \text{m/s}^2$ and lower it with a constant angular acceleration $\ddot{\theta} = -0.5\ \text{rad/s}^2$. At $t = 2$ s, determine (*a*) the velocity of point B, (*b*) the acceleration of point B, (*c*) the radius of curvature of the path.

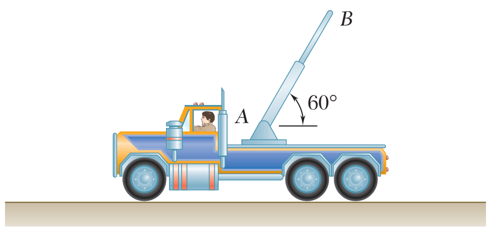


Fig. *P11.192*

11.193 A telemetry system is used to quantify kinematic values of a ski jumper immediately before she leaves the ramp. According to the system $r = 500$ ft, $\dot{r} = -105$ ft/s, $\ddot{r} = -10\ \text{ft/s}^2$, $\theta = 25°$, $\dot{\theta} = 0.07$ rad/s, $\ddot{\theta} = 0.06\ \text{rad/s}^2$. Determine (*a*) the velocity of the skier immediately before she leaves the jump, (*b*) the acceleration of the skier at this instant, (*c*) the distance of the jump d neglecting lift and air resistance.

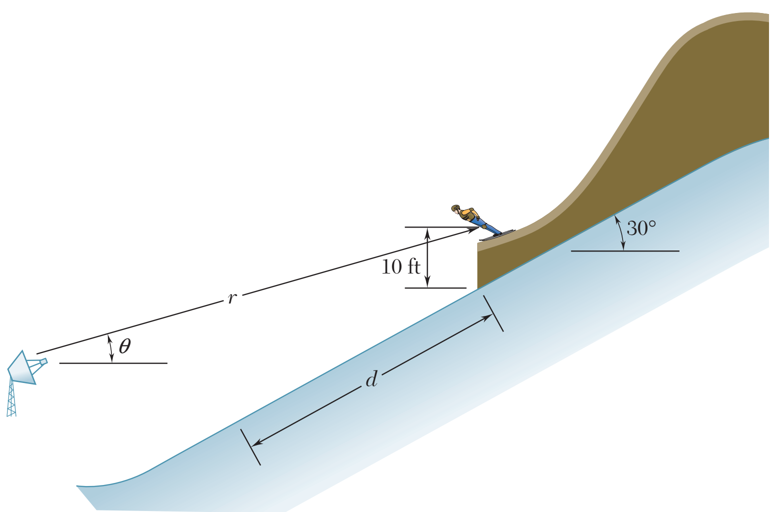


Fig. *P11.193*

COMPUTER PROBLEMS

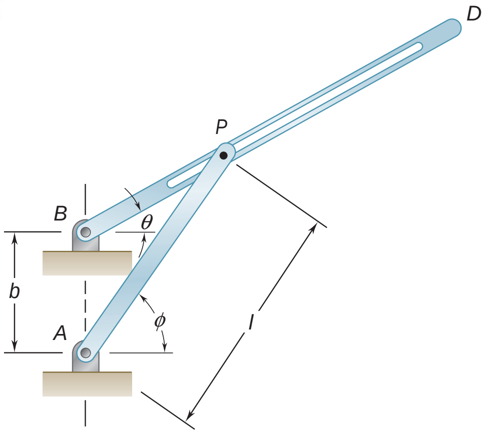


Fig. P11.C1

11.C1 The mechanism shown is known as a Whitworth quick-return mechanism. The input rod AP rotates at a constant rate $\dot{\phi}$, and the pin P is free to slide in the slot of the output rod BD. Plot θ versus ϕ and $\dot{\theta}$ versus ϕ for one revolution of rod AP. Assume $\dot{\phi} = 1$ rad/s, $l = 4$ in., and (*a*) $b = 2.5$ in., (*b*) $b = 3$ in., (*c*) $b = 3.5$ in.

11.C2 A ball is dropped with a velocity $\mathbf{v}_0$ at an angle α with the vertical onto the top step of a flight of stairs consisting of 8 steps. The ball rebounds and bounces down the steps as shown. Each time the ball bounces, at points $A, B, C, \ldots,$ the horizontal component of its velocity remains constant and the magnitude of the vertical component of its velocity is reduced by k percent. Use computational software to determine (*a*) if the ball bounces down the steps without skipping any step, (*b*) if the ball bounces down the steps without bouncing twice on the same step, (*c*) the first step on which the ball bounces twice. Use values of v_0 from 1.8 m/s to 3.0 m/s in 0.6-m/s increments, values of α from 18° to 26° in 4° increments, and values of k equal to 40 and 50.

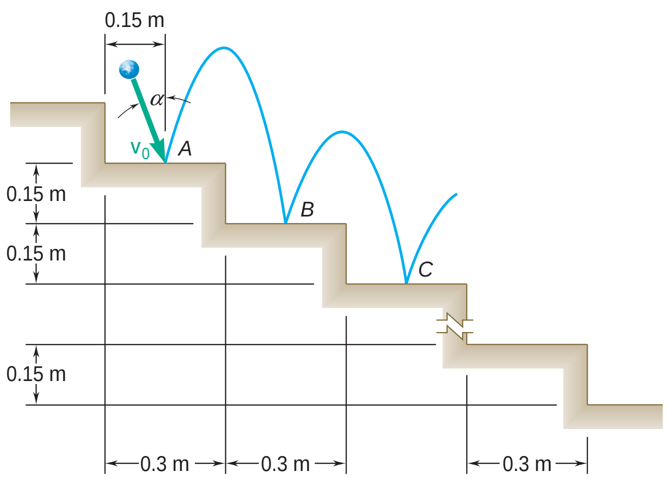


Fig. P11.C2

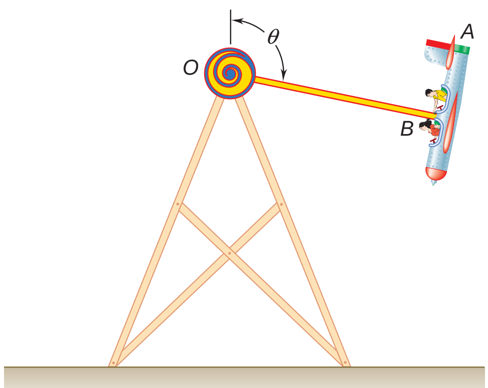


Fig. P11.C3

11.C3 In an amusement park ride, "airplane" A is attached to the 10-m-long rigid member OB. To operate the ride, the airplane and OB are rotated so that $70° \leq \theta_0 \leq 130°$ and then are allowed to swing freely about O. The airplane is subjected to the acceleration of gravity and to a deceleration due to air resistance, $-kv^2$, which acts in a direction opposite to that of its velocity $\mathbf{v}$. Neglecting the mass and the aerodynamic drag of OB and the friction in the bearing at O, use computational software or write a computer program to determine the speed of the airplane for given values of θ_0 and θ and the value of θ at which the airplane first comes to rest after being released. Use values of θ_0 from 70° to 130° in 30° increments, and determine the maximum speed of the airplane and the first two values of θ at which $v = 0$. For each value of θ_0, let (*a*) $k = 0$, (*b*) $k = 2 \times 10^{-4}$ m^{-1}, (*c*) $k = 4 \times 10^{-2}$ m^{-1}. (*Hint:* Express the tangential acceleration of the airplane in terms of g, k, and θ. Recall that $v_\theta = r\dot{\theta}$.)

11.C4 A motorist traveling on a highway at a speed of 60 mi/h exits onto an ice-covered exit ramp. Wishing to stop, he applies his brakes until his automobile comes to rest. Knowing that the magnitude of the total acceleration of the automobile cannot exceed 10 ft/s^2, use computational software to determine the minimum time required for the automobile to come to rest and the distance it travels on the exit ramp during that time if the exit ramp (*a*) is straight, (*b*) has a constant radius of curvature of 800 ft. Solve each part assuming that the driver applies his brakes so that dv/dt, during each time interval, (1) remains constant, (2) varies linearly.

11.C5 An oscillating garden sprinkler discharges water with an initial velocity $\mathbf{v}_0$ of 10 m/s. (*a*) Knowing that the sides but not the top of arbor *BCDE* are open, use computational software to calculate the distance *d* to the point *F* that will be watered for values of a from 20° to 80°. (*b*) Determine the maximum value of *d* and the corresponding value of a.

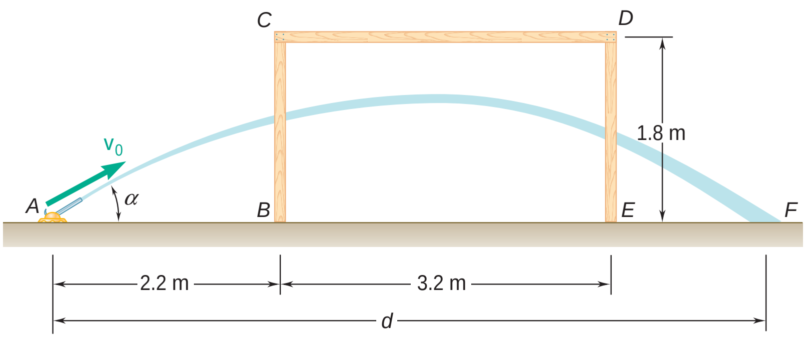


Fig. P11.C5

The forces experienced by the passengers on a roller coaster will depend on whether the roller-coaster car is traveling up a hill or down a hill, in a straight line, or along a horizontal or vertical curved path. The relation existing among force, mass, and acceleration will be studied in this chapter.

CHAPTER 12

Kinetics of Particles: Newton's Second Law

Chapter 12 Kinetics of Particles: Newton's Second Law

12.1 Introduction
12.2 Newton's Second Law of Motion
12.3 Linear Momentum of a Particle. Rate of Change of Linear Momentum
12.4 Systems of Units
12.5 Equations of Motion
12.6 Dynamic Equilibrium
12.7 Angular Momentum of a Particle. Rate of Change of Angular Momentum
12.8 Equations of Motion in Terms of Radial and Transverse Components
12.9 Motion Under a Central Force. Conservation of Angular Momentum
12.10 Newton's Law of Gravitation
12.11 Trajectory of a Particle Under a Central Force
12.12 Application to Space Mechanics
12.13 Kepler's Laws of Planetary Motion

12.1 INTRODUCTION

Newton's first and third laws of motion were used extensively in statics to study bodies at rest and the forces acting upon them. These two laws are also used in dynamics; in fact, they are sufficient for the study of the motion of bodies which have no acceleration. However, when bodies are accelerated, i.e., when the magnitude or the direction of their velocity changes, it is necessary to use Newton's second law of motion to relate the motion of the body with the forces acting on it.

In this chapter we will discuss Newton's second law and apply it to the analysis of the motion of particles. As we state in Sec. 12.2, if the resultant of the forces acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force. Moreover, the ratio of the magnitudes of the resultant force and of the acceleration can be used to define the *mass* of the particle.

In Sec. 12.3, the *linear momentum* of a particle is defined as the product $\mathbf{L} = m\mathbf{v}$ of the mass m and velocity $\mathbf{v}$ of the particle, and it is demonstrated that Newton's second law can be expressed in an alternative form relating the rate of change of the linear momentum with the resultant of the forces acting on that particle.

Section 12.4 stresses the need for consistent units in the solution of dynamics problems and provides a review of the International System of Units (SI units) and the system of U.S. customary units.

In Secs. 12.5 and 12.6 and in the Sample Problems which follow, Newton's second law is applied to the solution of engineering problems, using either rectangular components or tangential and normal components of the forces and accelerations involved. We recall that an actual body—including bodies as large as a car, rocket, or airplane—can be considered as a particle for the purpose of analyzing its motion as long as the effect of a rotation of the body about its mass center can be ignored.

The second part of the chapter is devoted to the solution of problems in terms of radial and transverse components, with particular emphasis on the motion of a particle under a central force. In Sec. 12.7, the *angular momentum* $\mathbf{H}_O$ of a particle about a point O is defined as the moment about O of the linear momentum of the particle: $\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$. It then follows from Newton's second law that the rate of change of the angular momentum $\mathbf{H}_O$ of a particle is equal to the sum of the moments about O of the forces acting on that particle.

Section 12.9 deals with the motion of a particle under a *central force,* i.e., under a force directed toward or away from a fixed point O. Since such a force has zero moment about O, it follows that the angular momentum of the particle about O is conserved. This property greatly simplifies the analysis of the motion of a particle under a central force; in Sec. 12.10 it is applied to the solution of problems involving the orbital motion of bodies under gravitational attraction.

Sections 12.11 through 12.13 are optional. They present a more extensive discussion of orbital motion and contain a number of problems related to space mechanics.

12.2 NEWTON'S SECOND LAW OF MOTION

Newton's second law can be stated as follows:

If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.

Newton's second law of motion is best understood by imagining the following experiment: A particle is subjected to a force $\mathbf{F}_1$ of constant direction and constant magnitude F_1. Under the action of that force, the particle is observed to move in a straight line and *in the direction of the force* (Fig. 12.1*a*). By determining the position of the particle at various instants, we find that its acceleration has a constant magnitude a_1. If the experiment is repeated with forces $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , of different magnitude or direction (Fig. 12.1*b* and *c*), we find each time that the particle moves in the direction of the force acting on it and that the magnitudes a_1, a_2, a_3, . . . , of the accelerations are proportional to the magnitudes F_1, F_2, F_3, . . . , of the corresponding forces:

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_3}{a_3} = \cdots = \text{constant}$$

The constant value obtained for the ratio of the magnitudes of the forces and accelerations is a characteristic of the particle under consideration; it is called the *mass* of the particle and is denoted by m. When a particle of mass m is acted upon by a force $\mathbf{F}$, the force $\mathbf{F}$ and the acceleration $\mathbf{a}$ of the particle must therefore satisfy the relation

$$\mathbf{F} = m\mathbf{a} \tag{12.1}$$

This relation provides a complete formulation of Newton's second law; it expresses not only that the magnitudes of $\mathbf{F}$ and $\mathbf{a}$ are proportional but also (since m is a positive scalar) that the vectors $\mathbf{F}$ and $\mathbf{a}$ have the same direction (Fig. 12.2). We should note that Eq. (12.1) still holds when $\mathbf{F}$ is not constant but varies with time in magnitude or direction. The magnitudes of $\mathbf{F}$ and $\mathbf{a}$ remain proportional, and the two vectors have the same direction at any given instant. However, they will not, in general, be tangent to the path of the particle.

When a particle is subjected simultaneously to several forces, Eq. (12.1) should be replaced by

$$\Sigma\mathbf{F} = m\mathbf{a} \tag{12.2}$$

where $\Sigma\mathbf{F}$ represents the sum, or resultant, of all the forces acting on the particle.

It should be noted that the system of axes with respect to which the acceleration $\mathbf{a}$ is determined is not arbitrary. These axes must have a constant orientation with respect to the stars, and their origin must either be attached to the sun† or move with a constant velocity

†More accurately, to the mass center of the solar system.

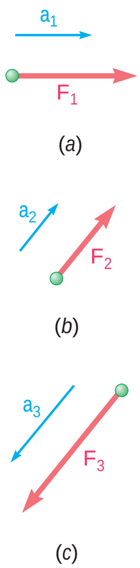


Fig. 12.1

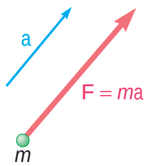


Fig. 12.2

Photo 12.1 When the racecar accelerates forward the rear tires have a friction force acting on them in the direction the car is moving.

with respect to the sun. Such a system of axes is called a *newtonian frame of reference*.† A system of axes attached to the earth does *not* constitute a newtonian frame of reference, since the earth rotates with respect to the stars and is accelerated with respect to the sun. However, in most engineering applications, the acceleration **a** can be determined with respect to axes attached to the earth and Eqs. (12.1) and (12.2) used without any appreciable error. On the other hand, these equations do not hold if **a** represents a relative acceleration measured with respect to moving axes, such as axes attached to an accelerated car or to a rotating piece of machinery.

We observe that if the resultant $\Sigma\mathbf{F}$ of the forces acting on the particle is zero, it follows from Eq. (12.2) that the acceleration **a** of the particle is also zero. If the particle is initially at rest ($\mathbf{v}_0 = 0$) with respect to the newtonian frame of reference used, it will thus remain at rest ($\mathbf{v} = 0$). If originally moving with a velocity $\mathbf{v}_0$, the particle will maintain a constant velocity $\mathbf{v} = \mathbf{v}_0$; that is, it will move with the constant speed v_0 in a straight line. This, we recall, is the statement of Newton's first law (Sec. 2.10). Thus, Newton's first law is a particular case of Newton's second law and can be omitted from the fundamental principles of mechanics.

12.3 LINEAR MOMENTUM OF A PARTICLE. RATE OF CHANGE OF LINEAR MOMENTUM

Replacing the acceleration **a** by the derivative $d\mathbf{v}/dt$ in Eq. (12.2), we write

$$\Sigma\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

or, since the mass m of the particle is constant,

$$\Sigma\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{12.3}$$

The vector $m\mathbf{v}$ is called the *linear momentum,* or simply the *momentum,* of the particle. It has the same direction as the velocity of the particle, and its magnitude is equal to the product of the mass m and the speed v of the particle (Fig. 12.3). Equation (12.3) expresses that *the resultant of the forces acting on the particle is equal to the rate of change of the linear momentum of the particle*. It is in this form that the second law of motion was originally stated by Newton. Denoting by **L** the linear momentum of the particle,

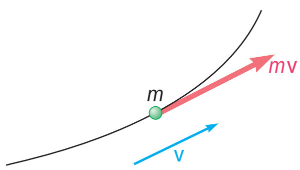


Fig. 12.3

$$\boxed{\mathbf{L} = m\mathbf{v}} \tag{12.4}$$

and by $\dot{\mathbf{L}}$ its derivative with respect to t, we can write Eq. (12.3) in the alternative form

$$\boxed{\Sigma\mathbf{F} = \dot{\mathbf{L}}} \tag{12.5}$$

†Since stars are not actually fixed, a more rigorous definition of a newtonian frame of reference (also called an *inertial system*) is *one with respect to which Eq. (12.2) holds*.

It should be noted that the mass m of the particle is assumed to be constant in Eqs. (12.3) to (12.5). Equation (12.3) or (12.5) should therefore not be used to solve problems involving the motion of bodies, such as rockets, which gain or lose mass. Problems of that type will be considered in Sec. 14.12.†

It follows from Eq. (12.3) that the rate of change of the linear momentum $m\mathbf{v}$ is zero when $\Sigma\mathbf{F} = 0$. Thus, *if the resultant force acting on a particle is zero, the linear momentum of the particle remains constant, in both magnitude and direction*. This is the principle of *conservation of linear momentum* for a particle, which can be recognized as an alternative statement of Newton's first law (Sec. 2.10).

12.4 SYSTEMS OF UNITS

In using the fundamental equation $\mathbf{F} = m\mathbf{a}$, the units of force, mass, length, and time cannot be chosen arbitrarily. If they are, the magnitude of the force $\mathbf{F}$ required to give an acceleration $\mathbf{a}$ to the mass m will *not* be numerically equal to the product ma; it will be only proportional to this product. Thus, we can choose three of the four units arbitrarily but must choose the fourth unit so that the equation $\mathbf{F} = m\mathbf{a}$ is satisfied. The units are then said to form a system of consistent kinetic units.

Two systems of consistent kinetic units are currently used by American engineers, the International System of Units (SI units‡) and the system of U.S. customary units. Both systems were discussed in detail in Sec. 1.3 and are described only briefly in this section.

International System of Units (SI Units).

In this system, the base units are the units of length, mass, and time, and are called, respectively, the *meter* (m), the *kilogram* (kg), and the *second* (s). All three are arbitrarily defined (Sec. 1.3). The unit of force is a derived unit. It is called the *newton* (N) and is defined as the force which gives an acceleration of 1 m/s^2 to a mass of 1 kg (Fig. 12.4). From Eq. (12.1) we write

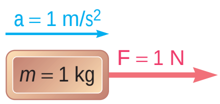


Fig. 12.4

$$1\ \text{N} = (1\ \text{kg})(1\ \text{m/s}^2) = 1\ \text{kg} \cdot \text{m/s}^2$$

The SI units are said to form an *absolute* system of units. This means that the three base units chosen are independent of the location where measurements are made. The meter, the kilogram, and the second may be used anywhere on the earth; they may even be used on another planet. They will always have the same significance.

The *weight* $\mathbf{W}$ of a body, or *force of gravity* exerted on that body, should, like any other force, be expressed in newtons. Since a body subjected to its own weight acquires an acceleration equal to the acceleration of gravity g, it follows from Newton's second law that the magnitude W of the weight of a body of mass m is

$$W = mg \tag{12.6}$$

†On the other hand, Eqs. (12.3) and (12.5) do hold in *relativistic mechanics*, where the mass m of the particle is assumed to vary with the speed of the particle.

‡SI stands for *Système International d'Unités* (French).

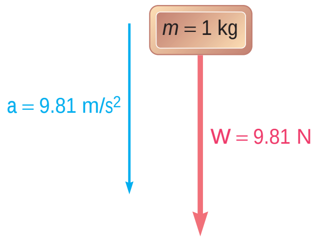


Fig. 12.5

Recalling that $g = 9.81\ \mathrm{m/s^2}$, we find that the weight of a body of mass 1 kg (Fig. 12.5) is

$$W = (1\ \mathrm{kg})(9.81\ \mathrm{m/s^2}) = 9.81\ \mathrm{N}$$

Multiples and submultiples of the units of length, mass, and force are frequently used in engineering practice. They are, respectively, the *kilometer* (km) and the *millimeter* (mm); the *megagram*† (Mg) and the *gram* (g); and the *kilonewton* (kN). By definition,

$$1\ \mathrm{km} = 1000\ \mathrm{m} \qquad 1\ \mathrm{mm} = 0.001\ \mathrm{m}$$
$$1\ \mathrm{Mg} = 1000\ \mathrm{kg} \qquad 1\ \mathrm{g} = 0.001\ \mathrm{kg}$$
$$1\ \mathrm{kN} = 1000\ \mathrm{N}$$

The conversion of these units to meters, kilograms, and newtons, respectively, can be effected simply by moving the decimal point three places to the right or to the left.

Units other than the units of mass, length, and time can all be expressed in terms of these three base units. For example, the unit of linear momentum can be obtained by recalling the definition of linear momentum and writing

$$mv = (\mathrm{kg})(\mathrm{m/s}) = \mathrm{kg \cdot m/s}$$

U.S. Customary Units. Most practicing American engineers still commonly use a system in which the base units are the units of length, force, and time. These units are, respectively, the *foot* (ft), the *pound* (lb), and the *second* (s). The second is the same as the corresponding SI unit. The foot is defined as 0.3048 m. The pound is defined as the *weight* of a platinum standard, called the *standard pound,* which is kept at the National Institute of Standards and Technology outside Washington and the mass of which is 0.453 592 43 kg. Since the weight of a body depends upon the gravitational attraction of the earth, which varies with location, it is specified that the standard pound should be placed at sea level and at a latitude of 45° to properly define a force of 1 lb. Clearly, the U.S. customary units do not form an absolute system of units. Because of their dependence upon the gravitational attraction of the earth, they are said to form a *gravitational* system of units.

While the standard pound also serves as the unit of mass in commercial transactions in the United States, it cannot be so used in engineering computations since such a unit would not be consistent with the base units defined in the preceding paragraph. Indeed, when acted upon by a force of 1 lb, that is, when subjected to its own weight, the standard pound receives the acceleration of gravity, $g = 32.2\ \mathrm{ft/s^2}$ (Fig. 12.6), and not the unit acceleration required by Eq. (12.1). The unit of mass consistent with the foot, the pound, and the second is the mass which receives an acceleration of $1\ \mathrm{ft/s^2}$ when a force of 1 lb is applied to it (Fig. 12.7). This unit, sometimes called a *slug*, can be derived from the equation $F = ma$ after substituting 1 lb and $1\ \mathrm{ft/s^2}$ for F and a, respectively. We write

$$F = ma \qquad 1\ \mathrm{lb} = (1\ \mathrm{slug})(1\ \mathrm{ft/s^2})$$

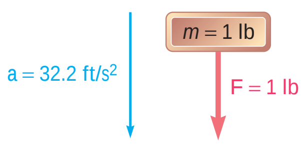


Fig. 12.6

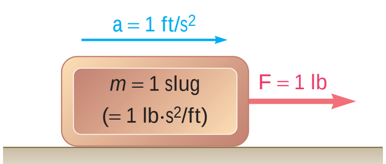


Fig. 12.7

†Also known as a *metric ton*.

and obtain

$$1 \text{ slug} = \frac{1 \text{ lb}}{1 \text{ ft/s}^2} = 1 \text{ lb} \cdot \text{s}^2/\text{ft}$$

Comparing Figs. 12.6 and 12.7, we conclude that the slug is a mass 32.2 times larger than the mass of the standard pound.

The fact that bodies are characterized in the U.S. customary system of units by their weight in pounds rather than by their mass in slugs was a convenience in the study of statics, where we were dealing for the most part with weights and other forces and only seldom with masses. However, in the study of kinetics, which involves forces, masses, and accelerations, it will be repeatedly necessary to express in slugs the mass m of a body, the weight W of which has been given in pounds. Recalling Eq. (12.6), we will write

$$m = \frac{W}{g} \tag{12.7}$$

where g is the acceleration of gravity ($g = 32.2 \text{ ft/s}^2$).

Units other than the units of force, length, and time can all be expressed in terms of these three base units. For example, the unit of linear momentum can be obtained by using the definition of linear momentum to write

$$mv = (\text{lb} \cdot \text{s}^2/\text{ft})(\text{ft/s}) = \text{lb} \cdot \text{s}$$

Conversion from One System of Units to Another. The conversion from U.S. customary units to SI units, and vice versa, was discussed in Sec. 1.4. You will recall that the conversion factors obtained for the units of length, force, and mass are, respectively,

Length: $1 \text{ ft} = 0.3048 \text{ m}$
Force: $1 \text{ lb} = 4.448 \text{ N}$
Mass: $1 \text{ slug} = 1 \text{ lb} \cdot \text{s}^2/\text{ft} = 14.59 \text{ kg}$

Although it cannot be used as a consistent unit of mass, the mass of the standard pound is, by definition,

$$1 \text{ pound-mass} = 0.4536 \text{ kg}$$

This constant can be used to determine the *mass* in SI units (kilograms) of a body which has been characterized by its *weight* in U.S. customary units (pounds).

12.5 EQUATIONS OF MOTION

Consider a particle of mass m acted upon by several forces. We recall from Sec. 12.2 that Newton's second law can be expressed by the equation

$$\Sigma \mathbf{F} = m\mathbf{a} \tag{12.2}$$

which relates the forces acting on the particle and the vector $m\mathbf{a}$ (Fig. 12.8). In order to solve problems involving the motion of a particle, however, it will be found more convenient to replace Eq. (12.2) by equivalent equations involving scalar quantities.

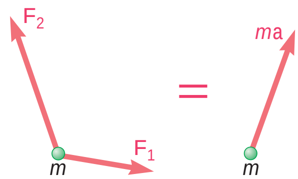


Fig. 12.8

Photo 12.2 The pilot of a fighter aircraft will experience very large normal forces when taking a sharp turn.

Rectangular Components. Resolving each force **F** and the acceleration **a** into rectangular components, we write

$$\Sigma(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) = m(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})$$

from which it follows that

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z \tag{12.8}$$

Recalling from Sec. 11.11 that the components of the acceleration are equal to the second derivatives of the coordinates of the particle, we have

$$\Sigma F_x = m\ddot{x} \qquad \Sigma F_y = m\ddot{y} \qquad \Sigma F_z = m\ddot{z} \tag{12.8$'$}$$

Consider, as an example, the motion of a projectile. If the resistance of the air is neglected, the only force acting on the projectile after it has been fired is its weight $\mathbf{W} = -W\mathbf{j}$. The equations defining the motion of the projectile are therefore

$$m\ddot{x} = 0 \qquad m\ddot{y} = -W \qquad m\ddot{z} = 0$$

and the components of the acceleration of the projectile are

$$\ddot{x} = 0 \qquad \ddot{y} = -\frac{W}{m} = -g \qquad \ddot{z} = 0$$

where g is 9.81 m/s^2 or 32.2 ft/s^2. The equations obtained can be integrated independently, as shown in Sec. 11.11, to obtain the velocity and displacement of the projectile at any instant.

When a problem involves two or more bodies, equations of motion should be written for each of the bodies (see Sample Probs. 12.3 and 12.4). You will recall from Sec. 12.2 that all accelerations should be measured with respect to a newtonian frame of reference. In most engineering applications, accelerations can be determined with respect to axes attached to the earth, but relative accelerations measured with respect to moving axes, such as axes attached to an accelerated body, cannot be substituted for **a** in the equations of motion.

Tangential and Normal Components. Resolving the forces and the acceleration of the particle into components along the tangent to the path (in the direction of motion) and the normal (toward the inside of the path) (Fig. 12.9), and substituting into Eq. (12.2), we obtain the two scalar equations

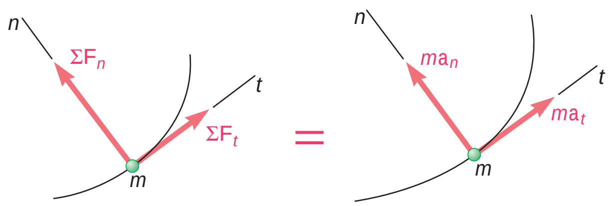


Fig. 12.9

$$\Sigma F_t = ma_t \qquad \Sigma F_n = ma_n \tag{12.9}$$

Substituting for a_t and a_n from Eqs. (11.40), we have

$$\Sigma F_t = m\frac{dv}{dt} \qquad \Sigma F_n = m\frac{v^2}{\rho} \tag{12.9$'$}$$

The equations obtained may be solved for two unknowns.

12.6 DYNAMIC EQUILIBRIUM

Returning to Eq. (12.2) and transposing the right-hand member, we write Newton's second law in the alternative form

$$\Sigma \mathbf{F} - m\mathbf{a} = 0 \tag{12.10}$$

which expresses that if we add the vector $-m\mathbf{a}$ to the forces acting on the particle, *we obtain a system of vectors equivalent to zero* (Fig. 12.10). The vector $-m\mathbf{a}$, of magnitude ma and of *direction opposite* to that of the acceleration, is called an *inertia vector*. The particle may thus be considered to be in equilibrium under the given forces and the inertia vector. The particle is said to be in *dynamic equilibrium*, and the problem under consideration can be solved by the methods developed earlier in statics.

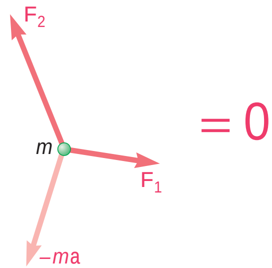


Fig. 12.10

In the case of coplanar forces, all the vectors shown in Fig. 12.10, *including the inertia vector*, can be drawn tip-to-tail to form a closed-vector polygon. Or the sums of the components of all the vectors in Fig. 12.10, again including the inertia vector, can be equated to zero. Using rectangular components, we therefore write

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \textbf{\textit{including inertia vector}} \tag{12.11}$$

When tangential and normal components are used, it is more convenient to represent the inertia vector by its two components $-m\mathbf{a}_t$ and $-m\mathbf{a}_n$ in the sketch itself (Fig. 12.11). The tangential component of the inertia vector provides a measure of the resistance the particle offers to a change in speed, while its normal component (also called *centrifugal force*) represents the tendency of the particle to leave its curved path. We should note that either of these two components may be zero under special conditions: (1) If the particle starts from rest, its initial velocity is zero and the normal component of the inertia vector is zero at $t = 0$; (2) if the particle moves at constant speed along its path, the tangential component of the inertia vector is zero and only its normal component needs to be considered.

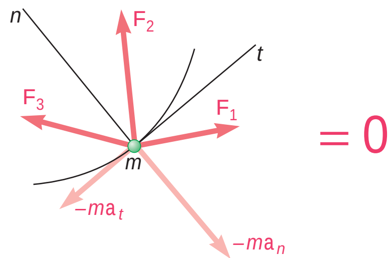
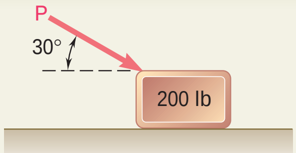


Fig. 12.11

Because they measure the resistance that particles offer when we try to set them in motion or when we try to change the conditions of their motion, inertia vectors are often called *inertia forces*. The inertia forces, however, are not forces like the forces found in statics, which are either contact forces or gravitational forces (weights). Many people, therefore, object to the use of the word "force" when referring to the vector $-m\mathbf{a}$ or even avoid altogether the concept of dynamic equilibrium. Others point out that inertia forces and actual forces, such as gravitational forces, affect our senses in the same way and cannot be distinguished by physical measurements. A man riding in an elevator which is accelerated upward will have the feeling that his weight has suddenly increased; and no measurement made within the elevator could establish whether the elevator is truly accelerated or whether the force of attraction exerted by the earth has suddenly increased.

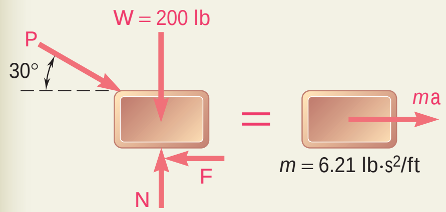
Sample problems have been solved in this text by the direct application of Newton's second law, as illustrated in Figs. 12.8 and 12.9, rather than by the method of dynamic equilibrium.

Photo 12.3 The angle each rider is with respect to the horizontal will depend on the weight of the rider and the speed of rotation.



SAMPLE PROBLEM 12.1

A 200-lb block rests on a horizontal plane. Find the magnitude of the force **P** required to give the block an acceleration of 10 ft/s^2 to the right. The coefficient of kinetic friction between the block and the plane is $\mu_k = 0.25$.



SOLUTION

The mass of the block is

$$m = \frac{W}{g} = \frac{200 \text{ lb}}{32.2 \text{ ft/s}^2} = 6.21 \text{ lb} \cdot \text{s}^2/\text{ft}$$

We note that $F = \mu_k N = 0.25N$ and that $a = 10$ ft/s^2. Expressing that the forces acting on the block are equivalent to the vector $m\mathbf{a}$, we write

$$\xrightarrow{+}\Sigma F_x = ma: \qquad P\cos 30° - 0.25N = (6.21 \text{ lb} \cdot \text{s}^2/\text{ft})(10 \text{ ft/s}^2)$$

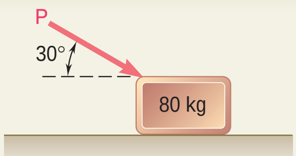
$$P\cos 30° - 0.25N = 62.1 \text{ lb} \qquad (1)$$

$$+\uparrow\Sigma F_y = 0: \qquad N - P\sin 30° - 200 \text{ lb} = 0 \qquad (2)$$

Solving (2) for N and substituting the result into (1), we obtain

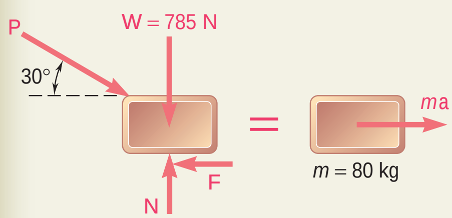
$$N = P\sin 30° + 200 \text{ lb}$$

$$P\cos 30° - 0.25(P\sin 30° + 200 \text{ lb}) = 62.1 \text{ lb} \qquad P = 151 \text{ lb} \blacktriangleleft$$



SAMPLE PROBLEM 12.2

An 80-kg block rests on a horizontal plane. Find the magnitude of the force **P** required to give the block an acceleration of 2.5 m/s^2 to the right. The coefficient of kinetic friction between the block and the plane is $\mu_k = 0.25$.



SOLUTION

The weight of the block is

$$W = mg = (80 \text{ kg})(9.81 \text{ m/s}^2) = 785 \text{ N}$$

We note that $F = \mu_k N = 0.25N$ and that $a = 2.5$ m/s^2. Expressing that the forces acting on the block are equivalent to the vector $m\mathbf{a}$, we write

$$\xrightarrow{+}\Sigma F_x = ma: \qquad P\cos 30° - 0.25N = (80 \text{ kg})(2.5 \text{ m/s}^2)$$

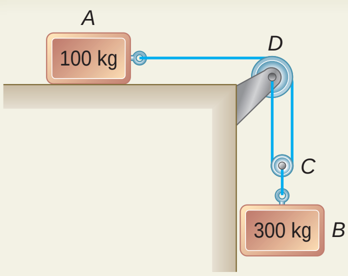
$$P\cos 30° - 0.25N = 200 \text{ N} \qquad (1)$$

$$+\uparrow\Sigma F_y = 0: \qquad N - P\sin 30° - 785 \text{ N} = 0 \qquad (2)$$

Solving (2) for N and substituting the result into (1), we obtain

$$N = P\sin 30° + 785 \text{ N}$$

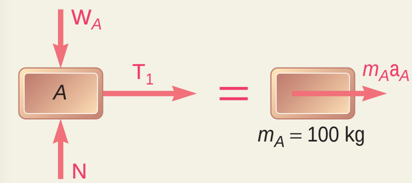
$$P\cos 30° - 0.25(P\sin 30° + 785 \text{ N}) = 200 \text{ N} \qquad P = 535 \text{ N} \blacktriangleleft$$

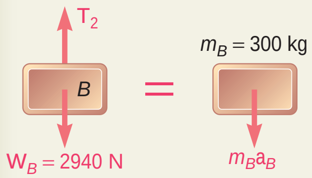


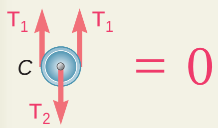
SAMPLE PROBLEM 12.3

The two blocks shown start from rest. The horizontal plane and the pulley are frictionless, and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in each cord.

SOLUTION







Kinematics. We note that if block A moves through x_A to the right, block B moves down through

$$x_B = \tfrac{1}{2}x_A$$

Differentiating twice with respect to t, we have

$$a_B = \tfrac{1}{2}a_A \qquad (1)$$

Kinetics. We apply Newton's second law successively to block A, block B, and pulley C.

Block A. Denoting by T_1 the tension in cord ACD, we write

$$\overset{+}{\rightarrow}\Sigma F_x = m_A a_A: \qquad T_1 = 100a_A \qquad (2)$$

Block B. Observing that the weight of block B is

$$W_B = m_B g = (300 \text{ kg})(9.81 \text{ m/s}^2) = 2940 \text{ N}$$

and denoting by T_2 the tension in cord BC, we write

$$+\downarrow\Sigma F_y = m_B a_B: \qquad 2940 - T_2 = 300a_B$$

or, substituting for a_B from (1),

$$2940 - T_2 = 300(\tfrac{1}{2}a_A)$$
$$T_2 = 2940 - 150a_A \qquad (3)$$

Pulley C. Since m_C is assumed to be zero, we have

$$+\downarrow\Sigma F_y = m_C a_C = 0: \qquad T_2 - 2T_1 = 0 \qquad (4)$$

Substituting for T_1 and T_2 from (2) and (3), respectively, into (4) we write

$$2940 - 150a_A - 2(100a_A) = 0$$
$$2940 - 350a_A = 0 \qquad a_A = 8.40 \text{ m/s}^2 \quad ◀$$

Substituting the value obtained for a_A into (1) and (2), we have

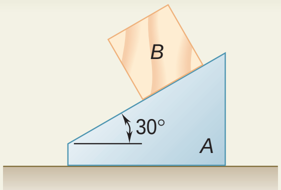
$$a_B = \tfrac{1}{2}a_A = \tfrac{1}{2}(8.40 \text{ m/s}^2) \qquad a_B = 4.20 \text{ m/s}^2 \quad ◀$$
$$T_1 = 100a_A = (100 \text{ kg})(8.40 \text{ m/s}^2) \qquad T_1 = 840 \text{ N} \quad ◀$$

Recalling (4), we write

$$T_2 = 2T_1 \qquad T_2 = 2(840 \text{ N}) \qquad T_2 = 1680 \text{ N} \quad ◀$$

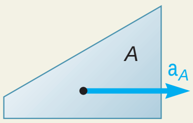
We note that the value obtained for T_2 is *not* equal to the weight of block B.

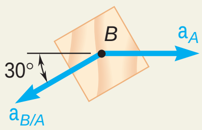
SAMPLE PROBLEM 12.4



The 12-lb block B starts from rest and slides on the 30-lb wedge A, which is supported by a horizontal surface. Neglecting friction, determine (*a*) the acceleration of the wedge, (*b*) the acceleration of the block relative to the wedge.

SOLUTION





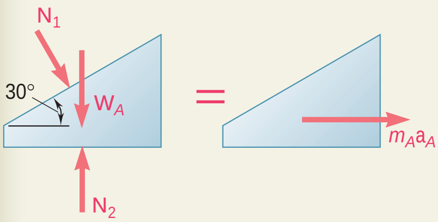
Kinematics. We first examine the acceleration of the wedge and the acceleration of the block.

Wedge A. Since the wedge is constrained to move on the horizontal surface, its acceleration $\mathbf{a}_A$ is horizontal. We will assume that it is directed to the right.

Block B. The acceleration $\mathbf{a}_B$ of block B can be expressed as the sum of the acceleration of A and the acceleration of B relative to A. We have

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

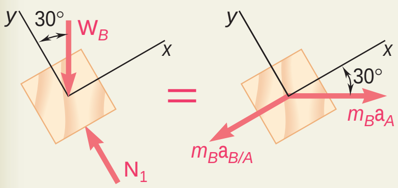
where $\mathbf{a}_{B/A}$ is directed along the inclined surface of the wedge.



Kinetics. We draw the free-body diagrams of the wedge and of the block and apply Newton's second law.

Wedge A. We denote the forces exerted by the block and the horizontal surface on wedge A by $\mathbf{N}_1$ and $\mathbf{N}_2$, respectively.

$$\xrightarrow{+} \Sigma F_x = m_A a_A: \qquad N_1 \sin 30° = m_A a_A$$
$$0.5 N_1 = (W_A/g) a_A \qquad (1)$$



Block B. Using the coordinate axes shown and resolving $\mathbf{a}_B$ into its components $\mathbf{a}_A$ and $\mathbf{a}_{B/A}$, we write

$$+\nearrow \Sigma F_x = m_B a_x: \qquad -W_B \sin 30° = m_B a_A \cos 30° - m_B a_{B/A}$$
$$-W_B \sin 30° = (W_B/g)(a_A \cos 30° - a_{B/A})$$
$$a_{B/A} = a_A \cos 30° + g \sin 30° \qquad (2)$$

$$+\nwarrow \Sigma F_y = m_B a_y: \qquad N_1 - W_B \cos 30° = -m_B a_A \sin 30°$$
$$N_1 - W_B \cos 30° = -(W_B/g) a_A \sin 30° \qquad (3)$$

a. Acceleration of Wedge *A*. Substituting for N_1 from Eq. (1) into Eq. (3), we have

$$2(W_A/g) a_A - W_B \cos 30° = -(W_B/g) a_A \sin 30°$$

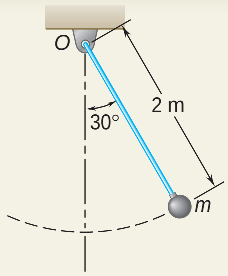
Solving for a_A and substituting the numerical data, we write

$$a_A = \frac{W_B \cos 30°}{2W_A + W_B \sin 30°} g = \frac{(12 \text{ lb}) \cos 30°}{2(30 \text{ lb}) + (12 \text{ lb}) \sin 30°} (32.2 \text{ ft/s}^2)$$
$$a_A = +5.07 \text{ ft/s}^2 \qquad \mathbf{a}_A = 5.07 \text{ ft/s}^2 \rightarrow \blacktriangleleft$$

b. Acceleration of Block *B* Relative to *A*. Substituting the value obtained for a_A into Eq. (2), we have

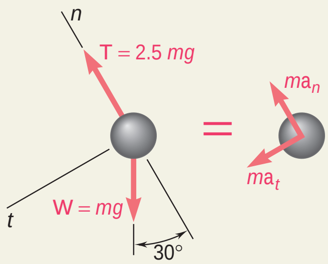
$$a_{B/A} = (5.07 \text{ ft/s}^2) \cos 30° + (32.2 \text{ ft/s}^2) \sin 30°$$
$$a_{B/A} = +20.5 \text{ ft/s}^2 \qquad \mathbf{a}_{B/A} = 20.5 \text{ ft/s}^2 \swarrow 30° \blacktriangleleft$$

SAMPLE PROBLEM 12.5



The bob of a 2-m pendulum describes an arc of circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, find the velocity and the acceleration of the bob in that position.

SOLUTION



The weight of the bob is $W = mg$; the tension in the cord is thus 2.5 mg. Recalling that $\mathbf{a}_n$ is directed toward O and assuming $\mathbf{a}_t$ as shown, we apply Newton's second law and obtain

$+\swarrow \Sigma F_t = ma_t$: $\qquad mg \sin 30° = ma_t$

$a_t = g \sin 30° = +4.90 \text{ m/s}^2 \qquad \mathbf{a}_t = 4.90 \text{ m/s}^2 \swarrow$ ◀

$+\nwarrow \Sigma F_n = ma_n$: $\qquad 2.5\, mg - mg \cos 30° = ma_n$

$a_n = 1.634\, g = +16.03 \text{ m/s}^2 \qquad \mathbf{a}_n = 16.03 \text{ m/s}^2 \nwarrow$ ◀

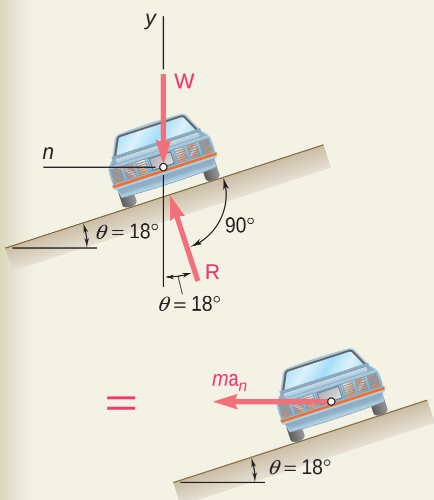
Since $a_n = v^2/\rho$, we have $v^2 = \rho a_n = (2 \text{ m})(16.03 \text{ m/s}^2)$

$v = \pm 5.66 \text{ m/s} \qquad \mathbf{v} = 5.66 \text{ m/s}$ (up or down) ◀

SAMPLE PROBLEM 12.6

Determine the rated speed of a highway curve of radius $\rho = 400$ ft banked through an angle $\theta = 18°$. The *rated speed* of a banked highway curve is the speed at which a car should travel if no lateral friction force is to be exerted on its wheels.

SOLUTION



The car travels in a *horizontal* circular path of radius ρ. The normal component $\mathbf{a}_n$ of the acceleration is directed toward the center of the path; its magnitude is $a_n = v^2/\rho$, where v is the speed of the car in ft/s. The mass m of the car is W/g, where W is the weight of the car. Since no lateral friction force is to be exerted on the car, the reaction $\mathbf{R}$ of the road is shown perpendicular to the roadway. Applying Newton's second law, we write

$+\uparrow \Sigma F_y = 0$: $\qquad R \cos \theta - W = 0 \qquad R = \dfrac{W}{\cos \theta}$ (1)

$\xleftarrow{+} \Sigma F_n = ma_n$: $\qquad R \sin \theta = \dfrac{W}{g} a_n$ (2)

Substituting for R from (1) into (2), and recalling that $a_n = v^2/\rho$,

$$\frac{W}{\cos \theta} \sin \theta = \frac{W}{g}\frac{v^2}{\rho} \qquad v^2 = g\rho \tan \theta$$

Substituting $\rho = 400$ ft and $\theta = 18°$ into this equation, we obtain

$$v^2 = (32.2 \text{ ft/s}^2)(400 \text{ ft}) \tan 18°$$

$v = 64.7 \text{ ft/s} \qquad v = 44.1 \text{ mi/h}$ ◀

SOLVING PROBLEMS ON YOUR OWN

In the problems for this lesson, you will apply *Newton's second law of motion,* $\Sigma\mathbf{F} = m\mathbf{a}$, to relate the forces acting on a particle to the motion of the particle.

1. Writing the equations of motion. When applying Newton's second law to the types of motion discussed in this lesson, you will find it most convenient to express the vectors **F** and **a** in terms of either their rectangular components or their tangential and normal components.

a. When using rectangular components, and recalling from Sec. 11.11 the expressions found for a_x, a_y, and a_z, you will write

$$\Sigma F_x = m\ddot{x} \qquad \Sigma F_y = m\ddot{y} \qquad \Sigma F_z = m\ddot{z}$$

b. When using tangential and normal components, and recalling from Sec. 11.13 the expressions found for a_t and a_n, you will write

$$\Sigma F_t = m\frac{dv}{dt} \qquad \Sigma F_n = m\frac{v^2}{\rho}$$

2. Drawing a free-body diagram showing the applied forces *and an equivalent diagram* showing the vector $m\mathbf{a}$ or its components will provide you with a pictorial representation of Newton's second law [Sample Probs. 12.1 through 12.6]. These diagrams will be of great help to you when writing the equations of motion. Note that when a problem involves two or more bodies, it is usually best to consider each body separately.

3. Applying Newton's second law. As we observed in Sec. 12.2, the acceleration used in the equation $\Sigma\mathbf{F} = m\mathbf{a}$ should always be the *absolute acceleration* of the particle (that is, it should be measured with respect to a newtonian frame of reference). Also, *if the sense of the acceleration* **a** *is unknown* or is not easily deduced, assume an arbitrary sense for **a** (usually the positive direction of a coordinate axis) and then let the solution provide the correct sense. Finally, note how the solutions of Sample Probs. 12.3 and 12.4 were divided into a *kinematics* portion and a *kinetics* portion, and how in Sample Prob. 12.4 we used two systems of coordinate axes to simplify the equations of motion.

4. When a problem involves dry friction, be sure to review the relevant sections of *Statics* [Secs. 8.1 to 8.3] before attempting to solve that problem. In particular, you should know when each of the equations $F = \mu_s N$ and $F = \mu_k N$ may be used.

You should also recognize that if the motion of a system is not specified, it is necessary first to assume a possible motion and then to check the validity of that assumption.

5. Solving problems involving relative motion. When a body B moves with respect to a body A, as in Sample Prob. 12.4, it is often convenient to express the acceleration of B as

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

where $\mathbf{a}_{B/A}$ is the acceleration of B relative to A, that is, the acceleration of B as observed from a frame of reference attached to A and in translation. If B is observed to move in a straight line, $\mathbf{a}_{B/A}$ will be directed along that line. On the other hand, if B is observed to move along a circular path, the relative acceleration $\mathbf{a}_{B/A}$ should be resolved into components tangential and normal to that path.

6. Finally, always consider the implications of any assumption you make. Thus, in a problem involving two cords, if you assume that the tension in one of the cords is equal to its maximum allowable value, check whether any requirements set for the other cord will then be satisfied. For instance, will the tension T in that cord satisfy the relation $0 \leq T \leq T_{\text{max}}$? That is, will the cord remain taut and will its tension be less than its maximum allowable value?

PROBLEMS

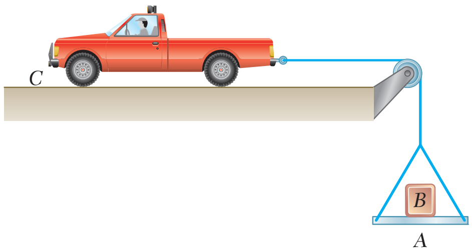


Fig. P12.CQ1

CONCEPT QUESTIONS

12.CQ1 A 1000-lb boulder *B* is resting on a 200-lb platform *A* when truck *C* accelerates to the left with a constant acceleration. Which of the following statements are true (more than one may be true)?

a. The tension in the cord connected to the truck is 200 lb.
b. The tension in the cord connected to the truck is 1200 lb.
c. The tension in the cord connected to the truck is greater than 1200 lb.
d. The normal force between *A* and *B* is 1000 lb.
e. The normal force between *A* and *B* is 1200 lb.
f. None of the above are true.

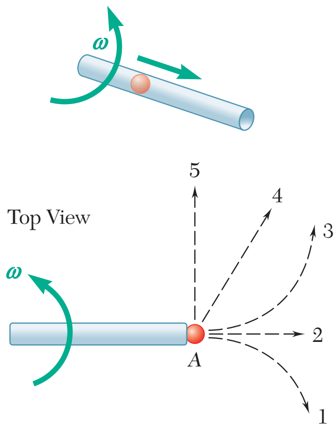


Fig. P12.CQ2

12.CQ2 Marble *A* is placed in a hollow tube, and the tube is swung in a horizontal plane causing the marble to be thrown out. As viewed from the top, which of the following choices best describes the path of the marble after leaving the tube?

a. 1 **b.** 2 **c.** 3 **d.** 4 **e.** 5

12.CQ3 The two systems shown start from rest. On the left, two 40-lb weights are connected by an inextensible cord, and on the right, a constant 40-lb force pulls on the cord. Neglecting all frictional forces, which of the following statements is true?

a. Blocks *A* and *C* will have the same acceleration.
b. Block *C* will have a larger acceleration than block *A*.
c. Block *A* will have a larger acceleration than block *C*.
d. Block *A* will not move.
e. None of the above are true.

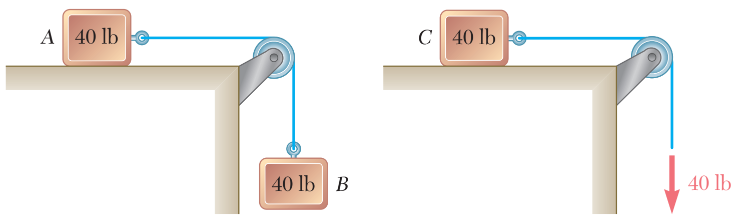


Fig. P12.CQ3

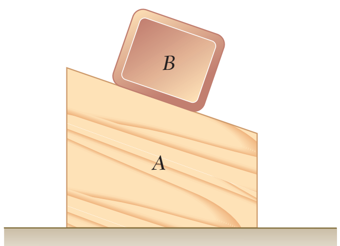


Fig. P12.CQ4

12.CQ4 Blocks *A* and *B* are released from rest in the position shown. Neglecting friction, the normal force between block *A* and the ground is:

a. Less than the weight of *A* plus the weight of *B*.
b. Equal to the weight of *A* plus the weight of *B*.
c. Greater than the weight of *A* plus the weight of *B*.

12.CQ5 People sit on a Ferris wheel at points A, B, C, and D. The Ferris wheel travels at a constant angular velocity. At the instant shown, which person experiences the largest force from his or her chair (back and seat)? Assume you can neglect the size of the chairs—that is, the people are located the same distance from the axis of rotation.

a. A
b. B
c. C
d. D
e. The force is the same for all the passengers.

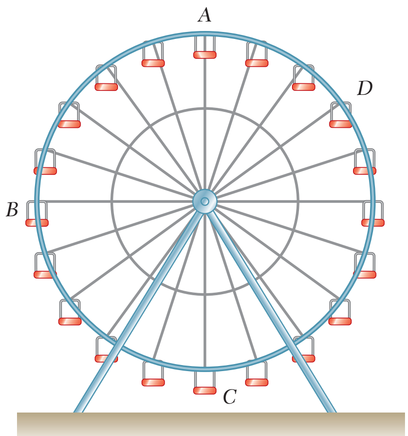


Fig. P12.CQ5

FREE BODY PRACTICE PROBLEMS

12.F1 Crate A is gently placed with zero initial velocity onto a moving conveyor belt. The coefficient of kinetic friction between the crate and the belt is μ_k. Draw the FBD and KD for A immediately after it contacts the belt.

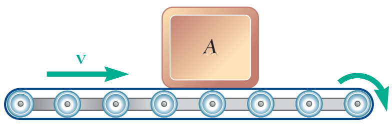


Fig. P12.F1

12.F2 Two blocks weighing W_A and W_B are at rest on a conveyor that is initially at rest. The belt is suddenly started in an upward direction so that slipping occurs between the belt and the boxes. Assuming the coefficient of friction between the boxes and the belt is μ_k, draw the FBDs and KDs for blocks A and B. How would you determine if A and B remain in contact?

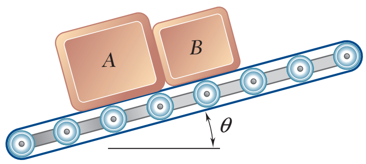


Fig. P12.F2

12.F3 Objects A, B, and C have masses m_A, m_B, and m_C, respectively. The coefficient of kinetic friction between A and B is μ_k, and the friction between A and the ground is negligible and the pulleys are massless and frictionless. Assuming B slides on A, draw the FBD and KD for each of the three masses A, B, and C.

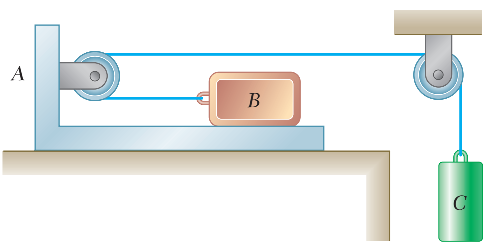


Fig. P12.F3

12.F4 Blocks A and B have masses m_A and m_B, respectively. Neglecting friction between all surfaces, draw the FBD and KD for each mass.

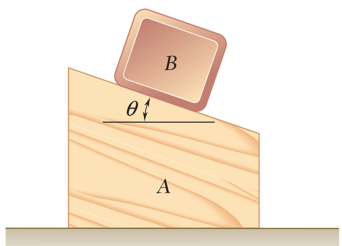


Fig. P12.F4

12.F5 Blocks A and B have masses m_A and m_B, respectively. Neglecting friction between all surfaces, draw the FBD and KD for the two systems shown.

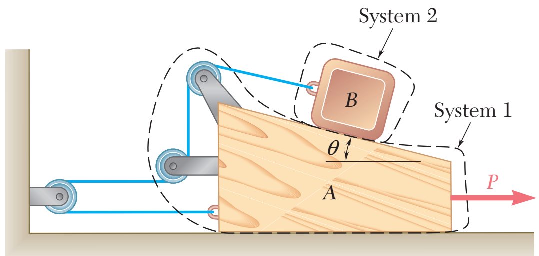


Fig. P12.F5

12.F6 A pilot of mass m flies a jet in a half-vertical loop of radius R so that the speed of the jet, v, remains constant. Draw a FBD and KD of the pilot at points A, B, and C.

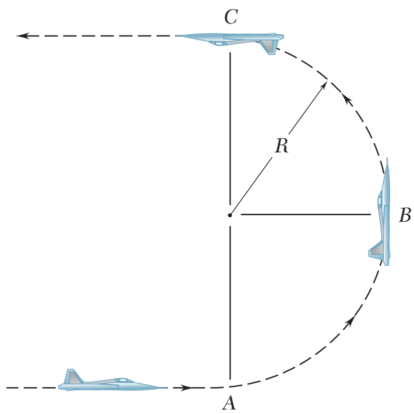


Fig. P12.F6

12.F7 Wires AC and BC are attached to a sphere which revolves at a constant speed v in the horizontal circle of radius r as shown. Draw a FBD and KD of C.

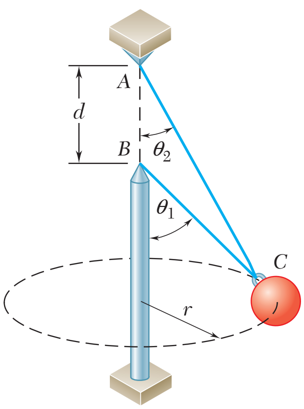


Fig. P12.F7

12.F8 A collar of mass m is attached to a spring and slides without friction along a circular rod in a vertical plane. The spring has an undeformed length of 5 in. and a constant k. Knowing that the collar has a speed v at point B, draw the FBD and KD of the collar at this point.

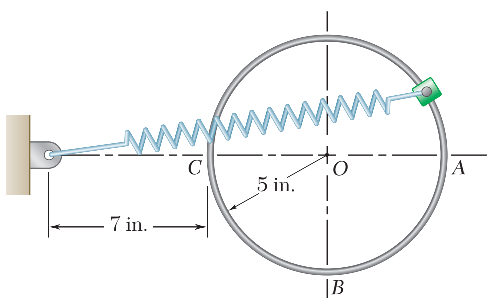


Fig. P12.F8

END-OF-SECTION PROBLEMS

12.1 Astronauts who landed on the moon during the Apollo 15, 16, and 17 missions brought back a large collection of rocks to the earth. Knowing the rocks weighed 139 lb when they were on the moon, determine (*a*) the weight of the rocks on the earth, (*b*) the mass of the rocks in slugs. The acceleration due to gravity on the moon is 5.30 ft/s^2.

12.2 The value of g at any latitude ϕ may be obtained from the formula

$$g = 32.09(1 + 0.0053 \sin^2 \phi)\text{ft/s}^2$$

which takes into account the effect of the rotation of the earth, as well as the fact that the earth is not truly spherical. Determine to four significant figures (*a*) the weight in pounds, (*b*) the mass in pounds, (*c*) the mass in lb · s^2/ft, at the latitudes of 0°, 45°, 60°, of a silver bar, the mass of which has been officially designated as 5 lb.

12.3 A 400-kg satellite has been placed in a circular orbit 1500 km above the surface of the earth. The acceleration of gravity at this elevation is 6.43 m/s^2. Determine the linear momentum of the satellite, knowing that its orbital speed is 25.6×10^3 km/h.

12.4 A spring scale A and a lever scale B having equal lever arms are fastened to the roof of an elevator, and identical packages are attached to the scales as shown. Knowing that when the elevator moves downward with an acceleration of 1 m/s^2 the spring scale indicates a load of 60 N, determine (*a*) the weight of the packages, (*b*) the load indicated by the spring scale and the mass needed to balance the lever scale when the elevator moves upward with an acceleration of 1 m/s^2.

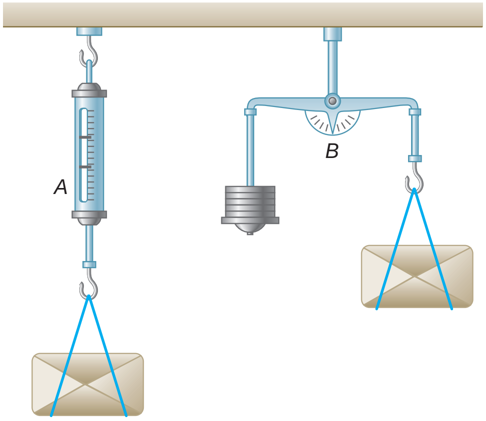


Fig. *P12.4*

12.5 In anticipation of a long 7° upgrade, a bus driver accelerates at a constant rate of 3 ft/s^2 while still on a level section of the highway. Knowing that the speed of the bus is 60 mi/h as it begins to climb the grade and that the driver does not change the setting of his throttle or shift gears, determine the distance traveled by the bus up the grade when its speed has decreased to 50 mi/h.

12.6 A hockey player hits a puck so that it comes to rest in 10 s after sliding 100 ft on the ice. Determine (*a*) the initial velocity of the puck, (*b*) the coefficient of friction between the puck and the ice.

12.7 The acceleration of a package sliding at point A is 3 m/s^2. Assuming that the coefficient of kinetic friction is the same for each section, determine the acceleration of the package at point B.

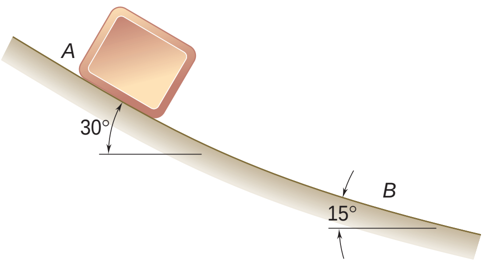


Fig. P12.7

12.8 Determine the maximum theoretical speed that may be achieved over a distance of 60 m by a car starting from rest, knowing that the coefficient of static friction is 0.80 between the tires and the pavement and that 60 percent of the weight of the car is distributed over its front wheels and 40 percent over its rear wheels. Assume (*a*) four-wheel drive, (*b*) front-wheel drive, (*c*) rear-wheel drive.

12.9 If an automobile's braking distance from 90 km/h is 45 m on level pavement, determine the automobile's braking distance from 90 km/h when it is (*a*) going up a 5° incline, (*b*) going down a 3-percent incline. Assume the braking force is independent of grade.

12.10 A mother and her child are skiing together, and the mother is holding the end of a rope tied to the child's waist. They are moving at a speed of 7.2 km/h on a gently sloping portion of the ski slope when the mother observes that they are approaching a steep descent. She pulls on the rope with an average force of 7 N. Knowing the coefficient of friction between the child and the ground is 0.1 and the angle of the rope does not change, determine (*a*) the time required for the child's speed to be cut in half, (*b*) the distance traveled in this time.

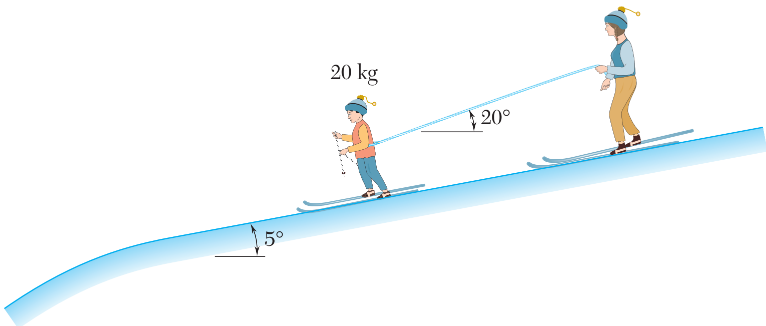


Fig. P12.10

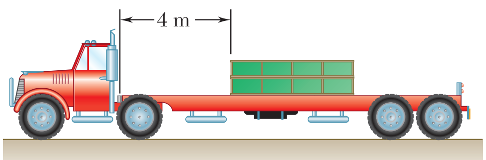


Fig. P12.11

12.11 The coefficients of friction between the load and the flatbed trailer shown are $\mu_s = 0.40$ and $\mu_k = 0.30$. Knowing that the speed of the rig is 72 km/h, determine the shortest distance in which the rig can be brought to a stop if the load is not to shift.

12.12 A light train made up of two cars is traveling at 90 km/h when the brakes are applied to both cars. Knowing that car *A* has a mass of 25 Mg and car *B* a mass of 20 Mg, and that the braking force is 30 kN on each car, determine (*a*) the distance traveled by the train before it comes to a stop, (*b*) the force in the coupling between the cars while the train is slowing down.

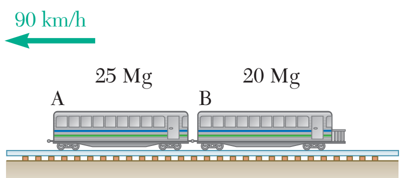


Fig. P12.12

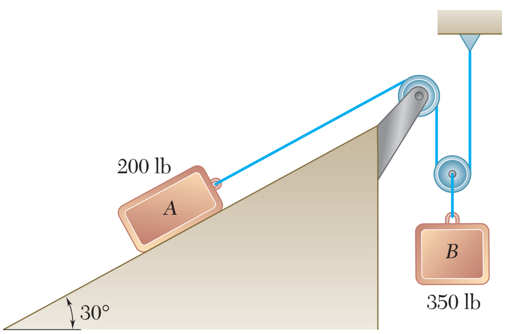


Fig. *P12.13*

12.13 The two blocks shown are originally at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys and between block *A* and the incline, determine (*a*) the acceleration of each block, (*b*) the tension in the cable.

12.14 Solve Prob. 12.13, assuming that the coefficients of friction between block A and the incline are $\mu_s = 0.25$ and $\mu_k = 0.20$.

12.15 Each of the systems shown is initially at rest. Neglecting axle friction and the masses of the pulleys, determine for each system (*a*) the acceleration of block A, (*b*) the velocity of block A after it has moved through 10 ft, (*c*) the time required for block A to reach a velocity of 20 ft/s.

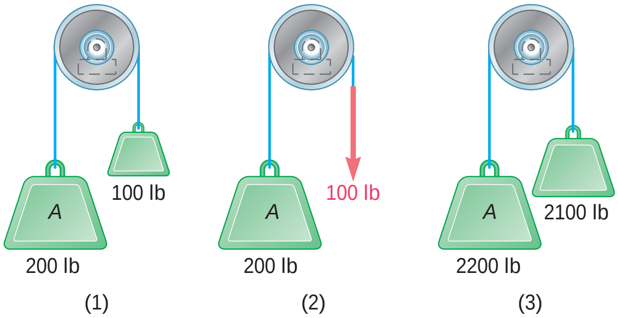


Fig. P12.15

12.16 Boxes A and B are at rest on a conveyor belt that is initially at rest. The belt is suddenly started in an upward direction so that slipping occurs between the belt and the boxes. Knowing that the coefficients of kinetic friction between the belt and the boxes are $(\mu_k)_A = 0.30$ and $(\mu_k)_B = 0.32$, determine the initial acceleration of each box.

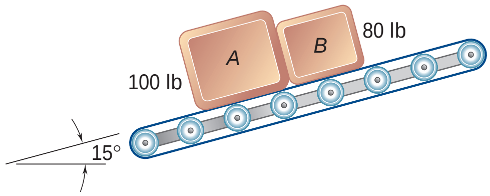


Fig. P12.16

12.17 A 5000-lb truck is being used to lift a 1000-lb boulder B that is on a 200-lb pallet A. Knowing the acceleration of the truck is 1 ft/s^2, determine (*a*) the horizontal force between the tires and the ground, (*b*) the force between the boulder and the pallet.

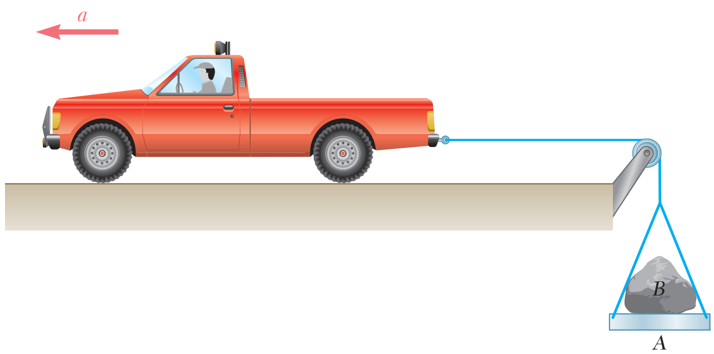


Fig. P12.17

12.18 Block A has a mass of 40 kg, and block B has a mass of 8 kg. The coefficients of friction between all surfaces of contact are $\mu_s = 0.20$ and $\mu_k = 0.15$. If $P = 0$, determine (*a*) the acceleration of block B, (*b*) the tension in the cord.

12.19 Block A has a mass of 40 kg, and block B has a mass of 8 kg. The coefficients of friction between all surfaces of contact are $\mu_s = 0.20$ and $\mu_k = 0.15$. If $P = 40$ N y, determine (*a*) the acceleration of block B, (*b*) the tension in the cord.

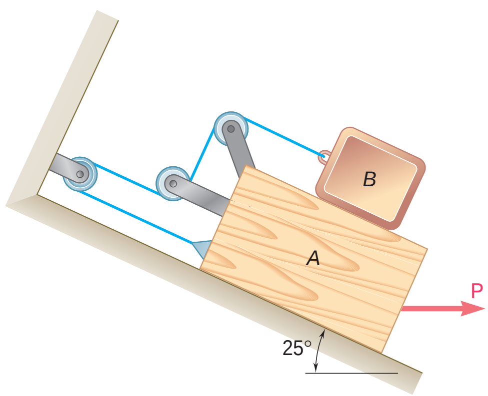


Fig. P12.18 and P12.19

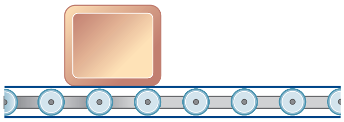


Fig. P12.20

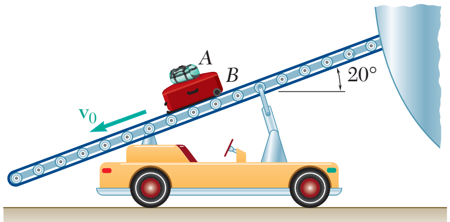


Fig. *P12.21*

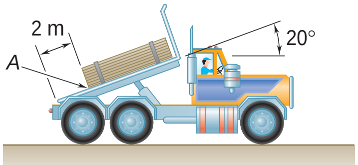


Fig. *P12.22*

12.20 A package is at rest on a conveyor belt which is initially at rest. The belt is started and moves to the right for 1.3 s with a constant acceleration of 2 m/s^2. The belt then moves with a constant deceleration $\mathbf{a}_2$ and comes to a stop after a total displacement of 2.2 m. Knowing that the coefficients of friction between the package and the belt are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine (*a*) the deceleration $\mathbf{a}_2$ of the belt, (*b*) the displacement of the package relative to the belt as the belt comes to a stop.

12.21 A baggage conveyor is used to unload luggage from an airplane. The 10-kg duffel bag *A* is sitting on top of the 20-kg suitcase *B*. The conveyor is moving the bags down at a constant speed of 0.5 m/s when the belt suddenly stops. Knowing that the coefficient of friction between the belt and *B* is 0.3 and that bag *A* does not slip on suitcase *B*, determine the smallest allowable coefficient of static friction between the bags.

12.22 To unload a bound stack of plywood from a truck, the driver first tilts the bed of the truck and then accelerates from rest. Knowing that the coefficients of friction between the bottom sheet of plywood and the bed are $\mu_s = 0.40$ and $\mu_k = 0.30$, determine (*a*) the smallest acceleration of the truck which will cause the stack of plywood to slide, (*b*) the acceleration of the truck which causes corner *A* of the stack to reach the end of the bed in 0.9 s.

12.23 To transport a series of bundles of shingles *A* to a roof, a contractor uses a motor-driven lift consisting of a horizontal platform *BC* which rides on rails attached to the sides of a ladder. The lift starts from rest and initially moves with a constant acceleration $\mathbf{a}_1$ as shown. The lift then decelerates at a constant rate $\mathbf{a}_2$ and comes to rest at *D*, near the top of the ladder. Knowing that the coefficient of static friction between a bundle of shingles and the horizontal platform is 0.30, determine the largest allowable acceleration $\mathbf{a}_1$ and the largest allowable deceleration $\mathbf{a}_2$ if the bundle is not to slide on the platform.

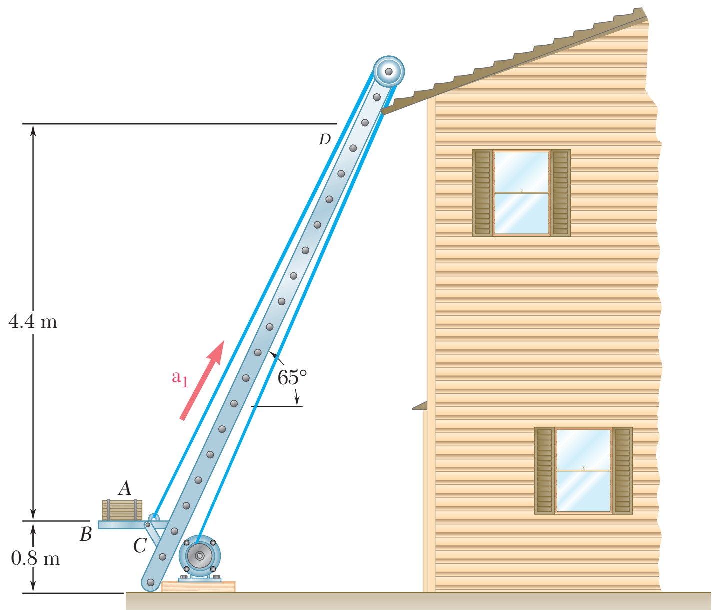


Fig. P12.23

12.24 An airplane has a mass of 25 Mg and its engines develop a total thrust of 40 kN during take-off. If the drag **D** exerted on the plane has a magnitude $D = 2.25\, v^2$, where v is expressed in meters per second and D in newtons, and if the plane becomes airborne at a speed of 240 km/h, determine the length of runway required for the plane to take off.

12.25 The propellers of a ship of weight W can produce a propulsive force $\mathbf{F}_0$; they produce a force of the same magnitude but of opposite direction when the engines are reversed. Knowing that the ship was proceeding forward at its maximum speed v_0 when the engines were put into reverse, determine the distance the ship travels before coming to a stop. Assume that the frictional resistance of the water varies directly with the square of the velocity.

12.26 A constant force **P** is applied to a piston and rod of total mass m to make them move in a cylinder filled with oil. As the piston moves, the oil is forced through orifices in the piston and exerts on the piston a force of magnitude kv in a direction opposite to the motion of the piston. Knowing that the piston starts from rest at $t = 0$ and $x = 0$, show that the equation relating x, v, and t, where x is the distance traveled by the piston and v is the speed of the piston, is linear in each of these variables.

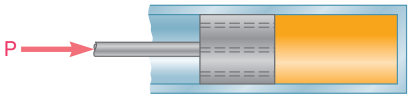


Fig. *P12.26*

12.27 A spring AB of constant k is attached to a support at A and to a collar of mass m. The unstretched length of the spring is l. Knowing that the collar is released from rest at $x = x_0$ and neglecting friction between the collar and the horizontal rod, determine the magnitude of the velocity of the collar as it passes through point C.

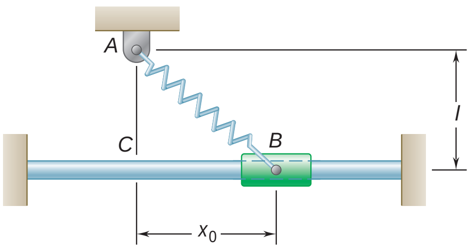


Fig. P12.27

12.28 Block A has a mass of 10 kg, and blocks B and C have masses of 5 kg each. Knowing that the blocks are initially at rest and that B moves through 3 m in 2 s, determine (*a*) the magnitude of the force **P**, (*b*) the tension in the cord AD. Neglect the masses of the pulleys and axle friction.

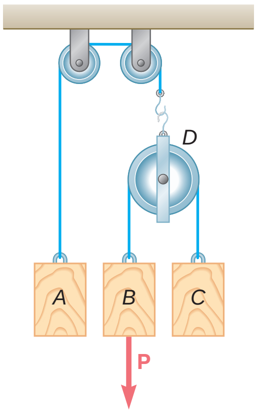


Fig. P12.28

12.29 A 40-lb sliding panel is supported by rollers at B and C. A 25-lb counterweight A is attached to a cable as shown and, in cases a and c, is initially in contact with a vertical edge of the panel. Neglecting friction, determine in each case shown the acceleration of the panel and the tension in the cord immediately after the system is released from rest.

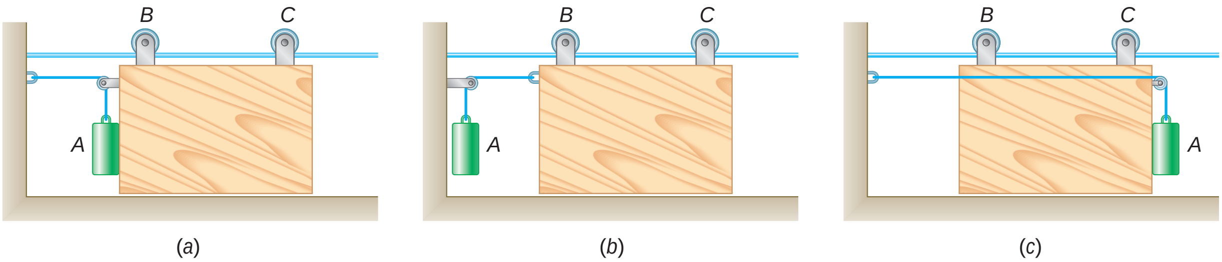


Fig. P12.29

12.30 The coefficients of friction between blocks A and C and the horizontal surfaces are $\mu_s = 0.24$ and $\mu_k = 0.20$. Knowing that $m_A = 5$ kg, $m_B = 10$ kg, and $m_C = 10$ kg, determine (*a*) the tension in the cord, (*b*) the acceleration of each block.

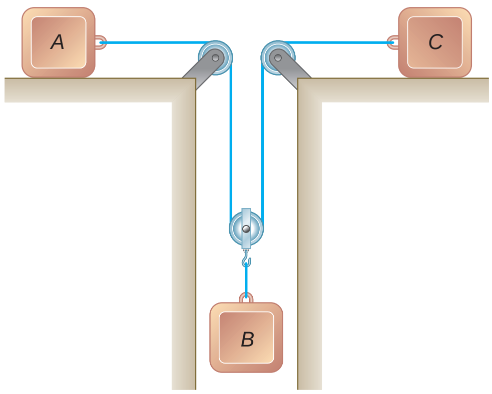


Fig. P12.30

12.31 A 10-lb block B rests as shown on a 20-lb bracket A. The coefficients of friction are $\mu_s = 0.30$ and $\mu_k = 0.25$ between block B and bracket A, and there is no friction in the pulley or between the bracket and the horizontal surface. (*a*) Determine the maximum weight of block C if block B is not to slide on bracket A. (*b*) If the weight of block C is 10 percent larger than the answer found in a, determine the accelerations of A, B, and C.

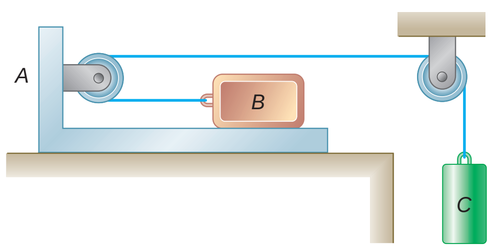


Fig. P12.31

12.32 The masses of blocks A, B, C, and D are 9 kg, 9 kg, 6 kg, and 7 kg, respectively. Knowing that a downward force of magnitude 120 N is applied to block D, determine (*a*) the acceleration of each block, (*b*) the tension in cord ABC. Neglect the weights of the pulleys and the effect of friction.

12.33 The masses of blocks A, B, C, and D are 9 kg, 9 kg, 6 kg, and 7 kg, respectively. Knowing that a downward force of magnitude 50 N is applied to block B and that the system starts from rest, determine at $t = 3$ s the velocity (*a*) of D relative to A, (*b*) of C relative to D. Neglect the weights of the pulleys and the effect of friction.

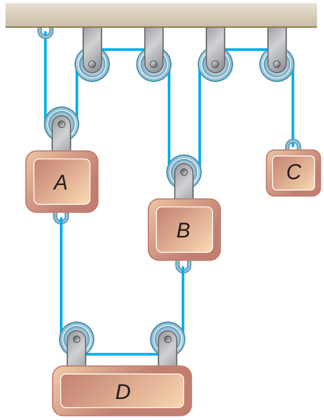


Fig. *P12.32* and *P12.33*

12.34 The 15-kg block B is supported by the 25-kg block A and is attached to a cord to which a 225-N horizontal force is applied as shown. Neglecting friction, determine (*a*) the acceleration of block A, (*b*) the acceleration of block B relative to A.

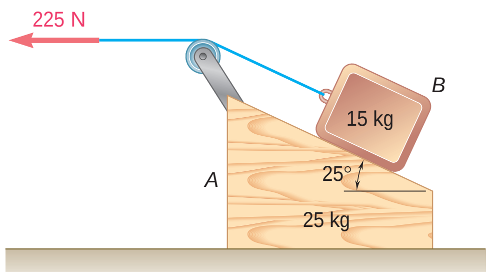


Fig. P12.34

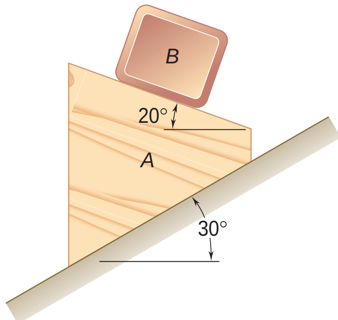


Fig. P12.35

12.35 Block B of mass 10 kg rests as shown on the upper surface of a 22-kg wedge A. Knowing that the system is released from rest and neglecting friction, determine (*a*) the acceleration of B, (*b*) the velocity of B relative to A at $t = 0.5$ s.

12.36 A 450-g tetherball A is moving along a horizontal circular path at a constant speed of 4 m/s. Determine (*a*) the angle u that the cord forms with pole BC, (*b*) the tension in the cord.

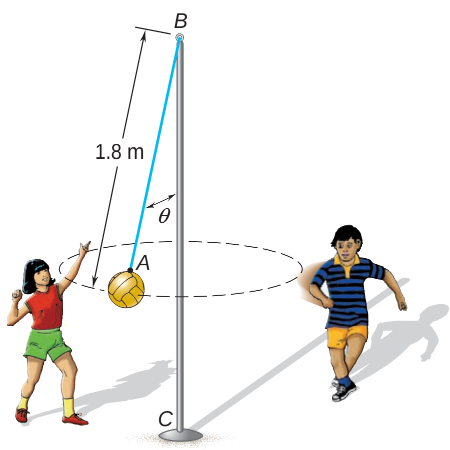


Fig. P12.36

12.37 During a hammer thrower's practice swings, the 7.1-kg head *A* of the hammer revolves at a constant speed *v* in a horizontal circle as shown. If r = 0.93 m and u = 60°, determine (*a*) the tension in wire *BC*, (*b*) the speed of the hammer's head.

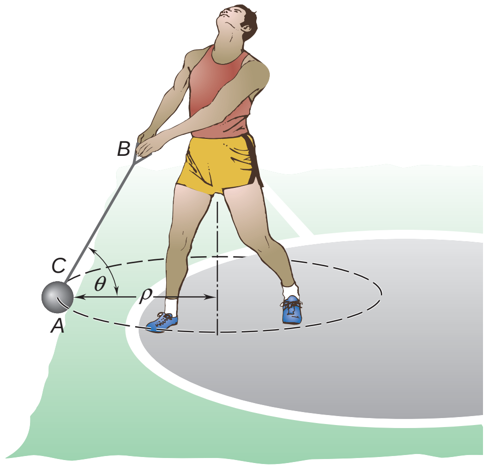


Fig. P12.37

12.38 A single wire *ACB* passes through a ring at *C* attached to a sphere which revolves at a constant speed *v* in the horizontal circle shown. Knowing that the tension is the same in both portions of the wire, determine the speed *v*.

12.39 Two wires *AC* and *BC* are tied at *C* to a sphere which revolves at a constant speed *v* in the horizontal circle shown. Determine the range of values of *v* for which both wires remain taut.

***12.40** Two wires *AC* and *BC* are tied at *C* to a sphere which revolves at a constant speed *v* in the horizontal circle shown. Determine the range of the allowable values of *v* if both wires are to remain taut and if the tension in either of the wires is not to exceed 60 N.

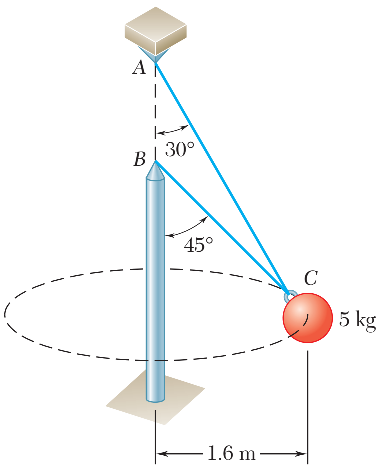


Fig. P12.38, P12.39, and P12.40

12.41 A 100-g sphere *D* is at rest relative to drum *ABC* which rotates at a constant rate. Neglecting friction, determine the range of the allowable values of the velocity *v* of the sphere if neither of the normal forces exerted by the sphere on the inclined surfaces of the drum is to exceed 1.1 N.

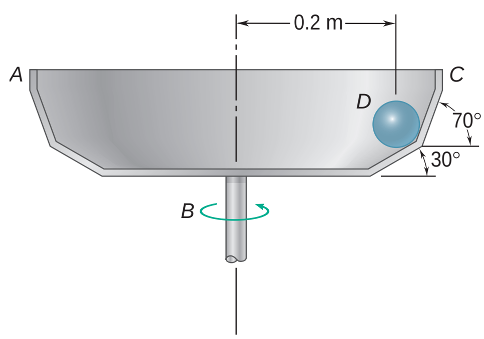


Fig. *P12.41*

***12.42** As part of an outdoor display, a 12-lb model C of the earth is attached to wires AC and BC and revolves at a constant speed v in the horizontal circle shown. Determine the range of the allowable values of v if both wires are to remain taut and if the tension in either of the wires is not to exceed 26 lb.

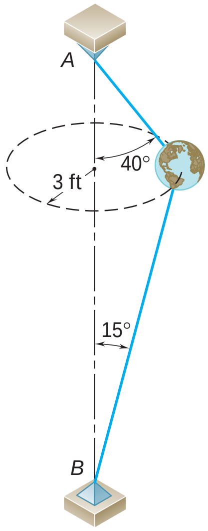


Fig. P12.42

***12.43** The 1.2-lb flyballs of a centrifugal governor revolve at a constant speed v in the horizontal circle of 6-in. radius shown. Neglecting the weights of links AB, BC, AD, and DE and requiring that the links support only tensile forces, determine the range of the allowable values of v so that the magnitudes of the forces in the links do not exceed 17 lb.

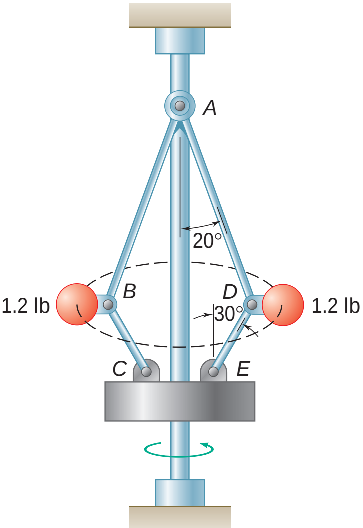


Fig. P12.43

12.44 A 130-lb wrecking ball B is attached to a 45-ft-long steel cable AB and swings in the vertical arc shown. Determine the tension in the cable (*a*) at the top C of the swing, (*b*) at the bottom D of the swing, where the speed of B is 13.2 ft/s.

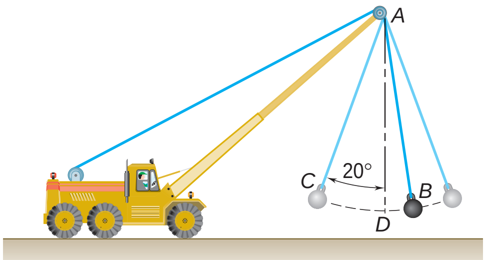


Fig. P12.44

12.45 During a high-speed chase, a 2400-lb sports car traveling at a speed of 100 mi/h just loses contact with the road as it reaches the crest A of a hill. (*a*) Determine the radius of curvature ρ of the vertical profile of the road at A. (*b*) Using the value of ρ found in part *a*, determine the force exerted on a 160-lb driver by the seat of his 3100-lb car as the car, traveling at a constant speed of 50 mi/h, passes through A.

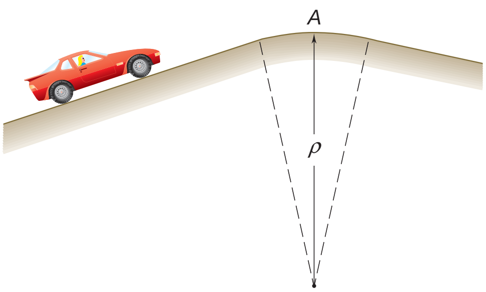


Fig. P12.45

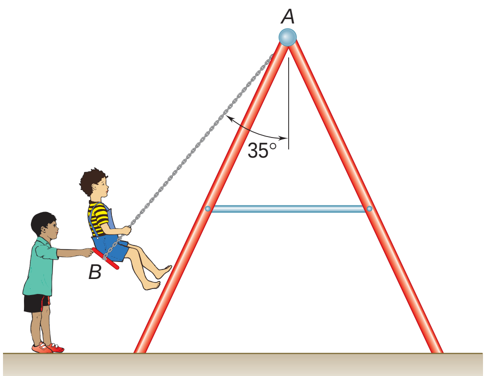


Fig. P12.46

12.46 A child having a mass of 22 kg sits on a swing and is held in the position shown by a second child. Neglecting the mass of the swing, determine the tension in rope *AB* (*a*) while the second child holds the swing with his arms outstretched horizontally, (*b*) immediately after the swing is released.

12.47 The roller-coaster track shown is contained in a vertical plane. The portion of track between *A* and *B* is straight and horizontal, while the portions to the left of *A* and to the right of *B* have radii of curvature as indicated. A car is traveling at a speed of 72 km/h when the brakes are suddenly applied, causing the wheels of the car to slide on the track (m_k = 0.20). Determine the initial deceleration of the car if the brakes are applied as the car (*a*) has almost reached *A*, (*b*) is traveling between *A* and *B*, (*c*) has just passed *B*.

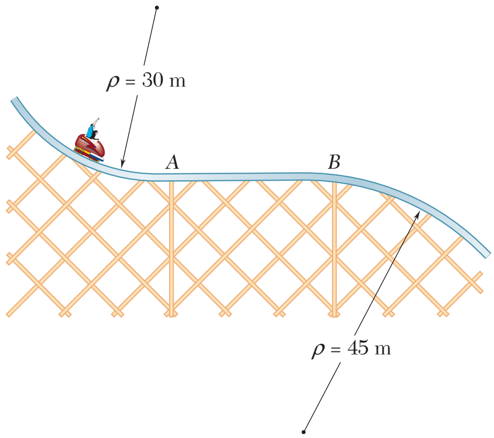


Fig. P12.47

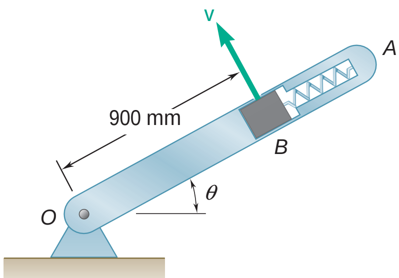


Fig. P12.48

12.48 A 250-g block fits inside a small cavity cut in arm *OA*, which rotates in the vertical plane at a constant rate such that v = 3 m/s. Knowing that the spring exerts on block *B* a force of magnitude P = 1.5 N and neglecting the effect of friction, determine the range of values of u for which block *B* is in contact with the face of the cavity closest to the axis of rotation *O*.

12.49 A series of small packages, each with a mass of 0.5 kg, are discharged from a conveyor belt as shown. Knowing that the coefficient of static friction between each package and the conveyor belt is 0.4, determine (*a*) the force exerted by the belt on the package just after it has passed point *A*, (*b*) the angle u defining the point *B* where the packages first *slip* relative to the belt.

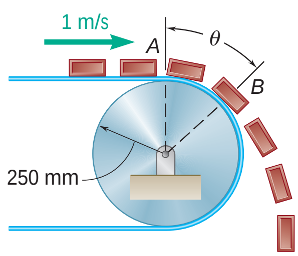


Fig. P12.49

12.50 A 54-kg pilot flies a jet trainer in a half-vertical loop of 1200-m radius so that the speed of the trainer decreases at a constant rate. Knowing that the pilot's apparent weights at points *A* and *C* are 1680 N and 350 N, respectively, determine the force exerted on her by the seat of the trainer when the trainer is at point *B*.

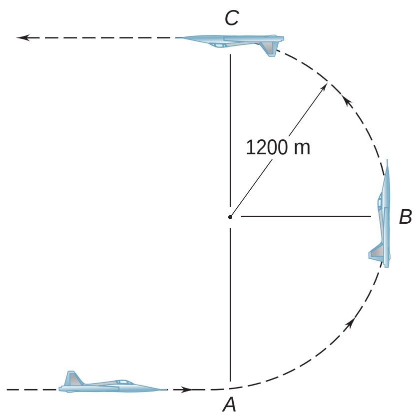


Fig. P12.50

12.51 A carnival ride is designed to allow the general public to experience high-acceleration motion. The ride rotates about point *O* in a horizontal circle such that the rider has a speed v_0. The rider reclines on a platform *A* which rides on rollers such that friction is negligible. A mechanical stop prevents the platform from rolling down the incline. Determine (*a*) the speed v_0 at which the platform *A* begins to roll upward, (*b*) the normal force experienced by an 80-kg rider at this speed.

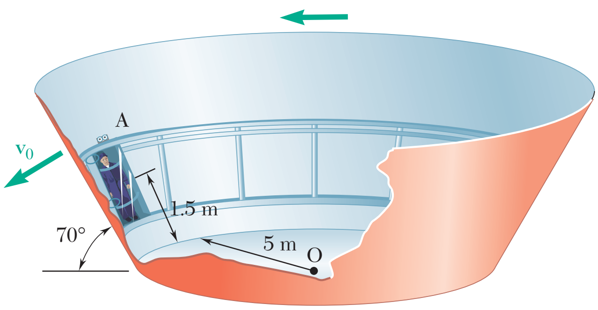


Fig. P12.51

12.52 A curve in a speed track has a radius of 1000 ft and a rated speed of 120 mi/h. (See Sample Prob. 12.6 for the definition of rated speed.) Knowing that a racing car starts skidding on the curve when traveling at a speed of 180 mi/h, determine (*a*) the banking angle u, (*b*) the coefficient of static friction between the tires and the track under the prevailing conditions, (*c*) the minimum speed at which the same car could negotiate the curve.

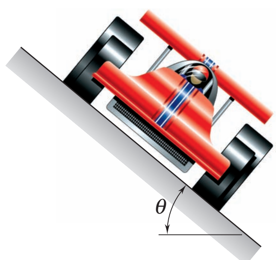


Fig. *P12.52*

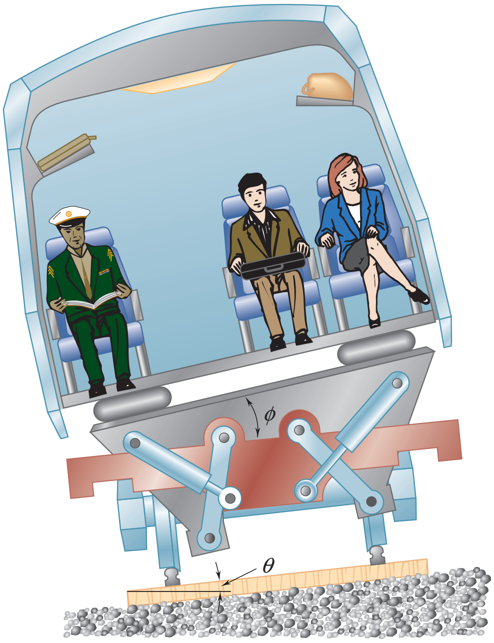


Fig. P12.53 and *P12.54*

12.53 Tilting trains, such as the *American Flyer* which will run from Washington to New York and Boston, are designed to travel safely at high speeds on curved sections of track which were built for slower, conventional trains. As it enters a curve, each car is tilted by hydraulic actuators mounted on its trucks. The tilting feature of the cars also increases passenger comfort by eliminating or greatly reducing the side force $\mathbf{F}_s$ (parallel to the floor of the car) to which passengers feel subjected. For a train traveling at 100 mi/h on a curved section of track banked through an angle u = 6° and with a rated speed of 60 mi/h, determine (*a*) the magnitude of the side force felt by a passenger of weight W in a standard car with no tilt (f = 0), (*b*) the required angle of tilt f if the passenger is to feel no side force. (See Sample Prob. 12.6 for the definition of rated speed.)

12.54 Tests carried out with the tilting trains described in Prob. 12.53 revealed that passengers feel queasy when they see through the car windows that the train is rounding a curve at high speed, yet do not feel any side force. Designers, therefore, prefer to reduce, but not eliminate that force. For the train of Prob. 12.53, determine the required angle of tilt f if passengers are to feel side forces equal to 10 percent of their weights.

12.55 A 3-kg block is at rest relative to a parabolic dish which rotates at a constant rate about a vertical axis. Knowing that the coefficient of static friction is 0.5 and that r = 2 m, determine the maximum allowable velocity v of the block.

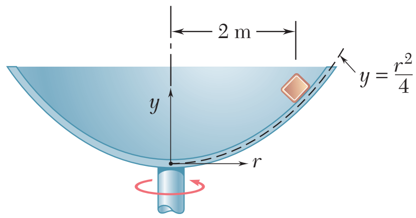


Fig. P12.55

12.56 Three seconds after a polisher is started from rest, small tufts of fleece from along the circumference of the 225-mm-diameter polishing pad are observed to fly free of the pad. If the polisher is started so that the fleece along the circumference undergoes a constant tangential acceleration of 4 m/s^2, determine (*a*) the speed v of a tuft as it leaves the pad, (*b*) the magnitude of the force required to free a tuft if the average mass of a tuft is 1.6 mg.

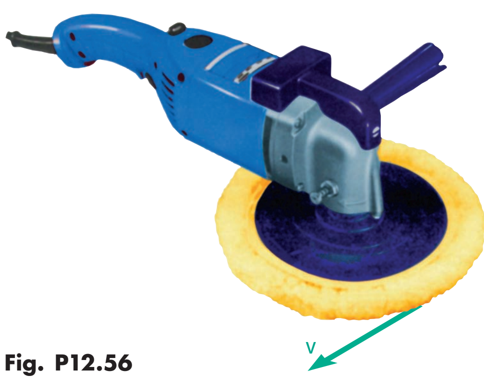


Fig. P12.56

12.57 A turntable A is built into a stage for use in a theatrical production. It is observed during a rehearsal that a trunk B starts to slide on the turntable 10 s after the turntable begins to rotate. Knowing that the trunk undergoes a constant tangential acceleration of 0.24 m/s^2, determine the coefficient of static friction between the trunk and the turntable.

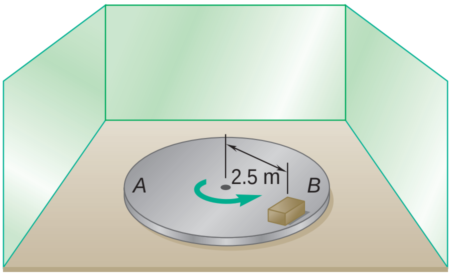


Fig. P12.57

12.58 A small, 300-g collar D can slide on portion AB of a rod which is bent as shown. Knowing that a = 40° and that the rod rotates about the vertical AC at a constant rate of 5 rad/s, determine the value of r for which the collar will not slide on the rod if the effect of friction between the rod and the collar is neglected.

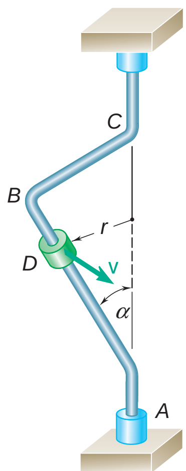


Fig. P12.58 and *P12.59*

12.59 A small, 200-g collar D can slide on portion AB of a rod which is bent as shown. Knowing that the rod rotates about the vertical AC at a constant rate and that a = 30° and r = 600 mm, determine the range of values of the speed v for which the collar will not slide on the rod if the coefficient of static friction between the rod and the collar is 0.30.

12.60 A semicircular slot of 10-in. radius is cut in a flat plate which rotates about the vertical AD at a constant rate of 14 rad/s. A small, 0.8-lb block E is designed to slide in the slot as the plate rotates. Knowing that the coefficients of friction are m$_s$ = 0.35 and m$_k$ = 0.25, determine whether the block will slide in the slot if it is released in the position corresponding to (*a*) u = 80°, (*b*) u = 40°. Also determine the magnitude and the direction of the friction force exerted on the block immediately after it is released.

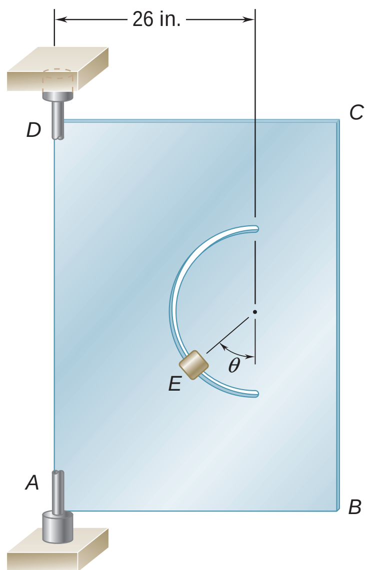


Fig. *P12.60*

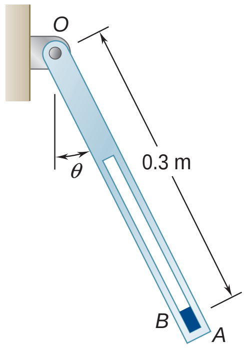


Fig. P12.61

12.61 A small block B fits inside a slot cut in arm OA which rotates in a vertical plane at a constant rate. The block remains in contact with the end of the slot closest to A and its speed is 1.4 m/s for $0 \leq u \leq 150°$. Knowing that the block begins to slide when $u = 150°$, determine the coefficient of static friction between the block and the slot.

12.62 The parallel-link mechanism $ABCD$ is used to transport a component I between manufacturing processes at stations E, F, and G by picking it up at a station when $u = 0$ and depositing it at the next station when $u = 180°$. Knowing that member BC remains horizontal throughout its motion and that links AB and CD rotate at a constant rate in a vertical plane in such a way that $v_B = 2.2$ ft/s, determine (*a*) the minimum value of the coefficient of static friction between the component and BC if the component is not to slide on BC while being transferred, (*b*) the values of u for which sliding is impending.

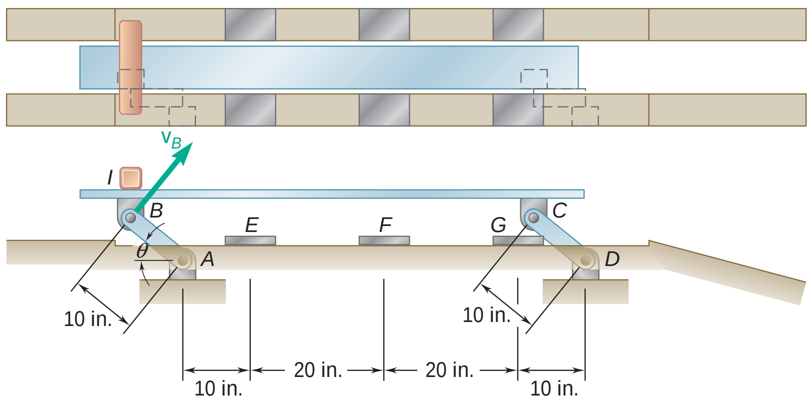


Fig. P12.62

12.63 Knowing that the coefficients of friction between the component I and member BC of the mechanism of Prob. 12.62 are $m_s = 0.35$ and $m_k = 0.25$, determine (*a*) the maximum allowable constant speed v_B if the component is not to slide on BC while being transferred, (*b*) the values of u for which sliding is impending.

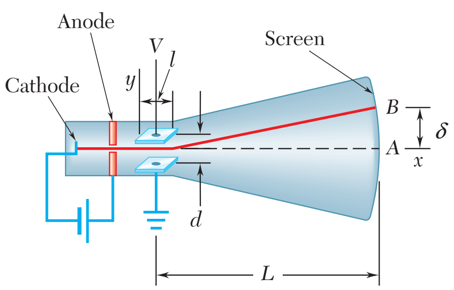


Fig. P12.64

12.64 In the cathode-ray tube shown, electrons emitted by the cathode and attracted by the anode pass through a small hole in the anode and then travel in a straight line with a speed v_0 until they strike the screen at A. However, if a difference of potential V is established between the two parallel plates, the electrons will be subjected to a force $\mathbf{F}$ perpendicular to the plates while they travel between the plates and will strike the screen at point B, which is at a distance d from A. The magnitude of the force $\mathbf{F}$ is $F = eV/d$, where $-e$ is the charge of an electron and d is the distance between the plates. Derive an expression for the deflection d in terms of V, v_0, the charge $-e$ and the mass m of an electron, and the dimensions d, l, and L.

12.65 In Prob. 12.64, determine the smallest allowable value of the ratio d/l in terms of e, m, v_0, and V if at $x = l$ the minimum permissible distance between the path of the electrons and the positive plate is $0.05d$.

12.7 ANGULAR MOMENTUM OF A PARTICLE. RATE OF CHANGE OF ANGULAR MOMENTUM

Consider a particle P of mass m moving with respect to a newtonian frame of reference $Oxyz$. As we saw in Sec. 12.3, the linear momentum of the particle at a given instant is defined as the vector $m\mathbf{v}$ obtained by multiplying the velocity $\mathbf{v}$ of the particle by its mass m. The moment about O of the vector $m\mathbf{v}$ is called the *moment of momentum,* or the *angular momentum,* of the particle about O at that instant and is denoted by $\mathbf{H}_O$. Recalling the definition of the moment of a vector (Sec. 3.6) and denoting by $\mathbf{r}$ the position vector of P, we write

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{12.12}$$

and note that $\mathbf{H}_O$ is a vector perpendicular to the plane containing $\mathbf{r}$ and $m\mathbf{v}$ and of magnitude

$$H_O = rmv \sin f \tag{12.13}$$

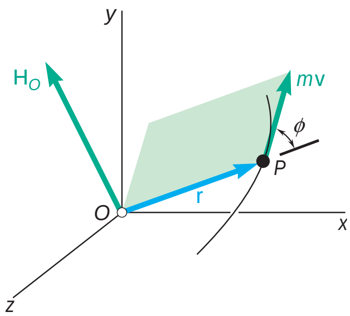


Fig. 12.12

where f is the angle between $\mathbf{r}$ and $m\mathbf{v}$ (Fig. 12.12). The sense of $\mathbf{H}_O$ can be determined from the sense of $m\mathbf{v}$ by applying the right-hand rule. The unit of angular momentum is obtained by multiplying the units of length and of linear momentum (Sec. 12.4). With SI units, we have

$$(\text{m})(\text{kg} \cdot \text{m/s}) = \text{kg} \cdot \text{m}^2/\text{s}$$

With U.S. customary units, we write

$$(\text{ft})(\text{lb} \cdot \text{s}) = \text{ft} \cdot \text{lb} \cdot \text{s}$$

Resolving the vectors $\mathbf{r}$ and $m\mathbf{v}$ into components and applying formula (3.10), we write

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix} \tag{12.14}$$

The components of $\mathbf{H}_O$, which also represent the moments of the linear momentum $m\mathbf{v}$ about the coordinate axes, can be obtained by expanding the determinant in (12.14). We have

$$\begin{aligned} H_x &= m(yv_z - zv_y) \\ H_y &= m(zv_x - xv_z) \\ H_z &= m(xv_y - yv_x) \end{aligned} \tag{12.15}$$

In the case of a particle moving in the xy plane, we have $z = v_z = 0$ and the components H_x and H_y reduce to zero. The angular momentum is thus perpendicular to the xy plane; it is then completely defined by the scalar

$$H_O = H_z = m(xv_y - yv_x) \tag{12.16}$$

which will be positive or negative according to the sense in which the particle is observed to move from O. If polar coordinates are used, we resolve the linear momentum of the particle into radial and transverse components (Fig. 12.13) and write

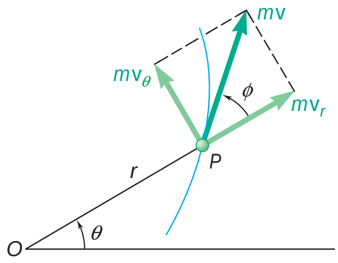


Fig. 12.13

$$H_O = rmv \sin \phi = rmv_\theta \tag{12.17}$$

or, recalling from (11.45) that $v_\theta = r\dot{\theta}$,

$$\boxed{H_O = mr^2\dot{\theta}} \tag{12.18}$$

Let us now compute the derivative with respect to t of the angular momentum $\mathbf{H}_O$ of a particle P moving in space. Differentiating both members of Eq. (12.12), and recalling the rule for the differentiation of a vector product (Sec. 11.10), we write

$$\dot{\mathbf{H}}_O = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{a}$$

Since the vectors $\mathbf{v}$ and $m\mathbf{v}$ are collinear, the first term of the expression obtained is zero; and, by Newton's second law, $m\mathbf{a}$ is equal to the sum $\Sigma\mathbf{F}$ of the forces acting on P. Noting that $\mathbf{r} \times \Sigma\mathbf{F}$ represents the sum $\Sigma\mathbf{M}_O$ of the moments about O of these forces, we write

$$\boxed{\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O} \tag{12.19}$$

Equation (12.19), which results directly from Newton's second law, states that *the sum of the moments about O of the forces acting on the particle is equal to the rate of change of the moment of momentum, or angular momentum, of the particle about O.*

12.8 EQUATIONS OF MOTION IN TERMS OF RADIAL AND TRANSVERSE COMPONENTS

Consider a particle P, of polar coordinates r and θ, which moves in a plane under the action of several forces. Resolving the forces and the acceleration of the particle into radial and transverse components (Fig. 12.14) and substituting into Eq. (12.2), we obtain the two scalar equations

Photo 12.4 The forces on the specimens used in a high speed centrifuge can be described in terms of radial and transverse components.

$$\Sigma F_r = ma_r \qquad \Sigma F_\theta = ma_\theta \tag{12.20}$$

Substituting for a_r and a_θ from Eqs. (11.46), we have

$$\Sigma F_r = m(\ddot{r} - r\dot{\theta}^2) \tag{12.21}$$

$$\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{12.22}$$

The equations obtained can be solved for two unknowns.

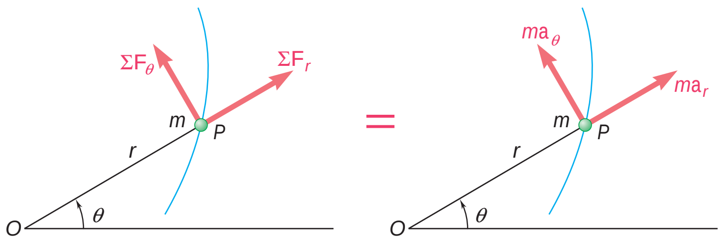


Fig. 12.14

Equation (12.22) could have been derived from Eq. (12.19). Recalling (12.18) and noting that $\Sigma M_O = r\Sigma F_u$, Eq. (12.19) yields

$$\begin{aligned} r\Sigma F_u &= \frac{d}{dt}(mr^2\dot{u}) \\ &= m(r^2\ddot{u} + 2r\dot{r}\dot{u}) \end{aligned}$$

and, after dividing both members by r,

$$\Sigma F_u = m(r\ddot{u} + 2\dot{r}\dot{u}) \tag{12.22}$$

12.9 MOTION UNDER A CENTRAL FORCE. CONSERVATION OF ANGULAR MOMENTUM

When the only force acting on a particle P is a force $\mathbf{F}$ directed toward or away from a fixed point O, the particle is said to be moving *under a central force*, and the point O is referred to as the *center of force* (Fig. 12.15). Since the line of action of $\mathbf{F}$ passes through O, we must have $\Sigma \mathbf{M}_O = 0$ at any given instant. Substituting into Eq. (12.19), we therefore obtain

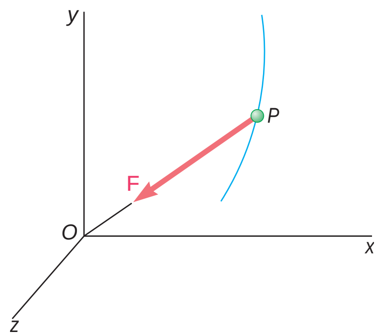


Fig. 12.15

$$\dot{\mathbf{H}}_O = 0$$

for all values of t and, integrating in t,

$$\mathbf{H}_O = \text{constant} \tag{12.23}$$

We thus conclude that *the angular momentum of a particle moving under a central force is constant, in both magnitude and direction.*

Recalling the definition of the angular momentum of a particle (Sec. 12.7), we write

$$\mathbf{r} \times m\mathbf{v} = \mathbf{H}_O = \text{constant} \tag{12.24}$$

from which it follows that the position vector $\mathbf{r}$ of the particle P must be perpendicular to the constant vector $\mathbf{H}_O$. Thus, a particle under

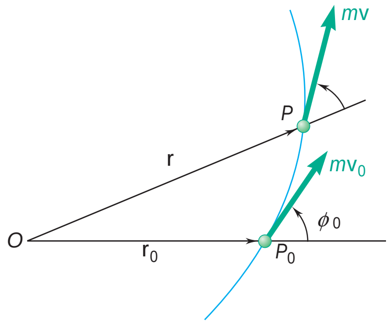


Fig. 12.16

a central force moves in a fixed plane perpendicular to $\mathbf{H}_O$. The vector $\mathbf{H}_O$ and the fixed plane are defined by the initial position vector $\mathbf{r}_0$ and the initial velocity $\mathbf{v}_0$ of the particle. For convenience, let us assume that the plane of the figure coincides with the fixed plane of motion (Fig. 12.16).

Since the magnitude H_O of the angular momentum of the particle P is constant, the right-hand member in Eq. (12.13) must be constant. We therefore write

$$rmv \sin \phi = r_0 m v_0 \sin \phi_0 \tag{12.25}$$

This relation applies to the motion of any particle under a central force. Since the gravitational force exerted by the sun on a planet is a central force directed toward the center of the sun, Eq. (12.25) is fundamental to the study of planetary motion. For a similar reason, it is also fundamental to the study of the motion of space vehicles in orbit about the earth.

Alternatively, recalling Eq. (12.18), we can express the fact that the magnitude H_O of the angular momentum of the particle P is constant by writing

$$mr^2\dot{\theta} = H_O = \text{constant} \tag{12.26}$$

or, dividing by m and denoting by h the angular momentum per unit mass H_O/m,

$$\boxed{r^2\dot{\theta} = h} \tag{12.27}$$

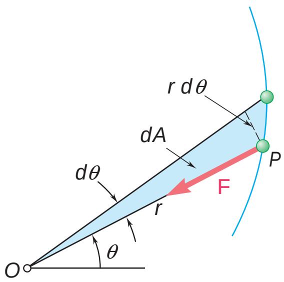


Fig. 12.17

Equation (12.27) can be given an interesting geometric interpretation. Observing from Fig. 12.17 that the radius vector OP sweeps an infinitesimal area $dA = \frac{1}{2}r^2\,d\theta$ as it rotates through an angle $d\theta$, and defining the *areal velocity* of the particle as the quotient dA/dt, we note that the left-hand member of Eq. (12.27) represents twice the areal velocity of the particle. We thus conclude that *when a particle moves under a central force, its areal velocity is constant*.

12.10 NEWTON'S LAW OF GRAVITATION

As you saw in the preceding section, the gravitational force exerted by the sun on a planet or by the earth on an orbiting satellite is an important example of a central force. In this section, you will learn how to determine the magnitude of a gravitational force.

In his *law of universal gravitation*, Newton states that two particles of masses M and m at a distance r from each other attract each other with equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ directed along the line joining the particles (Fig. 12.18). The common magnitude F of the two forces is

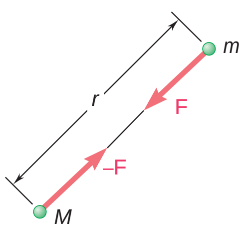


Fig. 12.18

$$\boxed{F = G\frac{Mm}{r^2}} \tag{12.28}$$

where G is a universal constant, called the *constant of gravitation*. Experiments show that the value of G is $(66.73 \pm 0.03) \times 10^{-12}\ \mathrm{m^3/kg \cdot s^2}$ in SI units or approximately $34.4 \times 10^{-9}\ \mathrm{ft^4/lb \cdot s^4}$ in U.S. customary units. Gravitational forces exist between any pair of bodies, but their effect is appreciable only when one of the bodies has a very large mass. The effect of gravitational forces is apparent in the cases of the motion of a planet about the sun, of satellites orbiting about the earth, or of bodies falling on the surface of the earth.

Since the force exerted by the earth on a body of mass m located on or near its surface is defined as the weight $\mathbf{W}$ of the body, we can substitute the magnitude $W = mg$ of the weight for F, and the radius R of the earth for r, in Eq. (12.28). We obtain

$$W = mg = \frac{GM}{R^2}m \qquad \text{or} \qquad g = \frac{GM}{R^2} \tag{12.29}$$

where M is the mass of the earth. Since the earth is not truly spherical, the distance R from the center of the earth depends upon the point selected on its surface, and the values of W and g will thus vary with the altitude and latitude of the point considered. Another reason for the variation of W and g with latitude is that a system of axes attached to the earth does not constitute a newtonian frame of reference (see Sec. 12.2). A more accurate definition of the weight of a body should therefore include a component representing the centrifugal force due to the rotation of the earth. Values of g at sea level vary from $9.781\ \mathrm{m/s^2}$, or $32.09\ \mathrm{ft/s^2}$, at the equator to $9.833\ \mathrm{m/s^2}$, or $32.26\ \mathrm{ft/s^2}$, at the poles.†

The force exerted by the earth on a body of mass m located in space at a distance r from its center can be found from Eq. (12.28). The computations will be somewhat simplified if we note that according to Eq. (12.29), the product of the constant of gravitation G and the mass M of the earth can be expressed as

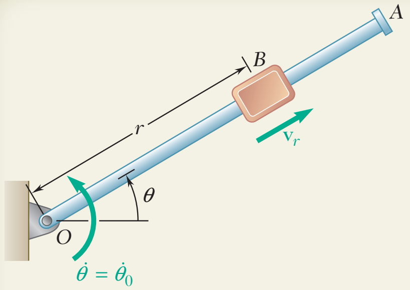
$$GM = gR^2 \tag{12.30}$$

where g and the radius R of the earth will be given their average values $g = 9.81\ \mathrm{m/s^2}$ and $R = 6.37 \times 10^6$ m in SI units‡ and $g = 32.2\ \mathrm{ft/s^2}$ and $R = (3960\ \mathrm{mi})(5280\ \mathrm{ft/mi})$ in U.S. customary units.

The discovery of the law of universal gravitation has often been attributed to the belief that, after observing an apple falling from a tree, Newton had reflected that the earth must attract an apple and the moon in much the same way. While it is doubtful that this incident actually took place, it may be said that Newton would not have formulated his law if he had not first perceived that the acceleration of a falling body must have the same cause as the acceleration which keeps the moon in its orbit. This basic concept of the continuity of gravitational attraction is more easily understood today, when the gap between the apple and the moon is being filled with artificial earth satellites.

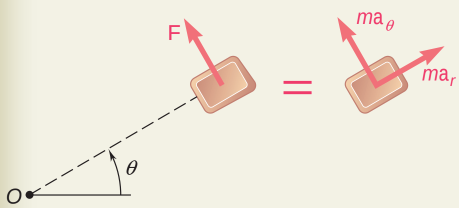
†A formula expressing g in terms of the latitude f was given in Prob. 12.2.

‡The value of R is easily found if one recalls that the circumference of the earth is $2\pi R = 40 \times 10^6$ m.



SAMPLE PROBLEM 12.7

A block B of mass m can slide freely on a frictionless arm OA which rotates in a horizontal plane at a constant rate $\dot{\theta}_0$. Knowing that B is released at a distance r_0 from O, express as a function of r, (*a*) the component v_r of the velocity of B along OA, (*b*) the magnitude of the horizontal force $\mathbf{F}$ exerted on B by the arm OA.



SOLUTION

Since all other forces are perpendicular to the plane of the figure, the only force shown acting on B is the force $\mathbf{F}$ perpendicular to OA.

Equations of Motion. Using radial and transverse components,

$$+\nearrow \Sigma F_r = ma_r: \qquad 0 = m(\ddot{r} - r\dot{\theta}^2) \qquad (1)$$
$$+\nwarrow \Sigma F_\theta = ma_\theta: \qquad F = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \qquad (2)$$

a. Component v_r of Velocity. Since $v_r = \dot{r}$, we have

$$\ddot{r} = \dot{v}_r = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}$$

Substituting for $\ddot{r}$ in (1), recalling that $\dot{\theta} = \dot{\theta}_0$, and separating the variables,

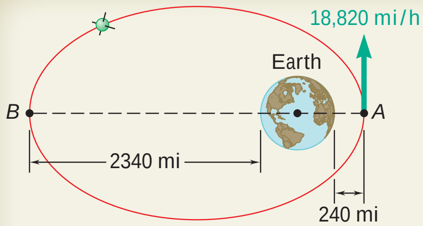
$$v_r\, dv_r = \dot{\theta}_0^2 r\, dr$$

Multiplying by 2, and integrating from 0 to v_r and from r_0 to r,

$$v_r^2 = \dot{\theta}_0^2(r^2 - r_0^2) \qquad v_r = \dot{\theta}_0(r^2 - r_0^2)^{1/2} \quad \blacktriangleleft$$

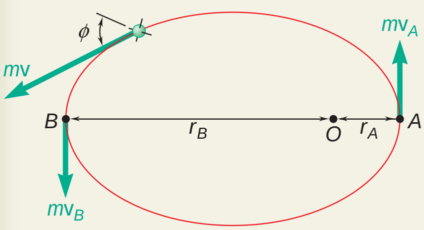
b. Horizontal Force F. Setting $\dot{\theta} = \dot{\theta}_0$, $\ddot{\theta} = 0$, $\dot{r} = v_r$ in Eq. (2), and substituting for v_r the expression obtained in part *a*,

$$F = 2m\dot{\theta}_0(r^2 - r_0^2)^{1/2}\dot{\theta}_0 \qquad F = 2m\dot{\theta}_0^2(r^2 - r_0^2)^{1/2} \quad \blacktriangleleft$$



SAMPLE PROBLEM 12.8

A satellite is launched in a direction parallel to the surface of the earth with a velocity of 18,820 mi/h from an altitude of 240 mi. Determine the velocity of the satellite as it reaches its maximum altitude of 2340 mi. It is recalled that the radius of the earth is 3960 mi.



SOLUTION

Since the satellite is moving under a central force directed toward the center O of the earth, its angular momentum $\mathbf{H}_O$ is constant. From Eq. (12.13) we have

$$rmv \sin \phi = H_O = \text{constant}$$

which shows that v is minimum at B, where both r and $\sin \phi$ are maximum. Expressing conservation of angular momentum between A and B,

$$r_A m v_A = r_B m v_B$$

$$v_B = v_A\frac{r_A}{r_B} = (18{,}820 \text{ mi/h})\frac{3960 \text{ mi} + 240 \text{ mi}}{3960 \text{ mi} + 2340 \text{ mi}}$$

$$v_B = 12{,}550 \text{ mi/h} \quad \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we continued our study of Newton's second law by expressing the force and the acceleration in terms of their *radial and transverse components*, where the corresponding equations of motion are

$$\Sigma F_r = ma_r: \qquad \Sigma F_r = m(\ddot{r} - r\dot{\theta}^2)$$
$$\Sigma F_\theta = ma_\theta: \qquad \Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

We introduced the *moment of the momentum,* or the *angular momentum,* $\mathbf{H}_O$ of a particle about O:

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \qquad (12.12)$$

and found that $\mathbf{H}_O$ is constant when the particle moves under a *central force* with its center located at O.

1. Using radial and transverse components. Radial and transverse components were introduced in the last lesson of Chap. 11 [Sec. 11.14]; you should review that material before attempting to solve the following problems. Also, our comments in the preceding lesson regarding the application of Newton's second law (drawing a free-body diagram and a $m\mathbf{a}$ diagram, etc.) still apply [Sample Prob. 12.7]. Finally, note that the solution of that sample problem depends on the application of techniques developed in Chap. 11—you will need to use similar techniques to solve some of the problems of this lesson.

2. Solving problems involving the motion of a particle under a central force. In problems of this type, the angular momentum $\mathbf{H}_O$ of the particle about the center of force O is conserved. You will find it convenient to introduce the constant $h = H_O/m$ representing the angular momentum per unit mass. Conservation of the angular momentum of the particle P about O can then be expressed by either of the following equations

$$rv \sin \phi = h \qquad \text{or} \qquad r^2\dot{\theta} = h$$

where r and θ are the polar coordinates of P, and ϕ is the angle that the velocity $\mathbf{v}$ of the particle forms with the line OP (Fig. 12.16). The constant h can be determined from the initial conditions and either of the above equations can be solved for one unknown.

(continued)

3. In space-mechanics problems involving the orbital motion of a planet about the sun, or a satellite about the earth, the moon, or some other planet, the central force **F** is the force of gravitational attraction; it is directed *toward* the center of force O and has the magnitude

$$F = G\frac{Mm}{r^2} \qquad (12.28)$$

Note that in the particular case of the gravitational force exerted by the earth, the product GM can be replaced by gR^2, where R is the radius of the earth [Eq. 12.30].

The following two cases of orbital motion are frequently encountered:

a. For a satellite in a circular orbit, the force **F** is normal to the orbit and you can write $F = ma_n$; substituting for F from Eq. (12.28) and observing that $a_n = v^2/\mathrm{r} = v^2/r$, you will obtain

$$G\frac{Mm}{r^2} = m\frac{v^2}{r} \qquad \text{or} \qquad v^2 = \frac{GM}{r}$$

b. For a satellite in an elliptic orbit, the radius vector **r** and the velocity **v** of the satellite are perpendicular to each other at the points A and B which are, respectively, farthest and closest to the center of force O [Sample Prob. 12.8]. Thus, conservation of angular momentum of the satellite between these two points can be expressed as

$$r_A m v_A = r_B m v_B$$

PROBLEMS

FREE-BODY PRACTICE PROBLEMS

12.F9 Four pins slide in four separate slots cut in a horizontal circular plate as shown. When the plate is at rest, each pin has a velocity directed as shown and of the same constant magnitude u. Each pin has a mass m and maintains the same velocity relative to the plate when the plate rotates about O with a constant counterclockwise angular velocity $\boldsymbol{\omega}$. Draw the FBDs and KDs to determine the forces on pins P_1 and P_2.

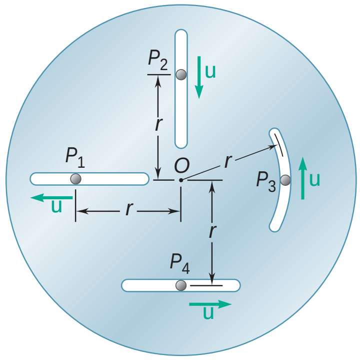


Fig. P12.F9

12.F10 At the instant shown, the length of the boom AB is being *decreased* at the constant rate of 0.2 m/s, and the boom is being lowered at the constant rate of 0.08 rad/s. If the mass of the men and lift connected to the boom at point B is m, draw the FBD and KD that could be used to determine the horizontal and vertical forces at B.

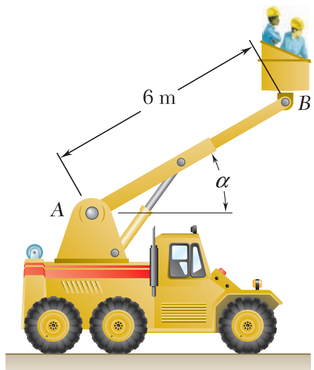


Fig. P12.F10

12.F11 Disk A rotates in a horizontal plane about a vertical axis at the constant rate $\dot{u}_0$. Slider B has a mass m and moves in a frictionless slot cut in the disk. The slider is attached to a spring of constant k, which is undeformed when $r = 0$. Knowing that the slider is released with no radial velocity in the position $r = r_0$, draw a FBD and KD at an arbitrary distance r from O.

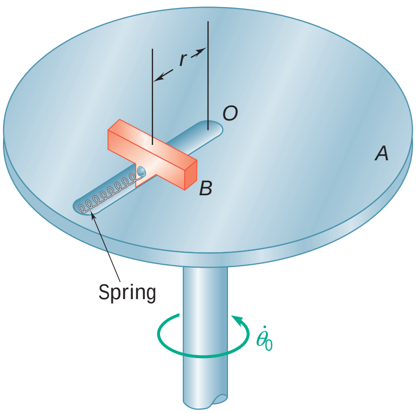


Fig. P12.F11

12.F12 Pin B has a mass m and slides along the slot in the rotating arm OC and along the slot DE which is cut in a fixed horizontal plate. Neglecting friction and knowing that rod OC rotates at the constant rate $\dot{u}_0$, draw a FBD and KD that can be used to determine the forces $\mathbf{P}$ and $\mathbf{Q}$ exerted on pin B by rod OC and the wall of slot DE, respectively.

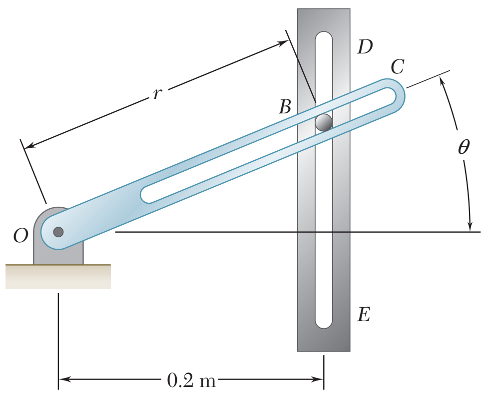


Fig. P12.F12

END-OF-SECTION PROBLEMS

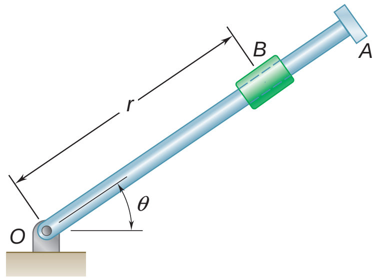


Fig. P12.66 and P12.67

12.66 Rod OA rotates about O in a horizontal plane. The motion of the 0.5-lb collar B is defined by the relations $r = 10 + 6\cos \pi t$ and $\theta = \pi(4t^2 - 8t)$, where r is expressed in inches, t in seconds, and θ in radians. Determine the radial and transverse components of the force exerted on the collar when (*a*) $t = 0$, (*b*) $t = 0.5$ s.

12.67 Rod OA oscillates about O in a horizontal plane. The motion of the 2-lb collar B is defined by the relations $r = 6(1 - e^{-2t})$ and $\theta = (3/\pi)(\sin \pi t)$, where r is expressed in inches, t in seconds, and θ in radians. Determine the radial and transverse components of the force exerted on the collar when (*a*) $t = 1$ s, (*b*) $t = 1.5$ s.

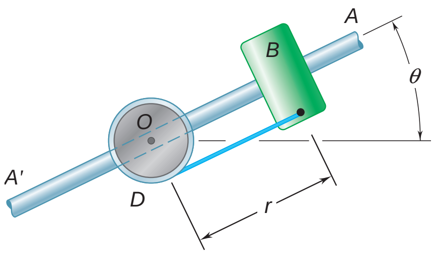


Fig. P12.68

12.68 The 3-kg collar B slides on the frictionless arm AA'. The arm is attached to drum D and rotates about O in a horizontal plane at the rate $\dot{\theta} = 0.75t$, where $\dot{\theta}$ and t are expressed in rad/s and seconds, respectively. As the arm-drum assembly rotates, a mechanism within the drum releases cord so that the collar moves outward from O with a constant speed of 0.5 m/s. Knowing that at $t = 0$, $r = 0$, determine the time at which the tension in the cord is equal to the magnitude of the horizontal force exerted on B by arm AA'.

12.69 The horizontal rod OA rotates about a vertical shaft according to the relation $\dot{\theta} = 10t$, where $\dot{\theta}$ and t are expressed in rad/s and seconds, respectively. A 250-g collar B is held by a cord with a breaking strength of 18 N. Neglecting friction, determine, immediately after the cord breaks, (*a*) the relative acceleration of the collar with respect to the rod, (*b*) the magnitude of the horizontal force exerted on the collar by the rod.

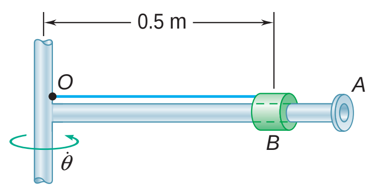


Fig. P12.69

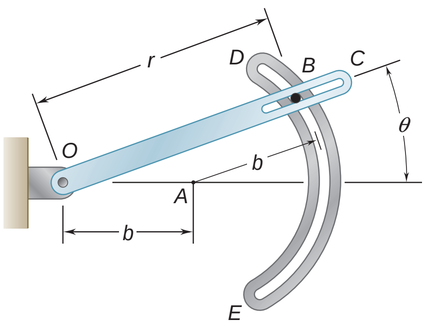


Fig. *P12.70*

12.70 Pin B weighs 4 oz and is free to slide in a horizontal plane along the rotating arm OC and along the circular slot DE of radius $b = 20$ in. Neglecting friction and assuming that $\dot{\theta} = 15$ rad/s and $\ddot{\theta} = 250$ rad/s^2 for the position $\theta = 20°$, determine for that position (*a*) the radial and transverse components of the resultant force exerted on pin B, (*b*) the forces $\mathbf{P}$ and $\mathbf{Q}$ exerted on pin B, respectively, by rod OC and the wall of slot DE.

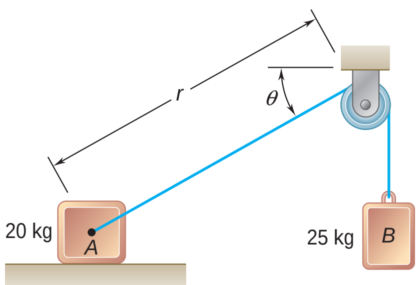


Fig. P12.71 and P12.72

12.71 The two blocks are released from rest when $r = 0.8$ m and $\theta = 30°$. Neglecting the mass of the pulley and the effect of friction in the pulley and between block A and the horizontal surface, determine (*a*) the initial tension in the cable, (*b*) the initial acceleration of block A, (*c*) the initial acceleration of block B.

12.72 The velocity of block A is 2 m/s to the right at the instant when $r = 0.8$ m and $\theta = 30°$. Neglecting the mass of the pulley and the effect of friction in the pulley and between block A and the horizontal surface, determine, at this instant, (*a*) the tension in the cable, (*b*) the acceleration of block A, (*c*) the acceleration of block B.

***12.73** Slider C has a weight of 0.5 lb and may move in a slot cut in arm AB, which rotates at the constant rate $\dot{u}_0 = 10$ rad/s in a horizontal plane. The slider is attached to a spring of constant $k = 2.5$ lb/ft, which is unstretched when $r = 0$. Knowing that the slider is released from rest with no radial velocity in the position $r = 18$ in. and neglecting friction, determine for the position $r = 12$ in. (*a*) the radial and transverse components of the velocity of the slider, (*b*) the radial and transverse components of its acceleration, (*c*) the horizontal force exerted on the slider by arm AB.

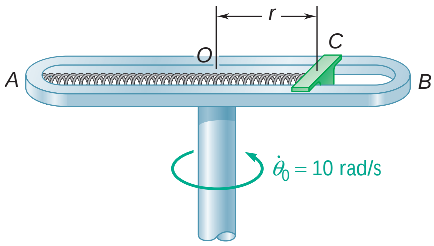


Fig. P12.73

12.74 A particle of mass m is projected from point A with an initial velocity $\mathbf{v}_0$ perpendicular to line OA and moves under a central force $\mathbf{F}$ directed away from the center of force O. Knowing that the particle follows a path defined by the equation $r = r_0/\sqrt{\cos 2u}$ and using Eq. (12.27), express the radial and transverse components of the velocity $\mathbf{v}$ of the particle as functions of u.

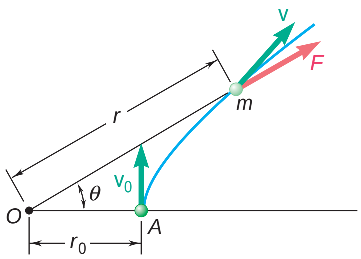


Fig. P12.74

12.75 For the particle of Prob. 12.74, show (*a*) that the velocity of the particle and the central force $\mathbf{F}$ are proportional to the distance r from the particle to the center of force O, (*b*) that the radius of curvature of the path is proportional to r^3.

12.76 A particle of mass m is projected from point A with an initial velocity $\mathbf{v}_0$ perpendicular to line OA and moves under a central force $\mathbf{F}$ along a semicircular path of diameter OA. Observing that $r = r_0 \cos u$ and using Eq. (12.27), show that the speed of the particle is $v = v_0/\cos^2 u$.

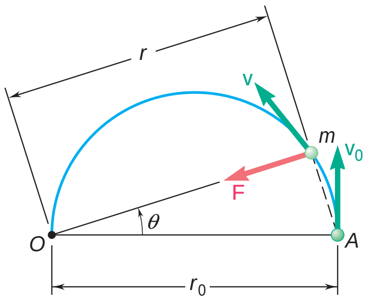


Fig. P12.76

12.77 For the particle of Prob. 12.76, determine the tangential component F_t of the central force $\mathbf{F}$ along the tangent to the path of the particle for (*a*) $u = 0$, (*b*) $u = 45°$.

12.78 Determine the mass of the earth knowing that the mean radius of the moon's orbit about the earth is 238,910 mi and that the moon requires 27.32 days to complete one full revolution about the earth.

12.79 Show that the radius r of the moon's orbit can be determined from the radius R of the earth, the acceleration of gravity g at the surface of the earth, and the time t required for the moon to complete one full revolution about the earth. Compute r knowing that $t = 27.3$ days, giving the answer in both SI and U.S. customary units.

12.80 Communication satellites are placed in a geosynchronous orbit, i.e., in a circular orbit such that they complete one full revolution about the earth in one sidereal day (23.934 h), and thus appear stationary with respect to the ground. Determine (*a*) the altitude of these satellites above the surface of the earth, (*b*) the velocity with which they describe their orbit. Give the answers in both SI and U.S. customary units.

12.81 Show that the radius r of the orbit of a moon of a given planet can be determined from the radius R of the planet, the acceleration of gravity at the surface of the planet, and the time τ required by the moon to complete one full revolution about the planet. Determine the acceleration of gravity at the surface of the planet Jupiter knowing that $R = 71\ 492$ km and that $\tau = 3.551$ days and $r = 670.9 \times 10^3$ km for its moon Europa.

12.82 The orbit of the planet Venus is nearly circular with an orbital velocity of 126.5×10^3 km/h. Knowing that the mean distance from the center of the sun to the center of Venus is 108×10^6 km and that the radius of the sun is 695.5×10^3 km, determine (*a*) the mass of the sun, (*b*) the acceleration of gravity at the surface of the sun.

12.83 A satellite is placed into a circular orbit about the planet Saturn at an altitude of 2100 mi. The satellite describes its orbit with a velocity of 54.7×10^3 mi/h. Knowing that the radius of the orbit about Saturn and the periodic time of Atlas, one of Saturn's moons, are 85.54×10^3 mi and 0.6017 days, respectively, determine (*a*) the radius of Saturn, (*b*) the mass of Saturn. (The *periodic time* of a satellite is the time it requires to complete one full revolution about the planet.)

12.84 The periodic times (see Prob. 12.83) of the planet Uranus's moons Juliet and Titania have been observed to be 0.4931 days and 8.706 days, respectively. Knowing that the radius of Juliet's orbit is 40,000 mi, determine (*a*) the mass of Uranus, (*b*) the radius of Titania's orbit.

12.85 A 500-kg spacecraft first is placed into a circular orbit about the earth at an altitude of 4500 km and then is transferred to a circular orbit about the moon. Knowing that the mass of the moon is 0.01230 times the mass of the earth and that the radius of the moon is 1737 km, determine (*a*) the gravitational force exerted on the spacecraft as it was orbiting the earth, (*b*) the required radius of the orbit of the spacecraft about the moon if the periodic times (see Prob. 12.83) of the two orbits are to be equal, (*c*) the acceleration of gravity at the surface of the moon.

12.86 A space vehicle is in a circular orbit of 2200-km radius around the moon. To transfer it to a smaller circular orbit of 2080-km radius, the vehicle is first placed on an elliptic path AB by reducing its speed by 26.3 m/s as it passes through A. Knowing that the mass of the moon is 73.49×10^{21} kg, determine (*a*) the speed of the vehicle as it approaches B on the elliptic path, (*b*) the amount by which its speed should be reduced as it approaches B to insert it into the smaller circular orbit.

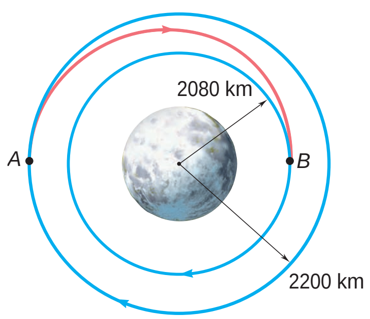


Fig. P12.86

12.87 Plans for an unmanned landing mission on the planet Mars called for the earth-return vehicle to first describe a circular orbit at an altitude $d_A = 2200$ km above the surface of the planet with a velocity of 2771 m/s. As it passed through point A, the vehicle was to be inserted into an elliptic transfer orbit by firing its engine and increasing its speed by $\Delta v_A = 1046$ m/s. As it passed through point B, at an altitude $d_B = 100\,000$ km, the vehicle was to be inserted into a second transfer orbit located in a slightly different plane, by changing the direction of its velocity and reducing its speed by $\Delta v_B = -22.0$ m/s. Finally, as the vehicle passed through point C, at an altitude $d_C = 1000$ km, its speed was to be increased by $\Delta v_C = 660$ m/s to insert it into its return trajectory. Knowing that the radius of the planet Mars is $R = 3400$ km, determine the velocity of the vehicle after completion of the last maneuver.

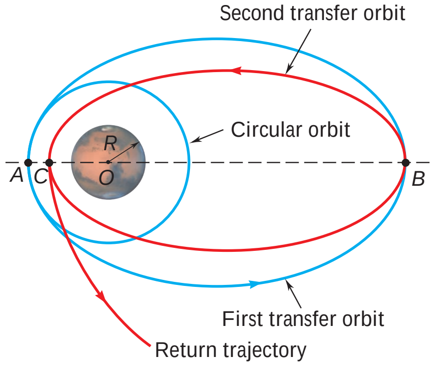


Fig. P12.87

12.88 To place a communications satellite into a geosynchronous orbit (see Prob. 12.80) at an altitude of 22,240 mi above the surface of the earth, the satellite first is released from a space shuttle, which is in a circular orbit at an altitude of 185 mi, and then is propelled by an upper-stage booster to its final altitude. As the satellite passes through A, the booster's motor is fired to insert the satellite into an elliptic transfer orbit. The booster is again fired at B to insert the satellite into a geosynchronous orbit. Knowing that the second firing increases the speed of the satellite by 4810 ft/s, determine (*a*) the speed of the satellite as it approaches B on the elliptic transfer orbit, (*b*) the increase in speed resulting from the first firing at A.

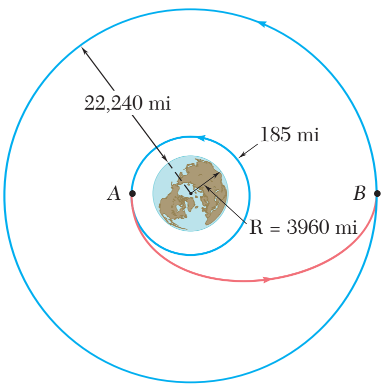


Fig. P12.88

12.89 A space shuttle S and a satellite A are in the circular orbits shown. In order for the shuttle to recover the satellite, the shuttle is first placed in an elliptic path BC by increasing its speed by $\Delta v_B = 280$ ft/s as it passes through B. As the shuttle approaches C, its speed is increased by $\Delta v_C = 260$ ft/s to insert it into a second elliptic transfer orbit CD. Knowing that the distance from O to C is 4289 mi, determine the amount by which the speed of the shuttle should be increased as it approaches D to insert it into the circular orbit of the satellite.

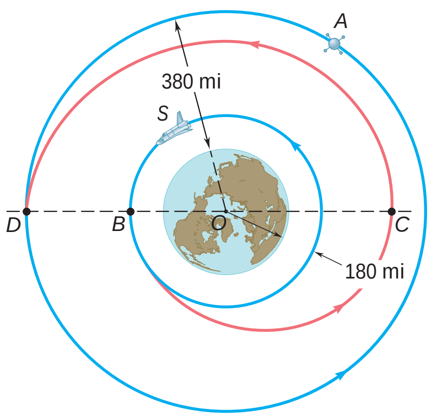


Fig. P12.89

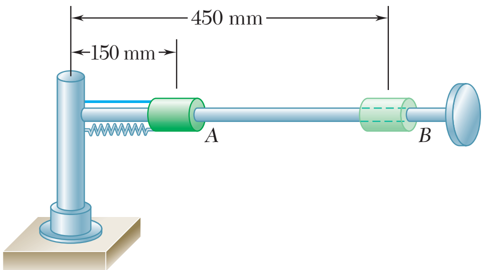


Fig. P12.90

12.90 A 1-kg collar can slide on a horizontal rod which is free to rotate about a vertical shaft. The collar is initially held at *A* by a cord attached to the shaft. A spring of constant 30 N/m is attached to the collar and to the shaft and is undeformed when the collar is at *A*. As the rod rotates at the rate $\dot{\theta}$ = 16 rad/s, the cord is cut and the collar moves out along the rod. Neglecting friction and the mass of the rod, determine (*a*) the radial and transverse components of the acceleration of the collar at *A*, (*b*) the acceleration of the collar relative to the rod at *A*, (*c*) the transverse component of the velocity of the collar at *B*.

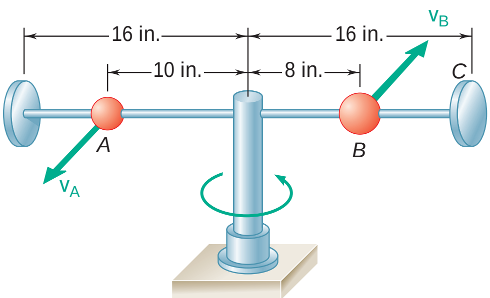


Fig. P12.91

12.91 A 1-lb ball *A* and a 2-lb ball *B* are mounted on a horizontal rod which rotates freely about a vertical shaft. The balls are held in the positions shown by pins. The pin holding *B* is suddenly removed and the ball moves to position *C* as the rod rotates. Neglecting friction and the mass of the rod and knowing that the initial speed of *A* is v_A = 8 ft/s, determine (*a*) the radial and transverse components of the acceleration of ball *B* immediately after the pin is removed, (*b*) the acceleration of ball *B* relative to the rod at that instant, (*c*) the speed of ball *A* after ball *B* has reached the stop at *C*.

12.92 Two 2.6-lb collars *A* and *B* can slide without friction on a frame, consisting of the horizontal rod *OE* and the vertical rod *CD*, which is free to rotate about *CD*. The two collars are connected by a cord running over a pulley that is attached to the frame at *O* and a stop prevents collar *B* from moving. The frame is rotating at the rate $\dot{\theta}$ = 12 rad/s and r = 0.6 ft when the stop is removed allowing collar *A* to move out along rod *OE*. Neglecting friction and the mass of the frame, determine, for the position r = 1.2 ft, (*a*) the transverse component of the velocity of collar *A*, (*b*) the tension in the cord and the acceleration of collar *A* relative to the rod *OE*.

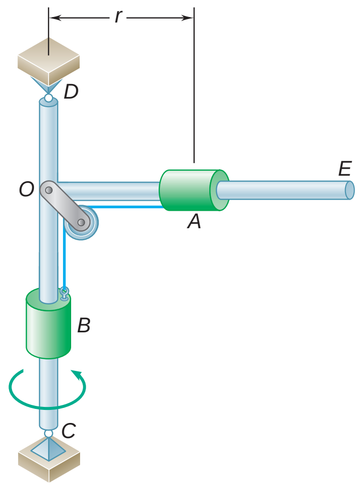


Fig. *P12.92*

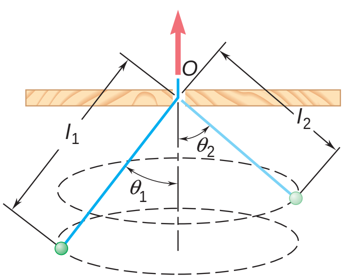


Fig. *P12.93*

12.93 A small ball swings in a horizontal circle at the end of a cord of length l_1, which forms an angle θ_1 with the vertical. The cord is then slowly drawn through the support at *O* until the length of the free end is l_2. (*a*) Derive a relation among l_1, l_2, θ_1, and θ_2. (*b*) If the ball is set in motion so that initially l_1 = 0.8 m and θ_1 = 35°, determine the angle θ_2 when l_2 = 0.6 m.

*12.11 TRAJECTORY OF A PARTICLE UNDER A CENTRAL FORCE

Consider a particle P moving under a central force $\mathbf{F}$. We propose to obtain the differential equation which defines its trajectory.

Assuming that the force $\mathbf{F}$ is directed toward the center of force O, we note that ΣF_r and ΣF_θ reduce, respectively, to $-F$ and zero in Eqs. (12.21) and (12.22). We therefore write

$$m(\ddot{r} - r\dot{\theta}^2) = -F \tag{12.31}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{12.32}$$

These equations define the motion of P. We will, however, replace Eq. (12.32) by Eq. (12.27), which is equivalent to Eq. (12.32), as can easily be checked by differentiating it with respect to t, but which is more convenient to use. We write

$$r^2\dot{\theta} = h \qquad \text{or} \qquad r^2\frac{d\theta}{dt} = h \tag{12.33}$$

Equation (12.33) can be used to eliminate the independent variable t from Eq. (12.31). Solving Eq. (12.33) for $\dot{\theta}$ or $d\theta/dt$, we have

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{h}{r^2} \tag{12.34}$$

from which it follows that

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right) \tag{12.35}$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta}$$

or, substituting for $\dot{r}$ from (12.35),

$$\ddot{r} = \frac{h}{r^2}\frac{d}{d\theta}\left[-h\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]$$

$$\ddot{r} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \tag{12.36}$$

Substituting for θ and $\ddot{r}$ from (12.34) and (12.36), respectively, in Eq. (12.31) and introducing the function $u = 1/r$, we obtain after reductions

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \tag{12.37}$$

In deriving Eq. (12.37), the force $\mathbf{F}$ was assumed directed toward O. The magnitude F should therefore be positive if $\mathbf{F}$ is actually directed toward O (attractive force) and negative if $\mathbf{F}$ is directed away from O (repulsive force). If F is a known function of r and thus of u, Eq. (12.37) is a differential equation in u and θ. This differential equation defines the trajectory followed by the particle under the central force $\mathbf{F}$. The equation of the trajectory can be obtained by solving the differential equation (12.37) for u as a function of θ and determining the constants of integration from the initial conditions.

Photo 12.5 The Hubble telescope was carried into orbit by the space shuttle in 1990 (first geosynchronous from NASA).

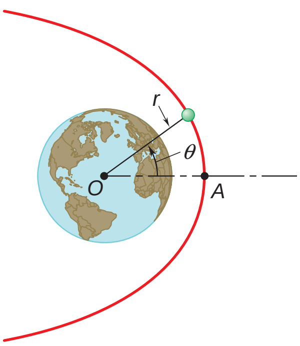


Fig. 12.19

*12.12 APPLICATION TO SPACE MECHANICS

After the last stages of their launching rockets have burned out, earth satellites and other space vehicles are subjected to only the gravitational pull of the earth. Their motion can therefore be determined from Eqs. (12.33) and (12.37), which govern the motion of a particle under a central force, after F has been replaced by the expression obtained for the force of gravitational attraction.† Setting in Eq. (12.37)

$$F = \frac{GMm}{r^2} = GMmu^2$$

where M = mass of earth
m = mass of space vehicle
r = distance from center of earth to vehicle
$u = 1/r$

we obtain the differential equation

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}} \tag{12.38}$$

where the right-hand member is observed to be a constant.

The solution of the differential equation (12.38) is obtained by adding the particular solution $u = GM/h^2$ to the general solution $u = C\cos(\theta - \theta_0)$ of the corresponding homogeneous equation (i.e., the equation obtained by setting the right-hand member equal to zero). Choosing the polar axis so that $\theta_0 = 0$, we write

$$\frac{1}{r} = u = \frac{GM}{h^2} + C\cos\theta \tag{12.39}$$

Equation (12.39) is the equation of a *conic section* (ellipse, parabola, or hyperbola) in the polar coordinates r and θ. The origin O of the coordinates, which is located at the center of the earth, is a *focus* of this conic section, and the polar axis is one of its axes of symmetry (Fig. 12.19).

The ratio of the constants C and GM/h^2 defines the *eccentricity* ε of the conic section; letting

$$\varepsilon = \frac{C}{GM/h^2} = \frac{Ch^2}{GM} \tag{12.40}$$

we can write Eq. (12.39) in the form

$$\frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon\cos\theta) \tag{12.39$'$}$$

This equation represents three possible trajectories.

1. $\varepsilon > 1$, or $C > GM/h^2$: There are two values θ_1 and $-\theta_1$ of the polar angle, defined by $\cos\theta_1 = -GM/Ch^2$, for which the

†It is assumed that the space vehicles considered here are attracted by the earth only and that their mass is negligible compared with the mass of the earth. If a vehicle moves very far from the earth, its path may be affected by the attraction of the sun, the moon, or another planet.

right-hand member of Eq. (12.39) becomes zero. For both these values, the radius vector r becomes infinite; the conic section is a *hyperbola* (Fig. 12.20).

2. $\varepsilon = 1$, or $C = GM/h^2$: The radius vector becomes infinite for $\theta = 180°$; the conic section is a *parabola.*

3. $\varepsilon < 1$, or $C < GM/h^2$: The radius vector remains finite for every value of θ; the conic section is an *ellipse.* In the particular case when $\varepsilon = C = 0$, the length of the radius vector is constant; the conic section is a circle.

Let us now see how the constants C and GM/h^2, which characterize the trajectory of a space vehicle, can be determined from the vehicle's position and velocity at the beginning of its free flight. We will assume that, as is generally the case, the powered phase of its flight has been programmed in such a way that as the last stage of the launching rocket burns out, the vehicle has a velocity parallel to the surface of the earth (Fig. 12.21). In other words, we will assume that the space vehicle begins its free flight at the vertex A of its trajectory.†

Denoting the radius vector and speed of the vehicle at the beginning of its free flight by r_0 and v_0, respectively, we observe that the velocity reduces to its transverse component and, thus, that $v_0 = r_0\dot{\theta}_0$. Recalling Eq. (12.27), we express the angular momentum per unit mass h as

$$h = r_0^2\dot{\theta}_0 = r_0 v_0 \tag{12.41}$$

The value obtained for h can be used to determine the constant GM/h^2. We also note that the computation of this constant will be simplified if we use the relation obtained in Sec. 12.10.

$$GM = gR^2 \tag{12.30}$$

where R is the radius of the earth ($R = 6.37 \times 10^6$ m or 3960 mi) and g is the acceleration of gravity at the surface of the earth.

The constant C is obtained by setting $\theta = 0$, $r = r_0$ in (12.39):

$$C = \frac{1}{r_0} - \frac{GM}{h^2} \tag{12.42}$$

Substituting for h from (12.41), we can then easily express C in terms of r_0 and v_0.

Let us now determine the initial conditions corresponding to each of the three fundamental trajectories indicated above. Considering first the parabolic trajectory, we set C equal to GM/h^2 in Eq. (12.42) and eliminate h between Eqs. (12.41) and (12.42). Solving for v_0, we obtain

$$v_0 = \sqrt{\frac{2GM}{r_0}}$$

We can easily check that a larger value of the initial velocity corresponds to a hyperbolic trajectory and a smaller value corresponds to an elliptic orbit. Since the value of v_0 obtained for the parabolic trajectory

†Problems involving oblique launchings will be considered in Sec. 13.9.

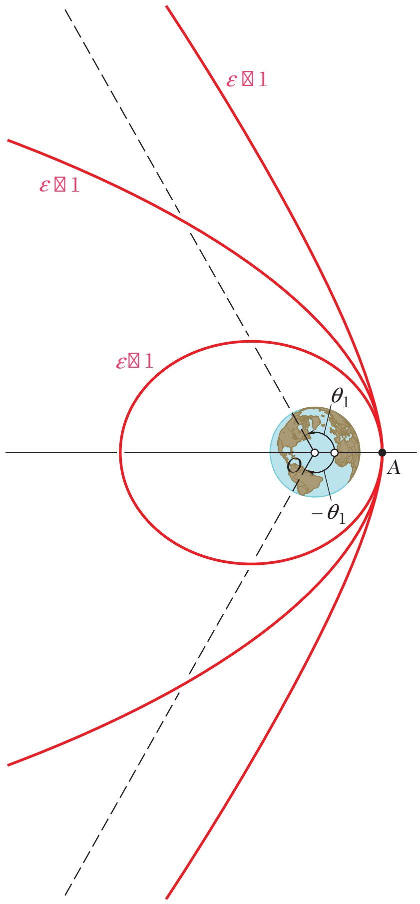


Fig. 12.20

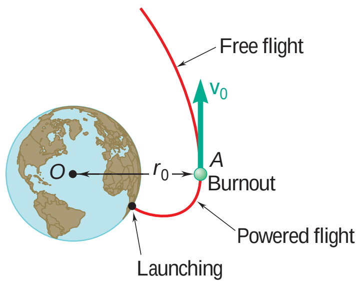


Fig. 12.21

is the smallest value for which the space vehicle does not return to its starting point, it is called the *escape velocity.* We write therefore

$$v_{esc} = \sqrt{\frac{2GM}{r_0}} \qquad \text{or} \qquad v_{esc} = \sqrt{\frac{2gR^2}{r_0}} \tag{12.43}$$

if we make use of Eq. (12.30). We note that the trajectory will be (1) hyperbolic if $v_0 > v_{esc}$, (2) parabolic if $v_0 = v_{esc}$, and (3) elliptic if $v_0 < v_{esc}$.

Among the various possible elliptic orbits, the one obtained when $C = 0$, the *circular orbit,* is of special interest. The value of the initial velocity corresponding to a circular orbit is easily found to be

$$v_{circ} = \sqrt{\frac{GM}{r_0}} \qquad \text{or} \qquad v_{circ} = \sqrt{\frac{gR^2}{r_0}} \tag{12.44}$$

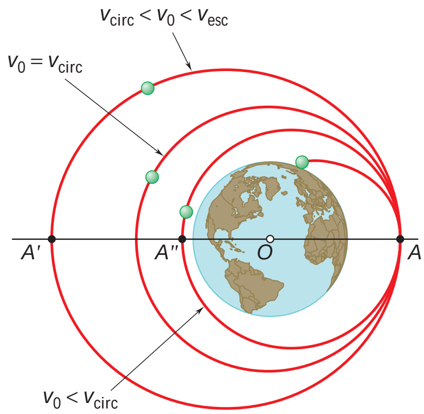


Fig. 12.22

if Eq. (12.30) is taken into account. We note from Fig. 12.22 that for values of v_0 larger than v_{circ} but smaller than v_{esc}, point A where free flight begins is the point of the orbit closest to the earth; this point is called the *perigee,* while point A', which is farthest away from the earth, is known as the *apogee.* For values of v_0 smaller than v_{circ}, point A is the apogee, while point A'', on the other side of the orbit, is the perigee. For values of v_0 much smaller than v_{circ}, the trajectory of the space vehicle intersects the surface of the earth; in such a case, the vehicle does not go into orbit.

Ballistic missiles, which were designed to hit the surface of the earth, also travel along elliptic trajectories. In fact, we should now realize that any object projected in vacuum with an initial velocity v_0 smaller than v_{esc} will move along an elliptic path. It is only when the distances involved are small that the gravitational field of the earth can be assumed uniform and that the elliptic path can be approximated by a parabolic path, as was done earlier (Sec. 11.11) in the case of conventional projectiles.

Periodic Time. An important characteristic of the motion of an earth satellite is the time required by the satellite to describe its orbit. This time, known as the *periodic time* of the satellite, is denoted by τ. We first observe, in view of the definition of areal velocity (Sec. 12.9), that τ can be obtained by dividing the area inside the orbit by the areal velocity. Noting that the area of an ellipse is equal to πab, where a and b denote the semimajor and semiminor axes, respectively, and that the areal velocity is equal to $h/2$, we write

$$\tau = \frac{2\pi ab}{h} \tag{12.45}$$

While h can be readily determined from r_0 and v_0 in the case of a satellite launched in a direction parallel to the surface of the earth, the semiaxes a and b are not directly related to the initial conditions. Since, on the other hand, the values r_0 and r_1 of r corresponding to the perigee and apogee of the orbit can easily be determined from Eq. (12.39), we will express the semiaxes a and b in terms of r_0 and r_1.

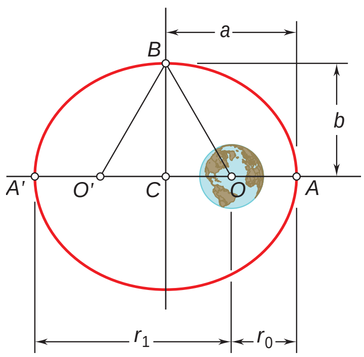


Fig. 12.23

Consider the elliptic orbit shown in Fig. 12.23. The earth's center is located at O and coincides with one of the two foci of the ellipse, while the points A and A' represent, respectively, the perigee and apogee of the orbit. We easily check that

$$r_0 + r_1 = 2a$$

and thus

$$a = \tfrac{1}{2}(r_0 + r_1) \tag{12.46}$$

Recalling that the sum of the distances from each of the foci to any point of the ellipse is constant, we write

$$O'B + BO = O'A + OA = 2a \qquad \text{or} \qquad BO = a$$

On the other hand, we have $CO = a - r_0$. We can therefore write

$$b^2 = (BC)^2 = (BO)^2 - (CO)^2 = a^2 - (a - r_0)^2$$
$$b^2 = r_0(2a - r_0) = r_0 r_1$$

and thus

$$b = \sqrt{r_0 r_1} \tag{12.47}$$

Formulas (12.46) and (12.47) indicate that the semimajor and semiminor axes of the orbit are equal, respectively, to the arithmetic and geometric means of the maximum and minimum values of the radius vector. Once r_0 and r_1 have been determined, the lengths of the semiaxes can be easily computed and substituted for a and b in formula (12.45).

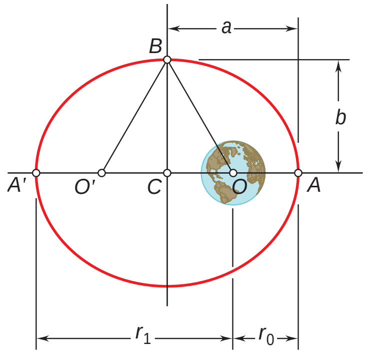


Fig. 12.23 (*repeated*)

*12.13 KEPLER'S LAWS OF PLANETARY MOTION

The equations governing the motion of an earth satellite can be used to describe the motion of the moon around the earth. In that case, however, the mass of the moon is not negligible compared with the mass of the earth, and the results obtained are not entirely accurate.

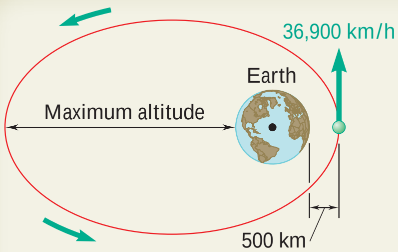
The theory developed in the preceding sections can also be applied to the study of the motion of the planets around the sun. Although another error is introduced by neglecting the forces exerted by the planets on one another, the approximation obtained is excellent. Indeed, even before Newton had formulated his fundamental theory, the properties expressed by Eq. (12.39), where M now represents the mass of the sun, and by Eq. (12.33) had been discovered by the German astronomer Johann Kepler (1571–1630) from astronomical observations of the motion of the planets.

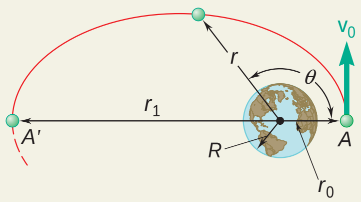
Kepler's three *laws of planetary motion* can be stated as follows:

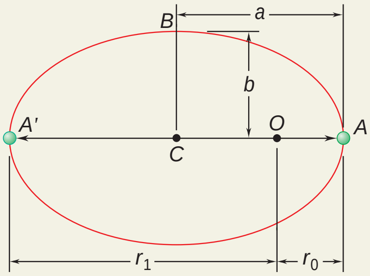
1. Each planet describes an ellipse, with the sun located at one of its foci.
2. The radius vector drawn from the sun to a planet sweeps equal areas in equal times.
3. The squares of the periodic times of the planets are proportional to the cubes of the semimajor axes of their orbits.

The first law states a particular case of the result established in Sec. 12.12, and the second law expresses that the areal velocity of each planet is constant (see Sec. 12.9). Kepler's third law can also be derived from the results obtained in Sec. 12.12.†

†See Prob. 12.120.







SAMPLE PROBLEM 12.9

A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36 900 km/h from an altitude of 500 km. Determine *(a)* the maximum altitude reached by the satellite, *(b)* the periodic time of the satellite.

SOLUTION

a. Maximum Altitude. After the satellite is launched, it is subjected only to the gravitational attraction of the earth; its motion is thus governed by Eq. (12.39),

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta \qquad (1)$$

Since the radial component of the velocity is zero at the point of launching A, we have $h = r_0 v_0$. Recalling that for the earth $R = 6370$ km, we compute

$$r_0 = 6370 \text{ km} + 500 \text{ km} = 6870 \text{ km} = 6.87 \times 10^6 \text{ m}$$

$$v_0 = 36\,900 \text{ km/h} = \frac{36.9 \times 10^6 \text{ m}}{3.6 \times 10^3 \text{ s}} = 10.25 \times 10^3 \text{ m/s}$$

$$h = r_0 v_0 = (6.87 \times 10^6 \text{ m})(10.25 \times 10^3 \text{ m/s}) = 70.4 \times 10^9 \text{ m}^2/\text{s}$$

$$h^2 = 4.96 \times 10^{21} \text{ m}^4/\text{s}^2$$

Since $GM = gR^2$, where R is the radius of the earth, we have

$$GM = gR^2 = (9.81 \text{ m/s}^2)(6.37 \times 10^6 \text{ m})^2 = 398 \times 10^{12} \text{ m}^3/\text{s}^2$$

$$\frac{GM}{h^2} = \frac{398 \times 10^{12} \text{ m}^3/\text{s}^2}{4.96 \times 10^{21} \text{ m}^4/\text{s}^2} = 80.3 \times 10^{-9} \text{ m}^{-1}$$

Substituting this value into (1), we obtain

$$\frac{1}{r} = 80.3 \times 10^{-9} \text{ m}^{-1} + C\cos\theta \qquad (2)$$

Noting that at point A we have $\theta = 0$ and $r = r_0 = 6.87 \times 10^6$ m, we compute the constant C:

$$\frac{1}{6.87 \times 10^6 \text{ m}} = 80.3 \times 10^{-9} \text{ m}^{-1} + C\cos 0° \qquad C = 65.3 \times 10^{-9} \text{ m}^{-1}$$

At A', the point on the orbit farthest from the earth, we have $\theta = 180°$. Using (2), we compute the corresponding distance r_1:

$$\frac{1}{r_1} = 80.3 \times 10^{-9} \text{ m}^{-1} + (65.3 \times 10^{-9} \text{ m}^{-1})\cos 180°$$

$$r_1 = 66.7 \times 10^6 \text{ m} = 66\,700 \text{ km}$$

Maximum altitude = 66 700 km − 6370 km = **60 300 km** ◀

b. Periodic Time. Since A and A' are the perigee and apogee, respectively, of the elliptic orbit, we use Eqs. (12.46) and (12.47) and compute the semimajor and semiminor axes of the orbit:

$$a = \tfrac{1}{2}(r_0 + r_1) = \tfrac{1}{2}(6.87 + 66.7)(10^6) \text{ m} = 36.8 \times 10^6 \text{ m}$$

$$b = \sqrt{r_0 r_1} = \sqrt{(6.87)(66.7)} \times 10^6 \text{ m} = 21.4 \times 10^6 \text{ m}$$

$$\tau = \frac{2\pi ab}{h} = \frac{2\pi(36.8 \times 10^6 \text{m})(21.4 \times 10^6 \text{m})}{70.4 \times 10^9 \text{ m}^2/\text{s}}$$

$$\tau = 70.3 \times 10^3 \text{ s} = 1171 \text{ min} = 19 \text{ h } 31 \text{ min}$$ ◀

In this lesson, we continued our study of the motion of a particle under a central force and applied the results to problems in space mechanics. We found that the trajectory of a particle under a central force is defined by the differential equation

$$\frac{d^2u}{d\mathsf{u}^2} + u = \frac{F}{mh^2u^2} \tag{12.37}$$

where u is the reciprocal of the distance r of the particle to the center of force ($u = 1/r$), F is the magnitude of the central force $\mathbf{F}$, and h is a constant equal to the angular momentum per unit mass of the particle. In space-mechanics problems, $\mathbf{F}$ is the force of gravitational attraction exerted on the satellite or spacecraft by the sun, earth, or other planet about which it travels. Substituting $F = GMm/r^2 = GMmu^2$ into Eq. (12.37), we obtain for that case

$$\frac{d^2u}{d\mathsf{u}^2} + u = \frac{GM}{h^2} \tag{12.38}$$

where the right-hand member is a constant.

1. Analyzing the motion of satellites and spacecraft. The solution of the differential equation (12.38) defines the trajectory of a satellite or spacecraft. It was obtained in Sec. 12.12 and was given in the alternative forms

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\mathsf{u} \qquad \text{or} \qquad \frac{1}{r} = \frac{GM}{h^2}(1 + \mathsf{e}\cos\mathsf{u}) \qquad (12.39,\ 12.39')$$

Remember when applying these equations that $\mathsf{u} = 0$ always corresponds to the perigee (the point of closest approach) of the trajectory (Fig. 12.19) and that h is a constant for a given trajectory. Depending on the value of the eccentricity ε, the trajectory will be a hyperbola, a parabola, or an ellipse.

a. $\mathsf{E} > 1$**: The trajectory is a hyperbola,** so that for this case the spacecraft never returns to its starting point.

b. $\mathsf{E} = 1$**: The trajectory is a parabola.** This is the limiting case between open (hyperbolic) and closed (elliptic) trajectories. We had observed for this case that the velocity v_0 at the perigee is equal to the escape velocity v_{esc},

$$v_0 = v_{esc} = \sqrt{\frac{2GM}{r_0}} \tag{12.43}$$

Note that the escape velocity is the smallest velocity for which the spacecraft does not return to its starting point.

c. $\mathsf{E} < 1$**: The trajectory is an elliptic orbit.** For problems involving elliptic orbits, you may find that the relation derived in Prob. 12.102,

$$\frac{1}{r_0} + \frac{1}{r_1} = \frac{2GM}{h^2}$$

(*continued*)

will be useful in the solution of subsequent problems. When you apply this equation, remember that r_0 and r_1 are the distances from the center of force to the perigee ($\theta = 0$) and apogee ($\theta = 180°$), respectively; that $h = r_0 v_0 = r_1 v_1$; and that, for a satellite orbiting the earth, $GM_{earth} = gR^2$, where R is the radius of the earth. Also recall that the trajectory is a circle when $\varepsilon = 0$.

2. Determining the point of impact of a descending spacecraft. For problems of this type, you may assume that the trajectory is elliptic and that the initial point of the descent trajectory is the apogee of the path (Fig. 12.22). Note that at the point of impact, the distance r in Eqs. (12.39) and (12.39′) is equal to the radius R of the body on which the spacecraft lands or crashes. In addition, we have $h = Rv_I \sin \phi_I$, where v_I is the speed of the spacecraft at impact and ϕ_I is the angle that its path forms with the vertical at the point of impact.

3. Calculating the time to travel between two points on a trajectory. For central force motion, the time t required for a particle to travel along a portion of its trajectory can be determined by recalling from Sec. 12.9 that the rate at which area is swept per unit time by the position vector $\mathbf{r}$ is equal to one-half of the angular momentum per unit mass h of the particle: $dA/dt = h/2$. It follows, since h is a constant for a given trajectory, that

$$t = \frac{2A}{h}$$

where A is the total area swept in the time t.

a. In the case of an elliptic trajectory, the time required to complete one orbit is called the *periodic time* and is expressed as

$$\tau = \frac{2(\pi ab)}{h} \tag{12.45}$$

where a and b are the semimajor and semiminor axes, respectively, of the ellipse and are related to the distances r_0 and r_1 by

$$a = \tfrac{1}{2}(r_0 + r_1) \qquad \text{and} \qquad b = \sqrt{r_0 r_1} \qquad (12.46, 12.47)$$

b. Kepler's third law provides a convenient relation between the periodic times of two satellites describing elliptic orbits about the same body [Sec. 12.13]. Denoting the semimajor axes of the two orbits by a_1 and a_2, respectively, and the corresponding periodic times by τ_1 and τ_2, we have

$$\frac{\tau_1^2}{\tau_2^2} = \frac{a_1^3}{a_2^3}$$

c. In the case of a parabolic trajectory, you may be able to use the expression given on the inside of the front cover of the book for a parabolic or a semiparabolic area to calculate the time required to travel between two points of the trajectory.

PROBLEMS

CONCEPTS QUESTIONS

12.CQ6 A uniform crate C with mass m_C is being transported to the left by a forklift with a constant speed v_1. What is the magnitude of the angular momentum of the crate about point D, that is, the upper left corner of the crate?

a. 0

b. mv_1a

c. mv_1b

d. $mv_1\sqrt{a^2 + b^2}$

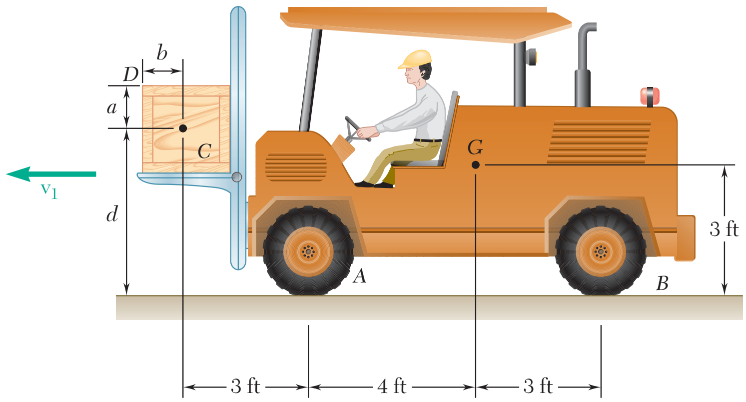


Fig. P12.CQ6 and P12.CQ7

12.CQ7 A uniform crate C with mass m_C is being transported to the left by a forklift with a constant speed v_1. What is the magnitude of the angular momentum of the crate about point A, that is, the point of contact between the front tire of the forklift and the ground?

a. 0

b. mv_1d

c. $3mv_1$

d. $mv_1\sqrt{3^2 + d^2}$

END-OF-SECTION PROBLEMS

12.94 A particle of mass m is projected from point A with an initial velocity $\mathbf{v}_0$ perpendicular to OA and moves under a central force $\mathbf{F}$ along an elliptic path defined by the equation $r = r_0/(2 - \cos\theta)$. Using Eq. (12.37), show that $\mathbf{F}$ is inversely proportional to the square of the distance r from the particle to the center of force O.

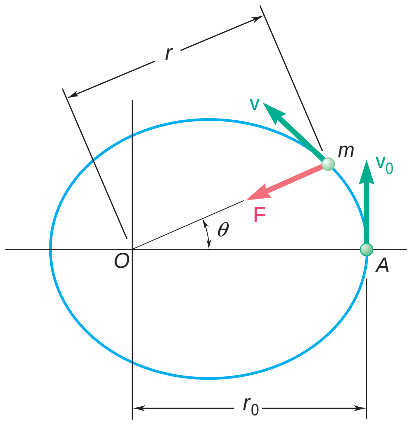


Fig. P12.94

12.95 A particle of mass m describes the logarithmic spiral $r = r_0 e^{bu}$ under a central force **F** directed toward the center of force O. Using Eq. (12.37), show that **F** is inversely proportional to the cube of the distance r from the particle to O.

12.96 For the particle of Prob. 12.74, and using Eq. (12.37), show that the central force **F** is proportional to the distance r from the particle to the center of force O.

12.97 A particle of mass m describes the path defined by the equation $r = r_0 \sin u$ under a central force **F** directed toward the center of force O. Using Eq. (12.37), show that **F** is inversely proportional to the fifth power of the distance r from the particle to O.

12.98 It was observed that during its second flyby of the earth, the Galileo spacecraft had a velocity of 14.1 km/s as it reached its minimum altitude of 303 km above the surface of the earth. Determine the eccentricity of the trajectory of the spacecraft during this portion of its flight.

12.99 It was observed that during the Galileo spacecraft's first flyby of the earth, its minimum altitude was 600 mi above the surface of the earth. Assuming that the trajectory of the spacecraft was parabolic, determine the maximum velocity of Galileo during its first flyby of the earth.

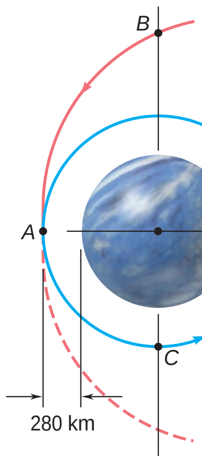


Fig. P12.100

12.100 As a space probe approaching the planet Venus on a parabolic trajectory reaches point A closest to the planet, its velocity is decreased to insert it into a circular orbit. Knowing that the mass and the radius of Venus are 4.87×10^{24} kg and 6052 km, respectively, determine (*a*) the velocity of the probe as it approaches A, (*b*) the decrease in velocity required to insert it into the circular orbit.

12.101 It was observed that as the Voyager I spacecraft reached the point of its trajectory closest to the planet Saturn, it was at a distance of 185×10^3 km from the center of the planet and had a velocity of 21.0 km/s. Knowing that Tethys, one of Saturn's moons, describes a circular orbit of radius 295×10^3 km at a speed of 11.35 km/s, determine the eccentricity of the trajectory of Voyager I on its approach to Saturn.

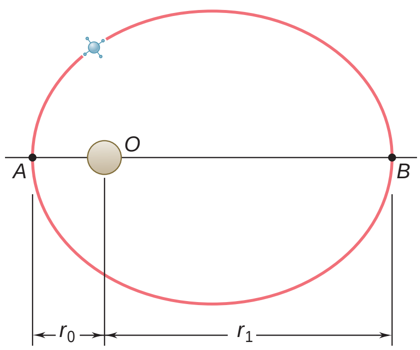


Fig. P12.102

12.102 A satellite describes an elliptic orbit about a planet of mass M. Denoting by r_0 and r_1, respectively, the minimum and maximum values of the distance r from the satellite to the center of the planet, derive the relation

$$\frac{1}{r_0} + \frac{1}{r_1} = \frac{2GM}{h^2}$$

where h is the angular momentum per unit mass of the satellite.

12.103 A space probe is describing a circular orbit about a planet of radius R. The altitude of the probe above the surface of the planet is aR and its speed is v_0. To place the probe in an elliptic orbit which will bring it closer to the planet, its speed is reduced from v_0 to bv_0, where $b < 1$, by firing its engine for a short interval of time. Determine the smallest permissible value of b if the probe is not to crash on the surface of the planet.

12.104 At main engine cutoff of its thirteenth flight, the space shuttle Discovery was in an elliptic orbit of minimum altitude 60 km and maximum altitude 500 km above the surface of the earth. Knowing that at point A the shuttle had a velocity $\mathbf{v}_0$ parallel to the surface of the earth and that the shuttle was transferred to a circular orbit as it passed through point B, determine (*a*) the speed v_0 of the shuttle at A, (*b*) the increase in speed required at B to insert the shuttle into the circular orbit.

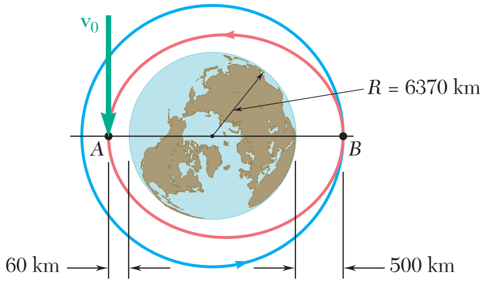


Fig. P12.104

12.105 A space probe is to be placed in a circular orbit of 5600-mi radius about the planet Venus in a specified plane. As the probe reaches A, the point of its original trajectory closest to Venus, it is inserted in a first elliptic transfer orbit by reducing its speed of Δv_A. This orbit brings it to point B with a much reduced velocity. There the probe is inserted in a second transfer orbit located in the specified plane by changing the direction of its velocity and further reducing its speed by Δv_B. Finally, as the probe reaches point C, it is inserted in the desired circular orbit by reducing its speed by Δv_C. Knowing that the mass of Venus is 0.82 times the mass of the earth, that $r_A = 9.3 \times 10^3$ mi and $r_B = 190 \times 10^3$ mi, and that the probe approaches A on a parabolic trajectory, determine by how much the velocity of the probe should be reduced (*a*) at A, (*b*) at B, (*c*) at C.

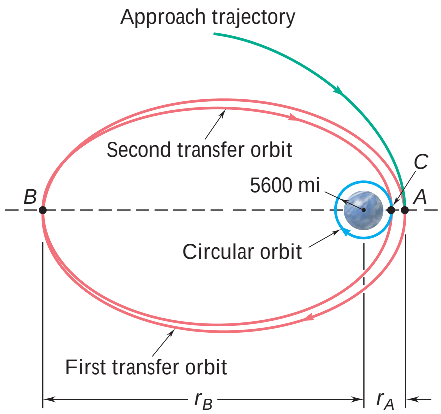


Fig. *P12.105*

12.106 For the space probe of Prob. 12.105, it is known that $r_A = 9.3 \times 10^3$ mi and that the velocity of the probe is reduced to 20,000 ft/s as it passes through A. Determine (*a*) the distance from the center of Venus to point B, (*b*) the amounts by which the velocity of the probe should be reduced at B and C, respectively.

12.107 As it describes an elliptic orbit about the sun, a spacecraft reaches a maximum distance of 202×10^6 mi from the center of the sun at point A (called the aphelion) and a minimum distance of 92×10^6 mi at point B (called the perihelion). To place the spacecraft in a smaller elliptic orbit with aphelion at A' and perihelion at B', where A' and B' are located 164.5×10^6 mi and 85.5×10^6 mi, respectively, from the center of the sun, the speed of the spacecraft is first reduced as it passes through A and then is further reduced as it passes through B'. Knowing that the mass of the sun is 332.8×10^3 times the mass of the earth, determine (*a*) the speed of the spacecraft at A, (*b*) the amounts by which the speed of the spacecraft should be reduced at A and B' to insert it into the desired elliptic orbit.

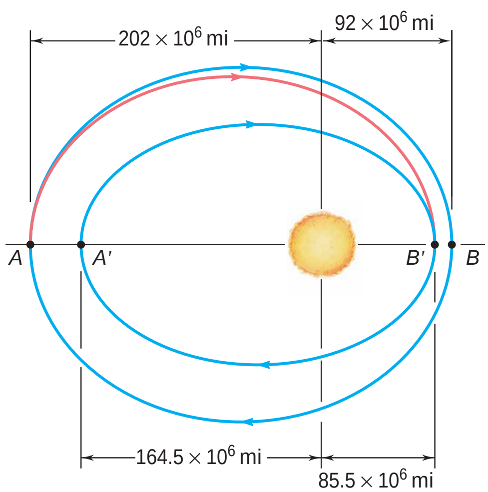


Fig. P12.107

12.108 Halley's comet travels in an elongated elliptic orbit for which the minimum distance from the sun is approximately $\frac{1}{2}r_E$, where $r_E = 150 \times 10^6$ km is the mean distance from the sun to the earth. Knowing that the periodic time of Halley's comet is about 76 years, determine the maximum distance from the sun reached by the comet.

12.109 Based on observations made during the 1996 sighting of comet Hyakutake, it was concluded that the trajectory of the comet is a highly elongated ellipse for which the eccentricity is approximately $e = 0.999887$. Knowing that for the 1996 sighting the minimum distance between the comet and the sun was $0.230R_E$, where R_E is the mean distance from the sun to the earth, determine the periodic time of the comet.

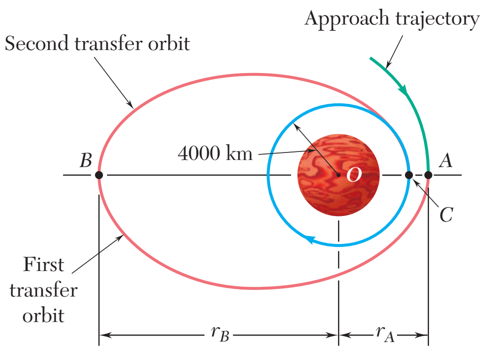


Fig. P12.110

12.110 A space probe is to be placed in a circular orbit of radius 4000 km about the planet Mars. As the probe reaches A, the point of its original trajectory closest to Mars, it is inserted into a first elliptic transfer orbit by reducing its speed. This orbit brings it to point B with a much-reduced velocity. There the probe is inserted into a second transfer orbit by further reducing its speed. Knowing that the mass of Mars is 0.1074 times the mass of the earth, that $r_A = 9000$ km and $r_B = 180\,000$ km, and that the probe approaches A on a parabolic trajectory, determine the time needed for the space probe to travel from A to B on its first transfer orbit.

12.111 A space shuttle is in an elliptic orbit of eccentricity 0.0356 and a minimum altitude of 300 km above the surface of the earth. Knowing that the radius of the earth is 6370 km, determine the periodic time for the orbit.

12.112 The Clementine spacecraft described an elliptic orbit of minimum altitude $h_A = 400$ km and maximum altitude $h_B = 2940$ km above the surface of the moon. Knowing that the radius of the moon is 1737 km and that the mass of the moon is 0.01230 times the mass of the earth, determine the periodic time of the spacecraft.

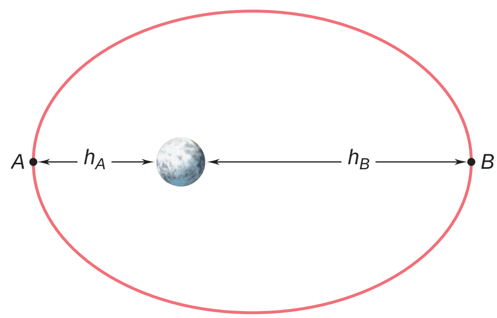


Fig. P12.112

12.113 Determine the time needed for the space probe of Prob. 12.100 to travel from B to C.

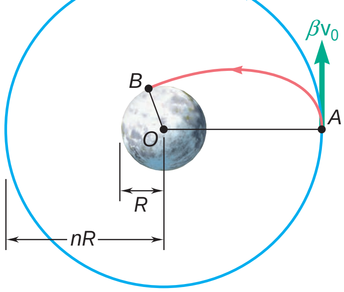


Fig. P12.114

12.114 A space probe is describing a circular orbit of radius nR with a velocity v_0 about a planet of radius R and center O. As the probe passes through point A, its velocity is reduced from v_0 to βv_0, where $\beta < 1$, to place the probe on a crash trajectory. Express in terms of n and β the angle AOB, where B denotes the point of impact of the probe on the planet.

12.115 A long-range ballistic trajectory between points A and B on the earth's surface consists of a portion of an ellipse with the apogee at point C. Knowing that point C is 1500 km above the surface of the earth and the range $R\phi$ of the trajectory is 6000 km, determine (*a*) the velocity of the projectile at C, (*b*) the eccentricity ε of the trajectory.

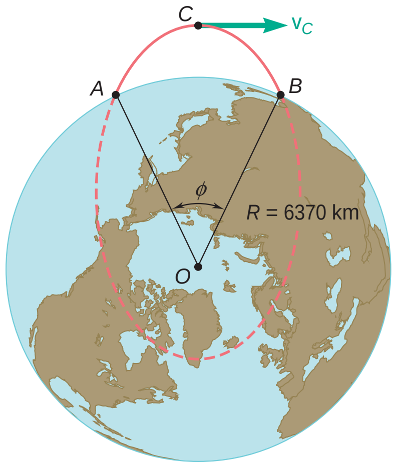


Fig. P12.115

12.116 A space shuttle is describing a circular orbit at an altitude of 563 km above the surface of the earth. As it passes through point A, it fires its engine for a short interval of time to reduce its speed by 152 m/s and begin its descent toward the earth. Determine the angle AOB so that the altitude of the shuttle at point B is 121 km. (*Hint:* Point A is the apogee of the elliptic descent trajectory.)

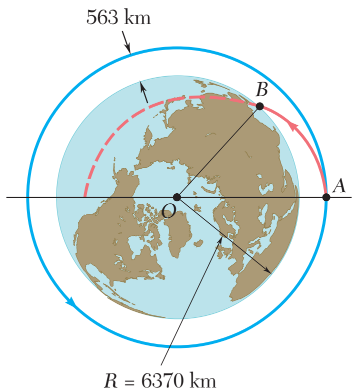


Fig. *P12.116*

12.117 As a spacecraft approaches the planet Jupiter, it releases a probe which is to enter the planet's atmosphere at point B at an altitude of 280 mi above the surface of the planet. The trajectory of the probe is a hyperbola of eccentricity $\varepsilon = 1.031$. Knowing that the radius and the mass of Jupiter are 44,423 mi and 1.30×10^{26} slug, respectively, and that the velocity $\mathbf{v}_B$ of the probe at B forms an angle of 82.9° with the direction of OA, determine (*a*) the angle AOB, (*b*) the speed v_B of the probe at B.

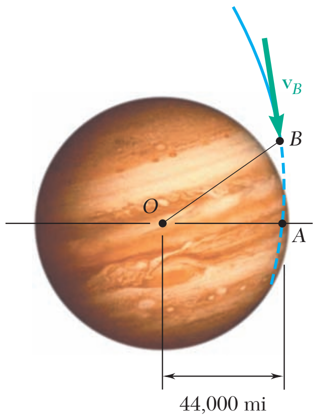


Fig. P12.117

12.118 A satellite describes an elliptic orbit about a planet. Denoting by r_0 and r_1 the distances corresponding, respectively, to the perigee and apogee of the orbit, show that the curvature of the orbit at each of these two points can be expressed as

$$\frac{1}{\rho} = \frac{1}{2}\left(\frac{1}{r_0} + \frac{1}{r_1}\right)$$

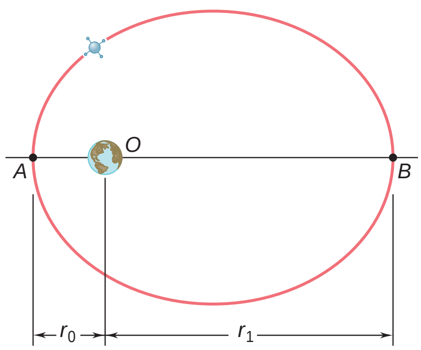


Fig. P12.118 and *P12.119*

12.119 (*a*) Express the eccentricity ε of the elliptic orbit described by a satellite about a planet in terms of the distances r_0 and r_1 corresponding, respectively, to the perigee and apogee of the orbit. (*b*) Use the result obtained in part *a* and the data given in Prob. 12.109, where $R_E = 149.6 \times 10^6$ km, to determine the approximate maximum distance from the sun reached by comet Hyakutake.

12.120 Derive Kepler's third law of planetary motion from Eqs. (12.39) and (12.45).

12.121 Show that the angular momentum per unit mass h of a satellite describing an elliptic orbit of semimajor axis a and eccentricity ε about a planet of mass M can be expressed as

$$h = \sqrt{GMa(1 - \varepsilon^2)}$$

REVIEW AND SUMMARY

This chapter was devoted to Newton's second law and its application to the analysis of the motion of particles.

Newton's second law

Denoting by m the mass of a particle, by $\Sigma\mathbf{F}$ the sum, or resultant, of the forces acting on the particle, and by $\mathbf{a}$ the acceleration of the particle relative to a *newtonian frame of reference* [Sec. 12.2], we wrote

$$\Sigma\mathbf{F} = m\mathbf{a} \tag{12.2}$$

Linear momentum

Introducing the *linear momentum* of a particle, $\mathbf{L} = m\mathbf{v}$ [Sec. 12.3], we saw that Newton's second law can also be written in the form

$$\Sigma\mathbf{F} = \dot{\mathbf{L}} \tag{12.5}$$

which expresses that *the resultant of the forces acting on a particle is equal to the rate of change of the linear momentum of the particle*.

Consistent systems of units

Equation (12.2) holds only if a consistent system of units is used. With SI units, the forces should be expressed in newtons, the masses in kilograms, and the accelerations in m/s^2; with U.S. customary units, the forces should be expressed in pounds, the masses in $lb \cdot s^2/ft$ (also referred to as *slugs*), and the accelerations in ft/s^2 [Sec. 12.4].

Equations of motion for a particle

To solve a problem involving the motion of a particle, Eq. (12.2) should be replaced by equations containing scalar quantities [Sec. 12.5]. Using *rectangular components* of $\mathbf{F}$ and $\mathbf{a}$, we wrote

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z \tag{12.8}$$

Using *tangential and normal components*, we had

$$\Sigma F_t = m\frac{dv}{dt} \qquad \Sigma F_n = m\frac{v^2}{\rho} \tag{12.9$'$}$$

Dynamic equilibrium

We also noted [Sec. 12.6] that the equations of motion of a particle can be replaced by equations similar to the equilibrium equations used in statics if a vector $-m\mathbf{a}$ of magnitude ma but of sense opposite to that of the acceleration is added to the forces applied to the particle; the particle is then said to be in *dynamic equilibrium*. For the sake of uniformity, however, all the Sample Problems were solved by using the equations of motion, first with rectangular components [Sample Probs. 12.1 through 12.4], then with tangential and normal components [Sample Probs. 12.5 and 12.6].

Angular momentum

In the second part of the chapter, we defined the *angular momentum* $\mathbf{H}_O$ of a particle about a point O as the moment about O of the linear momentum $m\mathbf{v}$ of that particle [Sec. 12.7]. We wrote

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{12.12}$$

and noted that $\mathbf{H}_O$ is a vector perpendicular to the plane containing $\mathbf{r}$ and $m\mathbf{v}$ (Fig. 12.24) and of magnitude

$$H_O = rmv \sin \phi \tag{12.13}$$

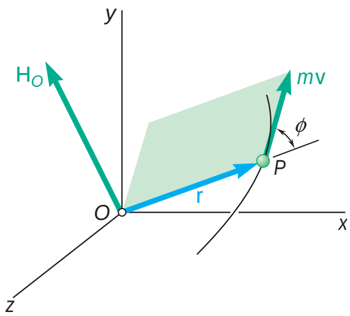


Fig. 12.24

Resolving the vectors $\mathbf{r}$ and $m\mathbf{v}$ into rectangular components, we expressed the angular momentum $\mathbf{H}_O$ in the determinant form

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix} \tag{12.14}$$

In the case of a particle moving in the xy plane, we have $z = v_z = 0$. The angular momentum is perpendicular to the xy plane and is completely defined by its magnitude. We wrote

$$H_O = H_z = m(xv_y - yv_x) \tag{12.16}$$

Rate of change of angular momentum

Computing the rate of change $\dot{\mathbf{H}}_O$ of the angular momentum $\mathbf{H}_O$, and applying Newton's second law, we wrote the equation

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \tag{12.19}$$

which states that *the sum of the moments about O of the forces acting on a particle is equal to the rate of change of the angular momentum of the particle about O.*

Radial and transverse components

In many problems involving the plane motion of a particle, it is found convenient to use *radial and transverse components* [Sec. 12.8, Sample Prob. 12.7] and to write the equations

$$\Sigma F_r = m(\ddot{r} - r\dot{\theta}^2) \tag{12.21}$$
$$\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{12.22}$$

Motion under a central force

When the only force acting on a particle P is a force $\mathbf{F}$ directed toward or away from a fixed point O, the particle is said to be moving *under a central force* [Sec. 12.9]. Since $\Sigma \mathbf{M}_O = 0$ at any given instant, it follows from Eq. (12.19) that $\dot{\mathbf{H}}_O = 0$ for all values of t and, thus, that

$$\mathbf{H}_O = \text{constant} \tag{12.23}$$

We concluded that *the angular momentum of a particle moving under a central force is constant, both in magnitude and direction*, and that the particle moves in a plane perpendicular to the vector $\mathbf{H}_O$.

Recalling Eq. (12.13), we wrote the relation

$$rmv \sin \phi = r_0 m v_0 \sin \phi_0 \tag{12.25}$$

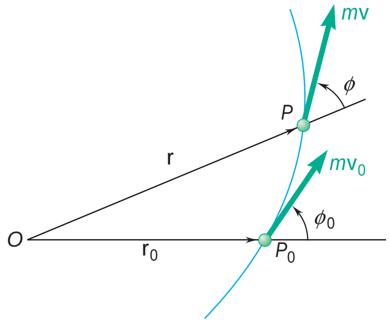


Fig. 12.25

for the motion of any particle under a central force (Fig. 12.25). Using polar coordinates and recalling Eq. (12.18), we also had

$$r^2\dot{\theta} = h \tag{12.27}$$

where h is a constant representing the angular momentum per unit mass, H_O/m, of the particle. We observed (Fig. 12.26) that the infinitesimal area dA swept by the radius vector OP as it rotates through $d\theta$ is equal to $\frac{1}{2}r^2 d\theta$ and, thus, that the left-hand member of Eq. (12.27) represents twice the *areal velocity* dA/dt of the particle. Therefore, *the areal velocity of a particle moving under a central force is constant*.

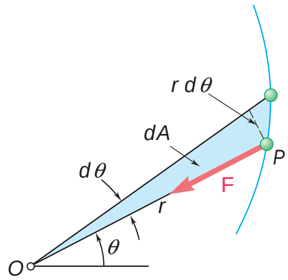


Fig. 12.26

Newton's law of universal gravitation

An important application of the motion under a central force is provided by the orbital motion of bodies under gravitational attraction [Sec. 12.10]. According to *Newton's law of universal gravitation*, two particles at a distance r from each other and of masses M and m, respectively, attract each other with equal and opposite forces **F** and −**F** directed along the line joining the particles (Fig. 12.27). The common magnitude F of the two forces is

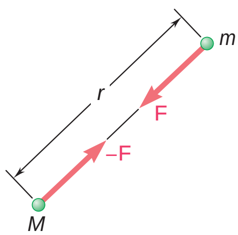


Fig. 12.27

$$F = G\frac{Mm}{r^2} \tag{12.28}$$

where G is the *constant of gravitation*. In the case of a body of mass m subjected to the gravitational attraction of the earth, the product GM, where M is the mass of the earth, can be expressed as

$$GM = gR^2 \tag{12.30}$$

where $g = 9.81 \text{ m/s}^2 = 32.2 \text{ ft/s}^2$ and R is the radius of the earth.

Orbital motion

It was shown in Sec. 12.11 that a particle moving under a central force describes a trajectory defined by the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \tag{12.37}$$

where $F > 0$ corresponds to an attractive force and $u = 1/r$. In the case of a particle moving under a force of gravitational attraction [Sec. 12.12], we substituted for F the expression given in Eq. (12.28). Measuring θ from the axis OA joining the focus O to the point A of the trajectory closest to O (Fig. 12.28), we found that the solution to Eq. (12.37) was

$$\frac{1}{r} = u = \frac{GM}{h^2} + C\cos\theta \qquad (12.39)$$

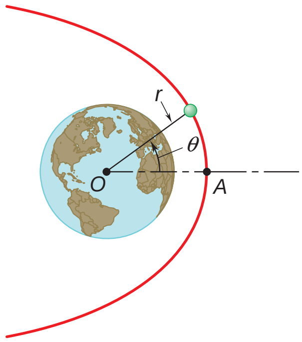


Fig. 12.28

This is the equation of a conic of eccentricity $\varepsilon = Ch^2/GM$. The conic is an *ellipse* if $\varepsilon < 1$, a *parabola* if $\varepsilon = 1$, and a hyperbola if $\varepsilon > 1$. The constants C and h can be determined from the initial conditions; if the particle is projected from point A ($\theta = 0$, $r = r_0$) with an initial velocity $\mathbf{v}_0$ perpendicular to OA, we have $h = r_0 v_0$ [Sample Prob. 12.9].

Escape velocity

It was also shown that the values of the initial velocity corresponding, respectively, to a parabolic and a circular trajectory were

$$v_{esc} = \sqrt{\frac{2GM}{r_0}} \qquad (12.43)$$

$$v_{circ} = \sqrt{\frac{GM}{r_0}} \qquad (12.44)$$

and that the first of these values, called the *escape velocity,* is the smallest value of v_0 for which the particle will not return to its starting point.

Periodic time

The *periodic time* τ of a planet or satellite was defined as the time required by that body to describe its orbit. It was shown that

$$\tau = \frac{2\pi ab}{h} \qquad (12.45)$$

where $h = r_0 v_0$ and where a and b represent the semimajor and semiminor axes of the orbit. It was further shown that these semiaxes are respectively equal to the arithmetic and geometric means of the maximum and minimum values of the radius vector r.

Kepler's laws

The last section of the chapter [Sec. 12.13] presented *Kepler's laws of planetary motion* and showed that these empirical laws, obtained from early astronomical observations, confirm Newton's laws of motion as well as his law of gravitation.

REVIEW PROBLEMS

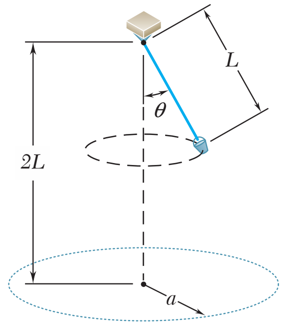


Fig. *P12.123*

12.122 In the braking test of a sports car its velocity is reduced from 70 mi/h to zero in a distance of 170 ft with slipping impending. Knowing that the coefficient of kinetic friction is 80 percent of the coefficient of static friction, determine (*a*) the coefficient of static friction, (*b*) the stopping distance for the same initial velocity if the car skids. Ignore air resistance and rolling resistance.

12.123 A bucket is attached to a rope of length $L = 1.2$ m and is made to revolve in a horizontal circle. Drops of water leaking from the bucket fall and strike the floor along the perimeter of a circle of radius a. Determine the radius a when $u = 30°$.

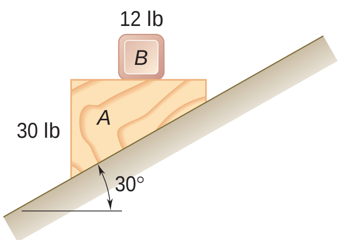


Fig. P12.124

12.124 A 12-lb block B rests as shown on the upper surface of a 30-lb wedge A. Neglecting friction, determine immediately after the system is released from rest (*a*) the acceleration of A, (*b*) the acceleration of B relative to A.

12.125 A 500-lb crate B is suspended from a cable attached to a 40-lb trolley A which rides on an inclined I-beam as shown. Knowing that at the instant shown the trolley has an acceleration of 1.2 ft/s^2 up and to the right, determine (*a*) the acceleration of B relative to A, (*b*) the tension in cable CD.

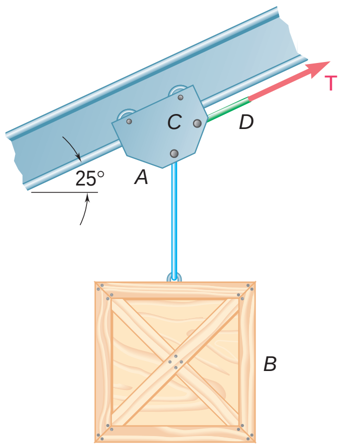


Fig. P12.125

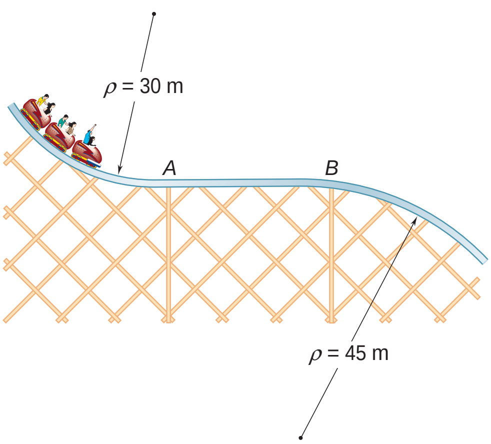


Fig. P12.126

12.126 The roller-coaster track shown is contained in a vertical plane. The portion of track between A and B is straight and horizontal, while the portions to the left of A and to the right of B have radii of curvature as indicated. A car is traveling at a speed of 72 km/h when the brakes are suddenly applied, causing the wheels of the car to slide on the track ($m_k = 0.25$). Determine the initial deceleration of the car if the brakes are applied as the car (*a*) has almost reached A, (*b*) is traveling between A and B, (*c*) has just passed B.

12.127 The 100-g pin B slides along the slot in the rotating arm OC and along the slot DE which is cut in a fixed horizontal plate. Neglecting friction and knowing that rod OC rotates at the constant rate $\dot{\theta}_0 = 12$ rad/s, determine for any given value of θ (*a*) the radial and transverse components of the resultant force $\mathbf{F}$ exerted on pin B, (*b*) the forces $\mathbf{P}$ and $\mathbf{Q}$ exerted on pin B by rod OC and the wall of slot DE, respectively.

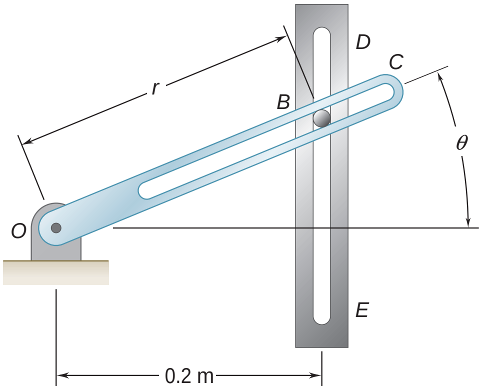


Fig. P12.127

12.128 A small 200-g collar C can slide on a semicircular rod which is made to rotate about the vertical AB at the constant rate of 6 rad/s. Determine the minimum required value of the coefficient of static friction between the collar and the rod if the collar is not to slide when (*a*) $\theta = 90°$, (*b*) $\theta = 75°$, (*c*) $\theta = 45°$. Indicate in each case the direction of the impending motion.

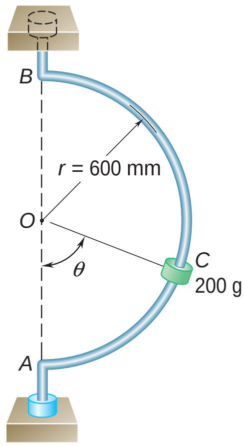


Fig. P12.128

12.129 Telemetry technology is used to quantify kinematic values of a 200-kg roller-coaster cart as it passes overhead. According to the system, $r = 25$ m, $\dot{r} = -10$ m/s, $\ddot{r} = -2$ m/s^2, $\theta = 90°$, $\dot{\theta} = -0.4$ rad/s, $\ddot{\theta} = -0.32$ rad/s^2. At this instant, determine (*a*) the normal force between the cart and the track, (*b*) the radius of curvature of the track.

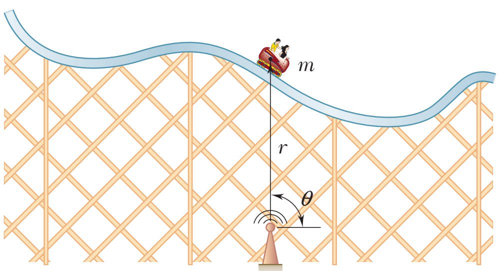


Fig. P12.129

12.130 The radius of the orbit of a moon of a given planet is equal to twice the radius of that planet. Denoting by ρ the mean density of the planet, show that the time required by the moon to complete one full revolution about the planet is $(24\pi/G\rho)^{1/2}$, where G is the constant of gravitation.

12.131 At engine burnout on a mission, a shuttle had reached point A at an altitude of 40 mi above the surface of the earth and had a horizontal velocity $\mathbf{v}_0$. Knowing that its first orbit was elliptic and that the shuttle was transferred to a circular orbit as it passed through point B at an altitude of 170 mi, determine (*a*) the time needed for the shuttle to travel from A to B on its original elliptic orbit, (*b*) the periodic time of the shuttle on its final circular orbit.

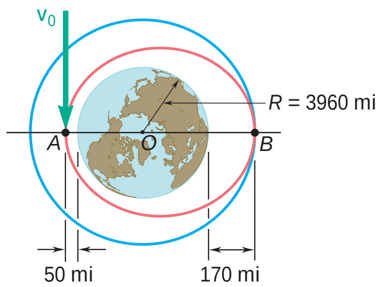


Fig. *P12.131*

12.132 It was observed that as the Galileo spacecraft reached the point on its trajectory closest to Io, a moon of the planet Jupiter, it was at a distance of 1750 mi from the center of Io and had a velocity of 49.4×10^3 ft/s. Knowing that the mass of Io is 0.01496 times the mass of the earth, determine the eccentricity of the trajectory of the spacecraft as it approached Io.

***12.133** Disk A rotates in a horizontal plane about a of vertical axis at the constant rate $\dot{u}_0 = 10$ rad/s. Slider B has mass 1 kg and moves in a frictionless slot cut in the disk. The slider is attached to a spring of constant k, which is undeformed when $r = 0$. Knowing that the slider is released with no radial velocity in the position $r =$ 500 mm, determine the position of the slider and the horizontal force exerted on it by the disk at $t = 0.1$ s for (*a*) $k = 100$ N/m, (*b*) $k = 200$ N/m.

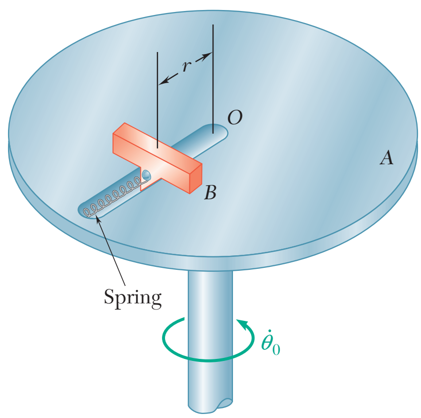


Fig. P12.133

COMPUTER PROBLEMS

12.C1 Block B of mass 10 kg is initially at rest as shown on the upper surface of a 20-kg wedge A which is supported by a horizontal surface. A 2-kg block C is connected to block B by a cord which passes over a pulley of negligible mass. Using computational software and denoting by m the coefficient of friction at all surfaces, use this program to determine the accelerations for values of m ≥ 0. Use 0.01 increments for m until the wedge does not move and then use 0.1 increments until no motion occurs.

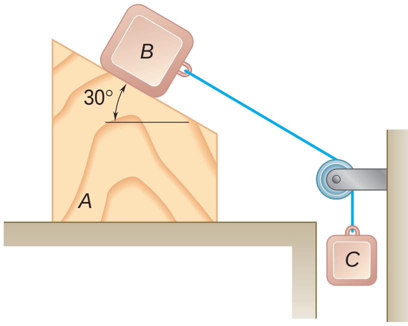


Fig. P12.C1

12.C2 A small, 1-lb block is at rest at the top of a cylindrical surface. The block is given an initial velocity $\mathbf{v}_0$ to the right of magnitude 10 ft/s, which causes it to slide on the cylindrical surface. Using computational software calculate and plot the values of u at which the block leaves the surface for values of m$_k$, the coefficient of kinetic friction between the block and the surface, from 0 to 0.4.

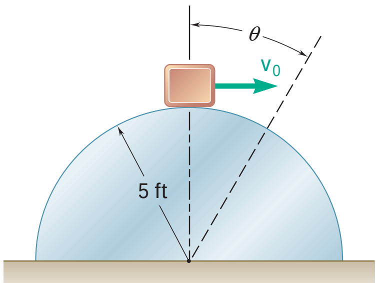


Fig. P12.C2

12.C3 A block of mass m is attached to a spring of constant k. The block is released from rest when the spring is in a horizontal and undeformed position. Use computational software to determine, for various selected values of k/m and r_0, (*a*) the length of the spring and the magnitude and direction of the velocity of the block as the block passes directly under the point of suspension of the spring, (*b*) the value of k/m when $r_0 = 1$ m for which that velocity is horizontal.

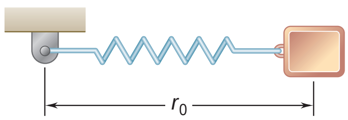


Fig. P12.C3

12.C4 Use computational software to determine the ranges of values of u for which the block E of Prob. 12.60 will not slide in the semicircular slot of the flat plate. Assuming a coefficient of static friction of 0.35, determine the ranges of values when the constant rate of rotation of the plate is (*a*) 14 rad/s, (*b*) 2 rad/s.

A golf ball will deform upon impact as shown by this high-speed photo. The maximum deformation will occur when the club head velocity and the ball velocity are the same. In this chapter impacts will be analyzed using the coefficient of restitution and conservation of linear momentum. The kinetics of particles using energy and momentum methods is the subject of this chapter.

CHAPTER 13

Kinetics of Particles: Energy and Momentum Methods

Chapter 13 Kinetics of Particles: Energy and Momentum Methods

13.1 Introduction
13.2 Work of a Force
13.3 Kinetic Energy of a Particle. Principle of Work and Energy
13.4 Applications of the Principle of Work and Energy
13.5 Power and Efficiency
13.6 Potential Energy
13.7 Conservative Forces
13.8 Conservation of Energy
13.9 Motion Under a Conservative Central Force. Application to Space Mechanics
13.10 Principle of Impulse and Momentum
13.11 Impulsive Motion
13.12 Impact
13.13 Direct Central Impact
13.14 Oblique Central Impact
13.15 Problems Involving Energy and Momentum

13.1 INTRODUCTION

In the preceding chapter, most problems dealing with the motion of particles were solved through the use of the fundamental equation of motion $\mathbf{F} = m\mathbf{a}$. Given a particle acted upon by a force $\mathbf{F}$, we could solve this equation for the acceleration $\mathbf{a}$; then, by applying the principles of kinematics, we could determine from $\mathbf{a}$ the velocity and position of the particle at any time.

Using the equation $\mathbf{F} = m\mathbf{a}$ together with the principles of kinematics allows us to obtain two additional methods of analysis, the *method of work and energy* and the *method of impulse and momentum*. The advantage of these methods lies in the fact that they make the determination of the acceleration unnecessary. Indeed, the method of work and energy directly relates force, mass, velocity, and displacement, while the method of impulse and momentum relates force, mass, velocity, and time.

The method of work and energy will be considered first. In Secs. 13.2 through 13.4, the *work of a force* and the *kinetic energy of a particle* are discussed and the principle of work and energy is applied to the solution of engineering problems. The concepts of *power* and *efficiency* of a machine are introduced in Sec. 13.5.

Sections 13.6 through 13.8 are devoted to the concept of *potential energy* of a conservative force and to the application of the principle of conservation of energy to various problems of practical interest. In Sec. 13.9, the principles of conservation of energy and of conservation of angular momentum are used jointly to solve problems of space mechanics.

The second part of the chapter is devoted to the *principle of impulse and momentum* and to its application to the study of the motion of a particle. As you will see in Sec. 13.11, this principle is particularly effective in the study of the *impulsive motion* of a particle, where very large forces are applied for a very short time interval.

In Secs. 13.12 through 13.14, the *central impact* of two bodies will be considered. It will be shown that a certain relation exists between the relative velocities of the two colliding bodies before and after impact. This relation, together with the fact that the total momentum of the two bodies is conserved, can be used to solve a number of problems of practical interest.

Finally, in Sec. 13.15, you will learn to select from the three fundamental methods presented in Chaps. 12 and 13 the method best suited for the solution of a given problem. You will also see how the principle of conservation of energy and the method of impulse and momentum can be combined to solve problems involving only conservative forces, except for a short impact phase during which impulsive forces must also be taken into consideration.

13.2 WORK OF A FORCE

We will first define the terms *displacement* and *work* as they are used in mechanics.† Consider a particle which moves from a point

†The definition of work was given in Sec. 10.2, and the basic properties of the work of a force were outlined in Secs. 10.2 and 10.6. For convenience, we repeat here the portions of this material which relate to the kinetics of particles.

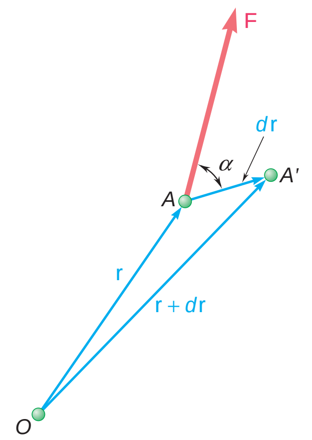


Fig. 13.1

A to a neighboring point A' (Fig. 13.1). If $\mathbf{r}$ denotes the position vector corresponding to point A, the small vector joining A and A' can be denoted by the differential $d\mathbf{r}$; the vector $d\mathbf{r}$ is called the *displacement* of the particle. Now, let us assume that a force $\mathbf{F}$ is acting on the particle. The *work of the force* $\mathbf{F}$ *corresponding to the displacement* $d\mathbf{r}$ is defined as the quantity

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{13.1}$$

obtained by forming the scalar product of the force $\mathbf{F}$ and the displacement $d\mathbf{r}$. Denoting by F and ds, respectively, the magnitudes of the force and of the displacement, and by α the angle formed by $\mathbf{F}$ and $d\mathbf{r}$, and recalling the definition of the scalar product of two vectors (Sec. 3.9), we write

$$dU = F\,ds \cos \alpha \tag{13.1$'$}$$

Using formula (3.30), we can also express the work dU in terms of the rectangular components of the force and of the displacement:

$$dU = F_x\,dx + F_y\,dy + F_z\,dz \tag{13.1$''$}$$

Being a *scalar quantity,* work has a magnitude and a sign but no direction. We also note that work should be expressed in units obtained by multiplying units of length by units of force. Thus, if U.S. customary units are used, work should be expressed in ft · lb or in · lb. If SI units are used, work should be expressed in N · m. The unit of work N · m is called a *joule* (J).† Recalling the conversion factors indicated in Sec. 12.4, we write

$$1 \text{ ft} \cdot \text{lb} = (1 \text{ ft})(1 \text{ lb}) = (0.3048 \text{ m})(4.448 \text{ N}) = 1.356 \text{ J}$$

It follows from (13.1′) that the work dU is positive if the angle α is acute and negative if α is obtuse. Three particular cases are of special

†The joule (J) is the SI unit of *energy,* whether in mechanical form (work, potential energy, kinetic energy) or in chemical, electrical, or thermal form. We should note that even though N · m = J, the moment of a force must be expressed in N · m and not in joules, since the moment of a force is not a form of energy.

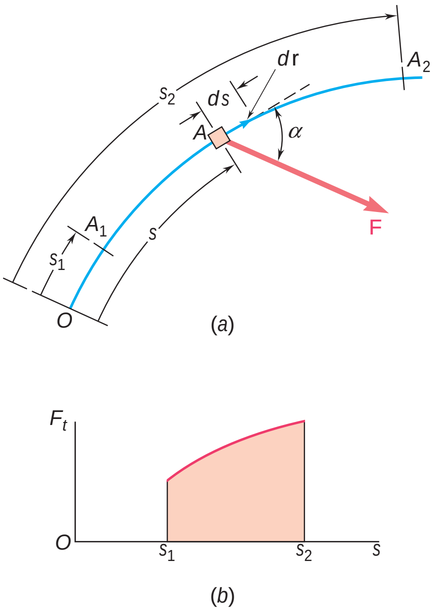


Fig. 13.2

interest. If the force $\mathbf{F}$ has the same direction as $d\mathbf{r}$, the work dU reduces to $F\,ds$. If $\mathbf{F}$ has a direction opposite to that of $d\mathbf{r}$, the work is $dU = -F\,ds$. Finally, if $\mathbf{F}$ is perpendicular to $d\mathbf{r}$, the work dU is zero.

The work of $\mathbf{F}$ during a *finite* displacement of the particle from A_1 to A_2 (Fig. 13.2*a*) is obtained by integrating Eq. (13.1) along the path described by the particle. This work, denoted by $U_{1\to 2}$, is

$$U_{1\to 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \tag{13.2}$$

Using the alternative expression (13.1′) for the elementary work dU, and observing that $F\cos\alpha$ represents the tangential component F_t of the force, we can also express the work $U_{1\to 2}$ as

$$U_{1\to 2} = \int_{s_1}^{s_2} (F\cos\alpha)\,ds = \int_{s_1}^{s_2} F_t\,ds \tag{13.2′}$$

where the variable of integration s measures the distance traveled by the particle along the path. The work $U_{1\to 2}$ is represented by the area under the curve obtained by plotting $F_t = F\cos\alpha$ against s (Fig. 13.2*b*).

When the force $\mathbf{F}$ is defined by its rectangular components, the expression (13.1″) can be used for the elementary work. We then write

$$U_{1\to 2} = \int_{A_1}^{A_2} (F_x\,dx + F_y\,dy + F_z\,dz) \tag{13.2″}$$

where the integration is to be performed along the path described by the particle.

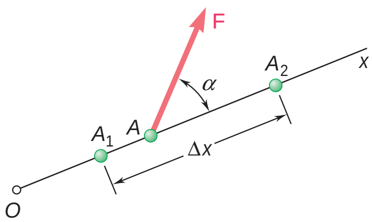


Fig. 13.3

Work of a Constant Force in Rectilinear Motion. When a particle moving in a straight line is acted upon by a force $\mathbf{F}$ of constant magnitude and of constant direction (Fig. 13.3), formula (13.2′) yields

$$U_{1\to 2} = (F\cos\alpha)\,\Delta x \tag{13.3}$$

where α = angle the force forms with direction of motion
Δx = displacement from A_1 to A_2

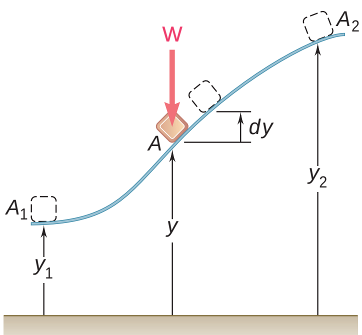


Fig. 13.4

Work of the Force of Gravity. The work of the weight $\mathbf{W}$ of a body, i.e., of the force of gravity exerted on that body, is obtained by substituting the components of $\mathbf{W}$ into (13.1″) and (13.2″). With the y axis chosen upward (Fig. 13.4), we have $F_x = 0$, $F_y = -W$, and $F_z = 0$, and we write

$$dU = -W\,dy$$

$$U_{1\to 2} = -\int_{y_1}^{y_2} W\,dy = Wy_1 - Wy_2 \tag{13.4}$$

or

$$U_{1\to 2} = -W(y_2 - y_1) = -W\,\Delta y \tag{13.4′}$$

where Δy is the vertical displacement from A_1 to A_2. The work of the weight $\mathbf{W}$ is thus equal to *the product of W and the vertical* *displacement of the center of gravity of the body*. The work is *positive* when $\Delta y < 0$, that is, *when the body moves down*.

Work of the Force Exerted by a Spring. Consider a body A attached to a fixed point B by a spring; it is assumed that the spring is undeformed when the body is at A_0 (Fig. 13.5*a*). Experimental evidence shows that the magnitude of the force $\mathbf{F}$ exerted by the spring on body A is proportional to the deflection x of the spring measured from the position A_0. We have

$$F = kx \tag{13.5}$$

where k is the *spring constant,* expressed in N/m or kN/m if SI units are used and in lb/ft or lb/in. if U.S. customary units are used.†

The work of the force $\mathbf{F}$ exerted by the spring during a finite displacement of the body from $A_1(x = x_1)$ to $A_2(x = x_2)$ is obtained by writing

$$dU = -F\,dx = -kx\,dx$$
$$U_{1\rightarrow 2} = -\int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \tag{13.6}$$

Care should be taken to express k and x in consistent units. For example, if U.S. customary units are used, k should be expressed in lb/ft and x in feet, or k in lb/in. and x in inches; in the first case, the work is obtained in ft · lb, in the second case, in in · lb. We note that the work of the force $\mathbf{F}$ exerted by the spring on the body is *positive* when $x_2 < x_1$, that is, *when the spring is returning to its undeformed position*.

Since Eq. (13.5) is the equation of a straight line of slope k passing through the origin, the work $U_{1\rightarrow 2}$ of $\mathbf{F}$ during the displacement from A_1 to A_2 can be obtained by evaluating the area of the trapezoid shown in Fig. 13.5*b*. This is done by computing F_1 and F_2 and multiplying the base Δx of the trapezoid by its mean height $\frac{1}{2}(F_1 + F_2)$. Since the work of the force $\mathbf{F}$ exerted by the spring is positive for a negative value of Δx, we write

$$U_{1\rightarrow 2} = -\tfrac{1}{2}(F_1 + F_2)\,\Delta x \tag{13.6$'$}$$

Formula (13.6′) is usually more convenient to use than (13.6) and affords fewer chances of confusing the units involved.

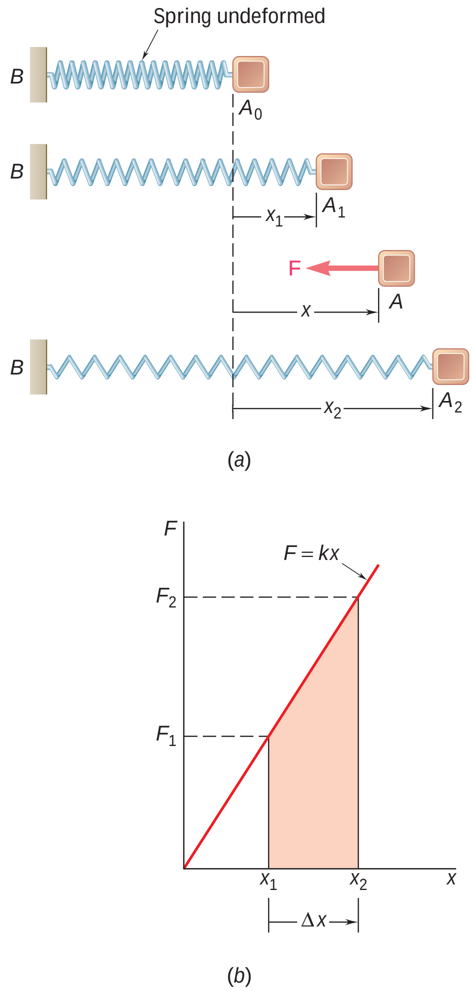


Fig. 13.5

Work of a Gravitational Force. We saw in Sec. 12.10 that two particles of mass M and m at a distance r from each other attract each other with equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$, directed along the line joining the particles and of magnitude

$$F = G\frac{Mm}{r^2}$$

†The relation $F = kx$ is correct under static conditions only. Under dynamic conditions, formula (13.5) should be modified to take the inertia of the spring into account. However, the error introduced by using the relation $F = kx$ in the solution of kinetics problems is small if the mass of the spring is small compared with the other masses in motion.

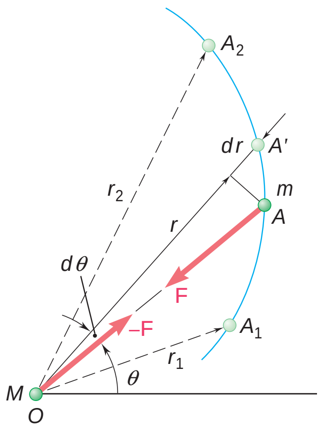


Fig. 13.6

Let us assume that the particle M occupies a fixed position O while the particle m moves along the path shown in Fig. 13.6. The work of the force $\mathbf{F}$ exerted on the particle m during an infinitesimal displacement of the particle from A to A' can be obtained by multiplying the magnitude F of the force by the radial component dr of the displacement. Since $\mathbf{F}$ is directed toward O, the work is negative and we write

$$dU = -F\,dr = -G\frac{Mm}{r^2}dr$$

The work of the gravitational force $\mathbf{F}$ during a finite displacement from $A_1(r = r_1)$ to $A_2(r = r_2)$ is therefore

$$U_{1\,y\,2} = -\int_{r_1}^{r_2}\frac{GMm}{r^2}dr = \frac{GMm}{r_2} - \frac{GMm}{r_1} \tag{13.7}$$

where M is the mass of the earth. This formula can be used to determine the work of the force exerted by the earth on a body of mass m at a distance r from the center of the earth, when r is larger than the radius R of the earth. Recalling the first of the relations (12.29), we can replace the product GMm in Eq. (13.7) by WR^2, where R is the radius of the earth ($R = 6.37 \times 10^6$ m or 3960 mi) and W is the weight of the body at the surface of the earth.

A number of forces frequently encountered in problems of kinetics *do no work*. They are forces applied to fixed points ($ds = 0$) or acting in a direction perpendicular to the displacement (cos a $= 0$). Among the forces which do no work are the following: the reaction at a frictionless pin when the body supported rotates about the pin, the reaction at a frictionless surface when the body in contact moves along the surface, the reaction at a roller moving along its track, and the weight of a body when its center of gravity moves horizontally.

13.3 KINETIC ENERGY OF A PARTICLE. PRINCIPLE OF WORK AND ENERGY

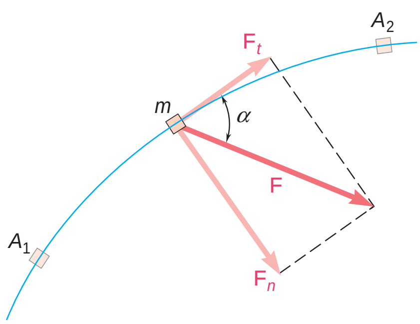


Fig. 13.7

Consider a particle of mass m acted upon by a force $\mathbf{F}$ and moving along a path which is either rectilinear or curved (Fig. 13.7). Expressing Newton's second law in terms of the tangential components of the force and of the acceleration (see Sec. 12.5), we write

$$F_t = ma_t \qquad \text{or} \qquad F_t = m\frac{dv}{dt}$$

where v is the speed of the particle. Recalling from Sec. 11.9 that $v = ds/dt$, we obtain

$$F_t = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$
$$F_t\,ds = mv\,dv$$

Integrating from A_1, where $s = s_1$ and $v = v_1$, to A_2, where $s = s_2$ and $v = v_2$, we write

$$\int_{s_1}^{s_2} F_t\,ds = m\int_{v_1}^{v_2} v\,dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{13.8}$$

The left-hand member of Eq. (13.8) represents the work $U_{1\,y\,2}$ of the force $\mathbf{F}$ exerted on the particle during the displacement from A_1 to

A_2; as indicated in Sec. 13.2, the work $U_{1 y 2}$ is a scalar quantity. The expression $\frac{1}{2}mv^2$ is also a scalar quantity; it is defined as the kinetic energy of the particle and is denoted by T. We write

$$\boxed{T = \tfrac{1}{2}mv^2} \tag{13.9}$$

Substituting into (13.8), we have

$$\boxed{U_{1 y 2} = T_2 - T_1} \tag{13.10}$$

which expresses that, when a particle moves from A_1 to A_2 under the action of a force **F**, *the work of the force* **F** *is equal to the change in kinetic energy of the particle*. This is known as the *principle of work and energy*. Rearranging the terms in (13.10), we write

$$\boxed{T_1 + U_{1 y 2} = T_2} \tag{13.11}$$

Thus, *the kinetic energy of the particle at A_2 can be obtained by adding to its kinetic energy at A_1 the work done during the displacement from A_1 to A_2 by the force* **F** *exerted on the particle*. Like Newton's second law from which it is derived, the principle of work and energy applies only with respect to a newtonian frame of reference (Sec. 12.2). The speed v used to determine the kinetic energy T should therefore be measured with respect to a newtonian frame of reference.

Since both work and kinetic energy are scalar quantities, their sum can be computed as an ordinary algebraic sum, the work $U_{1 y 2}$ being considered as positive or negative according to the direction of **F**. When several forces act on the particle, the expression $U_{1 y 2}$ represents the total work of the forces acting on the particle; it is obtained by adding algebraically the work of the various forces.

As noted above, the kinetic energy of a particle is a scalar quantity. It further appears from the definition $T = \frac{1}{2}mv^2$ that regardless of the direction of motion of the particle the kinetic energy is always positive. Considering the particular case when $v_1 = 0$ and $v_2 = v$, and substituting $T_1 = 0$ and $T_2 = T$ into (13.10), we observe that the work done by the forces acting on the particle is equal to T. Thus, the kinetic energy of a particle moving with a speed v represents the work which must be done to bring the particle from rest to the speed v. Substituting $T_1 = T$ and $T_2 = 0$ into (13.10), we also note that when a particle moving with a speed v is brought to rest, the work done by the forces acting on the particle is $-T$. Assuming that no energy is dissipated into heat, we conclude that the work done by the forces exerted *by the particle* on the bodies which cause it to come to rest is equal to T. Thus, the kinetic energy of a particle also represents *the capacity to do work associated with the speed of the particle*.

The kinetic energy is measured in the same units as work, i.e., in joules if SI units are used and in ft · lb if U.S. customary units are used. We check that, in SI units,

$$T = \tfrac{1}{2}mv^2 = \text{kg}(\text{m/s})^2 = (\text{kg} \cdot \text{m/s}^2)\text{m} = \text{N} \cdot \text{m} = \text{J}$$

while, in customary units,

$$T = \tfrac{1}{2}mv^2 = (\text{lb} \cdot \text{s}^2/\text{ft})(\text{ft/s})^2 = \text{ft} \cdot \text{lb}$$

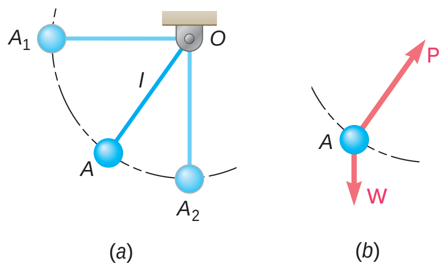


Fig. 13.8

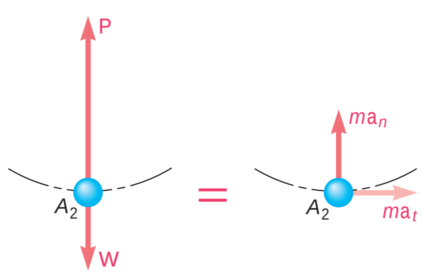


Fig. 13.9

13.4 APPLICATIONS OF THE PRINCIPLE OF WORK AND ENERGY

The application of the principle of work and energy greatly simplifies the solution of many problems involving forces, displacements, and velocities. Consider, for example, the pendulum OA consisting of a bob A of weight W attached to a cord of length l (Fig. 13.8*a*). The pendulum is released with no initial velocity from a horizontal position OA_1 and allowed to swing in a vertical plane. We wish to determine the speed of the bob as it passes through A_2, directly under O.

We first determine the work done during the displacement from A_1 to A_2 by the forces acting on the bob. We draw a free-body diagram of the bob, showing all the *actual* forces acting on it, i.e., the weight **W** and the force **P** exerted by the cord (Fig. 13.8*b*). (An inertia vector is not an actual force and *should not* be included in the free-body diagram.) We note that the force **P** does no work, since it is normal to the path; the only force which does work is thus the weight **W**. The work of **W** is obtained by multiplying its magnitude W by the vertical displacement l (Sec. 13.2); since the displacement is downward, the work is positive. We therefore write $U_{1\rightarrow 2} = Wl$.

Now considering the kinetic energy of the bob, we find $T_1 = 0$ at A_1 and $T_2 = \frac{1}{2}(W/g)v_2^2$ at A_2. We can now apply the principle of work and energy; recalling formula (13.11), we write

$$T_1 + U_{1\rightarrow 2} = T_2 \qquad 0 + Wl = \frac{1}{2}\frac{W}{g}v_2^2$$

Solving for v_2, we find $v_2 = \sqrt{2gl}$. We note that the speed obtained is that of a body falling freely from a height l.

The example we have considered illustrates the following advantages of the method of work and energy:

1. In order to find the speed at A_2, there is no need to determine the acceleration in an intermediate position A and to integrate the expression obtained from A_1 to A_2.
2. All quantities involved are scalars and can be added directly, without using x and y components.
3. Forces which do no work are eliminated from the solution of the problem.

What is an advantage in one problem, however, may be a disadvantage in another. It is evident, for instance, that the method of work and energy cannot be used to directly determine an acceleration. It is also evident that in determining a force which is normal to the path of the particle, a force which does no work, the method of work and energy must be supplemented by the direct application of Newton's second law. Suppose, for example, that we wish to determine the tension in the cord of the pendulum of Fig. 13.8*a* as the bob passes through A_2. We draw a free-body diagram of the bob in that position (Fig. 13.9) and express Newton's second law in terms of tangential and normal components. The equations $\Sigma F_t = ma_t$ and $\Sigma F_n = ma_n$ yield, respectively, $a_t = 0$ and

$$P - W = ma_n = \frac{W}{g}\frac{v_2^2}{l}$$

But the speed at A_2 was determined earlier by the method of work and energy. Substituting $v_2^2 = 2gl$ and solving for P, we write

$$P = W + \frac{W}{g}\frac{2gl}{l} = 3W$$

When a problem involves two particles or more, the principle of work and energy can be applied to each particle separately. Adding the kinetic energies of the various particles, and considering the work of all the forces acting on them, we can also write a single equation of work and energy for all the particles involved. We have

$$T_1 + U_{1 \rightarrow 2} = T_2 \tag{13.11}$$

where T represents the arithmetic sum of the kinetic energies of the particles involved (all terms are positive) and $U_{1 \rightarrow 2}$ is the work of all the forces acting on the particles, *including the forces of action and reaction exerted by the particles on each other*. In problems involving bodies connected by *inextensible cords or links,* however, the work of the forces exerted by a given cord or link on the two bodies it connects cancels out, since the points of application of these forces move through equal distances (see Sample Prob. 13.2).†

Since friction forces have a direction opposite of that of the displacement of the body on which they act, *the work of friction forces is always negative*. This work represents energy dissipated into heat and always results in a decrease in the kinetic energy of the body involved (see Sample Prob. 13.3).

13.5 POWER AND EFFICIENCY

Power is defined as the time rate at which work is done. In the selection of a motor or engine, power is a much more important criterion than is the actual amount of work to be performed. Either a small motor or a large power plant can be used to do a given amount of work; but the small motor may require a month to do the work done by the power plant in a matter of minutes. If ΔU is the work done during the time interval Δt, then the average power during that time interval is

$$\text{Average power} = \frac{\Delta U}{\Delta t}$$

Letting Δt approach zero, we obtain at the limit

$$\text{Power} = \frac{dU}{dt} \tag{13.12}$$

†The application of the method of work and energy to a system of particles is discussed in detail in Chap. 14.

Substituting the scalar product $\mathbf{F} \cdot d\mathbf{r}$ for dU, we can also write

$$\text{Power} = \frac{dU}{dt} = \frac{\mathbf{F} \cdot d\mathbf{r}}{dt}$$

and, recalling that $d\mathbf{r}/dt$ represents the velocity $\mathbf{v}$ of the point of application of $\mathbf{F}$,

$$\boxed{\text{Power} = \mathbf{F} \cdot \mathbf{v}} \tag{13.13}$$

Since power was defined as the time rate at which work is done, it should be expressed in units obtained by dividing units of work by the unit of time. Thus, if SI units are used, power should be expressed in J/s; this unit is called a *watt* (W). We have

$$1 \text{ W} = 1 \text{ J/s} = 1 \text{ N} \cdot \text{m/s}$$

If U.S. customary units are used, power should be expressed in ft · lb/s or in *horsepower* (hp), with the latter defined as

$$1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s}$$

Recalling from Sec. 13.2 that 1 ft · lb = 1.356 J, we verify that

$$1 \text{ ft} \cdot \text{lb/s} = 1.356 \text{ J/s} = 1.356 \text{ W}$$
$$1 \text{ hp} = 550(1.356 \text{ W}) = 746 \text{ W} = 0.746 \text{ kW}$$

The *mechanical efficiency* of a machine was defined in Sec. 10.5 as the ratio of the output work to the input work:

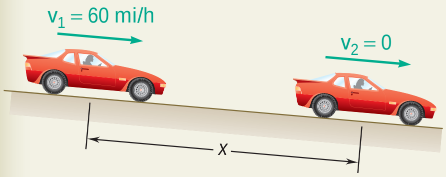
$$\eta = \frac{\text{output work}}{\text{input work}} \tag{13.14}$$

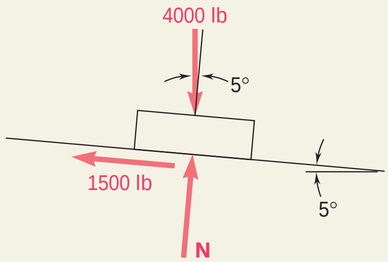
This definition is based on the assumption that work is done at a constant rate. The ratio of the output to the input work is therefore equal to the ratio of the rates at which output and input work are done, and we have

$$\boxed{\eta = \frac{\text{power output}}{\text{power input}}} \tag{13.15}$$

Because of energy losses due to friction, the output work is always smaller than the input work, and consequently the power output is always smaller than the power input. The mechanical efficiency of a machine is therefore always less than 1.

When a machine is used to transform mechanical energy into electric energy, or thermal energy into mechanical energy, its *overall efficiency* can be obtained from formula (13.15). The overall efficiency of a machine is always less than 1; it provides a measure of all the various energy losses involved (losses of electric or thermal energy as well as frictional losses). Note that it is necessary to express the power output and the power input in the same units before using formula (13.15).





SAMPLE PROBLEM 13.1

An automobile weighing 4000 lb is driven down a 5° incline at a speed of 60 mi/h when the brakes are applied, causing a constant total braking force (applied by the road on the tires) of 1500 lb. Determine the distance traveled by the automobile as it comes to a stop.

SOLUTION

Kinetic Energy

Position 1: $v_1 = \left(60 \frac{\text{mi}}{\text{h}}\right)\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 88 \text{ ft/s}$

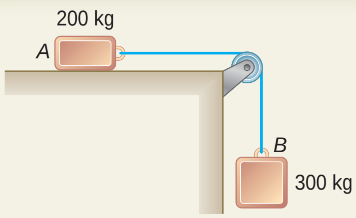
$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(4000/32.2)(88)^2 = 481{,}000 \text{ ft} \cdot \text{lb}$$

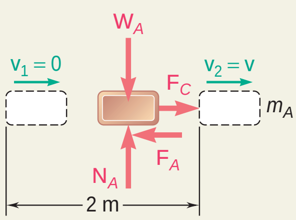
Position 2: $v_2 = 0 \qquad T_2 = 0$

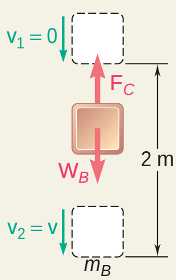
Work $\quad U_{1\to 2} = -1500x + (4000 \sin 5°)x = -1151x$

Principle of Work and Energy

$$T_1 + U_{1\to 2} = T_2$$
$$481{,}000 - 1151x = 0 \qquad x = 418 \text{ ft} \quad \blacktriangleleft$$







SAMPLE PROBLEM 13.2

Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block A after it has moved 2 m. Assume that the coefficient of kinetic friction between block A and the plane is $\mu_k = 0.25$ and that the pulley is weightless and frictionless.

SOLUTION

Work and Energy for Block *A*. We denote the friction force by $\mathbf{F}_A$ and the force exerted by the cable by $\mathbf{F}_C$, and write

$$m_A = 200 \text{ kg} \qquad W_A = (200 \text{ kg})(9.81 \text{ m/s}^2) = 1962 \text{ N}$$
$$F_A = \mu_k N_A = \mu_k W_A = 0.25(1962 \text{ N}) = 490 \text{ N}$$
$$T_1 + U_{1\to 2} = T_2: \qquad 0 + F_C(2 \text{ m}) - F_A(2 \text{ m}) = \tfrac{1}{2}m_A v^2$$
$$F_C(2 \text{ m}) - (490 \text{ N})(2 \text{ m}) = \tfrac{1}{2}(200 \text{ kg})v^2 \qquad (1)$$

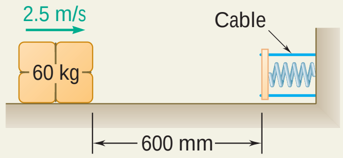
Work and Energy for Block *B*. We write

$$m_B = 300 \text{ kg} \qquad W_B = (300 \text{ kg})(9.81 \text{ m/s}^2) = 2940 \text{ N}$$
$$T_1 + U_{1\to 2} = T_2: \qquad 0 + W_B(2 \text{ m}) - F_C(2 \text{ m}) = \tfrac{1}{2}m_B v^2$$
$$(2940 \text{ N})(2 \text{ m}) - F_C(2 \text{ m}) = \tfrac{1}{2}(300 \text{ kg})v^2 \qquad (2)$$

Adding the left-hand and right-hand members of (1) and (2), we observe that the work of the forces exerted by the cable on A and B cancels out:

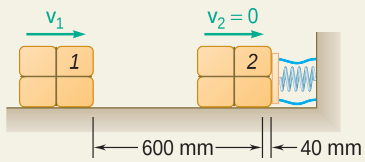
$$(2940 \text{ N})(2 \text{ m}) - (490 \text{ N})(2 \text{ m}) = \tfrac{1}{2}(200 \text{ kg} + 300 \text{ kg})v^2$$
$$4900 \text{ J} = \tfrac{1}{2}(500 \text{ kg})v^2 \qquad v = 4.43 \text{ m/s} \quad \blacktriangleleft$$

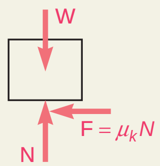
SAMPLE PROBLEM 13.3

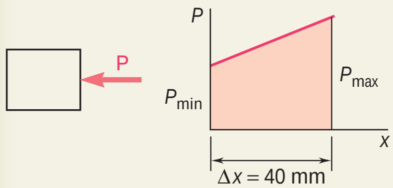


A spring is used to stop a 60-kg package which is sliding on a horizontal surface. The spring has a constant $k = 20$ kN/m and is held by cables so that it is initially compressed 120 mm. Knowing that the package has a velocity of 2.5 m/s in the position shown and that the maximum additional deflection of the spring is 40 mm, determine (*a*) the coefficient of kinetic friction between the package and the surface, (*b*) the velocity of the package as it passes again through the position shown.

SOLUTION







a. Motion from Position 1 to Position 2

Kinetic Energy ***Position 1:*** $v_1 = 2.5$ m/s

$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(60\text{ kg})(2.5\text{ m/s})^2 = 187.5\text{ N}\cdot\text{m} = 187.5\text{ J}$$

Position 2: (maximum spring deflection): $v_2 = 0 \qquad T_2 = 0$

Work

Friction Force **F**. We have

$$F = \mu_k N = \mu_k W = \mu_k mg = \mu_k(60\text{ kg})(9.81\text{ m/s}^2) = (588.6\text{ N})\mu_k$$

The work of **F** is negative and equal to

$$(U_{1\to 2})_f = -Fx = -(588.6\text{ N})\mu_k(0.600\text{ m} + 0.040\text{ m}) = -(377\text{ J})\mu_k$$

Spring Force **P**. The variable force **P** exerted by the spring does an amount of negative work equal to the area under the force-deflection curve of the spring force. We have

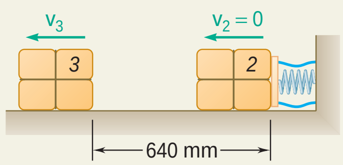
$$P_{min} = kx_0 = (20\text{ kN/m})(120\text{ mm}) = (20\,000\text{ N/m})(0.120\text{ m}) = 2400\text{ N}$$
$$P_{max} = P_{min} + k\,\Delta x = 2400\text{ N} + (20\text{ kN/m})(40\text{ mm}) = 3200\text{ N}$$
$$(U_{1\to 2})_e = -\tfrac{1}{2}(P_{min} + P_{max})\,\Delta x = -\tfrac{1}{2}(2400\text{ N} + 3200\text{ N})(0.040\text{ m}) = -112.0\text{ J}$$

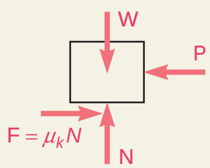
The total work is thus

$$U_{1\to 2} = (U_{1\to 2})_f + (U_{1\to 2})_e = -(377\text{ J})\mu_k - 112.0\text{ J}$$

Principle of Work and Energy

$$T_1 + U_{1\to 2} = T_2: \qquad 187.5\text{ J} - (377\text{ J})\mu_k - 112.0\text{ J} = 0 \qquad \mu_k = 0.20$$ ◀





b. Motion from Position 2 to Position 3

Kinetic Energy. ***Position 2:*** $v_2 = 0 \qquad T_2 = 0$

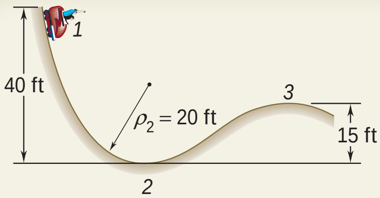
Position 3: $T_3 = \tfrac{1}{2}mv_3^2 = \tfrac{1}{2}(60\text{ kg})v_3^2$

Work. Since the distances involved are the same, the numerical values of the work of the friction force **F** and of the spring force **P** are the same as above. However, while the work of **F** is still negative, the work of **P** is now positive.

$$U_{2\to 3} = -(377\text{ J})\mu_k + 112.0\text{ J} = -75.5\text{ J} + 112.0\text{ J} = +36.5\text{ J}$$

Principle of Work and Energy

$$T_2 + U_{2\to 3} = T_3: \qquad 0 + 36.5\text{ J} = \tfrac{1}{2}(60\text{ kg})v_3^2$$
$$v_3 = 1.103\text{ m/s} \qquad \mathbf{v}_3 = 1.103\text{ m/s} \leftarrow$$ ◀



SAMPLE PROBLEM 13.4

A 2000-lb car starts from rest at point *1* and moves without friction down the track shown. (*a*) Determine the force exerted by the track on the car at point 2, where the radius of curvature of the track is 20 ft. (*b*) Determine the minimum safe value of the radius of curvature at point 3.

SOLUTION

a. Force Exerted by the Track at Point *2*. The principle of work and energy is used to determine the velocity of the car as it passes through point *2*.

Kinetic Energy. $T_1 = 0 \qquad T_2 = \tfrac{1}{2}mv_2^2 = \dfrac{1}{2}\dfrac{W}{g}v_2^2$

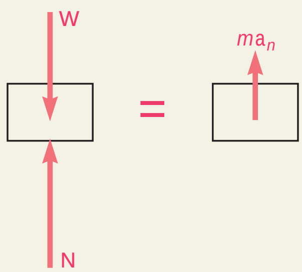
Work. The only force which does work is the weight **W**. Since the vertical displacement from point *1* to point *2* is 40 ft downward, the work of the weight is

$$U_{1\rightarrow 2} = +W(40\text{ ft})$$

Principle of Work and Energy

$$T_1 + U_{1\rightarrow 2} = T_2 \qquad 0 + W(40\text{ ft}) = \frac{1}{2}\frac{W}{g}v_2^2$$

$$v_2^2 = 80g = 80(32.2) \qquad v_2 = 50.8\text{ ft/s}$$



Newton's Second Law at Point 2. The acceleration $\mathbf{a}_n$ of the car at point *2* has a magnitude $a_n = v_2^2/\rho$ and is directed upward. Since the external forces acting on the car are **W** and **N**, we write

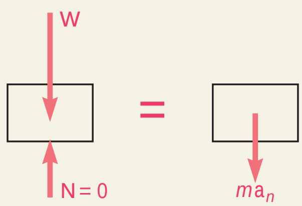
$$+\uparrow \Sigma F_n = ma_n: \qquad -W + N = ma_n$$
$$= \frac{W}{g}\frac{v_2^2}{\rho}$$
$$= \frac{W}{g}\frac{80g}{20}$$

$$N = 5W \qquad \mathbf{N} = 10{,}000\text{ lb}\uparrow \blacktriangleleft$$

b. Minimum Value of ρ at Point *3*. ***Principle of Work and Energy.*** Applying the principle of work and energy between point *1* and point *3*, we obtain

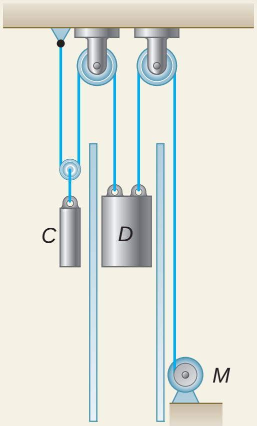
$$T_1 + U_{1\rightarrow 3} = T_3 \qquad 0 + W(25\text{ ft}) = \frac{1}{2}\frac{W}{g}v_3^2$$

$$v_3^2 = 50g = 50(32.2) \qquad v_3 = 40.1\text{ ft/s}$$



Newton's Second Law at Point 3. The minimum safe value of ρ occurs when $\mathbf{N} = 0$. In this case, the acceleration $\mathbf{a}_n$, of magnitude $a_n = v_3^2/\rho$, is directed downward, and we write

$$+\downarrow \Sigma F_n = ma_n: \qquad W = \frac{W}{g}\frac{v_3^2}{\rho}$$
$$= \frac{W}{g}\frac{50g}{\rho} \qquad \rho = 50\text{ ft} \blacktriangleleft$$

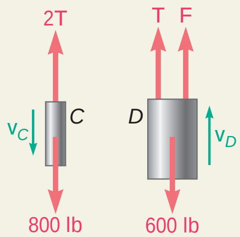


SAMPLE PROBLEM 13.5

The dumbwaiter D and its load have a combined weight of 600 lb, while the counterweight C weighs 800 lb. Determine the power delivered by the electric motor M when the dumbwaiter (a) is moving up at a constant speed of 8 ft/s, (b) has an instantaneous velocity of 8 ft/s and an acceleration of 2.5 ft/s^2, both directed upward.

SOLUTION

Since the force $\mathbf{F}$ exerted by the motor cable has the same direction as the velocity $\mathbf{v}_D$ of the dumbwaiter, the power is equal to Fv_D, where $v_D = 8$ ft/s. To obtain the power, we must first determine $\mathbf{F}$ in each of the two given situations.



a. Uniform Motion. We have $\mathbf{a}_C = \mathbf{a}_D = 0$; both bodies are in equilibrium.

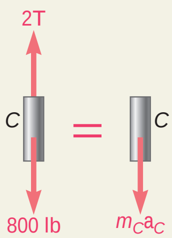
Free Body C: $\quad +\uparrow\Sigma F_y = 0$: $\quad 2T - 800 \text{ lb} = 0 \qquad T = 400 \text{ lb}$

Free Body D: $\quad +\uparrow\Sigma F_y = 0$: $\quad F + T - 600 \text{ lb} = 0$

$$F = 600 \text{ lb} - T = 600 \text{ lb} - 400 \text{ lb} = 200 \text{ lb}$$

$$Fv_D = (200 \text{ lb})(8 \text{ ft/s}) = 1600 \text{ ft} \cdot \text{lb/s}$$

$$\text{Power} = (1600 \text{ ft} \cdot \text{lb/s})\frac{1 \text{ hp}}{550 \text{ ft} \cdot \text{lb/s}} = 2.91 \text{ hp} \quad \blacktriangleleft$$



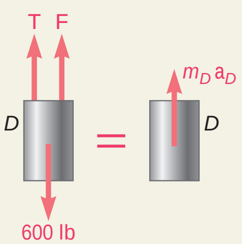
b. Accelerated Motion. We have

$$\mathbf{a}_D = 2.5 \text{ ft/s}^2\uparrow \qquad \mathbf{a}_C = -\tfrac{1}{2}\mathbf{a}_D = 1.25 \text{ ft/s}^2\downarrow$$

The equations of motion are

Free Body C: $\quad +\downarrow\Sigma F_y = m_C a_C$: $\quad 800 - 2T = \dfrac{800}{32.2}(1.25) \qquad T = 384.5 \text{ lb}$

Free Body D: $\quad +\uparrow\Sigma F_y = m_D a_D$: $\quad F + T - 600 = \dfrac{600}{32.2}(2.5)$



$$F + 384.5 - 600 = 46.6 \qquad F = 262.1 \text{ lb}$$

$$Fv_D = (262.1 \text{ lb})(8 \text{ ft/s}) = 2097 \text{ ft} \cdot \text{lb/s}$$

$$\text{Power} = (2097 \text{ ft} \cdot \text{lb/s})\frac{1 \text{ hp}}{550 \text{ ft} \cdot \text{lb/s}} = 3.81 \text{ hp} \quad \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In the preceding chapter, you solved problems dealing with the motion of a particle by using the fundamental equation $\mathbf{F} = m\mathbf{a}$ to determine the acceleration $\mathbf{a}$. By applying the principles of kinematics you were then able to determine from $\mathbf{a}$ the velocity and displacement of the particle at any time. In this lesson we combined $\mathbf{F} = m\mathbf{a}$ and the principles of kinematics to obtain an additional method of analysis called the *method of work and energy*. This eliminates the need to calculate the acceleration and will enable you to relate the velocities of the particle at two points along its path of motion. To solve a problem by the method of work and energy you will follow these steps:

1. Computing the work of each of the forces. The work $U_{1\,y\,2}$ of a given force $\mathbf{F}$ during the finite displacement of the particle from A_1 to A_2 is defined as

$$U_{1\,y\,2} = \int \mathbf{F} \cdot d\mathbf{r} \qquad \text{or} \qquad U_{1\,y\,2} = \int (F \cos \mathsf{a})\, ds \qquad (13.2, 13.2')$$

where a is the angle between $\mathbf{F}$ and the displacement $d\mathbf{r}$. The work $U_{1\,y\,2}$ is a scalar quantity and is expressed in ft · lb or in · lb in the U.S. customary system of units and in N · m or joules (J) in the SI system of units. Note that the work done is zero for a force perpendicular to the displacement ($\mathsf{a} = 90°$). Negative work is done for $90° < \mathsf{a} < 180°$ and in particular for a friction force, which is always opposite in direction to the displacement ($\mathsf{a} = 180°$).

The work $U_{1\,y\,2}$ can be easily evaluated in the following cases that you will encounter:

a. Work of a constant force in rectilinear motion

$$U_{1\,y\,2} = (F \cos \mathsf{a})\, \Delta x \qquad (13.3)$$

where a = angle the force forms with the direction of motion
Δx = displacement from A_1 to A_2 (Fig. 13.3)

b. Work of the force of gravity

$$U_{1\,y\,2} = -W\, \Delta y \qquad (13.4')$$

where Δy is the vertical displacement of the center of gravity of the body of weight W. Note that the work is positive when Δy is negative, that is, when the body moves down (Fig. 13.4).

c. Work of the force exerted by a spring

$$U_{1\,y\,2} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \qquad (13.6)$$

where k is the spring constant and x_1 and x_2 are the elongations of the spring corresponding to the positions A_1 and A_2 (Fig. 13.5).

(continued)

d. Work of a gravitational force

$$U_{1 y 2} = \frac{GMm}{r_2} - \frac{GMm}{r_1} \tag{13.7}$$

for a displacement of the body from $A_1(r = r_1)$ to $A_2(r = r_2)$ (Fig. 13.6).

2. Calculate the kinetic energy at A_1 and A_2. The kinetic energy T is

$$T = \tfrac{1}{2}mv^2 \tag{13.9}$$

where m is the mass of the particle and v is the magnitude of its velocity. The units of kinetic energy are the same as the units of work, that is, ft · lb or in · lb if U.S. customary units are used and N · m or joules (J) if SI units are used.

3. Substitute the values for the work done $U_{1 y 2}$ and the kinetic energies T_1 and T_2 into the equation

$$T_1 + U_{1 y 2} = T_2 \tag{13.11}$$

You will now have *one equation* which you can solve for *one unknown*. Note that this equation does not yield the time of travel or the acceleration directly. However, if you know the radius of curvature r of the path of the particle at a point where you have obtained the velocity v, you can express the normal component of the acceleration as $a_n = v^2/\mathrm{r}$ and obtain the normal component of the force exerted on the particle by writing $F_n = mv^2/\mathrm{r}$.

4. Power was introduced in this lesson as the time rate at which work is done, $P = dU/dt$. Power is measured in ft · lb/s or *horsepower* (hp) in U.S. customary units and in J/s or *watts* (W) in the SI system of units. To calculate the power, you can use the equivalent formula,

$$P = \mathbf{F} \cdot \mathbf{v} \tag{13.13}$$

where **F** and **v** denote the force and the velocity, respectively, at a given time [Sample Prob. 13.5]. In some problems [see, e.g., Prob. 13.47], you will be asked for the *average power*, which can be obtained by dividing the total work by the time interval during which the work is done.

PROBLEMS

CONCEPT QUESTION

13.CQ1 Block A is traveling with a speed v_0 on a smooth surface when the surface suddenly becomes rough with a coefficient of friction of μ causing the block to stop after a distance d. If block A were traveling twice as fast, that is, at a speed $2v_0$, how far will it travel on the rough surface before stopping?

a. $d/2$
b. d
c. $\sqrt{2}d$
d. $2d$
e. $4d$

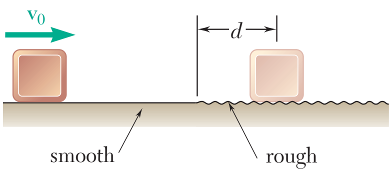


Fig. P13.CQ1

END-OF-SECTION PROBLEMS

13.1 A 400-kg satellite was placed in a circular orbit 1500 km above the surface of the earth. At this elevation the acceleration of gravity is 6.43 m/s^2. Determine the kinetic energy of the satellite, knowing that its orbital speed is 25.6×10^3 km/h.

13.2 A 1-lb stone is dropped down the "bottomless pit" at Carlsbad Caverns and strikes the ground with a speed of 95 ft/s. Neglecting air resistance, (*a*) determine the kinetic energy of the stone as it strikes the ground and the height h from which it was dropped. (*b*) Solve part *a* assuming that the same stone is dropped down a hole on the moon. (Acceleration of gravity on the moon = 5.31 ft/s^2.)

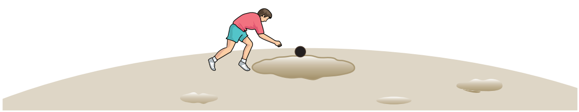


Fig. P13.2

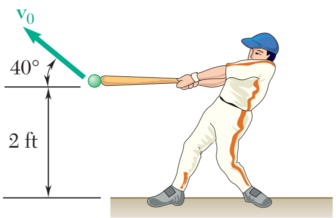


Fig. *P13.3*

13.3 A baseball player hits a 5.1-oz baseball with an initial velocity of 140 ft/s at an angle of 40° with the horizontal as shown. Determine (*a*) the kinetic energy of the ball immediately after it is hit, (*b*) the kinetic energy of the ball when it reaches its maximum height, (*c*) the maximum height above the ground reached by the ball.

13.4 A 500-kg communications satellite is in a circular geosynchronous orbit and completes one revolution about the earth in 23 h and 56 min at an altitude of 35 800 km above the surface of the earth. Knowing that the radius of the earth is 6370 km, determine the kinetic energy of the satellite.

13.5 In an ore-mixing operation, a bucket full of ore is suspended from a traveling crane which moves along a stationary bridge. The bucket is to swing no more than 10 ft horizontally when the crane is brought to a sudden stop. Determine the maximum allowable speed v of the crane.

13.6 In an ore-mixing operation, a bucket full of ore is suspended from a traveling crane which moves along a stationary bridge. The crane is traveling at a speed of 10 ft/s when it is brought to a sudden stop. Determine the maximum horizontal distance through which the bucket will swing.

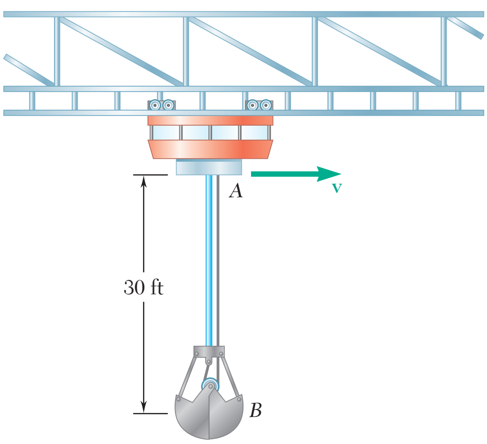


Fig. P13.5 and P13.6

13.7 Determine the maximum theoretical speed that may be achieved over a distance of 110 m by a car starting from rest assuming there is no slipping. The coefficient of static friction between the tires and pavement is 0.75, and 60 percent of the weight of the car is distributed over its front wheels and 40 percent over its rear wheels. Assume (*a*) front-wheel drive, (*b*) rear-wheel drive.

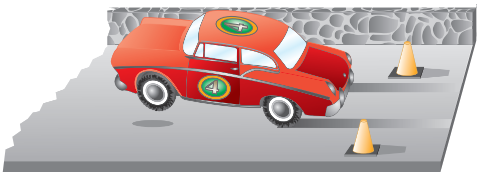


Fig. P13.8

13.8 Skid marks on a drag racetrack indicate that the rear (drive) wheels of a car slip for the first 20 m of the 400-m track. (*a*) Knowing that the coefficient of kinetic friction is 0.60, determine the speed of the car at the end of the first 20-m portion of the track if it starts from rest and the front wheels are just off the ground. (*b*) What is the maximum theoretical speed of the car at the finish line if, after skidding for 20 m, it is driven without the wheels slipping for the remainder of the race? Assume that while the car is rolling without slipping, 60 percent of the weight of the car is on the rear wheels and the coefficient of static friction is 0.75. Ignore air resistance and rolling resistance.

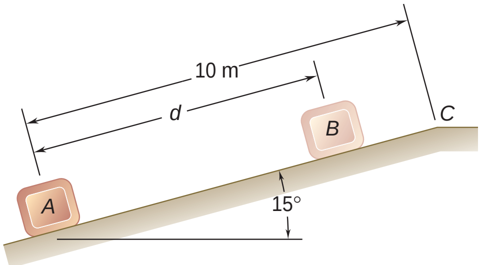


Fig. P13.9

13.9 A package is projected up a 15° incline at A with an initial velocity of 8 m/s. Knowing that the coefficient of kinetic friction between the package and the incline is 0.12, determine (*a*) the maximum distance d that the package will move up the incline, (*b*) the velocity of the package as it returns to its original position.

13.10 A 1.4-kg model rocket is launched vertically from rest with a constant thrust of 25 N until the rocket reaches an altitude of 15 m and the thrust ends. Neglecting air resistance, determine (*a*) the speed of the rocket when the thrust ends, (*b*) the maximum height reached by the rocket, (*c*) the speed of the rocket when it returns to the ground.

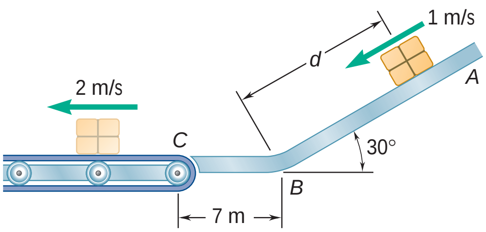


Fig. P13.11 and P13.12

13.11 Packages are thrown down an incline at A with a velocity of 1 m/s. The packages slide along the surface ABC to a conveyor belt which moves with a velocity of 2 m/s. Knowing that $\mu_k = 0.25$ between the packages and the surface ABC, determine the distance d if the packages are to arrive at C with a velocity of 2 m/s.

13.12 Packages are thrown down an incline at A with a velocity of 1 m/s. The packages slide along the surface ABC to a conveyor belt which moves with a velocity of 2 m/s. Knowing that $d = 7.5$ m and $\mu_k = 0.25$ between the packages and all surfaces, determine (*a*) the speed of the package at C, (*b*) the distance a package will slide on the conveyor belt before it comes to rest relative to the belt.

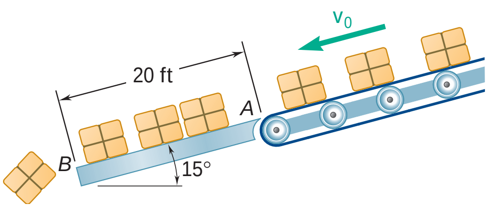


Fig. *P13.13* and *P13.14*

13.13 Boxes are transported by a conveyor belt with a velocity $\mathbf{v}_0$ to a fixed incline at A where they slide and eventually fall off at B. Knowing that $\mu_k = 0.40$, determine the velocity of the conveyor belt if the boxes leave the incline at B with a velocity of 8 ft/s.

13.14 Boxes are transported by a conveyor belt with a velocity $\mathbf{v}_0$ to a fixed incline at A where they slide and eventually fall off at B. Knowing that $\mu_k = 0.40$, determine the velocity of the conveyor belt if the boxes are to have zero velocity at B.

13.15 A 1200-kg trailer is hitched to a 1400-kg car. The car and trailer are traveling at 72 km/h when the driver applies the brakes on both the car and the trailer. Knowing that the braking forces exerted on the car and the trailer are 5000 N and 4000 N, respectively, determine (*a*) the distance traveled by the car and trailer before they come to a stop, (*b*) the horizontal component of the force exerted by the trailer hitch on the car.

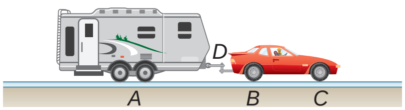


Fig. P13.15

13.16 A trailer truck enters a 2 percent uphill grade traveling at 72 km/h and reaches a speed of 108 km/h in 300 m. The cab has a mass of 1800 kg and the trailer 5400 kg. Determine (*a*) the average force at the wheels of the cab, (*b*) the average force in the coupling between the cab and the trailer.

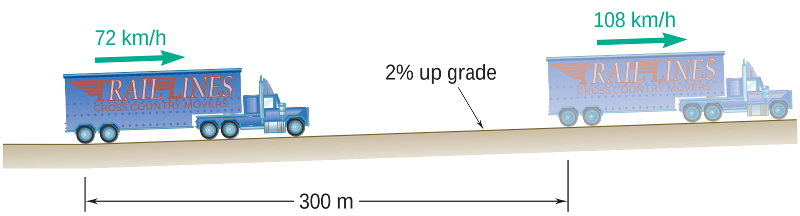


Fig. P13.16

13.17 The subway train shown is traveling at a speed of 30 mi/h when the brakes are fully applied on the wheels of cars *B* and *C*, causing them to slide on the track, but are not applied on the wheels of car *A*. Knowing that the coefficient of kinetic friction is 0.35 between the wheels and the track, determine (*a*) the distance required to bring the train to a stop, (*b*) the force in each coupling.

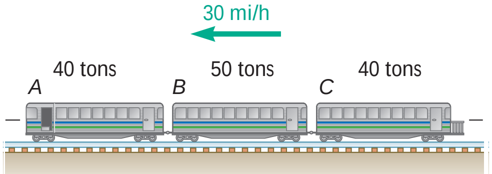


Fig. P13.17 and P13.18

13.18 The subway train shown is traveling at a speed of 30 mi/h when the brakes are fully applied on the wheels of car *A*, causing it to slide on the track, but are not applied on the wheels of cars *B* or *C*. Knowing that the coefficient of kinetic friction is 0.35 between the wheels and the track, determine (*a*) the distance required to bring the train to a stop, (*b*) the force in each coupling.

13.19 Blocks *A* and *B* weigh 25 lb and 10 lb, respectively, and they are both at a height 6 ft above the ground when the system is released from rest. Just before hitting the ground block *A* is moving at a speed of 9 ft/s. Determine (*a*) the amount of energy dissipated in friction by the pulley, (*b*) the tension in each portion of the cord during the motion.

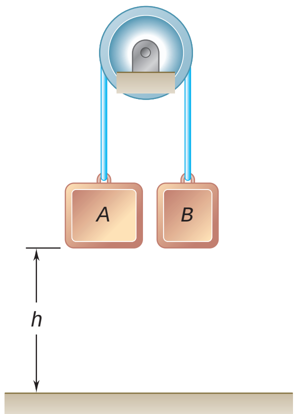


Fig. P13.19

13.20 The system shown is at rest when a constant 30-lb force is applied to collar B. (*a*) If the force acts through the entire motion, determine the speed of collar B as it strikes the support at C. (*b*) After what distance d should the 30-lb force be removed if the collar is to reach support C with zero velocity?

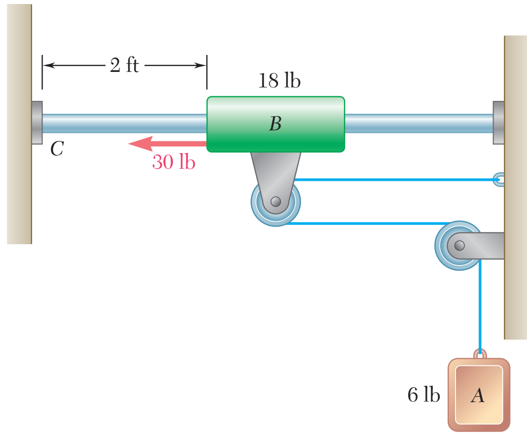


Fig. P13.20

13.21 Car B is towing car A at a constant speed of 10 m/s on an uphill grade when the brakes of car A are fully applied causing all four wheels to skid. The driver of car B does not change the throttle setting or change gears. The masses of the cars A and B are 1400 kg and 1200 kg, respectively, and the coefficient of kinetic friction is 0.8. Neglecting air resistance and rolling resistance, determine (*a*) the distance traveled by the cars before they come to a stop, (*b*) the tension in the cable.

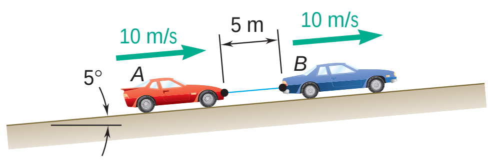


Fig. *P13.21*

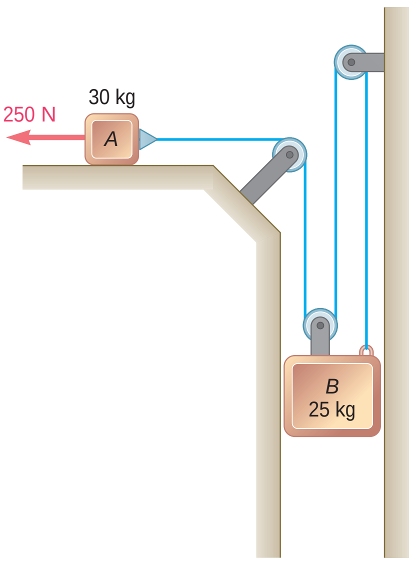


Fig. *P13.22* and P13.23

13.22 The system shown is at rest when a constant 250-N force is applied to block A. Neglecting the masses of the pulleys and the effect of friction in the pulleys and between block A and the horizontal surface, determine (*a*) the velocity of block B after block A has moved 2 m, (*b*) the tension in the cable.

13.23 The system shown is at rest when a constant 250-N force is applied to block A. Neglecting the masses of the pulleys and the effect of friction in the pulleys and assuming that the coefficients of friction between block A and the horizontal surface are $\mu_s = 0.25$ and $\mu_k = 0.20$, determine (*a*) the velocity of block B after block A has moved 2 m, (*b*) the tension in the cable.

13.24 Two blocks A and B, of mass 4 kg and 5 kg, respectively, are connected by a cord which passes over pulleys as shown. A 3-kg collar C is placed on block A and the system is released from rest. After the blocks have moved 0.9 m, collar C is removed and blocks A and B continue to move. Determine the speed of block A just before it strikes the ground.

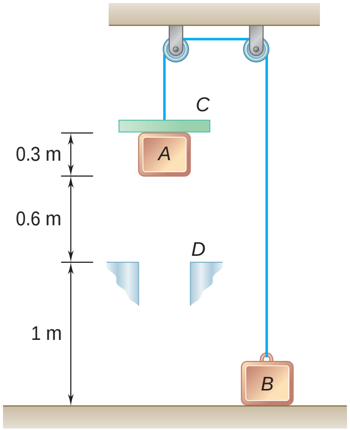


Fig. P13.24

13.25 Four packages, each weighing 6 lb, are held in place by friction on a conveyor which is disengaged from its drive motor. When the system is released from rest, package *1* leaves the belt at A just as package *4* comes onto the inclined portion of the belt at B. Determine (*a*) the speed of package *2* as it leaves the belt at A, (*b*) the speed of package *3* as it leaves the belt at A. Neglect the mass of the belt and rollers.

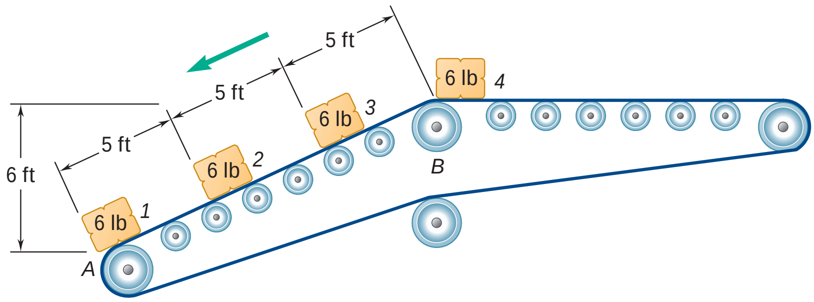


Fig. P13.25

13.26 A 3-kg block rests on top of a 2-kg block supported by but not attached to a spring of constant 40 N/m. The upper block is suddenly removed. Determine (*a*) the maximum speed reached by the 2-kg block, (*b*) the maximum height reached by the 2-kg block.

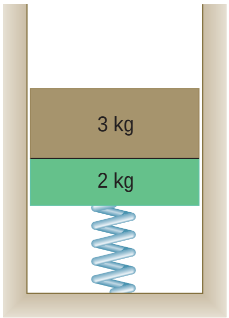


Fig. P13.26

13.27 Solve Prob. 13.26, assuming that the 2-kg block is attached to the spring.

13.28 An 8-lb collar C slides on a horizontal rod between springs A and B. If the collar is pushed to the right until spring B is compressed 2 in. and released, determine the distance through which the collar will travel assuming (*a*) no friction between the collar and the rod, (*b*) a coefficient of friction $\mu_k = 0.35$.

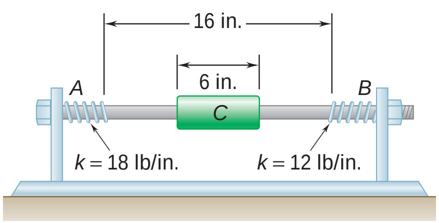


Fig. P13.28

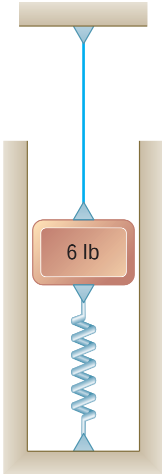


Fig. P13.29

13.29 A 6-lb block is attached to a cable and to a spring as shown. The constant of the spring is $k = 8$ lb/in. and the tension in the cable is 3 lb. If the cable is cut, determine (*a*) the maximum displacement of the block, (*b*) the maximum speed of the block.

13.30 A 10-kg block is attached to spring *A* and connected to spring *B* by a cord and pulley. The block is held in the position shown with both springs unstretched when the support is removed and the block is released with no initial velocity. Knowing that the constant of each spring is 2 kN/m, determine (*a*) the velocity of the block after it has moved down 50 mm, (*b*) the maximum velocity achieved by the block.

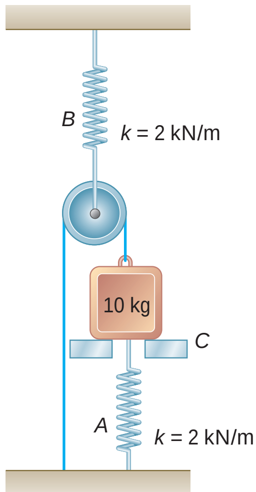


Fig. *P13.30*

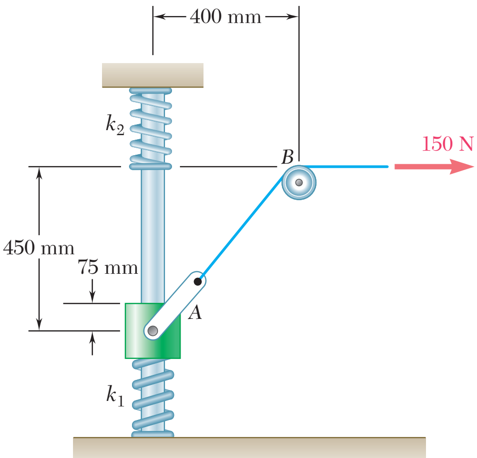


Fig. *P13.31*

13.31 A 5-kg collar *A* is at rest on top of, but not attached to, a spring with stiffness $k_1 = 400$ N/m when a constant 150-N force is applied to the cable. Knowing *A* has a speed of 1 m/s when the upper spring is compressed 75 mm, determine the spring stiffness k_2. Ignore friction and the mass of the pulley.

13.32 A piston of mass m and cross-sectional area A is in equilibrium under the pressure p at the center of a cylinder closed at both ends. Assuming that the piston is moved to the left a distance $a/2$ and released, and knowing that the pressure on each side of the piston varies inversely with the volume, determine the velocity of the piston as it again reaches the center of the cylinder. Neglect friction between the piston and the cylinder and express your answer in terms of m, a, p, and A.

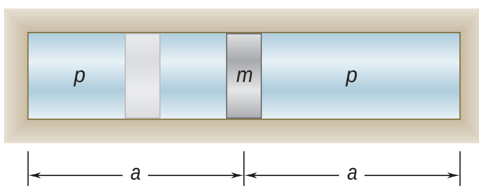


Fig. P13.32

13.33 An uncontrolled automobile traveling at 65 mph strikes squarely a highway crash cushion of the type shown in which the automobile is brought to rest by successively crushing steel barrels. The magnitude F of the force required to crush the barrels is shown as a function of the distance x the automobile has moved into the cushion. Knowing that the weight of the automobile is 2250 lb and neglecting the effect of friction, determine (*a*) the distance the automobile will move into the cushion before it comes to rest, (*b*) the maximum deceleration of the automobile.

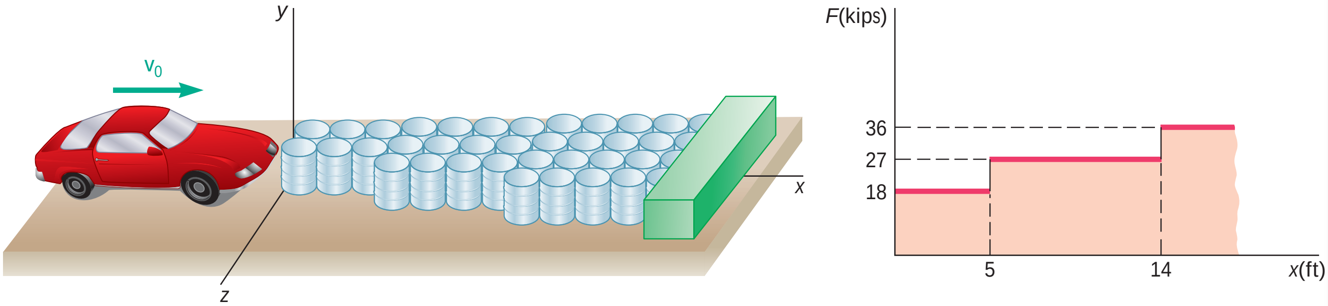


Fig. P13.33

13.34 Two types of energy-absorbing fenders designed to be used on a pier are statically loaded. The force-deflection curve for each type of fender is given in the graph. Determine the maximum deflection of each fender when a 90-ton ship moving at 1 mi/h strikes the fender and is brought to rest.

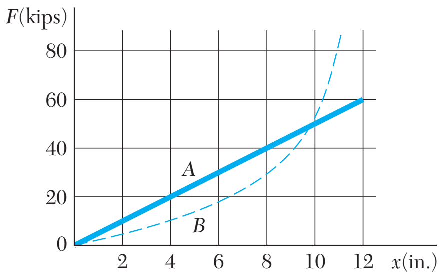


Fig. P13.34

13.35 Nonlinear springs are classified as hard or soft, depending upon the curvature of their force-deflection curve (see figure). If a delicate instrument having a mass of 5 kg is placed on a spring of length l so that its base is just touching the undeformed spring and then inadvertently released from that position, determine the maximum deflection x_m of the spring and the maximum force F_m exerted by the spring, assuming (*a*) a linear spring of constant $k = 3$ kN/m, (*b*) a hard, nonlinear spring, for which $F = (3 \text{ kN/m})(x + 160x^3)$.

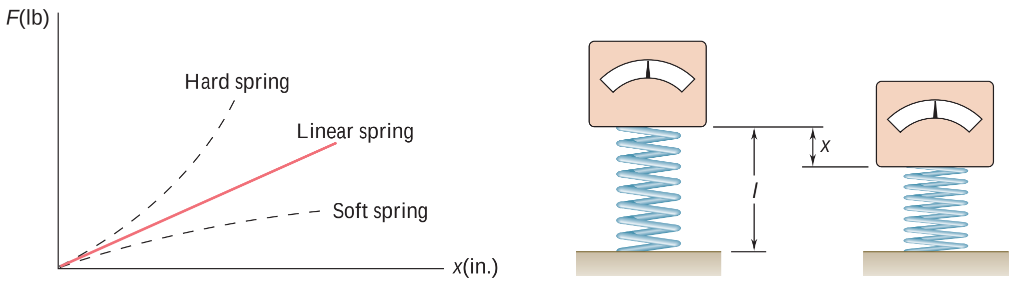


Fig. *P13.35*

13.36 A rocket is fired vertically from the surface of the moon with a speed v_0. Derive a formula for the ratio h_n/h_u of heights reached with a speed v, if Newton's law of gravitation is used to calculate h_n and a uniform gravitational field is used to calculate h_u. Express your answer in terms of the acceleration of gravity g_m on the surface of the moon, the radius R_m of the moon, and the speeds v and v_0.

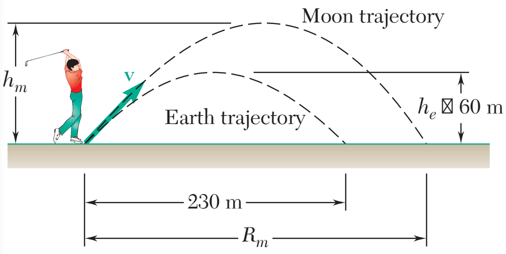


Fig. P13.38

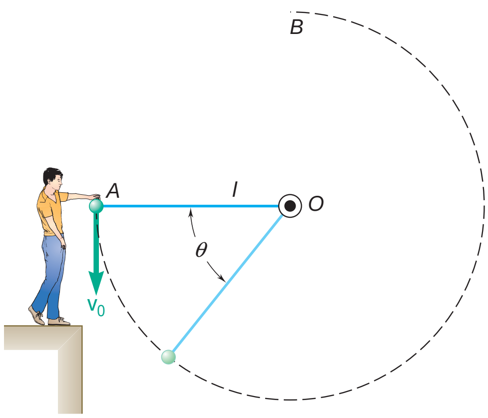


Fig. P13.39 and P13.40

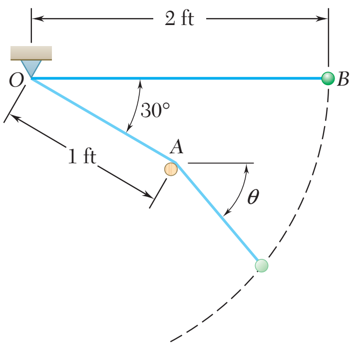


Fig. P13.41

13.37 Express the acceleration of gravity g_h at an altitude h above the surface of the earth in terms of the acceleration of gravity g_0 at the surface of the earth, the altitude h, and the radius R of the earth. Determine the percent error if the weight that an object has on the surface of earth is used as its weight at an altitude of (*a*) 1 km, (*b*) 1000 km.

13.38 A golf ball struck on earth rises to a maximum height of 60 m and hits the ground 230 m away. How high will the same golf ball travel on the moon if the magnitude and direction of its velocity are the same as they were on earth immediately after the ball was hit? Assume that the ball is hit and lands at the same elevation in both cases and that the effect of the atmosphere on the earth is neglected, so that the trajectory in both cases is a parabola. The acceleration of gravity on the moon is 0.165 times that on earth.

13.39 The sphere at A is given a downward velocity $\mathbf{v}_0$ of magnitude 5 m/s and swings in a vertical plane at the end of a rope of length l = 2 m attached to a support at O. Determine the angle u at which the rope will break, knowing that it can withstand a maximum tension equal to twice the weight of the sphere.

13.40 The sphere at A is given a downward velocity $\mathbf{v}_0$ and swings in a vertical circle of radius l and center O. Determine the smallest velocity $\mathbf{v}_0$ for which the sphere will reach point B as it swings about point O (*a*) if AO is a rope, (*b*) if AO is a slender rod of negligible mass.

13.41 A small sphere B of weight W is released from rest in the position shown and swings freely in a vertical plane, first about O and then about the peg A after the cord comes in contact with the peg. Determine the tension in the cord (*a*) just before the sphere comes in contact with the peg, (*b*) just after it comes in contact with the peg.

13.42 A roller coaster starts from rest at A, rolls down the track to B, describes a circular loop of 40-ft diameter, and moves up and down past point E. Knowing that h = 60 ft and assuming no energy loss due to friction, determine (*a*) the force exerted by his seat on a 160-lb rider at B and D, (*b*) the minimum value of the radius of curvature at E if the roller coaster is not to leave the track at that point.

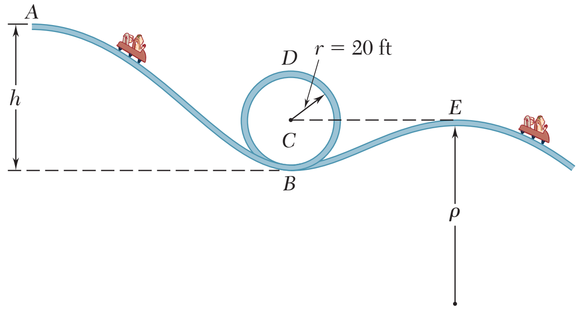


Fig. *P13.42*

13.43 In Prob. 13.42, determine the range of values of h for which the roller coaster will not leave the track at D or E, knowing that the radius of curvature at E is r = 75 ft. Assume no energy loss due to friction.

13.44 A small block slides at a speed v on a horizontal surface. Knowing that $h = 0.9$ m, determine the required speed of the block if it is to leave the cylindrical surface BCD when u = 30°.

13.45 A small block slides at a speed $v = 8$ ft/s on a horizontal surface at a height $h = 3$ ft above the ground. Determine (*a*) the angle u at which it will leave the cylindrical surface BCD, (*b*) the distance x at which it will hit the ground. Neglect friction and air resistance.

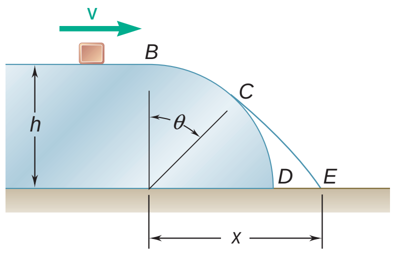


Fig. P13.44 and P13.45

13.46 A chair-lift is designed to transport 1000 skiers per hour from the base A to the summit B. The average mass of a skier is 70 kg and the average speed of the lift is 75 m/min. Determine (*a*) the average power required, (*b*) the required capacity of the motor if the mechanical efficiency is 85 percent and if a 300-percent overload is to be allowed.

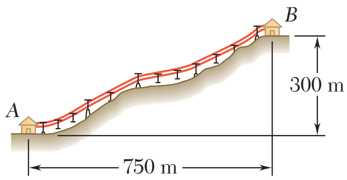


Fig. P13.46

13.47 It takes 15 s to raise a 1200-kg car and the supporting 300-kg hydraulic car-lift platform to a height of 2.8 m. Determine (*a*) the average output power delivered by the hydraulic pump to lift the system, (*b*) the average electric power required, knowing that the overall conversion efficiency from electric to mechanical power for the system is 82 percent.

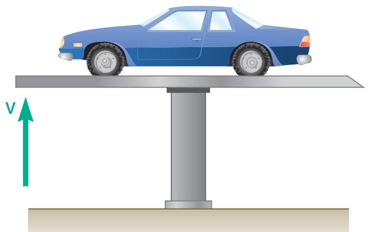


Fig. P13.47

13.48 The velocity of the lift of Prob. 13.47 increases uniformly from zero to its maximum value at mid-height in 7.5 s and then decreases uniformly to zero in 7.5 s. Knowing that the peak power output of the hydraulic pump is 6 kW when the velocity is maximum, determine the maximum lift force provided by the pump.

13.49 (*a*) A 120-lb woman rides a 15-lb bicycle up a 3-percent slope at a constant speed of 5 ft/s. How much power must be developed by the woman? (*b*) A 180-lb man on an 18-lb bicycle starts down the same slope and maintains a constant speed of 20 ft/s by braking. How much power is dissipated by the brakes? Ignore air resistance and rolling resistance.

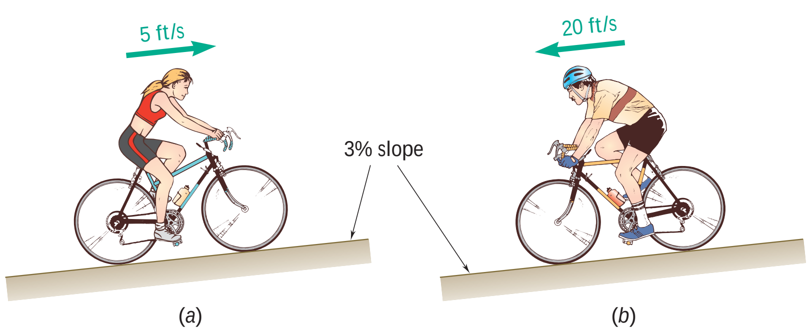


Fig. *P13.49*

13.50 A power specification formula is to be derived for electric motors which drive conveyor belts moving solid material at different rates to different heights and distances. Denoting the efficiency of a motor by η and neglecting the power needed to drive the belt itself, derive a formula (*a*) in the SI system of units for the power P in kW, in terms of the mass flow rate m in kg/h, the height b and horizontal distance l in meters and (*b*) in U.S. customary units, for the power in hp, in terms of the material flow rate w in tons/h, and the height b and horizontal distance l in feet.

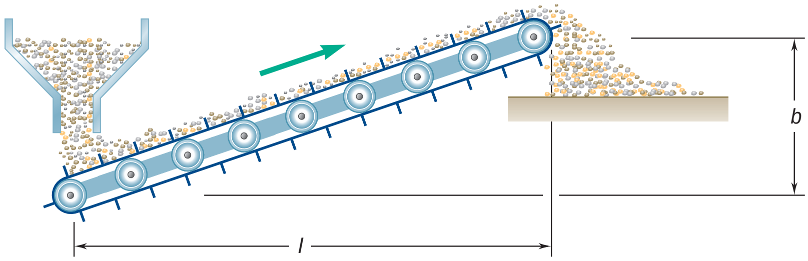


Fig. *P13.50*

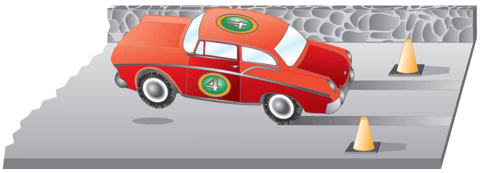


Fig. P13.51

13.51 In an automobile drag race, the rear (drive) wheels of a 1000-kg car skid for the first 20 m and roll with sliding impending during the remaining 380 m. The front wheels of the car are just off the ground for the first 20 m, and for the remainder of the race 80 percent of the weight is on the rear wheels. Knowing that the coefficients of friction are $\mu_s = 0.90$ and $\mu_k = 0.68$, determine the power developed by the car at the drive wheels (*a*) at the end of the 20-m portion of the race, (*b*) at the end of the race. Give your answer in kW and in hp. Ignore the effect of air resistance and rolling friction.

13.52 The frictional resistance of a ship is known to vary directly as the 1.75 power of the speed v of the ship. A single tugboat at full power can tow the ship at a constant speed of 4.5 km/h by exerting a constant force of 300 kN. Determine (*a*) the power developed by the tugboat, (*b*) the maximum speed at which two tugboats, capable of delivering the same power, can tow the ship.

13.53 A train of total mass equal to 500 Mg starts from rest and accelerates uniformely to a speed of 90 km/h in 50 s. After reaching this speed, the train travels with a constant velocity. The track is horizontal and axle friction and rolling resistance result in a total force of 15 kN in a direction opposite to the direction of motion. Determine the power required as a function of time.

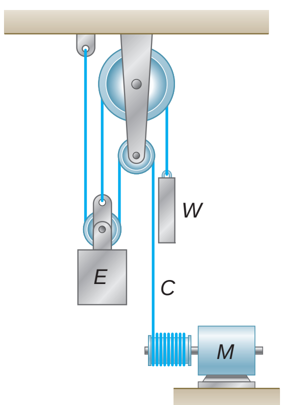


Fig. P13.54

13.54 The elevator E has a weight of 6600 lb when fully loaded and is connected as shown to a counterweight W of weight of 2200 lb. Determine the power in hp delivered by the motor (*a*) when the elevator is moving down at a constant speed of 1 ft/s, (*b*) when it has an upward velocity of 1 ft/s and a deceleration of 0.18 ft/s^2.

13.6 POTENTIAL ENERGY†

Let us consider again a body of weight **W** which moves along a curved path from a point A_1 of elevation y_1 to a point A_2 of elevation y_2 (Fig. 13.4). We recall from Sec. 13.2 that the work of the force of gravity **W** during this displacement is

$$U_{1 y 2} = Wy_1 - Wy_2 \qquad (13.4)$$

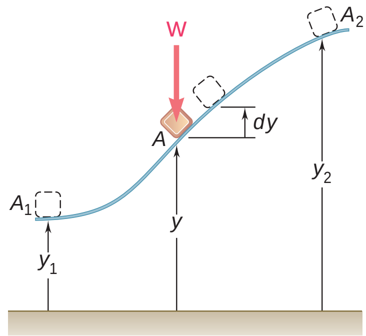


Fig. 13.4 *(repeated)*

The work of **W** may thus be obtained by subtracting the value of the function Wy corresponding to the second position of the body from its value corresponding to the first position. The work of **W** is independent of the actual path followed; it depends only upon the initial and final values of the function Wy. This function is called the *potential energy* of the body with respect to the *force of gravity* **W** and is denoted by V_g. We write

$$\boxed{U_{1 y 2} = (V_g)_1 - (V_g)_2 \qquad \text{with } V_g = Wy} \qquad (13.16)$$

We note that if $(V_g)_2 > (V_g)_1$, that is, *if the potential energy increases* during the displacement (as in the case considered here), *the work* $U_{1 y 2}$ *is negative*. If, on the other hand, the work of **W** is positive, the potential energy decreases. Therefore, the potential energy V_g of the body provides a measure of the work which can be done by its weight **W**. Since only the *change* in potential energy, and not the actual value of V_g, is involved in formula (13.16), an arbitrary constant can be added to the expression obtained for V_g. In other words, the level, or datum, from which the elevation y is measured can be chosen arbitrarily. Note that potential energy is expressed in the same units as work, i.e., in joules if SI units are used and in ft · lb or in · lb if U.S. customary units are used.

It should be noted that the expression just obtained for the potential energy of a body with respect to gravity is valid only as long as the weight **W** of the body can be assumed to remain constant, i.e., as long as the displacements of the body are small compared with the radius of the earth. In the case of a space vehicle, however, we should take into consideration the variation of the force of gravity with the distance r from the center of the earth. Using the expression obtained in Sec. 13.2 for the work of a gravitational force, we write (Fig. 13.6)

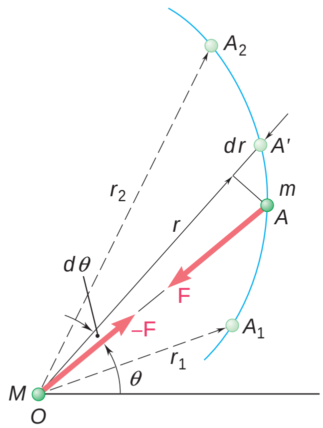


Fig. 13.6 *(repeated)*

$$U_{1 y 2} = \frac{GMm}{r_2} - \frac{GMm}{r_1} \qquad (13.7)$$

The work of the force of gravity can therefore be obtained by subtracting the value of the function $-GMm/r$ corresponding to the second position of the body from its value corresponding to the first position. Thus, the expression which should be used for the potential energy V_g when the variation in the force of gravity cannot be neglected is

$$\boxed{V_g = -\frac{GMm}{r}} \qquad (13.17)$$

†Some of the material in this section has already been considered in Sec. 10.7.

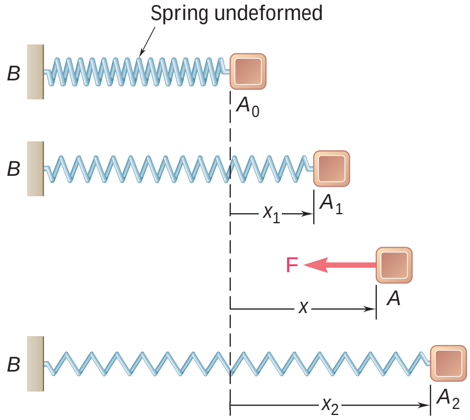


Fig. 13.5 *(repeated)*

Taking the first of the relations (12.29) into account, we write V_g in the alternative form

$$V_g = -\frac{WR^2}{r} \tag{13.17$'$}$$

where R is the radius of the earth and W is the value of the weight of the body at the surface of the earth. When either of the relations (13.17) or (13.17′) is used to express V_g, the distance r should, of course, be measured from the center of the earth.† Note that V_g is always negative and that it approaches zero for very large values of r.

Consider now a body attached to a spring and moving from a position A_1, corresponding to a deflection x_1 of the spring, to a position A_2, corresponding to a deflection x_2 of the spring (Fig. 13.5). We recall from Sec. 13.2 that the work of the force **F** exerted by the spring on the body is

$$U_{1\rightarrow 2} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \tag{13.6}$$

The work of the elastic force is thus obtained by subtracting the value of the function $\frac{1}{2}kx^2$ corresponding to the second position of the body from its value corresponding to the first position. This function is denoted by V_e and is called the *potential energy* of the body with respect to the *elastic force* **F**. We write

$$U_{1\rightarrow 2} = (V_e)_1 - (V_e)_2 \qquad \text{with } V_e = \tfrac{1}{2}kx^2 \tag{13.18}$$

and observe that during the displacement considered, the work of the force **F** exerted by the spring on the body is negative and the potential energy V_e increases. You should note that the expression obtained for V_e is valid only if the deflection of the spring is measured from its undeformed position. On the other hand, formula (13.18) can be used even when the spring is rotated about its fixed end (Fig. 13.10*a*). The work of the elastic force depends only upon the initial and final deflections of the spring (Fig. 13.10*b*).

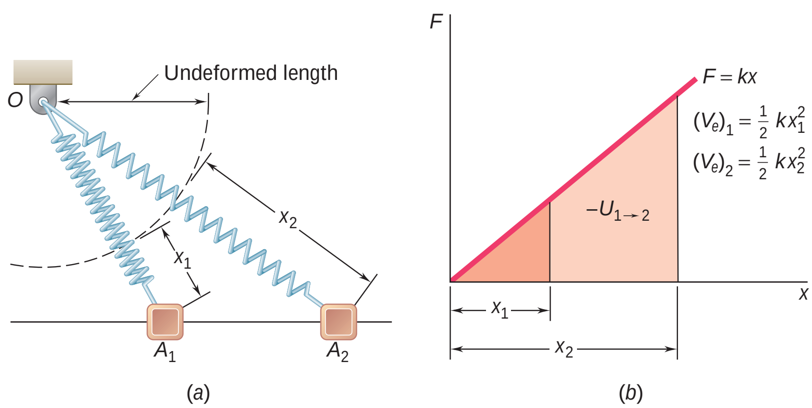


Fig. 13.10

†The expressions given for V_g in (13.17) and (13.17′) are valid only when $r \geq R$, that is, when the body considered is above the surface of the earth.

The concept of potential energy can be used when forces other than gravity forces and elastic forces are involved. Indeed, it remains valid as long as the work of the force considered is independent of the path followed by its point of application as this point moves from a given position A_1 to a given position A_2. Such forces are said to be *conservative forces;* the general properties of conservative forces are studied in the following section.

*13.7 CONSERVATIVE FORCES

As indicated in the preceding section, a force **F** acting on a particle A is said to be conservative *if its work $U_{1\rightarrow 2}$ is independent of the path followed by the particle A as it moves from A_1 to A_2* (Fig. 13.11*a*). We can then write

$$U_{1\rightarrow 2} = V(x_1, y_1, z_1) - V(x_2, y_2, z_2) \tag{13.19}$$

or, for short,

$$U_{1\rightarrow 2} = V_1 - V_2 \tag{13.19$'$}$$

The function $V(x, y, z)$ is called the potential energy, or *potential function,* of **F**.

We note that if A_2 is chosen to coincide with A_1, that is, if the particle describes a closed path (Fig. 13.11*b*), we have $V_1 = V_2$ and the work is zero. Thus for any conservative force **F** we can write

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \tag{13.20}$$

where the circle on the integral sign indicates that the path is closed.

Let us now apply (13.19) between two neighboring points $A(x, y, z)$ and $A'(x + dx, y + dy, z + dz)$. The elementary work dU corresponding to the displacement $d\mathbf{r}$ from A to A' is

$$dU = V(x, y, z) - V(x + dx, y + dy, z + dz)$$

or

$$dU = -dV(x, y, z) \tag{13.21}$$

Thus, the elementary work of a conservative force is an *exact differential.*

Substituting for dU in (13.21) the expression obtained in (13.1$''$) and recalling the definition of the differential of a function of several variables, we write

$$F_x\,dx + F_y\,dy + F_z\,dz = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right)$$

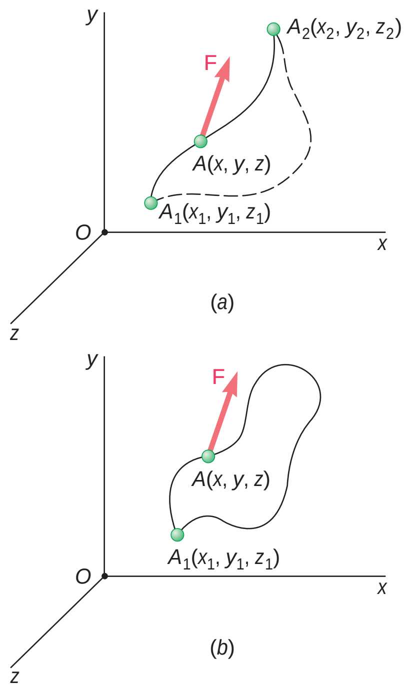


Fig. 13.11

from which it follows that

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z} \tag{13.22}$$

It is clear that the components of **F** must be functions of the coordinates x, y, and z. Thus, a *necessary* condition for a conservative force is that it depend only upon the position of its point of application. The relations (13.22) can be expressed more concisely if we write

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right)$$

The vector in parentheses is known as the *gradient of the scalar function* V and is denoted by **grad** V. We thus write for any conservative force

$$\mathbf{F} = -\mathbf{grad}\ V \tag{13.23}$$

The relations (13.19) to (13.23) were shown to be satisfied by any conservative force. It can also be shown that if a force **F** satisfies one of these relations, **F** must be a conservative force.

13.8 CONSERVATION OF ENERGY

We saw in the preceding two sections that the work of a conservative force, such as the weight of a particle or the force exerted by a spring, can be expressed as a change in potential energy. When a particle moves under the action of conservative forces, the principle of work and energy stated in Sec. 13.3 can be expressed in a modified form. Substituting for $U_{1 \rightarrow 2}$ from (13.19′) into (13.10), we write

$$V_1 - V_2 = T_2 - T_1$$

$$T_1 + V_1 = T_2 + V_2 \tag{13.24}$$

Formula (13.24) indicates that when a particle moves under the action of conservative forces, *the sum of the kinetic energy and of the potential energy of the particle remains constant*. The sum $T + V$ is called the *total mechanical energy* of the particle and is denoted by E.

Consider, for example, the pendulum analyzed in Sec. 13.4, which is released with no velocity from A_1 and allowed to swing in a vertical plane (Fig. 13.12). Measuring the potential energy from the level of A_2, we have, at A_1,

$$T_1 = 0 \qquad V_1 = Wl \qquad T_1 + V_1 = Wl$$

Recalling that at A_2 the speed of the pendulum is $v_2 = \sqrt{2gl}$, we have

$$T_2 = \tfrac{1}{2}mv_2^2 = \frac{1}{2}\frac{W}{g}(2gl) = Wl \qquad V_2 = 0$$
$$T_2 + V_2 = Wl$$

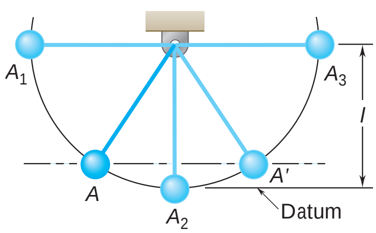


Fig. 13.12

We thus check that the total mechanical energy $E = T + V$ of the pendulum is the same at A_1 and A_2. Whereas the energy is entirely potential at A_1, it becomes entirely kinetic at A_2, and as the pendulum keeps swinging to the right, the kinetic energy is transformed back into potential energy. At A_3, $T_3 = 0$ and $V_3 = Wl$.

Since the total mechanical energy of the pendulum remains constant and since its potential energy depends only upon its elevation, the kinetic energy of the pendulum will have the same value at any two points located on the same level. Thus, the speed of the pendulum is the same at A and at A' (Fig. 13.12). This result can be extended to the case of a particle moving along any given path, regardless of the shape of the path, as long as the only forces acting on the particle are its weight and the normal reaction of the path. The particle of Fig. 13.13, for example, which slides in a vertical plane along a frictionless track, will have the same speed at A, A', and A''.

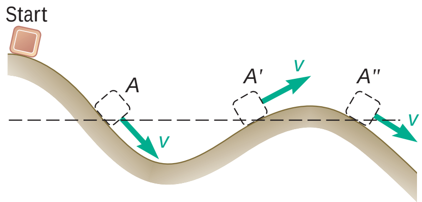


Fig. 13.13

While the weight of a particle and the force exerted by a spring are conservative forces, *friction forces are nonconservative forces*. In other words, *the work of a friction force cannot be expressed as a change in potential energy*. The work of a friction force depends upon the path followed by its point of application; and while the work $U_{1 y 2}$ defined by (13.19) is positive or negative according to the sense of motion, *the work of a friction force,* as we noted in Sec. 13.4, *is always negative*. It follows that when a mechanical system involves friction, its total mechanical energy does not remain constant but decreases. The energy of the system, however, is not lost; it is transformed into heat, and the sum of the *mechanical energy* and of the *thermal energy* of the system remains constant.

Other forms of energy can also be involved in a system. For instance, a generator converts mechanical energy into *electric energy;* a gasoline engine converts *chemical energy* into mechanical energy; a nuclear reactor converts *mass* into thermal energy. If all forms of energy are considered, the energy of any system can be considered as constant and the principle of conservation of energy remains valid under all conditions.

13.9 MOTION UNDER A CONSERVATIVE CENTRAL FORCE. APPLICATION TO SPACE MECHANICS

We saw in Sec. 12.9 that when a particle P moves under a central force $\mathbf{F}$, the angular momentum $\mathbf{H}_O$ of the particle about the center of force O is constant. If the force $\mathbf{F}$ is also conservative, there exists a potential energy V associated with $\mathbf{F}$, and the total energy $E = T + V$ of the particle is constant (Sec. 13.8). Thus, when a particle moves under a conservative central force, both the principle of conservation of angular momentum and the principle of conservation of energy can be used to study its motion.

Consider, for example, a space vehicle of mass m moving under the earth's gravitational force. Let us assume that it begins its free flight at point P_0 at a distance r_0 from the center of the earth, with a velocity $\mathbf{v}_0$ forming an angle f_0 with the radius vector OP_0

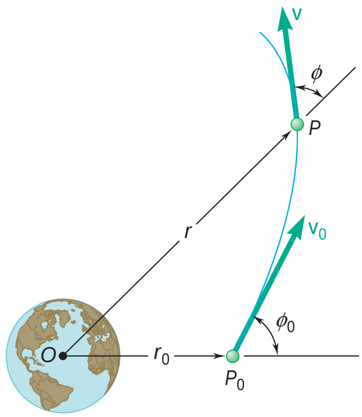


Fig. 13.14

(Fig. 13.14). Let P be a point of the trajectory described by the vehicle; we denote by r the distance from O to P, by $\mathbf{v}$ the velocity of the vehicle at P, and by ϕ the angle formed by $\mathbf{v}$ and the radius vector OP. Applying the principle of conservation of angular momentum about O between P_0 and P (Sec. 12.9), we write

$$r_0 m v_0 \sin \phi_0 = r m v \sin \phi \tag{13.25}$$

Recalling the expression (13.17) obtained for the potential energy due to a gravitational force, we apply the principle of conservation of energy between P_0 and P and write

$$T_0 + V_0 = T + V$$
$$\tfrac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2} m v^2 - \frac{GMm}{r} \tag{13.26}$$

where M is the mass of the earth.

Equation (13.26) can be solved for the magnitude v of the velocity of the vehicle at P when the distance r from O to P is known; Eq. (13.25) can then be used to determine the angle ϕ that the velocity forms with the radius vector OP.

Equations (13.25) and (13.26) can also be used to determine the maximum and minimum values of r in the case of a satellite launched from P_0 in a direction forming an angle ϕ_0 with the vertical OP_0 (Fig. 13.15). The desired values of r are obtained by making $\phi = 90°$ in (13.25) and eliminating v between Eqs. (13.25) and (13.26).

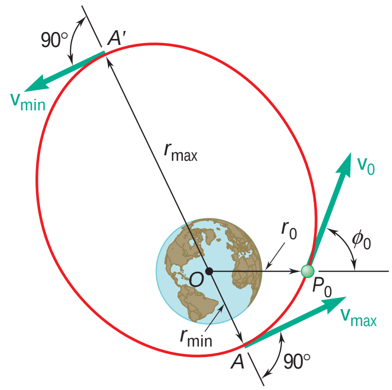
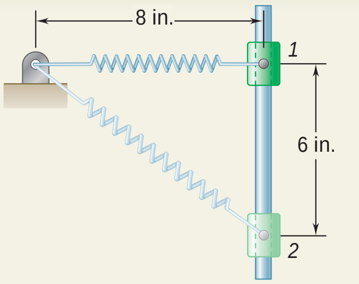


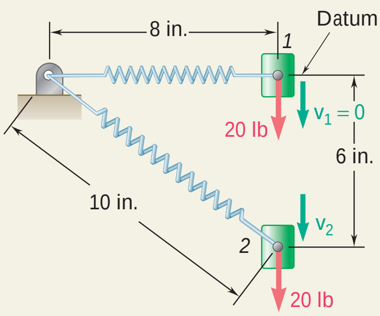
Fig. 13.15

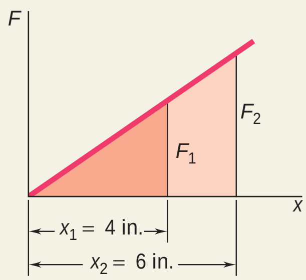
It should be noted that the application of the principles of conservation of energy and of conservation of angular momentum leads to a more fundamental formulation of the problems of space mechanics than does the method indicated in Sec. 12.12. In all cases involving oblique launchings, it will also result in much simpler computations. And while the method of Sec. 12.12 must be used when the actual trajectory or the periodic time of a space vehicle is to be determined, the calculations will be simplified if the conservation principles are first used to compute the maximum and minimum values of the radius vector r.



SAMPLE PROBLEM 13.6

A 20-lb collar slides without friction along a vertical rod as shown. The spring attached to the collar has an undeformed length of 4 in. and a constant of 3 lb/in. If the collar is released from rest in position *1*, determine its velocity after it has moved 6 in. to position *2*.





SOLUTION

Position *1*. ***Potential Energy.*** The elongation of the spring is

$$x_1 = 8 \text{ in.} - 4 \text{ in.} = 4 \text{ in.}$$

and we have

$$V_e = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(3 \text{ lb/in.})(4 \text{ in.})^2 = 24 \text{ in} \cdot \text{lb}$$

Choosing the datum as shown, we have $V_g = 0$. Therefore,

$$V_1 = V_e + V_g = 24 \text{ in} \cdot \text{lb} = 2 \text{ ft} \cdot \text{lb}$$

Kinetic Energy. Since the velocity in position *1* is zero, $T_1 = 0$.

Position *2*. ***Potential Energy.*** The elongation of the spring is

$$x_2 = 10 \text{ in.} - 4 \text{ in.} = 6 \text{ in.}$$

and we have

$$V_e = \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}(3 \text{ lb/in.})(6 \text{ in.})^2 = 54 \text{ in} \cdot \text{lb}$$
$$V_g = Wy = (20 \text{ lb})(-6 \text{ in.}) = -120 \text{ in} \cdot \text{lb}$$

Therefore,

$$\begin{aligned} V_2 = V_e + V_g = 54 - 120 &= -66 \text{ in} \cdot \text{lb} \\ &= -5.5 \text{ ft} \cdot \text{lb} \end{aligned}$$

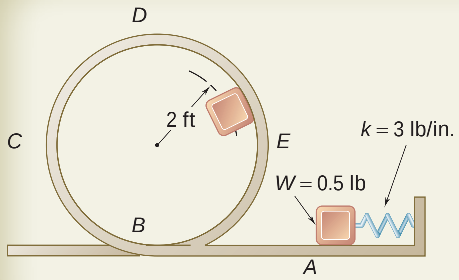
Kinetic Energy

$$T_2 = \tfrac{1}{2}mv_2^2 = \frac{1}{2}\frac{20}{32.2}v_2^2 = 0.311v_2^2$$

Conservation of Energy. Applying the principle of conservation of energy between positions *1* and *2*, we write

$$\begin{gathered} T_1 + V_1 = T_2 + V_2 \\ 0 + 2 \text{ ft} \cdot \text{lb} = 0.311v_2^2 - 5.5 \text{ ft} \cdot \text{lb} \\ v_2 = \pm 4.91 \text{ ft/s} \end{gathered}$$

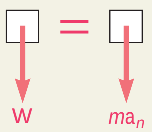
$\mathbf{v}_2 = 4.91 \text{ ft/s w}$ ◀



SAMPLE PROBLEM 13.7

The 0.5-lb pellet is pushed against the spring at A and released from rest. Neglecting friction, determine the smallest deflection of the spring for which the pellet will travel around the loop $ABCDE$ and remain at all times in contact with the loop.

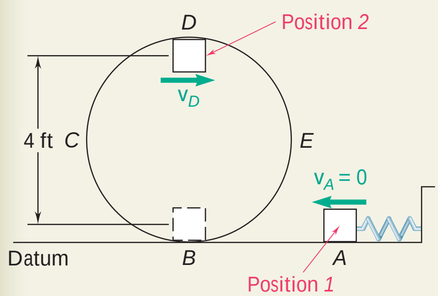
SOLUTION



Required Speed at Point *D*. As the pellet passes through the highest point D, its potential energy with respect to gravity is maximum and, thus, its kinetic energy and speed are minimum. Since the pellet must remain in contact with the loop, the force $\mathbf{N}$ exerted on the pellet by the loop must be equal to or greater than zero. Setting $\mathbf{N} = 0$, we compute the smallest possible speed v_D.

$$+\downarrow \Sigma F_n = ma_n: \qquad W = ma_n \qquad mg = ma_n \qquad a_n = g$$

$$a_n = \frac{v_D^2}{r}: \qquad v_D^2 = ra_n = rg = (2 \text{ ft})(32.2 \text{ ft/s}^2) = 64.4 \text{ ft}^2/\text{s}^2$$



Position *1*. ***Potential Energy.*** Denoting by x the deflection of the spring and noting that $k = 3$ lb/in. $= 36$ lb/ft, we write

$$V_e = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(36 \text{ lb/ft})x^2 = 18x^2$$

Choosing the datum at A, we have $V_g = 0$; therefore

$$V_1 = V_e + V_g = 18x^2$$

Kinetic Energy. Since the pellet is released from rest, $v_A = 0$ and we have $T_1 = 0$.

Position *2*. ***Potential Energy.*** The spring is now undeformed; thus $V_e = 0$. Since the pellet is 4 ft above the datum, we have

$$V_g = Wy = (0.5 \text{ lb})(4 \text{ ft}) = 2 \text{ ft} \cdot \text{lb}$$
$$V_2 = V_e + V_g = 2 \text{ ft} \cdot \text{lb}$$

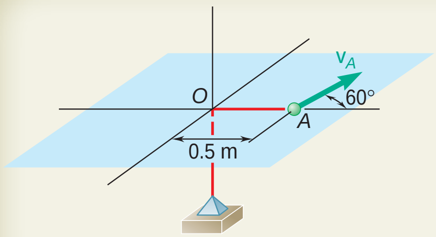
Kinetic Energy. Using the value of v_D^2 obtained above, we write

$$T_2 = \tfrac{1}{2}mv_D^2 = \frac{1}{2}\,\frac{0.5 \text{ lb}}{32.2 \text{ ft/s}^2}(64.4 \text{ ft}^2/\text{s}^2) = 0.5 \text{ ft} \cdot \text{lb}$$

Conservation of Energy. Applying the principle of conservation of energy between positions *1* and *2*, we write

$$T_1 + V_1 = T_2 + V_2$$
$$0 + 18x^2 = 0.5 \text{ ft} \cdot \text{lb} + 2 \text{ ft} \cdot \text{lb}$$
$$x = 0.3727 \text{ ft} \qquad\qquad x = 4.47 \text{ in.} \blacktriangleleft$$

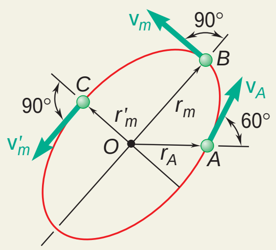
SAMPLE PROBLEM 13.8



A sphere of mass $m = 0.6$ kg is attached to an elastic cord of constant $k = 100$ N/m, which is undeformed when the sphere is located at the origin O. Knowing that the sphere may slide without friction on the horizontal surface and that in the position shown its velocity $\mathbf{v}_A$ has a magnitude of 20 m/s, determine (*a*) the maximum and minimum distances from the sphere to the origin O, (*b*) the corresponding values of its speed.

SOLUTION

The force exerted by the cord on the sphere passes through the fixed point O, and its work can be expressed as a change in potential energy. It is therefore a conservative central force, and both the total energy of the sphere and its angular momentum about O are conserved.



Conservation of Angular Momentum About *O*. At point B, where the distance from O is maximum, the velocity of the sphere is perpendicular to OB and the angular momentum is $r_m m v_m$. A similar property holds at point C, where the distance from O is minimum. Expressing conservation of angular momentum between A and B, we write

$$r_A m v_A \sin 60° = r_m m v_m$$
$$(0.5 \text{ m})(0.6 \text{ kg})(20 \text{ m/s}) \sin 60° = r_m(0.6 \text{ kg})v_m$$
$$v_m = \frac{8.66}{r_m} \qquad (1)$$

Conservation of Energy

At point A: $\quad T_A = \frac{1}{2}mv_A^2 = \frac{1}{2}(0.6 \text{ kg})(20 \text{ m/s})^2 = 120 \text{ J}$

$\quad V_A = \frac{1}{2}kr_A^2 = \frac{1}{2}(100 \text{ N/m})(0.5 \text{ m})^2 = 12.5 \text{ J}$

At point B: $\quad T_B = \frac{1}{2}mv_m^2 = \frac{1}{2}(0.6 \text{ kg})v_m^2 = 0.3v_m^2$

$\quad V_B = \frac{1}{2}kr_m^2 = \frac{1}{2}(100 \text{ N/m})r_m^2 = 50r_m^2$

Applying the principle of conservation of energy between points A and B, we write

$$T_A + V_A = T_B + V_B$$
$$120 + 12.5 = 0.3v_m^2 + 50r_m^2 \qquad (2)$$

a. Maximum and Minimum Values of Distance. Substituting for v_m from Eq. (1) into Eq. (2) and solving for r_m^2, we obtain

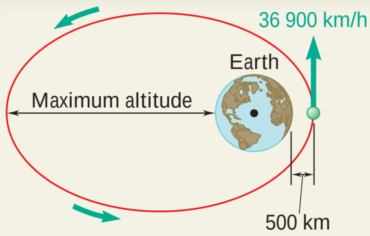
$$r_m^2 = 2.468 \text{ or } 0.1824 \qquad r_m = 1.571 \text{ m}, \; r'_m = 0.427 \text{ m} \blacktriangleleft$$

b. Corresponding Values of Speed. Substituting the values obtained for r_m and r'_m into Eq. (1), we have

$$v_m = \frac{8.66}{1.571} \qquad v_m = 5.51 \text{ m/s} \blacktriangleleft$$

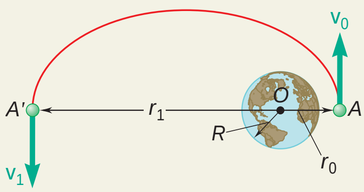
$$v'_m = \frac{8.66}{0.427} \qquad v'_m = 20.3 \text{ m/s} \blacktriangleleft$$

Note. It can be shown that the path of the sphere is an ellipse of *center O*.



SAMPLE PROBLEM 13.9

A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36 900 km/h from an altitude of 500 km. Determine (*a*) the maximum altitude reached by the satellite, (*b*) the maximum allowable error in the direction of launching if the satellite is to go into orbit and come no closer than 200 km to the surface of the earth.



SOLUTION

a. Maximum Altitude. We denote by A' the point of the orbit farthest from the earth and by r_1 the corresponding distance from the center of the earth. Since the satellite is in free flight between A and A', we apply the principle of conservation of energy:

$$T_A + V_A = T_{A'} + V_{A'}$$

$$\tfrac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2}mv_1^2 - \frac{GMm}{r_1} \qquad (1)$$

Since the only force acting on the satellite is the force of gravity, which is a central force, the angular momentum of the satellite about O is conserved. Considering points A and A', we write

$$r_0 m v_0 = r_1 m v_1 \qquad v_1 = v_0 \frac{r_0}{r_1} \qquad (2)$$

Substituting this expression for v_1 into Eq. (1), dividing each term by the mass m, and rearranging the terms, we obtain

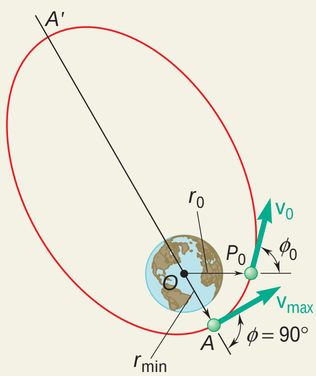
$$\tfrac{1}{2}v_0^2\left(1 - \frac{r_0^2}{r_1^2}\right) = \frac{GM}{r_0}\left(1 - \frac{r_0}{r_1}\right) \qquad 1 + \frac{r_0}{r_1} = \frac{2GM}{r_0 v_0^2} \qquad (3)$$

Recalling that the radius of the earth is $R = 6370$ km, we compute

$$r_0 = 6370 \text{ km} + 500 \text{ km} = 6870 \text{ km} = 6.87 \times 10^6 \text{ m}$$
$$v_0 = 36\,900 \text{ km/h} = (36.9 \times 10^6 \text{ m})/(3.6 \times 10^3 \text{ s}) = 10.25 \times 10^3 \text{ m/s}$$
$$GM = gR^2 = (9.81 \text{ m/s}^2)(6.37 \times 10^6 \text{ m})^2 = 398 \times 10^{12} \text{ m}^3/\text{s}^2$$

Substituting these values into (3), we obtain $r_1 = 66.8 \times 10^6$ m.

Maximum altitude $= 66.8 \times 10^6 \text{ m} - 6.37 \times 10^6 \text{ m} = 60.4 \times 10^6 \text{ m} =$
60 400 km ◀



b. Allowable Error in Direction of Launching. The satellite is launched from P_0 in a direction forming an angle ϕ_0 with the vertical OP_0. The value of ϕ_0 corresponding to $r_{min} = 6370 \text{ km} + 200 \text{ km} = 6570$ km is obtained by applying the principles of conservation of energy and of conservation of angular momentum between P_0 and A:

$$\tfrac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2}mv_{max}^2 - \frac{GMm}{r_{min}} \qquad (4)$$

$$r_0 m v_0 \sin \phi_0 = r_{min} m v_{max} \qquad (5)$$

Solving (5) for v_{max} and then substituting for v_{max} into (4), we can solve (4) for $\sin \phi_0$. Using the values of v_0 and GM computed in part *a* and noting that $r_0/r_{min} = 6870/6570 = 1.0457$, we find

$\sin \phi_0 = 0.9801 \qquad \phi_0 = 90° \pm 11.5° \qquad$ Allowable error $= \pm 11.5°$ ◀

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned that when the work done by a force **F** acting on a particle A *is independent of the path followed by the particle* as it moves from a given position A_1 to a given position A_2 (Fig. 13.11*a*), then a function V, called *potential energy,* can be defined for the force **F**. Such forces are said to be *conservative forces,* and you can write

$$U_{1 y 2} = V(x_1, y_1, z_1) - V(x_2, y_2, z_2) \tag{13.19}$$

or, for short,

$$U_{1 y 2} = V_1 - V_2 \tag{13.19$'$}$$

Note that the work is negative when the change in the potential energy is positive, i.e., when $V_2 > V_1$.

Substituting the above expression into the equation for work and energy, you can write

$$T_1 + V_1 = T_2 + V_2 \tag{13.24}$$

which shows that when a particle moves under the action of a conservative force *the sum of the kinetic and potential energies of the particle remains constant.*

Your solution of problems using the above formula will consist of the following steps.

1. Determine whether all the forces involved are conservative. If some of the forces are not conservative, for example if friction is involved, you must use the method of work and energy from the previous lesson, since the work done by such forces depends upon the path followed by the particle and a potential function does not exist. If there is no friction and if all the forces are conservative, you can proceed as follows.

2. Determine the kinetic energy $T = \frac{1}{2}mv^2$ at each end of the path.

3. Compute the potential energy for all the forces involved at each end of the path. You will recall that the following expressions for the potential energy were derived in this lesson.

a. The potential energy of a weight *W* close to the surface of the earth and at a height y above a given datum,

$$V_g = Wy \tag{13.16}$$

b. The potential energy of a mass *m* located at a distance *r* from the center of the earth, large enough so that the variation of the force of gravity must be taken into account,

$$V_g = -\frac{GMm}{r} \tag{13.17}$$

where the distance r is measured from the center of the earth and V_g is equal to zero at $r = \infty$.

c. The potential energy of a body with respect to an elastic force *F* = *kx*,

$$V_e = \tfrac{1}{2}kx^2 \tag{13.18}$$

where the distance x is the deflection of the elastic spring measured from its *undeformed* position and k is the spring constant. Note that V_e *depends only upon the deflection* x and not upon the path of the body attached to the spring. Also, V_e is always positive, whether the spring is compressed or elongated.

4. Substitute your expressions for the kinetic and potential energies into Eq. (13.24). You will be able to solve this equation for one unknown, for example, for a velocity [Sample Prob. 13.6]. If more than one unknown is involved, you will have to search for another condition or equation, such as the minimum speed [Sample Prob. 13.7] or the minimum potential energy of the particle. For problems involving a central force, a second equation can be obtained by using conservation of angular momentum [Sample Prob. 13.8]. This is especially useful in applications to space mechanics [Sec. 13.9].

PROBLEMS

CONCEPT QUESTIONS

13.CQ2 Two small balls A and B with masses $2m$ and m, respectively, are released from rest at a height h above the ground. Neglecting air resistance, which of the following statements is true when the two balls hit the ground?

a. The kinetic energy of A is the same as the kinetic energy of B.
b. The kinetic energy of A is half the kinetic energy of B.
c. The kinetic energy of A is twice the kinetic energy of B.
d. The kinetic energy of A is four times the kinetic energy of B.

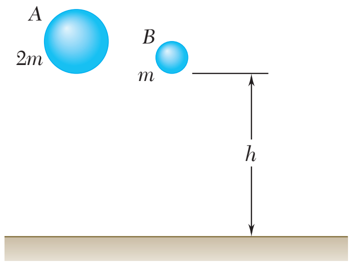


Fig. P13.CQ2

13.CQ3 A small block A is released from rest and slides down the frictionless ramp to the loop. The maximum height h of the loop is the same as the initial height of the block. Will A make it completely around the loop without losing contact with the track?

a. Yes
b. No
c. Need more information

Fig. P13.CQ3

END-OF-SECTION PROBLEMS

13.55 A force **P** is slowly applied to a plate that is attached to two springs and causes a deflection x_0. In each of the two cases shown, derive an expression for the constant k_e, in terms of k_1 and k_2, of the single spring equivalent to the given system, that is, of the single spring which will undergo the same deflection x_0 when subjected to the same force **P**.

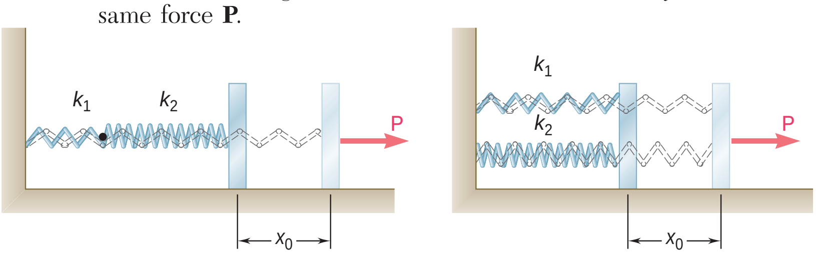


Fig. P13.55 (*a*) (*b*)

13.56 A loaded railroad car of mass m is rolling at a constant velocity $\mathbf{v_0}$ when it couples with a massless bumper system. Determine the maximum deflection of the bumper assuming the two springs are (*a*) in series (as shown), (*b*) in parallel.

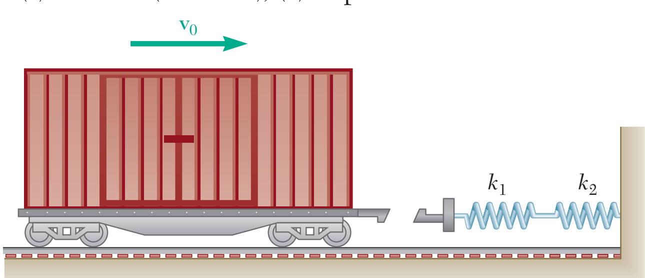


Fig. P13.56

13.57 A 600-g collar C may slide along a horizontal, semicircular rod ABD. The spring CE has an undeformed length of 250 mm and a spring constant of 135 N/m. Knowing that the collar is released from rest at A and neglecting friction, determine the speed of the collar (*a*) at B, (*b*) at D.

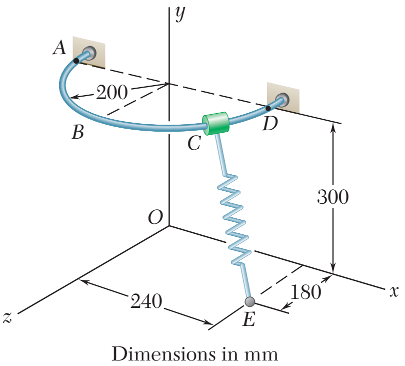


Fig. P13.57

13.58 A 3-lb collar is attached to a spring and slides without friction along a circular rod in a *horizontal* plane. The spring has an undeformed length of 7 in. and a constant $k = 1.5$ lb/in. Knowing that the collar is in equilibrium at A and is given a slight push to get it moving, determine the velocity of the collar (*a*) as it passes through B, (*b*) as it passes through C.

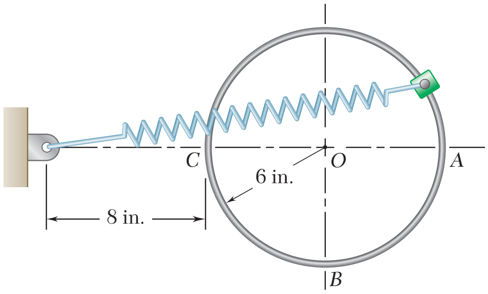


Fig. P13.58

13.59 A 3-lb collar C may slide without friction along a horizontal rod. It is attached to three springs, each of constant $k = 2$ lb/in. and 6-in. undeformed length. Knowing that the collar is released from rest in the position shown, determine the maximum speed it will reach in the ensuing motion.

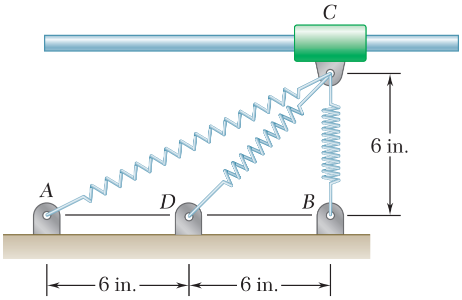


Fig. P13.59

13.60 A 500-g collar can slide without friction on the curved rod BC in a *horizontal* plane. Knowing that the undeformed length of the spring is 80 mm and that $k = 400$ kN/m, determine (*a*) the velocity that the collar should be given at A to reach B with zero velocity, (*b*) the velocity of the collar when it eventually reaches C.

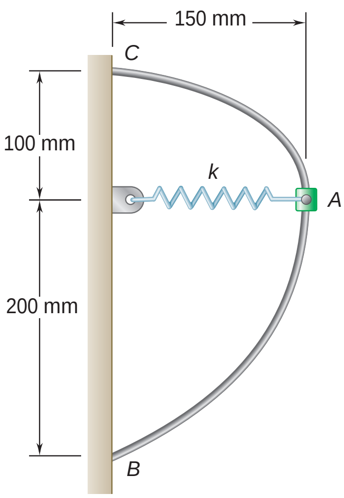


Fig. *P13.60*

13.61 An elastic cord is stretched between two points A and B, located 800 mm apart in the same horizontal plane. When stretched directly between A and B, the tension is 40 N. The cord is then stretched as shown until its midpoint C has moved through 300 mm to C'; a force of 240 N is required to hold the cord at C'. A 0.1-kg pellet is placed at C', and the cord is released. Determine the speed of the pellet as it passes through C.

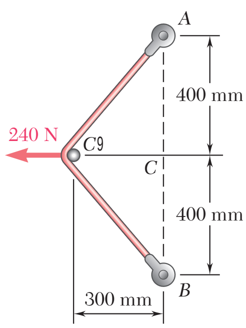


Fig. *P13.61*

13.62 An elastic cable is to be designed for bungee jumping from a tower 130 ft high. The specifications call for the cable to be 85 ft long when unstretched, and to stretch to a total length of 100 ft when a 600-lb weight is attached to it and dropped from the tower. Determine (*a*) the required spring constant k of the cable, (*b*) how close to the ground a 186-lb man will come if he uses this cable to jump from the tower.

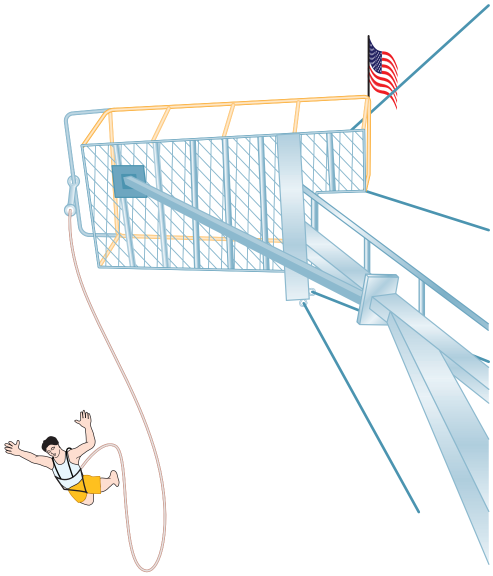


Fig. P13.62

13.63 It is shown in mechanics of materials that the stiffness of an elastic cable is $k = AE/L$, where A is the cross-sectional area of the cable, E is the modulus of elasticity, and L is the length of the cable. A winch is lowering a 4000-lb piece of machinery using a constant speed of 3 ft/s when the winch suddenly stops. Knowing that the steel cable has a diameter of 0.4 in., $E = 29 \times 10^6$ lb/in^2, and when the winch stops $L = 30$ ft, determine the maximum downward displacement of the piece of machinery from the point it was when the winch stopped.

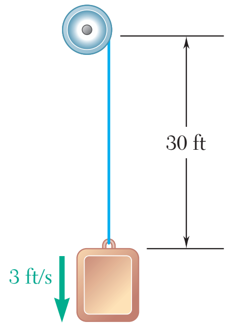


Fig. *P13.63*

13.64 A 2-kg collar is attached to a spring and slides without friction in a vertical plane along the curved rod ABC. The spring is undeformed when the collar is at C and its constant is 600 N/m. If the collar is released at A with no initial velocity, determine its velocity (*a*) as it passes through B, (*b*) as it reaches C.

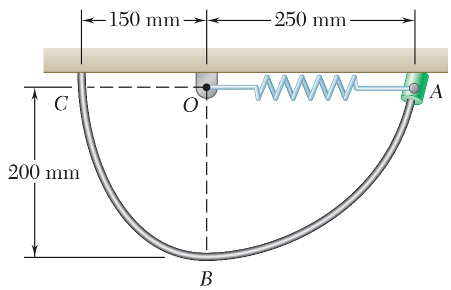


Fig. P13.64

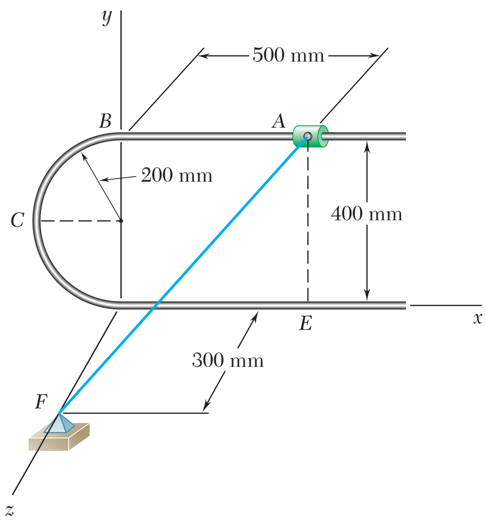


Fig. P13.65

13.65 A 1-kg collar can slide along the rod shown. It is attached to an elastic cord anchored at F, which has an undeformed length of 250 mm and spring constant of 75 N/m. Knowing that the collar is released from rest at A and neglecting friction, determine the speed of the collar (*a*) at B, (*b*) at E.

13.66 A thin circular rod is supported in a *vertical plane* by a bracket at A. Attached to the bracket and loosely wound around the rod is a spring of constant $k = 3$ lb/ft and undeformed length equal to the arc of circle AB. An 8-oz collar C, not attached to the spring, can slide without friction along the rod. Knowing that the collar is released from rest at an angle u with the vertical, determine (*a*) the smallest value of u for which the collar will pass through D and reach point A, (*b*) the velocity of the collar as it reaches point A.

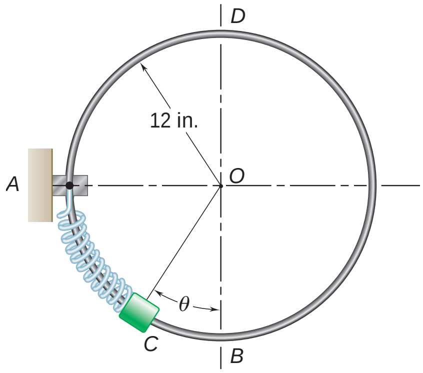


Fig. P13.66

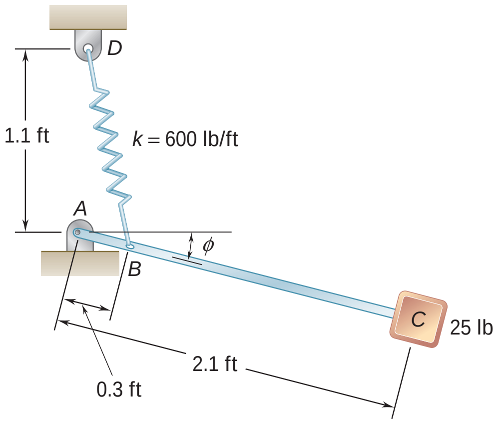


Fig. P13.67

13.67 The system shown is in equilibrium when f = 0. Knowing that initially f = 90° and that block C is given a slight nudge when the system is in that position, determine the speed of the block as it passes through the equilibrium position f = 0. Neglect the weight of the rod.

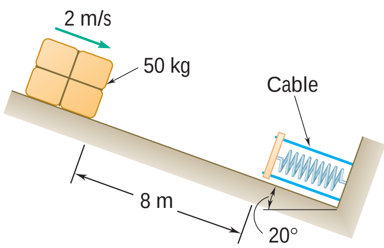


Fig. P13.68

13.68 A spring is used to stop a 50-kg package which is moving down a 20° incline. The spring has a constant $k = 30$ kN/m and is held by cables so that it is initially compressed 50 mm. Knowing that the velocity of the package is 2 m/s when it is 8 m from the spring and neglecting friction, determine the maximum additional deformation of the spring in bringing the package to rest.

13.69 Solve Prob. 13.68 assuming the kinetic coefficient of friction between the package and the incline is 0.2.

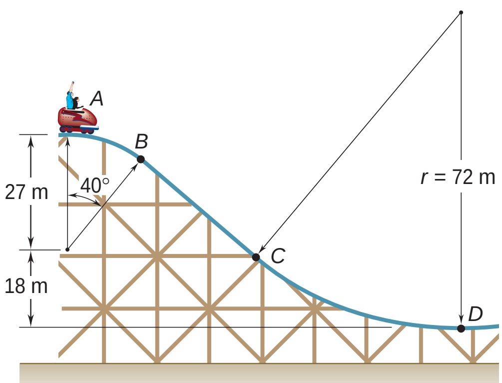


Fig. P13.70 and P13.71

13.70 A section of track for a roller coaster consists of two circular arcs AB and CD joined by a straight portion BC. The radius of AB is 27 m and the radius of CD is 72 m. The car and its occupants, of total mass 250 kg, reach point A with practically no velocity and then drop freely along the track. Determine the normal force exerted by the track on the car as the car reaches point B. Ignore air resistance and rolling resistance.

13.71 A section of track for a roller coaster consists of two circular arcs AB and CD joined by a straight portion BC. The radius of AB is 27 m and the radius of CD is 72 m. The car and its occupants, of total mass 250 kg, reach point A with practically no velocity and then drop freely along the track. Determine the maximum and minimum values of the normal force exerted by the track on the car as the car travels from A to D. Ignore air resistance and rolling resistance.

13.72 A 1-lb collar is attached to a spring and slides without friction along a circular rod in a *vertical* plane. The spring has an undeformed length of 5 in. and a constant $k = 10$ lb/ft. Knowing that the collar is released from being held at A, determine the speed of the collar and the normal force between the collar and the rod as the collar passes through B.

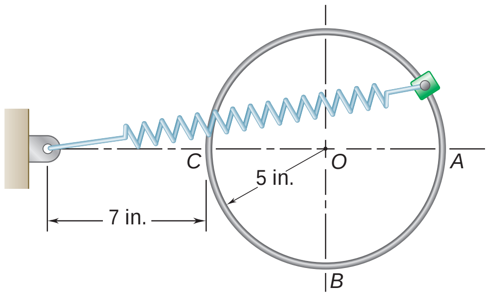


Fig. P13.72

13.73 A 10-lb collar is attached to a spring and slides without friction along a fixed rod in a vertical plane. The spring has an undeformed length of 14 in. and a constant $k = 4$ lb/in. Knowing that the collar is released from rest in the position shown, determine the force exerted by the rod on the collar at (*a*) point A, (*b*) point B. Both these points are on the curved portion of the rod.

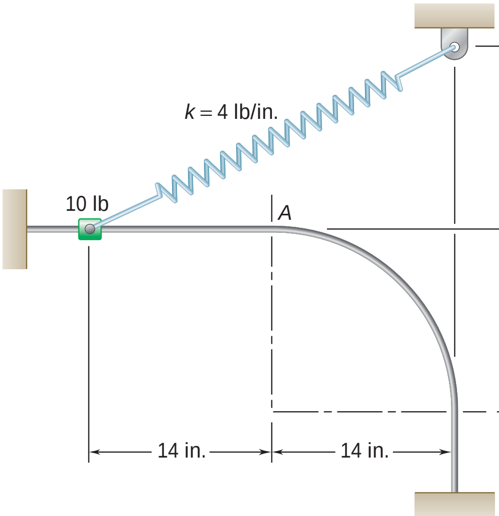


Fig. *P13.73*

13.74 An 8-oz package is projected upward with a velocity $\mathbf{v}_0$ by a spring at A; it moves around a frictionless loop and is deposited at C. For each of the two loops shown, determine (*a*) the smallest velocity $\mathbf{v}_0$ for which the package will reach C, (*b*) the corresponding force exerted by the package on the loop just before the package leaves the loop at C.

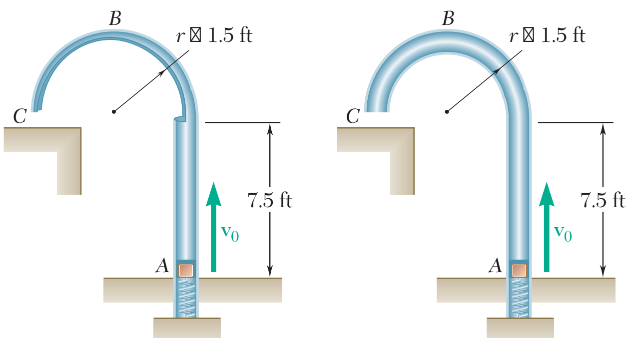


Fig. P13.74 and *P13.75*

13.75 If the package of Prob. 13.74 is not to hit the horizontal surface at C with a speed greater than 10 ft/s, (*a*) show that this requirement can be satisfied only by the second loop, (*b*) determine the largest allowable initial velocity $\mathbf{v}_0$ when the second loop is used.

13.76 A small package of weight W is projected into a vertical return loop at A with a velocity $\mathbf{v}_0$. The package travels without friction along a circle of radius r and is deposited on a horizontal surface at C. For each of the two loops shown, determine (*a*) the smallest velocity $\mathbf{v}_0$ for which the package will reach the horizontal surface at C, (*b*) the corresponding force exerted by the loop on the package as it passes point B.

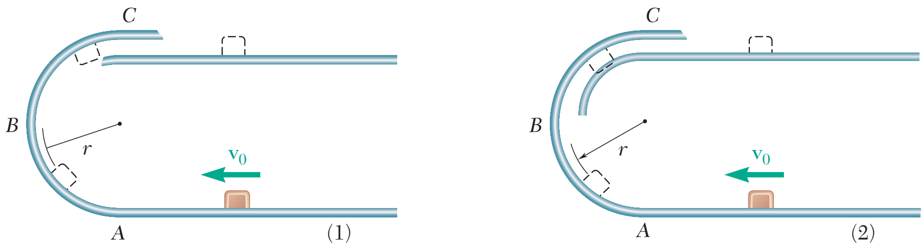


Fig. P13.76

13.77 The 1-kg ball at A is suspended by an inextensible cord and given an initial horizontal velocity of 5 m/s. If l = 0.6 m and x_B = 0, determine y_B so that the ball will enter the basket.

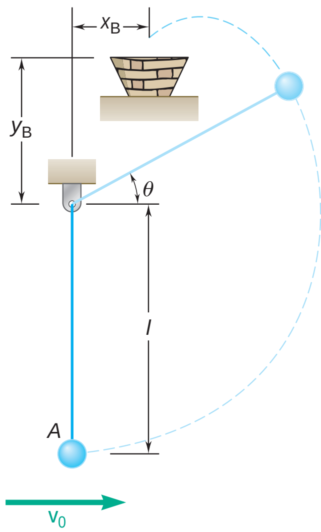


Fig. P13.77

***13.78** Packages are moved from point A on the upper floor of a warehouse to point B on the lower floor, 12 ft directly below A, by means of a chute, the centerline of which is in the shape of a helix of vertical axis y and radius R = 8 ft. The cross section of the chute is to be banked in such a way that each package, after being released at A with no velocity, will slide along the centerline of the chute without ever touching its edges. Neglecting friction, (*a*) express as a function of the elevation y of a given point P of the centerline the angle f formed by the normal to the surface of the chute at P and the principal normal of the centerline at that point, (*b*) determine the magnitude and direction of the force exerted by the chute on a 20-lb package as it reaches point B. *Hint:* The principal normal to the helix at any point P is horizontal and directed toward the y axis, and the radius of curvature of the helix is r = $R[1 + (h/2\text{p}R)^2]$.

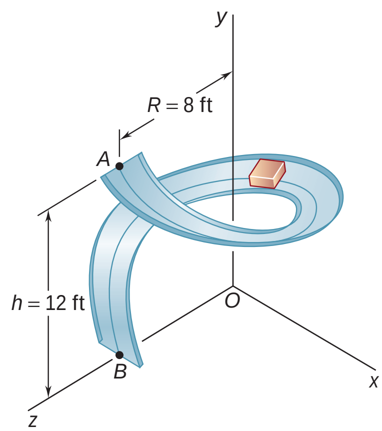


Fig. P13.78

***13.79** Prove that a force $F(x, y, z)$ is conservative if, and only if, the following relations are satisfied:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \qquad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

13.80 The force $\mathbf{F} = (yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k})/xyz$ acts on the particle $P(x, y, z)$ which moves in space. (*a*) Using the relation derived in Prob. 13.79, show that this force is a conservative force. (*b*) Determine the potential function associated with $\mathbf{F}$.

***13.81** A force F acts on a particle $P(x, y)$ which moves in the xy plane. Determine whether $\mathbf{F}$ is a conservative force and compute the work of $\mathbf{F}$ when P describes in a clockwise sense the path A, B, C, A including the quarter circle $x^2 + y^2 = a^2$, if (*a*) $\mathbf{F} = ky\mathbf{i}$, (*b*) $\mathbf{F} = k(y\mathbf{i} + x\mathbf{j})$.

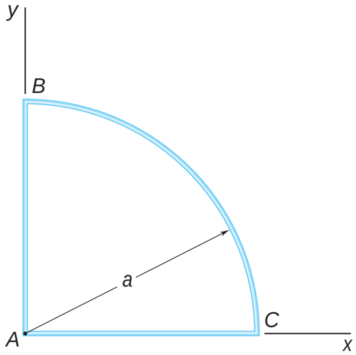


Fig. P13.81

***13.82** The potential function associated with a force $\mathbf{P}$ in space is known to be $V(x, y, z) = -(x^2 + y^2 + z^2)^{1/2}$. (*a*) Determine the x, y, and z components of $\mathbf{P}$. (*b*) Calculate the work done by $\mathbf{P}$ from O to D by integrating along the path $OABD$, and show that it is equal to the negative of the change in potential from O to D.

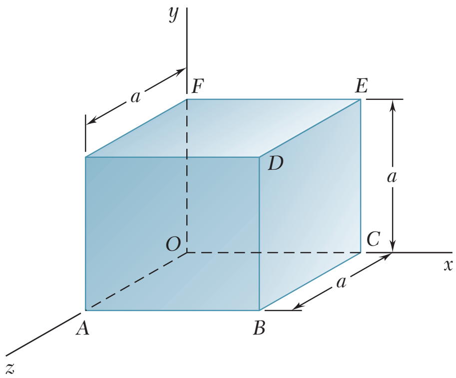


Fig. P13.82

***13.83** (*a*) Calculate the work done from D to O by the force $\mathbf{P}$ of Prob. 13.82 by integrating along the diagonal of the cube. (*b*) Using the result obtained and the answer to part b of Prob. 13.82, verify that the work done by a conservative force around the closed path $OABDO$ is zero.

***13.84** The force $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$ acts on the particle $P(x, y, z)$ which moves in space. (*a*) Using the relations derived in Prob. 13.79, prove that $\mathbf{F}$ is a conservative force. (*b*) Determine the potential function $V(x, y, z)$ associated with $\mathbf{F}$.

13.85 (*a*) Determine the kinetic energy per unit mass which a missile must have after being fired from the surface of the earth if it is to reach an infinite distance from the earth. (*b*) What is the initial velocity of the missile (called the *escape velocity*)? Give your answers in SI units and show that the answer to part b is independent of the firing angle.

13.86 A satellite describes an elliptic orbit of minimum altitude 606 km above the surface of the earth. The semimajor and semiminor axes are 17 440 km and 13 950 km, respectively. Knowing that the speed of the satellite at point C is 4.78 km/s, determine (*a*) the speed at point A, the perigee, (*b*) the speed at point B, the apogee.

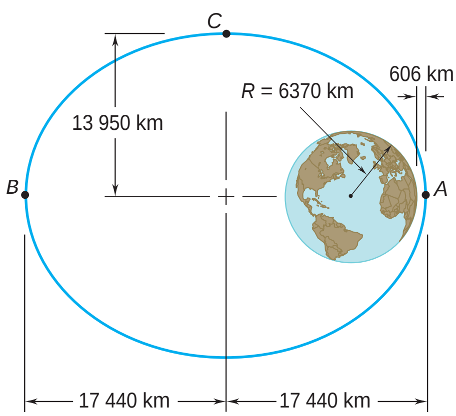


Fig. P13.86

13.87 While describing a circular orbit 200 mi above the earth a space vehicle launches a 6000-lb communications satellite. Determine (*a*) the additional energy required to place the satellite in a geosynchronous orbit at an altitude of 22,000 mi above the surface of the earth, (*b*) the energy required to place the satellite in the same orbit by launching it from the surface of the earth, excluding the energy needed to overcome air resistance. (A *geosynchronous orbit* is a circular orbit in which the satellite appears stationary with respect to the ground.)

13.88 A lunar excursion module (LEM) was used in the Apollo moon-landing missions to save fuel by making it unnecessary to launch the entire Apollo spacecraft from the moon's surface on its return trip to earth. Check the effectiveness of this approach by computing the energy per pound required for a spacecraft (as weighed on the earth) to escape the moon's gravitational field if the spacecraft starts from (*a*) the moon's surface, (*b*) a circular orbit 50 mi above the moon's surface. Neglect the effect of the earth's gravitational field. (The radius of the moon is 1081 mi and its mass is 0.0123 times the mass of the earth.)

13.89 Knowing that the velocity of an experimental space probe fired from the earth has a magnitude v_A = 32.5 Mm/h at point A, determine the speed of the probe as it passes through point B.

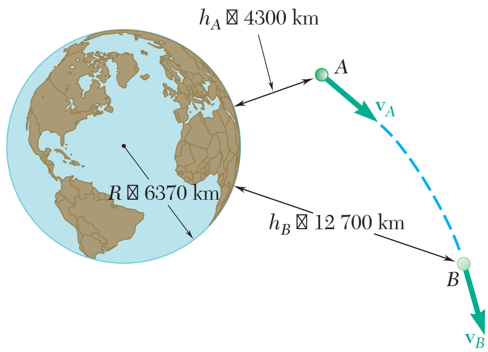


Fig. P13.89

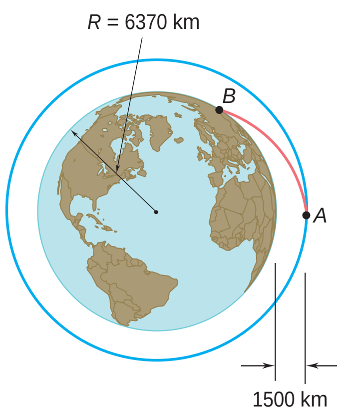


Fig. P13.90

13.90 A spacecraft is describing a circular orbit at an altitude of 1500 km above the surface of the earth. As it passes through point A, its speed is reduced by 40 percent and it enters an elliptic crash trajectory with the apogee at point A. Neglecting air resistance, determine the speed of the spacecraft when it reaches the earth's surface at point B.

13.91 Observations show that a celestial body traveling at 1.2×10^6 mi/h appears to be describing about point B a circle of radius equal to 60 light years. Point B is suspected of being a very dense concentration of mass called a black hole. Determine the ratio M_B/M_S of the mass at B to the mass of the sun. (The mass of the sun is 330,000 times the mass of the earth, and a light year is the distance traveled by light in 1 year at 186,300 mi/s.)

13.92 (*a*) Show that, by setting $r = R + y$ in the right-hand member of Eq. (13.17′) and expanding that member in a power series in y/R, the expression in Eq. (13.16) for the potential energy V_g due to gravity is a first-order approximation for the expression given in Eq. (13.17′). (*b*) Using the same expansion, derive a second-order approximation for V_g.

13.93 Collar A has a mass of 3 kg and is attached to a spring of constant 1200 N/m and of undeformed length equal to 0.5 m. The system is set in motion with $r = 0.3$ m, $v_\theta = 2$ m/s, and $v_r = 0$. Neglecting the mass of the rod and the effect of friction, determine the radial and transverse components of the velocity of the collar when $r = 0.6$ m.

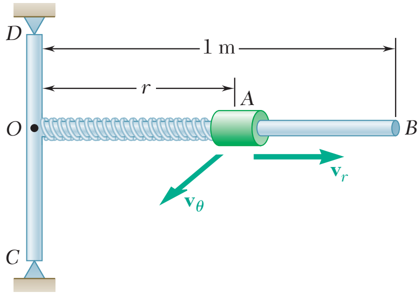


Fig. P13.93 and P13.94

13.94 Collar A has a mass of 3 kg and is attached to a spring of constant 1200 N/m and of undeformed length equal to 0.5 m. The system is set in motion with $r = 0.3$ m, $v_\theta = 2$ m/s, and $v_r = 0$. Neglecting the mass of the rod and the effect of friction, determine (*a*) the maximum distance between the origin and the collar, (*b*) the corresponding speed. (*Hint:* Solve the equation obtained for r by trial and error.)

13.95 A 4-lb collar A and a 1.5-lb collar B can slide without friction on a frame, consisting of the horizontal rod OE and the vertical rod CD, which is free to rotate about CD. The two collars are connected by a cord running over a pulley that is attached to the frame at O. At the instant shown, the velocity $\mathbf{v}_A$ of collar A has a magnitude of 6 ft/s and a stop prevents collar B from moving. If the stop is suddenly removed, determine (*a*) the velocity of collar A when it is 8 in. from O, (*b*) the velocity of collar A when collar B comes to rest. (Assume that collar B does not hit O, that collar A does not come off rod OE, and that the mass of the frame is negligible.)

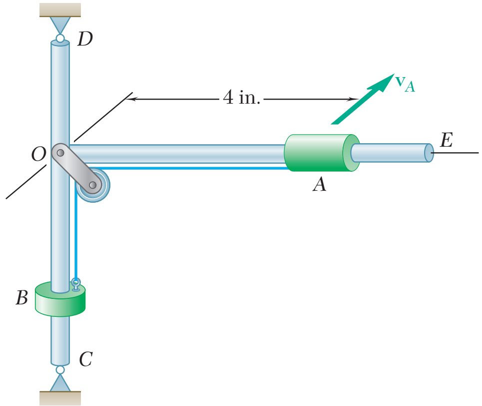


Fig. P13.95

13.96 A 1.5-lb ball that can slide on a *horizontal* frictionless surface is attached to a fixed point O by means of an elastic cord of constant $k = 1$ lb/in. and undeformed length 2 ft. The ball is placed at point A, 3 ft from O, and given an initial velocity $\mathbf{v}_0$ perpendicular to OA. Determine (*a*) the smallest allowable value of the initial speed v_0 if the cord is not to become slack, (*b*) the closest distance d that the ball will come to point O if it is given half the initial speed found in part *a*.

13.97 A 1.5-lb ball that can slide on a *horizontal* frictionless surface is attached to a fixed point O by means of an elastic cord of constant $k = 1$ lb/in. and undeformed length 2 ft. The ball is placed at point A, 3 ft from O, and given an initial velocity $\mathbf{v}_0$ perpendicular to OA, allowing the ball to come within a distance $d = 9$ in. of point O after the cord has become slack. Determine (*a*) the initial speed v_0 of the ball, (*b*) its maximum speed.

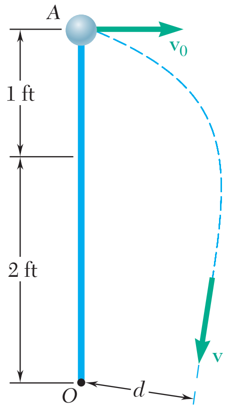


Fig. P13.96 and P13.97

13.98 Using the principles of conservation of energy and conservation of angular momentum, solve part *a* of Sample Prob. 12.9.

13.99 Solve Sample Prob. 13.8, assuming that the elastic cord is replaced by a central force $\mathbf{F}$ of magnitude $(80/r^2)$ N directed toward O.

13.100 A spacecraft is describing an elliptic orbit of minimum altitude $h_A = 2400$ km and maximum altitude $h_B = 9600$ km above the surface of the earth. Determine the speed of the spacecraft at A.

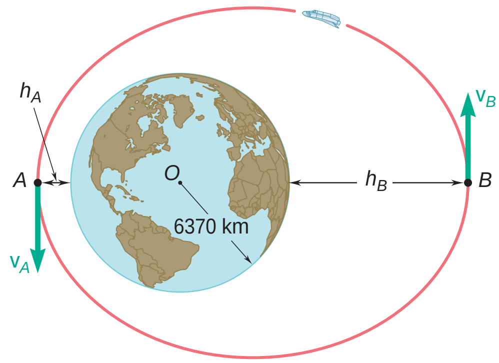


Fig. P13.100

13.101 While describing a circular orbit, 185 mi above the surface of the earth, a space shuttle ejects at point A an inertial upper stage (IUS) carrying a communications satellite to be placed in a geosynchronous orbit (see Prob. 13.87) at an altitude of 22,230 mi above the surface of the earth. Determine (*a*) the velocity of the IUS relative to the shuttle after its engine has been fired at A, (*b*) the increase in velocity required at B to place the satellite in its final orbit.

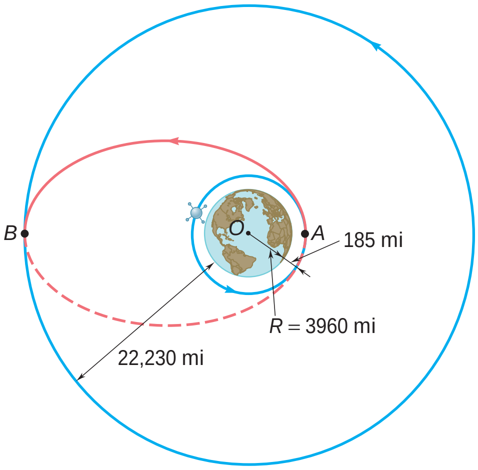


Fig. P13.101

13.102 A spacecraft approaching the planet Saturn reaches point A with a velocity $\mathbf{v}_A$ of magnitude 68.8×10^3 ft/s. It is to be placed in an elliptic orbit about Saturn so that it will be able to periodically examine Tethys, one of Saturn's moons. Tethys is in a circular orbit of radius 183×10^3 mi about the center of Saturn, traveling at a speed of 37.2×10^3 ft/s. Determine (*a*) the decrease in speed required by the spacecraft at A to achieve the desired orbit, (*b*) the speed of the spacecraft when it reaches the orbit of Tethys at B.

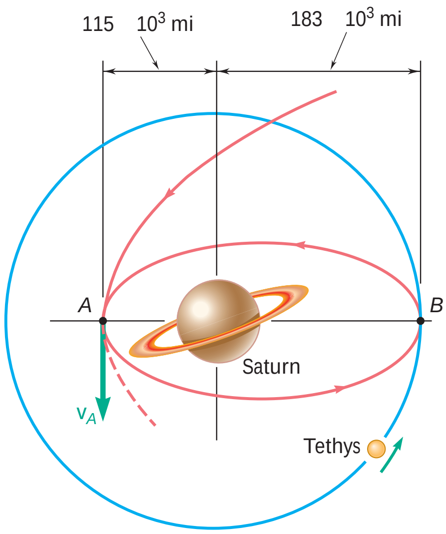


Fig. P13.102

13.103 A spacecraft traveling along a parabolic path toward the planet Jupiter is expected to reach point A with a velocity $\mathbf{v}_A$ of magnitude 26.9 km/s. Its engines will then be fired to slow it down, placing it into an elliptic orbit which will bring it to within 100×10^3 km of Jupiter. Determine the decrease in speed Δv at point A which will place the spacecraft into the required orbit. The mass of Jupiter is 319 times the mass of the earth.

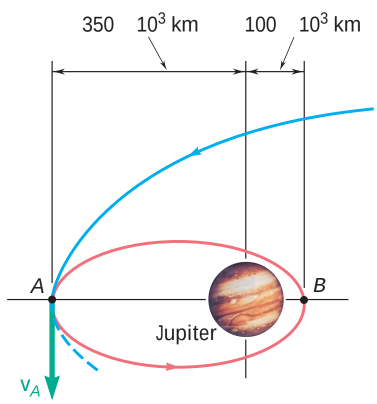


Fig. P13.103

13.104 As a first approximation to the analysis of a space flight from the earth to Mars, it is assumed that the orbits of the earth and Mars are circular and coplanar. The mean distances from the sun to the earth and to Mars are 149.6×10^6 km and 227.8×10^6 km, respectively. To place the spacecraft into an elliptical transfer orbit at point A, its speed is increased over a short interval of time to v_A which is faster than the earth's orbital speed. When the spacecraft reaches point B on the elliptical transfer orbit, its speed v_B is increased to the orbital speed of Mars. Knowing that the mass of the sun is 332.8×10^3 times the mass of the earth, determine the increase in velocity required (*a*) at A, (*b*) at B.

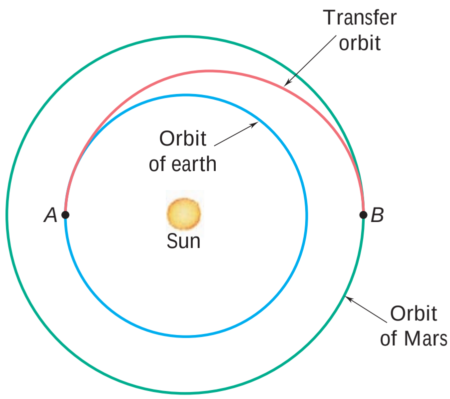


Fig. *P13.104*

13.105 The optimal way of transferring a space vehicle from an inner circular orbit to an outer coplanar circular orbit is to fire its engines as it passes through A to increase its speed and place it in an elliptic transfer orbit. Another increase in speed as it passes through B will place it in the desired circular orbit. For a vehicle in a circular orbit about the earth at an altitude $h_1 =$ 200 mi, which is to be transferred to a circular orbit at an altitude $h_2 =$ 500 mi, determine (*a*) the required increases in speed at A and at B, (*b*) the total energy per unit mass required to execute the transfer.

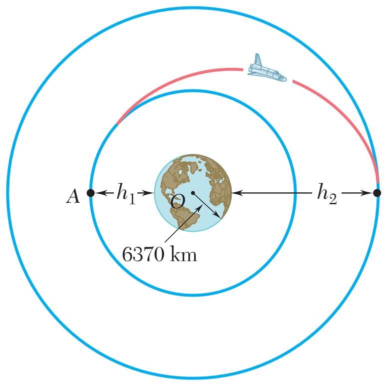


Fig. *P13.105*

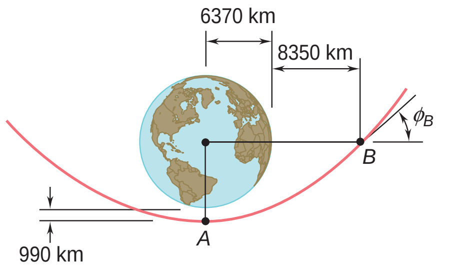


Fig. P13.106

13.106 During a flyby of the earth, the velocity of a spacecraft is 10.4 km/s as it reaches its minimum altitude of 990 km above the surface at point A. At point B the spacecraft is observed to have an altitude of 8350 km. Determine (*a*) the magnitude of the velocity at point B, (*b*) the angle ϕ_B.

13.107 A space platform is in a circular orbit about the earth at an altitude of 300 km. As the platform passes through A, a rocket carrying a communications satellite is launched from the platform with a relative velocity of magnitude 3.44 km/s in a direction tangent to the orbit of the platform. This was intended to place the rocket in an elliptic transfer orbit bringing it to point B, where the rocket would again be fired to place the satellite in a geosynchronous orbit of radius 42 140 km. After launching, it was discovered that the relative velocity imparted to the rocket was too large. Determine the angle γ at which the rocket will cross the intended orbit at point C.

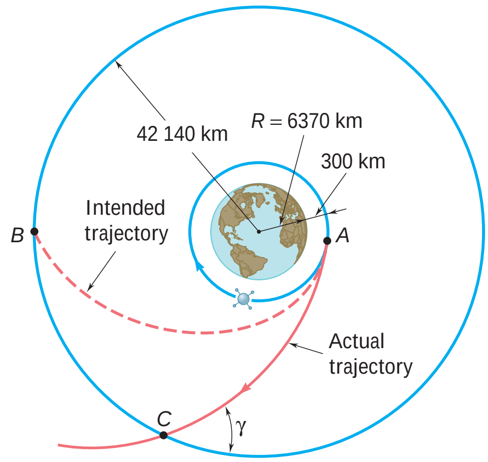


Fig. P13.107

13.108 A satellite is projected into space with a velocity $\mathbf{v}_0$ at a distance r_0 from the center of the earth by the last stage of its launching rocket. The velocity $\mathbf{v}_0$ was designed to send the satellite into a circular orbit of radius r_0. However, owing to a malfunction of control, the satellite is not projected horizontally but at an angle α with the horizontal and, as a result, is propelled into an elliptic orbit. Determine the maximum and minimum values of the distance from the center of the earth to the satellite.

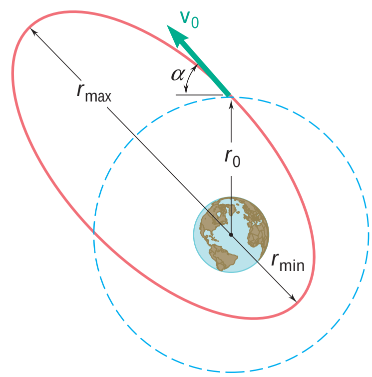


Fig. P13.108

13.109 Upon the LEM's return to the command module, the Apollo spacecraft was turned around so that the LEM faced to the rear. The LEM was then cast adrift with a velocity of 200 m/s relative to the command module. Determine the magnitude and direction (angle ϕ formed with the vertical OC) of the velocity $\mathbf{v}_C$ of the LEM just before it crashed at C on the moon's surface.

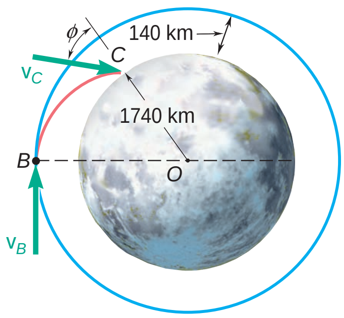


Fig. P13.109

13.110 A space vehicle is in a circular orbit at an altitude of 225 mi above the earth. To return to earth, it decreases its speed as it passes through A by firing its engine for a short interval of time in a direction opposite to the direction of its motion. Knowing that the velocity of the space vehicle should form an angle $\phi_B = 60°$ with the vertical as it reaches point B at an altitude of 40 mi, determine (*a*) the required speed of the vehicle as it leaves its circular orbit at A, (*b*) its speed at point B.

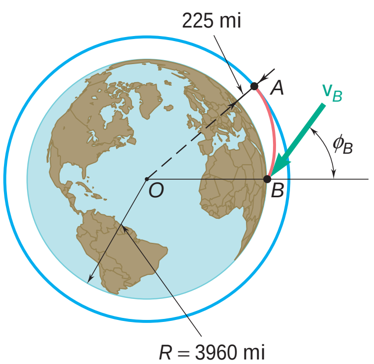


Fig. P13.110

***13.111** In Prob. 13.110, the speed of the space vehicle was decreased as it passed through A by firing its engine in a direction opposite to the direction of motion. An alternative strategy for taking the space vehicle out of its circular orbit would be to turn it around so that its engine would point away from the earth and then give it an incremental velocity $\Delta\mathbf{v}_A$ toward the center O of the earth. This would likely require a smaller expenditure of energy when firing the engine at A, but might result in too fast a descent at B. Assuming this strategy is used with only 50 percent of the energy expenditure used in Prob. 13.110, determine the resulting values of ϕ_B and v_B.

13.112 Show that the values v_A and v_P of the speed of an earth satellite at the apogee A and the perigee P of an elliptic orbit are defined by the relations

$$v_A^2 = \frac{2GM}{r_A + r_P}\frac{r_P}{r_A} \qquad v_P^2 = \frac{2GM}{r_A + r_P}\frac{r_A}{r_P}$$

where M is the mass of the earth, and r_A and r_P represent, respectively, the maximum and minimum distances of the orbit to the center of the earth.

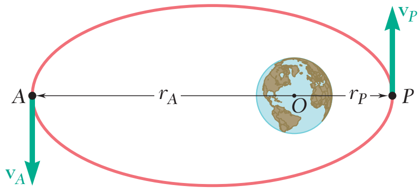


Fig. *P13.112* and *P13.113*

13.113 Show that the total energy E of an earth satellite of mass m describing an elliptic orbit is $E = -GMm/(r_A + r_P)$, where M is the mass of the earth, and r_A and r_P represent, respectively, the maximum and minimum distances of the orbit to the center of the earth. (Recall that the gravitational potential energy of a satellite was defined as being zero at an infinite distance from the earth.)

***13.114** A space probe describes a circular orbit of radius nR with a velocity $\mathbf{v}_0$ about a planet of radius R and center O. Show that (*a*) in order for the probe to leave its orbit and hit the planet at an angle θ with the vertical, its velocity must be reduced to $\alpha\mathbf{v}_0$, where

$$\alpha = \sin\theta\sqrt{\frac{2(n-1)}{n^2 - \sin^2\theta}}$$

(*b*) the probe will not hit the planet if α is larger than $\sqrt{2/(1+n)}$.

13.115 A missile is fired from the ground with an initial velocity $\mathbf{v}_0$ forming an angle ϕ_0 with the vertical. If the missile is to reach a maximum altitude equal to αR, where R is the radius of the earth, (*a*) show that the required angle ϕ_0 is defined by the relation

$$\sin\phi_0 = (1+\alpha)\sqrt{1 - \frac{\alpha}{1+\alpha}\left(\frac{v_{esc}}{v_0}\right)^2}$$

where v_{esc} is the escape velocity, (*b*) determine the range of allowable values of v_0.

13.116 A spacecraft of mass m describes a circular orbit of radius r_1 around the earth. (*a*) Show that the additional energy ΔE which must be imparted to the spacecraft to transfer it to a circular orbit of larger radius r_2 is

$$\Delta E = \frac{GMm(r_2 - r_1)}{2r_1r_2}$$

where M is the mass of the earth. (*b*) Further show that if the transfer from one circular orbit to the other is executed by placing the spacecraft on a transitional semielliptic path AB, the amounts of energy ΔE_A and ΔE_B which must be imparted at A and B are, respectively, proportional to r_2 and r_1:

$$\Delta E_A = \frac{r_2}{r_1 + r_2}\Delta E \qquad \Delta E_B = \frac{r_1}{r_1 + r_2}\Delta E$$

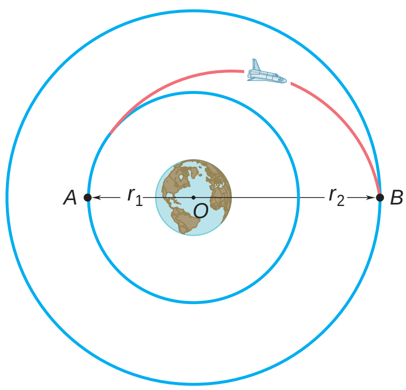


Fig. *P13.116*

***13.117** Using the answers obtained in Prob. 13.108, show that the intended circular orbit and the resulting elliptic orbit intersect at the ends of the minor axis of the elliptic orbit.

***13.118** (*a*) Express in terms of $r_{\min}$ and $v_{\max}$ the angular momentum per unit mass, h, and the total energy per unit mass, E/m, of a space vehicle moving under the gravitational attraction of a planet of mass M (Fig. 13.15). (*b*) Eliminating $v_{\max}$ between the equations obtained, derive the formula

$$\frac{1}{r_{\min}} = \frac{GM}{h^2}\left[1 + \sqrt{1 + \frac{2E}{m}\left(\frac{h}{GM}\right)^2}\right]$$

(*c*) Show that the eccentricity ε of the trajectory of the vehicle can be expressed as

$$\varepsilon = \sqrt{1 + \frac{2E}{m}\left(\frac{h}{GM}\right)^2}$$

(*d*) Further show that the trajectory of the vehicle is a hyperbola, an ellipse, or a parabola, depending on whether E is positive, negative, or zero.

13.10 PRINCIPLE OF IMPULSE AND MOMENTUM

A third basic method for the solution of problems dealing with the motion of particles will be considered now. This method is based on the principle of impulse and momentum and can be used to solve problems involving force, mass, velocity, and time. It is of particular interest in the solution of problems involving impulsive motion and problems involving impact (Secs. 13.11 and 13.12).

Consider a particle of mass m acted upon by a force $\mathbf{F}$. As we saw in Sec. 12.3, Newton's second law can be expressed in the form

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{13.27}$$

Photo 13.1

where $m\mathbf{v}$ is the linear momentum of the particle. Multiplying both sides of Eq. (13.27) by dt and integrating from a time t_1 to a time t_2, we write

$$\mathbf{F}\,dt = d(m\mathbf{v})$$

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = m\mathbf{v}_2 - m\mathbf{v}_1$$

or, transposing the last term,

$$m\mathbf{v}_1 + \int_{t_1}^{t_2} \mathbf{F}\,dt = m\mathbf{v}_2 \tag{13.28}$$

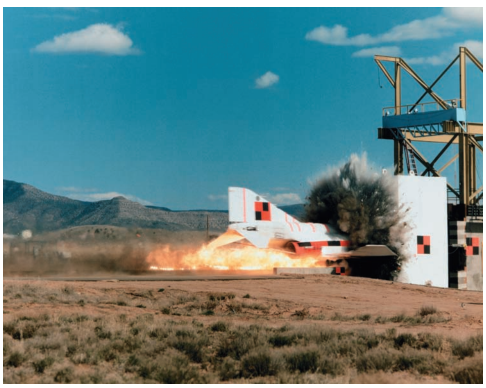


Photo 13.2 This impact test between an F-4 Phantom and a rigid reinforced target was to determine the impact force as a function of time.

The integral in Eq. (13.28) is a vector known as the *linear impulse*, or simply the *impulse*, of the force $\mathbf{F}$ during the interval of time considered. Resolving $\mathbf{F}$ into rectangular components, we write

$$\begin{aligned}\mathbf{Imp}_{1\rightarrow 2} &= \int_{t_1}^{t_2} \mathbf{F}\,dt \\ &= \mathbf{i}\int_{t_1}^{t_2} F_x\,dt + \mathbf{j}\int_{t_1}^{t_2} F_y\,dt + \mathbf{k}\int_{t_1}^{t_2} F_z\,dt\end{aligned} \tag{13.29}$$

and note that the components of the impulse of the force **F** are, respectively, equal to the areas under the curves obtained by plotting the components F_x, F_y, and F_z against t (Fig. 13.16). In the case of a force **F** of constant magnitude and direction, the impulse is represented by the vector $\mathbf{F}(t_2 - t_1)$, which has the same direction as **F**.

If SI units are used, the magnitude of the impulse of a force is expressed in N · s. But, recalling the definition of the newton, we have

$$\mathrm{N \cdot s = (kg \cdot m/s^2) \cdot s = kg \cdot m/s}$$

which is the unit obtained in Sec. 12.4 for the linear momentum of a particle. We thus check that Eq. (13.28) is dimensionally correct. If U.S. customary units are used, the impulse of a force is expressed in lb · s, which is also the unit obtained in Sec. 12.4 for the linear momentum of a particle.

Equation (13.28) expresses that when a particle is acted upon by a force **F** during a given time interval, *the final momentum* $m\mathbf{v}_2$ *of the particle can be obtained by adding vectorially its initial momentum* $m\mathbf{v}_1$ *and the impulse of the force* **F** *during the time interval considered*

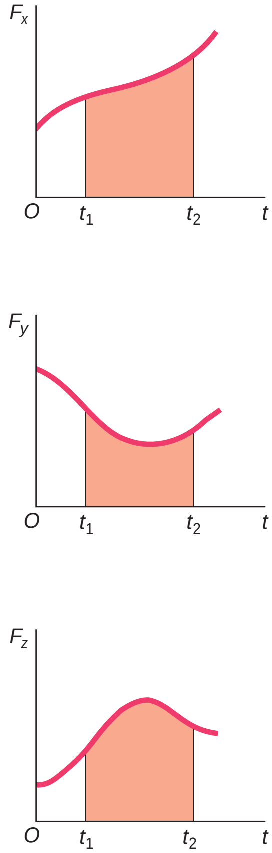


Fig. 13.16

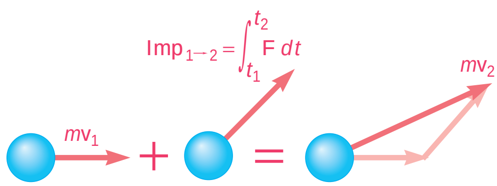


Fig. 13.17

(Fig. 13.17). We write

$$m\mathbf{v}_1 + \mathbf{Imp}_{1\,y\,2} = m\mathbf{v}_2 \tag{13.30}$$

We note that while kinetic energy and work are scalar quantities, momentum and impulse are vector quantities. To obtain an analytic solution, it is thus necessary to replace Eq. (13.30) by the corresponding component equations

$$\begin{aligned} (mv_x)_1 + \int_{t_1}^{t_2} F_x\,dt &= (mv_x)_2 \\ (mv_y)_1 + \int_{t_1}^{t_2} F_y\,dt &= (mv_y)_2 \\ (mv_z)_1 + \int_{t_1}^{t_2} F_z\,dt &= (mv_z)_2 \end{aligned} \tag{13.31}$$

When several forces act on a particle, the impulse of each of the forces must be considered. We have

$$m\mathbf{v}_1 + \Sigma\,\mathbf{Imp}_{1\,y\,2} = m\mathbf{v}_2 \tag{13.32}$$

Again, the equation obtained represents a relation between vector quantities; in the actual solution of a problem, it should be replaced by the corresponding component equations.

When a problem involves two particles or more, each particle can be considered separately and Eq. (13.32) can be written for each particle. We can also add vectorially the momenta of all the particles and the impulses of all the forces involved. We write then

$$\Sigma m\mathbf{v}_1 + \Sigma\ \mathbf{Imp}_{1\rightarrow 2} = \Sigma m\mathbf{v}_2 \tag{13.33}$$

Since the forces of action and reaction exerted by the particles on each other form pairs of equal and opposite forces, and since the time interval from t_1 to t_2 is common to all the forces involved, the impulses of the forces of action and reaction cancel out and only the impulses of the external forces need be considered.†

If no external force is exerted on the particles or, more generally, if the sum of the external forces is zero, the second term in Eq. (13.33) vanishes and Eq. (13.33) reduces to

$$\Sigma m\mathbf{v}_1 = \Sigma m\mathbf{v}_2 \tag{13.34}$$

which expresses that *the total momentum of the particles is conserved*. Consider, for example, two boats, of mass m_A and m_B, initially at rest, which are being pulled together (Fig. 13.18). If the resistance

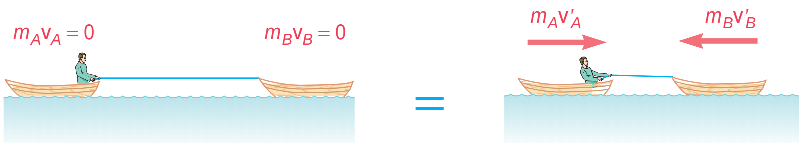


Fig. 13.18

of the water is neglected, the only external forces acting on the boats are their weights and the buoyant forces exerted on them. Since these forces are balanced, we write

$$\Sigma m\mathbf{v}_1 = \Sigma m\mathbf{v}_2$$
$$0 = m_A\mathbf{v}'_A + m_B\mathbf{v}'_B$$

where $\mathbf{v}'_A$ and $\mathbf{v}'_B$ represent the velocities of the boats after a finite interval of time. The equation obtained indicates that the boats move in opposite directions (toward each other) with velocities inversely proportional to their masses.‡

†We should note the difference between this statement and the corresponding statement made in Sec. 13.4 regarding the work of the forces of action and reaction between several particles. While the sum of the impulses of these forces is always zero, the sum of their work is zero only under special circumstances, e.g., when the various bodies involved are connected by inextensible cords or links and are thus constrained to move through equal distances.

‡Blue equals signs are used in Fig. 13.18 and throughout the remainder of this chapter to express that two systems of vectors are *equipollent*, i.e., that they have the same resultant and moment resultant (cf. Sec. 3.19). Red equals signs will continue to be used to indicate that two systems of vectors are *equivalent*, i.e., that they have the same effect. This and the concept of conservation of momentum for a system of particles will be discussed in greater detail in Chap. 14.

13.11 IMPULSIVE MOTION

A force acting on a particle during a very short time interval that is large enough to produce a definite change in momentum is called an *impulsive force* and the resulting motion is called an *impulsive motion*. For example, when a baseball is struck, the contact between bat and ball takes place during a very short time interval Δt. But the average value of the force $\mathbf{F}$ exerted by the bat on the ball is very large, and the resulting impulse $\mathbf{F}\,\Delta t$ is large enough to change the sense of motion of the ball (Fig. 13.19).

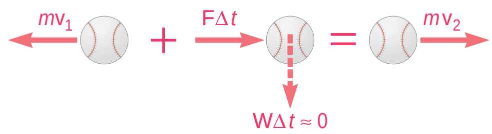


Fig. 13.19

When impulsive forces act on a particle, Eq. (13.32) becomes

$$m\mathbf{v}_1 + \Sigma\mathbf{F}\,\Delta t = m\mathbf{v}_2 \tag{13.35}$$

Any force which is not an impulsive force may be neglected, since the corresponding impulse $\mathbf{F}\,\Delta t$ is very small. *Nonimpulsive forces* include the weight of the body, the force exerted by a spring, or any other force which is *known* to be small compared with an impulsive force. Unknown reactions may or may not be impulsive; their impulses should therefore be included in Eq. (13.35) as long as they have not been proved negligible. The impulse of the weight of the baseball considered above, for example, may be neglected. If the motion of the bat is analyzed, the impulse of the weight of the bat can also be neglected. The impulses of the reactions of the player's hands on the bat, however, should be included; these impulses will not be negligible if the ball is incorrectly hit.

We note that the method of impulse and momentum is particularly effective in the analysis of the impulsive motion of a particle, since it involves only the initial and final velocities of the particle and the impulses of the forces exerted on the particle. The direct application of Newton's second law, on the other hand, would require the determination of the forces as functions of the time and the integration of the equations of motion over the time interval Δt.

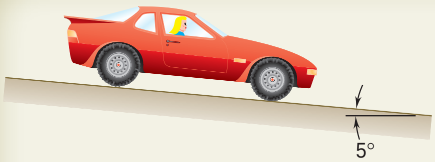
In the case of the impulsive motion of several particles, Eq. (13.33) can be used. It reduces to

$$\Sigma m\mathbf{v}_1 + \Sigma\mathbf{F}\,\Delta t = \Sigma m\mathbf{v}_2 \tag{13.36}$$

where the second term involves only impulsive, external forces. If all the external forces acting on the various particles are nonimpulsive, the second term in Eq. (13.36) vanishes and this equation reduces to Eq. (13.34). We write

$$\Sigma m\mathbf{v}_1 = \Sigma m\mathbf{v}_2 \tag{13.34}$$

which expresses that the total momentum of the particles is conserved. This situation occurs, for example, when two particles which are moving freely collide with one another. We should note, however, that while the total momentum of the particles is conserved, their total energy is generally *not* conserved. Problems involving the collision or *impact* of two particles will be discussed in detail in Secs. 13.12 through 13.14.

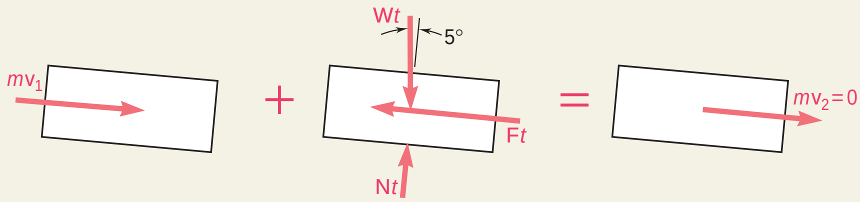


SAMPLE PROBLEM 13.10

An automobile weighing 4000 lb is driven down a 5° incline at a speed of 60 mi/h when the brakes are applied, causing a constant total braking force (applied by the road on the tires) of 1500 lb. Determine the time required for the automobile to come to a stop.

SOLUTION

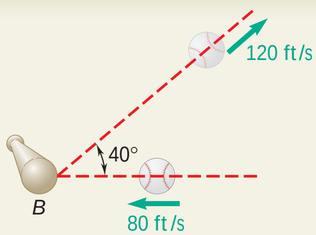
We apply the principle of impulse and momentum. Since each force is constant in magnitude and direction, each corresponding impulse is equal to the product of the force and of the time interval t.



$$m\mathbf{v}_1 + \Sigma\,\mathbf{Imp}_{1\rightarrow 2} = m\mathbf{v}_2$$

$$+\searrow \text{ components:} \quad mv_1 + (W \sin 5°)t - Ft = 0$$

$$(4000/32.2)(88 \text{ ft/s}) + (4000 \sin 5°)t - 1500t = 0 \qquad t = 9.49 \text{ s} \quad ◀$$

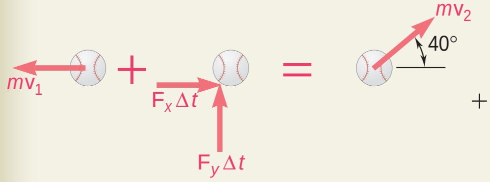


SAMPLE PROBLEM 13.11

A 4-oz baseball is pitched with a velocity of 80 ft/s toward a batter. After the ball is hit by the bat B, it has a velocity of 120 ft/s in the direction shown. If the bat and ball are in contact 0.015 s, determine the average impulsive force exerted on the ball during the impact.

SOLUTION

We apply the principle of impulse and momentum to the ball. Since the weight of the ball is a nonimpulsive force, it can be neglected.



$$m\mathbf{v}_1 + \Sigma\,\mathbf{Imp}_{1\rightarrow 2} = m\mathbf{v}_2$$

$$\overset{+}{\rightarrow} x \text{ components:} \quad -mv_1 + F_x\,\Delta t = mv_2 \cos 40°$$

$$-\frac{\frac{4}{16}}{32.2}(80 \text{ ft/s}) + F_x(0.015 \text{ s}) = \frac{\frac{4}{16}}{32.2}(120 \text{ ft/s}) \cos 40°$$

$$F_x = +89.0 \text{ lb}$$

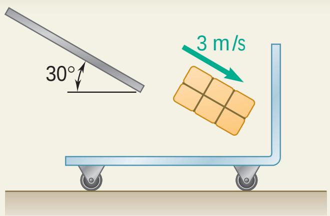
$$+\uparrow y \text{ components:} \quad 0 + F_y\,\Delta t = mv_2 \sin 40°$$

$$F_y(0.015 \text{ s}) = \frac{\frac{4}{16}}{32.2}(120 \text{ ft/s}) \sin 40°$$

$$F_y = +39.9 \text{ lb}$$

From its components F_x and F_y we determine the magnitude and direction of the force $\mathbf{F}$:

$$\mathbf{F} = 97.5 \text{ lb} \angle 24.2° \quad ◀$$



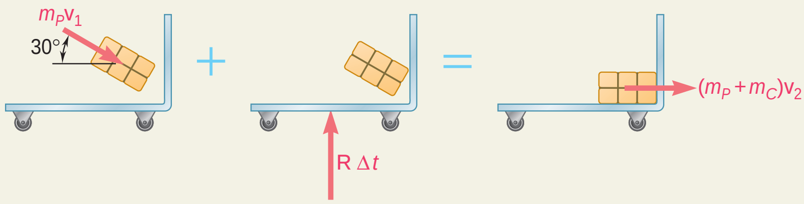
SAMPLE PROBLEM 13.12

A 10-kg package drops from a chute into a 25-kg cart with a velocity of 3 m/s. Knowing that the cart is initially at rest and can roll freely, determine (*a*) the final velocity of the cart, (*b*) the impulse exerted by the cart on the package, (*c*) the fraction of the initial energy lost in the impact.

SOLUTION

We first apply the principle of impulse and momentum to the package-cart system to determine the velocity $\mathbf{v}_2$ of the cart and package. We then apply the same principle to the package alone to determine the impulse $\mathbf{F}\ \Delta t$ exerted on it.

a. Impulse-Momentum Principle: Package and Cart



$$m_P\mathbf{v}_1 + \Sigma\,\mathbf{Imp}_{1\rightarrow 2} = (m_P + m_C)\mathbf{v}_2$$

$\xrightarrow{+}$ x components:

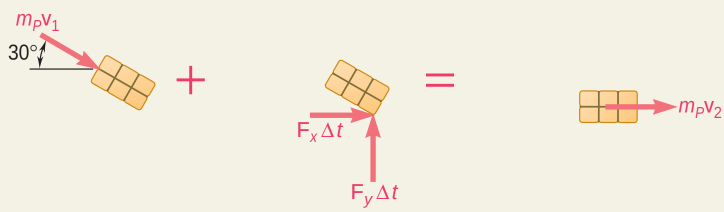
$$m_P v_1 \cos 30° + 0 = (m_P + m_C)v_2$$

$$(10\text{ kg})(3\text{ m/s})\cos 30° = (10\text{ kg} + 25\text{ kg})v_2$$

$$\mathbf{v}_2 = 0.742\text{ m/s} \rightarrow \quad ◀$$

We note that the equation used expresses conservation of momentum in the x direction.

b. Impulse-Momentum Principle: Package



$$m_P\mathbf{v}_1 + \Sigma\,\mathbf{Imp}_{1\rightarrow 2} = m_P\mathbf{v}_2$$

$\xrightarrow{+}$ x components:

$$(10\text{ kg})(3\text{ m/s})\cos 30° + F_x\,\Delta t = (10\text{ kg})(0.742\text{ m/s})$$

$$F_x\,\Delta t = -18.56\text{ N}\cdot\text{s}$$

$+\uparrow y$ components:

$$-m_P v_1 \sin 30° + F_y\,\Delta t = 0$$

$$-(10\text{ kg})(3\text{ m/s})\sin 30° + F_y\,\Delta t = 0$$

$$F_y\,\Delta t = +15\text{ N}\cdot\text{s}$$

The impulse exerted on the package is $\quad \mathbf{F}\,\Delta t = 23.9\text{ N}\cdot\text{s} \angle\ 38.9° \quad ◀$

c. Fraction of Energy Lost. The initial and final energies are

$$T_1 = \tfrac{1}{2}m_P v_1^2 = \tfrac{1}{2}(10\text{ kg})(3\text{ m/s})^2 = 45\text{ J}$$

$$T_2 = \tfrac{1}{2}(m_P + m_C)v_2^2 = \tfrac{1}{2}(10\text{ kg} + 25\text{ kg})(0.742\text{ m/s})^2 = 9.63\text{ J}$$

The fraction of energy lost is

$$\frac{T_1 - T_2}{T_1} = \frac{45\text{ J} - 9.63\text{ J}}{45\text{ J}} = 0.786 \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we integrated Newton's second law to derive the *principle of impulse and momentum* for a particle. Recalling that the *linear momentum* of a particle was defined as the product of its mass m and its velocity $\mathbf{v}$ [Sec. 12.3], we wrote

$$m\mathbf{v}_1 + \Sigma\, \mathbf{Imp}_{1\,y\,2} = m\mathbf{v}_2 \tag{13.32}$$

This equation expresses that the linear momentum $m\mathbf{v}_2$ of a particle at time t_2 can be obtained by adding to its linear momentum $m\mathbf{v}_1$ at time t_1 the *impulses* of the forces exerted on the particle during the time interval t_1 to t_2. For computing purposes, the momenta and impulses may be expressed in terms of their rectangular components, and Eq. (13.32) can be replaced by the equivalent scalar equations. The units of momentum and impulse are N · s in the SI system of units and lb · s in U.S. customary units. To solve problems using this equation you can follow these steps:

1. Draw a diagram showing the particle, its momentum at t_1 and at t_2, and the impulses of the forces exerted on the particle during the time interval t_1 to t_2.

2. Calculate the impulse of each force, expressing it in terms of its rectangular components if more than one direction is involved. You may encounter the following cases:

a. The time interval is finite and the force is constant.

$$\mathbf{Imp}_{1\,y\,2} = \mathbf{F}(t_2 - t_1)$$

b. The time interval is finite and the force is a function of *t*.

$$\mathbf{Imp}_{1\,y\,2} = \int_{t_1}^{t_2} \mathbf{F}(t)\, dt$$

c. The time interval is very small and the force is very large. The force is called an *impulsive force* and its impulse over the time interval $t_2 - t_1 = \Delta t$ is

$$\mathbf{Imp}_{1\,y\,2} = \mathbf{F}\, \Delta t$$

Note that this impulse is *zero for a nonimpulsive force* such as the *weight* of a body, the force exerted by a *spring*, or any other force which is known to be small by comparison with the impulsive forces. Unknown reactions, however, *cannot be assumed* to be nonimpulsive and their impulses should be taken into account.

3. Substitute the values obtained for the impulses into Eq. (13.32) or into the equivalent scalar equations. You will find that the forces and velocities in the problems of this lesson are contained in a plane. You will, therefore, write two scalar equations and solve these equations for *two unknowns*. These unknowns may be a *time* [Sample Prob. 13.10], a *velocity* and an *impulse* [Sample Prob. 13.12], or an *average impulsive force* [Sample Prob. 13.11].

4. When several particles are involved, a separate diagram should be drawn for each particle, showing the initial and final momentum of the particle, as well as the impulses of the forces exerted on the particle.

a. It is usually convenient, however, to first consider a diagram including all the particles. This diagram leads to the equation

$$\Sigma m\mathbf{v}_1 + \Sigma\ \mathbf{Imp}_{1\,y\,2} = \Sigma m\mathbf{v}_2 \tag{13.33}$$

where the impulses of *only the forces external to the system* need be considered. Therefore, the two equivalent scalar equations will not contain any of the impulses of the unknown internal forces.

b. If the sum of the impulses of the external forces is zero, Eq. (13.33) reduces to

$$\Sigma m\mathbf{v}_1 = \Sigma m\mathbf{v}_2 \tag{13.34}$$

which expresses that *the total momentum of the particles is conserved*. This occurs either if the resultant of the external forces is zero or, when the time interval Δt is very short (impulsive motion), if all the external forces are nonimpulsive. Keep in mind, however, that the total momentum may be conserved *in one direction,* but not in another [Sample Prob. 13.12].

PROBLEMS

CONCEPT QUESTIONS

13.CQ4 A large insect impacts the front windshield of a sports car traveling down a road. Which of the following statements is true during the collision?

- **a.** The car exerts a greater force on the insect than the insect exerts on the car.
- **b.** The insect exerts a greater force on the car than the car exerts on the insect.
- **c.** The car exerts a force on the insect, but the insect does not exert a force on the car.
- **d.** The car exerts the same force on the insect as the insect exerts on the car.
- **e.** Neither exerts a force on the other; the insect gets smashed simply because it gets in the way of the car.

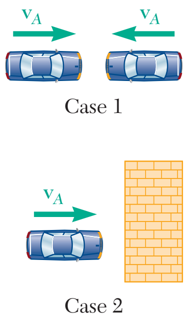


Fig. P13.CQ5

13.CQ5 The expected damages associated with two types of perfectly plastic collisions are to be compared. In the first case, two identical cars traveling at the same speed impact each other head-on. In the second case, the car impacts a massive concrete wall. In which case would you expect the car to be more damaged?

- **a.** Case 1
- **b.** Case 2
- **c.** The same damage in each case

IMPULSE-MOMENTUM PRACTICE PROBLEMS

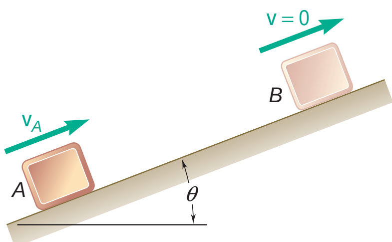


Fig. P13.F1

13.F1 The initial velocity of the block in position A is 30 ft/s. The coefficient of kinetic friction between the block and the plane is $m_k = 0.30$. Draw the impulse-momentum diagram that can be used to determine the time it takes for the block to reach B with zero velocity, if u = 20°.

13.F2 A 4-lb collar which can slide on a frictionless vertical rod is acted upon by a force **P** which varies in magnitude as shown. Knowing that the collar is initially at rest, draw the impulse-momentum diagram that can be used to determine its velocity at $t = 3$ s.

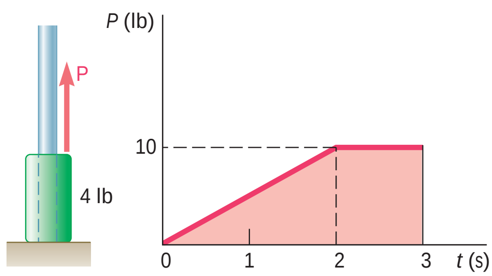


Fig. P13.F2

13.F3 The 15-kg suitcase A has been propped up against one end of a 40-kg luggage carrier B and is prevented from sliding down by other luggage. When the luggage is unloaded and the last heavy trunk is removed from the carrier, the suitcase is free to slide down, causing the 40-kg carrier to move to the left with a velocity v_B of magnitude 0.8 m/s. Neglecting friction, draw the impulse-momentum diagrams that can be used to determine (*a*) the velocity of A as it rolls on the carrier, (*b*) the velocity of the carrier after the suitcase hits the right side of the carrier without bouncing back.

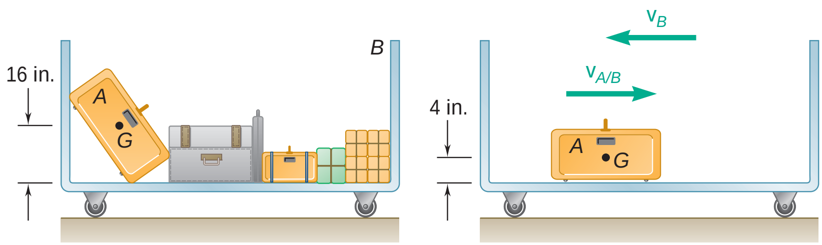


Fig. P13.F3

13.F4 Car A was traveling west at a speed of 15 m/s and car B was traveling north at an unknown speed when they slammed into each other at an intersection. Upon investigation it was found that after the crash the two cars got stuck and skidded off at an angle of 50° north of east. Knowing the masses of A and B are m_A and m_B, respectively, draw the impulse-momentum diagram that can be used to determine the velocity of B before impact.

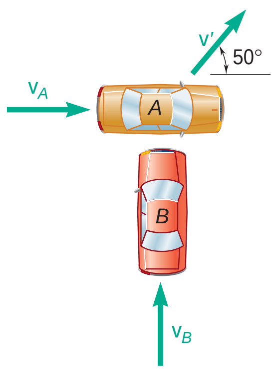


Fig. P13.F4

13.F5 Two identical spheres A and B, each of mass m, are attached to an inextensible inelastic cord of length L and are resting at a distance a from each other on a frictionless horizontal surface. Sphere B is given a velocity $\mathbf{v}_0$ in a direction perpendicular to line AB and moves it without friction until it reaches B' where the cord becomes taut. Draw the impulse-momentum diagram that can be used to determine the magnitude of the velocity of each sphere immediately after the cord has become taut.

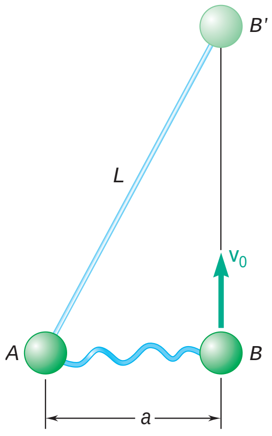


Fig. P13.F5

END-OF-SECTION PROBLEMS

13.119 A 35 000-Mg ocean liner has an initial velocity of 4 km/h. Neglecting the frictional resistance of the water, determine the time required to bring the liner to rest by using a single tugboat which exerts a constant force of 150 kN.

13.120 A 2500-lb automobile is moving at a speed of 60 mi/h when the brakes are fully applied, causing all four wheels to skid. Determine the time required to stop the automobile (*a*) on dry pavement ($\mu_k = 0.75$), (*b*) on an icy road ($\mu_k = 0.10$).

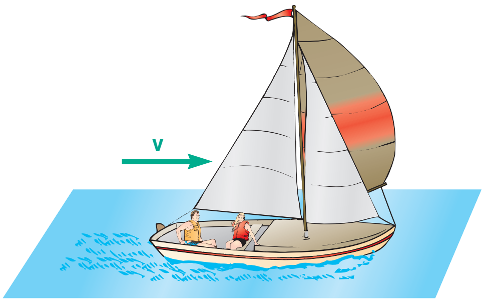


Fig. P13.121

13.121 A sailboat weighing 980 lb with its occupants is running down wind at 8 mi/h when its spinnaker is raised to increase its speed. Determine the net force provided by the spinnaker over the 10-s interval that it takes for the boat to reach a speed of 12 mi/h.

13.122 A truck is hauling a 300-kg log out of a ditch using a winch attached to the back of the truck. Knowing the winch applies a constant force of 2500 N and the coefficient of kinetic friction between the ground and the log is 0.45, determine the time for the log to reach a speed of 0.5 m/s.

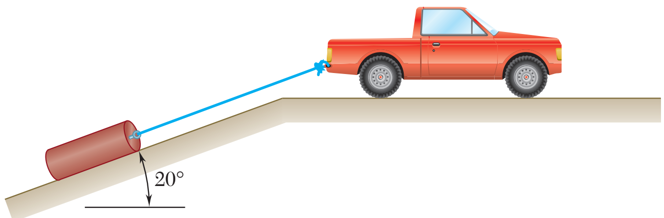


Fig. *P13.122*

13.123 A truck is traveling down a road with a 3-percent grade at a speed of 55 mi/h when the brakes are applied. Knowing the coefficients of friction between the load and the flatbed trailer shown are $\mu_s = 0.40$ and $\mu_k = 0.35$, determine the shortest time in which the rig can be brought to a stop if the load is not to shift.

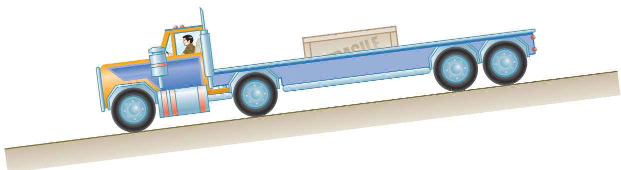


Fig. P13.123

Fig. P13.124

13.124 Steep safety ramps are built beside mountain highways to enable vehicles with defective brakes to stop. A 10-ton truck enters a 15° ramp at a high speed $v_0 = 108$ ft/s and travels for 6 s before its speed is reduced to 36 ft/s. Assuming constant deceleration, determine (*a*) the magnitude of the braking force, (*b*) the additional time required for the truck to stop. Neglect air resistance and rolling resistance.

13.125 Baggage on the floor of the baggage car of a high-speed train is not prevented from moving other than by friction. The train is traveling down a 5-percent grade when it decreases its speed at a constant rate from 120 mi/h to 60 mi/h in a time interval of 12 s. Determine the smallest allowable value of the coefficient of static friction between a trunk and the floor of the baggage car if the trunk is not to slide.

13.126 A 2-kg particle is acted upon by the force, expressed in newtons, $\mathbf{F} = (8 - 6t)\mathbf{i} + (4 - t^2)\mathbf{j} + (4 + t)\mathbf{k}$. Knowing that the velocity of the particle is $\mathbf{v} = (150 \text{ m/s})\mathbf{i} + (100 \text{ m/s})\mathbf{j} - (250 \text{ m/s})\mathbf{k}$ at $t = 0$, determine (*a*) the time at which the velocity of the particle is parallel to the yz plane, (*b*) the corresponding velocity of the particle.

13.127 A truck is traveling down a road with a 4-percent grade at a speed of 60 mi/h when its brakes are applied to slow it down to 20 mi/h. An antiskid braking system limits the braking force to a value at which the wheels of the truck are just about to slide. Knowing that the coefficient of static friction between the road and the wheels is 0.60, determine the shortest time needed for the truck to slow down.

13.128 Skid marks on a drag race track indicate that the rear (drive) wheels of a car slip for the first 20 m of the 400-m track. (*a*) Knowing that the coefficient of kinetic friction is 0.60, determine the shortest possible time for the car to travel the initial 20-m portion of the track if it starts from rest with its front wheels just off the ground. (*b*) Determine the minimum time for the car to run the whole race if, after skidding for 20 m, the wheels roll without sliding for the remainder of the race. Assume for the rolling portion of the race that 65 percent of the weight is on the rear wheels and that the coefficient of static friction is 0.85. Ignore air resistance and rolling resistance.

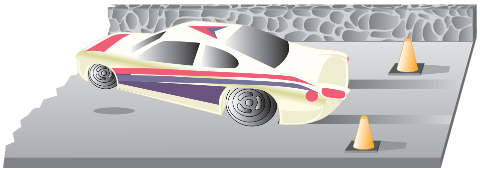


Fig. *P13.128*

13.129 The subway train shown is traveling at a speed of 30 mi/h when the brakes are fully applied on the wheels of cars *B* and *C*, causing them to slide on the track, but are not applied on the wheels of car *A*. Knowing that the coefficient of kinetic friction is 0.35 between the wheels and the track, determine (*a*) the time required to bring the train to a stop, (*b*) the force in each coupling.

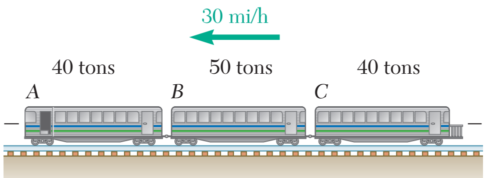


Fig. P13.129

13.130 Solve Prob. 13.129, assuming that the brakes are applied only on the wheels of car *A*.

13.131 A trailer truck with a 2000-kg cab and an 8000-kg trailer is traveling on a level road at 90 km/h. The brakes on the trailer fail and the antiskid system of the cab provides the largest possible force which will not cause the wheels of the cab to slide. Knowing that the coefficient of static friction is 0.65, determine (*a*) the shortest time for the rig to come to a stop, (*b*) the force in the coupling during that time.

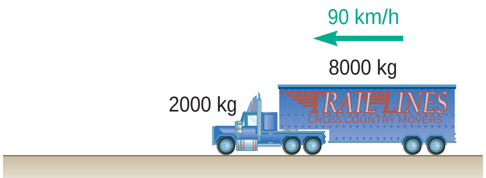


Fig. P13.131

13.132 The system shown is at rest when a constant 150-N force is applied to collar *B*. Neglecting the effect of friction, determine (*a*) the time at which the velocity of collar *B* will be 2.5 m/s to the left, (*b*) the corresponding tension in the cable.

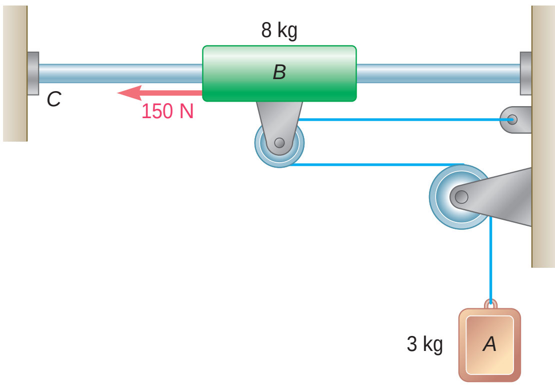


Fig. P13.132

13.133 An 8-kg cylinder C rests on a 4-kg platform A supported by a cord which passes over the pulleys D and E and is attached to a 4-kg block B. Knowing that the system is released from rest, determine (*a*) the velocity of block B after 0.8 s, (*b*) the force exerted by the cylinder on the platform.

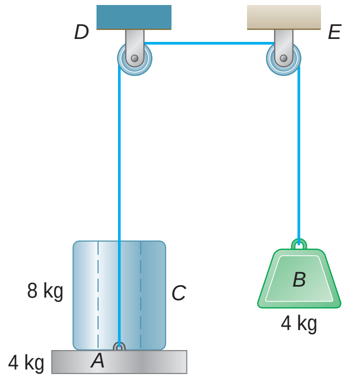


Fig. P13.133

13.134 An estimate of the expected load on over-the-shoulder seat belts is to be made before designing prototype belts that will be evaluated in automobile crash tests. Assuming that an automobile traveling at 45 mi/h is brought to a stop in 110 ms, determine (*a*) the average impulsive force exerted by a 200-lb man on the belt, (*b*) the maximum force F_m exerted on the belt if the force-time diagram has the shape shown.

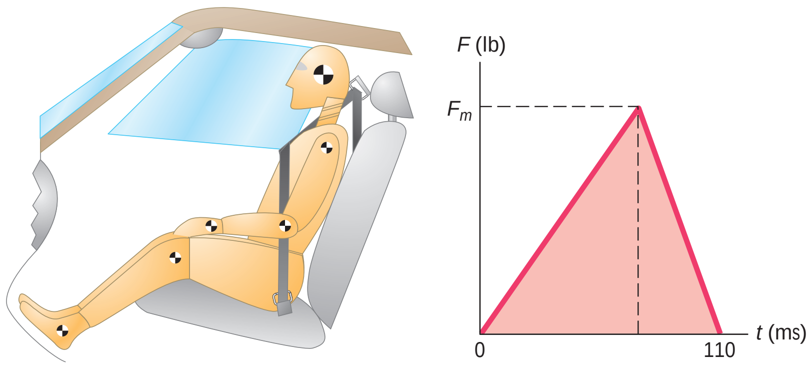


Fig. P13.134

13.135 A 60-g model rocket is fired vertically. The engine applies a thrust $\mathbf{P}$ which varies in magnitude as shown. Neglecting air resistance and the change in mass of the rocket, determine (*a*) the maximum speed of the rocket as it goes up, (*b*) the time for the rocket to reach its maximum elevation.

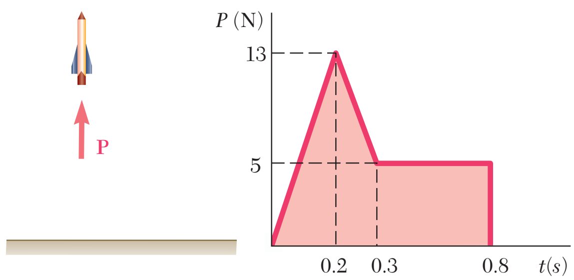


Fig. P13.135

13.136 A simplified model consisting of a single straight line is to be obtained for the variation of pressure inside the 10-mm-diameter barrel of a rifle as a 20-g bullet is fired. Knowing that it takes 1.6 ms for the bullet to travel the length of the barrel and that the velocity of the bullet upon exit is 700 m/s, determine the value of p_0.

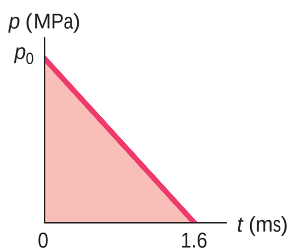


Fig. P13.136

13.137 A 125-lb block initially at rest is acted upon by a force **P** which varies as shown. Knowing that the coefficients of friction between the block and the horizontal surface are $\mu_s = 0.50$ and $\mu_k = 0.40$, determine (*a*) the time at which the block will start moving, (*b*) the maximum speed reached by the block, (*c*) the time at which the block will stop moving.

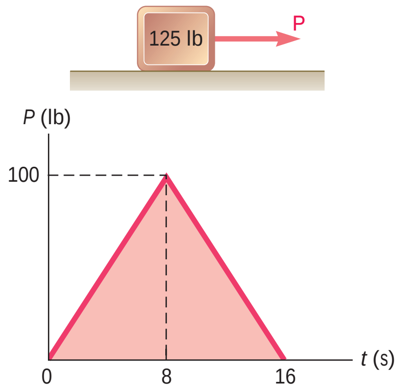


Fig. *P13.137*

13.138 Solve Prob. 13.137, assuming that the weight of the block is 175 lb.

13.139 A baseball player catching a ball can soften the impact by pulling his hand back. Assuming that a 5-oz ball reaches his glove at 90 mi/h and that the player pulls his hand back during the impact at an average speed of 30 ft/s over a distance of 6 in., bringing the ball to a stop, determine the average impulsive force exerted on the player's hand.

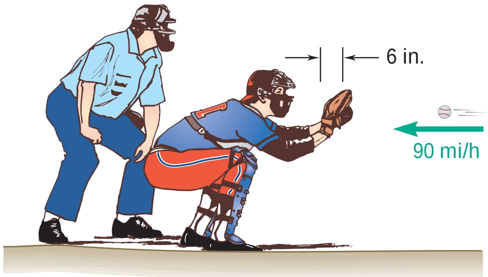


Fig. P13.139

13.140 A 1.62-oz golf ball is hit with a golf club and leaves it with a velocity of 100 mi/h. We assume that for $0 \leq t \leq t_0$, where t_0 is the duration of the impact, the magnitude F of the force exerted on the ball can be expressed as $F = F_m \sin(\pi t/t_0)$. Knowing that $t_0 = 0.5$ ms, determine the maximum value F_m of the force exerted on the ball.

13.141 The triple jump is a track-and-field event in which an athlete gets a running start and tries to leap as far as he can with a hop, step, and jump. Shown in the figure is the initial hop of the athlete. Assuming that he approaches the takeoff line from the left with a horizontal velocity of 10 m/s, remains in contact with the ground for 0.18 s, and takes off at a 50° angle with a velocity of 12 m/s, determine the vertical component of the average impulsive force exerted by the ground on his foot. Give your answer in terms of the weight W of the athlete.

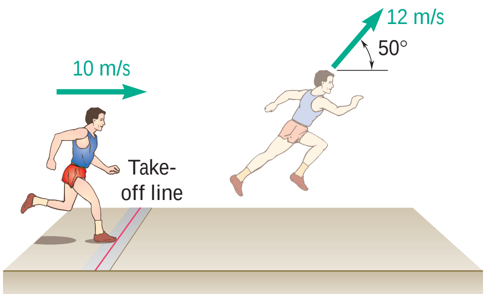
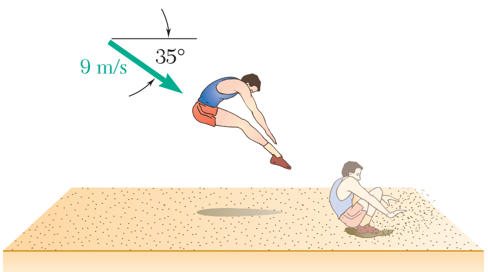


Fig. P13.141

13.142 The last segment of the triple jump track-and-field event is the jump, in which the athlete makes a final leap, landing in a sand-filled pit. Assuming that the velocity of a 80-kg athlete just before landing is 9 m/s at an angle of 35° with the horizontal and that the athlete comes to a complete stop in 0.22 s after landing, determine the horizontal component of the average impulsive force exerted on his feet during landing.



Landing pit

Fig. P13.142

13.143 The design for a new cementless hip implant is to be studied using an instrumented implant and a fixed simulated femur. Assuming the punch applies an average force of 2 kN over a time of 2 ms to the 200-g implant, determine (*a*) the velocity of the implant immediately after impact, (*b*) the average resistance of the implant to penetration if the implant moves 1 mm before coming to rest.

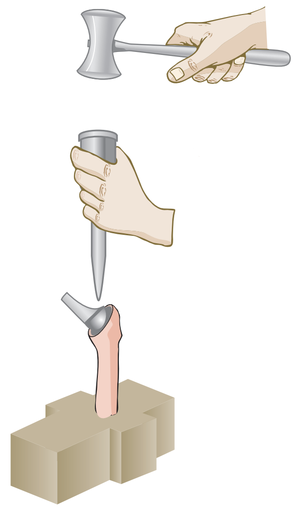


Fig. *P13.143*

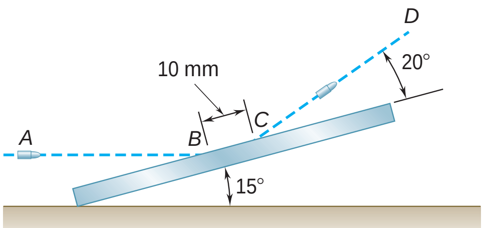


Fig. *P13.144*

13.144 A 25-g steel-jacketed bullet is fired horizontally with a velocity of 600 m/s and ricochets off a steel plate along the path *CD* with a velocity of 400 m/s. Knowing that the bullet leaves a 10-mm scratch on the plate and assuming that its average speed is 500 m/s while it is in contact with the plate, determine the magnitude and direction of the average impulsive force exerted by the bullet on the plate.

13.145 A 25-ton railroad car moving at 2.5 mi/h is to be coupled to a 50-ton car which is at rest with locked wheels ($\mu_k = 0.30$). Determine (*a*) the velocity of both cars after the coupling is completed, (*b*) the time it takes for both cars to come to rest.

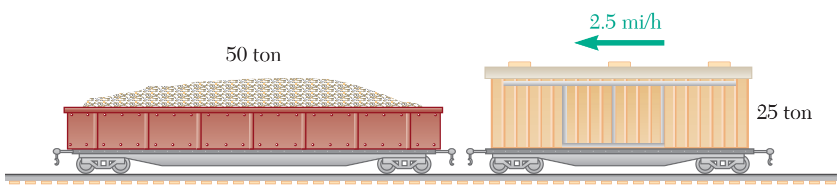


Fig. P13.145

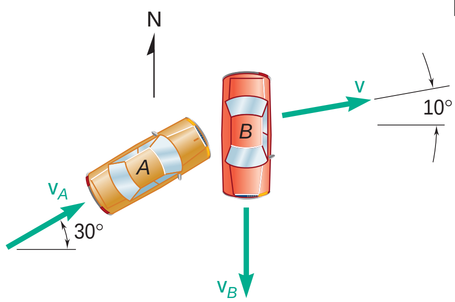


Fig. P13.146

13.146 At an intersection car *B* was traveling south and car *A* was traveling 30° north of east when they slammed into each other. Upon investigation it was found that after the crash the two cars got stuck and skidded off at an angle of 10° north of east. Each driver claimed that he was going at the speed limit of 50 km/h and that he tried to slow down but couldn't avoid the crash because the other driver was going a lot faster. Knowing that the masses of cars *A* and *B* were 1500 kg and 1200 kg, respectively, determine (*a*) which car was going faster, (*b*) the speed of the faster of the two cars if the slower car was traveling at the speed limit.

13.147 The 650-kg hammer of a drop-hammer pile driver falls from a height of 1.2 m onto the top of a 140-kg pile, driving it 110 mm into the ground. Assuming perfectly plastic impact ($e = 0$), determine the average resistance of the ground to penetration.

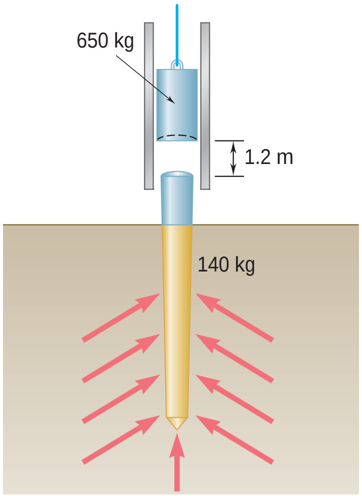


Fig. P13.147

13.148 A small rivet connecting two pieces of sheet metal is being clinched by hammering. Determine the impulse exerted on the rivet and the energy absorbed by the rivet under each blow, knowing that the head of the hammer has a weight of 1.5 lb and that it strikes the rivet with a velocity of 20 ft/s. Assume that the hammer does not rebound and that the anvil is supported by springs and (*a*) has an infinite mass (rigid support), (*b*) has a weight of 9 lb.

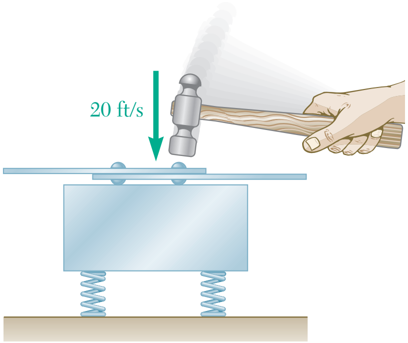


Fig. P13.148

13.149 Bullet *B* weighs 0.5 oz and blocks *A* and *C* both weigh 3 lb. The coefficient of friction between the blocks and the plane is $\mu_k = 0.25$. Initially the bullet is moving at v_0 and blocks *A* and *C* are at rest (Fig. 1). After the bullet passes through *A* it becomes embedded in block *C* and all three objects come to stop in the positions shown (Fig. 2). Determine the initial speed of the bullet v_0.

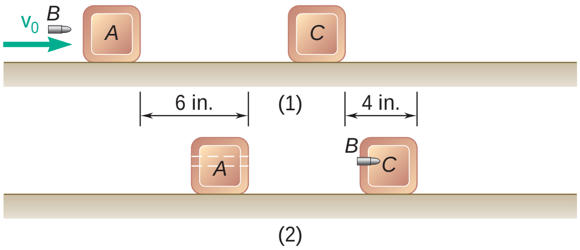


Fig. P13.149

13.150 A 180-lb man and a 120-lb woman stand at opposite ends of a 300-lb boat, ready to dive, each with a 16-ft/s velocity relative to the boat. Determine the velocity of the boat after they have both dived, if (*a*) the woman dives first, (*b*) the man dives first.

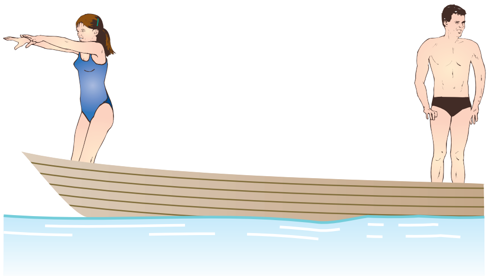


Fig. P13.150

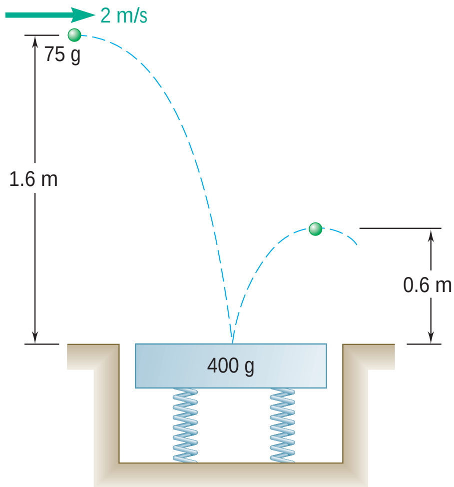


Fig. P13.151

13.151 A 75-g ball is projected from a height of 1.6 m with a horizontal velocity of 2 m/s and bounces from a 400-g smooth plate supported by springs. Knowing that the height of the rebound is 0.6 m, determine (*a*) the velocity of the plate immediately after the impact, (*b*) the energy lost due to the impact.

13.152 A 2-kg sphere *A* is connected to a fixed point *O* by an inextensible cord of length 1.2 m. The sphere is resting on a frictionless horizontal surface at a distance of 0.5 m from *O* when it is given a velocity $\mathbf{v}_0$ in a direction perpendicular to line *OA*. It moves freely until it reaches position *A*′, when the cord becomes taut. Determine the maximum allowable velocity $\mathbf{v}_0$ if the impulse of the force exerted on the cord is not to exceed 3 N · s.

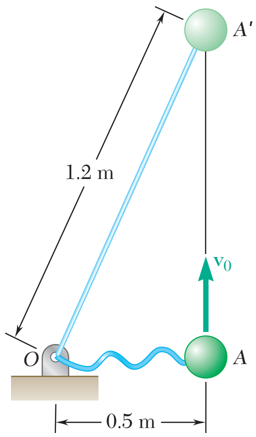


Fig. P13.152

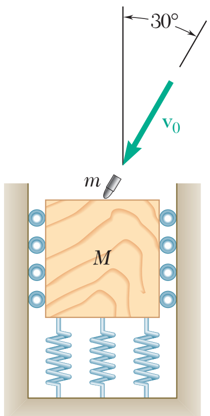


Fig. *P13.153*

13.153 A 1-oz bullet is traveling with a velocity of 1400 ft/s when it impacts and becomes embedded in a 5-lb wooden block. The block can move vertically without friction. Determine (*a*) the velocity of the bullet and block immediately after the impact, (*b*) the horizontal and vertical components of the impulse exerted by the block on the bullet.

13.154 In order to test the resistance of a chain to impact, the chain is suspended from a 240-lb rigid beam supported by two columns. A rod attached to the last link is then hit by a 60-lb block dropped from a 5-ft height. Determine the initial impulse exerted on the chain and the energy absorbed by the chain, assuming that the block does not rebound from the rod and that the columns supporting the beam are (*a*) perfectly rigid, (*b*) equivalent to two perfectly elastic springs.

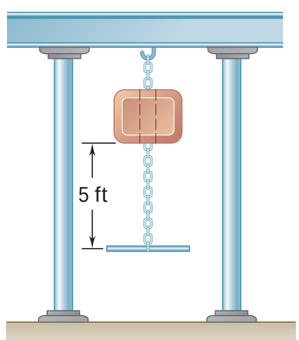
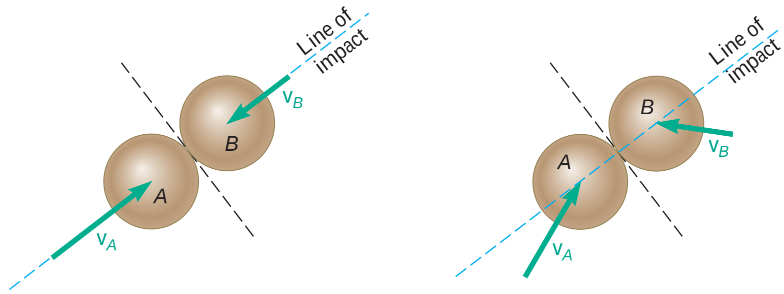


Fig. *P13.154*

13.12 IMPACT

A collision between two bodies which occurs in a very small interval of time and during which the two bodies exert relatively large forces on each other is called an *impact*. The common normal to the surfaces in contact during the impact is called the *line of impact*. If the mass centers on the two colliding bodies are located on this line, the impact is a *central impact*. Otherwise, the impact is said to be *eccentric*. Our present study will be limited to the central impact of two particles. The analysis of the eccentric impact of two rigid bodies will be considered later, in Sec. 17.12.



(*a*) Direct central impact (*b*) Oblique central impact

Fig. 13.20

If the velocities of the two particles are directed along the line of impact, the impact is said to be a *direct impact* (Fig. 13.20*a*). If either or both particles move along a line other than the line of impact, the impact is said to be an *oblique impact* (Fig. 13.20*b*).

13.13 DIRECT CENTRAL IMPACT

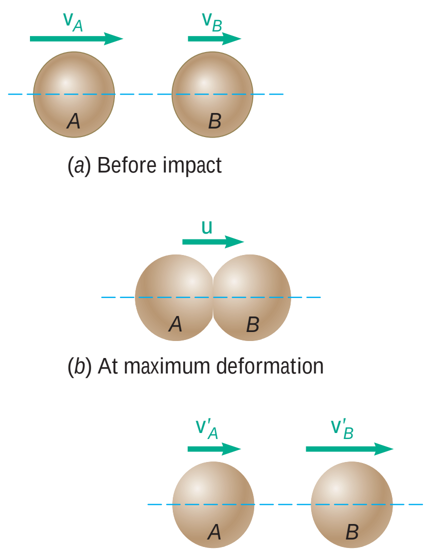
Consider two particles A and B, of mass m_A and m_B, which are moving in the same straight line and to the right with known velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ (Fig. 13.21*a*). If $\mathbf{v}_A$ is larger than $\mathbf{v}_B$, particle A will eventually strike particle B. Under the impact, the two particles will *deform* and, at the end of the period of deformation, they will have the same velocity $\mathbf{u}$ (Fig. 13.21*b*). A period of *restitution* will then take place, at the end of which, depending upon the magnitude of the impact forces and upon the materials involved, the two particles either will have regained their original shape or will stay permanently deformed. Our purpose here is to determine the velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ of the particles at the end of the period of restitution (Fig. 13.21*c*).

Considering first the two particles as a single system, we note that there is no impulsive, external force. Thus, the total momentum of the two particles is conserved, and we write

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m_A\mathbf{v}'_A + m_B\mathbf{v}'_B$$

Since all the velocities considered are directed along the same axis, we can replace the equation obtained by the following relation involving only scalar components:

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \tag{13.37}$$



(*a*) Before impact

(*b*) At maximum deformation

(*c*) After impact

Fig. 13.21

A positive value for any of the scalar quantities v_A, v_B, v'_A, or v'_B means that the corresponding vector is directed to the right; a negative value indicates that the corresponding vector is directed to the left.

To obtain the velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$, it is necessary to establish a second relation between the scalars v'_A and v'_B. For this purpose, let us now consider the motion of particle A during the period of deformation and apply the principle of impulse and momentum. Since the only impulsive force acting on A during this period is the force $\mathbf{P}$ exerted by B (Fig. 13.22*a*), we write, using again scalar components,

$$m_A v_A - \int P\,dt = m_A u \tag{13.38}$$

where the integral extends over the period of deformation. Considering now the motion of A during the period of restitution, and denoting by $\mathbf{R}$ the force exerted by B on A during this period (Fig. 13.22*b*), we write

$$m_A u - \int R\,dt = m_A v'_A \tag{13.39}$$

where the integral extends over the period of restitution.

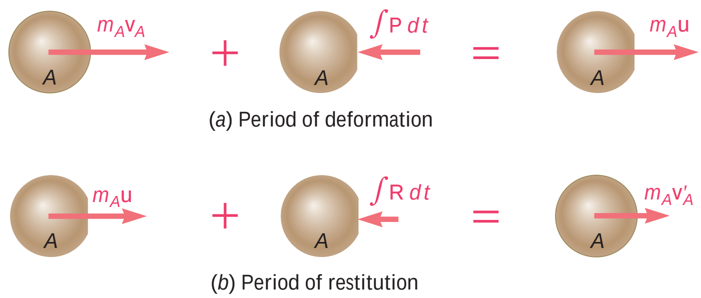


Fig. 13.22

In general, the force $\mathbf{R}$ exerted on A during the period of restitution differs from the force $\mathbf{P}$ exerted during the period of deformation, and the magnitude $\int R\,dt$ of its impulse is smaller than the magnitude $\int P\,dt$ of the impulse of $\mathbf{P}$. The ratio of the magnitudes of the impulses corresponding, respectively, to the period of restitution and to the period of deformation is called the *coefficient of restitution* and is denoted by e. We write

$$e = \frac{\int R\,dt}{\int P\,dt} \tag{13.40}$$

The value of the coefficient e is always between 0 and 1. It depends to a large extent on the two materials involved, but it also varies considerably with the impact velocity and the shape and size of the two colliding bodies.

Solving Eqs. (13.38) and (13.39) for the two impulses and substituting into (13.40), we write

$$e = \frac{u - v'_A}{v_A - u} \tag{13.41}$$

A similar analysis of particle B leads to the relation

$$e = \frac{v'_B - u}{u - v_B} \tag{13.42}$$

Since the quotients in (13.41) and (13.42) are equal, they are also equal to the quotient obtained by adding, respectively, their numerators and their denominators. We have, therefore,

$$e = \frac{(u - v'_A) + (v'_B - u)}{(v_A - u) + (u - v_B)} = \frac{v'_B - v'_A}{v_A - v_B}$$

and

$$v'_B - v'_A = e(v_A - v_B) \tag{13.43}$$

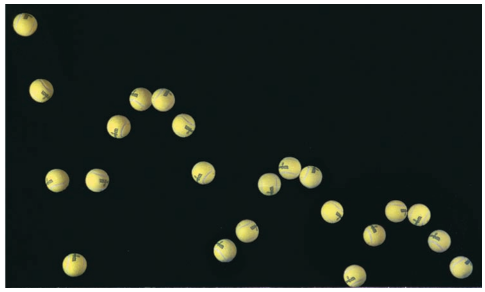


Photo 13.3 The height the tennis ball bounces decreases after each impact because it has a coefficient of restitution less than one and energy is lost with each bounce.

Since $v'_B - v'_A$ represents the relative velocity of the two particles after impact and $v_A - v_B$ represents their relative velocity before impact, formula (13.43) expresses that *the relative velocity of the two particles after impact can be obtained by multiplying their relative velocity before impact by the coefficient of restitution*. This property is used to determine experimentally the value of the coefficient of restitution of two given materials.

The velocities of the two particles after impact can now be obtained by solving Eqs. (13.37) and (13.43) simultaneously for v'_A and v'_B. It is recalled that the derivation of Eqs. (13.37) and (13.43) was based on the assumption that particle B is located to the right of A, and that both particles are initially moving to the right. If particle B is initially moving to the left, the scalar v_B should be considered negative. The same sign convention holds for the velocities after impact: A positive sign for v'_A will indicate that particle A moves to the right after impact, and a negative sign will indicate that it moves to the left.

Two particular cases of impact are of special interest:

1. $e = 0$, *Perfectly Plastic Impact*. When $e = 0$, Eq. (13.43) yields $v'_B = v'_A$. There is no period of restitution, and both particles stay together after impact. Substituting $v'_B = v'_A = v'$ into Eq. (13.37), which expresses that the total momentum of the particles is conserved, we write

$$m_A v_A + m_B v_B = (m_A + m_B)v' \tag{13.44}$$

This equation can be solved for the common velocity v' of the two particles after impact.

2. $e = 1$, *Perfectly Elastic Impact*. When $e = 1$, Eq. (13.43) reduces to

$$v'_B - v'_A = v_A - v_B \tag{13.45}$$

which expresses that the relative velocities before and after impact are equal. The impulses received by each particle during the period of deformation and during the period of restitution are equal. The particles move away from each other after impact with the same velocity with which they approached each other before impact. The velocities v'_A and v'_B can be obtained by solving Eqs. (13.37) and (13.45) simultaneously.

It is worth noting that *in the case of a perfectly elastic impact, the total energy of the two particles,* as well as their total momentum, *is conserved*. Equations (13.37) and (13.45) can be written as follows:

$$m_A(v_A - v'_A) = m_B(v'_B - v_B) \qquad (13.37')$$
$$v_A + v'_A = v_B + v'_B \qquad (13.45')$$

Multiplying (13.37′) and (13.45′) member by member, we have

$$m_A(v_A - v'_A)(v_A + v'_A) = m_B(v'_B - v_B)(v'_B + v_B)$$
$$m_A v_A^2 - m_A(v'_A)^2 = m_B(v'_B)^2 - m_B v_B^2$$

Rearranging the terms in the equation obtained and multiplying by $\frac{1}{2}$, we write

$$\tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2 = \tfrac{1}{2}m_A(v'_A)^2 + \tfrac{1}{2}m_B(v'_B)^2 \qquad (13.46)$$

which expresses that the kinetic energy of the particles is conserved. It should be noted, however, that *in the general case of impact,* i.e., when e is not equal to 1, *the total energy of the particles is not conserved*. This can be shown in any given case by comparing the kinetic energies before and after impact. The lost kinetic energy is in part transformed into heat and in part spent in generating elastic waves within the two colliding bodies.

13.14 OBLIQUE CENTRAL IMPACT

Let us now consider the case when the velocities of the two colliding particles are *not* directed along the line of impact (Fig. 13.23). As indicated in Sec. 13.12, the impact is said to be *oblique*. Since the

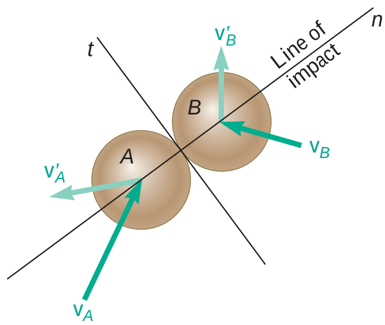


Fig. 13.23

velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ of the particles after impact are unknown in direction as well as in magnitude, their determination will require the use of four independent equations.

We choose as coordinate axes the n axis along the line of impact, i.e., along the common normal to the surfaces in contact, and the t axis along their common tangent. Assuming that the particles are perfectly *smooth and frictionless,* we observe that the only impulses

Photo 13.4 When pool balls strike each other there is a transfer of momentum.

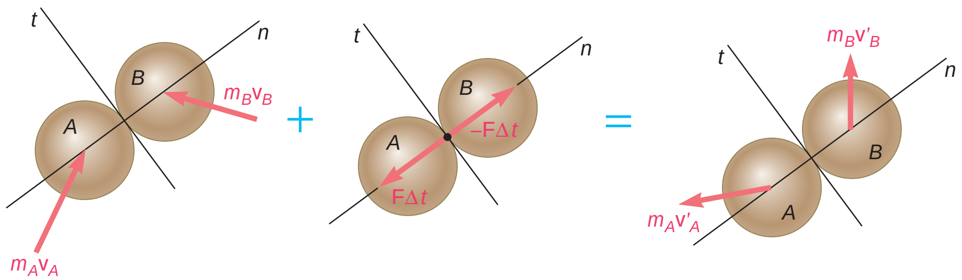


Fig. 13.24

exerted on the particles during the impact are due to internal forces directed along the line of impact, i.e., along the n axis (Fig. 13.24). It follows that

1. The component along the t axis of the momentum of each particle, considered separately, is conserved; hence the t component of the velocity of each particle remains unchanged. We write

$$(v_A)_t = (v'_A)_t \qquad (v_B)_t = (v'_B)_t \tag{13.47}$$

2. The component along the n axis of the total momentum of the two particles is conserved. We write

$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n \tag{13.48}$$

3. The component along the n axis of the relative velocity of the two particles after impact is obtained by multiplying the n component of their relative velocity before impact by the coefficient of restitution. Indeed, a derivation similar to that given in Sec. 13.13 for direct central impact yields

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{13.49}$$

We have thus obtained four independent equations which can be solved for the components of the velocities of A and B after impact. This method of solution is illustrated in Sample Prob. 13.15.

Our analysis of the oblique central impact of two particles has been based so far on the assumption that both particles moved freely before and after the impact. Let us now examine the case when one or both of the colliding particles is constrained in its motion. Consider, for instance, the collision between block A, which is constrained to move on a horizontal surface, and ball B, which is free to move in the plane of the figure (Fig. 13.25). Assuming no friction between the block and the ball, or between the block and the horizontal surface, we note that the impulses exerted on the system consist of the impulses of the internal forces $\mathbf{F}$ and $-\mathbf{F}$ directed along the line of impact, i.e., along the n axis, and of the impulse of the external force $\mathbf{F}_{\text{ext}}$ exerted by the horizontal surface on block A and directed along the vertical (Fig. 13.26).

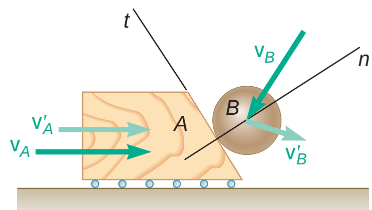


Fig. 13.25

The velocities of block A and ball B immediately after the impact are represented by three unknowns: the magnitude of the velocity $\mathbf{v}'_A$ of block A, which is known to be horizontal, and the magnitude and

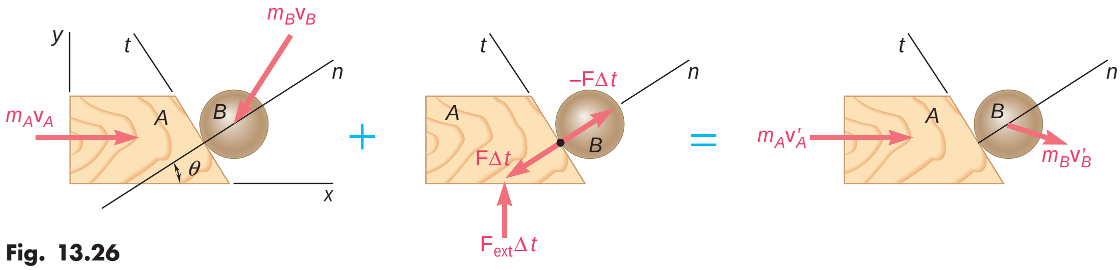


Fig. 13.26

direction of the velocity $\mathbf{v}'_B$ of ball B. We must therefore write three equations by expressing that

1. The component along the t axis of the momentum of ball B is conserved; hence the t component of the velocity of ball B remains unchanged. We write

$$(v_B)_t = (v'_B)_t \tag{13.50}$$

2. The component along the horizontal x axis of the total momentum of block A and ball B is conserved. We write

$$m_Av_A + m_B(v_B)_x = m_Av'_A + m_B(v'_B)_x \tag{13.51}$$

3. The component along the n axis of the relative velocity of block A and ball B after impact is obtained by multiplying the n component of their relative velocity before impact by the coefficient of restitution. We write again

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{13.49}$$

We should note, however, that in the case considered here, the validity of Eq. (13.49) cannot be established through a mere extension of the derivation given in Sec. 13.13 for the direct central impact of two particles moving in a straight line. Indeed, these particles were not subjected to any external impulse, while block A in the present analysis is subjected to the impulse exerted by the horizontal surface. To prove that Eq. (13.49) is still valid, we will first apply the principle of impulse and momentum to block A over the period of deformation (Fig. 13.27). Considering only the horizontal components, we write

$$m_Av_A - (\textstyle\int P\,dt)\cos \mathsf{u} = m_Au \tag{13.52}$$

where the integral extends over the period of deformation and where $\mathbf{u}$ represents the velocity of block A at the end of that period. Considering now the period of restitution, we write in a similar way

$$m_Au - (\textstyle\int R\,dt)\cos \mathsf{u} = m_Av'_A \tag{13.53}$$

where the integral extends over the period of restitution.

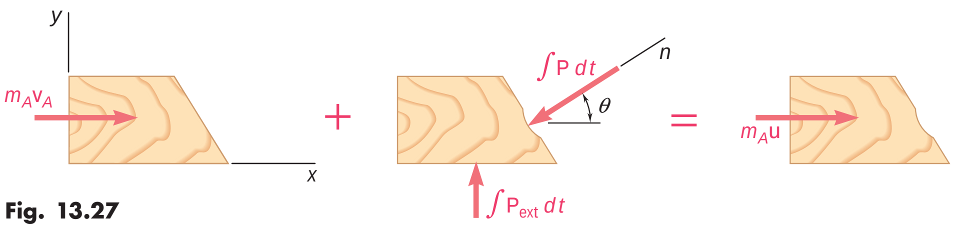


Fig. 13.27

Recalling from Sec. 13.13 the definition of the coefficient of restitution, we write

$$e = \frac{\int R\,dt}{\int P\,dt} \tag{13.40}$$

Solving Eqs. (13.52) and (13.53) for the integrals $\int P\,dt$ and $\int R\,dt$, and substituting into Eq. (13.40), we have, after reductions,

$$e = \frac{u - v'_A}{v_A - u}$$

or, multiplying all velocities by cos u to obtain their projections on the line of impact.

$$e = \frac{u_n - (v'_A)_n}{(v_A)_n - u_n} \tag{13.54}$$

We note that Eq. (13.54) is identical to Eq. (13.41) of Sec. 13.13, except for the subscripts n which are used here to indicate that we are considering velocity components along the line of impact. Since the motion of ball B is unconstrained, the proof of Eq. (13.49) can be completed in the same manner as the derivation of Eq. (13.43) of Sec. 13.13. Thus, we conclude that the relation (13.49) between the components along the line of impact of the relative velocities of two colliding particles remains valid when one of the particles is constrained in its motion. The validity of this relation is easily extended to the case when both particles are constrained in their motion.

13.15 PROBLEMS INVOLVING ENERGY AND MOMENTUM

You now have at your disposal three different methods for the solution of kinetics problems: the direct application of Newton's second law, $\Sigma\mathbf{F} = m\mathbf{a}$; the method of work and energy; and the method of impulse and momentum. To derive maximum benefit from these three methods, you should be able to choose the method best suited for the solution of a given problem. You should also be prepared to use different methods for solving the various parts of a problem when such a procedure seems advisable.

You have already seen that the method of work and energy is in many cases more expeditious than the direct application of Newton's second law. As indicated in Sec. 13.4, however, the method of work and energy has limitations, and it must sometimes be supplemented by the use of $\Sigma\mathbf{F} = m\mathbf{a}$. This is the case, for example, when you wish to determine an acceleration or a normal force.

For the solution of problems involving no impulsive forces, it will usually be found that the equation $\Sigma\mathbf{F} = m\mathbf{a}$ yields a solution just as fast as the method of impulse and momentum and that the method of work and energy, if it applies, is more rapid and more convenient. However, in problems of impact, the method of impulse and momentum is the only practicable method. A solution based on the direct application of $\Sigma\mathbf{F} = m\mathbf{a}$ would be unwieldy, and the method of work and energy cannot be used since impact (unless perfectly elastic) involves a loss of mechanical energy.

Many problems involve only conservative forces, except for a short impact phase during which impulsive forces act. The solution of such problems can be divided into several parts. The part corresponding to the impact phase calls for the use of the method of impulse and momentum and of the relation between relative velocities, and the other parts can usually be solved by the method of work and energy. If the problem involves the determination of a normal force, however, the use of $\Sigma\mathbf{F} = m\mathbf{a}$ is necessary.

Consider, for example, a pendulum A, of mass m_A and length l, which is released with no velocity from a position A_1 (Fig. 13.28*a*). The pendulum swings freely in a vertical plane and hits a second pendulum B, of mass m_B and same length l, which is initially at rest. After the impact (with coefficient of restitution e), pendulum B swings through an angle u that we wish to determine.

The solution of the problem can be divided into three parts:

1. *Pendulum A Swings from A_1 to A_2.* The principle of conservation of energy can be used to determine the velocity $(\mathbf{v}_A)_2$ of the pendulum at A_2 (Fig. 13.28*b*).
2. *Pendulum A Hits Pendulum B.* Using the fact that the total momentum of the two pendulums is conserved and the relation between their relative velocities, we determine the velocities $(\mathbf{v}_A)_3$ and $(\mathbf{v}_B)_3$ of the two pendulums after impact (Fig. 13.28*c*).
3. *Pendulum B Swings from B_3 to B_4.* Applying the principle of conservation of energy to pendulum B, we determine the maximum elevation y_4 reached by that pendulum (Fig. 13.28*d*). The angle u can then be determined by trigonometry.

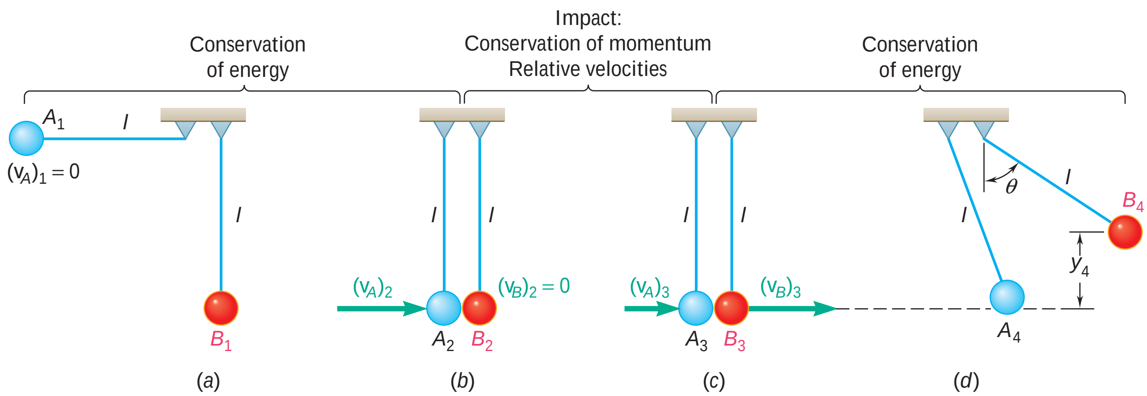


Fig. 13.28

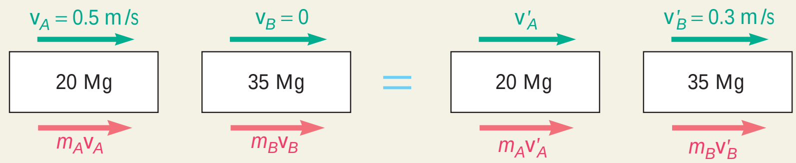
We note that if the tensions in the cords holding the pendulums are to be determined, the method of solution just described should be supplemented by the use of $\Sigma\mathbf{F} = m\mathbf{a}$.

SAMPLE PROBLEM 13.13

A 20-Mg railroad car moving at a speed of 0.5 m/s to the right collides with a 35-Mg car which is at rest. If after the collision the 35-Mg car is observed to move to the right at a speed of 0.3 m/s, determine the coefficient of restitution between the two cars.

SOLUTION

We express that the total momentum of the two cars is conserved.



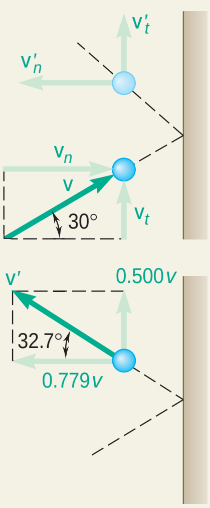
$$m_A \mathbf{v}_A + m_B \mathbf{v}_B = m_A \mathbf{v}'_A + m_B \mathbf{v}'_B$$
$$(20 \text{ Mg})(+0.5 \text{ m/s}) + (35 \text{ Mg})(0) = (20 \text{ Mg})v'_A + (35 \text{ Mg})(+0.3 \text{ m/s})$$
$$v'_A = -0.025 \text{ m/s} \qquad \mathbf{v}'_A = 0.025 \text{ m/s z}$$

The coefficient of restitution is obtained by writing

$$e = \frac{v'_B - v'_A}{v_A - v_B} = \frac{+0.3 - (-0.025)}{+0.5 - 0} = \frac{0.325}{0.5} \qquad e = 0.65 \quad ◀$$

SAMPLE PROBLEM 13.14

A ball is thrown against a frictionless, vertical wall. Immediately before the ball strikes the wall, its velocity has a magnitude v and forms an angle of 30° with the horizontal. Knowing that $e = 0.90$, determine the magnitude and direction of the velocity of the ball as it rebounds from the wall.



SOLUTION

We resolve the initial velocity of the ball into components respectively perpendicular and parallel to the wall:

$$v_n = v \cos 30° = 0.866v \qquad v_t = v \sin 30° = 0.500v$$

Motion Parallel to the Wall. Since the wall is frictionless, the impulse it exerts on the ball is perpendicular to the wall. Thus, the component parallel to the wall of the momentum of the ball is conserved and we have

$$\mathbf{v}'_t = \mathbf{v}_t = 0.500v \text{ x}$$

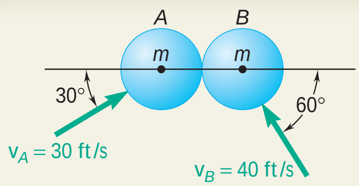
Motion Perpendicular to the Wall. Since the mass of the wall (and earth) is essentially infinite, expressing that the total momentum of the ball and wall is conserved would yield no useful information. Using the relation (13.49) between relative velocities, we write

$$0 - v'_n = e(v_n - 0)$$
$$v'_n = -0.90(0.866v) = -0.779v \qquad \mathbf{v}'_n = 0.779v \text{ z}$$

Resultant Motion. Adding vectorially the components $\mathbf{v}'_n$ and $\mathbf{v}'_t$,

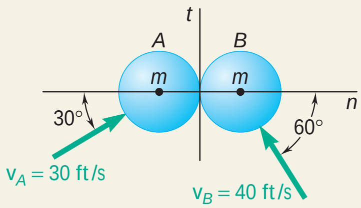
$$\mathbf{v}' = 0.926v \text{ b } 32.7° \quad ◀$$

SAMPLE PROBLEM 13.15



The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are as shown. Assuming $e = 0.90$, determine the magnitude and direction of the velocity of each ball after the impact.

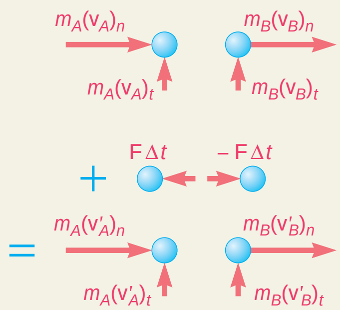
SOLUTION



The impulsive forces that the balls exert on each other during the impact are directed along a line joining the centers of the balls called the *line of impact*. Resolving the velocities into components directed, respectively, along the line of impact and along the common tangent to the surfaces in contact, we write

$$\begin{aligned}(v_A)_n &= v_A \cos 30° = +26.0 \text{ ft/s}\\ (v_A)_t &= v_A \sin 30° = +15.0 \text{ ft/s}\\ (v_B)_n &= -v_B \cos 60° = -20.0 \text{ ft/s}\\ (v_B)_t &= v_B \sin 60° = +34.6 \text{ ft/s}\end{aligned}$$

Principle of Impulse and Momentum. In the adjoining sketches we show in turn the initial momenta, the impulses, and the final momenta.



Motion Along the Common Tangent. Considering only the t components, we apply the principle of impulse and momentum to each ball *separately*. Since the impulsive forces are directed along the line of impact, the t component of the momentum, and hence the t component of the velocity of each ball, is unchanged. We have

$$(\mathbf{v}'_A)_t = 15.0 \text{ ft/s x} \qquad (\mathbf{v}'_B)_t = 34.6 \text{ ft/s x}$$

Motion Along the Line of Impact. In the n direction, we consider the two balls as a single system and note that by Newton's third law, the internal impulses are, respectively, $\mathbf{F}\,\Delta t$ and $-\mathbf{F}\,\Delta t$ and cancel. We thus write that the total momentum of the balls is conserved:

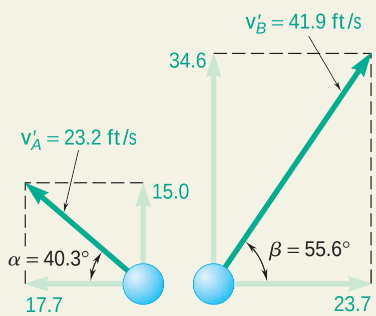
$$\begin{aligned}m_A(v_A)_n + m_B(v_B)_n &= m_A(v'_A)_n + m_B(v'_B)_n\\ m(26.0) + m(-20.0) &= m(v'_A)_n + m(v'_B)_n\\ (v'_A)_n + (v'_B)_n &= 6.0 \qquad (1)\end{aligned}$$

Using the relation (13.49) between relative velocities, we write

$$\begin{aligned}(v'_B)_n - (v'_A)_n &= e[(v_A)_n - (v_B)_n]\\ (v'_B)_n - (v'_A)_n &= (0.90)[26.0 - (-20.0)]\\ (v'_B)_n - (v'_A)_n &= 41.4 \qquad (2)\end{aligned}$$

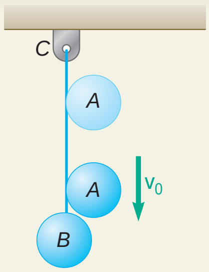
Solving Eqs. (1) and (2) simultaneously, we obtain

$$(v'_A)_n = -17.7 \qquad (v'_B)_n = +23.7$$
$$(\mathbf{v}'_A)_n = 17.7 \text{ ft/s z} \qquad (\mathbf{v}'_B)_n = 23.7 \text{ ft/s y}$$



Resultant Motion. Adding vectorially the velocity components of each ball, we obtain

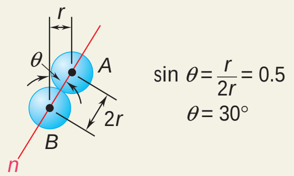
$$\mathbf{v}'_A = 23.2 \text{ ft/s b } 40.3° \qquad \mathbf{v}'_B = 41.9 \text{ ft/s a } 55.6° \quad ◀$$



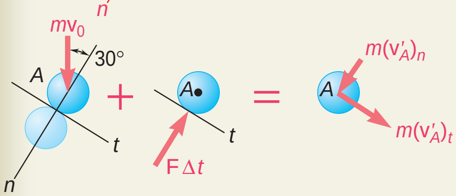
SAMPLE PROBLEM 13.16

Ball B is hanging from an inextensible cord BC. An identical ball A is released from rest when it is just touching the cord and acquires a velocity $\mathbf{v}_0$ before striking ball B. Assuming perfectly elastic impact ($e = 1$) and no friction, determine the velocity of each ball immediately after impact.

SOLUTION



Since ball B is constrained to move in a circle of center C, its velocity $\mathbf{v}_B$ after impact must be horizontal. Thus the problem involves three unknowns: the magnitude v'_B of the velocity of B, and the magnitude and direction of the velocity $\mathbf{v}'_A$ of A after impact.



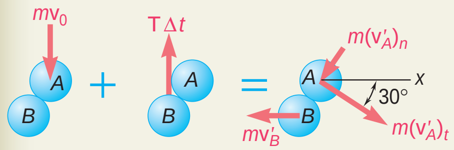
Impulse-Momentum Principle: Ball *A*

$$m\mathbf{v}_A + \mathbf{F}\,\Delta t = m\mathbf{v}'_A$$

$+\searrow t$ components: $\quad mv_0 \sin 30° + 0 = m(v'_A)_t$

$$(v'_A)_t = 0.5v_0 \qquad (1)$$

We note that the equation used expresses conservation of the momentum of ball A along the common tangent to balls A and B.



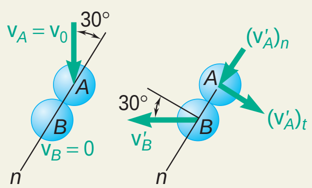
Impulse-Momentum Principle: Balls *A* and *B*

$$m\mathbf{v}_A + \mathbf{T}\,\Delta t = m\mathbf{v}'_A + m\mathbf{v}'_B$$

$\overset{+}{\rightarrow} x$ components: $\quad 0 = m(v'_A)_t \cos 30° - m(v'_A)_n \sin 30° - mv'_B$

We note that the equation obtained expresses conservation of the total momentum in the x direction. Substituting for $(v'_A)_t$ from Eq. (1) and rearranging terms, we write

$$0.5(v'_A)_n + v'_B = 0.433v_0 \qquad (2)$$



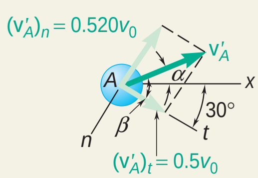
Relative Velocities Along the Line of Impact. Since $e = 1$, Eq. (13.49) yields

$$(v'_B)_n - (v'_A)_n = (v_A)_n - (v_B)_n$$
$$v'_B \sin 30° - (v'_A)_n = v_0 \cos 30° - 0$$
$$0.5v'_B - (v'_A)_n = 0.866v_0 \qquad (3)$$

Solving Eqs. (2) and (3) simultaneously, we obtain

$$(v'_A)_n = -0.520v_0 \qquad v'_B = 0.693v_0$$

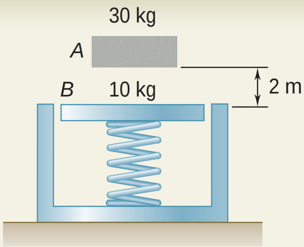
$$\mathbf{v}'_B = 0.693v_0 \leftarrow \quad ◀$$



Recalling Eq. (1) we draw the adjoining sketch and obtain by trigonometry

$$v'_A = 0.721v_0 \qquad \beta = 46.1° \qquad \alpha = 46.1° - 30° = 16.1°$$

$$\mathbf{v}'_A = 0.721v_0 \measuredangle 16.1° \quad ◀$$

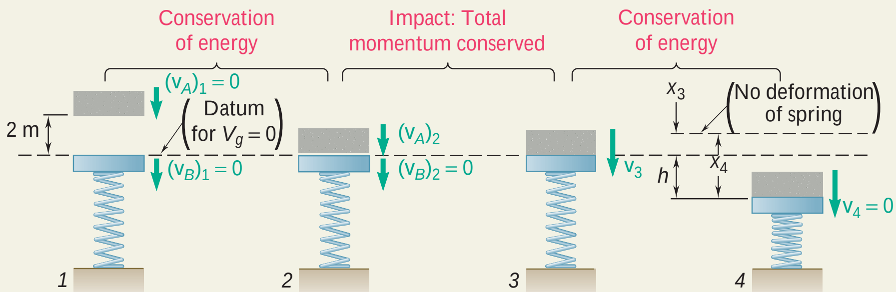


SAMPLE PROBLEM 13.17

A 30-kg block is dropped from a height of 2 m onto the 10-kg pan of a spring scale. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. The constant of the spring is $k = 20$ kN/m.

SOLUTION

The impact between the block and the pan *must* be treated separately; therefore we divide the solution into three parts.



Conservation of Energy. Block: $W_A = (30 \text{ kg})(9.81 \text{ m/s}^2) = 294$ N

$$T_1 = \tfrac{1}{2}m_A(v_A)_1^2 = 0 \qquad V_1 = W_A y = (294 \text{ N})(2 \text{ m}) = 588 \text{ J}$$
$$T_2 = \tfrac{1}{2}m_A(v_A)_2^2 = \tfrac{1}{2}(30 \text{ kg})(v_A)_2^2 \qquad V_2 = 0$$
$$T_1 + V_1 = T_2 + V_2: \qquad 0 + 588 \text{ J} = \tfrac{1}{2}(30 \text{ kg})(v_A)_2^2 + 0$$
$$(v_A)_2 = +6.26 \text{ m/s} \qquad (\mathbf{v}_A)_2 = 6.26 \text{ m/s}\text{w}$$

Impact: Conservation of Momentum. Since the impact is perfectly plastic, $e = 0$; the block and pan move together after the impact.

$$m_A(v_A)_2 + m_B(v_B)_2 = (m_A + m_B)v_3$$
$$(30 \text{ kg})(6.26 \text{ m/s}) + 0 = (30 \text{ kg} + 10 \text{ kg})v_3$$
$$v_3 = +4.70 \text{ m/s} \qquad \mathbf{v}_3 = 4.70 \text{ m/s}\text{w}$$

Conservation of Energy. Initially the spring supports the weight W_B of the pan; thus the initial deflection of the spring is

$$x_3 = \frac{W_B}{k} = \frac{(10 \text{ kg})(9.81 \text{ m/s}^2)}{20 \times 10^3 \text{ N/m}} = \frac{98.1 \text{ N}}{20 \times 10^3 \text{ N/m}} = 4.91 \times 10^{-3} \text{ m}$$

Denoting by x_4 the total maximum deflection of the spring, we write

$$T_3 = \tfrac{1}{2}(m_A + m_B)v_3^2 = \tfrac{1}{2}(30 \text{ kg} + 10 \text{ kg})(4.70 \text{ m/s})^2 = 442 \text{ J}$$
$$V_3 = V_g + V_e = 0 + \tfrac{1}{2}kx_3^2 = \tfrac{1}{2}(20 \times 10^3)(4.91 \times 10^{-3})^2 = 0.241 \text{ J}$$
$$T_4 = 0$$
$$V_4 = V_g + V_e = (W_A + W_B)(-h) + \tfrac{1}{2}kx_4^2 = -(392)h + \tfrac{1}{2}(20 \times 10^3)x_4^2$$

Noting that the displacement of the pan is $h = x_4 - x_3$, we write

$$T_3 + V_3 = T_4 + V_4:$$
$$442 + 0.241 = 0 - 392(x_4 - 4.91 \times 10^{-3}) + \tfrac{1}{2}(20 \times 10^3)x_4^2$$
$$x_4 = 0.230 \text{ m} \qquad h = x_4 - x_3 = 0.230 \text{ m} - 4.91 \times 10^{-3} \text{ m}$$
$$h = 0.225 \text{ m} \qquad h = 225 \text{ mm} \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

This lesson deals with the *impact of two bodies,* i.e., with a collision occurring in a very small interval of time. You will solve a number of impact problems by expressing that the total momentum of the two bodies is conserved and noting the relationship which exists between the relative velocities of the two bodies before and after impact.

1. As a first step in your solution you should select and draw the following coordinate axes: the t axis, which is tangent to the surfaces of contact of the two colliding bodies, and the n axis, which is normal to the surfaces of contact and defines the *line of impact*. In all the problems of this lesson the line of impact passes through the mass centers of the colliding bodies, and the impact is referred to as a *central impact*.

2. Next you will draw a diagram showing the momenta of the bodies before impact, the impulses exerted on the bodies during impact, and the final momenta of the bodies after impact (Fig. 13.24). You will then observe whether the impact is a *direct central impact* or an *oblique central impact*.

3. Direct central impact. This occurs when the velocities of bodies A and B before impact are *both directed along the line of impact* (Fig. 13.20*a*).

a. Conservation of momentum. Since the impulsive forces are internal to the system, you can write that the *total momentum of* A *and* B *is conserved*,

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \tag{13.37}$$

where v_A and v_B denote the velocities of bodies A and B before impact and v'_A and v'_B denote their velocities after impact.

b. Coefficient of restitution. You can also write the following relation between the *relative velocities* of the two bodies before and after impact,

$$v'_B - v'_A = e(v_A - v_B) \tag{13.43}$$

where e is the coefficient of restitution between the two bodies.

Note that Eqs. (13.37) and (13.43) are scalar equations which can be solved for two unknowns. Also, be careful to adopt a consistent sign convention for all velocities.

4. Oblique central impact. This occurs when *one or both* of the initial velocities of the two bodies is *not directed* along the line of impact (Fig. 13.20*b*). To solve problems of this type, you should *first resolve into components* along the t axis and the n axis the momenta and impulses shown in your diagram.

(continued)

a. Conservation of momentum. Since the impulsive forces act along the line of impact, i.e., along the n axis, the component along the t axis of the momentum *of each body* is conserved. Therefore, you can write for each body that the t components of its velocity before and after impact are equal,

$$(v_A)_t = (v'_A)_t \qquad (v_B)_t = (v'_B)_t \tag{13.47}$$

Also, the component along the n axis of the *total momentum* of the system is conserved,

$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n \tag{13.48}$$

b. Coefficient of restitution. The relation between the relative velocities of the two bodies before and after impact can be written in the n direction only,

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{13.49}$$

You now have four equations that you can solve for four unknowns. Note that after finding all the velocities, you can determine the impulse exerted by body A on body B by drawing an impulse-momentum diagram for B alone and equating components in the n direction.

c. When the motion of one of the colliding bodies is constrained, you must include the impulses of the external forces in your diagram. You will then observe that some of the above relations do not hold. However, in the example shown in Fig. 13.26 the total momentum of the system is conserved in a direction perpendicular to the external impulse. You should also note that when a body A bounces off a fixed surface B, the only conservation of momentum equation which can be used is the first of Eqs. (13.47) [Sample Prob. 13.14].

5. Remember that energy is lost during most impacts. The only exception is for *perfectly elastic* impacts ($e = 1$), where energy is conserved. Thus, in the general case of impact, where $e < 1$, the energy is not conserved. Therefore, be careful *not to apply* the principle of conservation of energy through an impact situation. Instead, apply this principle separately to the motions preceding and following the impact [Sample Prob. 13.17].

PROBLEMS

CONCEPT QUESTION

13.CQ6 A 5-kg ball A strikes a 1-kg ball B that is initially at rest. Is it possible that after the impact A is not moving and B has a speed of $5v$?

a. Yes

b. No

Explain your answer.

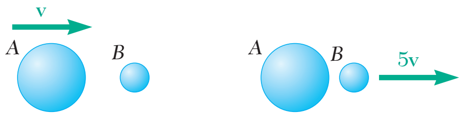


Fig. P13.CQ6

IMPULSE-MOMENTUM PRACTICE PROBLEMS

13.F6 A sphere with a speed v_0 rebounds after striking a frictionless inclined plane as shown. Draw the impulse-momentum diagram that can be used to find the velocity of the sphere after the impact.

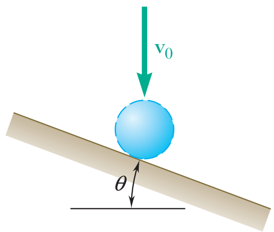


Fig. P13.F6

13.F7 An 80-Mg railroad engine A coasting at 6.5 km/h strikes a 20-Mg flatcar C carrying a 30-Mg load B which can slide along the floor of the car ($\mu_k = 0.25$). The flatcar was at rest with its brakes released. Instead of A and C coupling as expected, it is observed that A rebounds with a speed of 2 km/h after the impact. Draw impulse-momentum diagrams that can be used to determine (a) the coefficient of restitution and the speed of the flatcar immediately after impact, (b) the time it takes the load to slide to a stop relative to the car.

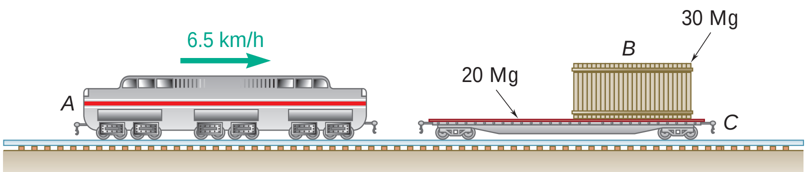


Fig. P13.F7

13.F8 Two frictionless balls strike each other as shown. The coefficient of restitution between the balls is e. Draw the impulse-momentum diagram that could be used to find the velocities of A and B after the impact.

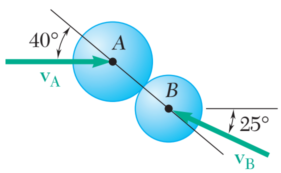


Fig. P13.F8

13.F9 A 10-kg ball A moving horizontally at 12 m/s strikes a 10-kg block B. The coefficient of restitution of the impact is 0.4 and the coefficient of kinetic friction between the block and the inclined surface is 0.5. Draw the impulse-momentum diagram that can be used to determine the speeds of A and B after the impact.

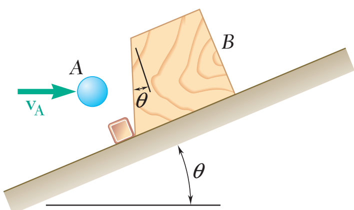


Fig. P13.F9

13.F10 Block A of mass m_A strikes ball B of mass m_B with a speed of v_A as shown. Draw the impulse-momentum diagram that can be used to determine the speeds of A and B after the impact and the impulse during the impact.

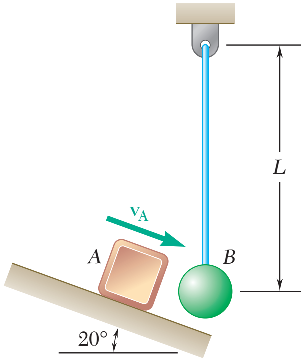


Fig. P13.F10

END-OF-SECTION PROBLEMS

13.155 The coefficient of restitution between the two collars is known to be 0.70. Determine (*a*) their velocities after impact, (*b*) the energy loss during impact.

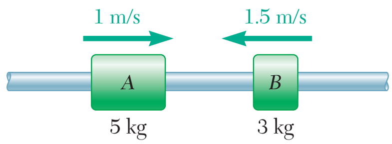


Fig. P13.155

13.156 Collars A and B, of the same mass m, are moving toward each other with identical speeds as shown. Knowing that the coefficient of restitution between the collars is e, determine the energy lost in the impact as a function of m, e, and v.

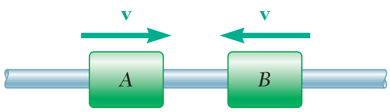


Fig. P13.156

13.157 One of the requirements for tennis balls to be used in official competition is that, when dropped onto a rigid surface from a height of 100 in., the height of the first bounce of the ball must be in the range 53 in. $\leq h \leq$ 58 in. Determine the range of the coefficients of restitution of the tennis balls satisfying this requirement.

13.158 Two disks sliding on a frictionless horizontal plane with opposite velocities of the same magnitude v_0 hit each other squarely. Disk A is known to have a weight of 6 lb and is observed to have zero velocity after impact. Determine (*a*) the weight of disk B, knowing that the coefficient of restitution between the two disks is 0.5, (*b*) the range of possible values of the weight of disk B if the coefficient of restitution between the two disks is unknown.

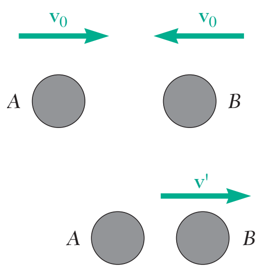


Fig. P13.158

13.159 To apply shock loading to an artillery shell, a 20-kg pendulum A is released from a known height and strikes impactor B at a known velocity $\mathbf{v}_0$. Impactor B then strikes the 1-kg artillery shell C. Knowing the coefficient of restitution between all objects is e, determine the mass of B to maximize the impulse applied to the artillery shell C.

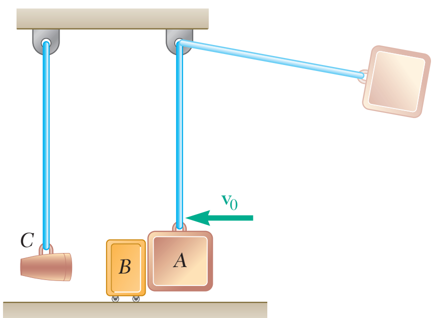


Fig. *P13.159*

13.160 Two identical cars A and B are at rest on a loading dock with brakes released. Car C, of a slightly different style but of the same weight, has been pushed by dockworkers and hits car B with a velocity of 1.5 m/s. Knowing that the coefficient of restitution is 0.8 between B and C and 0.5 between A and B, determine the velocity of each car after all collisions have taken place.

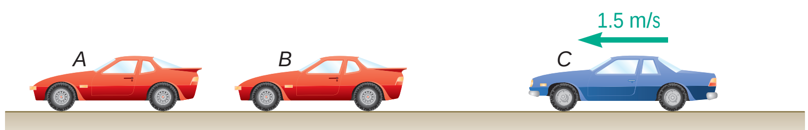


Fig. *P13.160*

13.161 Three steel spheres of equal weight are suspended from the ceiling by cords of equal length which are spaced at a distance slightly greater than the diameter of the spheres. After being pulled back and released, sphere A hits sphere B, which then hits sphere C. Denoting by e the coefficient of restitution between the spheres and by $\mathbf{v}_0$ the velocity of A just before it hits B, determine (*a*) the velocities of A and B immediately after the first collision, (*b*) the velocities of B and C immediately after the second collision. (*c*) Assuming now that n spheres are suspended from the ceiling and that the first sphere is pulled back and released as described above, determine the velocity of the last sphere after it is hit for the first time. (*d*) Use the result of part c to obtain the velocity of the last sphere when $n = 5$ and $e = 0.9$.

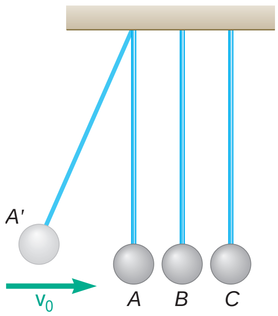


Fig. P13.161

13.162 At an amusement park there are 200-kg bumper cars A, B, and C that have riders with masses of 40 kg, 60 kg, and 35 kg, respectively. Car A is moving to the right with a velocity $\mathbf{v}_A = 2$ m/s and car C has a velocity $\mathbf{v}_B = 1.5$ m/s to the left, but car B is initially at rest. The coefficient of restitution between each car is 0.8. Determine the final velocity of each car, after all impacts, assuming (*a*) cars A and C hit car B at the same time, (*b*) car A hits car B before car C does.

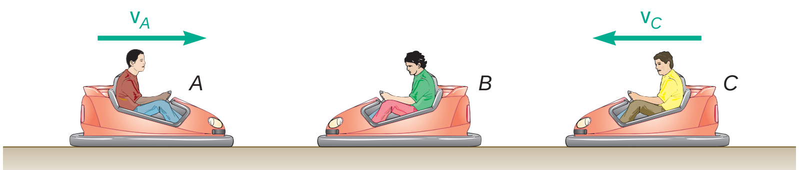


Fig. *P13.162* and P13.163

13.163 At an amusement park there are 200-kg bumper cars A, B, and C that have riders with masses of 40 kg, 60 kg, and 35 kg, respectively. Car A is moving to the right with a velocity $\mathbf{v}_A = 2$ m/s when it hits stationary car B. The coefficient of restitution between each car is 0.8. Determine the velocity of car C so that after car B collides with car C the velocity of car B is zero.

13.164 Two identical billiard balls can move freely on a horizontal table. Ball A has a velocity $\mathbf{v}_0$ as shown and hits ball B, which is at rest, at a point C defined by u = 45°. Knowing that the coefficient of restitution between the two balls is $e = 0.8$ and assuming no friction, determine the velocity of each ball after impact.

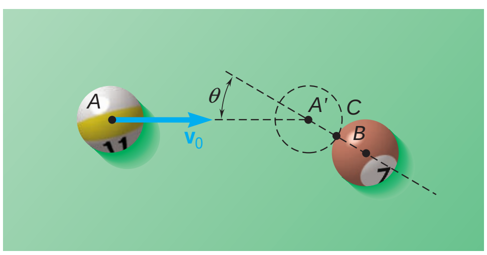


Fig. P13.164

13.165 The coefficient of restitution is 0.9 between the two 2.37-in.-diameter billiard balls A and B. Ball A is moving in the direction shown with a velocity of 3 ft/s when it strikes ball B, which is at rest. Knowing that after impact B is moving in the x direction, determine (*a*) the angle u, (*b*) the velocity of B after impact.

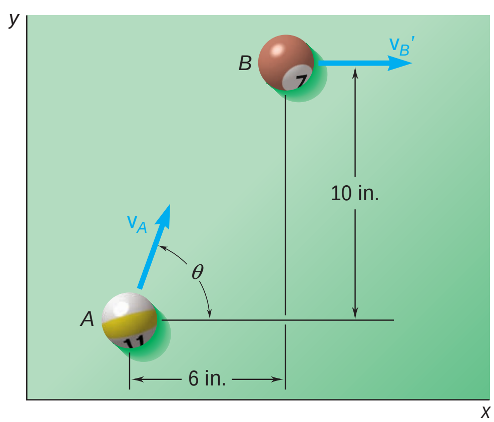


Fig. P13.165

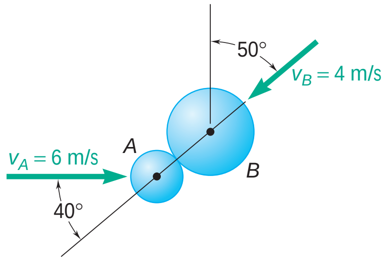


Fig. P13.166

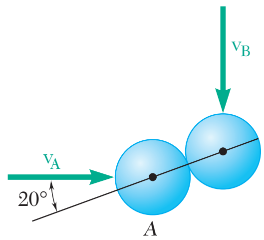


Fig. P13.167

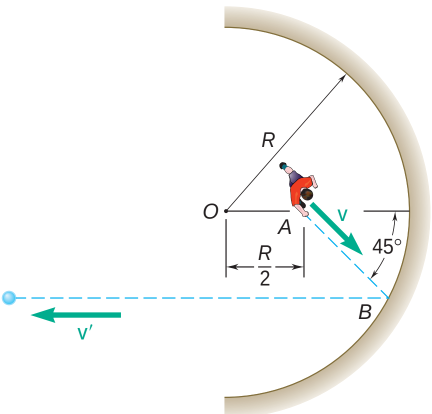


Fig. P13.169

13.166 A 600-g ball A is moving with a velocity of magnitude 6 m/s when it is hit as shown by a 1-kg ball B which has a velocity of magnitude 4 m/s. Knowing that the coefficient of restitution is 0.8 and assuming no friction, determine the velocity of each ball after impact.

13.167 Two identical hockey pucks are moving on a hockey rink at the same speed of 3 m/s and in perpendicular directions when they strike each other as shown. Assuming a coefficient of restitution $e = 0.9$, determine the magnitude and direction of the velocity of each puck after impact.

13.168 Two identical pool balls of 57.15 mm diameter may move freely on a pool table. Ball B is at rest and ball A has an initial velocity $\mathbf{v} = v_0\mathbf{i}$. (*a*) Knowing that $b = 50$ mm and $e = 0.7$, determine the velocity of each ball after impact. (*b*) Show that if $e = 1$, the final velocities of the balls form a right angle for all values of b.

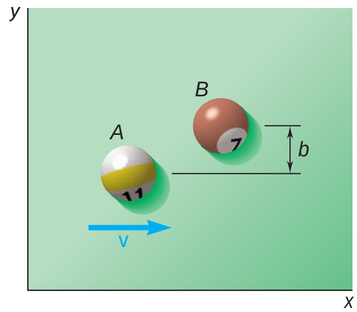


Fig. P13.168

13.169 A boy located at point A halfway between the center O of a semicircular wall and the wall itself throws a ball at the wall in a direction forming an angle of 45° with OA. Knowing that after hitting the wall the ball rebounds in a direction parallel to OA, determine the coefficient of restitution between the ball and the wall.

13.170 The Mars Pathfinder spacecraft used large airbags to cushion its impact with the planet's surface when landing. Assuming the spacecraft had an impact velocity of 18.5 m/s at an angle of 45° with respect to the horizontal, the coefficient of restitution is 0.85 and neglecting friction, determine (*a*) the height of the first bounce, (*b*) the length of the first bounce. (Acceleration of gravity on Mars = 3.73 m/s^2.)

Fig. *P13.170*

13.171 A girl throws a ball at an inclined wall from a height of 3 ft, hitting the wall at A with a horizontal velocity $\mathbf{v}_0$ of magnitude 25 ft/s. Knowing that the coefficient of restitution between the ball and the wall is 0.9 and neglecting friction, determine the distance d from the foot of the wall to the point B where the ball will hit the ground after bouncing off the wall.

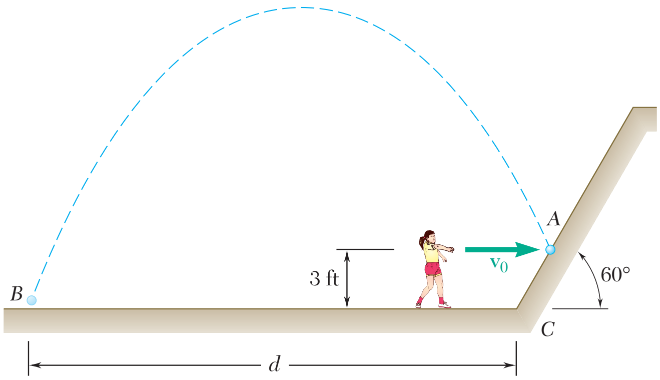


Fig. *P13.171*

13.172 A sphere rebounds as shown after striking an inclined plane with a vertical velocity $\mathbf{v}_0$ of magnitude $v_0 = 5$ m/s. Knowing that a = 30° and $e = 0.8$ between the sphere and the plane, determine the height h reached by the sphere.

13.173 A sphere rebounds as shown after striking an inclined plane with a vertical velocity $\mathbf{v}_0$ of magnitude $v_0 = 6$ m/s. Determine the value of a that will maximize the horizontal distance the ball travels before reaching its maximum height h assuming the coefficient of restitution between the ball and the ground is (*a*) $e = 1$, (*b*) $e = 0.8$.

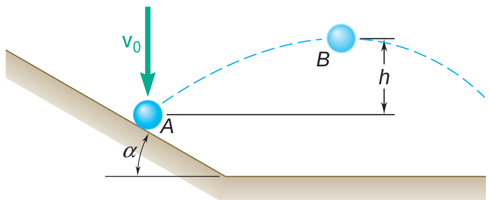


Fig. *P13.172* and P13.173

13.174 Two cars of the same mass run head-on into each other at C. After the collision, the cars skid with their brakes locked and come to a stop in the positions shown in the lower part of the figure. Knowing that the speed of car A just before impact was 5 mi/h and that the coefficient of kinetic friction between the pavement and the tires of both cars is 0.30, determine (*a*) the speed of car B just before impact, (*b*) the effective coefficient of restitution between the two cars.

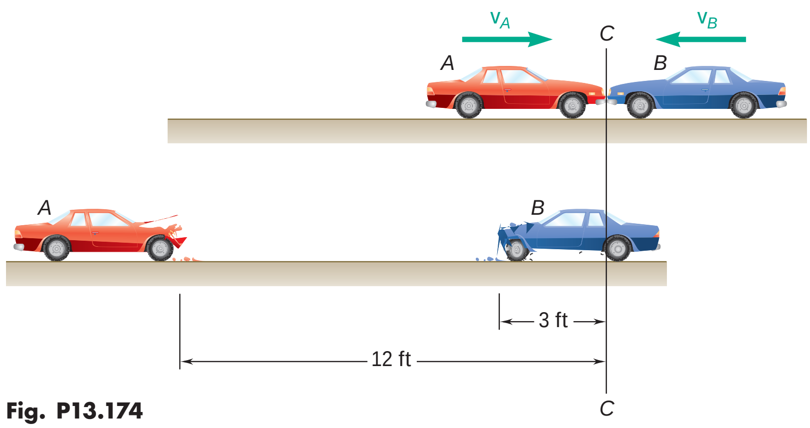


Fig. P13.174

13.175 A 1-kg block B is moving with a velocity $\mathbf{v}_0$ of magnitude $v_0 = 2$ m/s as it hits the 0.5-kg sphere A, which is at rest and hanging from a cord attached at O. Knowing that $\mu_k = 0.6$ between the block and the horizontal surface and $e = 0.8$ between the block and the sphere, determine after impact (*a*) the maximum height h reached by the sphere, (*b*) the distance x traveled by the block.

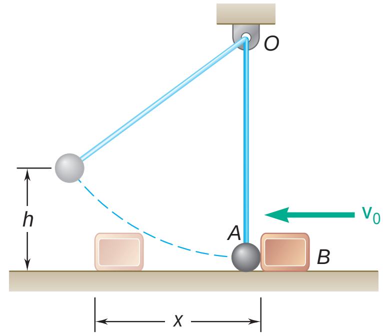


Fig. P13.175

13.176 A 0.25-lb ball thrown with a horizontal velocity $\mathbf{v}_0$ strikes a 1.5-lb plate attached to a vertical wall at a height of 36 in. above the ground. It is observed that after rebounding, the ball hits the ground at a distance of 24 in. from the wall when the plate is rigidly attached to the wall (Fig. 1) and at a distance of 10 in. when a foam-rubber mat is placed between the plate and the wall (Fig. 2). Determine (*a*) the coefficient of restitution e between the ball and the plate, (*b*) the initial velocity $\mathbf{v}_0$ of the ball.

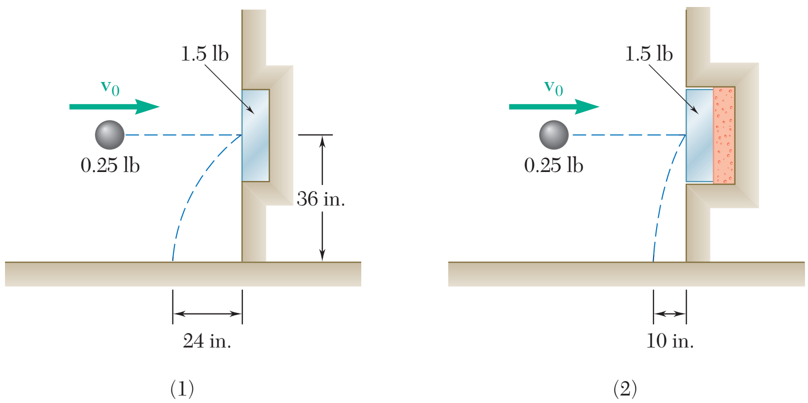


Fig. P13.176

13.177 After having been pushed by an airline employee, an empty 40-kg luggage carrier A hits with a velocity of 5 m/s an identical carrier B containing a 15-kg suitcase equipped with rollers. The impact causes the suitcase to roll into the left wall of carrier B. Knowing that the coefficient of restitution between the two carriers is 0.80 and that the coefficient of restitution between the suitcase and the wall of carrier B is 0.30, determine (*a*) the velocity of carrier B after the suitcase hits its wall for the first time, (*b*) the total energy lost in that impact.

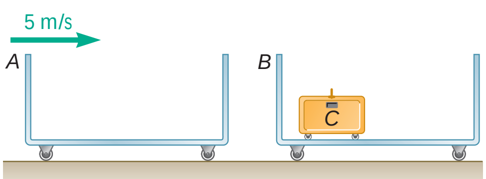


Fig. P13.177

13.178 Blocks A and B each weigh 0.8 lb and block C weighs 2.4 lb. The coefficient of friction between the blocks and the plane is $\mu_k = 0.30$. Initially block A is moving at a speed $v_0 = 15$ ft/s and blocks B and C are at rest (Fig. 1). After A strikes B and B strikes C, all three blocks come to a stop in the positions shown (Fig. 2). Determine (*a*) the coefficients of restitution between A and B and between B and C, (*b*) the displacement x of block C.

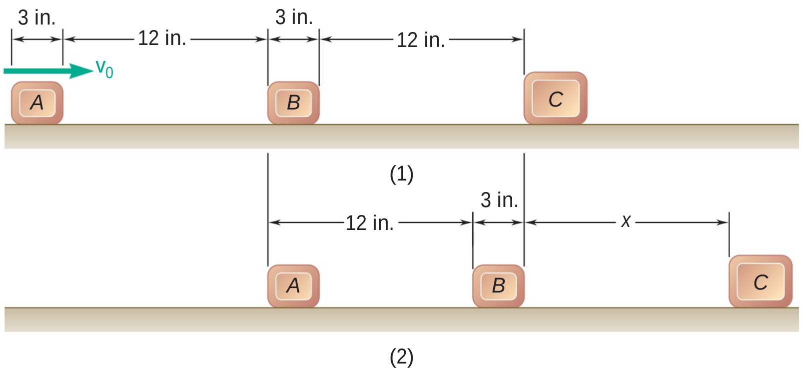


Fig. *P13.178*

13.179 A 0.5-kg sphere A is dropped from a height of 0.6 m onto a 1.0-kg plate B, which is supported by a nested set of springs and is initially at rest. Knowing that the coefficient of restitution between the sphere and the plate is $e = 0.8$, determine (*a*) the height h reached by the sphere after rebound, (*b*) the constant k of the single spring equivalent to the given set if the maximum deflection of the plate is observed to be equal to $3h$.

13.180 A 0.5-kg sphere A is dropped from a height of 0.6 m onto 1.0-kg plate B, which is supported by a nested set of springs and is initially at rest. Knowing that the set of springs is equivalent to a single spring of constant $k = 900$ N/m, determine (*a*) the value of the coefficient of restitution between the sphere and the plate for which the height h reached by the sphere after rebound is maximum, (*b*) the corresponding value of h, (c) the corresponding value of the maximum deflection of the plate.

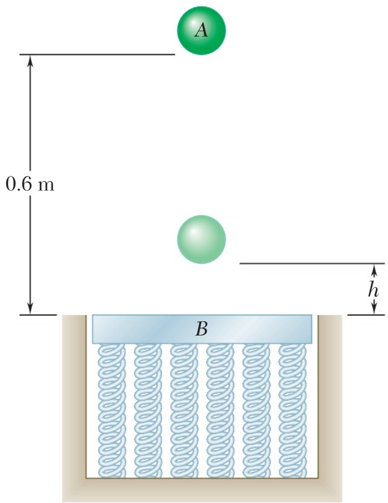


Fig. P13.179 and P13.180

13.181 The three blocks shown are identical. Blocks B and C are at rest when block B is hit by block A, which is moving with a velocity $\mathbf{v}_A$ of 3 ft/s. After the impact, which is assumed to be perfectly plastic ($e = 0$), the velocity of blocks A and B decreases due to friction, while block C picks up speed, until all three blocks are moving with the same velocity $\mathbf{v}$. Knowing that the coefficient of kinetic friction between all surfaces is $\mu_k = 0.20$, determine (*a*) the time required for the three blocks to reach the same velocity, (*b*) the total distance traveled by each block during that time.

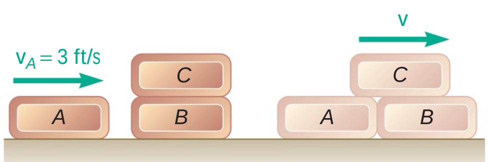


Fig. *P13.181*

13.182 Block A is released from rest and slides down the frictionless surface of B until it hits a bumper on the right end of B. Block A has a mass of 10 kg and object B has a mass of 30 kg and B can roll freely on the ground. Determine the velocities of A and B immediately after impact when (*a*) $e = 0$, (*b*) $e = 0.7$.

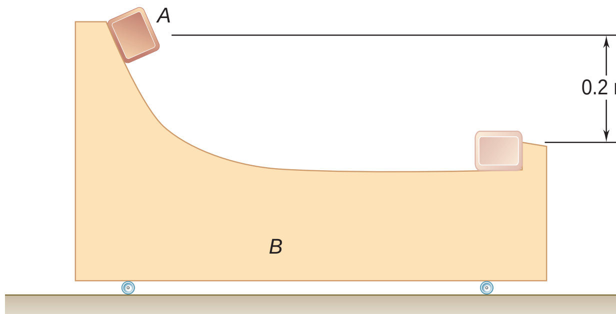


Fig. P13.182

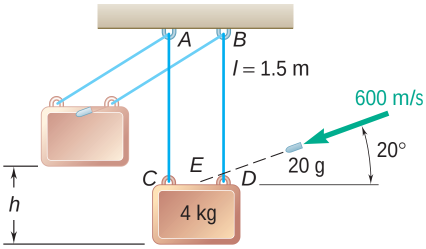


Fig. P13.183

13.183 A 20-g bullet fired into a 4-kg wooden block suspended from cords AC and BD penetrates the block at point E, halfway between C and D, without hitting cord BD. Determine (*a*) the maximum height h to which the block and the embedded bullet will swing after impact, (*b*) the total impulse exerted on the block by the two cords during the impact.

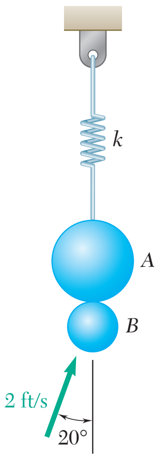


Fig. P13.184

13.184 A 2-lb ball A is suspended from a spring of constant 10 lb/in. and is initially at rest when it is struck by 1-lb ball B as shown. Neglecting friction and knowing the coefficient of restitution between the balls is 0.6, determine (*a*) the velocities of A and B after the impact, (*b*) the maximum height reached by A.

13.185 Ball B is hanging from an inextensible cord. An identical ball A is released from rest when it is just touching the cord and drops through the vertical distance $h_A = 8$ in. before striking ball B. Assuming $e = 0.9$ and no friction, determine the resulting maximum vertical displacement h_B of the ball B.

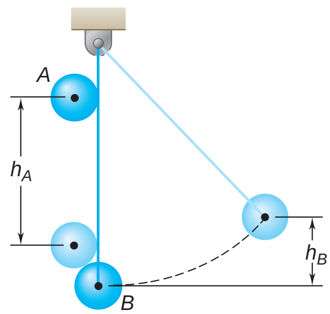


Fig. P13.185

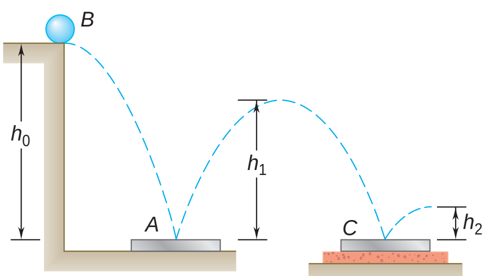


Fig. P13.186

13.186 A 70-g ball B dropped from a height $h_0 = 1.5$ m reaches a height $h_2 = 0.25$ m after bouncing twice from identical 210-g plates. Plate A rests directly on hard ground, while plate C rests on a foam-rubber mat. Determine (*a*) the coefficient of restitution between the ball and the plates, (*b*) the height h_1 of the ball's first bounce.

13.187 A 700-g sphere A moving with a velocity $\mathbf{v}_0$ parallel to the ground strikes the inclined face of a 2.1-kg wedge B which can roll freely on the ground and is initially at rest. After impact the sphere is observed from the ground to be moving straight up. Knowing that the coefficient of restitution between the sphere and the wedge is $e = 0.6$, determine (*a*) the angle u that the inclined face of the wedge makes with the horizontal, (*b*) the energy lost due to the impact.

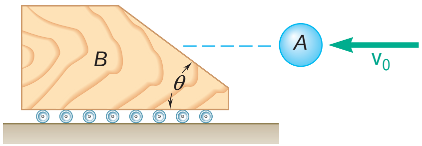


Fig. *P13.187*

13.188 When the rope is at an angle of a = 30° the 1-lb sphere A has a speed $v_0 = 4$ ft/s. The coefficient of restitution between A and the 2-lb wedge B is 0.7 and the length of rope $l = 2.6$ ft. The spring constant has a value of 2 lb/in. and u = 20°. Determine (*a*) the velocities of A and B immediately after the impact, (*b*) the maximum deflection of the spring assuming A does not strike B again before this point.

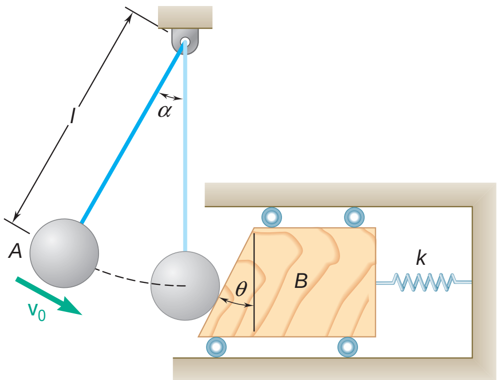


Fig. P13.188 and *P13.189*

13.189 When the rope is at an angle of a = 30° the 1-kg sphere A has a speed $v_0 = 0.6$ m/s. The coefficient of restitution between A and the 2-kg wedge B is 0.8 and the length of rope $l = 0.9$ m. The spring constant has a value of 1500 N/m and u = 20°. Determine (*a*) the velocities of A and B immediately after the impact, (*b*) the maximum deflection of the spring assuming A does not strike B again before this point.

REVIEW AND SUMMARY

This chapter was devoted to the method of work and energy and to the method of impulse and momentum. In the first half of the chapter we studied the method of work and energy and its application to the analysis of the motion of particles.

Work of a force

We first considered a force **F** acting on a particle A and defined the *work of* **F** *corresponding to the small displacement* $d\mathbf{r}$ [Sec. 13.2] as the quantity

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{13.1}$$

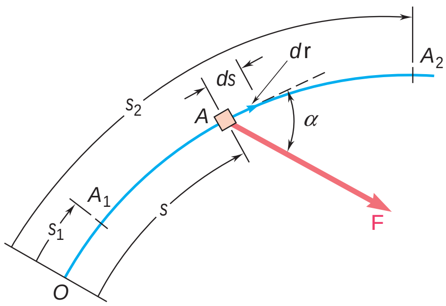


Fig. 13.29

or, recalling the definition of the scalar product of two vectors,

$$dU = F\, ds \cos a \tag{13.1$'$}$$

where a is the angle between **F** and $d\mathbf{r}$ (Fig. 13.29). The work of **F** during a finite displacement from A_1 to A_2, denoted by $U_{1 y 2}$, was obtained by integrating Eq. (13.1) along the path described by the particle:

$$U_{1 y 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \tag{13.2}$$

For a force defined by its rectangular components, we wrote

$$U_{1 y 2} = \int_{A_1}^{A_2} (F_x\, dx + F_y\, dy + F_z\, dz) \tag{13.2$''$}$$

Work of a weight

The work of the weight **W** of a body as its center of gravity moves from the elevation y_1 to y_2 (Fig. 13.30) was obtained by substituting $F_x = F_z = 0$ and $F_y = -W$ into Eq. (13.2″) and integrating. We found

$$U_{1 y 2} = -\int_{y_1}^{y_2} W\, dy = Wy_1 - Wy_2 \tag{13.4}$$

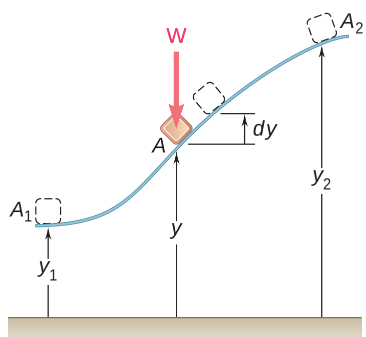


Fig. 13.30

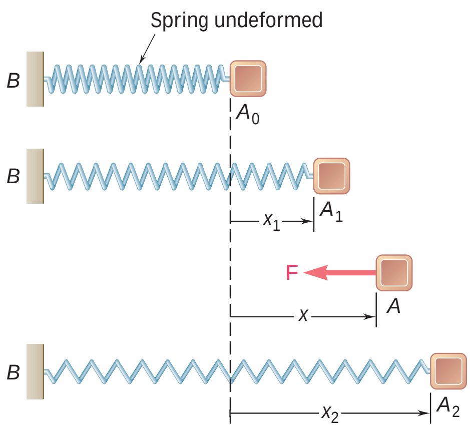


Fig. 13.31

Work of the force exerted by a spring

The work of a force **F** exerted by a spring on a body A during a finite displacement of the body (Fig. 13.31) from $A_1(x = x_1)$ to $A_2(x = x_2)$ was obtained by writing

$$dU = -F\,dx = -kx\,dx$$

$$U_{1\rightarrow 2} = -\int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \qquad (13.6)$$

The work of **F** is therefore positive *when the spring is returning to its undeformed position*.

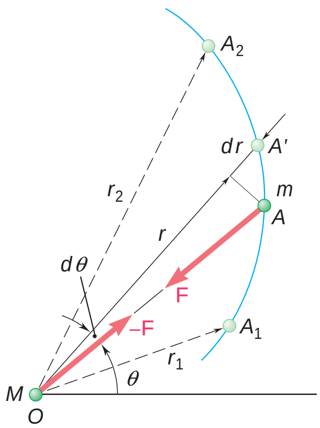


Fig. 13.32

Work of the gravitational force

The *work of the gravitational force* **F** exerted by a particle of mass M located at O on a particle of mass m as the latter moves from A_1 to A_2 (Fig. 13.32) was obtained by recalling from Sec. 12.10 the expression for the magnitude of **F** and writing

$$U_{1\rightarrow 2} = -\int_{r_1}^{r_2} \frac{GMm}{r^2}\,dr = \frac{GMm}{r_2} - \frac{GMm}{r_1} \qquad (13.7)$$

Kinetic energy of a particle

The *kinetic energy of a particle* of mass m moving with a velocity **v** [Sec. 13.3] was defined as the scalar quantity

$$T = \tfrac{1}{2}mv^2 \qquad (13.9)$$

Principle of work and energy

From Newton's second law we derived the *principle of work and energy,* which states that *the kinetic energy of a particle at* A_2 *can be obtained by adding to its kinetic energy at* A_1 *the work done during the displacement from* A_1 to A_2 *by the force* **F** *exerted on the particle:*

$$T_1 + U_{1 y 2} = T_2 \tag{13.11}$$

Method of work and energy

The method of work and energy simplifies the solution of many problems dealing with forces, displacements, and velocities, since it does not require the determination of accelerations [Sec. 13.4]. We also note that it involves only scalar quantities and that forces which do no work need not be considered [Sample Probs. 13.1 and 13.3]. However, this method should be supplemented by the direct application of Newton's second law to determine a force normal to the path of the particle [Sample Prob. 13.4].

Power and mechanical efficiency

The power developed by a machine and its mechanical efficiency were discussed in Sec. 13.5. Power was defined as the time rate at which work is done:

$$\text{Power} = \frac{dU}{dt} = \mathbf{F} \cdot \mathbf{v} \qquad (13.12,\ 13.13)$$

where **F** is the force exerted on the particle and **v** the velocity of the particle [Sample Prob. 13.5]. The *mechanical efficiency,* denoted by h, was expressed as

$$\mathrm{h} = \frac{\text{power output}}{\text{power input}} \tag{13.15}$$

Conservative force. Potential energy

When the work of a force **F** is independent of the path followed [Secs. 13.6 and 13.7], the force **F** is said to be a *conservative force,* and its work is equal to *minus the change in the potential energy* V associated with **F**:

$$U_{1 y 2} = V_1 - V_2 \tag{13.19$'$}$$

The following expressions were obtained for the potential energy associated with each of the forces considered earlier:

Force of gravity (weight): $$V_g = Wy \tag{13.16}$$

Gravitational force: $$V_g = -\frac{GMm}{r} \tag{13.17}$$

Elastic force exerted by a spring: $$V_e = \tfrac{1}{2}kx^2 \tag{13.18}$$

Substituting for $U_{1 \rightarrow 2}$ from Eq. (13.19′) into Eq. (13.11) and rearranging the terms [Sec. 13.8], we obtained

$$T_1 + V_1 = T_2 + V_2 \tag{13.24}$$

Principle of conservation of energy

This is the *principle of conservation of energy,* which states that when a particle moves under the action of conservative forces, *the sum of its kinetic and potential energies remains constant*. The application of this principle facilitates the solution of problems involving only conservative forces [Sample Probs. 13.6 and 13.7].

Motion under a gravitational force

Recalling from Sec. 12.9 that, when a particle moves under a central force **F**, its angular momentum about the center of force O remains constant, we observed [Sec. 13.9] that, if the central force **F** is also conservative, the principles of conservation of angular momentum and of conservation of energy can be used jointly to analyze the motion of the particle [Sample Prob. 13.8]. Since the gravitational force exerted by the earth on a space vehicle is both central and conservative, this approach was used to study the motion of such vehicles [Sample Prob. 13.9] and was found particularly effective in the case of an *oblique launching*. Considering the initial position P_0 and an arbitrary position P of the vehicle (Fig. 13.33), we wrote

$$(H_O)_0 = H_O: \qquad r_0 m v_0 \sin \phi_0 = r m v \sin \phi \tag{13.25}$$

$$T_0 + V_0 = T + V: \qquad \tfrac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2} m v^2 - \frac{GMm}{r} \tag{13.26}$$

where m was the mass of the vehicle and M the mass of the earth.

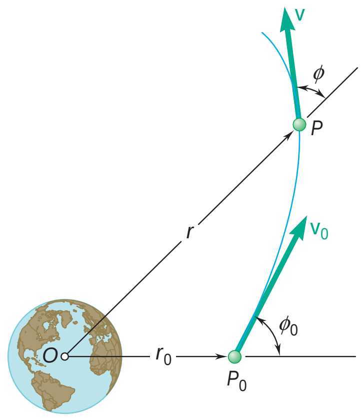


Fig. 13.33

Principle of impulse and momentum for a particle

The second half of the chapter was devoted to the method of impulse and momentum and to its application to the solution of various types of problems involving the motion of particles.

The *linear momentum of a particle* was defined [Sec. 13.10] as the product $m\mathbf{v}$ of the mass m of the particle and its velocity $\mathbf{v}$. From Newton's second law, $\mathbf{F} = m\mathbf{a}$, we derived the relation

$$m\mathbf{v}_1 + \int_{t_1}^{t_2} \mathbf{F}\,dt = m\mathbf{v}_2 \tag{13.28}$$

where $m\mathbf{v}_1$ and $m\mathbf{v}_2$ represent the momentum of the particle at a time t_1 and a time t_2, respectively, and where the integral defines the *linear impulse of the force* **F** during the corresponding time interval. We wrote therefore

$$m\mathbf{v}_1 + \mathbf{Imp}_{1 \rightarrow 2} = m\mathbf{v}_2 \tag{13.30}$$

which expresses the principle of impulse and momentum for a particle.

When the particle considered is subjected to several forces, the sum of the impulses of these forces should be used; we had

$$m\mathbf{v}_1 + \Sigma\,\mathbf{Imp}_{1\ y\ 2} = m\mathbf{v}_2 \qquad (13.32)$$

Since Eqs. (13.30) and (13.32) involve *vector quantities,* it is necessary to consider their x and y components separately when applying them to the solution of a given problem [Sample Probs. 13.10 and 13.11].

Impulsive motion

The method of impulse and momentum is particularly effective in the study of the *impulsive motion* of a particle, when very large forces, called *impulsive forces,* are applied for a very short interval of time Δt, since this method involves the impulses $\mathbf{F}\,\Delta t$ of the forces, rather than the forces themselves [Sec. 13.11]. Neglecting the impulse of any nonimpulsive force, we wrote

$$m\mathbf{v}_1 + \Sigma\mathbf{F}\,\Delta t = m\mathbf{v}_2 \qquad (13.35)$$

In the case of the impulsive motion of several particles, we had

$$\Sigma m\mathbf{v}_1 + \Sigma\mathbf{F}\,\Delta t = \Sigma m\mathbf{v}_2 \qquad (13.36)$$

where the second term involves only impulsive, external forces [Sample Prob. 13.12].

In the particular case *when the sum of the impulses of the external forces is zero,* Eq. (13.36) reduces to $\Sigma m\mathbf{v}_1 = \Sigma m\mathbf{v}_2$; that is, *the total momentum of the particles is conserved.*

Direct central impact

In Secs. 13.12 through 13.14, we considered the *central impact* of two colliding bodies. In the case of a *direct central impact* [Sec. 13.13], the two colliding bodies A and B were moving along the *line of impact* with velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively (Fig. 13.34). Two equations could be used to determine their velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ after the impact.

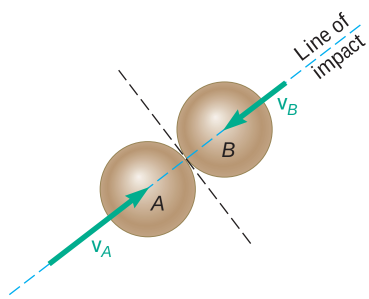


Fig. 13.34

The first expressed conservation of the total momentum of the two bodies,

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \qquad (13.37)$$

where a positive sign indicates that the corresponding velocity is directed to the right, while the second related the *relative velocities* of the two bodies before and after the impact,

$$v'_B - v'_A = e(v_A - v_B) \qquad (13.43)$$

The constant e is known as the *coefficient of restitution;* its value lies between 0 and 1 and depends in a large measure on the materials involved. When $e = 0$, the impact is said to be *perfectly plastic;* when $e = 1$, it is said to be *perfectly elastic* [Sample Prob. 13.13].

Oblique central impact

In the case of an *oblique central impact* [Sec. 13.14], the velocities of the two colliding bodies before and after the impact were resolved into n components along the line of impact and t components along the common tangent to the surfaces in contact (Fig. 13.35). We observed that the t component of the velocity of each body remained

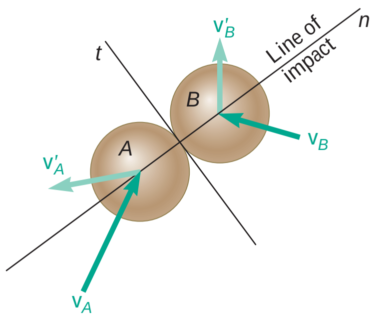


Fig. 13.35

unchanged, while the n components satisfied equations similar to Eqs. (13.37) and (13.43) [Sample Probs. 13.14 and 13.15]. It was shown that although this method was developed for bodies moving freely before and after the impact, it could be extended to the case when one or both of the colliding bodies is constrained in its motion [Sample Prob. 13.16].

Using the three fundamental methods of kinetic analysis

In Sec. 13.15, we discussed the relative advantages of the three fundamental methods presented in this chapter and the preceding one, namely, Newton's second law, work and energy, and impulse and momentum. We noted that the method of work and energy and the method of impulse and momentum can be combined to solve problems involving a short impact phase during which impulsive forces must be taken into consideration [Sample Prob. 13.17].

REVIEW PROBLEMS

13.190 A 32,000-lb airplane lands on an aircraft carrier and is caught by an arresting cable. The cable is inextensible and is paid out at A and B from mechanisms located below deck and consisting of pistons moving in long oil-filled cylinders. Knowing that the piston-cylinder system maintains a constant tension of 85 kips in the cable during the entire landing, determine the landing speed of the airplane if it travels a distance $d = 95$ ft after being caught by the cable.

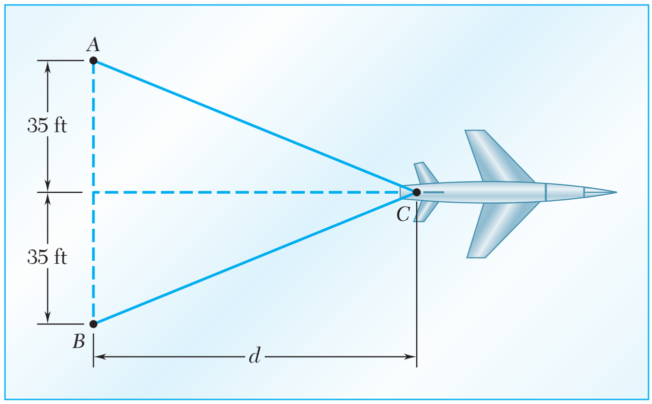


Fig. P13.190

13.191 A 2-oz pellet shot vertically from a spring-loaded pistol on the surface of the earth rises to a height of 300 ft. The same pellet shot from the same pistol on the surface of the moon rises to a height of 1900 ft. Determine the energy dissipated by aerodynamic drag when the pellet is shot on the surface of the earth. (The acceleration of gravity on the surface of the moon is 0.165 times that on the surface of the earth.)

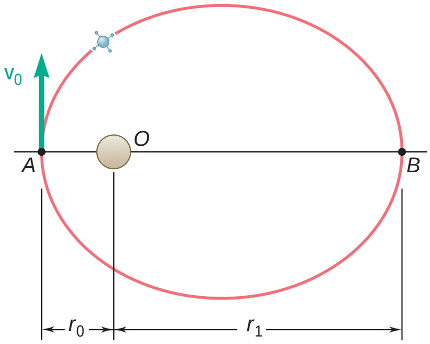


Fig. P13.192

13.192 A satellite describes an elliptic orbit about a planet of mass M. The minimum and maximum values of the distance r from the satellite to the center of the planet are, respectively, r_0 and r_1. Use the principles of conservation of energy and conservation of angular momentum to derive the relation

$$\frac{1}{r_0} + \frac{1}{r_1} = \frac{2GM}{h^2}$$

where h is the angular momentum per unit mass of the satellite and G is the constant of gravitation.

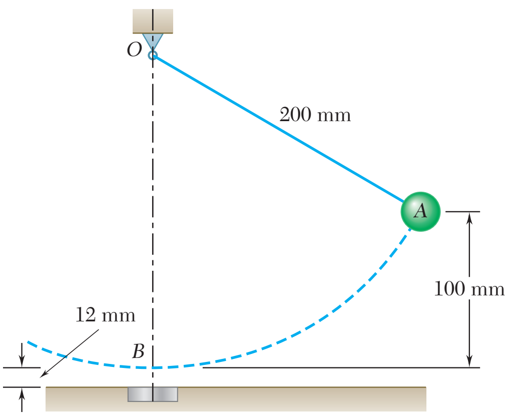


Fig. *P13.193*

13.193 A 60-g steel sphere attached to a 200-mm cord can swing about point O in a vertical plane. It is subjected to its own weight and to a force $\mathbf{F}$ exerted by a small magnet embedded in the ground. The magnitude of that force expressed in newtons is $F = 3000/r^2$, where r is the distance from the magnet to the sphere expressed in millimeters. Knowing that the sphere is released from rest at A, determine its speed as it passes through point B.

13.194 A shuttle is to rendezvous with a space station which is in a circular orbit at an altitude of 250 mi above the surface of the earth. The shuttle has reached an altitude of 40 mi when its engine is turned off at point B. Knowing that at that time the velocity $\mathbf{v}_0$ of the shuttle forms an angle $f_0 = 55°$ with the vertical, determine the required magnitude of $\mathbf{v}_0$ if the trajectory of the shuttle is to be tangent at A to the orbit of the space station.

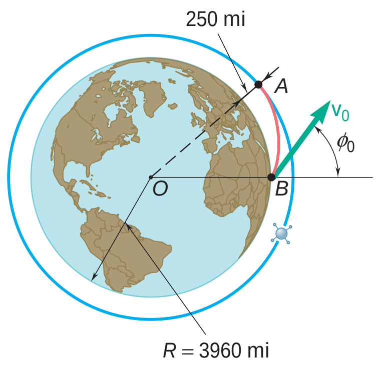


Fig. P13.194

13.195 A 300-g block is released from rest after a spring of constant $k = 600$ N/m has been compressed 160 mm. Determine the force exerted by the loop $ABCD$ on the block as the block passes through (a) point A, (b) point B, (c) point C. Assume no friction.

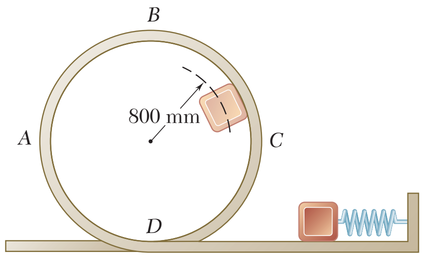


Fig. P13.195

13.196 A small sphere B of mass m is attached to an inextensible cord of length $2a$, which passes around the fixed peg A and is attached to a fixed support at O. The sphere is held close to the support at O and released with no initial velocity. It drops freely to point C, where the cord becomes taut, and swings in a vertical plane, first about A and then about O. Determine the vertical distance from line OD to the highest point C'' that the sphere will reach.

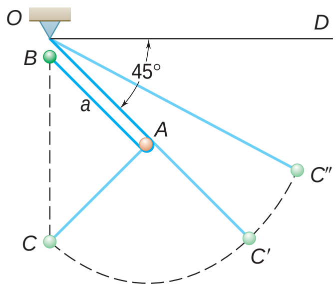


Fig. *P13.196*

13.197 A 300-g collar A is released from rest, slides down a frictionless rod, and strikes a 900-g collar B which is at rest and supported by a spring of constant 500 N/m. Knowing that the coefficient of restitution between the two collars is 0.9, determine (a) the maximum distance collar A moves up the rod after impact, (b) the maximum distance collar B moves down the rod after impact.

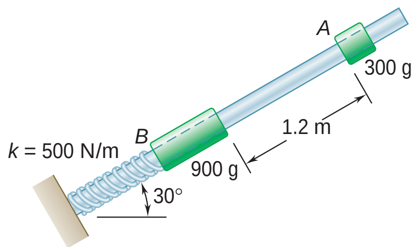


Fig. P13.197

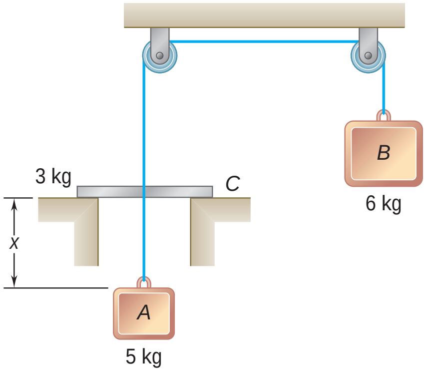


Fig. P13.198

13.198 Blocks A and B are connected by a cord which passes over pulleys and through a collar C. The system is released from rest when $x = 1.7$ m. As block A rises, it strikes collar C with perfectly plastic impact ($e = 0$). After impact, the two blocks and the collar keep moving until they come to a stop and reverse their motion. As A and C move down, C hits the ledge and blocks A and B keep moving until they come to another stop. Determine (*a*) the velocity of the blocks and collar immediately after A hits C, (*b*) the distance the blocks and collar move after the impact before coming to a stop, (*c*) the value of x at the end of one complete cycle.

13.199 A 2-kg ball B is traveling horizontally at 10 m/s when it strikes 2-kg ball A. Ball A is initially at rest and is attached to a spring with constant 100 N/m and an unstretched length of 1.2 m. Knowing the coefficient of restitution between A and B is 0.8 and friction between all surfaces is negligible, determine the normal force between A and the ground when it is at the bottom of the hill.

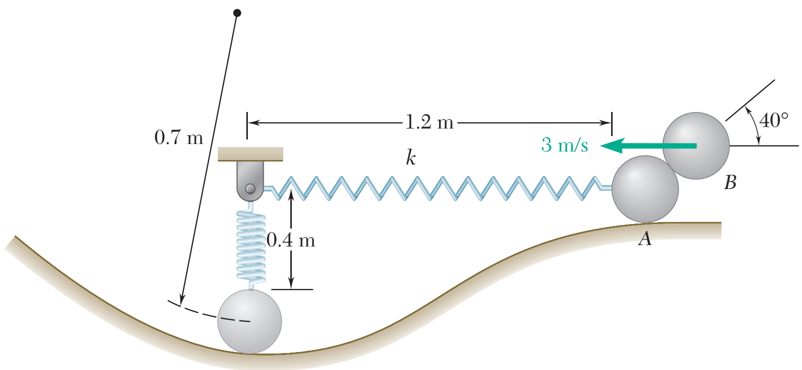


Fig. *P13.199*

13.200 A 2-kg block A is pushed up against a spring compressing it a distance $x = 0.1$ m. The block is then released from rest and slides down the 20° incline until it strikes a 1-kg sphere B which is suspended from a 1-m inextensible rope. The spring constant $k = 800$ N/m, the coefficient of friction between A and the ground is 0.2, the distance A slides from the unstretched length of the spring $d = 1.5$ m, and the coefficient of restitution between A and B is 0.8. When $\alpha = 40°$, determine (*a*) the speed of B, (*b*) the tension in the rope.

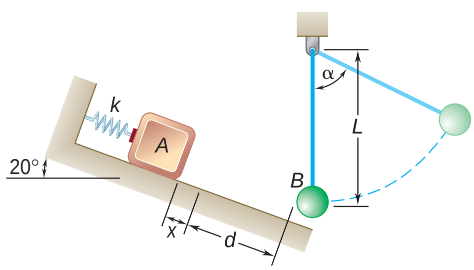


Fig. P13.200

***13.201** The 2-lb ball at A is suspended by an inextensible cord and given an initial horizontal velocity of $\mathbf{v}_0$. If $l = 2$ ft, $x_B = 0.3$ ft, and $y_B = 0.4$ ft, determine the initial velocity $\mathbf{v}_0$ so that the ball will enter the basket. (*Hint:* Use a computer to solve the resulting set of equations.)

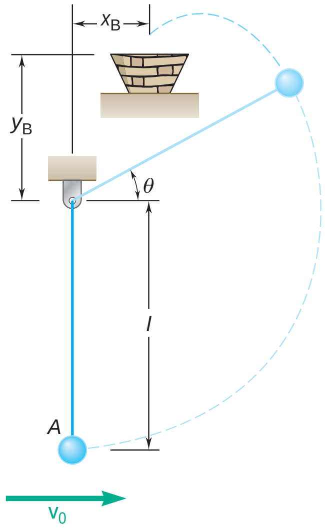


Fig. ***P13.201***

COMPUTER PROBLEMS

13.C1 A 12-lb collar is attached to a spring anchored at point C and can slide on a frictionless rod forming an angle of 30° with the vertical. The spring is of constant k and is unstretched when the collar is at A. Knowing that the collar is released from rest at A, use computational software to determine the velocity of the collar at point B for values of k from 0.1 to 2.0 lb/in.

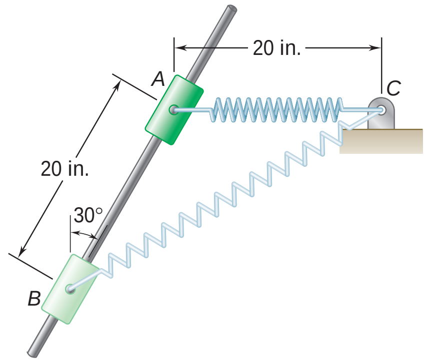


Fig. P13.C1

13.C2 Skid marks on a drag race track indicate that the rear (drive) wheels of a 2000-lb car slip with the front wheels just off the ground for the first 60 ft of the 1320-ft track. The car is driven with slipping impending, with 60 percent of its weight on the rear wheels, for the remaining 1260 ft of the race. Knowing that the coefficients of kinetic and static friction are 0.60 and 0.85, respectively, and that the force due to the aerodynamic drag is $F_d = 0.0098v^2$, where the speed v is expressed in ft/s and the force F_d in lb, use computational software to determine the time elapsed and the speed of the car at various points along the track, (*a*) taking the force F_d into account, (*b*) ignoring the force F_d. If you write a computer program use increments of distance $\Delta x = 0.1$ ft in your calculations, and tabulate your results every 5 ft for the first 60 ft and every 90 ft for the remaining 1260 ft. [*Hint:* The time Δt_i required for the car to move through the increment of distance Δx_i can be obtained by dividing Δx_i by the average velocity $\frac{1}{2}(v_i + v_{i+1})$ of the car over Δx_i if the acceleration of the car is assumed to remain constant over Δx.]

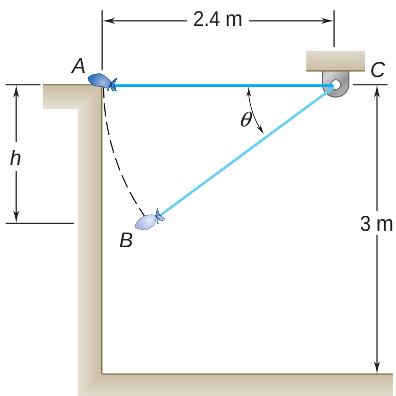


Fig. P13.C3

13.C3 A 5-kg bag is gently pushed off the top of a wall and swings in a vertical plane at the end of a 2.4-m rope which can withstand a maximum tension F_m. For F_m from 40 to 140 N use computational software to determine (*a*) the difference in elevation h between point A and point B where the rope will break, (*b*) the distance d from the vertical wall to the point where the bag will strike the floor.

13.C4 Use computational software to determine (*a*) the time required for the system of Prob. 13.198 to complete 10 successive cycles of the motion described in that problem, starting with $x = 1.7$ m, (*b*) the value of x at the end of the tenth cycle.

13.C5 A 700-g ball B is hanging from an inextensible cord attached to a support at C. A 350-g ball A strikes B with a velocity $\mathbf{v}_0$ at an angle θ_0 with the vertical. Assuming no friction and denoting by e the coefficient of restitution, use computational software to determine the magnitudes v'_A and v'_B of the velocities of the balls immediately after impact and the percentage of energy lost in the collision for $v_0 = 6$ m/s and values of θ_0 from 20° to 150°, assuming (*a*) $e = 1$, (*b*) $e = 0.75$, (*c*) $e = 0$.

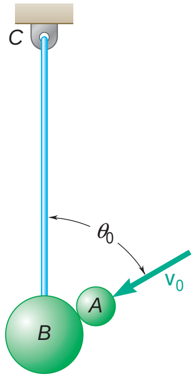


Fig. P13.C5

13.C6 In Prob. 13.110, a space vehicle was in a circular orbit at an altitude of 225 mi above the surface of the earth. To return to earth it decreased its speed as it passed through A by firing its engine for a short interval of time in a direction opposite to the direction of its motion. Its resulting velocity as it reached point B at an altitude of 40 mi formed an angle $\phi_B = 60°$ with the vertical. An alternative strategy for taking the space vehicle out of its circular orbit would be to turn it around so that its engine pointed away from the earth and then give it an incremental velocity $\Delta\mathbf{v}_A$ toward the center O of the earth. This would likely require a smaller expenditure of energy when firing the engine at A, but might result in too fast a descent at B. Assuming that this strategy is used, use computational software to determine the values of ϕ_B and v_B for an energy expenditure ranging from 5 to 100 percent of that needed in Prob. 13.110.

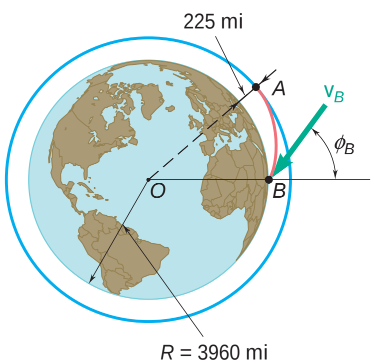


Fig. P13.C6