The thrust for this XR-5M15 prototype engine is produced by gas particles being ejected at a high velocity. The determination of the forces on the test stand is based on the analysis of the motion of a *variable system of particles,* i.e., the motion of a large number of air particles considered together rather than separately.

CHAPTER 14

Systems of Particles

14.1 INTRODUCTION

In this chapter you will study the motion of *systems of particles,* i.e., the motion of a large number of particles considered together. The first part of the chapter is devoted to systems consisting of well-defined particles; the second part considers the motion of variable systems, i.e., systems which are continually gaining or losing particles, or doing both at the same time.

In Sec. 14.2, Newton's second law will first be applied to each particle of the system. Defining the *effective force* of a particle as the product $m_i\mathbf{a}_i$ of its mass m_i and its acceleration $\mathbf{a}_i$, we will show that the *external forces* acting on the various particles form a system equipollent to the system of the effective forces, i.e., both systems have the same resultant and the same moment resultant about any given point. In Sec. 14.3, it will be further shown that the resultant and moment resultant of the external forces are equal, respectively, to the rate of change of the total linear momentum and of the total angular momentum of the particles of the system.

In Sec. 14.4, the *mass center* of a system of particles is defined and the motion of that point is described, while in Sec. 14.5 the motion of the particles about their mass center is analyzed. The conditions under which the linear momentum and the angular momentum of a system of particles are conserved are discussed in Sec. 14.6, and the results obtained in that section are applied to the solution of various problems.

Sections 14.7 and 14.8 deal with the application of the work-energy principle to a system of particles, and Sec. 14.9 with the application of the impulse-momentum principle. These sections also contain a number of problems of practical interest.

It should be noted that while the derivations given in the first part of this chapter are carried out for a system of independent particles, they remain valid when the particles of the system are rigidly connected, i.e., when they form a rigid body. In fact, the results obtained here will form the foundation of our discussion of the kinetics of rigid bodies in Chaps. 16 through 18.

The second part of this chapter is devoted to the study of variable systems of particles. In Sec. 14.11 you will consider steady streams of particles, such as a stream of water diverted by a fixed vane, or the flow of air through a jet engine, and learn to determine the force exerted by the stream on the vane and the thrust developed by the engine. Finally, in Sec. 14.12, you will learn how to analyze systems which gain mass by continually absorbing particles or lose mass by continually expelling particles. Among the various practical applications of this analysis will be the determination of the thrust developed by a rocket engine.

14.2 APPLICATION OF NEWTON'S LAWS TO THE MOTION OF A SYSTEM OF PARTICLES. EFFECTIVE FORCES

In order to derive the equations of motion for a system of n particles, let us begin by writing Newton's second law for each individual particle of the system. Consider the particle P_i, where $1 \leq i \leq n$. Let

m_i be the mass of P_i and $\mathbf{a}_i$ its acceleration with respect to the newtonian frame of reference $Oxyz$. The force exerted on P_i by another particle P_j of the system (Fig. 14.1), called an *internal force,* will be denoted by $\mathbf{f}_{ij}$. The resultant of the internal forces exerted on P_i by all the other particles of the system is thus $\sum_{j=1}^{n} \mathbf{f}_{ij}$ (where $\mathbf{f}_{ii}$ has no meaning and is assumed to be equal to zero). Denoting, on the other hand, by $\mathbf{F}_i$ the resultant of all the *external forces* acting on P_i, we write Newton's second law for the particle P_i as follows:

$$\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{f}_{ij} = m_i \mathbf{a}_i \tag{14.1}$$

Denoting by $\mathbf{r}_i$ the position vector of P_i and taking the moments about O of the various terms in Eq. (14.1), we also write

$$\mathbf{r}_i \times \mathbf{F}_i + \sum_{j=1}^{n} (\mathbf{r}_i \times \mathbf{f}_{ij}) = \mathbf{r}_i \times m_i \mathbf{a}_i \tag{14.2}$$

Repeating this procedure for each particle P_i of the system, we obtain n equations of the type (14.1) and n equations of the type (14.2), where i takes successively the values $1, 2, \ldots, n$. The vectors $m_i\mathbf{a}_i$ are referred to as the *effective forces* of the particles.† Thus the equations obtained express the fact that the external forces $\mathbf{F}_i$ and the internal forces $\mathbf{f}_{ij}$ acting on the various particles form a system equivalent to the system of the effective forces $m_i\mathbf{a}_i$ (i.e., one system may be replaced by the other) (Fig. 14.2).

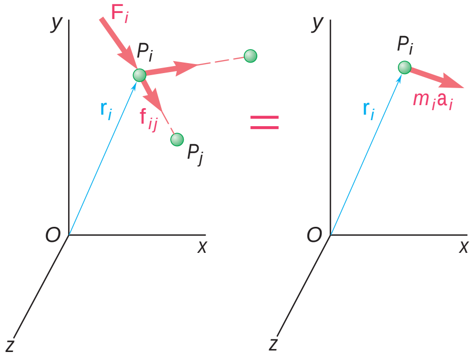


Fig. 14.1

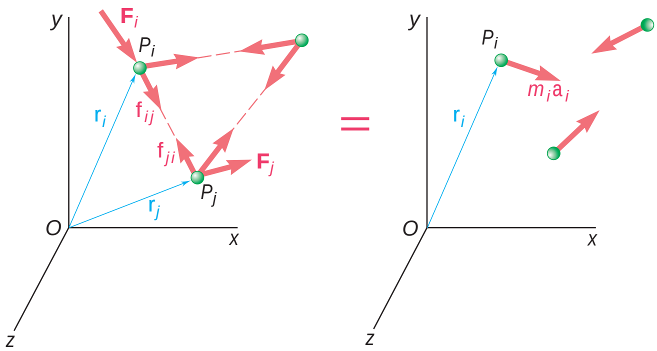


Fig. 14.2

Before proceeding further with our derivation, let us examine the internal forces $\mathbf{f}_{ij}$. We note that these forces occur in pairs $\mathbf{f}_{ij}$, $\mathbf{f}_{ji}$, where $\mathbf{f}_{ij}$ represents the force exerted by the particle P_j on the particle P_i and $\mathbf{f}_{ji}$ represents the force exerted by P_i on P_j (Fig. 14.2). Now, according to Newton's third law (Sec. 6.1), as extended by Newton's law of gravitation to particles acting at a distance (Sec. 12.10), the forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ are equal and opposite and have the same line of action. Their sum is therefore $\mathbf{f}_{ij} + \mathbf{f}_{ji} = 0$, and the sum of their moments about O is

$$\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = \mathbf{r}_i \times (\mathbf{f}_{ij} + \mathbf{f}_{ji}) + (\mathbf{r}_j - \mathbf{r}_i) \times \mathbf{f}_{ji} = 0$$

†Since these vectors represent the resultants of the forces acting on the various particles of the system, they can truly be considered as forces.

since the vectors $\mathbf{r}_j - \mathbf{r}_i$ and $\mathbf{f}_{ji}$ in the last term are collinear. Adding all the internal forces of the system and summing their moments about O, we obtain the equations

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{f}_{ij} = 0 \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathbf{r}_i \times \mathbf{f}_{ij}) = 0 \tag{14.3}$$

which express the fact that the resultant and the moment resultant of the internal forces of the system are zero.

Returning now to the n equations (14.1), where $i = 1, 2, \ldots, n$, we sum their left-hand members and sum their right-hand members. Taking into account the first of Eqs. (14.3), we obtain

$$\sum_{i=1}^{n} \mathbf{F}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i \tag{14.4}$$

Proceeding similarly with Eq. (14.2) and taking into account the second of Eqs. (14.3), we have

$$\sum_{i=1}^{n} (\mathbf{r}_i \times \mathbf{F}_i) = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{a}_i) \tag{14.5}$$

Equations (14.4) and (14.5) express the fact that the system of the external forces $\mathbf{F}_i$ and the system of the effective forces $m_i\mathbf{a}_i$ have the same resultant and the same moment resultant. Referring to the definition given in Sec. 3.19 for two equipollent systems of vectors, we can therefore state that *the system of the external forces acting on the particles and the system of the effective forces of the particles are equipollent*† (Fig. 14.3).

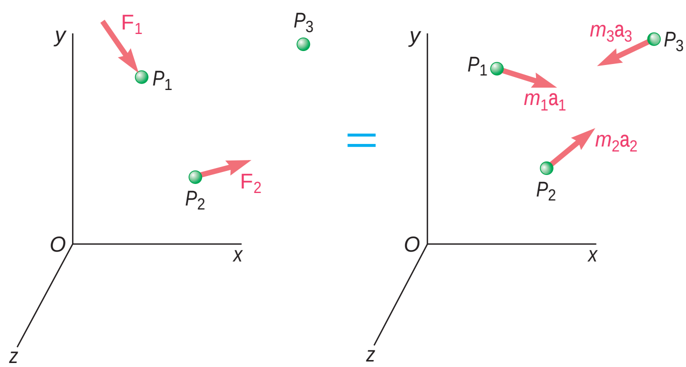


Fig. 14.3

†The result just obtained is often referred to as *d'Alembert's principle*, after the French mathematician Jean le Rond d'Alembert (1717–1783). However, d'Alembert's original statement refers to the motion of a system of connected bodies, with $\mathbf{f}_{ij}$ representing constraint forces which if applied by themselves will not cause the system to move. Since, as it will now be shown, this is in general not the case for the internal forces acting on a system of free particles, the consideration of d'Alembert's principle will be postponed until the motion of rigid bodies is considered (Chap. 16).

Equations (14.3) express the fact that the system of the internal forces $\mathbf{f}_{ij}$ is equipollent to zero. Note, however, that it does *not* follow that the internal forces have no effect on the particles under consideration. Indeed, the gravitational forces that the sun and the planets exert on one another are internal to the solar system and equipollent to zero. Yet these forces are alone responsible for the motion of the planets about the sun.

Similarly, it does not follow from Eqs. (14.4) and (14.5) that two systems of external forces which have the same resultant and the same moment resultant will have the same effect on a given system of particles. Clearly, the systems shown in Figs. 14.4*a* and 14.4*b* have

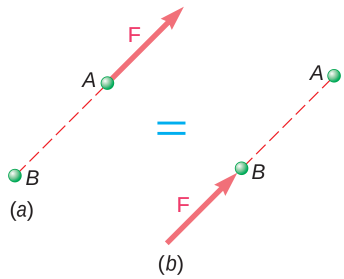


Fig. 14.4

the same resultant and the same moment resultant; yet the first system accelerates particle *A* and leaves particle *B* unaffected, while the second accelerates *B* and does not affect *A*. It is important to recall that when we stated in Sec. 3.19 that two equipollent systems of forces acting on a rigid body are also equivalent, we specifically noted that this property could *not* be extended to a system of forces acting on a set of independent particles such as those considered in this chapter.

In order to avoid any confusion, blue equals signs are used to connect equipollent systems of vectors, such as those shown in Figs. 14.3 and 14.4. These signs indicate that the two systems of vectors have the same resultant and the same moment resultant. Red equals signs will continue to be used to indicate that two systems of vectors are equivalent, i.e., that one system can actually be replaced by the other (Fig. 14.2).

14.3 LINEAR AND ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

Equations (14.4) and (14.5), obtained in the preceding section for the motion of a system of particles, can be expressed in a more condensed form if we introduce the linear and the angular momentum of the system of particles. Defining the linear momentum $\mathbf{L}$ of the system of particles as the sum of the linear momenta of the various particles of the system (Sec. 12.3), we write

$$\mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{v}_i \tag{14.6}$$

Defining the angular momentum $\mathbf{H}_O$ about O of the system of particles in a similar way (Sec. 12.7), we have

$$\mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{v}_i) \tag{14.7}$$

Differentiating both members of Eqs. (14.6) and (14.7) with respect to t, we write

$$\dot{\mathbf{L}} = \sum_{i=1}^{n} m_i \dot{\mathbf{v}}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i \tag{14.8}$$

and

$$\begin{aligned} \dot{\mathbf{H}}_O &= \sum_{i=1}^{n} (\dot{\mathbf{r}}_i \times m_i \mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \dot{\mathbf{v}}_i) \\ &= \sum_{i=1}^{n} (\mathbf{v}_i \times m_i \mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{a}_i) \end{aligned}$$

which reduces to

$$\dot{\mathbf{H}}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{a}_i) \tag{14.9}$$

since the vectors $\mathbf{v}_i$ and $m_i\mathbf{v}_i$ are collinear.

We observe that the right-hand members of Eqs. (14.8) and (14.9) are respectively identical with the right-hand members of Eqs. (14.4) and (14.5). It follows that the left-hand members of these equations are respectively equal. Recalling that the left-hand member of Eq. (14.5) represents the sum of the moments $\mathbf{M}_O$ about O of the external forces acting on the particles of the system, and omitting the subscript i from the sums, we write

$$\Sigma \mathbf{F} = \dot{\mathbf{L}} \tag{14.10}$$

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \tag{14.11}$$

These equations express that *the resultant and the moment resultant about the fixed point O of the external forces are respectively equal to the rates of change of the linear momentum and of the angular momentum about O of the system of particles*.

14.4 MOTION OF THE MASS CENTER OF A SYSTEM OF PARTICLES

Equation (14.10) may be written in an alternative form if the *mass center* of the system of particles is considered. The mass center of the system is the point G defined by the position vector $\bar{\mathbf{r}}$, which satisfies the relation

$$m\bar{\mathbf{r}} = \sum_{i=1}^{n} m_i \mathbf{r}_i \tag{14.12}$$

where m represents the total mass $\sum_{i=1}^{n} m_i$ of the particles. Resolving the position vectors $\bar{\mathbf{r}}$ and $\mathbf{r}_i$ into rectangular components, we obtain the following three scalar equations, which can be used to determine the coordinates $\bar{x}, \bar{y}, \bar{z}$ of the mass center:

$$m\bar{x} = \sum_{i=1}^{n} m_i x_i \qquad m\bar{y} = \sum_{i=1}^{n} m_i y_i \qquad m\bar{z} = \sum_{i=1}^{n} m_i z_i \tag{14.12$'$}$$

Since $m_i g$ represents the weight of the particle P_i, and mg the total weight of the particles, G is also the center of gravity of the system of particles. However, in order to avoid any confusion, G will be referred to as the *mass center* of the system of particles when properties associated with the *mass* of the particles are being discussed, and as the *center of gravity* of the system when properties associated with the *weight* of the particles are being considered. Particles located outside the gravitational field of the earth, for example, have a mass but no weight. We can then properly refer to their mass center, but obviously not to their center of gravity.†

Differentiating both members of Eq. (14.12) with respect to t, we write

$$m\dot{\bar{\mathbf{r}}} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i$$

or

$$m\bar{\mathbf{v}} = \sum_{i=1}^{n} m_i \mathbf{v}_i \tag{14.13}$$

where $\bar{\mathbf{v}}$ represents the velocity of the mass center G of the system of particles. But the right-hand member of Eq. (14.13) is, by definition, the linear momentum $\mathbf{L}$ of the system (Sec. 14.3). We therefore have

$$\mathbf{L} = m\bar{\mathbf{v}} \tag{14.14}$$

and, differentiating both members with respect to t,

$$\dot{\mathbf{L}} = m\bar{\mathbf{a}} \tag{14.15}$$

†It may also be pointed out that the mass center and the center of gravity of a system of particles do not exactly coincide, since the weights of the particles are directed toward the center of the earth and thus do not truly form a system of parallel forces.

where $\bar{\mathbf{a}}$ represents the acceleration of the mass center G. Substituting for $\dot{\mathbf{L}}$ from (14.15) into (14.10), we write the equation

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{14.16}$$

which defines the motion of the mass center G of the system of particles.

We note that Eq. (14.16) is identical with the equation we would obtain for a particle of mass m equal to the total mass of the particles of the system, acted upon by all the external forces. We therefore state that *the mass center of a system of particles moves as if the entire mass of the system and all the external forces were concentrated at that point*.

This principle is best illustrated by the motion of an exploding shell. We know that if air resistance is neglected, it can be assumed that a shell will travel along a parabolic path. After the shell has exploded, the mass center G of the fragments of shell will continue to travel along the same path. Indeed, point G must move as if the mass and the weight of all fragments were concentrated at G; it must, therefore, move as if the shell had not exploded.

It should be noted that the preceding derivation does not involve the moments of the external forces. Therefore, *it would be wrong to assume* that the external forces are equipollent to a vector $m\bar{\mathbf{a}}$ attached at the mass center G. This is not in general the case since, as you will see in the next section, the sum of the moments about G of the external forces is not in general equal to zero.

14.5 ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES ABOUT ITS MASS CENTER

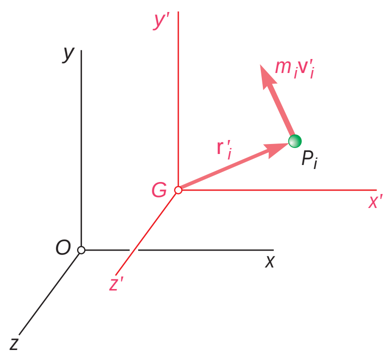


Fig. 14.5

In some applications (for example, in the analysis of the motion of a rigid body) it is convenient to consider the motion of the particles of the system with respect to a centroidal frame of reference $Gx'y'z'$ which translates with respect to the newtonian frame of reference $Oxyz$ (Fig. 14.5). Although a centroidal frame is not, in general, a newtonian frame of reference, it will be seen that the fundamental relation (14.11) holds when the frame $Oxyz$ is replaced by $Gx'y'z'$.

Denoting, respectively, by $\mathbf{r}'_i$ and $\mathbf{v}'_i$ the position vector and the velocity of the particle P_i relative to the moving frame of reference $Gx'y'z'$, we define the *angular momentum* $\mathbf{H}'_G$ of the system of particles *about the mass center* G as follows:

$$\mathbf{H}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}'_i) \tag{14.17}$$

We now differentiate both members of Eq. (14.17) with respect to t. This operation is similar to that performed in Sec. 14.3 on Eq. (14.7), and so we write immediately

$$\dot{\mathbf{H}}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{a}'_i) \tag{14.18}$$

where $\mathbf{a}'_i$ denotes the acceleration of P_i relative to the moving frame of reference. Referring to Sec. 11.12, we write

$$\mathbf{a}_i = \overline{\mathbf{a}} + \mathbf{a}'_i$$

where $\mathbf{a}_i$ and $\overline{\mathbf{a}}$ denote, respectively, the accelerations of P_i and G relative to the frame $Oxyz$. Solving for $\mathbf{a}'_i$ and substituting into (14.18), we have

$$\dot{\mathbf{H}}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{a}_i) - \left(\sum_{i=1}^{n} m_i\mathbf{r}'_i\right) \times \overline{\mathbf{a}} \tag{14.19}$$

But, by (14.12), the second sum in Eq. (14.19) is equal to $m\overline{\mathbf{r}}'$ and thus to zero, since the position vector $\overline{\mathbf{r}}'$ of G relative to the frame $Gx'y'z'$ is clearly zero. On the other hand, since $\mathbf{a}_i$ represents the acceleration of P_i relative to a newtonian frame, we can use Eq. (14.1) and replace $m_i\mathbf{a}_i$ by the sum of the internal forces $\mathbf{f}_{ij}$ and of the resultant $\mathbf{F}_i$ of the external forces acting on P_i. But a reasoning similar to that used in Sec. 14.2 shows that the moment resultant about G of the internal forces $\mathbf{f}_{ij}$ of the entire system is zero. The first sum in Eq. (14.19) therefore reduces to the moment resultant about G of the external forces acting on the particles of the system, and we write

$$\boxed{\Sigma\mathbf{M}_G = \dot{\mathbf{H}}'_G} \tag{14.20}$$

which expresses that *the moment resultant about G of the external forces is equal to the rate of change of the angular momentum about G of the system of particles.*

It should be noted that in Eq. (14.17) we defined the angular momentum $\mathbf{H}'_G$ as the sum of the moments about G of the momenta of the particles $m_i\mathbf{v}'_i$ *in their motion relative to the centroidal frame of reference* $Gx'y'z'$. We may sometimes want to compute the sum $\mathbf{H}_G$ of the moments about G of the momenta of the particles $m_i\mathbf{v}_i$ *in their absolute motion,* i.e., in their motion as observed from the newtonian frame of reference $Oxyz$ (Fig. 14.6):

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}_i) \tag{14.21}$$

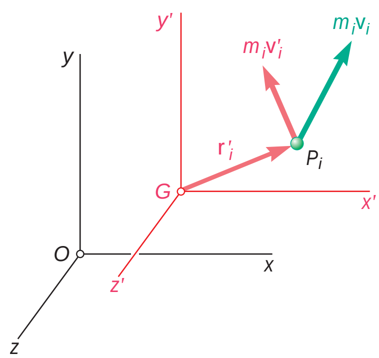


Fig. 14.6

Remarkably, the angular momenta $\mathbf{H}'_G$ and $\mathbf{H}_G$ are identically equal. This can be verified by referring to Sec. 11.12 and writing

$$\mathbf{v}_i = \overline{\mathbf{v}} + \mathbf{v}'_i \tag{14.22}$$

Substituting for $\mathbf{v}_i$ from (14.22) into Eq. (14.21), we have

$$\mathbf{H}_G = \left(\sum_{i=1}^{n} m_i\mathbf{r}'_i\right) \times \overline{\mathbf{v}} + \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}'_i)$$

But, as observed earlier, the first sum is equal to zero. Thus $\mathbf{H}_G$ reduces to the second sum, which, by definition, is equal to $\mathbf{H}'_G$.†

†Note that this property is peculiar to the centroidal frame $Gx'y'z'$ and does not, in general, hold for other frames of reference (see Prob. 14.29).

Taking advantage of the property we have just established, we simplify our notation by dropping the prime (′) from Eq. (14.20) and writing

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{14.23}$$

where it is understood that the angular momentum $\mathbf{H}_G$ can be computed by forming the moments about G of the momenta of the particles in their motion with respect to either the newtonian frame $Oxyz$ or the centroidal frame $Gx'y'z'$:

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i \mathbf{v}_i) = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i \mathbf{v}'_i) \tag{14.24}$$

14.6 CONSERVATION OF MOMENTUM FOR A SYSTEM OF PARTICLES

If no external force acts on the particles of a system, the left-hand members of Eqs. (14.10) and (14.11) are equal to zero and these equations reduce to $\dot{\mathbf{L}} = 0$ and $\dot{\mathbf{H}}_O = 0$. We conclude that

$$\mathbf{L} = \text{constant} \qquad \mathbf{H}_O = \text{constant} \tag{14.25}$$

The equations obtained express that the linear momentum of the system of particles and its angular momentum about the fixed point O are conserved.

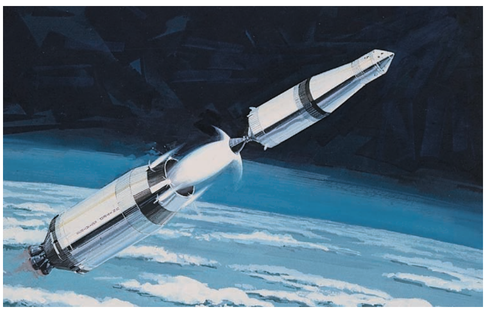


Photo 14.1 If no external forces are acting on the two stages of this rocket, the linear and angular momentum of the system will be conserved.

In some applications, such as problems involving central forces, the moment about a fixed point O of each of the external forces can be zero without any of the forces being zero. In such cases, the second of Eqs. (14.25) still holds; the angular momentum of the system of particles about O is conserved.

The concept of conservation of momentum can also be applied to the analysis of the motion of the mass center G of a system of particles and to the analysis of the motion of the system about G. For example, if the sum of the external forces is zero, the first of Eqs. (14.25) applies. Recalling Eq. (14.14), we write

$$\bar{\mathbf{v}} = \text{constant} \tag{14.26}$$

which expresses that the mass center G of the system moves in a straight line and at a constant speed. On the other hand, if the sum of the moments about G of the external forces is zero, it follows from Eq. (14.23) that the angular momentum of the system about its mass center is conserved:

$$\mathbf{H}_G = \text{constant} \tag{14.27}$$

SAMPLE PROBLEM 14.1

A 200-kg space vehicle is observed at $t = 0$ to pass through the origin of a newtonian reference frame $Oxyz$ with velocity $\mathbf{v}_0 = (150 \text{ m/s})\mathbf{i}$ relative to the frame. Following the detonation of explosive charges, the vehicle separates into three parts A, B, and C, of mass 100 kg, 60 kg, and 40 kg, respectively. Knowing that at $t = 2.5$ s the positions of parts A and B are observed to be $A(555, -180, 240)$ and $B(255, 0, -120)$, where the coordinates are expressed in meters, determine the position of part C at that time.

SOLUTION

Since there is no external force, the mass center G of the system moves with the constant velocity $\mathbf{v}_0 = (150 \text{ m/s})\mathbf{i}$. At $t = 2.5$ s, its position is

$$\bar{\mathbf{r}} = \mathbf{v}_0 t = (150\,\text{m/s})\mathbf{i}(2.5\,\text{s}) = (375\,\text{m})\mathbf{i}$$

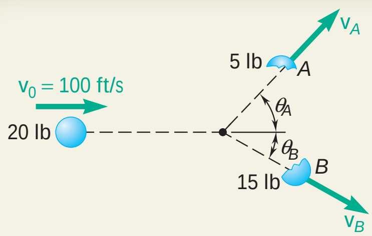
Recalling Eq. (14.12), we write

$$m\bar{\mathbf{r}} = m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C$$

$$(200 \text{ kg})(375 \text{ m})\mathbf{i} = (100 \text{ kg})[(555 \text{ m})\mathbf{i} - (180 \text{ m})\mathbf{j} + (240 \text{ m})\mathbf{k}] + (60 \text{ kg})[(255 \text{ m})\mathbf{i} - (120 \text{ m})\mathbf{k}] + (40 \text{ kg})\mathbf{r}_C$$

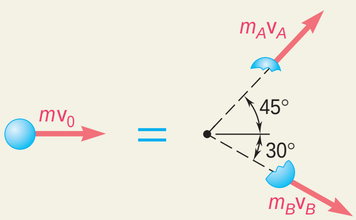
$$\mathbf{r}_C = (105 \text{ m})\mathbf{i} + (450 \text{ m})\mathbf{j} - (420 \text{ m})\mathbf{k} \quad ◀$$

SAMPLE PROBLEM 14.2



A 20-lb projectile is moving with a velocity of 100 ft/s when it explodes into two fragments A and B, weighing 5 lb and 15 lb, respectively. Knowing that immediately after the explosion, fragments A and B travel in directions defined respectively by $\mathrm{u}_A = 45°$ and $\mathrm{u}_B = 30°$, determine the velocity of each fragment.

SOLUTION



Since there is no external force, the linear momentum of the system is conserved, and we write

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m\mathbf{v}_0$$

$$(5/g)\mathbf{v}_A + (15/g)\mathbf{v}_B = (20/g)\mathbf{v}_0$$

$\overset{+}{y}$ x components: $\quad 5v_A \cos 45° + 15v_B \cos 30° = 20(100)$

$+xy$ components: $\quad 5v_A \sin 45° - 15v_B \sin 30° = 0$

Solving simultaneously the two equations for v_A and v_B, we have

$$v_A = 207 \text{ ft/s} \qquad v_B = 97.6 \text{ ft/s}$$

$$\mathbf{v}_A = 207 \text{ ft/s a } 45° \qquad \mathbf{v}_B = 97.6 \text{ ft/s c } 30° \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

This chapter deals with the motion of *systems of particles,* that is, with the motion of a large number of particles considered together, rather than separately. In this first lesson you learned to compute the *linear momentum* and the *angular momentum* of a system of particles. We defined the linear momentum $\mathbf{L}$ of a system of particles as the sum of the linear momenta of the particles and we defined the angular momentum $\mathbf{H}_O$ of the system as the sum of the angular momenta of the particles about O:

$$\mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{v}_i \qquad \mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{v}_i) \tag{14.6, 14.7}$$

In this lesson, you will solve a number of problems of practical interest, either by observing that the linear momentum of a system of particles is conserved or by considering the motion of the mass center of a system of particles.

1. Conservation of the linear momentum of a system of particles. This occurs *when the resultant of the external forces acting on the particles of the system is zero*. You may encounter such a situation in the following types of problems.

a. Problems involving the rectilinear motion of objects such as colliding automobiles and railroad cars. After you have checked that the resultant of the external forces is zero, equate the algebraic sums of the initial momenta and final momenta to obtain an equation which can be solved for one unknown.

b. Problems involving the two-dimensional or three-dimensional motion of objects such as exploding shells, or colliding aircraft, automobiles, or billiard balls. After you have checked that the resultant of the external forces is zero, add vectorially the initial momenta of the objects, add vectorially their final momenta, and equate the two sums to obtain a vector equation expressing that the linear momentum of the system is conserved.

In the case of a two-dimensional motion, this equation can be replaced by two scalar equations which can be solved for two unknowns, while in the case of a three-dimensional motion it can be replaced by three scalar equations which can be solved for three unknowns.

2. Motion of the mass center of a system of particles. You saw in Sec. 14.4 that *the mass center of a system of particles moves as if the entire mass of the system and all of the external forces were concentrated at that point*.

a. In the case of a body exploding while in motion, it follows that the mass center of the resulting fragments moves as the body itself would have moved if the explosion had not occurred. Problems of this type can be solved by writing the equation of motion of the mass center of the system in vectorial form and expressing the position vector of the mass center in terms of the position vectors of the various fragments [Eq. (14.12)]. You can then rewrite the vector equation as two or three scalar equations and solve the equations for an equivalent number of unknowns.

b. In the case of the collision of several moving bodies, it follows that the motion of the mass center of the various bodies is unaffected by the collision. Problems of this type can be solved by writing the equation of motion of the mass center of the system in vectorial form and expressing its position vector before and after the collision in terms of the position vectors of the relevant bodies [Eq. (14.12)]. You can then rewrite the vector equation as two or three scalar equations and solve these equations for an equivalent number of unknowns.

PROBLEMS

14.1 A 30-g bullet is fired with a horizontal velocity of 450 m/s and becomes embedded in block *B* which has a mass of 3 kg. After the impact, block *B* slides on 30-kg carrier *C* until it impacts the end of the carrier. Knowing the impact between *B* and *C* is perfectly plastic and the coefficient of kinetic friction between *B* and *C* is 0.2, determine (*a*) the velocity of the bullet and *B* after the first impact, (*b*) the final velocity of the carrier.

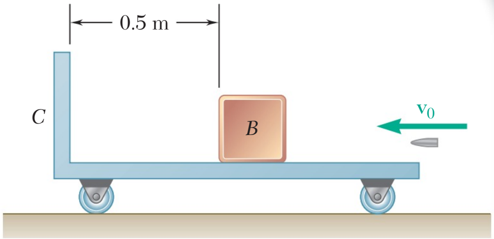


Fig. P14.1

14.2 A 30-g bullet is fired with a horizontal velocity of 450 m/s through 3-kg block *B* and becomes embedded in carrier *C* which has a mass of 30 kg. After the impact, block *B* slides 0.3 m on *C* before coming to rest relative to the carrier. Knowing the coefficient of kinetic friction between *B* and *C* is 0.2, determine (*a*) the velocity of the bullet immediately after passing through *B*, (*b*) the final velocity of the carrier.

14.3 Car *A* weighing 4000 lb and car *B* weighing 3700 lb are at rest on a 22-ton flatcar which is also at rest. Cars *A* and *B* then accelerate and quickly reach constant speeds relative to the flatcar of 7 ft/s and 3.5 ft/s, respectively, before decelerating to a stop at the opposite end of the flatcar. Neglecting friction and rolling resistance, determine the velocity of the flatcar when the cars are moving at constant speeds.

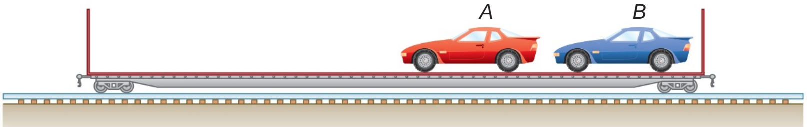


Fig. P14.3

14.4 A bullet is fired with a horizontal velocity of 1500 ft/s through a 6-lb block *A* and becomes embedded in a 4.95-lb block *B*. Knowing that blocks *A* and *B* start moving with velocities of 5 ft/s and 9 ft/s, respectively, determine (*a*) the weight of the bullet, (*b*) its velocity as it travels from block *A* to block *B*.

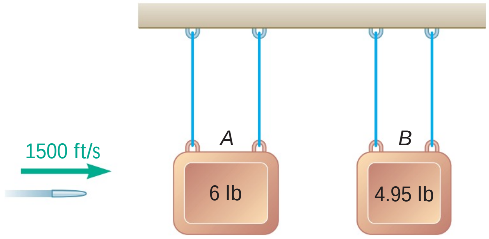


Fig. P14.4

14.5 Two swimmers *A* and *B*, of weight 190 lb and 125 lb, respectively, are at diagonally opposite corners of a floating raft when they realize that the raft has broken away from its anchor. Swimmer *A* immediately starts walking toward *B* at a speed of 2 ft/s relative to the raft. Knowing that the raft weighs 300 lb, determine (*a*) the speed of the raft if *B* does not move, (*b*) the speed with which *B* must walk toward *A* if the raft is not to move.

Fig. *P14.5*

14.6 A 180-lb man and a 120-lb woman stand side by side at the same end of a 300-lb boat, ready to dive, each with a 16-ft/s velocity relative to the boat. Determine the velocity of the boat after they have both dived, if (*a*) the woman dives first, (*b*) the man dives first.

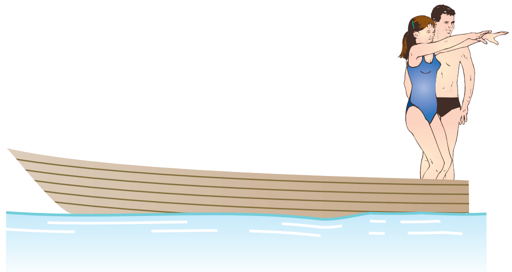


Fig. *P14.6*

14.7 A 40-Mg boxcar A is moving in a railroad switchyard with a velocity of 9 km/h toward cars B and C, which are both at rest with their brakes off at a short distance from each other. Car B is a 25-Mg flatcar supporting a 30-Mg container, and car C is a 35-Mg boxcar. As the cars hit each other they get automatically and tightly coupled. Determine the velocity of car A immediately after each of the two couplings, assuming that the container (*a*) does not slide on the flatcar, (*b*) slides after the first coupling but hits a stop before the second coupling occurs, (*c*) slides and hits the stop only after the second coupling has occurred.

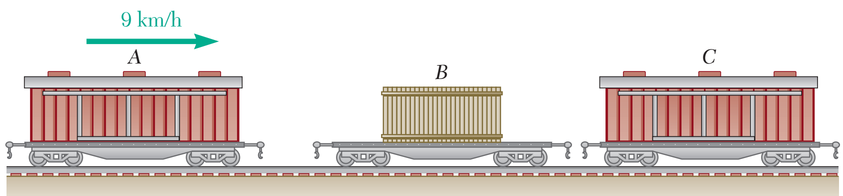


Fig. P14.7

14.8 Packages in an automobile parts supply house are transported to the loading dock by pushing them along on a roller track with very little friction. At the instant shown packages B and C are at rest and package A has a velocity of 2 m/s. Knowing that the coefficient of restitution between the packages is 0.3, determine (*a*) the velocity of package C after A hits B and B hits C, (*b*) the velocity of A after it hits B for the second time.

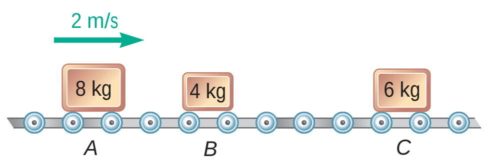


Fig. P14.8

14.9 A system consists of three particles A, B, and C. We know that $m_A = 3$ kg, $m_B = 2$ kg, and $m_C = 4$ kg and that the velocities of the particles expressed in m/s are, respectively, $\mathbf{v}_A = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v}_B = 4\mathbf{i} + 3\mathbf{j}$, and $\mathbf{v}_C = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Determine the angular momentum $\mathbf{H}_O$ of the system about O.

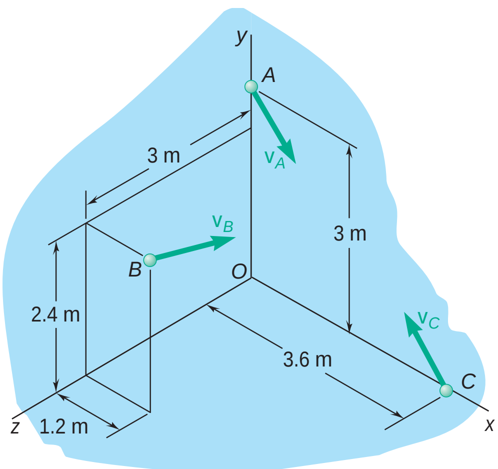


Fig. P14.9

14.10 For the system of particles of Prob. 14.9, determine (*a*) the position vector $\bar{\mathbf{r}}$ of the mass center G of the system, (*b*) the linear momentum $m\bar{\mathbf{v}}$ of the system, (*c*) the angular momentum $\mathbf{H}_G$ of the system about G. Also verify that the answers to this problem and to Prob. 14.9 satisfy the equation given in Prob. 14.27.

14.11 A system consists of three particles A, B, and C. We know that $W_A = 5$ lb, $W_B = 4$ lb, and $W_C = 3$ lb and that the velocities of the particles expressed in ft/s are, respectively, $\mathbf{v}_A = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{v}_B = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$, and $\mathbf{v}_C = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Determine (*a*) the components v_x and v_y of the velocity of particle B for which the angular momentum $\mathbf{H}_O$ of the system about O is parallel to the x axis, (*b*) the value of $\mathbf{H}_O$.

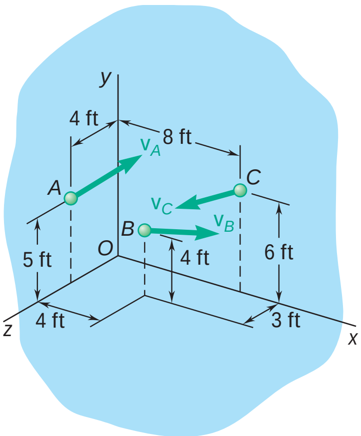


Fig. P14.11

14.12 For the system of particles of Prob. 14.11, determine (*a*) the components v_x and v_z of the velocity of particle B for which the angular momentum $\mathbf{H}_O$ of the system about O is parallel to the z axis, (*b*) the value of $\mathbf{H}_O$.

14.13 A system consists of three particles A, B, and C. We know that $m_A = 3$ kg, $m_B = 4$ kg, and $m_c = 5$ kg and that the velocities of the particles expressed in m/s are, respectively, $\mathbf{v}_A = -4\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, $\mathbf{v}_B = -6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$, and $\mathbf{v}_C = 2\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$. Determine the angular momentum $\mathbf{H}_O$ of the system about O.

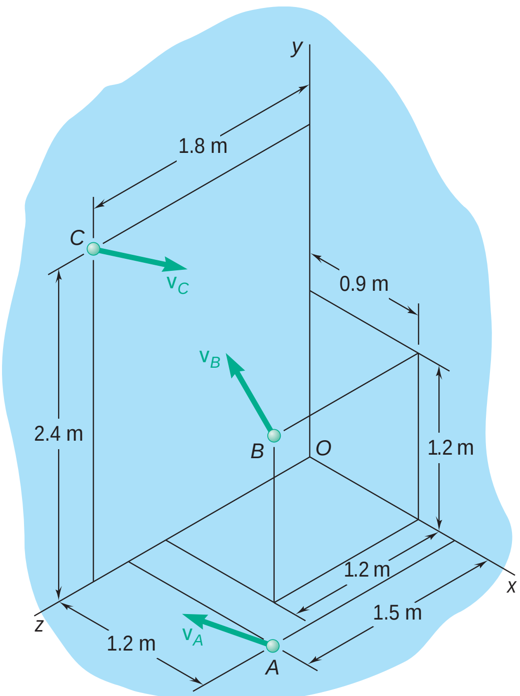


Fig. *P14.13*

14.14 For the system of particles of Prob. 14.13, determine (*a*) the position vector $\bar{\mathbf{r}}$ of the mass center G of the system, (*b*) the linear momentum $m\bar{\mathbf{v}}$ of the system, (*c*) the angular momentum $\mathbf{H}_G$ of the system about G. Also verify that the answers to this problem and to Prob. 14.13 satisfy the equation given in Prob. 14.27.

14.15 A 13-kg projectile is passing through the origin O with a velocity $\mathbf{v}_0 = (35 \text{ m/s})\mathbf{i}$ when it explodes into two fragments A and B, of mass 5 kg and 8 kg, respectively. Knowing that 3 s later the position of fragment A is (90 m, 7 m, −14 m), determine the position of fragment B at the same instant. Assume $a_y = -g = -9.81 \text{ m/s}^2$ and neglect air resistance.

14.16 A 300-kg space vehicle traveling with a velocity $\mathbf{v}_0 = (360 \text{ m/s})\mathbf{i}$ passes through the origin O at $t = 0$. Explosive charges then separate the vehicle into three parts A, B, and C, with mass, respectively, 150 kg, 100 kg, and 50 kg. Knowing that at $t = 4$ s, the positions of parts A and B are observed to be A (1170 m, −290 m, –585 m) and B (1975 m, 365 m, 800 m), determine the corresponding position of part C. Neglect the effect of gravity.

14.17 A 2-kg model rocket is launched vertically and reaches an altitude of 70 m with a speed of 30 m/s at the end of powered flight, time $t = 0$. As the rocket approaches its maximum altitude it explodes into two parts of masses $m_A = 0.7$ kg and $m_B = 1.3$ kg. Part A is observed to strike the ground 80 m west of the launch point at $t = 6$ s. Determine the position of part B at that time.

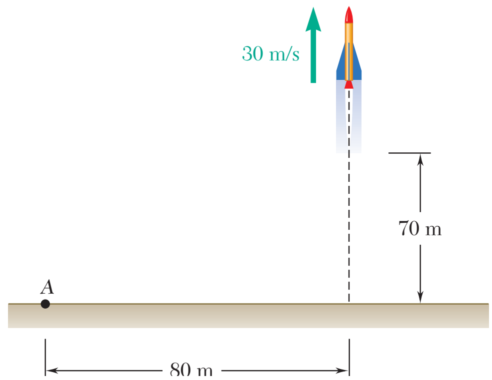


Fig. ***P14.17***

14.18 An 18-kg cannonball and a 12-kg cannonball are chained together and fired horizontally with a velocity of 165 m/s from the top of a 15-m wall. The chain breaks during the flight of the cannonballs and the 12-kg cannonball strikes the ground at $t = 1.5$ s, at a distance of 240 m from the foot of the wall, and 7 m to the right of the line of fire. Determine the position of the other cannonball at that instant. Neglect the resistance of the air.

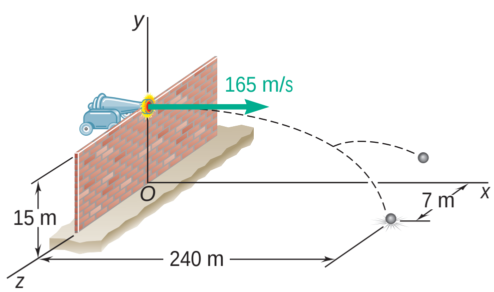


Fig. ***P14.18***

14.19 and 14.20 Car A was traveling east at high speed when it collided at point O with car B, which was traveling north at 45 mi/h. Car C, which was traveling west at 60 mi/h, was 32 ft east and 10 ft north of point O at the time of the collision. Because the pavement was wet, the driver of car C could not prevent his car from sliding into the other two cars, and the three cars, stuck together, kept sliding until they hit the utility pole P. Knowing that the weights of cars A, B, and C are, respectively, 3000 lb, 2600 lb, and 2400 lb, and neglecting the forces exerted on the cars by the wet pavement, solve the problems indicated.

14.19 Knowing that the speed of car A was 75 mi/h and that the time elapsed from the first collision to the stop at P was 2.4 s, determine the coordinates of the utility pole P.

14.20 Knowing that the coordinates of the utility pole are $x_p = 46$ ft and $y_p = 59$ ft, determine (*a*) the time elapsed from the first collision to the stop at P, (*b*) the speed of car A.

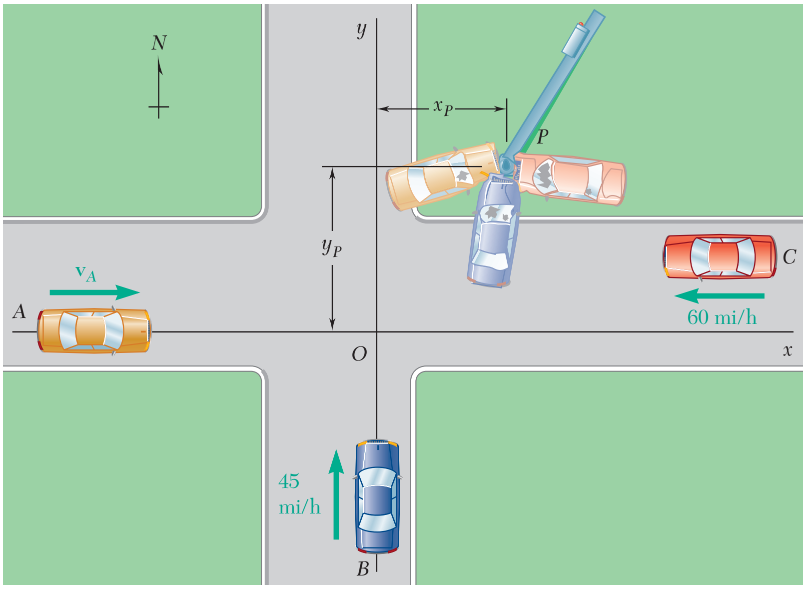


Fig. P14.19 and P14.20

14.21 An expert archer demonstrates his ability by hitting tennis balls thrown by an assistant. A 2-oz tennis ball has a velocity of (32 ft/s)$\mathbf{i}$ − (7 ft/s)$\mathbf{j}$ and is 33 ft above the ground when it is hit by a 1.2-oz arrow traveling with a velocity of (165 ft/s)$\mathbf{j}$ + (230 ft/s)$\mathbf{k}$ where $\mathbf{j}$ is directed upwards. Determine the position P where the ball and arrow will hit the ground, relative to point O located directly under the point of impact.

14.22 Two spheres, each of mass m, can slide freely on a frictionless, horizontal surface. Sphere A is moving at a speed v_0 = 16 ft/s when it strikes sphere B which is at rest and the impact causes sphere B to break into two pieces, each of mass $m/2$. Knowing that 0.7 s after the collision one piece reaches point C and 0.9 s after the collision the other piece reaches point D, determine (*a*) the velocity of sphere A after the collision, (*b*) the angle u and the speeds of the two pieces after the collision.

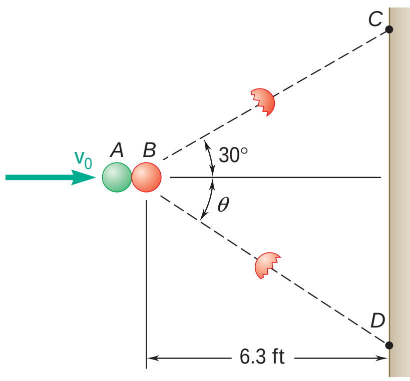


Fig. P14.22

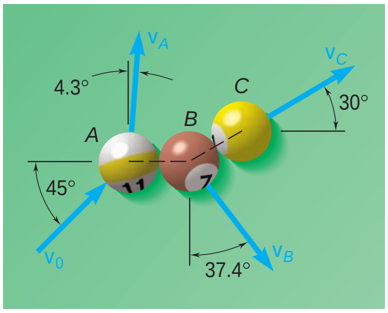


Fig. *P14.23*

14.23 In a game of pool, ball A is moving with a velocity $\mathbf{v}_0$ when it strikes balls B and C which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and that $v_0 = 12$ ft/s and $v_C = 6.29$ ft/s, determine the magnitude of the velocity of (*a*) ball A, (*b*) ball B.

14.24 A 6-kg shell moving with a velocity $\mathbf{v}_0 = (12 \text{ m/s})\mathbf{i} - (9 \text{ m/s})\mathbf{j} - (360 \text{ m/s})\mathbf{k}$ explodes at point D into three fragments A, B, and C of mass, respectively, 3 kg, 2 kg, and 1 kg. Knowing that the fragments hit the vertical wall at the points indicated, determine the speed of each fragment immediately after the explosion. Assume that elevation changes due to gravity may be neglected.

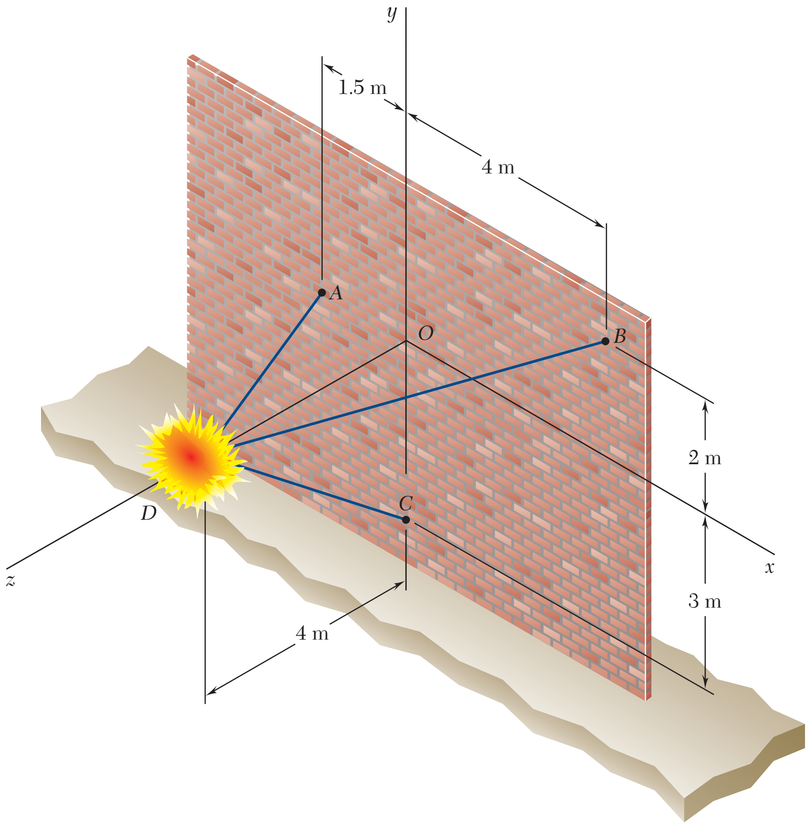


Fig. P14.24 and P14.25

14.25 A 6-kg shell moving with a velocity $\mathbf{v}_0 = (12 \text{ m/s})\mathbf{i} - (9 \text{ m/s})\mathbf{j} - (360 \text{ m/s})\mathbf{k}$ explodes at point D into three fragments A, B, and C of mass, respectively, 2 kg, 1 kg, and 3 kg. Knowing that the fragments hit the vertical wall at the points indicated, determine the speed of each fragment immediately after the explosion. Assume that elevation changes due to gravity may be neglected.

14.26 In a scattering experiment, an alpha particle A is projected with the velocity $\mathbf{u}_0 = -(600 \text{ m/s})\mathbf{i} + (750 \text{ m/s})\mathbf{j} - (800 \text{ m/s})\mathbf{k}$ into a stream of oxygen nuclei moving with a common velocity $\mathbf{v}_0 = (600 \text{ m/s})\mathbf{j}$. After colliding successively with the nuclei B and C, particle A is observed to move along the path defined by the points A_1 (280, 240, 120) and A_2 (360, 320, 160), while nuclei B and C are observed to move along paths defined, respectively, by B_1 (147, 220, 130) and B_2 (114, 290, 120), and by C_1 (240, 232, 90) and C_2 (240, 280, 75). All paths are along straight lines and all coordinates are expressed in millimeters. Knowing that the mass of an oxygen nucleus is four times that of an alpha particle, determine the speed of each of the three particles after the collisions.

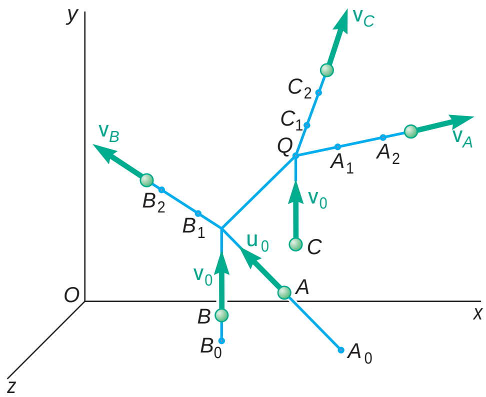


Fig. P14.26

14.27 Derive the relation

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + H_G$$

between the angular momenta $\mathbf{H}_O$ and $\mathbf{H}_G$ defined in Eqs. (14.7) and (14.24), respectively. The vectors $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ define, respectively, the position and velocity of the mass center G of the system of particles relative to the newtonian frame of reference $Oxyz$, and m represents the total mass of the system.

14.28 Show that Eq. (14.23) may be derived directly from Eq. (14.11) by substituting for $\mathbf{H}_O$ the expression given in Prob. 14.27.

14.29 Consider the frame of reference $Ax'y'z'$ in translation with respect to the newtonian frame of reference $Oxyz$. We define the angular momentum $\mathbf{H}'_A$ of a system of n particles about A as the sum

$$\mathbf{H}'_A = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i\mathbf{v}'_i \qquad (1)$$

of the moments about A of the momenta $m_i\mathbf{v}'_i$ of the particles in their motion relative to the frame $Ax'y'z'$. Denoting by $\mathbf{H}_A$ the sum

$$\mathbf{H}_A = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i\mathbf{v}_i$$

of the moments about A of the momenta $m_i\mathbf{v}_i$ of the particles in their motion relative to the newtonian frame $Oxyz$, show that $\mathbf{H}_A = \mathbf{H}'_A$ at a given instant if, and only if, one of the following conditions is satisfied at that instant: (*a*) A has zero velocity with respect to the frame $Oxyz$, (*b*) A coincides with the mass center G of the system, (*c*) the velocity $\mathbf{v}_A$ relative to $Oxyz$ is directed along the line AG.

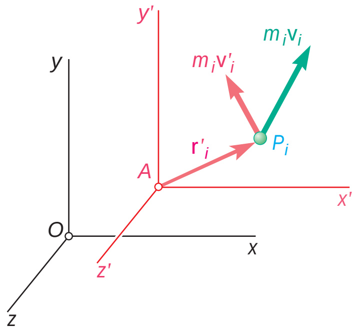


Fig. P14.29

14.30 Show that the relation $\Sigma\mathbf{M}_A = \dot{\mathbf{H}}'_A$, where $\mathbf{H}'_A$ is defined by Eq. (1) of Prob. 14.29 and where $\Sigma\mathbf{M}_A$ represents the sum of the moments about A of the external forces acting on the system of particles, is valid if, and only if, one of the following conditions is satisfied: (*a*) the frame $Ax'y'z'$ is itself a newtonian frame of reference, (*b*) A coincides with the mass center G, (*c*) the acceleration $\mathbf{a}_A$ of A relative to $Oxyz$ is directed along the line AG.

14.7 KINETIC ENERGY OF A SYSTEM OF PARTICLES

The kinetic energy T of a system of particles is defined as the sum of the kinetic energies of the various particles of the system. Referring to Sec. 13.3, we therefore write

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \tag{14.28}$$

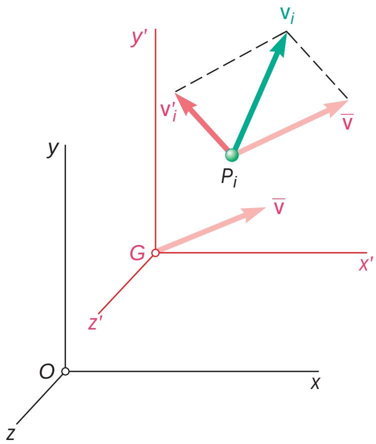


Fig. 14.7

Using a Centroidal Frame of Reference. It is often convenient when computing the kinetic energy of a system comprising a large number of particles (as in the case of a rigid body) to consider separately the motion of the mass center G of the system and the motion of the system relative to a moving frame attached to G.

Let P_i be a particle of the system, $\mathbf{v}_i$ its velocity relative to the newtonian frame of reference $Oxyz$, and $\mathbf{v}'_i$ its velocity relative to the moving frame $Gx'y'z'$ which is in translation with respect to $Oxyz$ (Fig. 14.7). We recall from the preceding section that

$$\mathbf{v}_i = \bar{\mathbf{v}} + \mathbf{v}'_i \tag{14.22}$$

where $\bar{\mathbf{v}}$ denotes the velocity of the mass center G relative to the newtonian frame $Oxyz$. Observing that v_i^2 is equal to the scalar product $\mathbf{v}_i \cdot \mathbf{v}_i$, we express the kinetic energy T of the system relative to the newtonian frame $Oxyz$ as follows:

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2} \sum_{i=1}^{n} (m_i \mathbf{v}_i \cdot \mathbf{v}_i)$$

or, substituting for $\mathbf{v}_i$ from (14.22),

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=1}^{n} [m_i(\bar{\mathbf{v}} + \mathbf{v}'_i) \cdot (\bar{\mathbf{v}} + \mathbf{v}'_i)] \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right)\bar{v}^2 + \bar{\mathbf{v}} \cdot \sum_{i=1}^{n} m_i \mathbf{v}'_i + \frac{1}{2} \sum_{i=1}^{n} m_i v_i'^2 \end{aligned}$$

The first sum represents the total mass m of the system. Recalling Eq. (14.13), we note that the second sum is equal to $m\bar{\mathbf{v}}'$ and thus to zero, since $\bar{\mathbf{v}}'$, which represents the velocity of G relative to the frame $Gx'y'z'$, is clearly zero. We therefore write

$$T = \tfrac{1}{2} m \bar{v}^2 + \frac{1}{2} \sum_{i=1}^{n} m_i v_i'^2 \tag{14.29}$$

This equation shows that the kinetic energy T of a system of particles can be obtained *by adding the kinetic energy of the mass center G* (assuming the entire mass concentrated at G) *and the kinetic energy of the system in its motion relative to the frame $Gx'y'z'$.*

14.8 WORK-ENERGY PRINCIPLE. CONSERVATION OF ENERGY FOR A SYSTEM OF PARTICLES

The principle of work and energy can be applied to each particle P_i of a system of particles. We write

$$T_1 + U_{1 y 2} = T_2 \tag{14.30}$$

for each particle P_i, where $U_{1 y 2}$ represents the work done by the internal forces $\mathbf{f}_{ij}$ and the resultant external force $\mathbf{F}_i$ acting on P_i. Adding the kinetic energies of the various particles of the system and considering the work of all the forces involved, we can apply Eq. (14.30) to the entire system. The quantities T_1 and T_2 now represent the kinetic energy of the entire system and can be computed from either Eq. (14.28) or Eq. (14.29). The quantity $U_{1 y 2}$ represents the work of all the forces acting on the particles of the system. Note that while the internal forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ are equal and opposite, the work of these forces will not, in general, cancel out, since the particles P_i and P_j on which they act will, in general, undergo different displacements. Therefore, in computing $U_{1 y 2}$, *we must consider the work of the internal forces* $\mathbf{f}_{ij}$ *as well as the work of the external forces* $\mathbf{F}_i$.

If all the forces acting on the particles of the system are conservative, Eq. (14.30) can be replaced by

$$T_1 + V_1 = T_2 + V_2 \tag{14.31}$$

where V represents the potential energy associated with the internal and external forces acting on the particles of the system. Equation (14.31) expresses the principle of *conservation of energy* for the system of particles.

14.9 PRINCIPLE OF IMPULSE AND MOMENTUM FOR A SYSTEM OF PARTICLES

Integrating Eqs. (14.10) and (14.11) in t from t_1 to t_2, we write

$$\sum \int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{L}_2 - \mathbf{L}_1 \tag{14.32}$$

$$\sum \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1 \tag{14.33}$$

Recalling the definition of the linear impulse of a force given in Sec. 13.10, we observe that the integrals in Eq. (14.32) represent the linear impulses of the external forces acting on the particles of the system. We shall refer in a similar way to the integrals in Eq. (14.33) as the *angular impulses* about O of the external forces. Thus, Eq. (14.32) expresses that the sum of the linear impulses of the external forces acting on the system is equal to the change in linear momentum of the system. Similarly, Eq. (14.33) expresses that the sum of the angular impulses about O of the external forces is equal to the change in angular momentum about O of the system.

Photo 14.2 When a golf ball is hit out of a sand trap, some of the momentum of the club is transferred to the golf ball and any sand that is hit.

In order to make clear the physical significance of Eqs. (14.32) and (14.33), we will rearrange the terms in these equations and write

$$\mathbf{L}_1 + \sum \int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{L}_2 \tag{14.34}$$

$$(\mathbf{H}_O)_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 \tag{14.35}$$

In parts *a* and *c* of Fig. 14.8 we have sketched the momenta of the particles of the system at times t_1 and t_2, respectively. In part *b* we have shown a vector equal to the sum of the linear impulses of the external forces and a couple of moment equal to the sum of the angular impulses about O of the external forces. For simplicity, the particles have been assumed to move in the plane of the figure, but the present discussion remains valid in the case of particles moving in space. Recalling from Eq. (14.6) that $\mathbf{L}$, by definition, is the resultant of the momenta $m_i\mathbf{v}_i$, we note that Eq. (14.34) expresses that the resultant of the vectors shown in parts *a* and *b* of Fig. 14.8 is equal to the resultant of the vectors shown in part *c* of the same figure. Recalling from Eq. (14.7) that $\mathbf{H}_O$ is the moment resultant of the momenta $m_i\mathbf{v}_i$, we note that Eq. (14.35) similarly expresses that the moment resultant of the vectors in parts *a* and *b* of Fig. 14.8 is equal to the moment resultant of the vectors in part *c*. Together, Eqs. (14.34) and (14.35) thus express that *the momenta of the particles at time t_1 and the impulses of the external forces from t_1 to t_2 form a system of vectors equipollent to the system of the momenta of the particles at time t_2*. This has been indicated in Fig. 14.8 by the use of blue plus and equals signs.

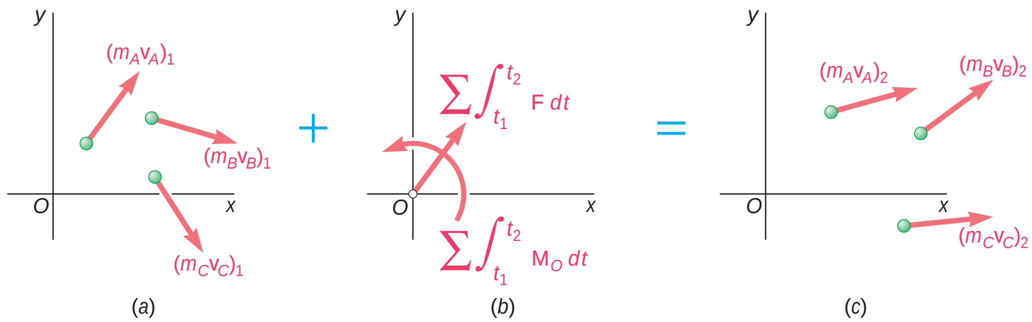


Fig. 14.8

If no external force acts on the particles of the system, the integrals in Eqs. (14.34) and (14.35) are zero, and these equations yield

$$\mathbf{L}_1 = \mathbf{L}_2 \tag{14.36}$$

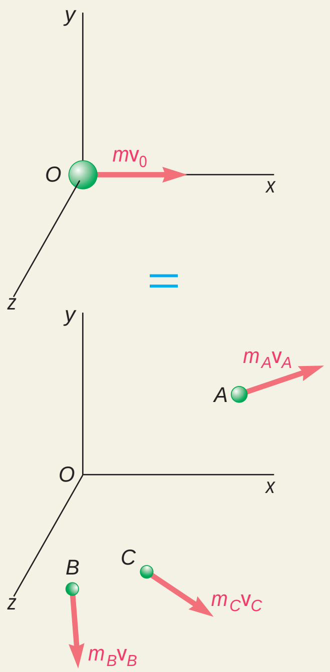
$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \tag{14.37}$$

We thus check the result obtained in Sec. 14.6: If no external force acts on the particles of a system, the linear momentum and the angular momentum about O of the system of particles are conserved. The system of the initial momenta is equipollent to the system of the final momenta, and it follows that the angular momentum of the system of particles about *any* fixed point is conserved.

SAMPLE PROBLEM 14.3

For the 200-kg space vehicle of Sample Prob. 14.1, it is known that at t = 2.5 s, the velocity of part A is $\mathbf{v}_A = (270 \text{ m/s})\mathbf{i} - (120 \text{ m/s})\mathbf{j} + (160 \text{ m/s})\mathbf{k}$ and the velocity of part B is parallel to the xz plane. Determine the velocity of part C.

SOLUTION



Since there is no external force, the initial momentum $m\mathbf{v}_0$ is equipollent to the system of the final momenta. Equating first the sums of the vectors in both parts of the adjoining sketch, and then the sums of their moments about O, we write

$$\mathbf{L}_1 = \mathbf{L}_2: \qquad m\mathbf{v}_0 = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C \tag{1}$$

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2: \qquad 0 = \mathbf{r}_A \times m_A\mathbf{v}_A + \mathbf{r}_B \times m_B\mathbf{v}_B + \mathbf{r}_C \times m_C\mathbf{v}_C \tag{2}$$

Recalling from Sample Prob. 14.1 that $\mathbf{v}_0 = (150 \text{ m/s})\mathbf{i}$,

$$m_A = 100 \text{ kg} \qquad m_B = 60 \text{ kg} \qquad m_C = 40 \text{ kg}$$
$$\mathbf{r}_A = (555 \text{ m})\mathbf{i} - (180 \text{ m})\mathbf{j} + (240 \text{ m})\mathbf{k}$$
$$\mathbf{r}_B = (255 \text{ m})\mathbf{i} - (120 \text{ m})\mathbf{k}$$
$$\mathbf{r}_C = (105 \text{ m})\mathbf{i} + (450 \text{ m})\mathbf{j} - (420 \text{ m})\mathbf{k}$$

and using the information given in the statement of this problem, we rewrite Eqs. (1) and (2) as follows:

$$200(150\mathbf{i}) = 100(270\mathbf{i} - 120\mathbf{j} + 160\mathbf{k}) + 60[(v_B)_x\mathbf{i} + (v_B)_z\mathbf{k}] + 40[(v_C)_x\mathbf{i} + (v_C)_y\mathbf{j} + (v_C)_z\mathbf{k}] \tag{1'}$$

$$0 = 100\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 555 & -180 & 240 \\ 270 & -120 & 160 \end{vmatrix} + 60\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 255 & 0 & -120 \\ (v_B)_x & 0 & (v_B)_z \end{vmatrix} + 40\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 105 & 450 & -420 \\ (v_C)_x & (v_C)_y & (v_C)_z \end{vmatrix} \tag{2'}$$

Equating to zero the coefficient of $\mathbf{j}$ in (1′) and the coefficients of $\mathbf{i}$ and $\mathbf{k}$ in (2′), we write, after reductions, the three scalar equations

$$(v_C)_y - 300 = 0$$
$$450(v_C)_z + 420(v_C)_y = 0$$
$$105(v_C)_y - 450(v_C)_x - 45\,000 = 0$$

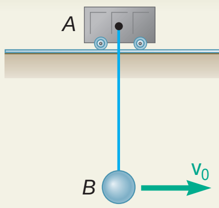
which yield, respectively,

$$(v_C)_y = 300 \qquad (v_C)_z = -280 \qquad (v_C)_x = -30$$

The velocity of part C is thus

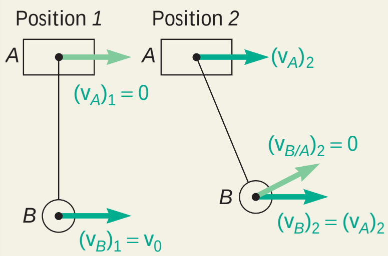
$$\mathbf{v}_C = -(30 \text{ m/s})\mathbf{i} + (300 \text{ m/s})\mathbf{j} - (280 \text{ m/s})\mathbf{k}$$ ◀

SAMPLE PROBLEM 14.4



Ball B, of mass m_B, is suspended from a cord of length l attached to cart A, of mass m_A, which can roll freely on a frictionless horizontal track. If the ball is given an initial horizontal velocity $\mathbf{v}_0$ while the cart is at rest, determine (*a*) the velocity of B as it reaches its maximum elevation, (*b*) the maximum vertical distance h through which B will rise. (It is assumed that $v_0^2 < 2gl$.)

SOLUTION

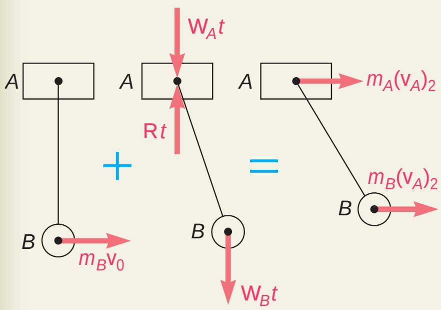


The impulse-momentum principle and the principle of conservation of energy will be applied to the cart-ball system between its initial position *1* and position *2*, when B reaches its maximum elevation.

Velocities ***Position 1:*** $(\mathbf{v}_A)_1 = 0 \qquad (\mathbf{v}_B)_1 = \mathbf{v}_0$ (1)

Position 2: When ball B reaches its maximum elevation, its velocity $(\mathbf{v}_{B/A})_2$ relative to its support A is zero. Thus, at that instant, its absolute velocity is

$$(\mathbf{v}_B)_2 = (\mathbf{v}_A)_2 + (\mathbf{v}_{B/A})_2 = (\mathbf{v}_A)_2 \tag{2}$$



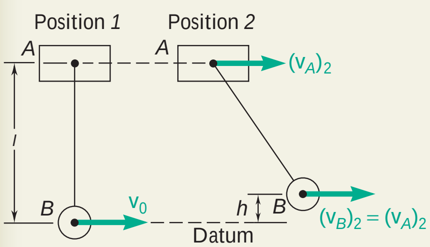
Impulse-Momentum Principle. Noting that the external impulses consist of $\mathbf{W}_At$, $\mathbf{W}_Bt$, and $\mathbf{R}t$, where $\mathbf{R}$ is the reaction of the track on the cart, and recalling (1) and (2), we draw the impulse-momentum diagram and write

$$\Sigma m\mathbf{v}_1 + \Sigma\ \mathbf{Ext\ Imp}_{1\rightarrow 2} = \Sigma m\mathbf{v}_2$$

$\xrightarrow{+}$ x components: $\qquad m_Bv_0 = (m_A + m_B)(v_A)_2$

which expresses that the linear momentum of the system is conserved in the horizontal direction. Solving for $(v_A)_2$:

$$(v_A)_2 = \frac{m_B}{m_A + m_B}v_0 \qquad (\mathbf{v}_B)_2 = (\mathbf{v}_A)_2 = \frac{m_B}{m_A + m_B}v_0 \rightarrow \quad ◀$$



Conservation of Energy

Position 1.	***Potential Energy:***	$V_1 = m_Agl$
	Kinetic Energy:	$T_1 = \frac{1}{2}m_Bv_0^2$
Position 2.	***Potential Energy:***	$V_2 = m_Agl + m_Bgh$
	Kinetic Energy:	$T_2 = \frac{1}{2}(m_A + m_B)(v_A)_2^2$

$$T_1 + V_1 = T_2 + V_2: \qquad \tfrac{1}{2}m_Bv_0^2 + m_Agl = \tfrac{1}{2}(m_A + m_B)(v_A)_2^2 + m_Agl + m_Bgh$$

Solving for h, we have

$$h = \frac{v_0^2}{2g} - \frac{m_A + m_B}{m_B}\frac{(v_A)_2^2}{2g}$$

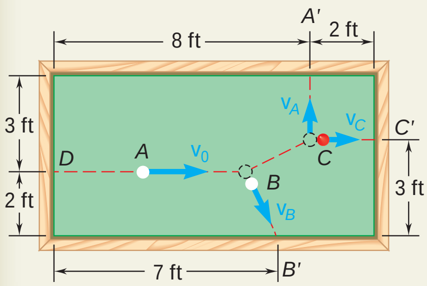
or, substituting for $(v_A)_2$ the expression found above,

$$h = \frac{v_0^2}{2g} - \frac{m_B}{m_A + m_B}\frac{v_0^2}{2g} \qquad h = \frac{m_A}{m_A + m_B}\frac{v_0^2}{2g} \quad ◀$$

Remarks. (1) Recalling that $v_0^2 < 2gl$, it follows from the last equation that $h < l$; we thus check that B stays below A as assumed in our solution.

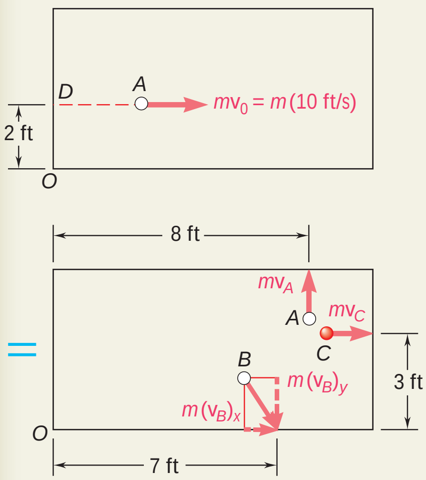
(2) For $m_A \gg m_B$, the answers obtained reduce to $(\mathbf{v}_B)_2 = (\mathbf{v}_A)_2 = 0$ and $h = v_0^2/2g$; B oscillates as a simple pendulum with A fixed. For $m_A \ll m_B$, they reduce to $(\mathbf{v}_B)_2 = (\mathbf{v}_A)_2 = \mathbf{v}_0$ and $h = 0$; A and B move with the same constant velocity $\mathbf{v}_0$.

SAMPLE PROBLEM 14.5



In a game of billiards, ball A is given an initial velocity $\mathbf{v}_0$ of magnitude $v_0 = 10$ ft/s along line DA parallel to the axis of the table. It hits ball B and then ball C, which are both at rest. Knowing that A and C hit the sides of the table squarely at points A' and C', respectively, that B hits the side obliquely at B', and assuming frictionless surfaces and perfectly elastic impacts, determine the velocities $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$ with which the balls hit the sides of the table. (*Remark:* In this sample problem and in several of the problems which follow, the billiard balls are assumed to be particles moving freely in a horizontal plane, rather than the rolling and sliding spheres they actually are.)

SOLUTION



Conservation of Momentum. Since there is no external force, the initial momentum $m\mathbf{v}_0$ is equipollent to the system of momenta after the two collisions (and before any of the balls hits the side of the table). Referring to the adjoining sketch, we write

$\xrightarrow{+}$ x components: $\quad m(10 \text{ ft/s}) = m(v_B)_x + mv_C \qquad (1)$

$+\uparrow y$ components: $\quad 0 = mv_A - m(v_B)_y \qquad (2)$

$+\circlearrowleft$ moments about O: $\quad -(2 \text{ ft})m(10 \text{ ft/s}) = (8 \text{ ft})mv_A - (7 \text{ ft})m(v_B)_y - (3 \text{ ft})mv_C \qquad (3)$

Solving the three equations for v_A, $(v_B)_x$, and $(v_B)_y$ in terms of v_C,

$$v_A = (v_B)_y = 3v_C - 20 \qquad (v_B)_x = 10 - v_C \tag{4}$$

Conservation of Energy. Since the surfaces are frictionless and the impacts are perfectly elastic, the initial kinetic energy $\frac{1}{2}mv_0^2$ is equal to the final kinetic energy of the system:

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2 + \tfrac{1}{2}m_C v_C^2$$
$$v_A^2 + (v_B)_x^2 + (v_B)_y^2 + v_C^2 = (10 \text{ ft/s})^2 \tag{5}$$

Substituting for v_A, $(v_B)_x$, and $(v_B)_y$ from (4) into (5), we have

$$2(3v_C - 20)^2 + (10 - v_C)^2 + v_C^2 = 100$$
$$20v_C^2 - 260v_C + 800 = 0$$

Solving for v_C, we find $v_C = 5$ ft/s and $v_C = 8$ ft/s. Since only the second root yields a positive value for v_A after substitution into Eqs. (4), we conclude that $v_C = 8$ ft/s and

$$v_A = (v_B)_y = 3(8) - 20 = 4 \text{ ft/s} \qquad (v_B)_x = 10 - 8 = 2 \text{ ft/s}$$

$$\mathbf{v}_A = 4 \text{ ft/s}\uparrow \qquad \mathbf{v}_B = 4.47 \text{ ft/s} \searrow 63.4° \qquad \mathbf{v}_C = 8 \text{ ft/s} \rightarrow \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In the preceding lesson we defined the linear momentum and the angular momentum of a system of particles. In this lesson we defined the *kinetic energy* T of a system of particles:

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \qquad (14.28)$$

The solutions of the problems in the preceding lesson were based on the conservation of the linear momentum of a system of particles or on the observation of the motion of the mass center of a system of particles. In this lesson you will solve problems involving the following:

1. Computation of the kinetic energy lost in collisions. The kinetic energy T_1 of the system of particles before the collisions and its kinetic energy T_2 after the collisions are computed from Eq. (14.28) and are subtracted from each other. Keep in mind that, while linear momentum and angular momentum are vector quantities, kinetic energy is a *scalar* quantity.

2. Conservation of linear momentum and conservation of energy. As you saw in the preceding lesson, when the resultant of the external forces acting on a system of particles is zero, the linear momentum of the system is conserved. In problems involving two-dimensional motion, expressing that the initial linear momentum and the final linear momentum of the system are equipollent yields two algebraic equations. Equating the initial total energy of the system of particles (including potential energy as well as kinetic energy) to its final total energy yields an additional equation. Thus, you can write three equations which can be solved for three unknowns [Sample Prob. 14.5]. Note that if the resultant of the external forces is not zero but has a fixed direction, the component of the linear momentum in a direction perpendicular to the resultant is still conserved; the number of equations which can be used is then reduced to two [Sample Prob. 14.4].

3. Conservation of linear and angular momentum. When no external forces act on a system of particles, both the linear momentum of the system and its angular momentum about some arbitrary point are conserved. In the case of three-dimensional motion, this will enable you to write as many as six equations, although you may need to solve only some of them to obtain the desired answers [Sample Prob. 14.3]. In the case of two-dimensional motion, you will be able to write three equations which can be solved for three unknowns.

4. Conservation of linear and angular momentum and conservation of energy. In the case of the two-dimensional motion of a system of particles which are not subjected to any external forces, you will obtain two algebraic equations by expressing that the linear momentum of the system is conserved, one equation by writing that the angular momentum of the system about some arbitrary point is conserved, and a fourth equation by expressing that the total energy of the system is conserved. These equations can be solved for four unknowns.

PROBLEMS

14.31 Determine the energy lost due to friction and the impacts for Prob. 14.1.

14.32 In Prob. 14.4, determine the energy lost as the bullet (*a*) passes through block A, (*b*) becomes embedded in block B.

14.33 In Prob. 14.6, determine the work done by the woman and by the man as each dives from the boat, assuming that the woman dives first.

14.34 Determine the energy lost as a result of the series of collisions described in Prob. 14.8.

14.35 Two automobiles A and B, of mass m_A and m_B, respectively, are traveling in opposite directions when they collide head on. The impact is assumed perfectly plastic, and it is further assumed that the energy absorbed by each automobile is equal to its loss of kinetic energy with respect to a moving frame of reference attached to the mass center of the two-vehicle system. Denoting by E_A and E_B, respectively, the energy absorbed by automobile A and by automobile B, (*a*) show that $E_A/E_B = m_B/m_A$, that is, the amount of energy absorbed by each vehicle is inversely proportional to its mass, (*b*) compute E_A and E_B, knowing that $m_A = 1600$ kg and $m_B = 900$ kg and that the speeds of A and B are, respectively, 90 km/h and 60 km/h.

Fig. P14.35

14.36 It is assumed that each of the two automobiles involved in the collision described in Prob. 14.35 had been designed to safely withstand a test in which it crashed into a solid, immovable wall at the speed v_0. The severity of the collision of Prob. 14.35 may then be measured for each vehicle by the ratio of the energy it absorbed in the collision to the energy it absorbed in the test. On that basis, show that the collision described in Prob. 14.35 is $(m_A/m_B)^2$ times more severe for automobile B than for automobile A.

14.37 Solve Sample Prob. 14.4, assuming that cart A is given an initial horizontal velocity $\mathbf{v}_0$ while ball B is at rest.

14.38 Two hemispheres are held together by a cord which maintains a spring under compression (the spring is not attached to the hemispheres). The potential energy of the compressed spring is 120 J and the assembly has an initial velocity $\mathbf{v}_0$ of magnitude $v_0 = 8$ m/s. Knowing that the cord is severed when $\theta = 30°$, causing the hemispheres to fly apart, determine the resulting velocity of each hemisphere.

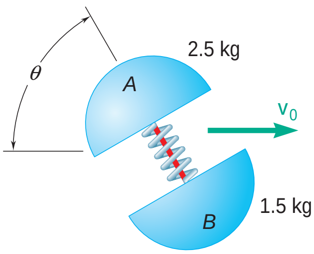


Fig. P14.38

14.39 A 15-lb block B starts from rest and slides on the 25-lb wedge A, which is supported by a horizontal surface. Neglecting friction, determine (*a*) the velocity of B relative to A after it has slid 3 ft down the inclined surface of the wedge, (*b*) the corresponding velocity of A.

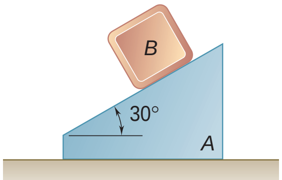


Fig. P14.39

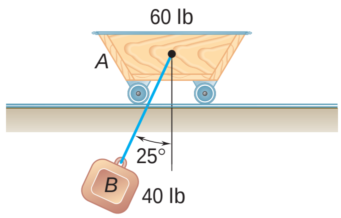


Fig. P14.40

14.40 A 40-lb block B is suspended from a 6-ft cord attached to a 60-lb cart A, which may roll freely on a frictionless, horizontal track. If the system is released from rest in the position shown, determine the velocities of A and B as B passes directly under A.

14.41 and 14.42 In a game of pool, ball A is moving with a velocity $\mathbf{v}_0$ of magnitude $v_0 = 15$ ft/s when it strikes balls B and C, which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and assuming frictionless surfaces and perfectly elastic impact (i.e., conservation of energy), determine the magnitudes of the velocities $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$.

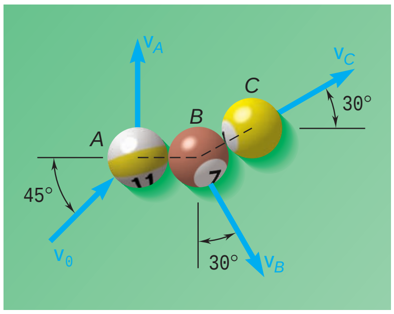


Fig. P14.41

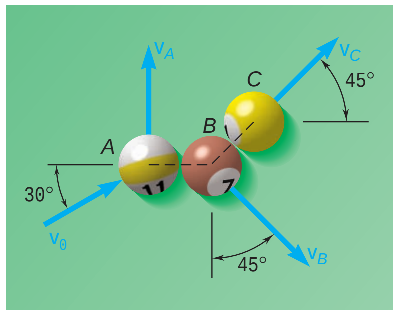


Fig. P14.42

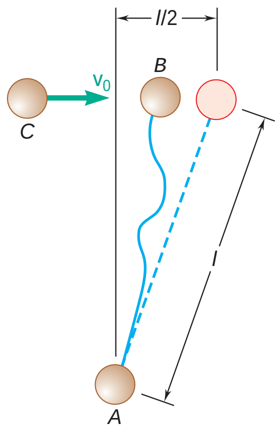


Fig. P14.43

14.43 Three spheres, each of mass m, can slide freely on a frictionless, horizontal surface. Spheres A and B are attached to an inextensible, inelastic cord of length l and are at rest in the position shown when sphere B is struck squarely by sphere C which is moving to the right with a velocity $\mathbf{v}_0$. Knowing that the cord is slack when sphere B is struck by sphere C and assuming perfectly elastic impact between B and C, determine (*a*) the velocity of each sphere immediately after the cord becomes taut, (*b*) the fraction of the initial kinetic energy of the system which is dissipated when the cord becomes taut.

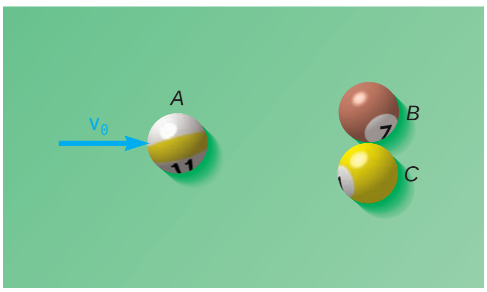


Fig. P14.44

14.44 In a game of pool, ball A is moving with the velocity $\mathbf{v}_0 = v_0\mathbf{i}$ when it strikes balls B and C, which are at rest side by side. Assuming frictionless surfaces and perfectly elastic impact (i.e., conservation of energy), determine the final velocity of each ball, assuming that the path of A is (*a*) perfectly centered and that A strikes B and C simultaneously, (*b*) not perfectly centered and that A strikes B slightly before it strikes C.

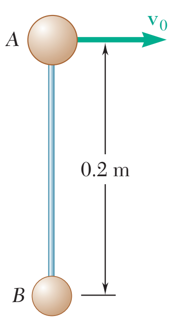


Fig. P14.45

14.45 Two small spheres A and B, of mass 2.5 kg and 1 kg, respectively, are connected by a rigid rod of negligible mass. The two spheres are resting on a horizontal, frictionless surface when A is suddenly given the velocity $\mathbf{v}_0 = (3.5 \text{ m/s})\mathbf{i}$. Determine (*a*) the linear momentum of the system and its angular momentum about its mass center G, (*b*) the velocities of A and B after the rod AB has rotated through 180°.

14.46 A 900-lb space vehicle traveling with a velocity $\mathbf{v}_0 = (1500 \text{ ft/s})\mathbf{k}$ passes through the origin O. Explosive charges then separate the vehicle into three parts A, B, and C, with masses of 150 lb, 300 lb, and 450 lb, respectively. Knowing that shortly thereafter the positions of the three parts are, respectively, $A(250, 250, 2250)$, $B(600, 1300, 3200)$, and $C(-475, -950, 1900)$, where the coordinates are expressed in feet, that the velocity of B is $\mathbf{v}_B = (500 \text{ ft/s})\mathbf{i} + (1100 \text{ ft/s})\mathbf{j} + (2100 \text{ ft/s})\mathbf{k}$, and that the x component of the velocity of C is -400 ft/s, determine the velocity of part A.

14.47 Four small disks A, B, C, and D can slide freely on a frictionless horizontal surface. Disks B, C, and D are connected by light rods and are at rest in the position shown when disk B is struck squarely by disk A which is moving to the right with a velocity $\mathbf{v}_0 = (38.5 \text{ ft/s})\mathbf{i}$. The weights of the disks are $W_A = W_B = W_C = 15$ lb, and $W_D = 30$ lb. Knowing that the velocities of the disks immediately after the impact are $\mathbf{v}_A = \mathbf{v}_B = (8.25 \text{ ft/s})\mathbf{i}$, $\mathbf{v}_C = v_C\mathbf{i}$, and $\mathbf{v}_D = v_D\mathbf{i}$, determine (*a*) the speeds v_C and v_D, (*b*) the fraction of the initial kinetic energy of the system which is dissipated during the collision.

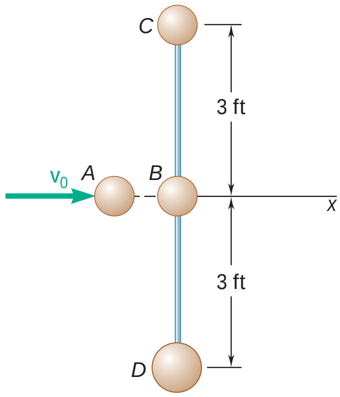


Fig. P14.47

14.48 In the scattering experiment of Prob. 14.26, it is known that the alpha particle is projected from $A_0(300, 0, 300)$ and that it collides with the oxygen nucleus C at $Q(240, 200, 100)$, where all coordinates are expressed in millimeters. Determine the coordinates of point B_0 where the original path of nucleus B intersects the zx plane. (*Hint.* Express that the angular momentum of the three particles about Q is conserved.)

14.49 Three identical small spheres, each of weight 2 lb, can slide freely on a horizontal frictionless surface. Spheres B and C are connected by a light rod and are at rest in the position shown when sphere B is struck squarely by sphere A which is moving to the right with a velocity $\mathbf{v}_0 = (8 \text{ ft/s})\mathbf{i}$. Knowing that u = 45° and that the velocities of spheres A and B immediately after the impact are $\mathbf{v}_A = 0$ and $\mathbf{v}_B = (6 \text{ ft/s})\mathbf{i} + (\mathbf{v}_B)_y\mathbf{j}$, determine $(v_B)_y$ and the velocity of C immediately after impact.

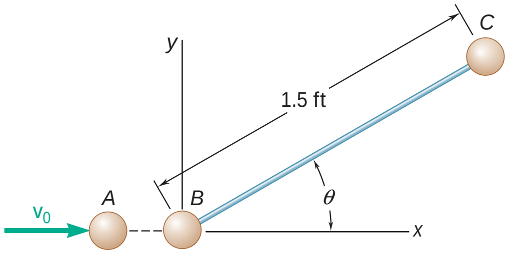


Fig. *P14.49*

14.50 Three small spheres A, B, and C, each of mass m, are connected to a small ring D of negligible mass by means of three inextensible, inelastic cords of length l. The spheres can slide freely on a frictionless horizontal surface and are rotating initially at a speed v_0 about ring D which is at rest. Suddenly the cord CD breaks. After the other two cords have again become taut, determine (*a*) the speed of ring D, (*b*) the relative speed at which spheres A and B rotate about D, (*c*) the fraction of the original energy of spheres A and B which is dissipated when cords AD and BD again became taut.

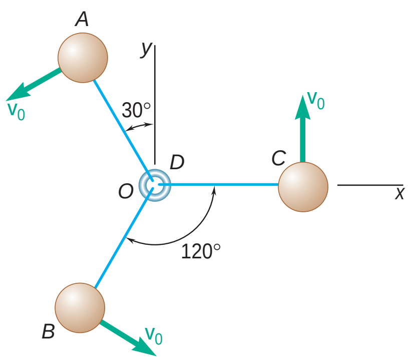


Fig. *P14.50*

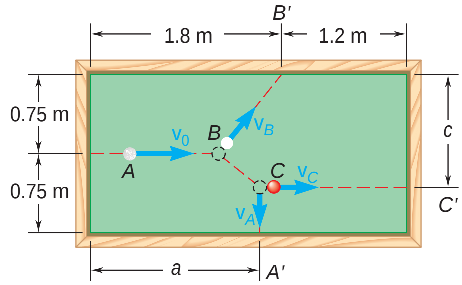


Fig. P14.51

14.51 In a game of billiards, ball A is given an initial velocity $\mathbf{v}_0$ along the longitudinal axis of the table. It hits ball B and then ball C, which are both at rest. Balls A and C are observed to hit the sides of the table squarely at A' and C', respectively, and ball B is observed to hit the side obliquely at B'. Knowing that $v_0 = 4$ m/s, $v_A = 1.92$ m/s, and $a = 1.65$ m, determine (*a*) the velocities $\mathbf{v}_B$ and $\mathbf{v}_C$ of balls B and C, (*b*) the point C' where ball C hits the side of the table. Assume frictionless surfaces and perfectly elastic impacts (i.e., conservation of energy).

14.52 For the game of billiards of Prob. 14.51, it is now assumed that $v_0 = 5$ m/s, $v_C = 3.2$ m/s, and $c = 1.22$ m. Determine (*a*) the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of balls A and B, (*b*) the point A' where ball A hits the side of the table.

14.53 Two small disks A and B, of mass 3 kg and 1.5 kg, respectively, may slide on a horizontal, frictionless surface. They are connected by a cord, 600 mm long, and spin counterclockwise about their mass center G at the rate of 10 rad/s. At $t = 0$, the coordinates of G are $\bar{x}_0 = 0$, $\bar{y}_0 = 2$ m, and its velocity $\bar{\mathbf{v}}_0 = (1.2 \text{ m/s})\mathbf{i} + (0.96 \text{ m/s})\mathbf{j}$. Shortly thereafter the cord breaks; disk A is then observed to move along a path parallel to the y axis and disk B along a path which intersects the x axis at a distance $b = 7.5$ m from O. Determine (*a*) the velocities of A and B after the cord breaks, (*b*) the distance a from the y axis to the path of A.

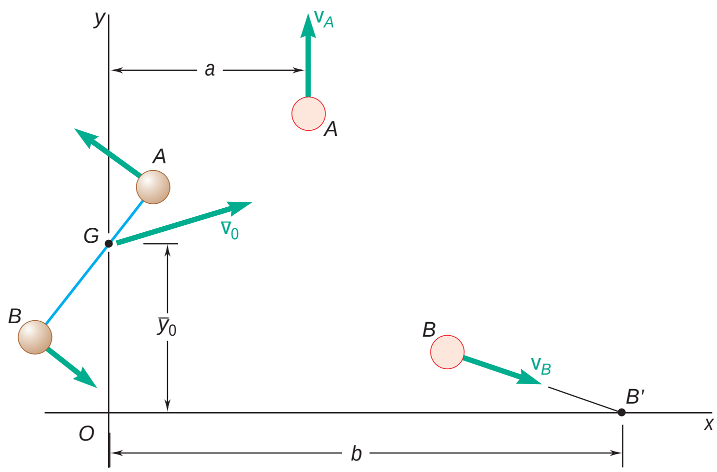


Fig. *P14.53* and *P14.54*

14.54 Two small disks A and B, of mass 2 kg and 1 kg, respectively, may slide on a horizontal and frictionless surface. They are connected by a cord of negligible mass and spin about their mass center G. At $t = 0$, G is moving with the velocity $\bar{\mathbf{v}}_0$ and its coordinates are $\bar{x}_0 = 0$, $\bar{y}_0 = 1.89$ m. Shortly thereafter, the cord breaks and disk A is observed to move with a velocity $\mathbf{v}_A = (5 \text{ m/s})\mathbf{j}$ in a straight line and at a distance $a = 2.56$ m from the y axis, while B moves with a velocity $\mathbf{v}_B = (7.2 \text{ m/s})\mathbf{i} - (4.6 \text{ m/s})\mathbf{j}$ along a path intersecting the x axis at a distance $b = 7.48$ m from the origin O. Determine (*a*) the initial velocity $\bar{\mathbf{v}}_0$ of the mass center G of the two disks, (*b*) the length of the cord initially connecting the two disks, (*c*) the rate in rad/s at which the disks were spinning about G.

14.55 Three small identical spheres A, B, and C, which can slide on a horizontal, frictionless surface, are attached to three 9-in.-long strings, which are tied to a ring G. Initially the spheres rotate clockwise about the ring with a relative velocity of 2.6 ft/s and the ring moves along the x axis with a velocity $\mathbf{v}_0 = (1.3 \text{ ft/s})\mathbf{i}$. Suddenly the ring breaks and the three spheres move freely in the xy plane with A and B following paths parallel to the y axis at a distance $a = 1.0$ ft from each other and C following a path parallel to the x axis. Determine (*a*) the velocity of each sphere, (*b*) the distance d.

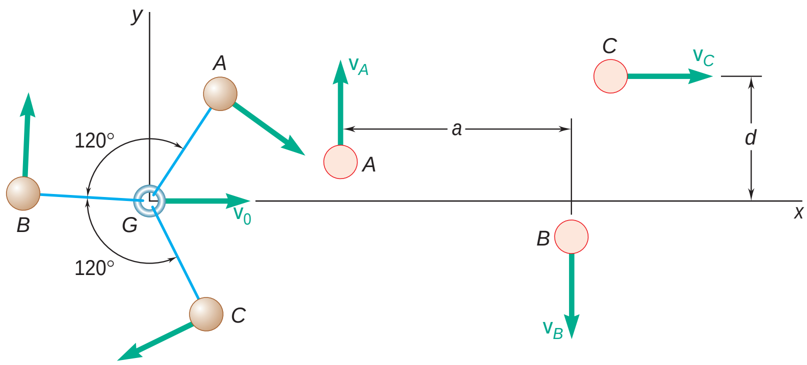


Fig. *P14.55* and *P14.56*

14.56 Three small identical spheres A, B, and C, which can slide on a horizontal, frictionless surface, are attached to three strings of length l which are tied to a ring G. Initially the spheres rotate clockwise about the ring which moves along the x axis with a velocity $\mathbf{v}_0$. Suddenly the ring breaks and the three spheres move freely in the xy plane. Knowing that $\mathbf{v}_A = (3.5 \text{ ft/s})\mathbf{j}$, $\mathbf{v}_C = (6.0 \text{ ft/s})\mathbf{i}$, $a = 16$ in., and $d = 9$ in., determine (*a*) the initial velocity of the ring, (*b*) the length l of the strings, (*c*) the rate in rad/s at which the spheres were rotating about G.

*14.10 VARIABLE SYSTEMS OF PARTICLES

All the systems of particles considered so far consisted of well-defined particles. These systems did not gain or lose any particles during their motion. In a large number of engineering applications, however, it is necessary to consider *variable systems of particles,* i.e., systems which are continually gaining or losing particles, or doing both at the same time. Consider, for example, a hydraulic turbine. Its analysis involves the determination of the forces exerted by a stream of water on rotating blades, and we note that the particles of water in contact with the blades form an everchanging system which continually acquires and loses particles. Rockets furnish another example of variable systems, since their propulsion depends upon the continual ejection of fuel particles.

We recall that all the kinetics principles established so far were derived for constant systems of particles, which neither gain nor lose particles. We must therefore find a way to reduce the analysis of a variable system of particles to that of an auxiliary constant system. The procedure to follow is indicated in Secs. 14.11 and 14.12 for two broad categories of applications: a steady stream of particles and a system that is gaining or losing mass.

*14.11 STEADY STREAM OF PARTICLES

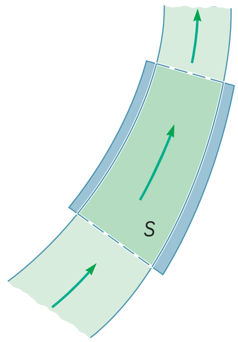


Fig. 14.9

Consider a steady stream of particles, such as a stream of water diverted by a fixed vane or a flow of air through a duct or through a blower. In order to determine the resultant of the forces exerted on the particles in contact with the vane, duct, or blower, we isolate these particles and denote by S the system thus defined (Fig. 14.9). We observe that S is a variable system of particles, since it continually gains particles flowing in and loses an equal number of particles flowing out. Therefore, the kinetics principles that have been established so far cannot be directly applied to S.

However, we can easily define an auxiliary system of particles which does remain constant for a short interval of time Δt. Consider at time t the system S *plus* the particles which will enter S during the interval at time Δt (Fig. 14.10*a*). Next, consider at time $t + \Delta t$ the system S *plus* the particles which have left S during the interval Δt (Fig. 14.10*c*). Clearly, *the same particles are involved in both cases,* and we can apply to those particles the principle of impulse and momentum. Since the total mass m of the system S remains constant, the particles entering the system and those leaving the system in the time Δt must have the same mass Δm. Denoting by $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively, the velocities of the particles entering S at A and leaving S at B, we represent the momentum of the particles entering S by $(\Delta m)\mathbf{v}_A$ (Fig. 14.10*a*) and the momentum of the particles leaving S by $(\Delta m)\mathbf{v}_B$ (Fig. 14.10*c*). We also represent by appropriate vectors the momenta $m_i\mathbf{v}_i$ of the particles forming S and the impulses of the forces exerted on S and indicate by blue plus and equals signs that the system of the momenta and impulses in parts *a* and *b* of Fig. 14.10 is equipollent to the system of the momenta in part *c* of the same figure.

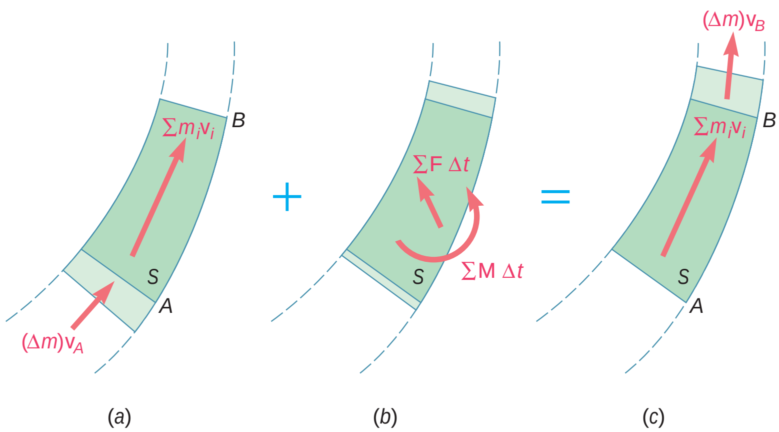


Fig. 14.10

The resultant $\Sigma m_i\mathbf{v}_i$ of the momenta of the particles of S is found on both sides of the equals sign and can thus be omitted. We conclude that *the system formed by the momentum* $(\Delta m)\mathbf{v}_A$ *of the particles entering* S *in the time* Δt *and the impulses of the forces exerted on* S *during that time is equipollent to the momentum* $(\Delta m)\mathbf{v}_B$ *of the particles leaving* S *in the same time* Δt. We can therefore write

$$(\Delta m)\mathbf{v}_A + \Sigma\mathbf{F}\,\Delta t = (\Delta m)\mathbf{v}_B \tag{14.38}$$

A similar equation can be obtained by taking the moments of the vectors involved (see Sample Prob. 14.5). Dividing all terms of Eq. (14.38) by Δt and letting Δt approach zero, we obtain at the limit

$$\Sigma\mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A) \tag{14.39}$$

where $\mathbf{v}_B - \mathbf{v}_A$ represents the difference between the *vector* $\mathbf{v}_B$ and the *vector* $\mathbf{v}_A$.

If SI units are used, dm/dt is expressed in kg/s and the velocities in m/s; we check that both members of Eq. (14.39) are expressed in the same units (newtons). If U.S. customary units are used, dm/dt must be expressed in slugs/s and the velocities in ft/s; we check again that both members of the equation are expressed in the same units (pounds).†

The principle we have established can be used to analyze a large number of engineering applications. Some of the more common of these applications will be considered next.

Fluid Stream Diverted by a Vane. If the vane is fixed, the method of analysis given above can be applied directly to find the force $\mathbf{F}$ exerted by the vane on the stream. We note that $\mathbf{F}$ is the only force which needs to be considered since the pressure in the stream is constant (atmospheric pressure). The force exerted by the stream on the vane will be equal and opposite to $\mathbf{F}$. If the vane moves with a constant velocity, the stream is not steady. However, it will appear steady to an observer moving with the vane. We should therefore choose a system of axes moving with the vane. Since this system of axes is not accelerated, Eq. (14.38) can still be used, but $\mathbf{v}_A$ and $\mathbf{v}_B$ must be replaced by the *relative velocities* of the stream with respect to the vane (see Sample Prob. 14.7).

Fluid Flowing Through a Pipe. The force exerted by the fluid on a pipe transition such as a bend or a contraction can be determined by considering the system of particles S in contact with the transition. Since, in general, the pressure in the flow will vary, the forces exerted on S by the adjoining portions of the fluid should also be considered.

†It is often convenient to express the mass rate of flow dm/dt as the product rQ, where r is the density of the stream (mass per unit volume) and Q its volume rate of flow (volume per unit time). If SI units are used, r is expressed in kg/m^3 (for instance, r $= 1000$ kg/m^3 for water) and Q in m^3/s. However, if U.S. customary units are used, r will generally have to be computed from the corresponding specific weight g (weight per unit volume), r = g/g. Since g is expressed in lb/ft^3 (for instance, g $= 62.4$ lb/ft^3 for water), r is obtained in slugs/ft^3. The volume rate of flow Q is expressed in ft^3/s.

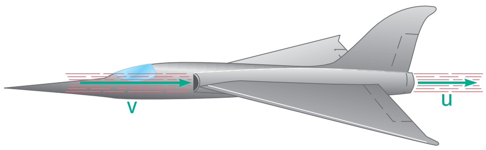


Fig. 14.11

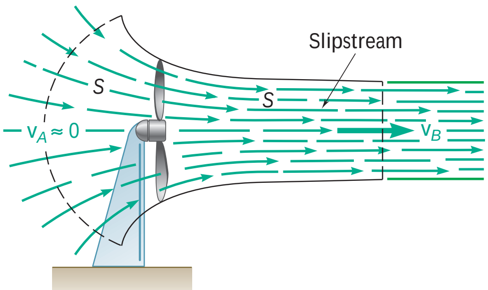


Fig. 14.12

Jet Engine. In a jet engine, air enters with no velocity through the front of the engine and leaves through the rear with a high velocity. The energy required to accelerate the air particles is obtained by burning fuel. The mass of the burned fuel in the exhaust gases will usually be small enough compared with the mass of the air flowing through the engine that it can be neglected. Thus, the analysis of a jet engine reduces to that of an airstream. This stream can be considered as a steady stream if all velocities are measured with respect to the airplane. It will be assumed, therefore, that the airstream enters the engine with a velocity $\mathbf{v}$ of magnitude equal to the speed of the airplane and leaves with a velocity $\mathbf{u}$ equal to the relative velocity of the exhaust gases (Fig. 14.11). Since the intake and exhaust pressures are nearly atmospheric, the only external force which needs to be considered is the force exerted by the engine on the airstream. This force is equal and opposite to the thrust.†

Fan. We consider the system of particles S shown in Fig. 14.12. The velocity $\mathbf{v}_A$ of the particles entering the system is assumed equal to zero, and the velocity $\mathbf{v}_B$ of the particles leaving the system is the velocity of the *slipstream*. The rate of flow can be obtained by multiplying v_B by the cross-sectional area of the slipstream. Since the pressure all around S is atmospheric, the only external force acting on S is the thrust of the fan.

Helicopter. The determination of the thrust created by the rotating blades of a hovering helicopter is similar to the determination of the thrust of a fan. The velocity $\mathbf{v}_A$ of the air particles as they approach the blades is assumed to be zero, and the rate of flow is obtained by multiplying the magnitude of the velocity $\mathbf{v}_B$ of the slipstream by its cross-sectional area.

*14.12 SYSTEMS GAINING OR LOSING MASS

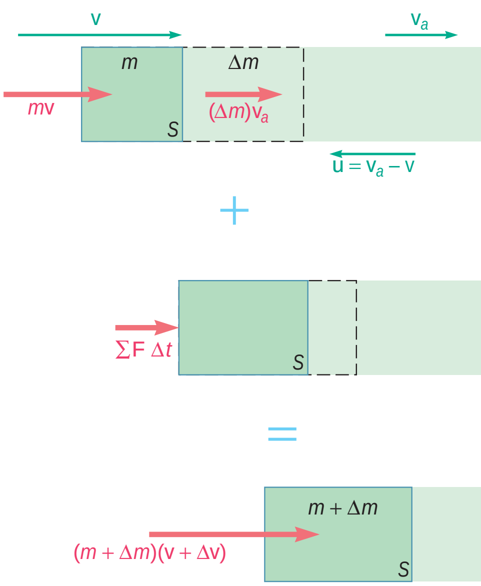


Fig. 14.13

Let us now analyze a different type of variable system of particles, namely, a system which gains mass by continually absorbing particles or loses mass by continually expelling particles. Consider the system S shown in Fig. 14.13. Its mass, equal to m at the instant t, increases by Δm in the interval of time Δt. In order to apply the principle of impulse and momentum to the analysis of this system, we must consider at time t the system S *plus* the particles of mass Δm which S absorbs during the time interval Δt. The velocity of S at time t is denoted by $\mathbf{v}$, the velocity of S at time $t + \Delta t$ is denoted by $\mathbf{v} + \Delta\mathbf{v}$, and the absolute velocity of the particles absorbed is denoted by $\mathbf{v}_a$. Applying the principle of impulse and momentum, we write

$$m\mathbf{v} + (\Delta m)\mathbf{v}_a + \Sigma\mathbf{F}\,\Delta t = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) \qquad (14.40)$$

†Note that if the airplane is accelerated, it cannot be used as a newtonian frame of reference. The same result will be obtained for the thrust, however, by using a reference frame at rest with respect to the atmosphere, since the air particles will then be observed to enter the engine with no velocity and to leave it with a velocity of magnitude $u - v$.

Solving for the sum $\Sigma\mathbf{F}\ \Delta t$ of the impulses of the external forces acting on S (excluding the forces exerted by the particles being absorbed), we have

$$\Sigma\mathbf{F}\ \Delta t = m\Delta\mathbf{v} + \Delta m(\mathbf{v} - \mathbf{v}_a) + (\Delta m)(\Delta\mathbf{v}) \tag{14.41}$$

Introducing the *relative velocity* $\mathbf{u}$ with respect to S of the particles which are absorbed, we write $\mathbf{u} = \mathbf{v}_a - \mathbf{v}$ and note, since $v_a < v$, that the relative velocity $\mathbf{u}$ is directed to the left, as shown in Fig. 14.13. Neglecting the last term in Eq. (14.41), which is of the second order, we write

$$\Sigma\mathbf{F}\ \Delta t = m\ \Delta\mathbf{v} - (\Delta m)\mathbf{u}$$

Dividing through by Δt and letting Δt approach zero, we have at the limit†

$$\Sigma\mathbf{F} = m\frac{d\mathbf{v}}{dt} - \frac{dm}{dt}\mathbf{u} \tag{14.42}$$

Rearranging the terms and recalling that $d\mathbf{v}/dt = \mathbf{a}$, where $\mathbf{a}$ is the acceleration of the system S, we write

$$\Sigma\mathbf{F} + \frac{dm}{dt}\mathbf{u} = m\mathbf{a} \tag{14.43}$$

which shows that the action on S of the particles being absorbed is equivalent to a thrust

$$\mathbf{P} = \frac{dm}{dt}\mathbf{u} \tag{14.44}$$

which tends to slow down the motion of S, since the relative velocity $\mathbf{u}$ of the particles is directed to the left. If SI units are used, dm/dt is expressed in kg/s, the relative velocity u in m/s, and the corresponding thrust in newtons. If U.S. customary units are used, dm/dt must be expressed in slugs/s, u in ft/s, and the corresponding thrust in pounds.‡

The equations obtained can also be used to determine the motion of a system S losing mass. In this case, the rate of change of mass is negative, and the action on S of the particles being expelled is equivalent to a thrust in the direction of $-\mathbf{u}$, that is, in the direction opposite to that in which the particles are being expelled. A *rocket* represents a typical case of a system continually losing mass (see Sample Prob. 14.8).

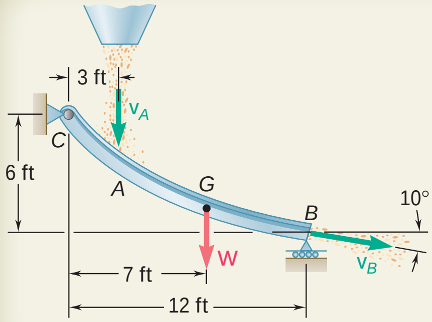
Photo 14.3 As the shuttle's booster rockets are fired, the gas particles they eject provide the thrust required for liftoff.

†When the absolute velocity $\mathbf{v}_a$ of the particles absorbed is zero, $\mathbf{u} = -\mathbf{v}$, and formula (14.42) becomes

$$\Sigma\mathbf{F} = \frac{d}{dt}(m\mathbf{v})$$

Comparing the formula obtained to Eq. (12.3) of Sec. 12.3, we observe that Newton's second law can be applied to a system gaining mass, *provided that the particles absorbed are initially at rest*. It may also be applied to a system losing mass, *provided that the velocity of the particles expelled is zero* with respect to the frame of reference selected.

‡See footnote on page 899.

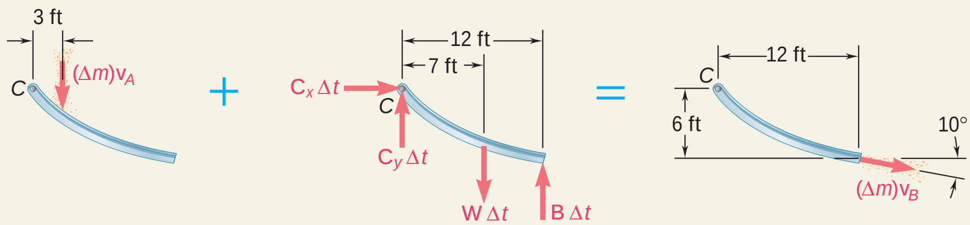


SAMPLE PROBLEM 14.6

Grain falls from a hopper onto a chute CB at the rate of 240 lb/s. It hits the chute at A with a velocity of 20 ft/s and leaves at B with a velocity of 15 ft/s, forming an angle of 10° with the horizontal. Knowing that the combined weight of the chute and of the grain it supports is a force $\mathbf{W}$ of magnitude 600 lb applied at G, determine the reaction at the roller support B and the components of the reaction at the hinge C.

SOLUTION

We apply the principle of impulse and momentum for the time interval Δt to the system consisting of the chute, the grain it supports, and the amount of grain which hits the chute in the interval Δt. Since the chute does not move, it has no momentum. We also note that the sum $\Sigma m_i \mathbf{v}_i$ of the momenta of the particles supported by the chute is the same at t and $t + \Delta t$ and can thus be omitted.



Since the system formed by the momentum $(\Delta m)\mathbf{v}_A$ and the impulses is equipollent to the momentum $(\Delta m)\mathbf{v}_B$, we write

$\xrightarrow{+}$ x components: $\quad C_x\,\Delta t = (\Delta m)v_B \cos 10° \qquad (1)$

$+\uparrow y$ components: $\quad -(\Delta m)v_A + C_y\,\Delta t - W\,\Delta t + B\,\Delta t$
$$= -(\Delta m)v_B \sin 10° \qquad (2)$$

$+\circlearrowleft$ moments about C: $\quad -3(\Delta m)v_A - 7(W\,\Delta t) + 12(B\,\Delta t)$
$$= 6(\Delta m)v_B \cos 10° - 12(\Delta m)v_B \sin 10° \qquad (3)$$

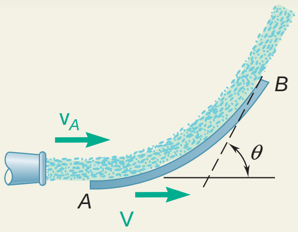
Using the given data, $W = 600$ lb, $v_A = 20$ ft/s, $v_B = 15$ ft/s, and $\Delta m/\Delta t = 240/32.2 = 7.45$ slugs/s, and solving Eq. (3) for B and Eq. (1) for C_x,

$$12B = 7(600) + 3(7.45)(20) + 6(7.45)(15)(\cos 10° - 2 \sin 10°)$$
$$12B = 5075 \qquad B = 423 \text{ lb} \qquad \mathbf{B} = 423 \text{ lb}\uparrow \quad ◀$$

$$C_x = (7.45)(15)\cos 10° = 110.1 \text{ lb} \qquad \mathbf{C}_x = 110.1 \text{ lb} \rightarrow \quad ◀$$

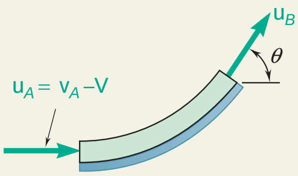
Substituting for B and solving Eq. (2) for C_y,

$$C_y = 600 - 423 + (7.45)(20 - 15 \sin 10°) = 307 \text{ lb}$$
$$\mathbf{C}_y = 307 \text{ lb}\uparrow \quad ◀$$



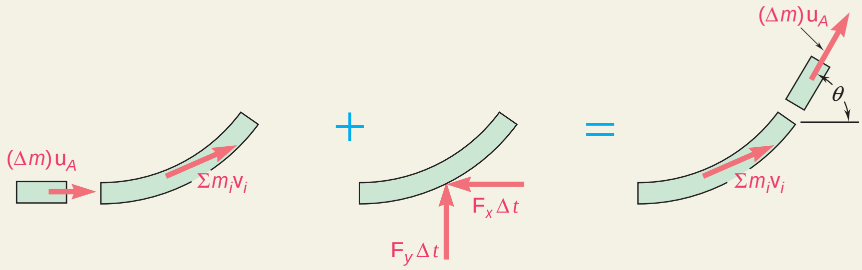
SAMPLE PROBLEM 14.7

A nozzle discharges a stream of water of cross-sectional area A with a velocity $\mathbf{v}_A$. The stream is deflected by a *single* blade which moves to the right with a constant velocity $\mathbf{V}$. Assuming that the water moves along the blade at constant speed, determine (*a*) the components of the force $\mathbf{F}$ exerted by the blade on the stream, (*b*) the velocity $\mathbf{V}$ for which maximum power is developed.



SOLUTION

a. Components of Force Exerted on Stream. We choose a coordinate system which moves with the blade at a constant velocity $\mathbf{V}$. The particles of water strike the blade with a relative velocity $\mathbf{u}_A = \mathbf{v}_A - \mathbf{V}$ and leave the blade with a relative velocity $\mathbf{u}_B$. Since the particles move along the blade at a constant speed, the relative velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ have the same magnitude u. Denoting the density of water by ρ, the mass of the particles striking the blade during the time interval Δt is $\Delta m = A\rho(v_A - V)\,\Delta t$; an equal mass of particles leaves the blade during Δt. We apply the principle of impulse and momentum to the system formed by the particles in contact with the blade and the particles striking the blade in the time Δt.



Recalling that $\mathbf{u}_A$ and $\mathbf{u}_B$ have the same magnitude u, and omitting the momentum $\Sigma m_i \mathbf{v}_i$ which appears on both sides, we write

$\xrightarrow{+}$ x components: $\quad (\Delta m)u - F_x\,\Delta t = (\Delta m)u \cos\theta$

$+\uparrow y$ components: $\quad +F_y\,\Delta t = (\Delta m)u \sin\theta$

Substituting $\Delta m = A\rho(v_A - V)\,\Delta t$ and $u = v_A - V$, we obtain

$$\mathbf{F}_x = A\rho(v_A - V)^2(1 - \cos\theta) \leftarrow \qquad \mathbf{F}_y = A\rho(v_A - V)^2 \sin\theta \uparrow \quad ◀$$

b. Velocity of Blade for Maximum Power. The power is obtained by multiplying the velocity V of the blade by the component F_x of the force exerted by the stream on the blade.

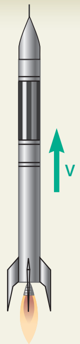
$$\text{Power} = F_x V = A\rho(v_A - V)^2(1 - \cos\theta)V$$

Differentiating the power with respect to V and setting the derivative equal to zero, we obtain

$$\frac{d(\text{power})}{dV} = A\rho(v_A^2 - 4v_A V + 3V^2)(1 - \cos\theta) = 0$$

$$V = v_A \qquad V = \tfrac{1}{3}v_A \qquad \text{For maximum power } \mathbf{V} = \tfrac{1}{3}v_A \rightarrow \quad ◀$$

Note. These results are valid only when a *single* blade deflects the stream. Different results are obtained when a series of blades deflects the stream, as in a Pelton-wheel turbine. (See Prob. 14.81.)

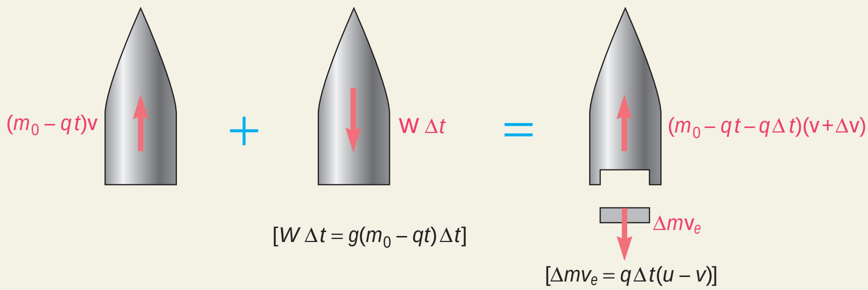


SAMPLE PROBLEM 14.8

A rocket of initial mass m_0 (including shell and fuel) is fired vertically at time $t = 0$. The fuel is consumed at a constant rate $q = dm/dt$ and is expelled at a constant speed u relative to the rocket. Derive an expression for the magnitude of the velocity of the rocket at time t, neglecting the resistance of the air.

SOLUTION

At time t, the mass of the rocket shell and remaining fuel is $m = m_0 - qt$, and the velocity is $\mathbf{v}$. During the time interval Δt, a mass of fuel $\Delta m = q\,\Delta t$ is expelled with a speed u relative to the rocket. Denoting by $\mathbf{v}_e$ the absolute velocity of the expelled fuel, we apply the principle of impulse and momentum between time t and time $t + \Delta t$.



We write

$$(m_0 - qt)v - g(m_0 - qt)\,\Delta t = (m_0 - qt - q\,\Delta t)(v + \Delta v) - q\,\Delta t(u - v)$$

Dividing through by Δt and letting Δt approach zero, we obtain

$$-g(m_0 - qt) = (m_0 - qt)\frac{dv}{dt} - qu$$

Separating variables and integrating from $t = 0$, $v = 0$ to $t = t$, $v = v$,

$$dv = \left(\frac{qu}{m_0 - qt} - g\right)dt \qquad \int_0^v dv = \int_0^t \left(\frac{qu}{m_0 - qt} - g\right)dt$$

$$v = [-u \ln(m_0 - qt) - gt]_0^t \qquad v = u \ln\frac{m_0}{m_0 - qt} - gt \quad ◀$$

Remark. The mass remaining at time t_f, after all the fuel has been expended, is equal to the mass of the rocket shell $m_s = m_0 - qt_f$, and the maximum velocity attained by the rocket is $v_m = u \ln (m_0/m_s) - gt_f$. Assuming that the fuel is expelled in a relatively short period of time, the term gt_f is small and we have $v_m \approx u \ln (m_0/m_s)$. In order to escape the gravitational field of the earth, a rocket must reach a velocity of 11.18 km/s. Assuming $u = 2200$ m/s and $v_m = 11.18$ km/s, we obtain $m_0/m_s = 161$. Thus, to project each kilogram of the rocket shell into space, it is necessary to consume more than 161 kg of fuel if a propellant yielding $u = 2200$ m/s is used.

SOLVING PROBLEMS ON YOUR OWN

This lesson is devoted to the study of the motion of *variable systems of particles*, i.e., systems which are continually *gaining or losing particles* or doing both at the same time. The problems you will be asked to solve will involve (1) *steady streams of particles* and (2) *systems gaining or losing mass*.

1. To solve problems involving a steady stream of particles, you will consider a portion S of the stream and express that the system formed by the momentum of the particles entering S at A in the time Δt and the impulses of the forces exerted on S during that time is equipollent to the momentum of the particles leaving S at B in the same time Δt (Fig. 14.10). Considering only the resultants of the vector systems involved, you can write the vector equation

$$(\Delta m)\mathbf{v}_A + \Sigma\mathbf{F}\,\Delta t = (\Delta m)\mathbf{v}_B \qquad (14.38)$$

You may want to consider as well the moments about a given point of the vector systems involved to obtain an additional equation [Sample Prob. 14.6], but many problems can be solved using Eq. (14.38) or the equation obtained by dividing all terms by Δt and letting Δt approach zero,

$$\Sigma\mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A) \qquad (14.39)$$

where $\mathbf{v}_B - \mathbf{v}_A$ represents a *vector subtraction* and where the mass rate of flow dm/dt can be expressed as the product ρQ of the density ρ of the stream (mass per unit volume) and the volume rate of flow Q (volume per unit time). If U.S. customary units are used, ρ is expressed as the ratio γ/g, where γ is the specific weight of the stream and g is the acceleration of gravity.

Typical problems involving a steady stream of particles have been described in Sec. 14.11. You may be asked to determine the following:

a. Thrust caused by a diverted flow. Equation (14.39) is applicable, but you will get a better understanding of the problem if you use a solution based on Eq. (14.38).

b. Reactions at supports of vanes or conveyor belts. First draw a diagram showing on one side of the equals sign the momentum $(\Delta m)\mathbf{v}_A$ of the particles impacting the vane or belt in the time Δt, as well as the impulses of the loads and reactions at the supports during that time, and showing on the other side the momentum $(\Delta m)\mathbf{v}_B$ of the particles leaving the vane or belt in the time Δt [Sample Prob. 14.6]. Equating the x components, y components, and moments of the quantities on both sides of the equals sign will yield three scalar equations which can be solved for three unknowns.

c. Thrust developed by a jet engine, a propeller, or a fan. In most cases, a single unknown is involved, and that unknown can be obtained by solving the scalar equation derived from Eq. (14.38) or Eq. (14.39).

(continued)

2. To solve problems involving systems gaining mass, you will consider the system S, which has a mass m and is moving with a velocity $\mathbf{v}$ at time t, and the particles of mass Δm with velocity $\mathbf{v}_a$ that S will absorb in the time interval Δt (Fig. 14.13). You will then express that the total momentum of S and of the particles that will be absorbed, *plus* the impulse of the external forces exerted on S, are equipollent to the momentum of S at time $t + \Delta t$. Noting that the mass of S and its velocity at that time are, respectively, $m + \Delta m$ and $\mathbf{v} + \Delta\mathbf{v}$, you will write the vector equation

$$m\mathbf{v} + (\Delta m)\mathbf{v}_a + \Sigma\mathbf{F}\,\Delta t = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) \qquad (14.40)$$

As was shown in Sec. 14.12, if you introduce the relative velocity $\mathbf{u} = \mathbf{v}_a - \mathbf{v}$ of the particles being absorbed, you obtain the following expression for the resultant of the external forces applied to S:

$$\Sigma\mathbf{F} = m\frac{d\mathbf{v}}{dt} - \frac{dm}{dt}\mathbf{u} \qquad (14.42)$$

Furthermore, it was shown that the action on S of the particles being absorbed is equivalent to a thrust

$$\mathbf{P} = \frac{dm}{dt}\mathbf{u} \qquad (14.44)$$

exerted in the direction of the relative velocity of the particles being absorbed.

Examples of systems gaining mass are conveyor belts and moving railroad cars being loaded with gravel or sand, and chains being pulled out of a pile.

3. To solve problems involving systems losing mass, such as rockets and rocket engines, you can use Eqs. (14.40) through (14.44), provided that you give negative values to the increment of mass Δm and to the rate of change of mass dm/dt. It follows that the thrust defined by Eq. (14.44) will be exerted in a direction opposite to the direction of the relative velocity of the particles being ejected.

PROBLEMS

14.57 A stream of water of cross-section area A_1 and velocity $\mathbf{v}_1$ strikes a circular plate which is held motionless by a force **P**. A hole in the circular plate of area A_2 results in a discharge jet having a velocity $\mathbf{v}_1$. Determine the magnitude of **P**.

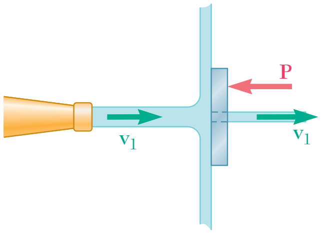


Fig. P14.57

14.58 A jet ski is placed in a channel and is tethered so that it is stationary. Water enters the jet ski with velocity $\mathbf{v}_1$ and exits with velocity $\mathbf{v}_2$. Knowing the inlet area is A_1 and the exit area is A_2, determine the tension in the tether.

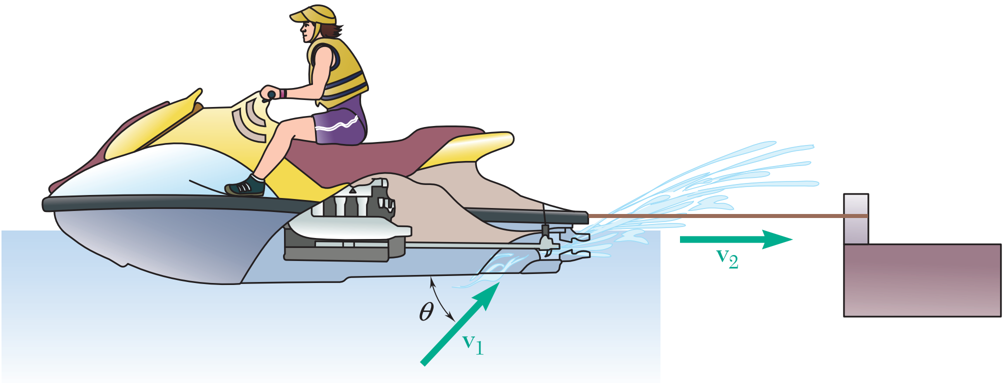


Fig. P14.58

14.59 A stream of water of cross-section area A and velocity $\mathbf{v}_1$ strikes a plate which is held motionless by a force **P**. Determine the magnitude of **P**, knowing that $A = 0.75$ in^2, $v_1 = 80$ ft/s, and $V = 0$.

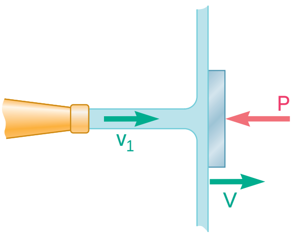


Fig. P14.59 and P14.60

14.60 A stream of water of cross-section area A and velocity $\mathbf{v}_1$ strikes a plate which moves to the right with a velocity **V**. Determine the magnitude of **V**, knowing that $A = 1$ in^2, $v_1 = 100$ ft/s, and $P = 90$ lb.

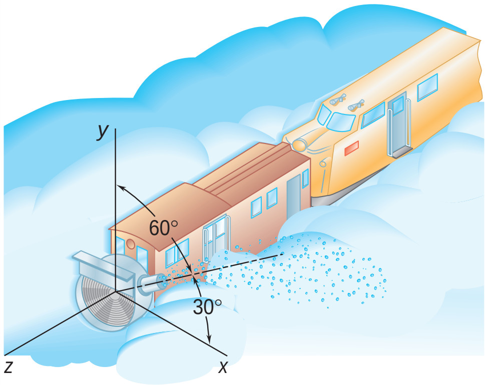


Fig. P14.61

14.61 A rotary power plow is used to remove snow from a level section of railroad track. The plow car is placed ahead of an engine which propels it at a constant speed of 20 km/h. The plow car clears 160 Mg of snow per minute, projecting it in the direction shown with a velocity of 12 m/s relative to the plow car. Neglecting friction, determine (*a*) the force exerted by the engine on the plow car, (*b*) the lateral force exerted by the track on the plow.

14.62 Tree limbs and branches are being fed at *A* at the rate of 5 kg/s into a shredder which spews the resulting wood chips at *C* with a velocity of 20 m/s. Determine the horizontal component of the force exerted by the shredder on the truck hitch at *D*.

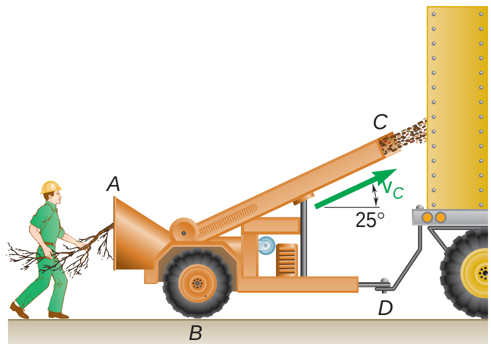


Fig. P14.62

14.63 Sand falls from three hoppers onto a conveyor belt at a rate of 90 lb/s for each hopper. The sand hits the belt with a vertical velocity $v_1 = 10$ ft/s and is discharged at *A* with a horizontal velocity $v_2 = 13$ ft/s. Knowing that the combined mass of the beam, belt system, and the sand it supports is 1300 lb with a mass center at *G*, determine the reaction at *E*.

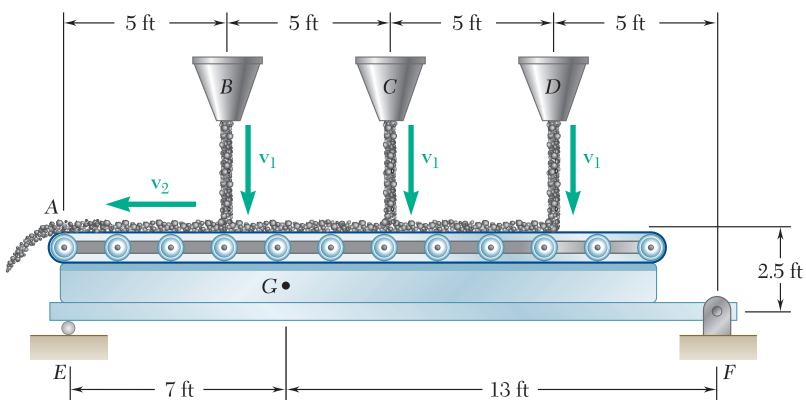


Fig. *P14.63*

14.64 The stream of water shown flows at a rate of 550 L/min and moves with a velocity of magnitude 18 m/s at both A and B. The vane is supported by a pin and bracket at C and by a load cell at D which can exert only a horizontal force. Neglecting the weight of the vane, determine the components of the reactions at C and D.

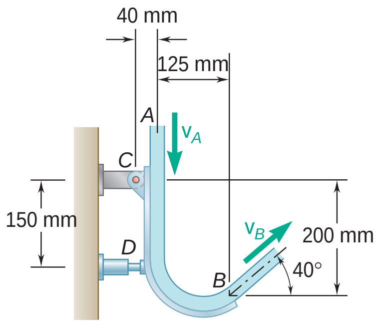


Fig. P14.64

14.65 The nozzle discharges water at the rate of 340 gal/min. Knowing the velocity of the water at both A and B has a magnitude of 65 ft/s and neglecting the weight of the vane, determine the components of the reactions at C and D ($1 \text{ ft}^3 = 7.48$ gal).

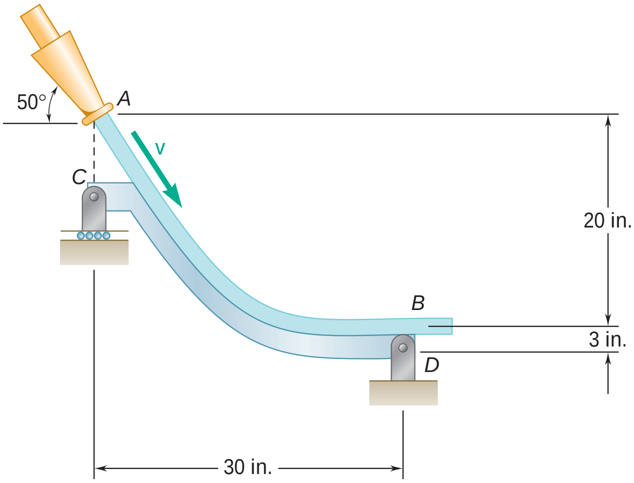


Fig. *P14.65*

14.66 A high-speed jet of air issues from nozzle A with a velocity of $\mathbf{v}_A$ and mass flow rate of 0.36 kg/s. The air impinges on a vane causing it to rotate to the position shown. The vane has a mass of 6 kg. Knowing that the magnitude of the air velocity is equal at A and B, determine (*a*) the magnitude of the velocity at A, (*b*) the components of the reactions at O.

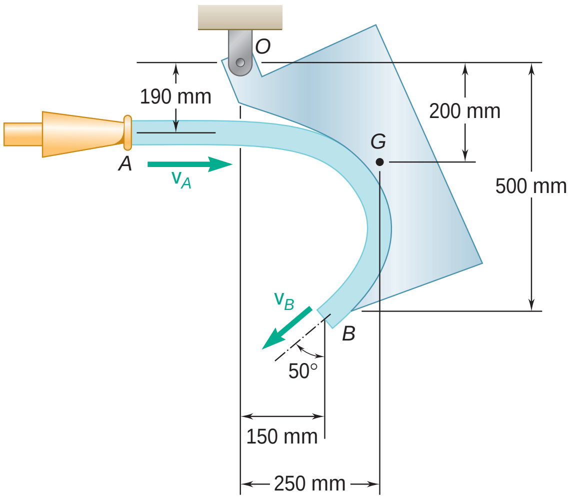


Fig. P14.66

14.67 Coal is being discharged from a first conveyor belt at the rate of 120 kg/s. It is received at A by a second belt which discharges it again at B. Knowing that $v_1 = 3$ m/s and $v_2 = 4.25$ m/s and that the second belt assembly and the coal it supports have a total mass of 472 kg, determine the components of the reactions at C and D.

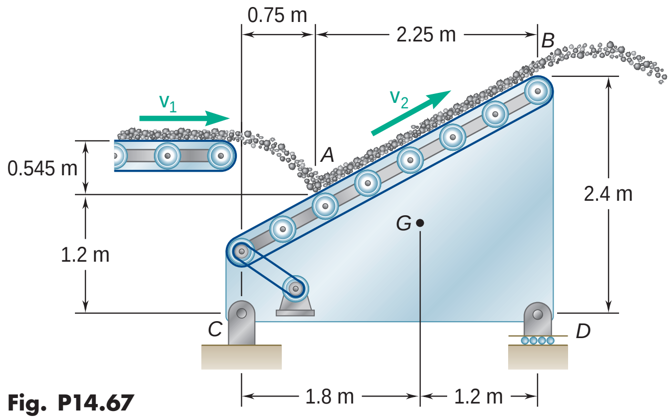


Fig. P14.67

14.68 A mass q of sand is discharged per unit time from a conveyor belt moving with a velocity $\mathbf{v}_0$. The sand is deflected by a plate at A so that it falls in a vertical stream. After falling a distance h the sand is again deflected by a curved plate at B. Neglecting the friction between the sand and the plates, determine the force required to hold in the position shown (*a*) plate A, (*b*) plate B.

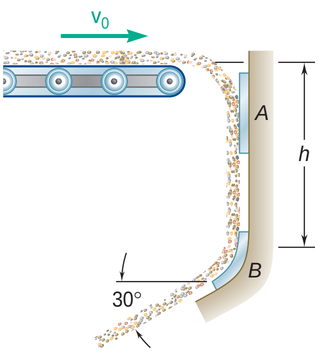


Fig. P14.68

14.69 The total drag due to air friction on a jet airplane traveling at 900 km/h is 35 kN. Knowing that the exhaust velocity is 600 m/s relative to the airplane, determine the mass of air which must pass through the engine per second to maintain the speed of 900 km/h in level flight.

14.70 While cruising in level flight at a speed of 600 mi/h, a jet plane scoops in air at the rate of 200 lb/s and discharges it with a velocity of 2100 ft/s relative to the airplane. Determine the total drag due to air friction on the airplane.

14.71 In order to shorten the distance required for landing, a jet airplane is equipped with movable vanes which partially reverse the direction of the air discharged by each of its engines. Each engine scoops in the air at a rate of 120 kg/s and discharges it with a velocity of 600 m/s relative to the engine. At an instant when the speed of the airplane is 270 km/h, determine the reverse thrust provided by each of the engines.

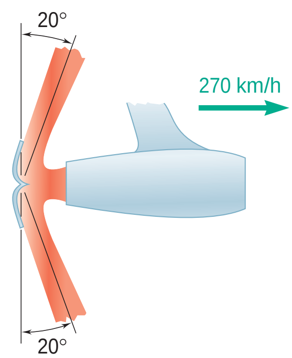


Fig. P14.71

14.72 The helicopter shown can produce a maximum downward air speed of 80 ft/s in a 30-ft-diameter slipstream. Knowing that the weight of the helicopter and its crew is 3500 lb and assuming $\gamma = 0.076 \text{ lb/ft}^3$ for air, determine the maximum load that the helicopter can lift while hovering in midair.

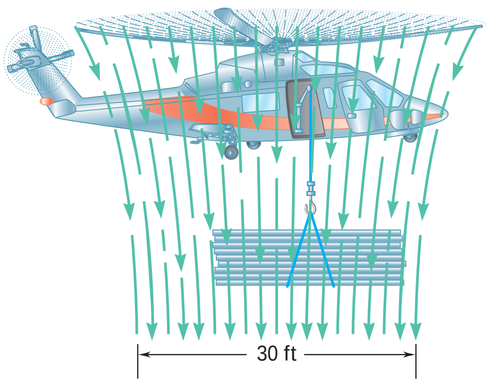


Fig. P14.72

14.73 A floor fan designed to deliver air at a maximum velocity of 6 m/s in a 400-mm-diameter slipstream is supported by a 200-mm-diameter circular base plate. Knowing that the total weight of the assembly is 60 N and that its center of gravity is located directly above the center of the base plate, determine the maximum height h at which the fan may be operated if it is not to tip over. Assume $\rho = 1.21 \text{ kg/m}^3$ for air and neglect the approach velocity of the air.

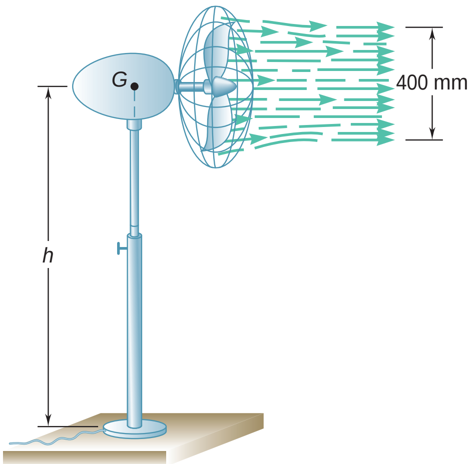


Fig. *P14.73*

14.74 The jet engine shown scoops in air at A at a rate of 200 lb/s and discharges it at B with a velocity of 2000 ft/s relative to the airplane. Determine the magnitude and line of action of the propulsive thrust developed by the engine when the speed of the airplane is (*a*) 300 mi/h, (*b*) 600 mi/h.

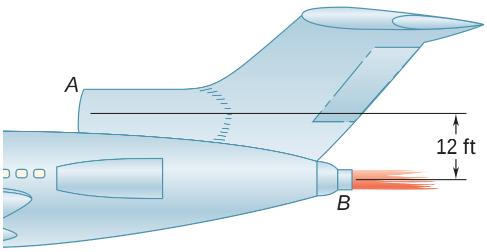


Fig. P14.74

14.75 A jet airliner is cruising at a speed of 900 km/h with each of its three engines discharging air with a velocity of 800 m/s relative to the plane. Determine the speed of the airliner after it has lost the use of (*a*) one of its engines, (*b*) two of its engines. Assume that the drag due to air friction is proportional to the square of the speed and that the remaining engines keep operating at the same rate.

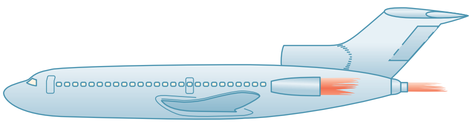


Fig. *P14.75*

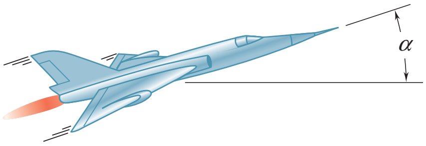


Fig. P14.76

14.76 A 16-Mg jet airplane maintains a constant speed of 774 km/h while climbing at an angle $\alpha = 18°$. The airplane scoops in air at a rate of 300 kg/s and discharges it with a velocity of 665 m/s relative to the airplane. If the pilot changes to a horizontal flight while maintaining the same engine setting, determine (*a*) the initial acceleration of the plane, (*b*) the maximum horizontal speed that will be attained. Assume that the drag due to air friction is proportional to the square of the speed.

14.77 The propeller of a small airplane has a 2-m-diameter slipstream and produces a thrust of 3600 N when the airplane is at rest on the ground. Assuming $\rho = 1.225$ kg/m^3 for air, determine (*a*) the speed of the air in the slipstream, (*b*) the volume of air passing through the propeller per second, (*c*) the kinetic energy imparted per second to the air in the slipstream.

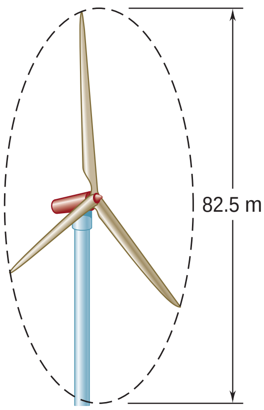


Fig. P14.78 and P14.79

14.78 The wind turbine–generator shown has an output-power rating of 1.5 MW for a wind speed of 36 km/h. For the given wind speed, determine (*a*) the kinetic energy of the air particles entering the 82.5-m-diameter circle per second, (*b*) the efficiency of this energy conversion system. Assume $\rho = 1.21$ kg/m^3 for air.

14.79 A wind turbine-generator system having a diameter of 82.5 m produces 1.5 MW at a wind speed of 12 m/s. Determine the diameter of blade necessary to produce 10 MW of power assuming the efficiency is the same for both designs and $\rho = 1.21$ kg/m^3 for air.

14.80 While cruising in level flight at a speed of 570 mi/h, a jet airplane scoops in air at a rate of 240 lb/s and discharges it with a velocity of 2200 ft/s relative to the airplane. Determine (*a*) the power actually used to propel the airplane, (*b*) the total power developed by the engine, (*c*) the mechanical efficiency of the airplane.

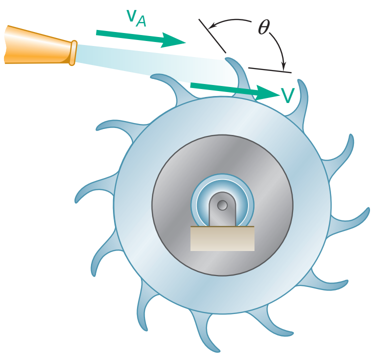


Fig. *P14.81*

14.81 In a Pelton-wheel turbine, a stream of water is deflected by a series of blades so that the rate at which water is deflected by the blades is equal to the rate at which water issues from the nozzle ($\Delta m/\Delta t = A\rho v_A$). Using the same notation as in Sample Prob. 14.7, (*a*) determine the velocity **V** of the blades for which maximum power is developed, (*b*) derive an expression for the maximum power, (*c*) derive an expression for the mechanical efficiency.

14.82 A circular reentrant orifice (also called Borda's mouthpiece) of diameter D is placed at a depth h below the surface of a tank. Knowing that the speed of the issuing stream is $v = \sqrt{2gh}$ and assuming that the speed of approach v_1 is zero, show that the diameter of the stream is $d = D/\sqrt{2}$. (*Hint:* Consider the section of water indicated, and note that P is equal to the pressure at a depth h multiplied by the area of the orifice.)

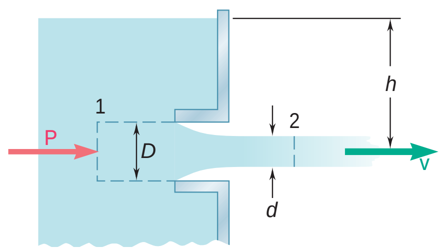


Fig. *P14.82*

14.83 Gravel falls with practically zero velocity onto a conveyor belt at the constant rate $q = dm/dt$. (*a*) Determine the magnitude of the force **P** required to maintain a constant belt speed v. (*b*) Show that the kinetic energy acquired by the gravel in a given time interval is equal to half the work done in that interval by the force **P**. Explain what happens to the other half of the work done by **P**.

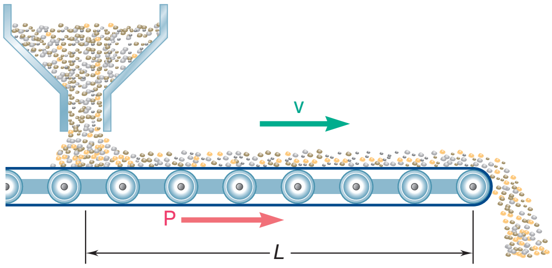


Fig. P14.83

***14.84** The depth of water flowing in a rectangular channel of width b at a speed v_1 and a depth d_1 increases to a depth d_2 at a *hydraulic jump*. Express the rate of flow Q in terms of b, d_1, and d_2.

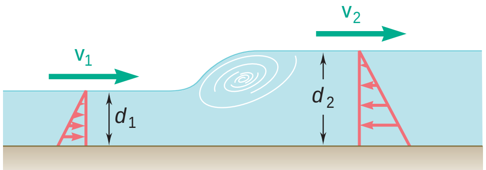


Fig. P14.84

***14.85** Determine the rate of flow in the channel of Prob. 14.84, knowing that $b = 12$ ft, $d_1 = 4$ ft, and $d_2 = 5$ ft.

14.86 A chain of length l and mass m lies in a pile on the floor. If its end A is raised vertically at a constant speed v, express in terms of the length y of chain which is off the floor at any given instant (*a*) the magnitude of the force **P** applied to A, (*b*) the reaction of the floor.

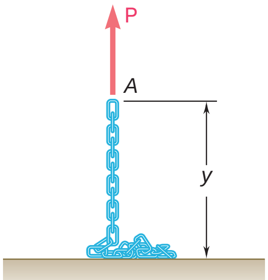


Fig. P14.86

14.87 Solve Prob. 14.86, assuming that the chain is being *lowered* to the floor at a constant speed v.

14.88 The ends of a chain lie in piles at A and C. When released from rest at time $t = 0$, the chain moves over the pulley at B, which has a negligible mass. Denoting by L the length of chain connecting the two piles and neglecting friction, determine the speed v of the chain at time t.

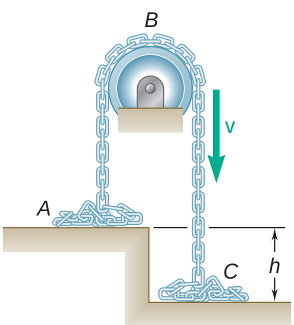


Fig. P14.88

14.89 A toy car is propelled by water that squirts from an internal tank at a constant 6 ft/s relative to the car. The weight of the empty car is 0.4 lb and it holds 2 lb of water. Neglecting other tangential forces, determine the top speed of the car.

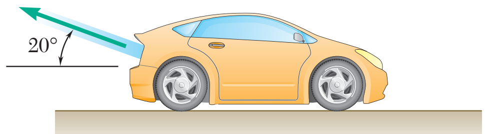


Fig. P14.89 and P14.90

14.90 A toy car is propelled by water that squirts from an internal tank. The weight of the empty car is 0.4 lb and it holds 2 lb of water. Knowing the top speed of the car is 8 ft/s determine the relative velocity of the water that is being ejected.

14.91 The main propulsion system of a space shuttle consists of three identical rocket engines which provide a total thrust of 6 MN. Determine the rate at which the hydrogen-oxygen propellant is burned by each of the three engines, knowing that it is ejected with a relative velocity of 3750 m/s.

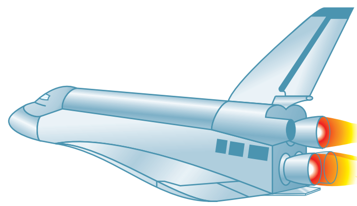


Fig. P14.91 and *P14.92*

14.92 The main propulsion system of a space shuttle consists of three identical rocket engines, each of which burns the hydrogen-oxygen propellant at the rate of 750 lb/s and ejects it with a relative velocity of 12,000 ft/s. Determine the total thrust provided by the three engines.

14.93 A rocket weighs 2600 lb, including 2200 lb of fuel, which is consumed at a rate of 25 lb/s and ejected with a relative velocity of 13,000 ft/s. Knowing that the rocket is fired vertically from the ground, determine its acceleration (*a*) as it is fired, (*b*) as the last particle of fuel is being consumed.

14.94 A space vehicle describing a circular orbit about the earth at a speed of 24×10^3 km/h releases at its front end a capsule which has a gross mass of 600 kg, including 400 kg of fuel. If the fuel is consumed at the rate of 18 kg/s and ejected with a relative velocity of 3000 m/s, determine (*a*) the tangential acceleration of the capsule as its engine is fired, (*b*) the maximum speed attained by the capsule.

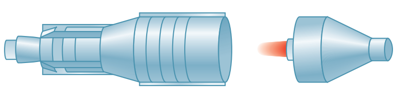


Fig. P14.94

14.95 A 540-kg spacecraft is mounted on top of a rocket with a mass of 19 Mg, including 17.8 Mg of fuel. Knowing that the fuel is consumed at a rate of 225 kg/s and ejected with a relative velocity of 3600 m/s, determine the maximum speed imparted to the spacecraft if the rocket is fired vertically from the ground.

Fig. P14.95

Fig. P14.96

14.96 The rocket used to launch the 540-kg spacecraft of Prob. 14.95 is redesigned to include two stages A and B, each of mass 9.5 Mg, including 8.9 Mg of fuel. The fuel is again consumed at a rate of 225 kg/s and ejected with a relative velocity of 3600 m/s. Knowing that when stage A expels its last particle of fuel, its casing is released and jettisoned, determine (*a*) the speed of the rocket at that instant, (*b*) the maximum speed imparted to the spacecraft.

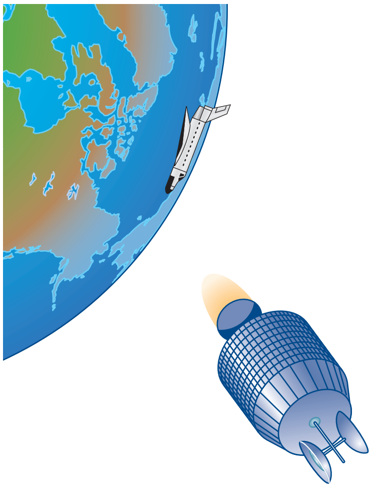


Fig. *P14.97*

14.97 A communications satellite weighing 10,000 lb, including fuel, has been ejected from a space shuttle describing a low circular orbit around the earth. After the satellite has slowly drifted to a safe distance from the shuttle, its engine is fired to increase its velocity by 8000 ft/s as a first step to its transfer to a geosynchronous orbit. Knowing that the fuel is ejected with a relative velocity of 13,750 ft/s, determine the weight of fuel consumed in this maneuver.

14.98 Determine the increase in velocity of the communications satellite of Prob. 14.97 after 2500 lb of fuel has been consumed.

14.99 Determine the distance separating the communications satellite of Prob. 14.97 from the space shuttle 60 s after its engine has been fired, knowing that the fuel is consumed at a rate of 37.5 lb/s.

14.100 For the rocket of Prob. 14.93, determine (*a*) the altitude at which all of the fuel has been consumed, (*b*) the velocity of the rocket at this time.

14.101 Determine the altitude reached by the spacecraft of Prob. 14.95 when all the fuel of its launching rocket has been consumed.

14.102 For the spacecraft and the two-stage launching rocket of Prob. 14.96, determine the altitude at which (*a*) stage A of the rocket is released, (*b*) the fuel of both stages has been consumed.

14.103 In a jet airplane, the kinetic energy imparted to the exhaust gases is wasted as far as propelling the airplane is concerned. The useful power is equal to the product of the force available to propel the airplane and the speed of the airplane. If v is the speed of the airplane and u is the relative speed of the expelled gases, show that the mechanical efficiency of the airplane is $\eta = 2v/(u + v)$. Explain why $\eta = 1$ when $u = v$.

14.104 In a rocket, the kinetic energy imparted to the consumed and ejected fuel is wasted as far as propelling the rocket is concerned. The useful power is equal to the product of the force available to propel the rocket and the speed of the rocket. If v is the speed of the rocket and u is the relative speed of the expelled fuel, show that the mechanical efficiency of the rocket is $\eta = 2uv/(u^2 + v^2)$. Explain why $\eta = 1$ when $u = v$.

REVIEW AND SUMMARY

In this chapter we analyzed the motion of *systems of particles,* i.e., the motion of a large number of particles considered together. In the first part of the chapter we considered systems consisting of well-defined particles, while in the second part we analyzed systems which are continually gaining or losing particles, or doing both at the same time.

Effective forces

We first defined the *effective force* of a particle P_i of a given system as the product $m_i\mathbf{a}_i$ of its mass m_i and its acceleration $\mathbf{a}_i$ with respect to a newtonian frame of reference centered at O [Sec. 14.2]. We then showed that *the system of the external forces acting on the particles and the system of the effective forces of the particles are equipollent;* i.e., both systems have the *same resultant* and the *same moment resultant* about O:

$$\sum_{i=1}^{n} \mathbf{F}_i = \sum_{i=1}^{n} m_i\mathbf{a}_i \tag{14.4}$$

$$\sum_{i=1}^{n} (\mathbf{r}_i \times \mathbf{F}_i) = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\mathbf{a}_i) \tag{14.5}$$

Linear and angular momentum of a system of particles

Defining the *linear momentum* $\mathbf{L}$ and the *angular momentum* $\mathbf{H}_O$ *about point O* of the system of particles [Sec. 14.3] as

$$\mathbf{L} = \sum_{i=1}^{n} m_i\mathbf{v}_i \qquad \mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\mathbf{v}_i) \tag{14.6, 14.7}$$

we showed that Eqs. (14.4) and (14.5) can be replaced by the equations

$$\Sigma\mathbf{F} = \dot{\mathbf{L}} \qquad \Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O \tag{14.10, 14.11}$$

which express that *the resultant and the moment resultant about O of the external forces are, respectively, equal to the rates of change of the linear momentum and of the angular momentum about O of the system of particles*.

Motion of the mass center of a system of particles

In Sec. 14.4, we defined the mass center of a system of particles as the point G whose position vector $\bar{\mathbf{r}}$ satisfies the equation

$$m\bar{\mathbf{r}} = \sum_{i=1}^{n} m_i\mathbf{r}_i \tag{14.12}$$

where m represents the total mass $\sum_{i=1}^{n} m_i$ of the particles. Differentiating both members of Eq. (14.12) twice with respect to t, we obtained the relations

$$\mathbf{L} = m\bar{\mathbf{v}} \qquad \dot{\mathbf{L}} = m\bar{\mathbf{a}} \qquad (14.14,\ 14.15)$$

where $\bar{\mathbf{v}}$ and $\bar{\mathbf{a}}$ represent, respectively, the velocity and the acceleration of the mass center G. Substituting for $\dot{\mathbf{L}}$ from (14.15) into (14.10), we obtained the equation

$$\Sigma\mathbf{F} = m\bar{\mathbf{a}} \qquad (14.16)$$

from which we concluded that *the mass center of a system of particles moves as if the entire mass of the system and all the external forces were concentrated at that point* [Sample Prob. 14.1].

Angular momentum of a system of particles about its mass center

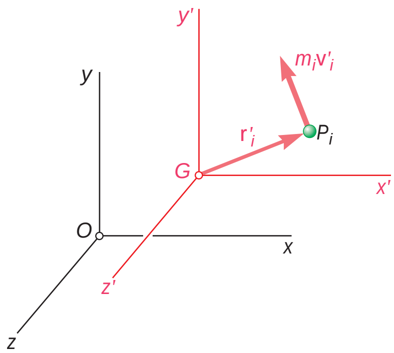


Fig. 14.14

In Sec. 14.5 we considered the motion of the particles of a system with respect to a centroidal frame $Gx'y'z'$ attached to the mass center G of the system and in translation with respect to the newtonian frame $Oxyz$ (Fig. 14.14). We defined the *angular momentum* of the system *about its mass center* G as the sum of the moments about G of the momenta $m_i\mathbf{v}'_i$ of the particles in their motion relative to the frame $Gx'y'z'$. We also noted that the same result can be obtained by considering the moments about G of the momenta $m_i\mathbf{v}_i$ of the particles in their absolute motion. We therefore wrote

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}_i) = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}'_i) \qquad (14.24)$$

and derived the relation

$$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G \qquad (14.23)$$

which expresses that *the moment resultant about G of the external forces is equal to the rate of change of the angular momentum about G of the system of particles*. As will be seen later, this relation is fundamental to the study of the motion of rigid bodies.

Conservation of momentum

When no external force acts on a system of particles [Sec. 14.6], it follows from Eqs. (14.10) and (14.11) that the linear momentum $\mathbf{L}$ and the angular momentum $\mathbf{H}_O$ of the system are conserved [Sample Probs. 14.2 and 14.3]. In problems involving central forces, the angular momentum of the system about the center of force O will also be conserved.

Kinetic energy of a system of particles

The kinetic energy T of a system of particles was defined as the sum of the kinetic energies of the particles [Sec. 14.7]:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 \qquad (14.28)$$

Using the centroidal frame of reference $Gx'y'z'$ of Fig. 14.14, we noted that the kinetic energy of the system can also be obtained by adding the kinetic energy $\frac{1}{2}m\bar{v}^2$ associated with the motion of the mass center G and the kinetic energy of the system in its motion relative to the frame $Gx'y'z'$:

$$T = \tfrac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2 \qquad (14.29)$$

Principle of work and energy

The *principle of work and energy* can be applied to a system of particles as well as to individual particles [Sec. 14.8]. We wrote

$$T_1 + U_{1 y 2} = T_2 \qquad (14.30)$$

and noted that $U_{1 y 2}$ represents the work of *all* the forces acting on the particles of the system, internal as well as external.

Conservation of energy

If all the forces acting on the particles of the system are *conservative*, we can determine the potential energy V of the system and write

$$T_1 + V_1 = T_2 + V_2 \qquad (14.31)$$

which expresses the *principle of conservation of energy* for a system of particles.

Principle of impulse and momentum

We saw in Sec. 14.9 that the *principle of impulse and momentum* for a system of particles can be expressed graphically as shown in Fig. 14.15. It states that the momenta of the particles at time t_1 and the impulses of the external forces from t_1 to t_2 form a system of vectors equipollent to the system of the momenta of the particles at time t_2.

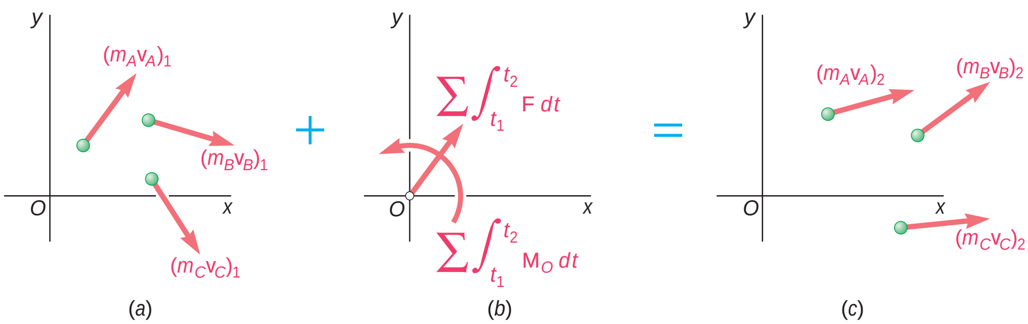


Fig. 14.15

If no external force acts on the particles of the system, the systems of momenta shown in parts *a* and *c* of Fig. 14.15 are equipollent and we have

$$\mathbf{L}_1 = \mathbf{L}_2 \qquad (\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \qquad (14.36, 14.37)$$

Use of conservation principles in the solution of problems involving systems of particles

Many problems involving the motion of systems of particles can be solved by applying simultaneously the principle of impulse and momentum and the principle of conservation of energy [Sample Prob. 14.4] or by expressing that the linear momentum, angular momentum, and energy of the system are conserved [Sample Prob. 14.5].

Variable systems of particles
Steady stream of particles

In the second part of the chapter, we considered *variable systems of particles*. First we considered a *steady stream of particles,* such as a stream of water diverted by a fixed vane or the flow of air through a jet engine [Sec. 14.11]. Applying the principle of impulse and momentum to a system S of particles during a time interval Δt, and including the particles which enter the system at A during that time interval and those (of the same mass Δm) which leave the system at B, we concluded that *the system formed by the momentum* $(\Delta m)\mathbf{v}_A$ *of the particles entering* S *in the time* Δt *and the impulses of the forces exerted on* S *during that time is equipollent to the momentum* $(\Delta m)\mathbf{v}_B$ *of the particles leaving* S *in the same time* Δt (Fig. 14.16). Equating

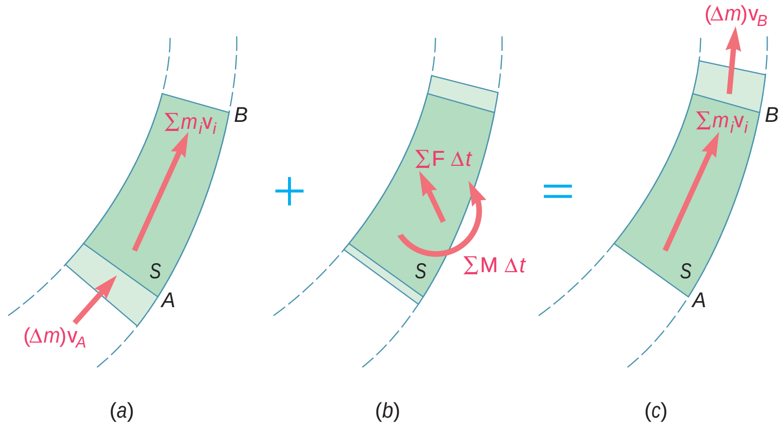


Fig. 14.16

the x components, y components, and moments about a fixed point of the vectors involved, we could obtain as many as three equations, which could be solved for the desired unknowns [Sample Probs. 14.6 and 14.7]. From this result, we could also derive the following expression for the resultant $\Sigma\mathbf{F}$ of the forces exerted on S,

$$\Sigma\mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A) \tag{14.39}$$

where $\mathbf{v}_B - \mathbf{v}_A$ represents the difference between the *vectors* $\mathbf{v}_B$ and $\mathbf{v}_A$ and where dm/dt is the mass rate of flow of the stream (see footnote, page 899).

Systems gaining or losing mass

Considering next a system of particles gaining mass by continually absorbing particles or losing mass by continually expelling particles [Sec. 14.12], as in the case of a rocket, we applied the principle of impulse and momentum to the system during a time interval Δt, being careful to include the particles gained or lost during that time interval [Sample Prob. 14.8]. We also noted that the action on a system S of the particles being *absorbed* by S was equivalent to a thrust

$$\mathbf{P} = \frac{dm}{dt}\mathbf{u} \tag{14.44}$$

where dm/dt is the rate at which mass is being absorbed, and $\mathbf{u}$ is the velocity of the particles *relative to* S. In the case of particles being *expelled* by S, the rate dm/dt is negative and the thrust $\mathbf{P}$ is exerted in a direction opposite to that in which the particles are being expelled.

REVIEW PROBLEMS

14.105 Three identical cars are being unloaded from an automobile carrier. Cars *B* and *C* have just been unloaded and are at rest with their brakes off when car *A* leaves the unloading ramp with a velocity of 5.76 ft/s and hits car *B*, which hits car *C*. Car *A* then again hits car *B*. Knowing that the velocity of car *B* is 5.04 ft/s after the first collision, 0.630 ft/s after the second collision, and 0.709 ft/s after the third collision, determine (*a*) the final velocities of cars *A* and *C*, (*b*) the coefficient of restitution for each of the collisions.

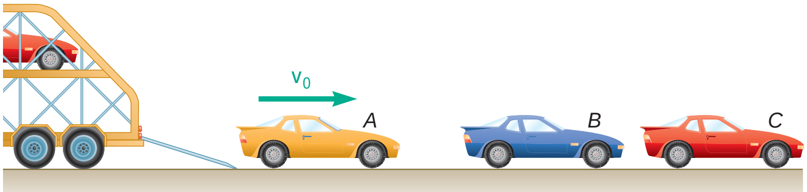


Fig. *P14.105*

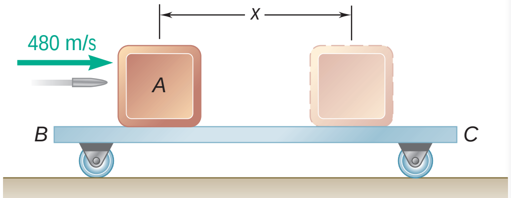


Fig. P14.106

14.106 A 30-g bullet is fired with a velocity of 480 m/s into block *A*, which has a mass of 5 kg. The coefficient of kinetic friction between block *A* and cart *BC* is 0.50. Knowing that the cart has a mass of 4 kg and can roll freely, determine (*a*) the final velocity of the cart and block, (*b*) the final position of the block on the cart.

14.107 An 80-Mg railroad engine *A* coasting at 6.5 km/h strikes a 20-Mg flatcar *C* carrying a 30-Mg load *B* which can slide along the floor of the car ($\mu_k = 0.25$). Knowing that the car was at rest with its brakes released and that it automatically coupled with the engine upon impact, determine the velocity of the car (*a*) immediately after impact, (*b*) after the load has slid to a stop relative to the car.

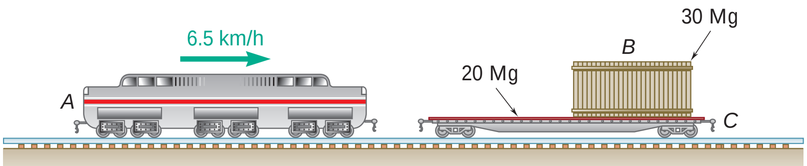


Fig. P14.107

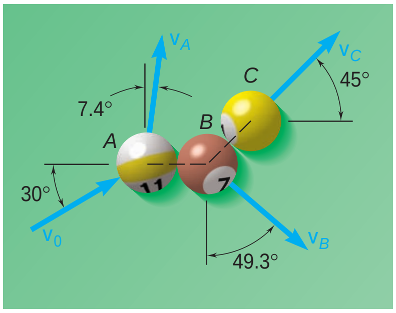


Fig. P14.108

14.108 In a game of pool, ball *A* is moving with a velocity $\mathbf{v}_0$ when it strikes balls *B* and *C* which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and that $v_0 = 12$ ft/s and $v_C = 6.29$ ft/s, determine the magnitude of the velocity of (*a*) ball *A*, (*b*) ball *B*.

14.109 Mass *C*, which has a mass of 4 kg, is suspended from a cord attached to cart *A*, which has a mass of 5 kg and can roll freely on a frictionless horizontal track. A 60-g bullet is fired with a speed $v_0 = 500$ m/s and gets lodged in block *C*. Determine (*a*) the velocity of *C* as it reaches its maximum elevation, (*b*) the maximum vertical distance *h* through which *C* will rise.

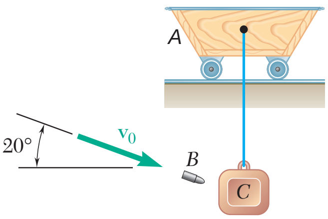


Fig. P14.109

14.110 A 15-lb block B is at rest and a spring of constant $k = 72$ lb/in is held compressed 3 in. by a cord. After 5-lb block A is placed against the end of the spring the cord is cut causing A and B to move. Neglecting friction, determine the velocities of blocks A and B immediately after A leaves B.

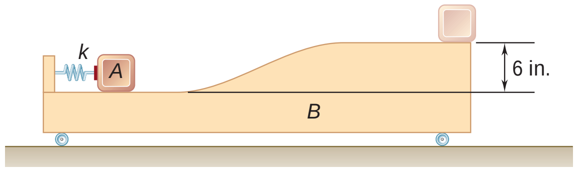


Fig. P14.110

14.111 Car A was at rest 9.28 m south of point O when it was struck in the rear by car B, which was traveling north at a speed v_B. Car C, which was traveling west at a speed v_C, was 40 m east of point O at the time of the collision. Cars A and B stuck together and, because the pavement was covered with ice, they slid into the intersection and were struck by car C which had not changed its speed. Measurements based on a photograph taken from a traffic helicopter shortly after the second collision indicated that the positions of the cars, expressed in meters, were $\mathbf{r}_A = -10.1\mathbf{i} + 16.9\mathbf{j}$, $\mathbf{r}_B = -10.1\mathbf{i} + 20.4\mathbf{j}$, and $\mathbf{r}_C = -19.8\mathbf{i} - 15.2\mathbf{j}$. Knowing that the masses of cars A, B, and C are, respectively, 1400 kg, 1800 kg, and 1600 kg, and that the time elapsed between the first collision and the time the photograph was taken was 3.4 s, determine the initial speeds of cars B and C.

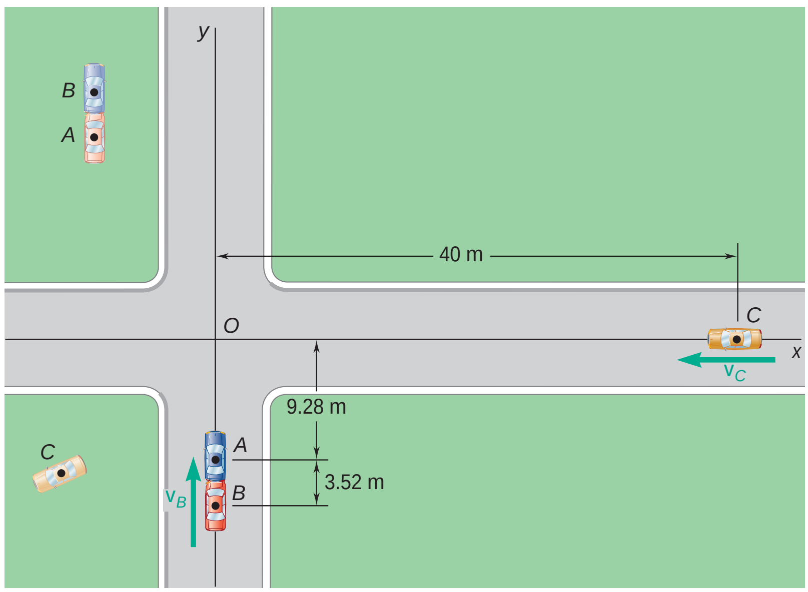


Fig. *P14.111*

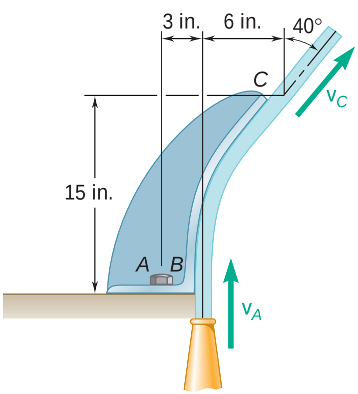


Fig. P14.112

14.112 The nozzle shown discharges water at the rate of 200 gal/min. Knowing that at both B and C the stream of water moves with a velocity of magnitude 100 ft/s, and neglecting the weight of the vane, determine the force-couple system which must be applied at A to hold the vane in place (1 ft^3 = 7.48 gal).

14.113 Prior to takeoff, the pilot of a 6000-lb twin-engine airplane tests the reversible-pitch propellers with the brakes at point B locked. Knowing that the velocity of the air in the two 6.6-ft-diameter slipstreams is 60 ft/s and that point G is the center of gravity of the airplane, determine the reactions at points A and B. Assume $g = 0.075$ lb/ft^3 and neglect the approach velocity of the air.

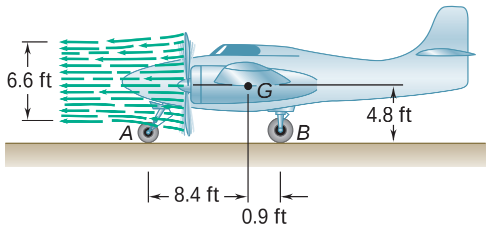


Fig. *P14.113*

14.114 A railroad car of length L and mass m_0 when empty is moving freely on a horizontal track while being loaded with sand from a stationary chute at a rate $dm/dt = q$. Knowing that the car was approaching the chute at a speed v_0, determine (a) the mass of the car and its load after the car has cleared the chute, (b) the speed of the car at that time.

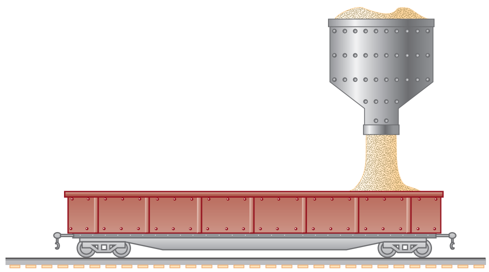


Fig. P14.114

14.115 A garden sprinkler has four rotating arms, each of which consists of two horizontal straight sections of pipe forming an angle of 120° with each other. Each arm discharges water at a rate of 20 L/min with a velocity of 18 m/s relative to the arm. Knowing that the friction between the moving and stationary parts of the sprinkler is equivalent to a couple of magnitude $M = 0.375$ N · m, determine the constant rate at which the sprinkler rotates.

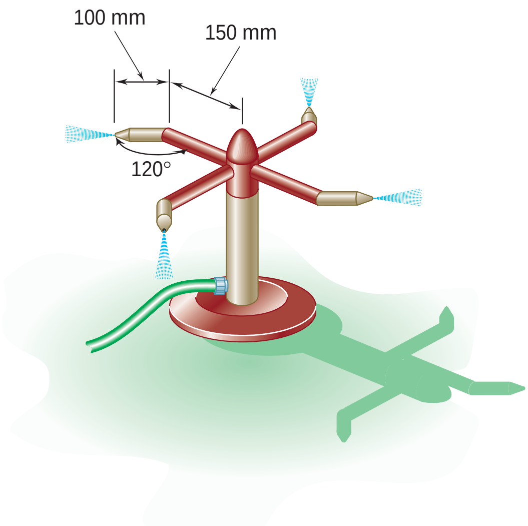


Fig. P14.115

14.116 A chain of length l and mass m falls through a small hole in a plate. Initially, when y is very small, the chain is at rest. In each case shown, determine (a) the acceleration of the first link A as a function of y, (b) the velocity of the chain as the last link passes through the hole. In case *1* assume that the individual links are at rest until they fall through the hole; in case *2* assume that at any instant all links have the same speed. Ignore the effect of friction.

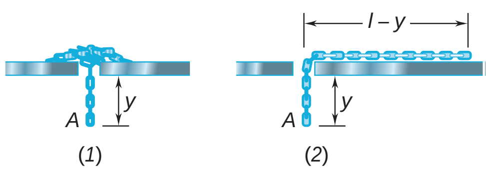


Fig. P14.116

COMPUTER PROBLEMS

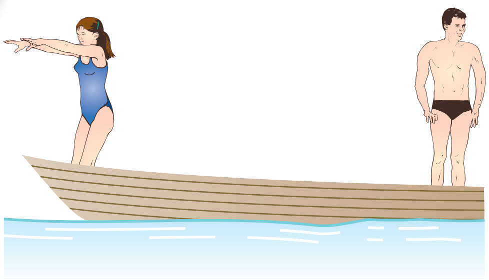


Fig. P14.C1

14.C1 A man and a woman, of weights 180 lb and 120 lb, respectively, stand at opposite ends of a stationary boat of weight 300 lb, ready to dive with velocities v_m and v_w, respectively, relative to the boat. Use computational software to determine the velocity of the boat after both swimmers have dived if (*a*) the woman dives first, (*b*) the man dives first. Solve that problem assuming that the velocities of the woman and the man relative to the boat are, respectively, (i) 14 ft/s and 18 ft/s, (ii) 18 ft/s and 14 ft/s.

14.C2 A system of particles consists of n particles A_i of mass m_i and coordinates x_i, y_i, and z_i, having velocities of components $(v_x)_i$, $(v_y)_i$, and $(v_z)_i$. Derive expressions for the components of the angular momentum of the system about the origin O of the coordinates. Use computational software to solve Probs. 14.11 and 14.13.

14.C3 A shell moving with a velocity of known components v_x, v_y, and v_z explodes into three fragments of weights W_1, W_2, and W_3 at point A_0 at a distance d from a vertical wall. Use computational software to determine the speed of each fragment immediately after the explosion, knowing the coordinates x_i and y_i of the points A_i ($i = 1, 2, 3$) where the fragments hit the wall. Use this program to solve (*a*) Prob. 14.24, (*b*) Prob. 14.25.

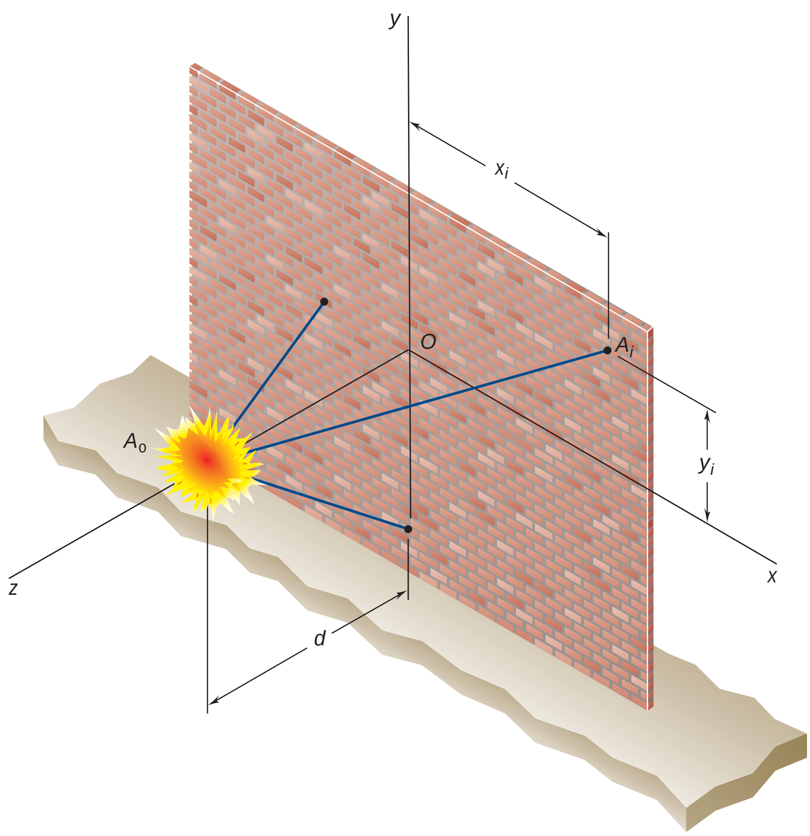


Fig. P14.C3

14.C4 As a 6000-kg training plane lands on an aircraft carrier at a speed of 180 km/h, its tail hooks into the end of an 80-m long chain which lies in a pile below deck. Knowing that the chain has a mass per unit length of 50 kg/m and assuming no other retarding force, use computational software to determine the distance traveled by the plane while the chain is being pulled out and the corresponding values of the time and of the velocity and deceleration of the plane.

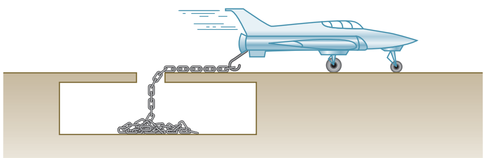


Fig. P14.C4

14.C5 A 16-Mg jet airplane maintains a constant speed of 774 km/h while climbing at an angle a = 18°. The airplane scoops in air at a rate of 300 kg/s and discharges it with a velocity of 665 m/s relative to the airplane. Knowing that the pilot changes the angle of climb a while maintaining the same engine setting, use computational software to calculate and plot values of a from 0 to 20° (*a*) the initial acceleration of the plane, (*b*) the maximum speed that will be attained. Assume that the drag due to air friction is proportional to the square of the speed.

14.C6 A rocket has a weight of 2400 lb, including 2000 lb of fuel, which is consumed at the rate of 25 lb/s and ejected with a relative velocity of 12,000 ft/s. Knowing that the rocket is fired vertically from the ground, assuming a constant value for the acceleration of gravity, and using 4-s time intervals, use computational software to determine and plot from the time of ignition to the time when the last particle of fuel is being consumed (*a*) the acceleration a of the rocket in ft/s^2, (*b*) its velocity v in ft/s, (*c*) its elevation h above the ground in miles. (*Hint:* Use for v the expression derived in Sample Prob. 14.8, and integrate this expression analytically to obtain h.)

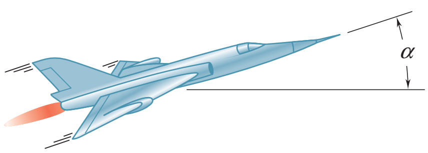


Fig. P14.C5

This huge crank belongs to a Wartsila-Sulzer RTA96-C turbocharged two-stroke diesel engine. In this chapter you will learn to perform the *kinematic* analysis of rigid bodies that undergo *translation, fixed axis rotation,* and *general plane motion.*

CHAPTER 15

Kinematics of Rigid Bodies

15.1 INTRODUCTION

In this chapter, the kinematics of *rigid bodies* will be considered. You will investigate the relations existing between the time, the positions, the velocities, and the accelerations of the various particles forming a rigid body. As you will see, the various types of rigid-body motion can be conveniently grouped as follows:

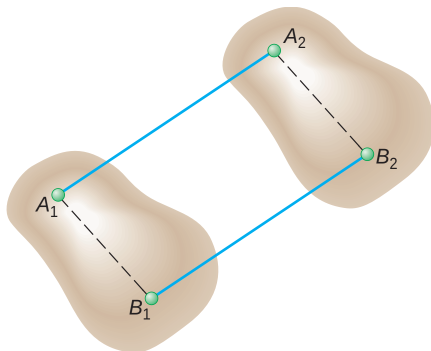


Fig. 15.1

1. *Translation*. A motion is said to be a translation if any straight line inside the body keeps the same direction during the motion. It can also be observed that in a translation all the particles forming the body move along parallel paths. If these paths are straight lines, the motion is said to be a *rectilinear translation* (Fig. 15.1); if the paths are curved lines, the motion is a *curvilinear translation* (Fig. 15.2).
2. *Rotation About a Fixed Axis*. In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis (Fig. 15.3). If this axis, called the *axis of rotation*, intersects the rigid body, the particles located on the axis have zero velocity and zero acceleration.

   Rotation should not be confused with certain types of curvilinear translation. For example, the plate shown in Fig. 15.4*a* is in curvilinear translation, with all its particles moving along *parallel* circles, while the plate shown in Fig. 15.4*b* is in rotation, with all its particles moving along *concentric* circles.

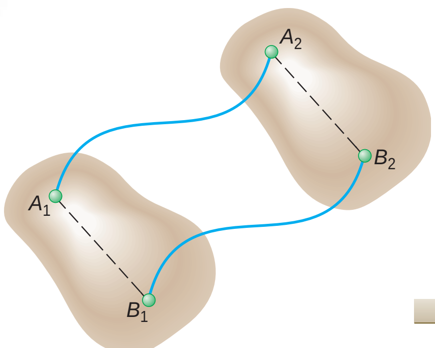


Fig. 15.2

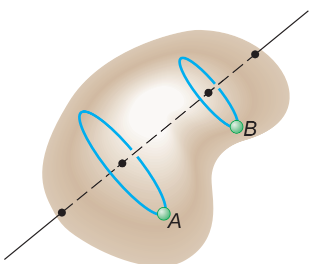
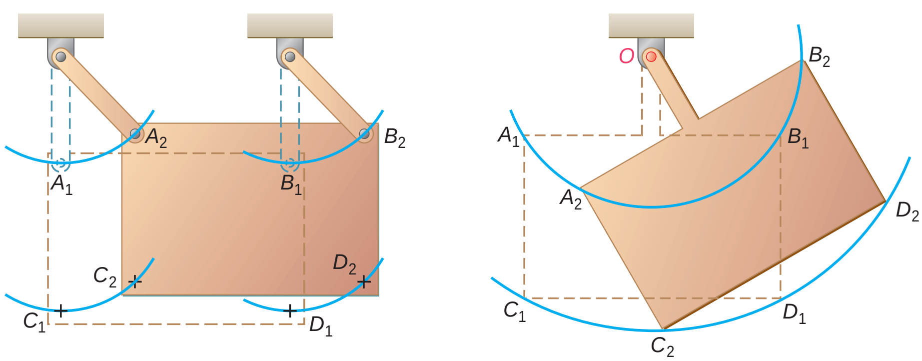


Fig. 15.3



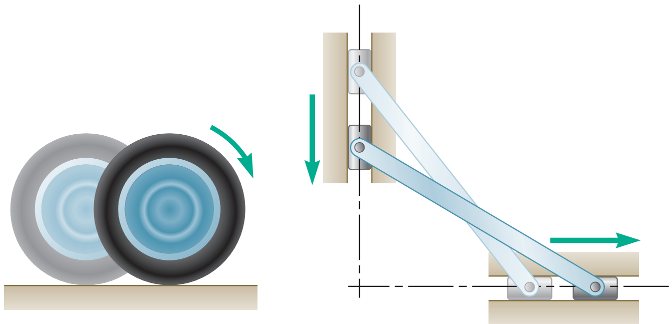
(*a*) Curvilinear translation (*b*) Rotation

Fig. 15.4

In the first case, any given straight line drawn on the plate will maintain the same direction, whereas in the second case, point O remains fixed.

Because each particle moves in a given plane, the rotation of a body about a fixed axis is said to be a *plane motion*.

3. *General Plane Motion*. There are many other types of plane motion, i.e., motions in which all the particles of the body move in parallel planes. Any plane motion that is neither a rotation nor a translation is referred to as a general plane motion. Two examples of general plane motion are given in Fig. 15.5.



(*a*) Rolling wheel

(*b*) Sliding rod

Fig. 15.5

4. *Motion About a Fixed Point*. The three-dimensional motion of a rigid body attached at a fixed point O, e.g., the motion of a top on a rough floor (Fig. 15.6), is known as motion about a fixed point.
5. *General Motion*. Any motion of a rigid body that does not fall in any of the categories above is referred to as a general motion.

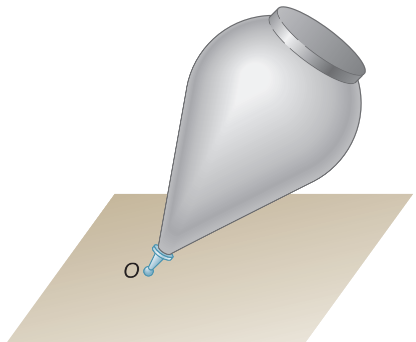


Fig. 15.6

After a brief discussion in Sec. 15.2 of the motion of translation, the rotation of a rigid body about a fixed axis is considered in Sec. 15.3. The *angular velocity* and the *angular acceleration* of a rigid body about a fixed axis will be defined, and you will learn to express the velocity and the acceleration of a given point of the body in terms of its position vector and the angular velocity and angular acceleration of the body.

The following sections are devoted to the study of the general plane motion of a rigid body and to its application to the analysis of mechanisms such as gears, connecting rods, and pin-connected linkages. Resolving the plane motion of a slab into a translation and a rotation (Secs. 15.5 and 15.6), we will then express the velocity of a point B of the slab as the sum of the velocity of a reference point A and of the velocity of B relative to a frame of reference translating with A (i.e., moving with A but not rotating). The same approach is used later in Sec. 15.8 to express the acceleration of B in terms of the acceleration of A and of the acceleration of B relative to a frame translating with A.

An alternative method for the analysis of velocities in plane motion, based on the concept of *instantaneous center of rotation*, is given in Sec. 15.7; and still another method of analysis, based on the use of parametric expressions for the coordinates of a given point, is presented in Sec. 15.9.

The motion of a particle relative to a rotating frame of reference and the concept of *Coriolis acceleration* are discussed in Secs. 15.10 and 15.11, and the results obtained are applied to the analysis of the plane motion of mechanisms containing parts which slide on each other.

The remaining part of the chapter is devoted to the analysis of the three-dimensional motion of a rigid body, namely, the motion of a rigid body with a fixed point and the general motion of a rigid body. In Secs. 15.12 and 15.13, a fixed frame of reference or a frame of reference in translation will be used to carry out this analysis; in Secs. 15.14 and 15.15, the motion of the body relative to a rotating frame or to a frame in general motion will be considered, and the concept of Coriolis acceleration will again be used.

Photo 15.1 This replica of a battering ram at *Château des Baux, France* undergoes curvilinear translation.

15.2 TRANSLATION

Consider a rigid body in translation (either rectilinear or curvilinear translation), and let A and B be any two of its particles (Fig. 15.7*a*). Denoting, respectively, by $\mathbf{r}_A$ and $\mathbf{r}_B$ the position vectors of A and B with respect to a fixed frame of reference and by $\mathbf{r}_{B/A}$ the vector joining A and B, we write

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \tag{15.1}$$

Let us differentiate this relation with respect to t. We note that from the very definition of a translation, the vector $\mathbf{r}_{B/A}$ must maintain a constant direction; its magnitude must also be constant, since A and B

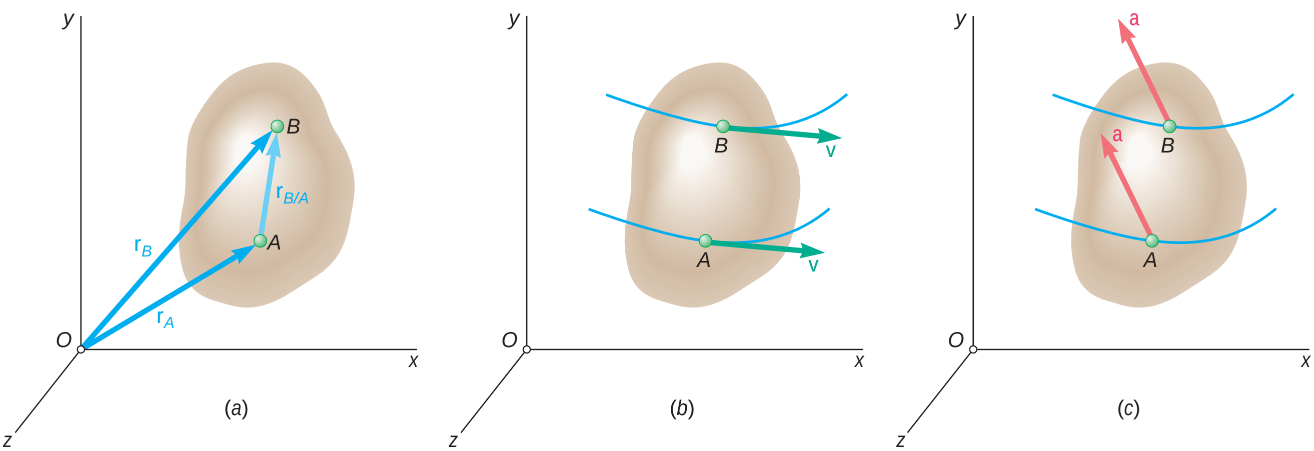


Fig. 15.7

belong to the same rigid body. Thus, the derivative of $\mathbf{r}_{B/A}$ is zero and we have

$$\mathbf{v}_B = \mathbf{v}_A \tag{15.2}$$

Differentiating once more, we write

$$\mathbf{a}_B = \mathbf{a}_A \tag{15.3}$$

Thus, *when a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant* (Fig. 15.7*b* and *c*). In the case of curvilinear translation, the velocity and acceleration change in direction as well as in magnitude at every instant. In the case of rectilinear translation, all particles of the body move along parallel straight lines, and their velocity and acceleration keep the same direction during the entire motion.

15.3 ROTATION ABOUT A FIXED AXIS

Consider a rigid body which rotates about a fixed axis AA'. Let P be a point of the body and $\mathbf{r}$ its position vector with respect to a fixed frame of reference. For convenience, let us assume that the frame is centered at point O on AA' and that the z axis coincides with AA' (Fig. 15.8). Let B be the projection of P on AA'; since P must remain at a constant distance from B, it will describe a circle of center B and of radius $r \sin \phi$, where ϕ denotes the angle formed by $\mathbf{r}$ and AA'.

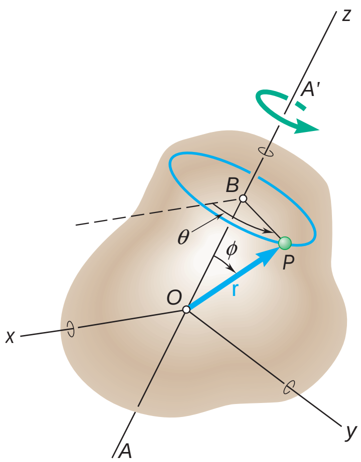


Fig. 15.8

The position of P and of the entire body is completely defined by the angle θ the line BP forms with the zx plane. The angle θ is known as the *angular coordinate* of the body and is defined as positive when viewed as counterclockwise from A'. The angular coordinate will be expressed in radians (rad) or, occasionally, in degrees (°) or revolutions (rev). We recall that

$$1 \text{ rev} = 2\pi \text{ rad} = 360°$$

We recall from Sec. 11.9 that the velocity $\mathbf{v} = d\mathbf{r}/dt$ of a particle P is a vector tangent to the path of P and of magnitude $v = ds/dt$. Observing that the length Δs of the arc described by P when the body rotates through $\Delta\theta$ is

$$\Delta s = (BP)\,\Delta\theta = (r \sin \phi)\,\Delta\theta$$

and dividing both members by Δt, we obtain at the limit, as Δt approaches zero,

$$v = \frac{ds}{dt} = r\dot{\theta} \sin \phi \tag{15.4}$$

where $\dot{\theta}$ denotes the time derivative of θ. (Note that the angle θ depends on the position of P within the body, but the rate of change $\dot{\theta}$ is itself independent of P.) We conclude that the velocity $\mathbf{v}$ of P is a vector perpendicular to the plane containing AA' and $\mathbf{r}$, and of

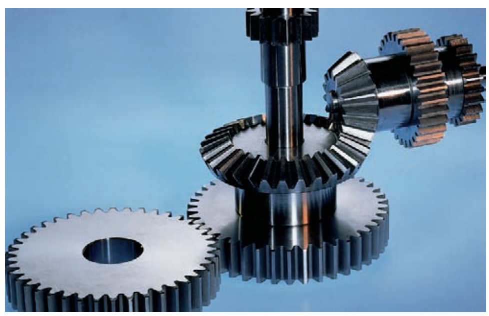


Photo 15.2 For the central gear rotating about a fixed axis, the angular velocity and angular acceleration of that gear are vectors directed along the vertical axis of rotation.

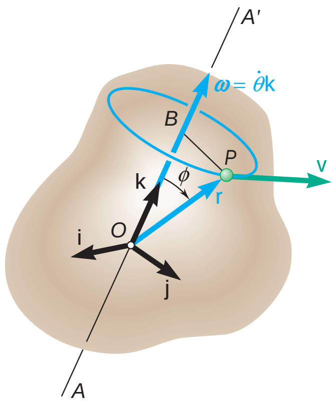


Fig. 15.9

magnitude v defined by (15.4). But this is precisely the result we would obtain if we drew along AA' a vector $\mathsf{V} = \dot{\mathsf{u}}\mathbf{k}$ and formed the vector product $\mathsf{V} \times \mathbf{r}$ (Fig. 15.9). We thus write

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathsf{V} \times \mathbf{r} \tag{15.5}$$

The vector

$$\mathsf{V} = \mathsf{v}\mathbf{k} = \dot{\mathsf{u}}\mathbf{k} \tag{15.6}$$

which is directed along the axis of rotation, is called the *angular velocity* of the body and is equal in magnitude to the rate of change $\dot{\mathsf{u}}$ of the angular coordinate; its sense may be obtained by the right-hand rule (Sec. 3.6) from the sense of rotation of the body.†

The acceleration $\mathbf{a}$ of the particle P will now be determined. Differentiating (15.5) and recalling the rule for the differentiation of a vector product (Sec. 11.10), we write

$$\begin{aligned}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathsf{V} \times \mathbf{r}) \\
&= \frac{d\mathsf{V}}{dt} \times \mathbf{r} + \mathsf{V} \times \frac{d\mathbf{r}}{dt} \\
&= \frac{d\mathsf{V}}{dt} \times \mathbf{r} + \mathsf{V} \times \mathbf{v}
\end{aligned} \tag{15.7}$$

The vector $d\mathsf{V}/dt$ is denoted by A and is called the *angular acceleration* of the body. Substituting also for $\mathbf{v}$ from (15.5), we have

$$\mathbf{a} = \mathsf{A} \times \mathbf{r} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r}) \tag{15.8}$$

Differentiating (15.6) and recalling that $\mathbf{k}$ is constant in magnitude and direction, we have

$$\mathsf{A} = \mathsf{a}\mathbf{k} = \dot{\mathsf{v}}\mathbf{k} = \ddot{\mathsf{u}}\mathbf{k} \tag{15.9}$$

Thus, the angular acceleration of a body rotating about a fixed axis is a vector directed along the axis of rotation, and is equal in magnitude to the rate of change $\dot{\mathsf{v}}$ of the angular velocity. Returning to (15.8), we note that the acceleration of P is the sum of two vectors. The first vector is equal to the vector product $\mathsf{A} \times \mathbf{r}$; it is tangent to the circle described by P and therefore represents the tangential component of the acceleration. The second vector is equal to the *vector triple product* $\mathsf{V} \times (\mathsf{V} \times \mathbf{r})$ obtained by forming the vector product of V and $\mathsf{V} \times \mathbf{r}$; since $\mathsf{V} \times \mathbf{r}$ is tangent to the circle described by P, the vector triple product is directed toward the center B of the circle and therefore represents the normal component of the acceleration.

†It will be shown in Sec. 15.12 in the more general case of a rigid body rotating simultaneously about axes having different directions that angular velocities obey the parallelogram law of addition and thus are actually vector quantities.

Rotation of a Representative Slab. The rotation of a rigid body about a fixed axis can be defined by the motion of a representative slab in a reference plane perpendicular to the axis of rotation. Let us choose the xy plane as the reference plane and assume that it coincides with the plane of the figure, with the z axis pointing out of the paper (Fig. 15.10). Recalling from (15.6) that $\mathsf{V} = \mathsf{v}\mathbf{k}$, we

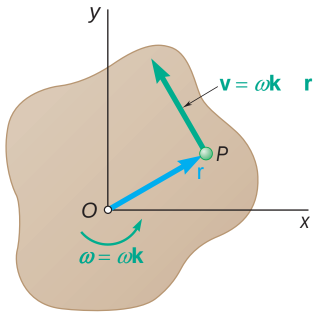


Fig. 15.10

note that a positive value of the scalar v corresponds to a counterclockwise rotation of the representative slab, and a negative value to a clockwise rotation. Substituting $\mathsf{v}\mathbf{k}$ for V into Eq. (15.5), we express the velocity of any given point P of the slab as

$$\boxed{\mathbf{v} = \mathsf{v}\mathbf{k} \times \mathbf{r}} \tag{15.10}$$

Since the vectors $\mathbf{k}$ and $\mathbf{r}$ are mutually perpendicular, the magnitude of the velocity $\mathbf{v}$ is

$$\boxed{v = r\mathsf{v}} \tag{15.10$'$}$$

and its direction can be obtained by rotating $\mathbf{r}$ through 90° in the sense of rotation of the slab.

Substituting $\mathsf{V} = \mathsf{v}\mathbf{k}$ and $\mathsf{A} = \mathsf{a}\mathbf{k}$ into Eq. (15.8), and observing that cross-multiplying $\mathbf{r}$ twice by $\mathbf{k}$ results in a 180° rotation of the vector $\mathbf{r}$, we express the acceleration of point P as

$$\boxed{\mathbf{a} = \mathsf{a}\mathbf{k} \times \mathbf{r} - \mathsf{v}^2\mathbf{r}} \tag{15.11}$$

Resolving $\mathbf{a}$ into tangential and normal components (Fig. 15.11), we write

$$\begin{aligned} \mathbf{a}_t &= \mathsf{a}\mathbf{k} \times \mathbf{r} \qquad & a_t &= r\mathsf{a} \\ \mathbf{a}_n &= -\mathsf{v}^2\mathbf{r} \qquad & a_n &= r\mathsf{v}^2 \end{aligned} \tag{15.11$'$}$$

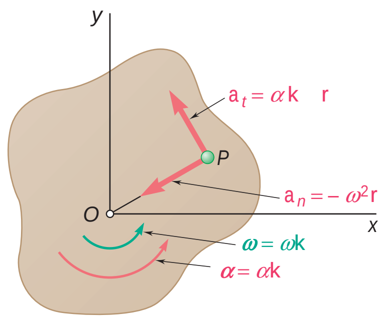


Fig. 15.11

The tangential component $\mathbf{a}_t$ points in the counterclockwise direction if the scalar a is positive, and in the clockwise direction if a is negative. The normal component $\mathbf{a}_n$ always points in the direction opposite to that of $\mathbf{r}$, that is, toward O.

Photo 15.3 If the lower roll has a constant angular velocity, the speed of the paper being wound onto it increases as the radius of the roll increases.

15.4 EQUATIONS DEFINING THE ROTATION OF A RIGID BODY ABOUT A FIXED AXIS

The motion of a rigid body rotating about a fixed axis AA' is said to be *known* when its angular coordinate u can be expressed as a known function of t. In practice, however, the rotation of a rigid body is seldom defined by a relation between u and t. More often, the conditions of motion will be specified by the type of angular acceleration that the body possesses. For example, a may be given as a function of t, as a function of u, or as a function of v. Recalling the relations (15.6) and (15.9), we write

$$\mathrm{v} = \frac{d\mathrm{u}}{dt} \tag{15.12}$$

$$\mathrm{a} = \frac{d\mathrm{v}}{dt} = \frac{d^2\mathrm{u}}{dt^2} \tag{15.13}$$

or, solving (15.12) for dt and substituting into (15.13),

$$\mathrm{a} = \mathrm{v}\frac{d\mathrm{v}}{d\mathrm{u}} \tag{15.14}$$

Since these equations are similar to those obtained in Chap. 11 for the rectilinear motion of a particle, their integration can be performed by following the procedure outlined in Sec. 11.3.

Two particular cases of rotation are frequently encountered:

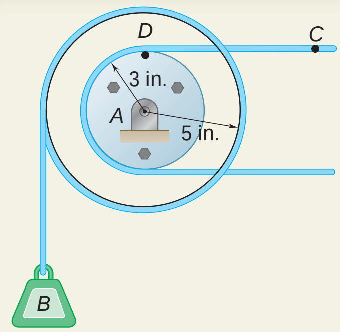
1. *Uniform Rotation*. This case is characterized by the fact that the angular acceleration is zero. The angular velocity is thus constant, and the angular coordinate is given by the formula

$$\mathrm{u} = \mathrm{u}_0 + \mathrm{v}t \tag{15.15}$$

2. *Uniformly Accelerated Rotation*. In this case, the angular acceleration is constant. The following formulas relating angular velocity, angular coordinate, and time can then be derived in a manner similar to that described in Sec. 11.5. The similarity between the formulas derived here and those obtained for the rectilinear uniformly accelerated motion of a particle is apparent.

$$\begin{aligned} \mathrm{v} &= \mathrm{v}_0 + \mathrm{a}t \\ \mathrm{u} &= \mathrm{u}_0 + \mathrm{v}_0 t + \tfrac{1}{2}\mathrm{a}t^2 \\ \mathrm{v}^2 &= \mathrm{v}_0^2 + 2\mathrm{a}(\mathrm{u} - \mathrm{u}_0) \end{aligned} \tag{15.16}$$

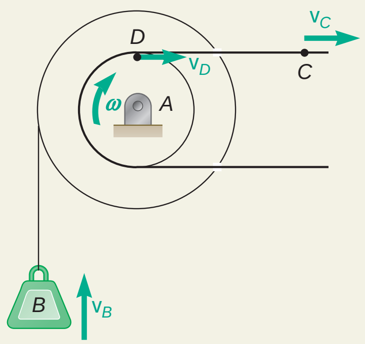
It should be emphasized that formula (15.15) can be used only when a = 0, and formulas (15.16) can be used only when a = constant. In any other case, the general formulas (15.12) to (15.14) should be used.



SAMPLE PROBLEM 15.1

Load B is connected to a double pulley by one of the two inextensible cables shown. The motion of the pulley is controlled by cable C, which has a constant acceleration of 9 in./s^2 and an initial velocity of 12 in./s, both directed to the right. Determine (*a*) the number of revolutions executed by the pulley in 2 s, (*b*) the velocity and change in position of the load B after 2 s, and (*c*) the acceleration of point D on the rim of the inner pulley at $t = 0$.

SOLUTION



a. Motion of Pulley. Since the cable is inextensible, the velocity of point D is equal to the velocity of point C and the tangential component of the acceleration of D is equal to the acceleration of C.

$$(\mathbf{v}_D)_0 = (\mathbf{v}_C)_0 = 12 \text{ in./s } \text{y} \qquad (\mathbf{a}_D)_t = \mathbf{a}_C = 9 \text{ in./s}^2 \text{ y}$$

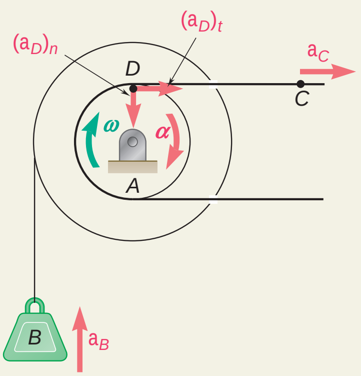
Noting that the distance from D to the center of the pulley is 3 in., we write

$$(v_D)_0 = r\text{v}_0 \qquad 12 \text{ in./s} = (3 \text{ in.})\text{v}_0 \qquad \text{V}_0 = 4 \text{ rad/s i}$$
$$(a_D)_t = r\text{a} \qquad 9 \text{ in./s}^2 = (3 \text{ in.})\text{a} \qquad \text{A} = 3 \text{ rad/s}^2 \text{ i}$$

Using the equations of uniformly accelerated motion, we obtain, for $t = 2$ s,

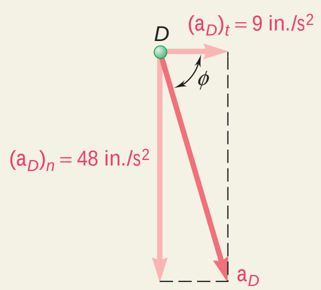
$$\text{v} = \text{v}_0 + \text{a}t = 4 \text{ rad/s} + (3 \text{ rad/s}^2)(2 \text{ s}) = 10 \text{ rad/s}$$
$$\text{V} = 10 \text{ rad/s i}$$
$$\text{u} = \text{v}_0 t + \tfrac{1}{2}\text{a}t^2 = (4 \text{ rad/s})(2 \text{ s}) + \tfrac{1}{2}(3 \text{ rad/s}^2)(2 \text{ s})^2 = 14 \text{ rad}$$
$$\text{u} = 14 \text{ rad i}$$

$$\text{Number of revolutions} = (14 \text{ rad})\left(\frac{1 \text{ rev}}{2\text{p rad}}\right) = 2.23 \text{ rev} \quad ◀$$



b. Motion of Load *B*. Using the following relations between linear and angular motion, with $r = 5$ in., we write

$$v_B = r\text{v} = (5 \text{ in.})(10 \text{ rad/s}) = 50 \text{ in./s} \qquad \mathbf{v}_B = 50 \text{ in./s x} \quad ◀$$
$$\Delta y_B = r\text{u} = (5 \text{ in.})(14 \text{ rad}) = 70 \text{ in.} \qquad \Delta y_B = 70 \text{ in. upward} \quad ◀$$



c. Acceleration of Point *D* at *t* = 0. The tangential component of the acceleration is

$$(\mathbf{a}_D)_t = \mathbf{a}_C = 9 \text{ in./s}^2 \text{ y}$$

Since, at $t = 0$, $\text{v}_0 = 4$ rad/s, the normal component of the acceleration is

$$(a_D)_n = r_D \text{v}_0^2 = (3 \text{ in.})(4 \text{ rad/s})^2 = 48 \text{ in./s}^2 \qquad (\mathbf{a}_D)_n = 48 \text{ in./s}^2 \text{ w}$$

The magnitude and direction of the total acceleration can be obtained by writing

$$\tan \text{f} = (48 \text{ in./s}^2)/(9 \text{ in./s}^2) \qquad \text{f} = 79.4°$$
$$a_D \sin 79.4° = 48 \text{ in./s}^2 \qquad a_D = 48.8 \text{ in./s}^2$$
$$\mathbf{a}_D = 48.8 \text{ in./s}^2 \text{ c } 79.4° \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we began the study of the motion of rigid bodies by considering two particular types of motion of rigid bodies: *translation* and *rotation* about a *fixed axis*.

1. Rigid body in translation. At any given instant, all the points of a rigid body in translation have the *same velocity* and the *same acceleration* (Fig. 15.7).

2. Rigid body rotating about a fixed axis. The position of a rigid body rotating about a fixed axis was defined at any given instant by the *angular coordinate* u, which is usually measured in *radians*. Selecting the unit vector $\mathbf{k}$ along the fixed axis and in such a way that the rotation of the body appears counterclockwise as seen from the tip of $\mathbf{k}$, we defined the *angular velocity* V and the *angular acceleration* A of the body:

$$\mathsf{V} = \dot{\mathsf{u}}\mathbf{k} \qquad \mathsf{A} = \ddot{\mathsf{u}}\mathbf{k} \tag{15.6, 15.9}$$

In solving problems, keep in mind that the vectors V and A are both directed along the fixed axis of rotation and that their sense can be obtained by the right-hand rule.

a. The velocity of a point *P* of a body rotating about a fixed axis was found to be

$$\mathbf{v} = \mathsf{V} \times \mathbf{r} \tag{15.5}$$

where V is the angular velocity of the body and $\mathbf{r}$ is the position vector drawn from any point on the axis of rotation to point *P* (Fig. 15.9).

b. The acceleration of point *P* was found to be

$$\mathbf{a} = \mathsf{A} \times \mathbf{r} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r}) \tag{15.8}$$

Since vector products are not commutative, *be sure to write the vectors in the order shown* when using either of the above two equations.

3. Rotation of a representative slab. In many problems, you will be able to reduce the analysis of the rotation of a three-dimensional body about a fixed axis to the study of the rotation of a representative slab in a plane perpendicular to the fixed axis. The *z* axis should be directed along the axis of rotation and point out of the paper. Thus, the representative slab will be rotating in the *xy* plane about the origin *O* of the coordinate system (Fig. 15.10).

To solve problems of this type you should do the following:

a. Draw a diagram of the representative slab, showing its dimensions, its angular velocity and angular acceleration, as well as the vectors representing the velocities and accelerations of the points of the slab for which you have or seek information.

b. Relate the rotation of the slab and the motion of points of the slab by writing the equations

$$v = r\omega \qquad (15.10')$$

$$a_t = r\alpha \qquad a_n = r\omega^2 \qquad (15.11')$$

Remember that the velocity $\mathbf{v}$ and the component $\mathbf{a}_t$ of the acceleration of a point P of the slab are tangent to the circular path described by P. The directions of $\mathbf{v}$ and $\mathbf{a}_t$ are found by rotating the position vector $\mathbf{r}$ through 90° in the sense indicated by $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, respectively. The normal component $\mathbf{a}_n$ of the acceleration of P is always directed toward the axis of rotation.

4. Equations defining the rotation of a rigid body. You must have been pleased to note the similarity existing between the equations defining the rotation of a rigid body about a fixed axis [Eqs. (15.12) through (15.16)] and those in Chap. 11 defining the rectilinear motion of a particle [Eqs. (11.1) through (11.8)]. All you have to do to obtain the new set of equations is to substitute θ, ω, and α for x, v, and a in the equations of Chap. 11.

PROBLEMS

CONCEPT QUESTIONS

15.CQ1 A rectangular plate swings from arms of equal length as shown. What is the magnitude of the angular velocity of the plate?

a. 0 rad/s
b. 1 rad/s
c. 2 rad/s
d. 3 rad/s
e. Need to know the location of the center of gravity.

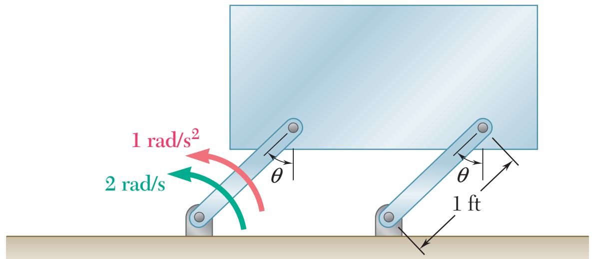


Fig. P15.CQ1

15.CQ2 Knowing that wheel *A* rotates with a constant angular velocity and that no slipping occurs between ring *C* and wheel *A* and wheel *B*, which of the following statements concerning the angular speeds of the three objects is true?

a. $\omega_a = \omega_b$
b. $\omega_a > \omega_b$
c. $\omega_a < \omega_b$
d. $\omega_a = \omega_c$
e. The contact points between *A* and *C* have the same acceleration.

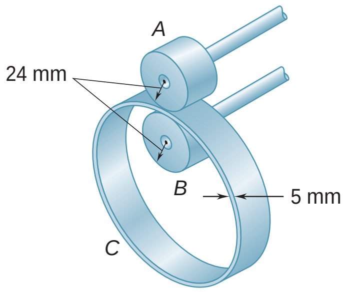


Fig. P15.CQ2

END-OF-SECTION PROBLEMS

15.1 The brake drum is attached to a larger flywheel that is not shown. The motion of the brake drum is defined by the relation $\theta = 36t - 1.6t^2$, where θ is expressed in radians and t in seconds. Determine (*a*) the angular velocity at $t = 2$ s, (*b*) the number of revolutions executed by the brake drum before coming to rest.

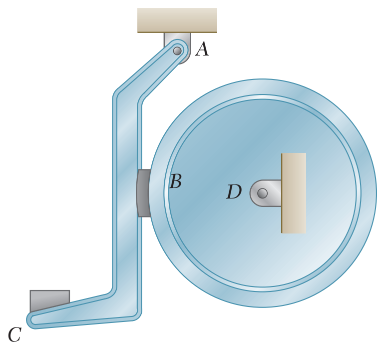


Fig. P15.1

15.2 The motion of an oscillating crank is defined by the relation $\theta = \theta_0 \sin(\pi t/T) - (0.5\theta_0)\sin(2\pi t/T)$, where θ is expressed in radians and t in seconds. Knowing that $\theta_0 = 6$ rad and $T = 4$ s, determine the angular coordinate, the angular velocity, and the angular acceleration of the crank when (*a*) $t = 0$, (*b*) $t = 2$ s.

15.3 The motion of a disk rotating in an oil bath is defined by the relation $\theta = \theta_0(1 - e^{-t/4})$, where θ is expressed in radians and t in seconds. Knowing that $\theta_0 = 0.40$ rad, determine the angular coordinate, velocity, and acceleration of the disk when (*a*) $t = 0$, (*b*) $t = 3$ s, (*c*) $t = \infty$.

15.4 The rotor of a gas turbine is rotating at a speed of 6900 rpm when the turbine is shut down. It is observed that 4 min is required for the rotor to coast to rest. Assuming uniformly accelerated motion, determine (*a*) the angular acceleration, (*b*) the number of revolutions that the rotor executes before coming to rest.

15.5 A small grinding wheel is attached to the shaft of an electric motor which has a rated speed of 3600 rpm. When the power is turned on, the unit reaches its rated speed in 5 s, and when the power is turned off, the unit coasts to rest in 70 s. Assuming uniformly accelerated motion, determine the number of revolutions that the motor executes (*a*) in reaching its rated speed, (*b*) in coasting to rest.

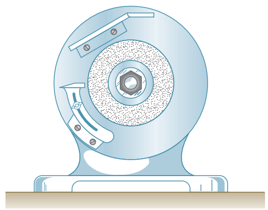


Fig. P15.5

15.6 A connecting rod is supported by a knife-edge at point *A*. For small oscillations the angular acceleration of the connecting rod is governed by the relation $a = -6u$ where a is expressed in rad/s^2 and u in radians. Knowing that the connecting rod is released from rest when $u = 20°$, determine (*a*) the maximum angular velocity, (*b*) the angular position when $t = 2$ s.

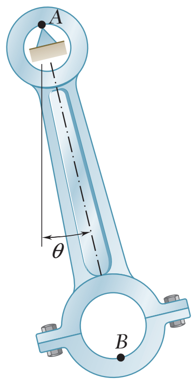


Fig. P15.6

15.7 When studying whiplash resulting from rear-end collisions, the rotation of the head is of primary interest. An impact test was performed, and it was found that the angular acceleration of the head is defined by the relation $a = 700 \cos u + 70 \sin u$, where a is expressed in rad/s^2 and u in radians. Knowing that the head is initially at rest, determine the angular velocity of the head when $u = 30°$.

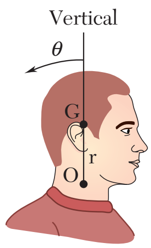


Fig. *P15.7*

15.8 The angular acceleration of an oscillating disk is defined by the relation $a = -ku$. Determine (*a*) the value of k for which $v = 8$ rad/s when $u = 0$ and $u = 4$ rad when $v = 0$, (*b*) the angular velocity of the disk when $u = 3$ rad.

15.9 The angular acceleration of a shaft is defined by the relation $a = -0.25v$, where a is expressed in rad/s^2 and v in rad/s. Knowing that at $t = 0$ the angular velocity of the shaft is 20 rad/s, determine (*a*) the number of revolutions the shaft will execute before coming to rest, (*b*) the time required for the shaft to come to rest, (*c*) the time required for the angular velocity of the shaft to be reduced to 1 percent of its initial value.

15.10 The bent rod *ABCDE* rotates about a line joining points *A* and *E* with a constant angular velocity of 9 rad/s. Knowing that the rotation is clockwise as viewed from *E*, determine the velocity and acceleration of corner *C*.

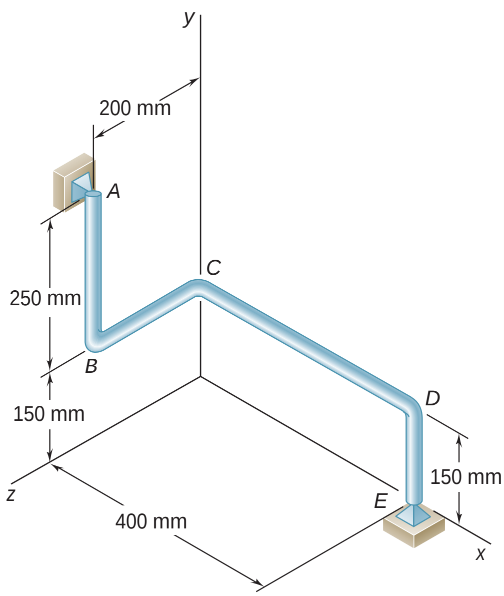


Fig. P15.10

15.11 In Prob. 15.10, determine the velocity and acceleration of corner *B*, assuming that the angular velocity is 9 rad/s and increases at the rate of 45 rad/s^2.

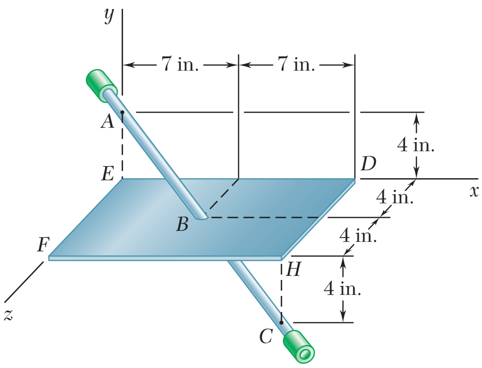


Fig. P15.12

15.12 The assembly shown consists of the straight rod *ABC* which passes through and is welded to the rectangular plate *DEFH*. The assembly rotates about the axis *AC* with a constant angular velocity of 9 rad/s. Knowing that the motion when viewed from *C* is counterclockwise, determine the velocity and acceleration of corner *F*.

15.13 In Prob. 15.12, determine the acceleration of corner *H*, assuming that the angular velocity is 9 rad/s and decreases at a rate of 18 rad/s^2.

15.14 A circular plate of 120-mm radius is supported by two bearings *A* and *B* as shown. The plate rotates about the rod joining *A* and *B* with a constant angular velocity of 26 rad/s. Knowing that, at the instant considered, the velocity of point *C* is directed to the right, determine the velocity and acceleration of point *E*.

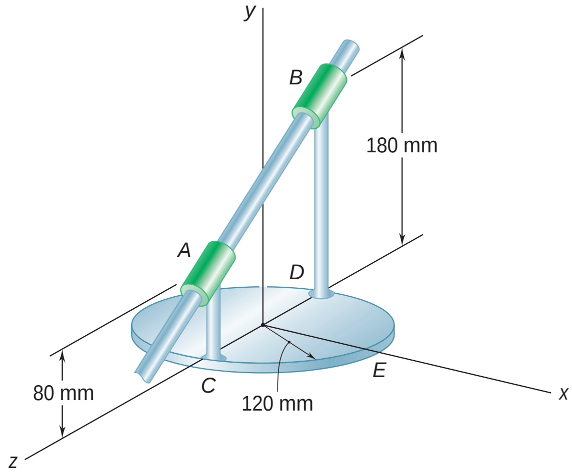


Fig. *P15.14*

15.15 In Prob. 15.14, determine the velocity and acceleration of point *E*, assuming that the angular velocity is 26 rad/s and increases at the rate of 65 rad/s^2.

15.16 The earth makes one complete revolution around the sun in 365.24 days. Assuming that the orbit of the earth is circular and has a radius of 93,000,000 mi, determine the velocity and acceleration of the earth.

15.17 The earth makes one complete revolution on its axis in 23 h 56 min. Knowing that the mean radius of the earth is 3960 mi, determine the linear velocity and acceleration of a point on the surface of the earth (*a*) at the equator, (*b*) at Philadelphia, latitude 40° north, (*c*) at the North Pole.

15.18 A series of small machine components being moved by a conveyor belt pass over a 120-mm-radius idler pulley. At the instant shown, the velocity of point *A* is 300 mm/s to the left and its acceleration is 180 mm/s^2 to the right. Determine (*a*) the angular velocity and angular acceleration of the idler pulley, (*b*) the total acceleration of the machine component at *B*.

15.19 A series of small machine components being moved by a conveyor belt pass over a 120-mm-radius idler pulley. At the instant shown, the angular velocity of the idler pulley is 4 rad/s clockwise. Determine the angular acceleration of the pulley for which the magnitude of the total acceleration of the machine component at *B* is 2400 mm/s^2.

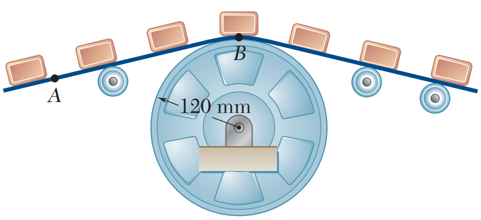


Fig. P15.18 and P15.19

15.20 The belt sander shown is initially at rest. If the driving drum B has a constant angular acceleration of 120 rad/s^2 counterclockwise, determine the magnitude of the acceleration of the belt at point C when (*a*) $t = 0.5$ s, (*b*) $t = 2$ s.

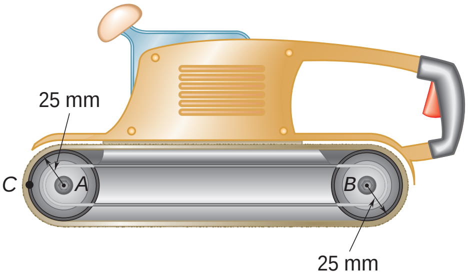


Fig. *P15.20* and *P15.21*

15.21 The rated speed of drum B of the belt sander shown is 2400 rpm. When the power is turned off, it is observed that the sander coasts from its rated speed to rest in 10 s. Assuming uniformly decelerated motion, determine the velocity and acceleration of point C of the belt, (*a*) immediately before the power is turned off, (*b*) 9 s later.

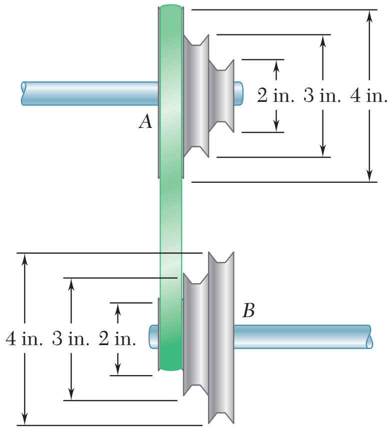


Fig. P15.22

15.22 The two pulleys shown may be operated with the V belt in any of three positions. If the angular acceleration of shaft A is 6 rad/s^2 and if the system is initially at rest, determine the time required for shaft B to reach a speed of 400 rpm with the belt in each of the three positions.

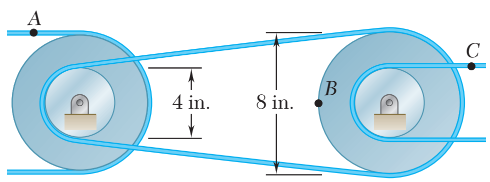


Fig. P15.23

15.23 Three belts move over two pulleys without slipping in the speed reduction system shown. At the instant shown, the velocity of point A on the input belt is 2 ft/s to the right, decreasing at the rate of 6 ft/s^2. Determine, at this instant, (*a*) the velocity and acceleration of point C on the output belt, (*b*) the acceleration of point B on the output pulley.

15.24 A gear reduction system consists of three gears A, B, and C. Knowing that gear A rotates clockwise with a constant angular velocity $\omega_A = 600$ rpm, determine (*a*) the angular velocities of gears B and C, (*b*) the accelerations of the points on gears B and C which are in contact.

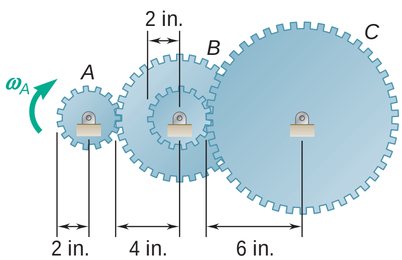


Fig. P15.24

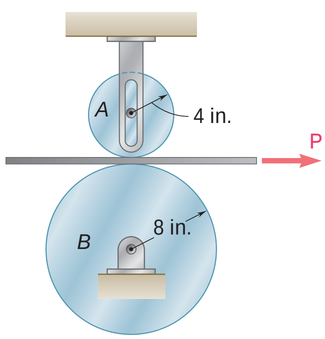


Fig. P15.25

15.25 A belt is pulled to the right between cylinders A and B. Knowing that the speed of the belt is a constant 5 ft/s and no slippage occurs, determine (*a*) the angular velocities of A and B, (*b*) the accelerations of the points which are in contact with the belt.

15.26 Ring C has an inside radius of 55 mm and an outside radius of 60 mm and is positioned between two wheels A and B, each of 24-mm outside radius. Knowing that wheel A rotates with a constant angular velocity of 300 rpm and that no slipping occurs, determine (*a*) the angular velocity of ring C and of wheel B, (*b*) the acceleration of the points A and B which are in contact with C.

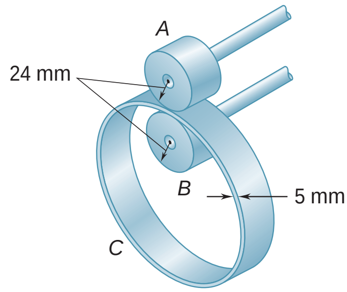


Fig. P15.26

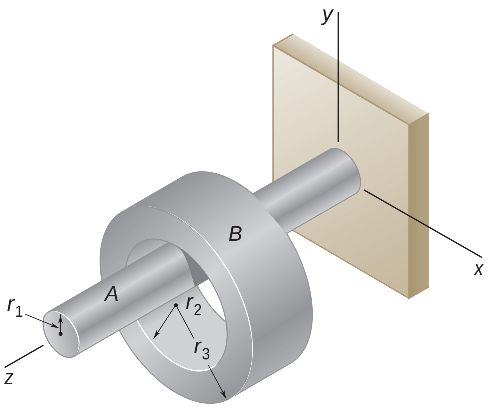


Fig. P15.27

15.27 Ring B has an inside radius r_2 and hangs from the horizontal shaft A as shown. Shaft A rotates with a constant angular velocity of 25 rad/s and no slipping occurs. Knowing that $r_1 = 12$ mm, $r_2 = 30$ mm, and $r_3 = 40$ mm, determine (*a*) the angular velocity of ring B, (*b*) the accelerations of the points of shaft A and ring B which are in contact, (*c*) the magnitude of the acceleration of a point on the outside surface of ring B.

15.28 A plastic film moves over two drums. During a 4-s interval the speed of the tape is increased uniformly from $v_0 = 2$ ft/s to $v_1 = 4$ ft/s. Knowing that the tape does not slip on the drums, determine (*a*) the angular acceleration of drum B, (*b*) the number of revolutions executed by drum B during the 4-s interval.

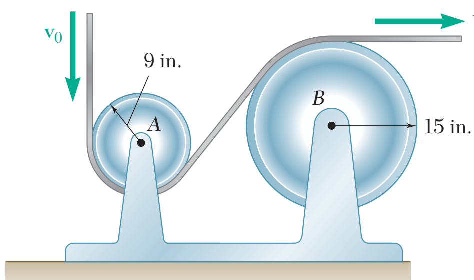


Fig. P15.28

15.29 A pulley and two loads are connected by inextensible cords as shown. Load A has a constant acceleration of 300 mm/s^2 and an initial velocity of 240 mm/s, both directed upward. Determine (*a*) the number of revolutions executed by the pulley in 3 s, (*b*) the velocity and position of load B after 3 s, (*c*) the acceleration of point D on the rim of the pulley at $t = 0$.

15.30 A pulley and two loads are connected by inextensible cords as shown. The pulley starts from rest at $t = 0$ and is accelerated at the uniform rate of 2.4 rad/s^2 clockwise. At $t = 4$ s, determine the velocity and position (*a*) of load A, (*b*) of load B.

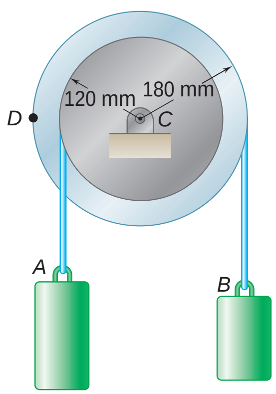


Fig. P15.29 and P15.30

15.31 A load is to be raised 20 ft by the hoisting system shown. Assuming gear *A* is initially at rest, accelerates uniformly to a speed of 120 rpm in 5 s, and then maintains a constant speed of 120 rpm, determine (*a*) the number of revolutions executed by gear *A* in raising the load, (*b*) the time required to raise the load.

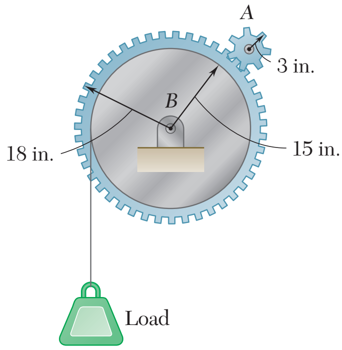


Fig. P15.31

15.32 Disk *B* is at rest when it is brought into contact with disk *A* which is rotating freely at 450 rpm clockwise. After 6 s of slippage, during which each disk has a constant angular acceleration, disk *A* reaches a final angular velocity of 140 rpm clockwise. Determine the angular acceleration of each disk during the period of slippage.

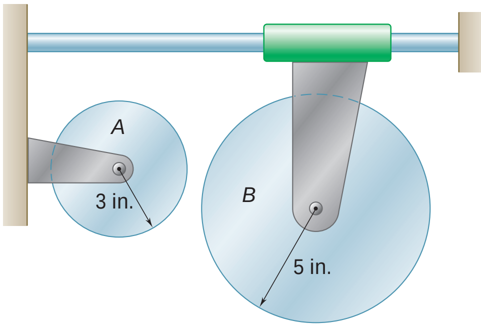


Fig. P15.32 and P15.33

15.33 and *15.34* A simple friction drive consists of two disks *A* and *B*. Initially, disk *A* has a clockwise angular velocity of 500 rpm and disk *B* is at rest. It is known that disk *A* will coast to rest in 60 s. However, rather than waiting until both disks are at rest to bring them together, disk *B* is given a constant angular acceleration of 2.5 rad/s^2 counterclockwise. Determine (*a*) at what time the disks can be brought together if they are not to slip, (*b*) the angular velocity of each disk as contact is made.

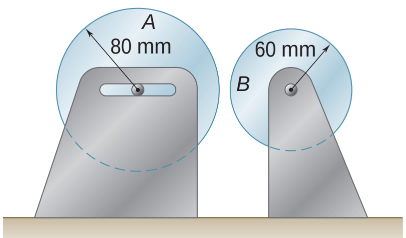


Fig. *P15.34* and *P15.35*

15.35 Two friction disks *A* and *B* are both rotating freely at 240 rpm counterclockwise when they are brought into contact. After 8 s of slippage, during which each disk has a constant angular acceleration, disk *A* reaches a final angular velocity of 60 rpm counterclockwise. Determine (*a*) the angular acceleration of each disk during the period of slippage, (*b*) the time at which the angular velocity of disk *B* is equal to zero.

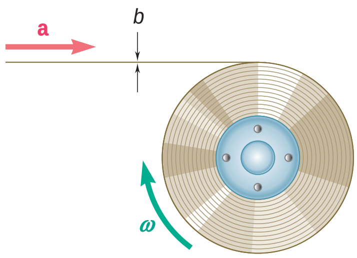


Fig. P15.36

***15.36** Steel tape is being wound onto a spool which rotates with a constant angular velocity V_0. Denoting by r the radius of the spool and tape at any given time and by b the thickness of the tape, derive an expression for the acceleration of the tape as it approaches the spool.

***15.37** In a continuous printing process, paper is drawn into the presses at a constant speed v. Denoting by r the radius of the paper roll at any given time and by b the thickness of the paper, derive an expression for the angular acceleration of the paper roll.

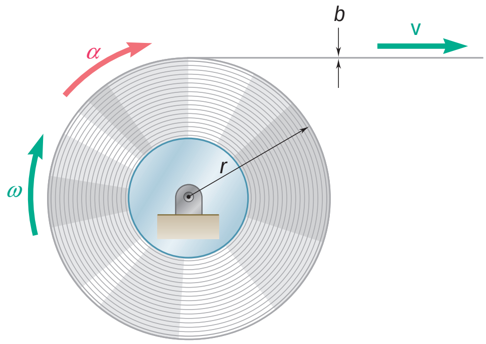


Fig. P15.37

15.5 GENERAL PLANE MOTION

As indicated in Sec. 15.1, we understand by general plane motion a plane motion which is neither a translation nor a rotation. As you will presently see, however, *a general plane motion can always be considered as the sum of a translation and a rotation*.

Consider, for example, a wheel rolling on a straight track (Fig. 15.12). Over a certain interval of time, two given points A and B will have moved, respectively, from A_1 to A_2 and from B_1 to B_2. The same result could be obtained through a translation which would bring A and B into A_2 and B_1' (the line AB remaining vertical), followed by a rotation about A bringing B into B_2. Although the original rolling motion differs from the combination of translation and rotation when these motions are taken in succession, the original motion can be exactly duplicated by a combination of simultaneous translation and rotation.

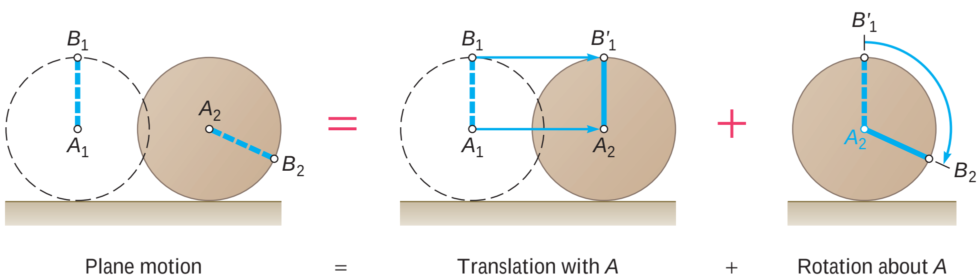


Fig. 15.12

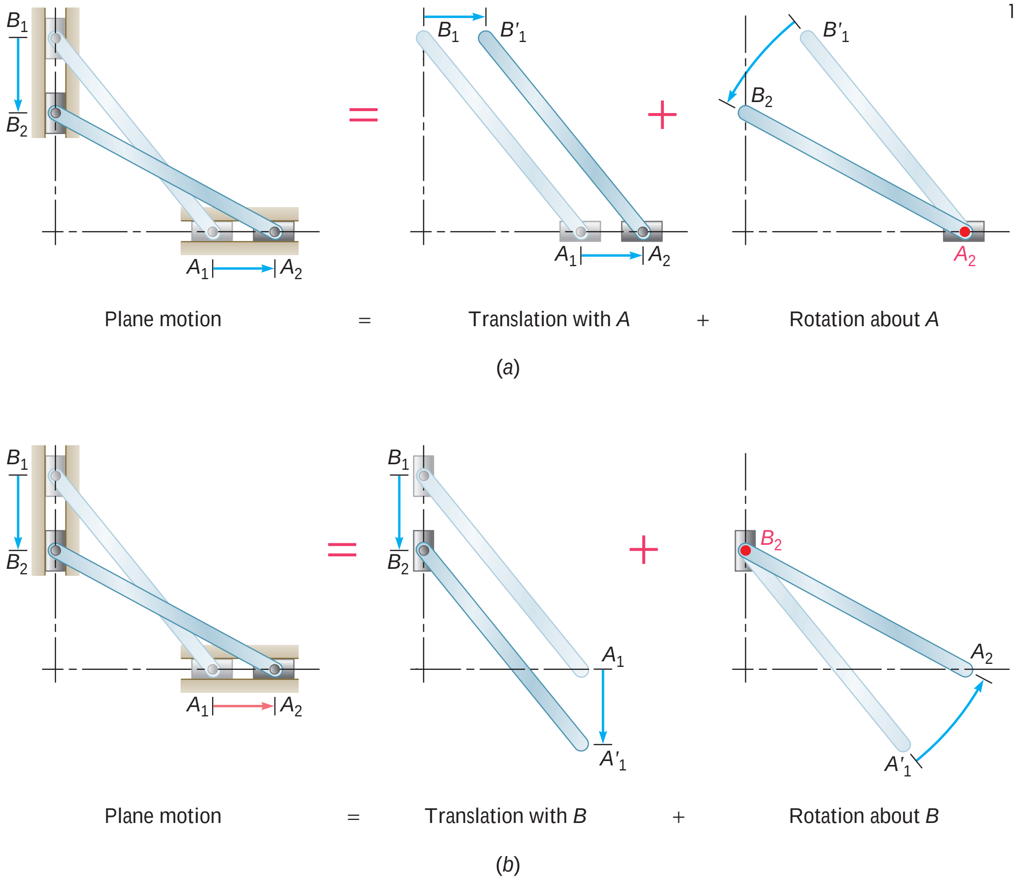


Fig. 15.13

Another example of plane motion is given in Fig. 15.13, which represents a rod whose extremities slide along a horizontal and a vertical track, respectively. This motion can be replaced by a translation in a horizontal direction and a rotation about A (Fig. 15.13*a*) or by a translation in a vertical direction and a rotation about B (Fig. 15.13*b*).

In the general case of plane motion, we will consider a small displacement which brings two particles A and B of a representative slab, respectively, from A_1 and B_1 into A_2 and B_2 (Fig. 15.14). This displacement can be divided into two parts: in one, the particles move into A_2 and B_1' while the line AB maintains the same direction; in the other, B moves into B_2 while A remains fixed. The first part of the motion is clearly a translation and the second part a rotation about A.

Recalling from Sec. 11.12 the definition of the relative motion of a particle with respect to a moving frame of reference—as opposed to its absolute motion with respect to a fixed frame of reference—we can restate as follows the result obtained above: Given two particles A and B of a rigid slab in plane motion, the relative motion of B with respect to a frame attached to A and of fixed orientation is a rotation. To an observer moving with A but not rotating, particle B will appear to describe an arc of circle centered at A.

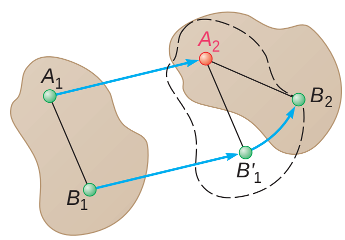


Fig. 15.14

Photo 15.4 Planetary gear systems are used to high reduction ratios with minimum space and weight. The small gears undergo general plane motion.

15.6 ABSOLUTE AND RELATIVE VELOCITY IN PLANE MOTION

We saw in the preceding section that any plane motion of a slab can be replaced by a translation defined by the motion of an arbitrary reference point A and a simultaneous rotation about A. The absolute velocity $\mathbf{v}_B$ of a particle B of the slab is obtained from the relative-velocity formula derived in Sec. 11.12,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.17}$$

where the right-hand member represents a vector sum. The velocity $\mathbf{v}_A$ corresponds to the translation of the slab with A, while the relative velocity $\mathbf{v}_{B/A}$ is associated with the rotation of the slab about A and is measured with respect to axes centered at A and of fixed orientation (Fig. 15.15). Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, and by $\omega\mathbf{k}$ the angular velocity of the slab with respect to axes of fixed orientation, we have from (15.10) and (15.10′)

$$\mathbf{v}_{B/A} = \omega\mathbf{k} \times \mathbf{r}_{B/A} \qquad v_{B/A} = r\omega \tag{15.18}$$

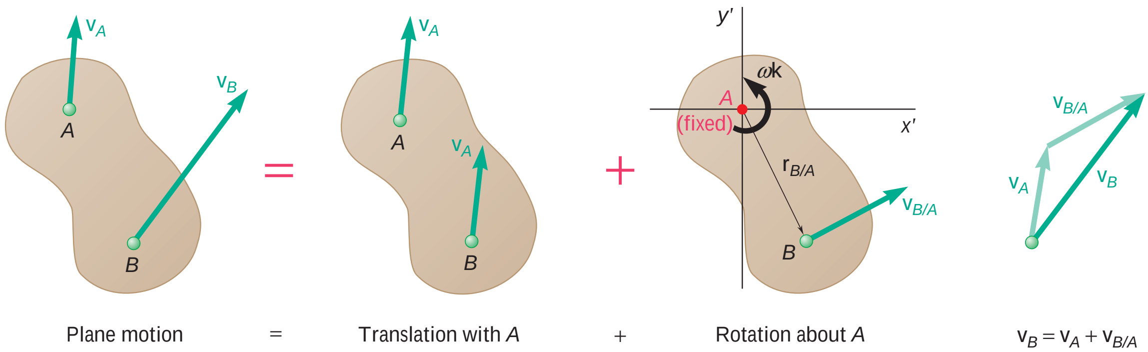


Fig. 15.15

where r is the distance from A to B. Substituting for $\mathbf{v}_{B/A}$ from (15.18) into (15.17), we can also write

$$\mathbf{v}_B = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{B/A} \tag{15.17′}$$

As an example, let us again consider the rod AB of Fig. 15.13. Assuming that the velocity $\mathbf{v}_A$ of end A is known, we propose to find the velocity $\mathbf{v}_B$ of end B and the angular velocity ω of the rod, in terms of the velocity $\mathbf{v}_A$, the length l, and the angle θ. Choosing A as a reference point, we express that the given motion is equivalent to a translation with A and a simultaneous rotation about A (Fig. 15.16). The absolute velocity of B must therefore be equal to the vector sum

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.17}$$

We note that while the direction of $\mathbf{v}_{B/A}$ is known, its magnitude $l\omega$ is unknown. However, this is compensated for by the fact that the direction of $\mathbf{v}_B$ is known. We can therefore complete the diagram of Fig. 15.16. Solving for the magnitudes v_B and ω, we write

$$v_B = v_A \tan\theta \qquad \omega = \frac{v_{B/A}}{l} = \frac{v_A}{l\cos\theta} \tag{15.19}$$

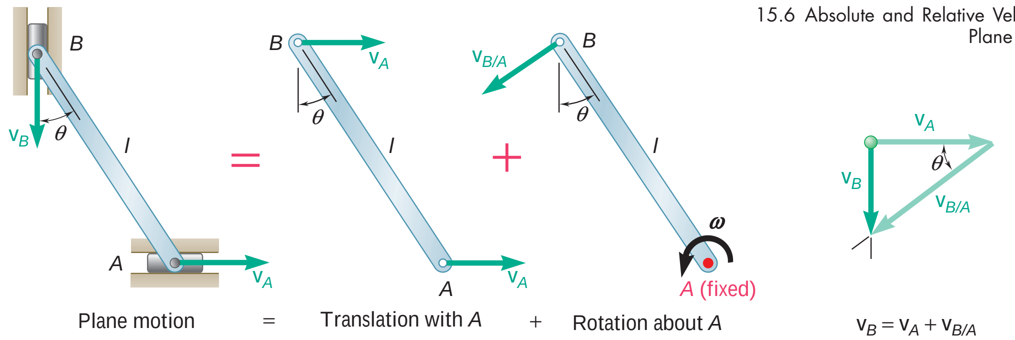


Fig. 15.16

The same result can be obtained by using B as a point of reference. Resolving the given motion into a translation with B and a simultaneous rotation about B (Fig. 15.17), we write the equation

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \tag{15.20}$$

which is represented graphically in Fig. 15.17. We note that $\mathbf{v}_{A/B}$ and $\mathbf{v}_{B/A}$ have the same magnitude lV but opposite sense. The sense of the relative velocity depends, therefore, upon the point of reference which has been selected and should be carefully ascertained from the appropriate diagram (Fig. 15.16 or 15.17).

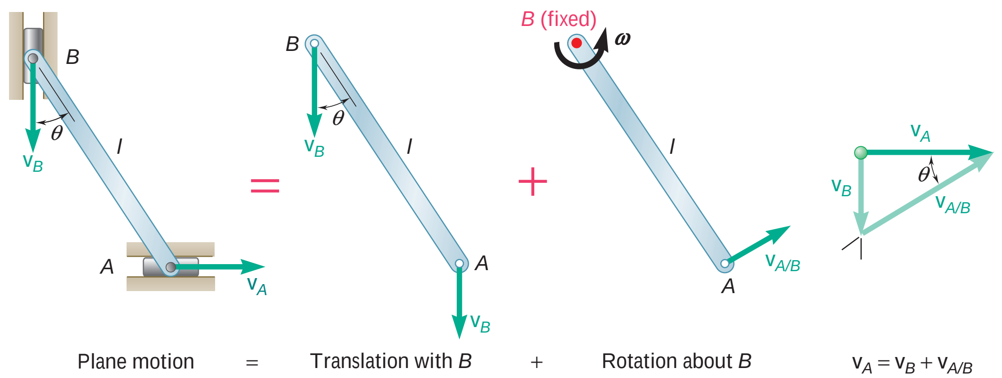
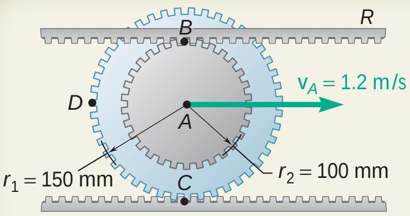


Fig. 15.17

Finally, we observe that the angular velocity V of the rod in its rotation about B is the same as in its rotation about A. It is measured in both cases by the rate of change of the angle u. This result is quite general; we should therefore bear in mind that *the angular velocity* V *of a rigid body in plane motion is independent of the reference point.*

Most mechanisms consist not of one but of *several* moving parts. When the various parts of a mechanism are pin-connected, the analysis of the mechanism can be carried out by considering each part as a rigid body, keeping in mind that the points where two parts are connected must have the same absolute velocity (see Sample Prob. 15.3). A similar analysis can be used when gears are involved, since the teeth in contact must also have the same absolute velocity. However, when a mechanism contains parts which slide on each other, the relative velocity of the parts in contact must be taken into account (see Secs. 15.10 and 15.11).



SAMPLE PROBLEM 15.2

The double gear shown rolls on the stationary lower rack; the velocity of its center A is 1.2 m/s directed to the right. Determine (a) the angular velocity of the gear, (b) the velocities of the upper rack R and of point D of the gear.

SOLUTION

a. Angular Velocity of the Gear. Since the gear rolls on the lower rack, its center A moves through a distance equal to the outer circumference $2\pi r_1$ for each full revolution of the gear. Noting that 1 rev $= 2\pi$ rad, and that when A moves to the right ($x_A > 0$) the gear rotates clockwise ($\theta < 0$), we write

$$\frac{x_A}{2\pi r_1} = -\frac{\theta}{2\pi} \qquad x_A = -r_1\theta$$

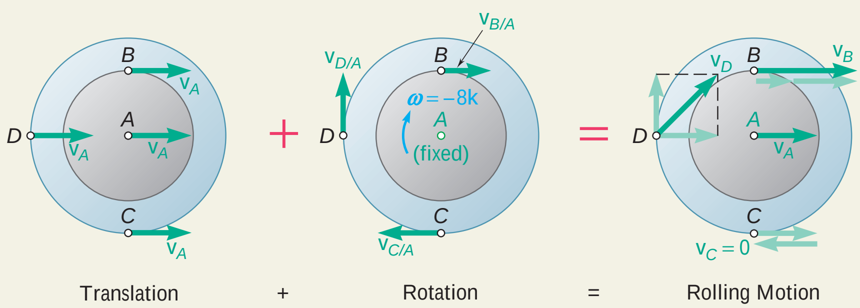
Differentiating with respect to the time t and substituting the known values $v_A = 1.2$ m/s and $r_1 = 150$ mm $= 0.150$ m, we obtain

$$v_A = -r_1\omega \qquad 1.2 \text{ m/s} = -(0.150 \text{ m})\omega \qquad \omega = -8 \text{ rad/s}$$

$$\boldsymbol{\omega} = \omega\mathbf{k} = -(8 \text{ rad/s})\mathbf{k} \quad ◀$$

where $\mathbf{k}$ is a unit vector pointing out of the paper.

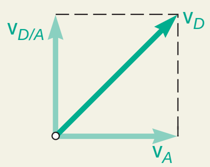
b. Velocities. The rolling motion is resolved into two component motions: a translation with the center A and a rotation about the center A. In the translation, all points of the gear move with the same velocity $\mathbf{v}_A$. In the rotation, each point P of the gear moves about A with a relative velocity $\mathbf{v}_{P/A} = \omega\mathbf{k} \times \mathbf{r}_{P/A}$, where $\mathbf{r}_{P/A}$ is the position vector of P relative to A.



Velocity of Upper Rack. The velocity of the upper rack is equal to the velocity of point B; we write

$$\begin{aligned}\mathbf{v}_R = \mathbf{v}_B &= \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{B/A}\\ &= (1.2 \text{ m/s})\mathbf{i} - (8 \text{ rad/s})\mathbf{k} \times (0.100 \text{ m})\mathbf{j}\\ &= (1.2 \text{ m/s})\mathbf{i} + (0.8 \text{ m/s})\mathbf{i} = (2 \text{ m/s})\mathbf{i}\end{aligned}$$

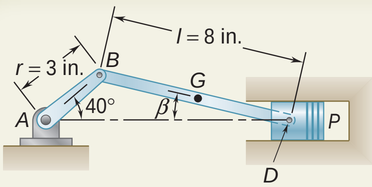
$$\mathbf{v}_R = 2 \text{ m/s} \rightarrow \quad ◀$$



Velocity of Point D

$$\begin{aligned}\mathbf{v}_D &= \mathbf{v}_A + \mathbf{v}_{D/A} = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{D/A}\\ &= (1.2 \text{ m/s})\mathbf{i} - (8 \text{ rad/s})\mathbf{k} \times (-0.150 \text{ m})\mathbf{i}\\ &= (1.2 \text{ m/s})\mathbf{i} + (1.2 \text{ m/s})\mathbf{j}\end{aligned}$$

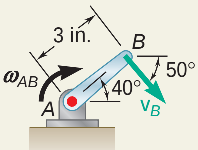
$$\mathbf{v}_D = 1.697 \text{ m/s} \angle 45° \quad ◀$$



SAMPLE PROBLEM 15.3

In the engine system shown, the crank AB has a constant clockwise angular velocity of 2000 rpm. For the crank position indicated, determine (*a*) the angular velocity of the connecting rod BD, (*b*) the velocity of the piston P.

SOLUTION



Motion of Crank *AB*. The crank AB rotates about point A. Expressing ω_{AB} in rad/s and writing $v_B = r\omega_{AB}$, we obtain

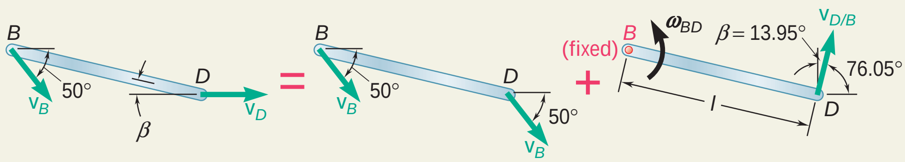
$$\omega_{AB} = \left(2000\frac{\text{rev}}{\text{min}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right)\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right) = 209.4\text{ rad/s}$$

$$v_B = (AB)\omega_{AB} = (3\text{ in.})(209.4\text{ rad/s}) = 628.3\text{ in./s}$$
$$\mathbf{v}_B = 628.3\text{ in./s} \searrow 50°$$

Motion of Connecting Rod *BD*. We consider this motion as a general plane motion. Using the law of sines, we compute the angle β between the connecting rod and the horizontal:

$$\frac{\sin 40°}{8\text{ in.}} = \frac{\sin\beta}{3\text{ in.}} \qquad \beta = 13.95°$$

The velocity $\mathbf{v}_D$ of the point D where the rod is attached to the piston must be horizontal, while the velocity of point B is equal to the velocity $\mathbf{v}_B$ obtained above. Resolving the motion of BD into a translation with B and a rotation about B, we obtain

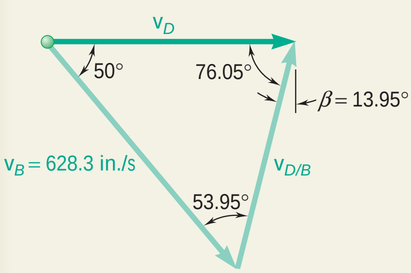


Plane motion = Translation + Rotation

Expressing the relation between the velocities $\mathbf{v}_D$, $\mathbf{v}_B$, and $\mathbf{v}_{D/B}$, we write

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$

We draw the vector diagram corresponding to this equation. Recalling that $\beta = 13.95°$, we determine the angles of the triangle and write



$$\frac{v_D}{\sin 53.95°} = \frac{v_{D/B}}{\sin 50°} = \frac{628.3\text{ in./s}}{\sin 76.05°}$$

$$v_{D/B} = 495.9\text{ in./s} \qquad \mathbf{v}_{D/B} = 495.9\text{ in./s} \nearrow 76.05°$$
$$v_D = 523.4\text{ in./s} = 43.6\text{ ft/s} \qquad \mathbf{v}_D = 43.6\text{ ft/s} \rightarrow$$
$$\mathbf{v}_P = \mathbf{v}_D = 43.6\text{ ft/s} \rightarrow \blacktriangleleft$$

Since $v_{D/B} = l\omega_{BD}$, we have

$$495.9\text{ in./s} = (8\text{ in.})\omega_{BD} \qquad \omega_{BD} = 62.0\text{ rad/s} \circlearrowleft \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to analyze the velocity of bodies in *general plane motion.* You found that a general plane motion can always be considered as the sum of the two motions you studied in the last lesson, namely, *a translation and a rotation.*

To solve a problem involving the velocity of a body in plane motion you should take the following steps.

1. Whenever possible determine the velocity of the points of the body where the body is connected to another body whose motion is known. That other body may be an arm or crank rotating with a given angular velocity [Sample Prob. 15.3].

2. Next start drawing a "diagram equation" to use in your solution (Figs. 15.15 and 15.16). This "equation" will consist of the following diagrams.

a. Plane motion diagram: Draw a diagram of the body including all dimensions and showing those points for which you know or seek the velocity.

b. Translation diagram: Select a reference point A for which you know the direction and/or the magnitude of the velocity $\mathbf{v}_A$, and draw a second diagram showing the body in translation with all of its points having the same velocity $\mathbf{v}_A$.

c. Rotation diagram: Consider point A as a fixed point and draw a diagram showing the body in rotation about A. Show the angular velocity $\mathsf{V} = \mathsf{v}\mathbf{k}$ of the body and the relative velocities with respect to A of the other points, such as the velocity $\mathbf{v}_{B/A}$ of B relative to A.

3. Write the relative-velocity formula

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

While you can solve this vector equation analytically by writing the corresponding scalar equations, you will usually find it easier to solve it by using a vector triangle (Fig. 15.16).

4. A different reference point can be used to obtain an equivalent solution. For example, if point B is selected as the reference point, the velocity of point A is expressed as

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

Note that the relative velocities $\mathbf{v}_{B/A}$ and $\mathbf{v}_{A/B}$ have the same magnitude but opposite sense. Relative velocities, therefore, depend upon the reference point that has been selected. The angular velocity, however, is independent of the choice of reference point.

PROBLEMS

CONCEPT QUESTIONS

15.CQ3 The ball rolls without slipping on the fixed surface as shown. What is the direction of the velocity of point A?

a. y **b.** ↗ **c.** ↑ **d.** ↓ **e.** ↘

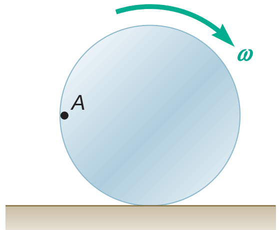


Fig. P15.CQ3

15.CQ4 Three uniform rods—ABC, DCE, and FGH—are connected as shown. Which of the following statements concerning the angular speed of the three objects is true?

a. $\mathrm{v}_{ABC} = \mathrm{v}_{DCE} = \mathrm{v}_{FGH}$

b. $\mathrm{v}_{DCE} > \mathrm{v}_{ABC} > \mathrm{v}_{FGH}$

c. $\mathrm{v}_{DCE} < \mathrm{v}_{ABC} < \mathrm{v}_{FGH}$

d. $\mathrm{v}_{ABC} > \mathrm{v}_{DCE} > \mathrm{v}_{FGH}$

e. $\mathrm{v}_{FGH} = \mathrm{v}_{DCE} < \mathrm{v}_{ABC}$

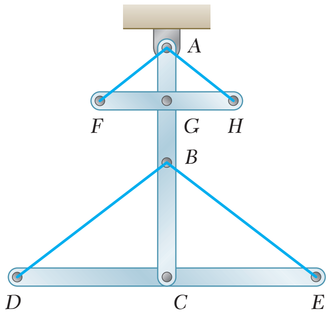


Fig. P15.CQ4

END-OF-SECTION PROBLEMS

15.38 An automobile travels to the right at a constant speed of 48 mi/h. If the diameter of a wheel is 22 in., determine the velocities of points B, C, D, and E on the rim of the wheel.

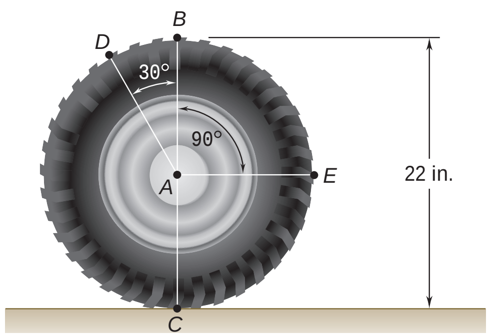


Fig. P15.38

15.39 The motion of rod AB is guided by pins attached at A and B which slide in the slots shown. At the instant shown, u = 40° and the pin at B moves upward to the left with a constant velocity of 6 in./s. Determine (*a*) the angular velocity of the rod, (*b*) the velocity of the pin at end A.

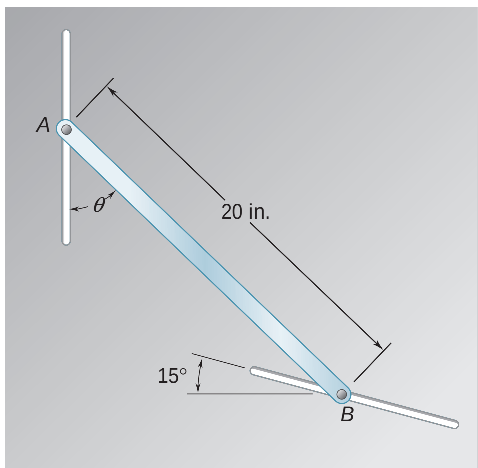


Fig. P15.39

15.40 Collar B moves upward with a constant velocity of 1.5 m/s. At the instant when u = 50°, determine (*a*) the angular velocity of rod AB, (*b*) the velocity of end A of the rod.

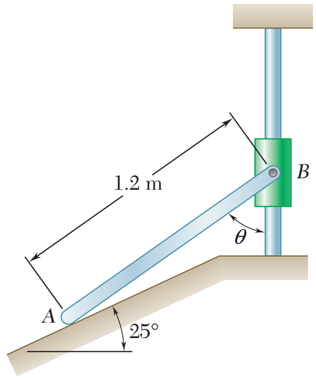


Fig. P15.40

15.41 Collar B moves downward to the left with a constant velocity of 1.6 m/s. At the instant shown when u = 40°, determine (*a*) the angular velocity of rod AB, (*b*) the velocity of collar A.

15.42 Collar A moves upward with a constant velocity of 1.2 m/s. At the instant shown when u = 25°, determine (*a*) the angular velocity of rod AB, (*b*) the velocity of collar B.

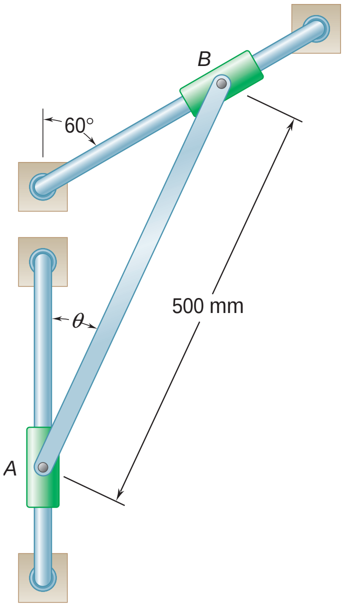


Fig. P15.41 and *P15.42*

15.43 Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity of 25 in./s. At the instant shown, determine (*a*) the angular velocity of the rod, (*b*) the velocity of end B of the rod.

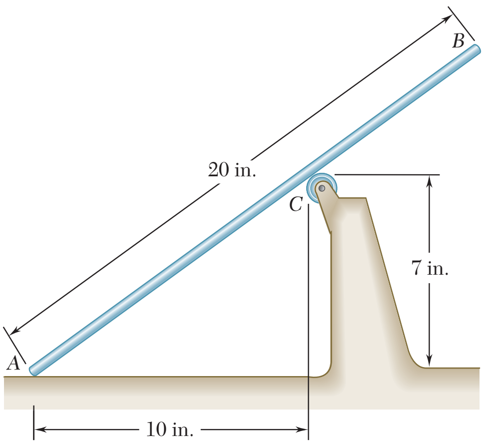


Fig. *P15.43*

15.44 The plate shown moves in the xy plane. Knowing that $(v_A)_x = 120$ mm/s, $(v_B)_y = 300$ mm/s, and $(v_C)_y = -60$ mm/s, determine (*a*) the angular velocity of the plate, (*b*) the velocity of point A.

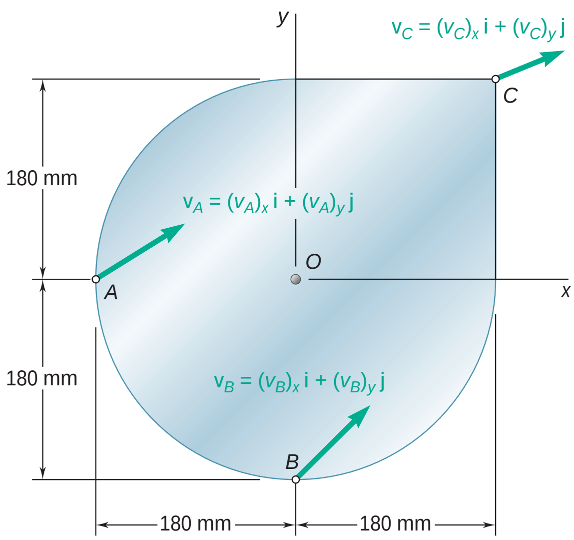


Fig. P15.44

15.45 In Prob. 15.44, determine (*a*) the velocity of point B, (*b*) the point of the plate with zero velocity.

15.46 The plate shown moves in the xy plane. Knowing that $(v_A)_x = 250$ mm/s, $(v_B)_y = -450$ mm/s, and $(v_C)_x = -500$ mm/s, determine (*a*) the angular velocity of the plate, (*b*) the velocity of point A.

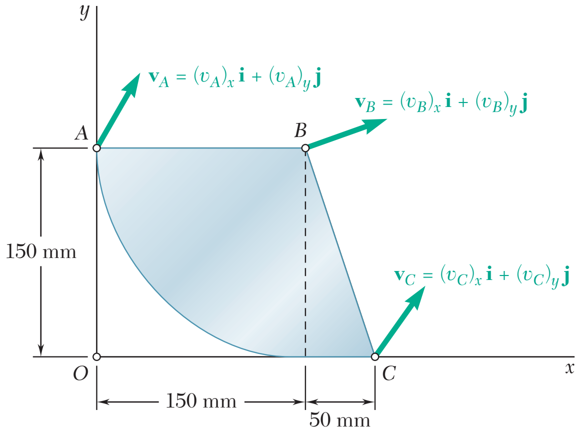


Fig. *P15.46*

15.47 The plate shown moves in the xy plane. Knowing that $(v_A)_x = 12$ in./s, $(v_B)_x = -4$ in./s, and $(v_C)_y = -24$ in./s, determine (*a*) the angular velocity of the plate, (*b*) the velocity of point B.

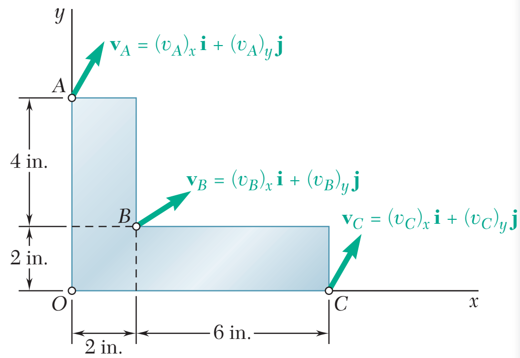


Fig. P15.47

15.48 In the planetary gear system shown, the radius of gears A, B, C, and D is a and the radius of the outer gear E is $3a$. Knowing that the angular velocity of gear A is ω_A clockwise and that the outer gear E is stationary, determine (*a*) the angular velocity of each planetary gear, (*b*) the angular velocity of the spider connecting the planetary gears.

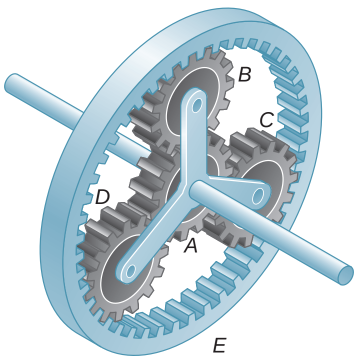


Fig. P15.48 and P15.49

15.49 In the planetary gear system shown, the radius of gears A, B, C, and D is 30 mm and the radius of the outer gear E is 90 mm. Knowing that gear E has an angular velocity of 180 rpm clockwise and that the central gear A has an angular velocity of 240 rpm clockwise, determine (*a*) the angular velocity of each planetary gear, (*b*) the angular velocity of the spider connecting the planetary gears.

15.50 Arm AB rotates with an angular velocity of 20 rad/s counterclockwise. Knowing that the outer gear C is stationary, determine (*a*) the angular velocity of gear B, (*b*) the velocity of the gear tooth located at point D.

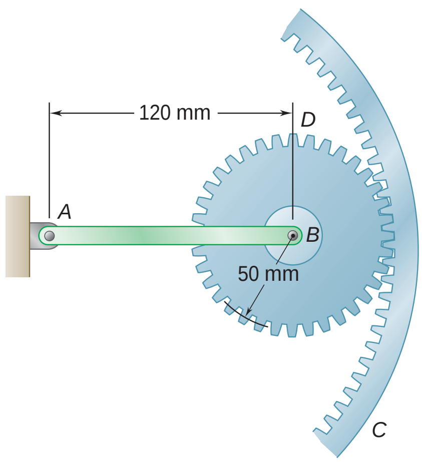


Fig. P15.50

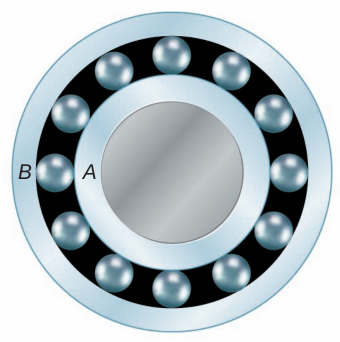


Fig. P15.51

15.51 In the simplified sketch of a ball bearing shown, the diameter of the inner race *A* is 60 mm and the diameter of each ball is 12 mm. The outer race *B* is stationary while the inner race has an angular velocity of 3600 rpm. Determine (*a*) the speed of the center of each ball, (*b*) the angular velocity of each ball, (*c*) the number of times per minute each ball describes a complete circle.

15.52 A simplified gear system for a mechanical watch is shown. Knowing that gear *A* has a constant angular velocity of 1 rev/h and gear *C* has a constant angular velocity of 1 rpm, determine (*a*) the radius *r*, (*b*) the magnitudes of the accelerations of the points on gear *B* that are in contact with gears *A* and *C*.

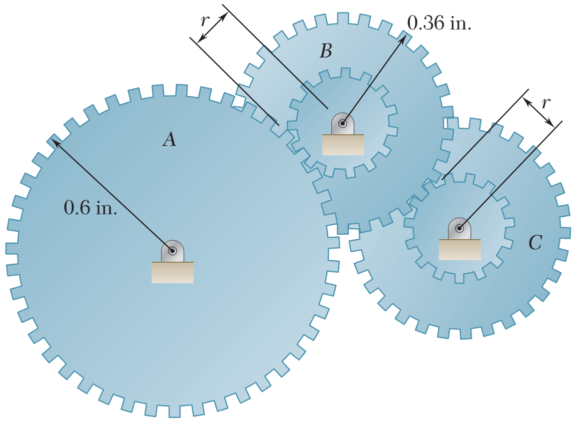


Fig. *P15.52*

15.53 and *15.54* Arm *ACB* rotates about point *C* with an angular velocity of 40 rad/s counterclockwise. Two friction disks *A* and *B* are pinned at their centers to arm *ACB* as shown. Knowing that the disks roll without slipping at surfaces of contact, determine the angular velocity of (*a*) disk *A*, (*b*) disk *B*.

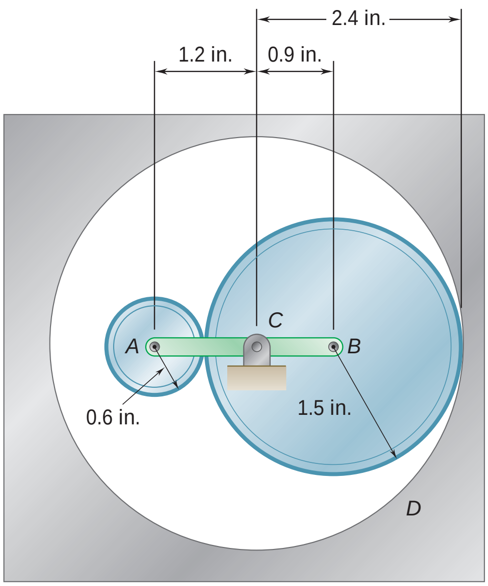


Fig. P15.53

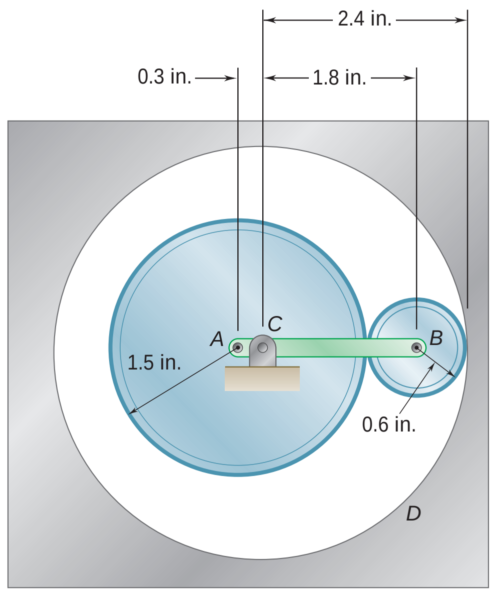


Fig. *P15.54*

15.55 Knowing that at the instant shown the velocity of collar A is 900 mm/s to the left, determine (*a*) the angular velocity of rod ADB, (*b*) the velocity of point B.

15.56 Knowing that at the instant shown the angular velocity of rod DE is 2.4 rad/s clockwise, determine (*a*) the velocity of collar A, (*b*) the velocity of point B.

15.57 A straight rack rests on a gear of radius r and is attached to a block B as shown. Denoting by v_D the clockwise angular velocity of gear D and by u the angle formed by the rack and the horizontal, derive expressions for the velocity of block B and the angular velocity of the rack in terms of r, u, and v_D.

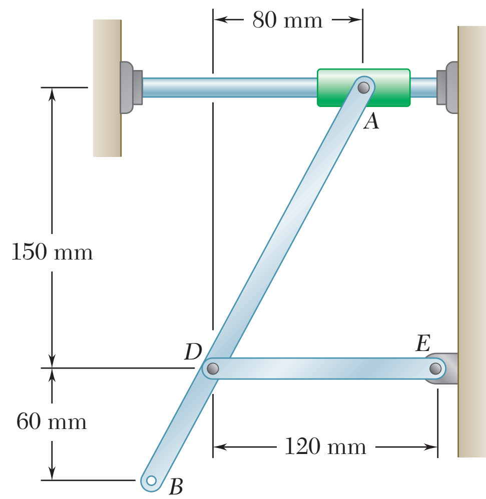


Fig. P15.55 and P15.56

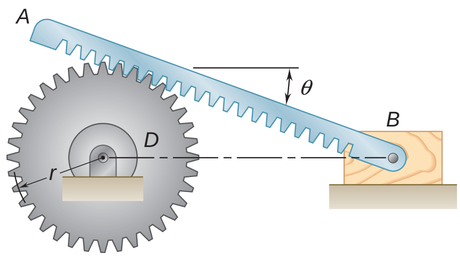


Fig. P15.57 and P15.58

15.58 A straight rack rests on a gear of radius r = 2.5 in. and is attached to a block B as shown. Knowing that at the instant shown the velocity of block B is 8 in./s to the right and u = 25°, determine (*a*) the angular velocity of gear D, (*b*) the angular velocity of the rack.

15.59 Knowing that at the instant shown the angular velocity of crank AB is 2.7 rad/s clockwise, determine (*a*) the angular velocity of link BD, (*b*) the velocity of collar D, (*c*) the velocity of the midpoint of link BD.

15.60 In the eccentric shown, a disk of 2-in. radius revolves about shaft O that is located 0.5 in. from the center A of the disk. The distance between the center A of the disk and the pin at B is 8 in. Knowing that the angular velocity of the disk is 900 rpm clockwise, determine the velocity of the block when u = 30°.

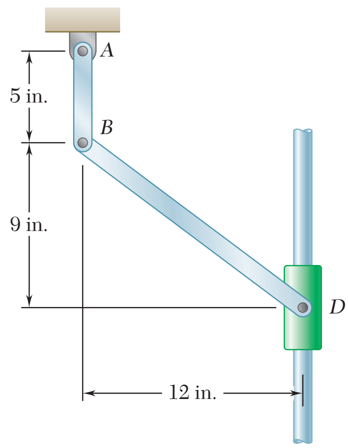


Fig. P15.59

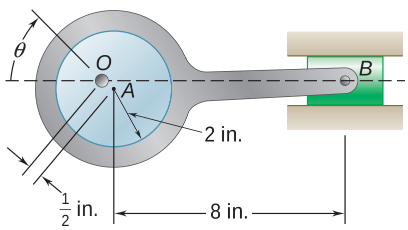


Fig. *P15.60*

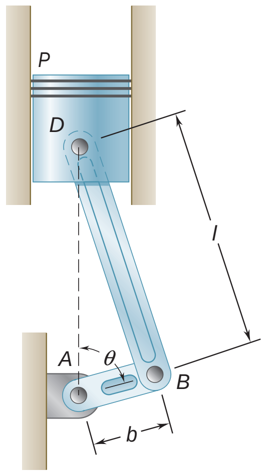


Fig. P15.61 and P15.62

15.61 In the engine system shown, $l = 160$ mm and $b = 60$ mm. Knowing that the crank AB rotates with a constant angular velocity of 1000 rpm clockwise, determine the velocity of the piston P and the angular velocity of the connecting rod when (*a*) u $= 0$, (*b*) u $= 90°$.

15.62 In the engine system shown, $l = 160$ mm and $b = 60$ mm. Knowing that crank AB rotates with a constant angular velocity of 1000 rpm clockwise, determine the velocity of the piston P and the angular velocity of the connecting rod when u $= 60°$.

15.63 Knowing that at the instant shown the angular velocity of rod AB is 15 rad/s clockwise, determine (*a*) the angular velocity of rod BD, (*b*) the velocity of the midpoint of rod BD.

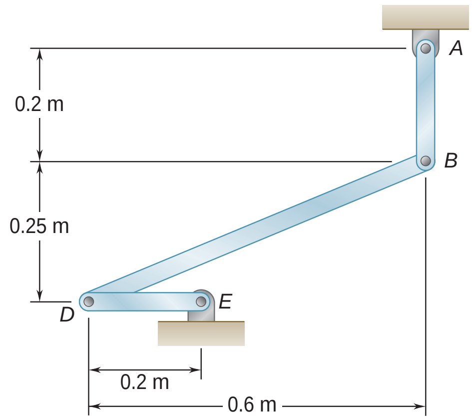


Fig. P15.63

15.64 and 15.65 In the position shown, bar AB has an angular velocity of 4 rad/s clockwise. Determine the angular velocity of bars BD and DE.

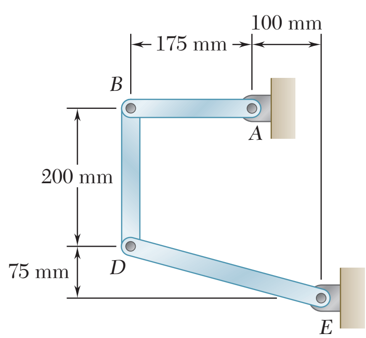


Fig. P15.64

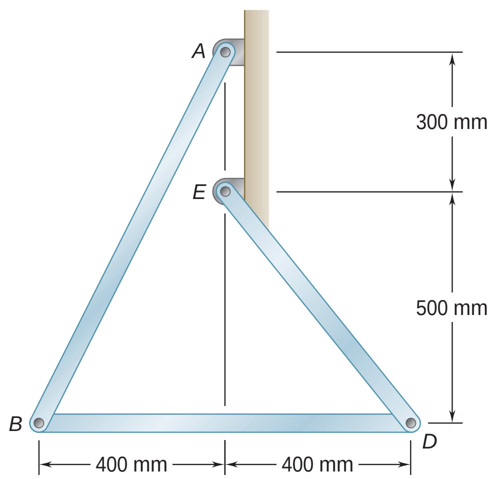


Fig. P15.65

15.66 Roberts linkage is named after Richard Roberts (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at point *F*. The distance *AB* is the same as *BF*, *DF*, and *DE*. Knowing that the angular velocity of bar *AB* is 5 rad/s clockwise in the position shown, determine (*a*) the angular velocity of bar *DE*, (*b*) the velocity of point *F*.

15.67 Roberts linkage is named after Richard Roberts (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at point *F*. The distance *AB* is the same as *BF*, *DF*, and *DE*. Knowing that the angular velocity of plate *BDF* is 2 rad/s counterclockwise when u = 90°, determine (*a*) the angular velocities of bars *AB* and *DE*, (*b*) the velocity of point *F*. When u = 90°, point *F* may be assumed to coincide with point *E*, with negligible error in the velocity analysis.

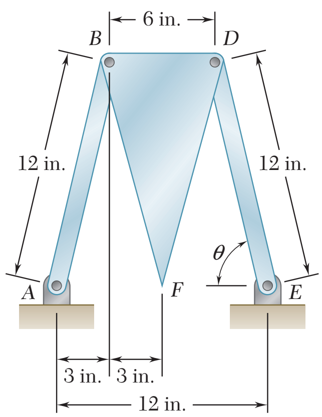


Fig. *P15.66* and *P15.67*

15.68 In the position shown, bar *DE* has a constant angular velocity of 10 rad/s clockwise. Knowing that h = 500 mm, determine (*a*) the angular velocity of bar *FBD*, (*b*) the velocity of point *F*.

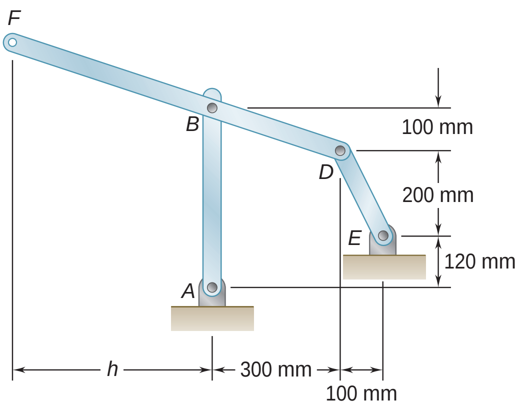


Fig. P15.68 and P15.69

15.69 In the position shown, bar *DE* has a constant angular velocity of 10 rad/s clockwise. Determine (*a*) the distance h for which the velocity of point *F* is vertical, (*b*) the corresponding velocity of point *F*.

15.70 Both 6-in.-radius wheels roll without slipping on the horizontal surface. Knowing that the distance *AD* is 5 in., the distance *BE* is 4 in., and *D* has a velocity of 6 in./s to the right, determine the velocity of point *E*.

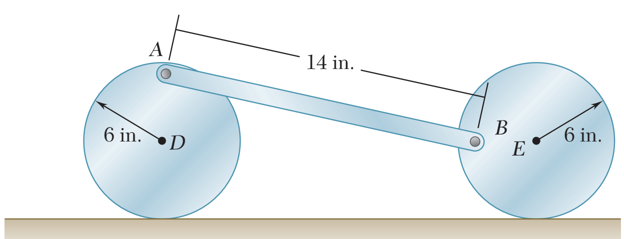


Fig. P15.70

15.71 The 80-mm-radius wheel shown rolls to the left with a velocity of 900 mm/s. Knowing that the distance *AD* is 50 mm, determine the velocity of the collar and the angular velocity of rod *AB* when (*a*) $\beta = 0$, (*b*) $\beta = 90°$.

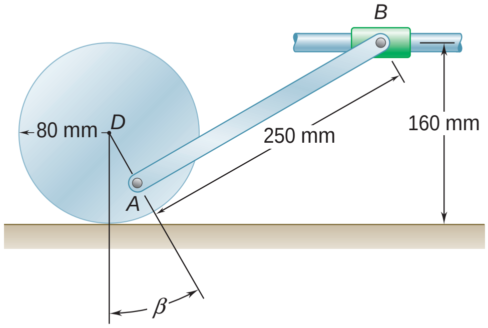


Fig. P15.71

***15.72** For the gearing shown, derive an expression for the angular velocity ω_C of gear *C* and show that ω_C is independent of the radius of gear *B*. Assume that point *A* is fixed and denote the angular velocities of rod *ABC* and gear *A* by ω_{ABC} and ω_A, respectively.

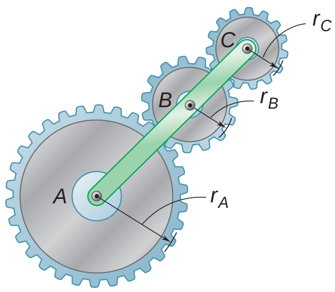


Fig. P15.72

15.7 INSTANTANEOUS CENTER OF ROTATION IN PLANE MOTION

Photo 15.5 If the tires of this car are rolling without sliding, the instantaneous center of rotation of a tire is the point of contact between the road and the tire.

Consider the general plane motion of a slab. We propose to show that at any given instant the velocities of the various particles of the slab are the same as if the slab were rotating about a certain axis perpendicular to the plane of the slab, called the *instantaneous axis of rotation.* This axis intersects the plane of the slab at a point *C*, called the *instantaneous center of rotation* of the slab.

We first recall that the plane motion of a slab can always be replaced by a translation defined by the motion of an arbitrary reference point *A* and by a rotation about *A*. As far as the velocities are concerned, the translation is characterized by the velocity $\mathbf{v}_A$ of the reference point *A* and the rotation is characterized by the angular velocity ω of the slab (which is independent of the choice of *A*). Thus, the velocity $\mathbf{v}_A$ of point *A* and the angular velocity ω of the slab define

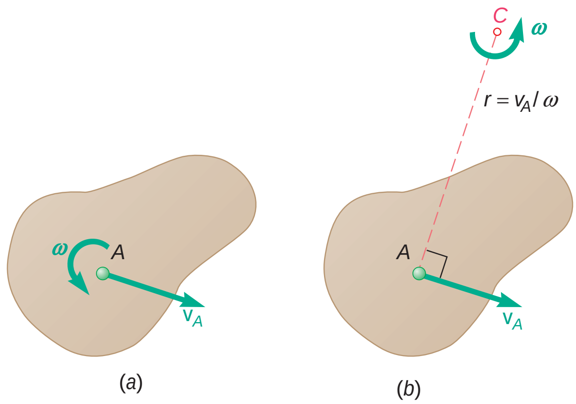


Fig. 15.18

completely the velocities of all the other particles of the slab (Fig. 15.18*a*). Now let us assume that $\mathbf{v}_A$ and V are known and that they are both different from zero. (If $\mathbf{v}_A = 0$, point A is itself the instantaneous center of rotation, and if V $= 0$, all the particles have the same velocity $\mathbf{v}_A$.) These velocities could be obtained by letting the slab rotate with the angular velocity V about a point C located on the perpendicular to $\mathbf{v}_A$ at a distance $r = v_A/$V from A as shown in Fig. 15.18*b*. We check that the velocity of A would be perpendicular to AC and that its magnitude would be rV $= (v_A/$V$)$V $= v_A$. Thus the velocities of all the other particles of the slab would be the same as originally defined. Therefore, *as far as the velocities are concerned, the slab seems to rotate about the instantaneous center* C at the instant considered.

The position of the instantaneous center can be defined in two other ways. If the directions of the velocities of two particles A and B of the slab are known and if they are different, the instantaneous center C is obtained by drawing the perpendicular to $\mathbf{v}_A$ through A and the perpendicular to $\mathbf{v}_B$ through B and determining the point in which these two lines intersect (Fig. 15.19*a*). If the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of two particles A and B are perpendicular to the line AB and if their magnitudes are known, the instantaneous center can be found by intersecting the line AB with the line joining the extremities of the vectors $\mathbf{v}_A$ and $\mathbf{v}_B$ (Fig. 15.19*b*). Note that if $\mathbf{v}_A$ and $\mathbf{v}_B$ were parallel

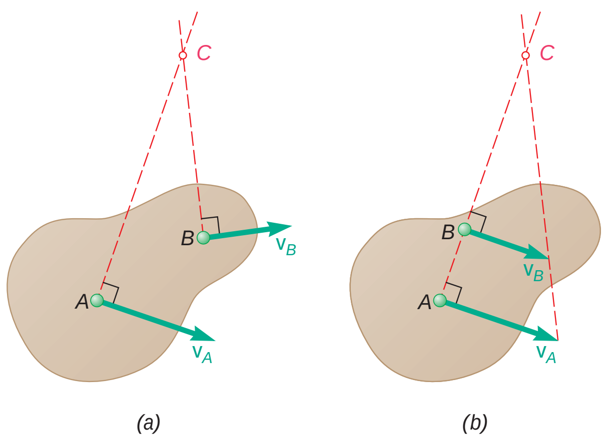


Fig. 15.19

in Fig. 15.19*a* or if $\mathbf{v}_A$ and $\mathbf{v}_B$ had the same magnitude in Fig. 15.19*b*, the instantaneous center C would be at an infinite distance and ω would be zero; all points of the slab would have the same velocity.

To see how the concept of instantaneous center of rotation can be put to use, let us consider again the rod of Sec. 15.6. Drawing the perpendicular to $\mathbf{v}_A$ through A and the perpendicular to $\mathbf{v}_B$ through B (Fig. 15.20), we obtain the instantaneous center C. At the

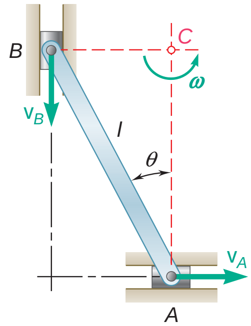


Fig. 15.20

instant considered, the velocities of all the particles of the rod are thus the same as if the rod rotated about C. Now, if the magnitude v_A of the velocity of A is known, the magnitude ω of the angular velocity of the rod can be obtained by writing

$$\omega = \frac{v_A}{AC} = \frac{v_A}{l \cos \theta}$$

The magnitude of the velocity of B can then be obtained by writing

$$v_B = (BC)\omega = l \sin \theta \frac{v_A}{l \cos \theta} = v_A \tan \theta$$

Note that only *absolute* velocities are involved in the computation.

The instantaneous center of a slab in plane motion can be located either on the slab or outside the slab. If it is located on the slab, the particle C coinciding with the instantaneous center at a given instant t must have zero velocity at that instant. However, it should be noted that the instantaneous center of rotation is valid only at a given instant. Thus, the particle C of the slab which coincides with the instantaneous center at time t will generally not coincide with the instantaneous center at time $t + \Delta t$; while its velocity is zero at time t, it will probably be different from zero at time $t + \Delta t$. This means that, in general, the particle C *does not have zero acceleration* and, therefore, that the *accelerations* of the various particles of the slab *cannot* be determined as if the slab were rotating about C.

As the motion of the slab proceeds, the instantaneous center moves in space. But it was just pointed out that the position of the instantaneous center on the slab keeps changing. Thus, the instantaneous center describes one curve in space, called the *space centrode*, and another curve on the slab, called the *body centrode* (Fig. 15.21). It can be shown that at any instant, these two curves are tangent at C and that as the slab moves, the body centrode appears to *roll* on the space centrode.

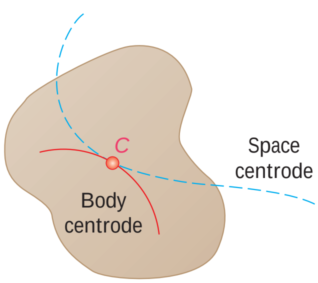
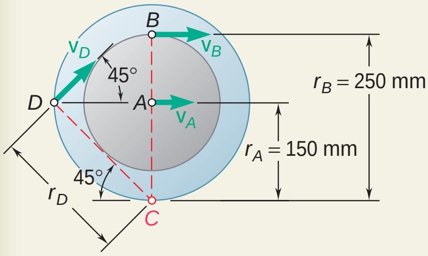


Fig. 15.21

SAMPLE PROBLEM 15.4



Solve Sample Prob. 15.2, using the method of the instantaneous center of rotation.

SOLUTION

a. Angular Velocity of the Gear. Since the gear rolls on the stationary lower rack, the point of contact C of the gear with the rack has no velocity; point C is therefore the instantaneous center of rotation. We write

$$v_A = r_A\omega \qquad 1.2 \text{ m/s} = (0.150 \text{ m})\omega$$

$$\omega = 8 \text{ rad/s} \circlearrowright \quad \blacktriangleleft$$

b. Velocities. As far as velocities are concerned, all points of the gear seem to rotate about the instantaneous center.

Velocity of Upper Rack. Recalling that $v_R = v_B$, we write

$$v_R = v_B = r_B\omega \qquad v_R = (0.250 \text{ m})(8 \text{ rad/s}) = 2 \text{ m/s}$$

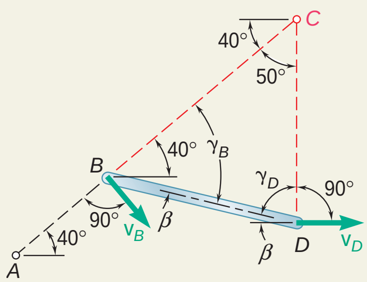
$$\mathbf{v}_R = 2 \text{ m/s} \rightarrow \quad \blacktriangleleft$$

Velocity of Point *D*. Since $r_D = (0.150 \text{ m})\sqrt{2} = 0.2121$ m, we write

$$v_D = r_D\omega \qquad v_D = (0.2121 \text{ m})(8 \text{ rad/s}) = 1.697 \text{ m/s}$$

$$\mathbf{v}_D = 1.697 \text{ m/s} \measuredangle 45° \quad \blacktriangleleft$$

SAMPLE PROBLEM 15.5



Solve Sample Prob. 15.3, using the method of the instantaneous center of rotation.

SOLUTION

Motion of Crank *AB*. Referring to Sample Prob. 15.3, we obtain the velocity of point B; $\mathbf{v}_B = 628.3$ in./s $\measuredangle$ 50°.

Motion of the Connecting Rod *BD*. We first locate the instantaneous center C by drawing lines perpendicular to the absolute velocities $\mathbf{v}_B$ and $\mathbf{v}_D$. Recalling from Sample Prob. 15.3 that $\beta = 13.95°$ and that $BD = 8$ in., we solve the triangle BCD.

$$\gamma_B = 40° + \beta = 53.95° \qquad \gamma_D = 90° - \beta = 76.05°$$

$$\frac{BC}{\sin 76.05°} = \frac{CD}{\sin 53.95°} = \frac{8 \text{ in.}}{\sin 50°}$$

$$BC = 10.14 \text{ in.} \qquad CD = 8.44 \text{ in.}$$

Since the connecting rod BD seems to rotate about point C, we write

$$v_B = (BC)\omega_{BD}$$

$$628.3 \text{ in./s} = (10.14 \text{ in.})\omega_{BD}$$

$$\omega_{BD} = 62.0 \text{ rad/s} \circlearrowleft \quad \blacktriangleleft$$

$$v_D = (CD)\omega_{BD} = (8.44 \text{ in.})(62.0 \text{ rad/s})$$

$$= 523 \text{ in./s} = 43.6 \text{ ft/s}$$

$$\mathbf{v}_P = \mathbf{v}_D = 43.6 \text{ ft/s} \rightarrow \quad \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced the *instantaneous center of rotation* in plane motion. This provides us with an alternative way for solving problems involving the *velocities* of the various points of a body in plane motion.

As its name suggests, the *instantaneous center of rotation* is the point about which you can assume a body is rotating at a given instant, as you determine the velocities of the points of the body at that instant.

A. To determine the instantaneous center of rotation of a body in plane motion, you should use one of the following procedures.

1. If the velocity $\mathbf{v}_A$ of a point *A* and the angular velocity ω of the body are both known (Fig. 15.18):

a. Draw a sketch of the body, showing point A, its velocity $\mathbf{v}_A$, and the angular velocity ω of the body.

b. From *A* draw a line perpendicular to $\mathbf{v}_A$ on the side of $\mathbf{v}_A$ from which this velocity is viewed as having *the same sense as* ω.

c. Locate the instantaneous center *C* on this line, at a distance $r = v_A/\omega$ from point A.

2. If the directions of the velocities of two points *A* and *B* are known and are different (Fig. 15.19*a*):

a. Draw a sketch of the body, showing points A and B and their velocities $\mathbf{v}_A$ and $\mathbf{v}_B$.

b. From *A* and *B* draw lines perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. The instantaneous center C is located at the point where the two lines intersect.

c. If the velocity of one of the two points is known, you can determine the angular velocity of the body. For example, if you know v_A, you can write $\omega = v_A/AC$, where AC is the distance from point A to the instantaneous center C.

3. If the velocities of two points *A* and *B* are known and are both perpendicular to the line *AB* (Fig. 15.19*b*):

a. Draw a sketch of the body, showing points A and B with their velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ *drawn to scale.*

b. Draw a line through points *A* and *B,* and another line through the tips of the vectors $\mathbf{v}_A$ and $\mathbf{v}_B$. The instantaneous center C is located at the point where the two lines intersect.

c. The angular velocity of the body is obtained by either dividing $\mathbf{v}_A$ by AC or $\mathbf{v}_B$ by BC.

d. If the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ have the same magnitude, the two lines drawn in part *b* do not intersect; the instantaneous center C is at an infinite distance. The angular velocity V is zero and *the body is in translation.*

B. Once you have determined the instantaneous center and the angular velocity of a body, you can determine the velocity $\mathbf{v}_P$ of any point P of the body in the following way.

1. Draw a sketch of the body, showing point P, the instantaneous center of rotation C, and the angular velocity V.

2. Draw a line from P to the instantaneous center C and measure or calculate the distance from P to C.

3. The velocity $\mathbf{v}_P$ is a vector perpendicular to the line PC, of the same sense as V, and of magnitude $v_P = (PC)$v.

Finally, keep in mind that the instantaneous center of rotation can be used *only* to determine velocities. *It cannot be used to determine accelerations.*

PROBLEMS

CONCEPT QUESTIONS

15.CQ5 The disk rolls without sliding on the fixed horizontal surface. At the instant shown, the instantaneous center of zero velocity for rod *AB* would be located in which region?

a. Region 1
b. Region 2
c. Region 3
d. Region 4
e. Region 5
f. Region 6

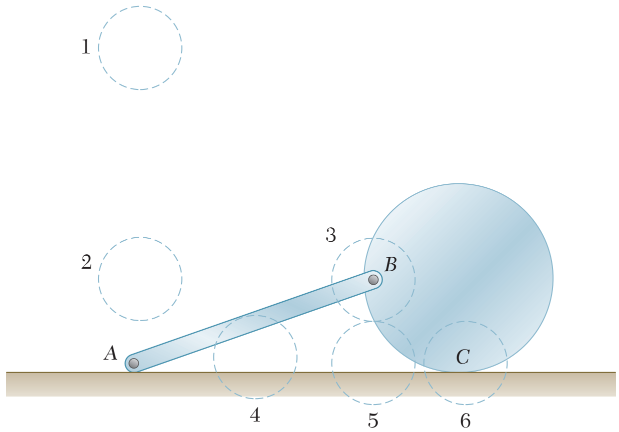


Fig. P15.CQ5

15.CQ6 Bar *BDE* is pinned to two links, *AB* and *CD*. At the instant shown, the angular velocities of link *AB*, link *CD*, and bar *BDE* are ω_{AB}, ω_{CD}, and ω_{BDE}, respectively. Which of the following statements concerning the angular speeds of the three objects is true at this instant?

a. $\omega_{AB} = \omega_{CD} = \omega_{BDE}$
b. $\omega_{BDE} > \omega_{AB} > \omega_{CD}$
c. $\omega_{AB} = \omega_{CD} > \omega_{BDE}$
d. $\omega_{AB} > \omega_{CD} > \omega_{BDE}$
e. $\omega_{CD} > \omega_{AB} > \omega_{BDE}$

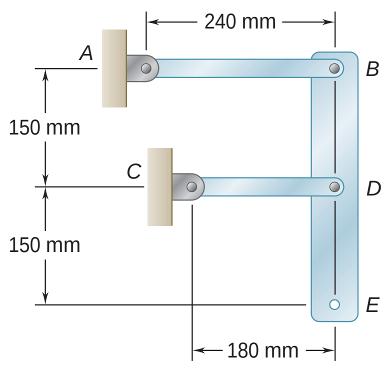


Fig. P15.CQ6

END-OF-SECTION PROBLEMS

15.73 A juggling club is thrown vertically into the air. The center of gravity *G* of the 20-in. club is located 12 in. from the knob. Knowing that at the instant shown, *G* has a velocity of 4 ft/s upwards and the club has an angular velocity of 30 rad/s counterclockwise, determine (*a*) the speeds of points *A* and *B*, (*b*) the location of the instantaneous center of rotation.

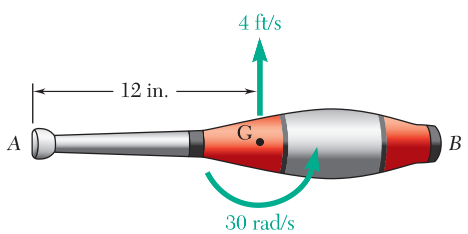


Fig. *P15.73*

15.74 A 10-ft beam AE is being lowered by means of two overhead cranes. At the instant shown, it is known that the velocity of point D is 24 in./s downward and the velocity of point E is 36 in./s downward. Determine (*a*) the instantaneous center of rotation of the beam, (*b*) the velocity of point A.

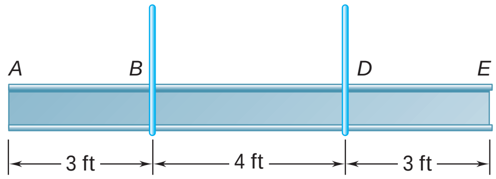


Fig. P15.74

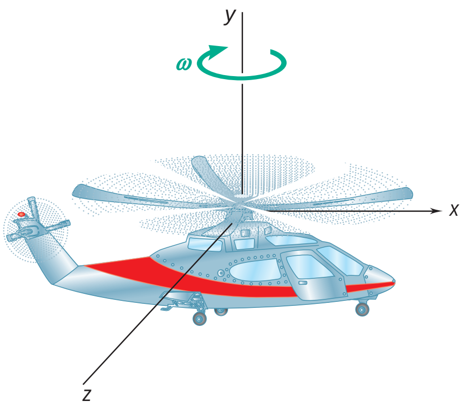


Fig. P15.75

15.75 A helicopter moves horizontally in the x direction at a speed of 120 mi/h. Knowing that the main blades rotate clockwise with an angular velocity of 180 rpm, determine the instantaneous axis of rotation of the main blades.

15.76 and 15.77 A 60-mm-radius drum is rigidly attached to a 100-mm-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that end E of the cord is pulled to the left with a velocity of 120 mm/s, determine (*a*) the angular velocity of the drums, (*b*) the velocity of the center of the drums, (*c*) the length of cord wound or unwound per second.

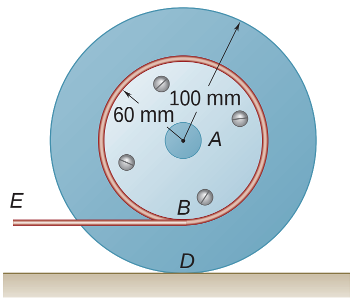


Fig. P15.76

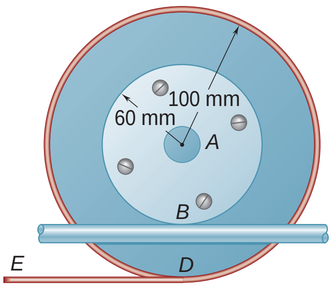


Fig. P15.77

15.78 The spool of tape shown and its frame assembly are pulled upward at a speed v_A = 750 mm/s. Knowing that the 80-mm-radius spool has an angular velocity of 15 rad/s clockwise and that at the instant shown the total thickness of the tape on the spool is 20 mm, determine (*a*) the instantaneous center of rotation of the spool, (*b*) the velocities of points B and D.

15.79 The spool of tape shown and its frame assembly are pulled upward at a speed v_A = 100 mm/s. Knowing that end B of the tape is pulled downward with a velocity of 300 mm/s and that at the instant shown the total thickness of the tape on the spool is 20 mm, determine (*a*) the instantaneous center of rotation of the spool, (*b*) the velocity of point D of the spool.

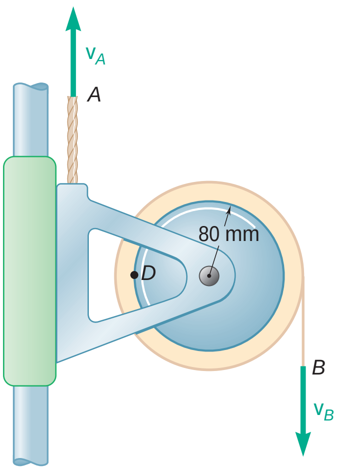


Fig. P15.78 and P15.79

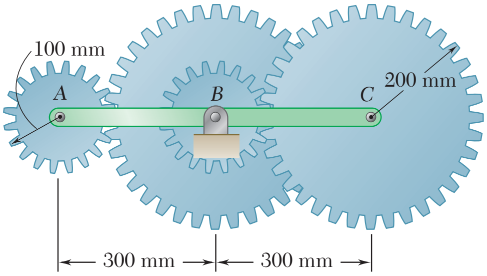


Fig. P15.80

15.80 The arm *ABC* rotates with an angular velocity of 4 rad/s counterclockwise. Knowing that the angular velocity of the intermediate gear *B* is 8 rad/s counterclockwise, determine (*a*) the instantaneous centers of rotation of gears *A* and *C*, (*b*) the angular velocities of gears *A* and *C*.

15.81 The double gear rolls on the stationary left rack *R*. Knowing that the rack on the right has a constant velocity of 2 ft/s, determine (*a*) the angular velocity of the gear, (*b*) the velocities of points *A* and *D*.

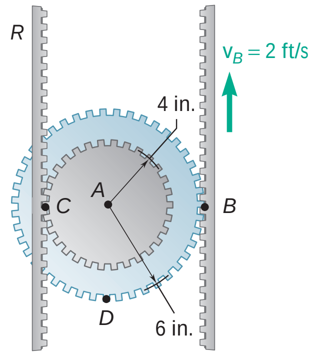


Fig. P15.81

15.82 An overhead door is guided by wheels at *A* and *B* that roll in horizontal and vertical tracks. Knowing that when $\theta = 40°$ the velocity of wheel *B* is 1.5 ft/s upward, determine (*a*) the angular velocity of the door, (*b*) the velocity of end *D* of the door.

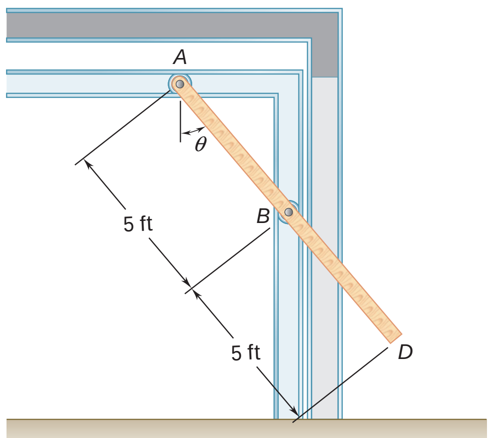


Fig. P15.82

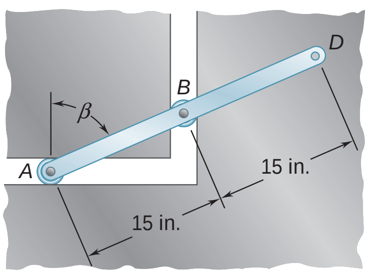


Fig. P15.83

15.83 Rod *ABD* is guided by wheels at *A* and *B* that roll in horizontal and vertical tracks. Knowing that at the instant $\beta = 60°$ and the velocity of wheel *B* is 40 in./s downward, determine (*a*) the angular velocity of the rod, (*b*) the velocity of point *D*.

15.84 Rod BDE is partially guided by a roller at D which moves in a vertical track. Knowing that at the instant shown the angular velocity of crank AB is 5 rad/s clockwise and that b = 25°, determine (*a*) the angular velocity of the rod, (*b*) the velocity of point E.

15.85 Rod BDE is partially guided by a roller at D which moves in a vertical track. Knowing that at the instant shown b = 30°, point E has a velocity of 2 m/s down and to the right, determine the angular velocities of rod BDE and crank AB.

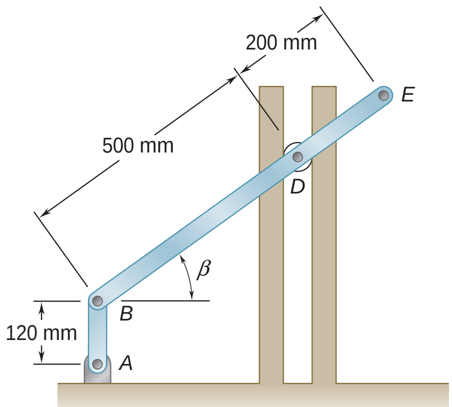


Fig. *P15.84* and *P15.85*

15.86 Knowing that at the instant shown, the velocity of collar D is 1.6 m/s upward, determine (*a*) the angular velocity of rod AD, (*b*) the velocity of point B, (*c*) the velocity of point A.

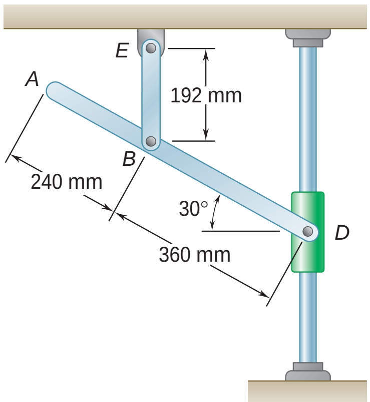


Fig. P15.86 and P15.87

15.87 Knowing that at the instant shown, the angular velocity of rod BE is 4 rad/s counterclockwise, determine (*a*) the angular velocity of rod AD, (*b*) the velocity of collar D, (*c*) the velocity of point A.

15.88 Rod AB can slide freely along the floor and the inclined plane. Denoting by $\mathbf{v}_A$ the velocity of point A, derive an expression for (*a*) the angular velocity of the rod, (*b*) the velocity of end B.

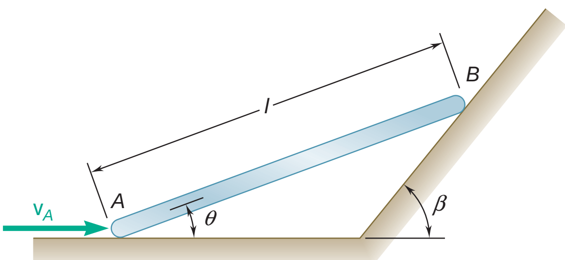


Fig. P15.88 and P15.89

15.89 Rod AB can slide freely along the floor and the inclined plane. Knowing that u = 20°, b = 50°, l = 2 ft, and v_A = 8 ft/s, determine (*a*) the angular velocity of the rod, (*b*) the velocity of end B.

15.90 Two slots have been cut in plate *FG* and the plate has been placed so that the slots fit two fixed pins *A* and *B*. Knowing that at the instant shown the angular velocity of crank *DE* is 6 rad/s clockwise, determine (*a*) the velocity of point *F*, (*b*) the velocity of point *G*.

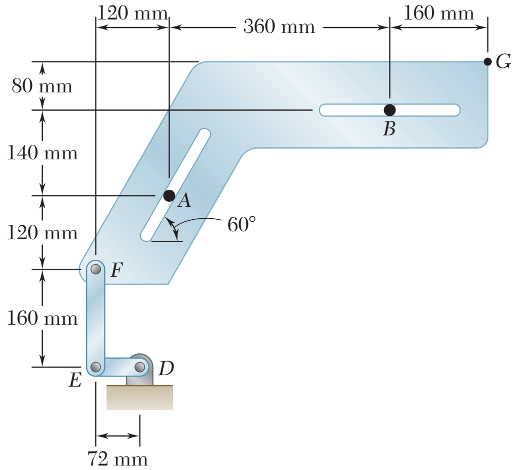


Fig. P15.90

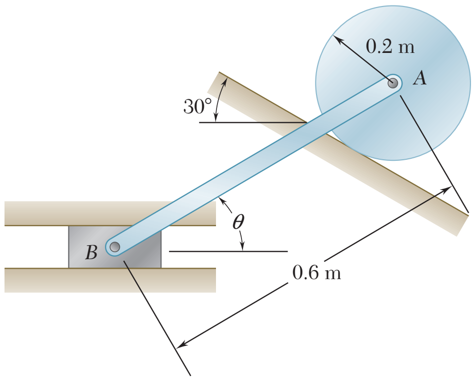


Fig. *P15.91*

15.91 The disk is released from rest and rolls down the incline. Knowing that the speed of *A* is 1.2 m/s when u = 0°, determine at that instant (*a*) the angular velocity of the rod, (*b*) the velocity of *B*. (Only portions of the two tracks are shown.)

15.92 Arm *ABD* is connected by pins to a collar at *B* and to crank *DE*. Knowing that the velocity of collar *B* is 400 mm/s upward, determine (*a*) the angular velocity of arm *ABD*, (*b*) the velocity of point *A*.

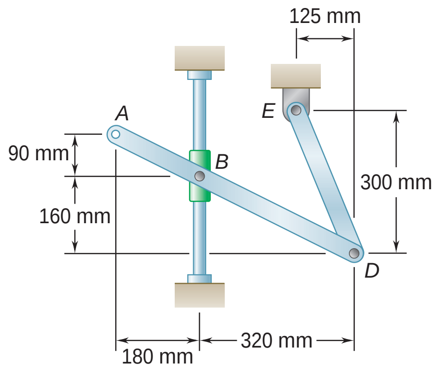


Fig. *P15.92* and *P15.93*

15.93 Arm *ABD* is connected by pins to a collar at *B* and to crank *DE*. Knowing that the angular velocity of crank *DE* is 1.2 rad/s counterclockwise, determine (*a*) the angular velocity of arm *ABD*, (*b*) the velocity of point *A*.

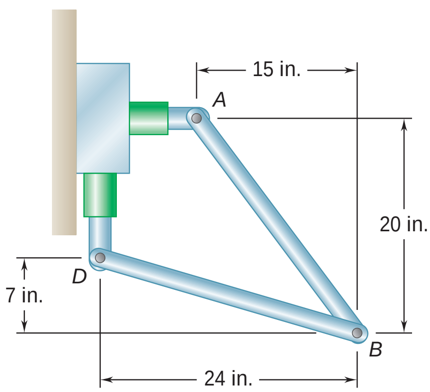


Fig. P15.94

15.94 Two links *AB* and *BD*, each 25 in. long, are connected at *B* and guided by hydraulic cylinders attached at *A* and *D*. Knowing that *D* is stationary and that the velocity of *A* is 30 in./s to the right, determine at the instant shown (*a*) the angular velocity of each link, (*b*) the velocity of *B*.

15.95 Two 25-in. rods are pin-connected at D as shown. Knowing that B moves to the left with a constant velocity of 24 in./s, determine at the instant shown (a) the angular velocity of each rod, (b) the velocity of E.

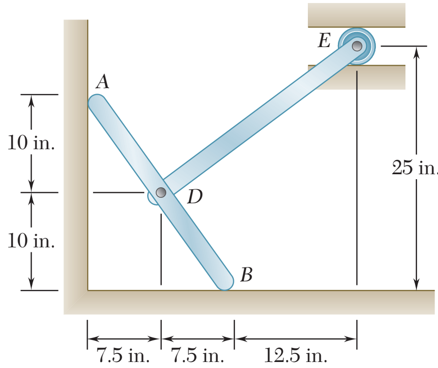


Fig. P15.95

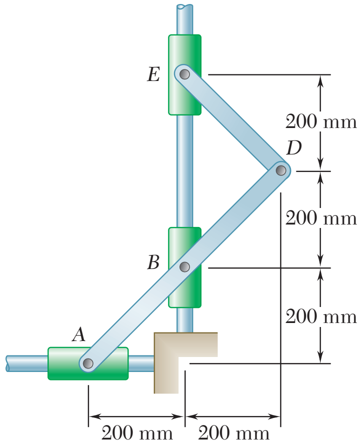


Fig. P15.96

15.96 Two rods ABD and DE are connected to three collars as shown. Knowing that the angular velocity of ABD is 5 rad/s clockwise, determine at the instant shown (a) the angular velocity of DE, (b) the velocity of collar E.

15.97 Two collars C and D move along the vertical rod shown. Knowing that the velocity of collar C is 660 mm/s downward, determine (a) the velocity of collar D, (b) the angular velocity of member AB.

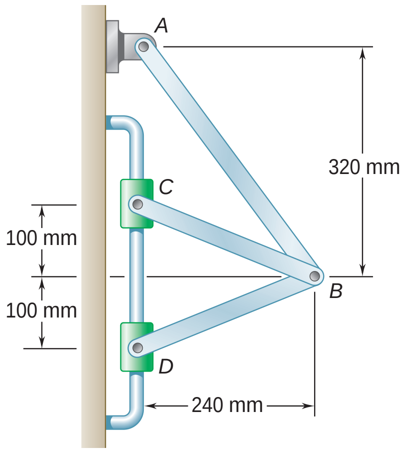


Fig. P15.97

15.98 Two rods AB and DE are connected as shown. Knowing that point D moves to the left with a velocity of 40 in./s, determine (a) the angular velocity of each rod, (b) the velocity of point A.

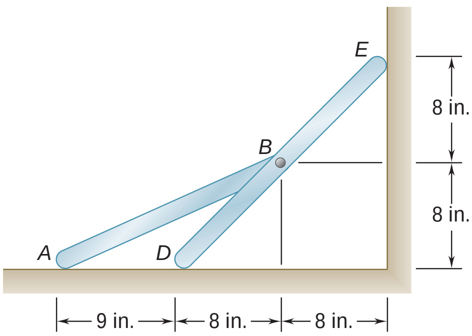


Fig. P15.98

15.99 Describe the space centrode and the body centrode of rod ABD of Prob. 15.83. (*Hint:* The body centrode need not lie on a physical portion of the rod.)

15.100 Describe the space centrode and the body centrode of the gear of Sample Prob. 15.2 as the gear rolls on the stationary horizontal rack.

15.101 Using the method of Sec. 15.7, solve Prob. 15.60.

15.102 Using the method of Sec. 15.7, solve Prob. 15.64.

15.103 Using the method of Sec. 15.7, solve Prob. 15.65.

15.104 Using the method of Sec. 15.7, solve Prob. 15.38.

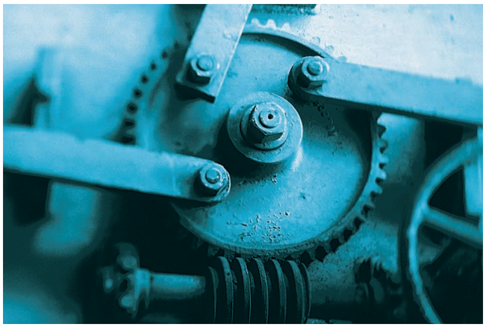


Photo 15.6 The central gear rotates about a fixed axis and is pin-connected to three bars which are in general plane motion.

15.8 ABSOLUTE AND RELATIVE ACCELERATION IN PLANE MOTION

We saw in Sec. 15.5 that any plane motion can be replaced by a translation defined by the motion of an arbitrary reference point A and a simultaneous rotation about A. This property was used in Sec. 15.6 to determine the velocity of the various points of a moving slab. The same property will now be used to determine the acceleration of the points of the slab.

We first recall that the absolute acceleration $\mathbf{a}_B$ of a particle of the slab can be obtained from the relative-acceleration formula derived in Sec. 11.12,

$$\boxed{\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}} \tag{15.21}$$

where the right-hand member represents a vector sum. The acceleration $\mathbf{a}_A$ corresponds to the translation of the slab with A, while the relative acceleration $\mathbf{a}_{B/A}$ is associated with the rotation of the slab about A and is measured with respect to axes centered at A and of fixed orientation. We recall from Sec. 15.3 that the relative acceleration $\mathbf{a}_{B/A}$ can be resolved into two components, a *tangential component* $(\mathbf{a}_{B/A})_t$ perpendicular to the line AB, and a *normal component* $(\mathbf{a}_{B/A})_n$ directed toward A (Fig. 15.22). Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A and, respectively, by $\omega\mathbf{k}$ and $\alpha\mathbf{k}$ the angular velocity and angular acceleration of the slab with respect to axes of fixed orientation, we have

$$\begin{aligned} (\mathbf{a}_{B/A})_t &= \alpha\mathbf{k} \times \mathbf{r}_{B/A} & \qquad (a_{B/A})_t &= r\alpha \\ (\mathbf{a}_{B/A})_n &= -\omega^2\mathbf{r}_{B/A} & \qquad (a_{B/A})_n &= r\omega^2 \end{aligned} \tag{15.22}$$

where r is the distance from A to B. Substituting into (15.21) the expressions obtained for the tangential and normal components of $\mathbf{a}_{B/A}$, we can also write

$$\mathbf{a}_B = \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A} \tag{15.21'}$$

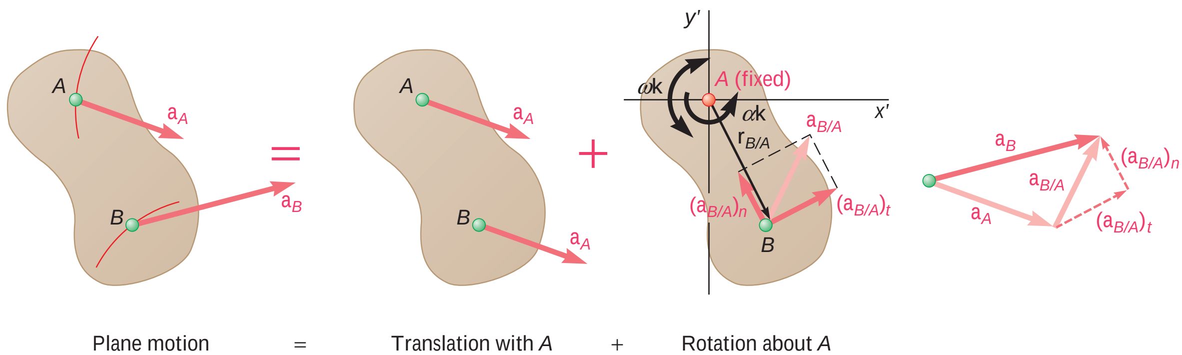


Fig. 15.22

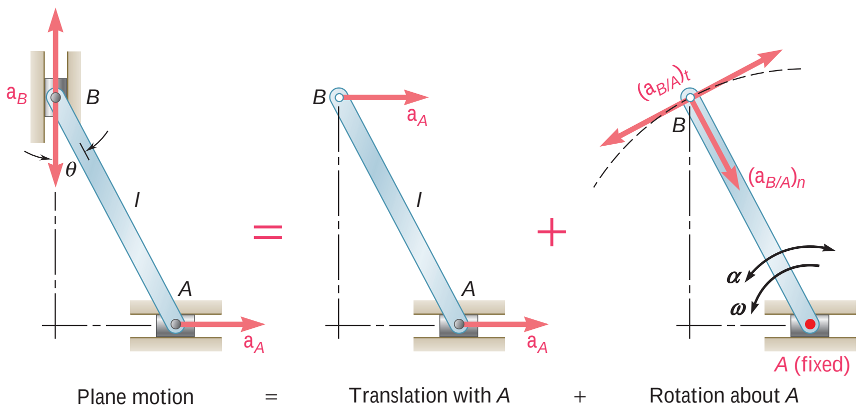


Fig. 15.23

As an example, let us again consider the rod AB whose extremities slide, respectively, along a horizontal and a vertical track (Fig. 15.23). Assuming that the velocity $\mathbf{v}_A$ and the acceleration $\mathbf{a}_A$ of A are known, we propose to determine the acceleration $\mathbf{a}_B$ of B and the angular acceleration α of the rod. Choosing A as a reference point, we express that the given motion is equivalent to a translation with A and a rotation about A. The absolute acceleration of B must be equal to the sum

$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} \\ &= \mathbf{a}_A + (\mathbf{a}_{B/A})_n + (\mathbf{a}_{B/A})_t\end{aligned} \tag{15.23}$$

where $(\mathbf{a}_{B/A})_n$ has the magnitude $l\omega^2$ and is *directed toward A*, while $(\mathbf{a}_{B/A})_t$ has the magnitude $l\alpha$ and is perpendicular to AB. Students should note that there is no way to tell whether the tangential component $(\mathbf{a}_{B/A})_t$ is directed to the left or to the right, and therefore both possible directions for this component are indicated in Fig. 15.23. Similarly, both possible senses for $\mathbf{a}_B$ are indicated, since it is not known whether point B is accelerated upward or downward.

Equation (15.23) has been expressed geometrically in Fig. 15.24. Four different vector polygons can be obtained, depending upon the sense of $\mathbf{a}_A$ and the relative magnitude of a_A and $(a_{B/A})_n$. If we are to determine a_B and α from one of these diagrams, we must know not only a_A and θ but also ω. The angular velocity of the rod should therefore be separately determined by one of the methods indicated in Secs. 15.6 and 15.7. The values of a_B and α can then be obtained by considering successively the x and y components of the vectors shown in Fig. 15.24. In the case of polygon *a*, for example, we write

$\xrightarrow{+}$ x components: $\qquad 0 = a_A + l\omega^2 \sin\theta - l\alpha\cos\theta$

$+\uparrow y$ components: $\qquad -a_B = -l\omega^2\cos\theta - l\alpha\sin\theta$

and solve for a_B and α. The two unknowns can also be obtained by direct measurement on the vector polygon. In that case, care should be taken to draw first the known vectors $\mathbf{a}_A$ and $(\mathbf{a}_{B/A})_n$.

It is quite evident that the determination of accelerations is considerably more involved than the determination of velocities. Yet

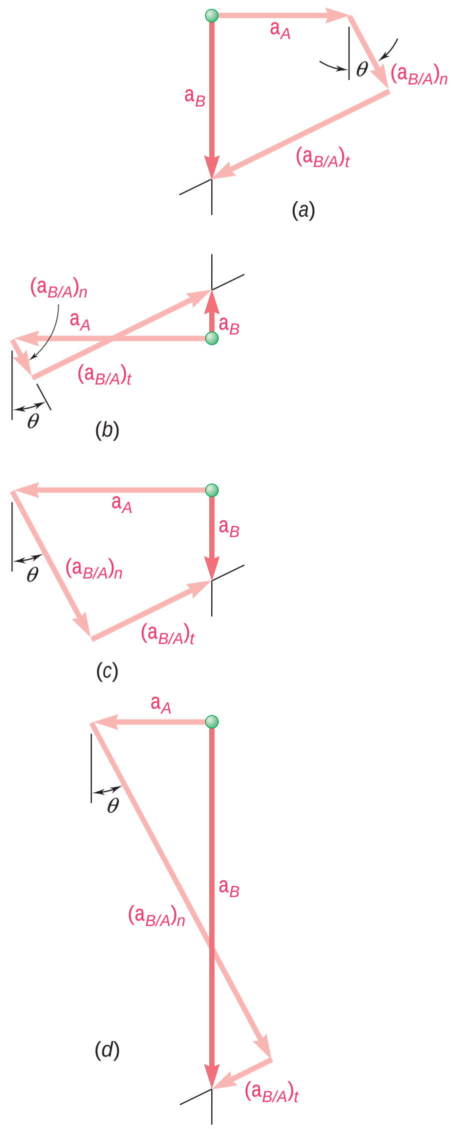


Fig. 15.24

in the example considered here, the extremities A and B of the rod were moving along straight tracks, and the diagrams drawn were relatively simple. If A and B had moved along curved tracks, it would have been necessary to resolve the accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$ into normal and tangential components and the solution of the problem would have involved six different vectors.

When a mechanism consists of several moving parts which are pin-connected, the analysis of the mechanism can be carried out by considering each part as a rigid body, keeping in mind that the points at which two parts are connected must have the same absolute acceleration (see Sample Prob. 15.7). In the case of meshed gears, the tangential components of the accelerations of the teeth in contact are equal, but their normal components are different.

*15.9 ANALYSIS OF PLANE MOTION IN TERMS OF A PARAMETER

In the case of certain mechanisms, it is possible to express the coordinates x and y of all the significant points of the mechanism by means of simple analytic expressions containing a single parameter. It is sometimes advantageous in such a case to determine the absolute velocity and the absolute acceleration of the various points of the mechanism directly, since the components of the velocity and of the acceleration of a given point can be obtained by differentiating the coordinates x and y of that point.

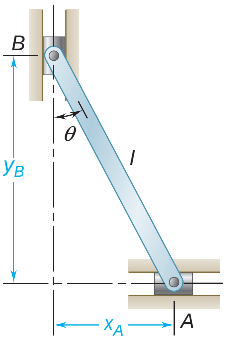


Fig. 15.25

Let us consider again the rod AB whose extremities slide, respectively, in a horizontal and a vertical track (Fig. 15.25). The coordinates x_A and y_B of the extremities of the rod can be expressed in terms of the angle θ the rod forms with the vertical:

$$x_A = l \sin \theta \qquad y_B = l \cos \theta \tag{15.24}$$

Differentiating Eqs. (15.24) twice with respect to t, we write

$$\begin{aligned} v_A &= \dot{x}_A = l\dot{\theta} \cos \theta \\ a_A &= \ddot{x}_A = -l\dot{\theta}^2 \sin \theta + l\ddot{\theta} \cos \theta \end{aligned}$$

$$\begin{aligned} v_B &= \dot{y}_B = -l\dot{\theta} \sin \theta \\ a_B &= \ddot{y}_B = -l\dot{\theta}^2 \cos \theta - l\ddot{\theta} \sin \theta \end{aligned}$$

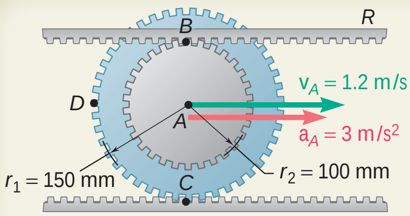
Recalling that $\dot{\theta} = \omega$ and $\ddot{\theta} = \alpha$, we obtain

$$v_A = l\omega \cos \theta \qquad v_B = -l\omega \sin \theta \tag{15.25}$$

$$a_A = -l\omega^2 \sin \theta + l\alpha \cos \theta \qquad a_B = -l\omega^2 \cos \theta - l\alpha \sin \theta \tag{15.26}$$

We note that a positive sign for v_A or a_A indicates that the velocity $\mathbf{v}_A$ or the acceleration $\mathbf{a}_A$ is directed to the right; a positive sign for v_B or a_B indicates that $\mathbf{v}_B$ or $\mathbf{a}_B$ is directed upward. Equations (15.25) can be used, for example, to determine v_B and ω when v_A and θ are known. Substituting for ω in (15.26), we can then determine a_B and α if a_A is known.

SAMPLE PROBLEM 15.6



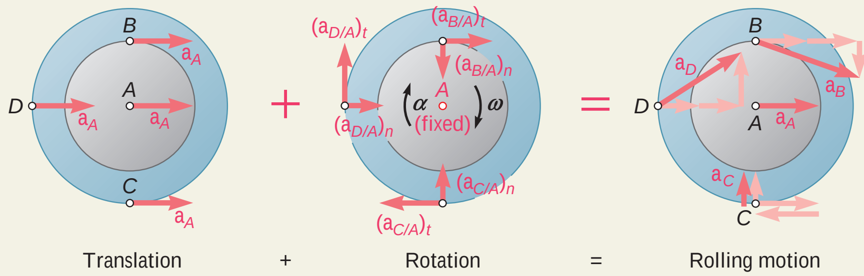
The center of the double gear of Sample Prob. 15.2 has a velocity of 1.2 m/s to the right and an acceleration of 3 m/s^2 to the right. Recalling that the lower rack is stationary, determine (*a*) the angular acceleration of the gear, (*b*) the acceleration of points *B*, *C*, and *D* of the gear.

SOLUTION

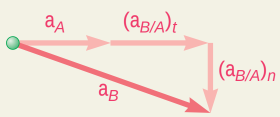
a. Angular Acceleration of the Gear. In Sample Prob. 15.2, we found that $x_A = -r_1\theta$ and $v_A = -r_1\omega$. Differentiating the latter with respect to time, we obtain $a_A = -r_1\alpha$.

$$v_A = -r_1\omega \qquad 1.2\text{ m/s} = -(0.150\text{ m})\omega \qquad \omega = -8\text{ rad/s}$$
$$a_A = -r_1\alpha \qquad 3\text{ m/s}^2 = -(0.150\text{ m})\alpha \qquad \alpha = -20\text{ rad/s}^2$$
$$\boldsymbol{\alpha} = \alpha\mathbf{k} = -(20\text{ rad/s}^2)\mathbf{k} \quad \blacktriangleleft$$

b. Accelerations. The rolling motion of the gear is resolved into a translation with *A* and a rotation about *A*.

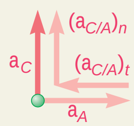


Acceleration of Point *B*. Adding vectorially the accelerations corresponding to the translation and to the rotation, we obtain



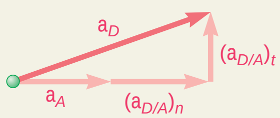
$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n\\ &= \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}\\ &= (3\text{ m/s}^2)\mathbf{i} - (20\text{ rad/s}^2)\mathbf{k} \times (0.100\text{ m})\mathbf{j} - (8\text{ rad/s})^2(0.100\text{ m})\mathbf{j}\\ &= (3\text{ m/s}^2)\mathbf{i} + (2\text{ m/s}^2)\mathbf{i} - (6.40\text{ m/s}^2)\mathbf{j}\end{aligned}$$
$$\mathbf{a}_B = 8.12\text{ m/s}^2 \searrow 52.0° \quad \blacktriangleleft$$

Acceleration of Point *C*

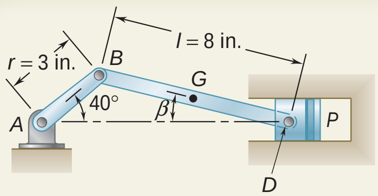


$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_A + \mathbf{a}_{C/A} = \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{C/A} - \omega^2\mathbf{r}_{C/A}\\ &= (3\text{ m/s}^2)\mathbf{i} - (20\text{ rad/s}^2)\mathbf{k} \times (-0.150\text{ m})\mathbf{j} - (8\text{ rad/s})^2(-0.150\text{ m})\mathbf{j}\\ &= (3\text{ m/s}^2)\mathbf{i} - (3\text{ m/s}^2)\mathbf{i} + (9.60\text{ m/s}^2)\mathbf{j}\end{aligned}$$
$$\mathbf{a}_C = 9.60\text{ m/s}^2 \uparrow \quad \blacktriangleleft$$

Acceleration of Point *D*



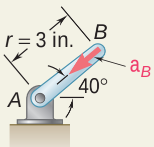
$$\begin{aligned}\mathbf{a}_D &= \mathbf{a}_A + \mathbf{a}_{D/A} = \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{D/A} - \omega^2\mathbf{r}_{D/A}\\ &= (3\text{ m/s}^2)\mathbf{i} - (20\text{ rad/s}^2)\mathbf{k} \times (-0.150\text{ m})\mathbf{i} - (8\text{ rad/s})^2(-0.150\text{ m})\mathbf{i}\\ &= (3\text{ m/s}^2)\mathbf{i} + (3\text{ m/s}^2)\mathbf{j} + (9.60\text{ m/s}^2)\mathbf{i}\end{aligned}$$
$$\mathbf{a}_D = 12.95\text{ m/s}^2 \nearrow 13.4° \quad \blacktriangleleft$$



SAMPLE PROBLEM 15.7

Crank AB of the engine system of Sample Prob. 15.3 has a constant clockwise angular velocity of 2000 rpm. For the crank position shown, determine the angular acceleration of the connecting rod BD and the acceleration of point D.

SOLUTION



Motion of Crank *AB*. Since the crank rotates about A with constant v_{AB} = 2000 rpm = 209.4 rad/s, we have $\mathsf{a}_{AB} = 0$. The acceleration of B is therefore directed toward A and has a magnitude

$$a_B = r\mathsf{v}_{AB}^2 = (\tfrac{3}{12}\text{ ft})(209.4\text{ rad/s})^2 = 10{,}962\text{ ft/s}^2$$
$$\mathbf{a}_B = 10{,}962\text{ ft/s}^2 \text{ d } 40°$$

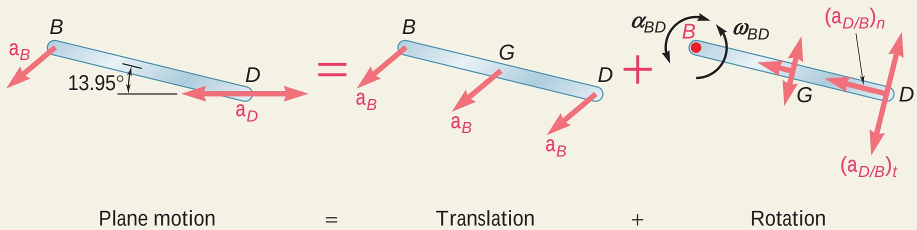
Motion of the Connecting Rod *BD*. The angular velocity V_{BD} and the value of b were obtained in Sample Prob. 15.3:

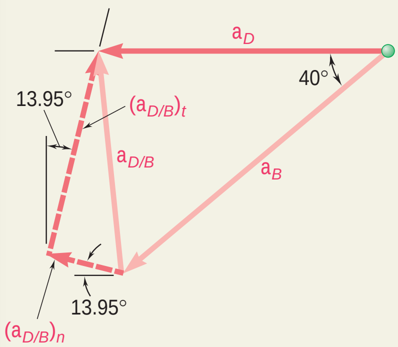
$$\mathsf{V}_{BD} = 62.0\text{ rad/s l} \qquad \mathsf{b} = 13.95°$$

The motion of BD is resolved into a translation with B and a rotation about B. The relative acceleration $\mathbf{a}_{D/B}$ is resolved into normal and tangential components:

$$(a_{D/B})_n = (BD)\mathsf{v}_{BD}^2 = (\tfrac{8}{12}\text{ ft})(62.0\text{ rad/s})^2 = 2563\text{ ft/s}^2$$
$$(\mathbf{a}_{D/B})_n = 2563\text{ ft/s}^2 \text{ b } 13.95°$$
$$(a_{D/B})_t = (BD)\mathsf{a}_{BD} = (\tfrac{8}{12})\mathsf{a}_{BD} = 0.6667\mathsf{a}_{BD}$$
$$(\mathbf{a}_{D/B})_t = 0.6667\mathsf{a}_{BD} \text{ za } 76.05°$$

While $(\mathbf{a}_{D/B})_t$ must be perpendicular to BD, its sense is not known.





Noting that the acceleration $\mathbf{a}_D$ must be horizontal, we write

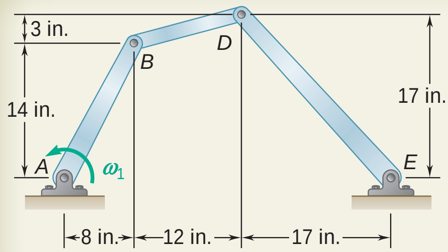
$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} = \mathbf{a}_B + (\mathbf{a}_{D/B})_n + (\mathbf{a}_{D/B})_t$$
$$[a_D \text{ G}] = [10{,}962 \text{ d } 40°] + [2563 \text{ b } 13.95°] + [0.6667\mathsf{a}_{BD} \text{ za } 76.05°]$$

Equating x and y components, we obtain the following scalar equations:

y x components:
$$-a_D = -10{,}962 \cos 40° - 2563 \cos 13.95° + 0.6667\mathsf{a}_{BD} \sin 13.95°$$
+xy components:
$$0 = -10{,}962 \sin 40° + 2563 \sin 13.95° + 0.6667\mathsf{a}_{BD} \cos 13.95°$$

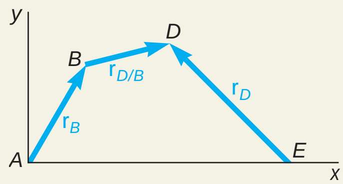
Solving the equations simultaneously, we obtain $\mathsf{a}_{BD} = +9940\text{ rad/s}^2$ and $a_D = +9290\text{ ft/s}^2$. The positive signs indicate that the senses shown on the vector polygon are correct; we write

$$\mathsf{a}_{BD} = 9940\text{ rad/s}^2 \text{ l} \quad ◀$$
$$\mathbf{a}_D = 9290\text{ ft/s}^2 \text{ z} \quad ◀$$



SAMPLE PROBLEM 15.8

The linkage *ABDE* moves in the vertical plane. Knowing that in the position shown crank *AB* has a constant angular velocity $\boldsymbol{\omega}_1$ of 20 rad/s counterclockwise, determine the angular velocities and angular accelerations of the connecting rod *BD* and of the crank *DE*.



$\mathbf{r}_B = 8\mathbf{i} + 14\mathbf{j}$
$\mathbf{r}_D = -17\mathbf{i} + 17\mathbf{j}$
$\mathbf{r}_{D/B} = 12\mathbf{i} + 3\mathbf{j}$

SOLUTION

This problem could be solved by the method used in Sample Prob. 15.7. In this case, however, the vector approach will be used. The position vectors $\mathbf{r}_B$, $\mathbf{r}_D$, and $\mathbf{r}_{D/B}$ are chosen as shown in the sketch.

Velocities. Since the motion of each element of the linkage is contained in the plane of the figure, we have

$$\boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k} = (20 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \qquad \boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$$

where $\mathbf{k}$ is a unit vector pointing out of the paper. We now write

$$\begin{aligned}
\mathbf{v}_D &= \mathbf{v}_B + \mathbf{v}_{D/B} \\
\omega_{DE}\mathbf{k} \times \mathbf{r}_D &= \omega_{AB}\mathbf{k} \times \mathbf{r}_B + \omega_{BD}\mathbf{k} \times \mathbf{r}_{D/B} \\
\omega_{DE}\mathbf{k} \times (-17\mathbf{i} + 17\mathbf{j}) &= 20\mathbf{k} \times (8\mathbf{i} + 14\mathbf{j}) + \omega_{BD}\mathbf{k} \times (12\mathbf{i} + 3\mathbf{j}) \\
-17\omega_{DE}\mathbf{j} - 17\omega_{DE}\mathbf{i} &= 160\mathbf{j} - 280\mathbf{i} + 12\omega_{BD}\mathbf{j} - 3\omega_{BD}\mathbf{i}
\end{aligned}$$

Equating the coefficients of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, we obtain the following two scalar equations:

$$\begin{aligned}
-17\omega_{DE} &= -280 - 3\omega_{BD} \\
-17\omega_{DE} &= +160 + 12\omega_{BD}
\end{aligned}$$

$$\boldsymbol{\omega}_{BD} = -(29.33 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{DE} = (11.29 \text{ rad/s})\mathbf{k} \quad \blacktriangleleft$$

Accelerations. Noting that at the instant considered crank *AB* has a constant angular velocity, we write

$$\boldsymbol{\alpha}_{AB} = 0 \qquad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k} \qquad \boldsymbol{\alpha}_{DE} = \alpha_{DE}\mathbf{k}$$

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} \tag{1}$$

Each term of Eq. (1) is evaluated separately:

$$\begin{aligned}
\mathbf{a}_D &= \alpha_{DE}\mathbf{k} \times \mathbf{r}_D - \omega_{DE}^2\mathbf{r}_D \\
&= \alpha_{DE}\mathbf{k} \times (-17\mathbf{i} + 17\mathbf{j}) - (11.29)^2(-17\mathbf{i} + 17\mathbf{j}) \\
&= -17\alpha_{DE}\mathbf{j} - 17\alpha_{DE}\mathbf{i} + 2170\mathbf{i} - 2170\mathbf{j} \\
\mathbf{a}_B &= \alpha_{AB}\mathbf{k} \times \mathbf{r}_B - \omega_{AB}^2\mathbf{r}_B = 0 - (20)^2(8\mathbf{i} + 14\mathbf{j}) \\
&= -3200\mathbf{i} - 5600\mathbf{j} \\
\mathbf{a}_{D/B} &= \alpha_{BD}\mathbf{k} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B} \\
&= \alpha_{BD}\mathbf{k} \times (12\mathbf{i} + 3\mathbf{j}) - (29.33)^2(12\mathbf{i} + 3\mathbf{j}) \\
&= 12\alpha_{BD}\mathbf{j} - 3\alpha_{BD}\mathbf{i} - 10{,}320\mathbf{i} - 2580\mathbf{j}
\end{aligned}$$

Substituting into Eq. (1) and equating the coefficients of $\mathbf{i}$ and $\mathbf{j}$, we obtain

$$\begin{aligned}
-17\alpha_{DE} + 3\alpha_{BD} &= -15{,}690 \\
-17\alpha_{DE} - 12\alpha_{BD} &= -6010
\end{aligned}$$

$$\boldsymbol{\alpha}_{BD} = -(645 \text{ rad/s}^2)\mathbf{k} \qquad \boldsymbol{\alpha}_{DE} = (809 \text{ rad/s}^2)\mathbf{k} \quad \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

This lesson was devoted to the determination of the *accelerations* of the points of a *rigid body in plane motion.* As you did previously for velocities, you will again consider the plane motion of a rigid body as the sum of two motions, namely, *a translation and a rotation.*

To solve a problem involving accelerations in plane motion you should use the following steps:

1. Determine the angular velocity of the body. To find V you can either

a. Consider the motion of the body as the sum of a translation and a rotation as you did in Sec. 15.6, or

b. Use the instantaneous center of rotation of the body as you did in Sec. 15.7. However, *keep in mind that you cannot use the instantaneous center to determine accelerations.*

2. Start drawing a "diagram equation" to use in your solution. This "equation" will involve the following diagrams (Fig. 15.22).

a. Plane motion diagram. Draw a sketch of the body, including all dimensions, as well as the angular velocity V. Show the angular acceleration A with its magnitude and sense if you know them. Also show those points for which you know or seek the accelerations, indicating all that you know about these accelerations.

b. Translation diagram. Select a reference point *A* for which you know the direction, the magnitude, or a component of the acceleration $\mathbf{a}_A$. Draw a second diagram showing the body in translation with each point having the same acceleration as point *A*.

c. Rotation diagram. Considering point *A* as a fixed reference point, draw a third diagram showing the body in rotation about *A*. Indicate the normal and tangential components of the relative accelerations of other points, such as the components $(\mathbf{a}_{B/A})_n$ and $(\mathbf{a}_{B/A})_t$ of the acceleration of point *B* with respect to point *A*.

3. Write the relative-acceleration formula

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \qquad \text{or} \qquad \mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_n + (\mathbf{a}_{B/A})_t$$

The sample problems illustrate three different ways to use this vector equation:

a. If A is given or can easily be determined, you can use this equation to determine the accelerations of various points of the body [Sample Prob. 15.6].

b. If A cannot easily be determined, select for point B a point for which you know the direction, the magnitude, or a component of the acceleration $\mathbf{a}_B$ and draw a vector diagram of the equation. Starting at the same point, draw all known acceleration components in tip-to-tail fashion for each member of the equation. Complete the diagram by drawing the two remaining vectors in appropriate directions and in such a way that the two sums of vectors end at a common point.

The magnitudes of the two remaining vectors can be found either graphically or analytically. Usually an analytic solution will require the solution of two simultaneous equations [Sample Prob. 15.7]. However, by first considering the components of the various vectors in a direction perpendicular to one of the unknown vectors, you may be able to obtain an equation in a single unknown.

One of the two vectors obtained by the method just described will be $(\mathbf{a}_{B/A})_t$, from which you can compute α. Once α has been found, the vector equation can be used to determine the acceleration of any other point of the body.

c. A full vector approach can also be used to solve the vector equation. This is illustrated in Sample Prob. 15.8.

4. The analysis of plane motion in terms of a parameter completed this lesson. This method should be used *only if it is possible* to express the coordinates x and y of all significant points of the body in terms of a single parameter (Sec. 15.9). By differentiating twice with respect to t the coordinates x and y of a given point, you can determine the rectangular components of the absolute velocity and absolute acceleration of that point.

PROBLEMS

CONCEPT QUESTION

15.CQ7 A rear-wheel-drive car starts from rest and accelerates to the left so that the tires do not slip on the road. What is the direction of the acceleration of the point on the tire in contact with the road, that is, point A?

a. ← **b.** ↖ **c.** ↑ **d.** ↓ **e.** ↙

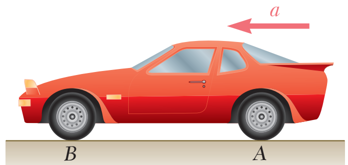


Fig. P15.CQ7

END-OF-SECTION PROBLEMS

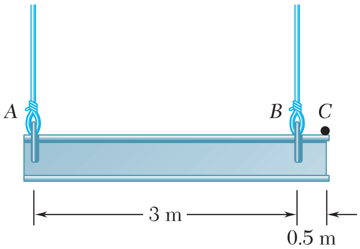


Fig. P15.105 and P15.106

15.105 A 3.5-m steel beam is lowered by means of two cables unwinding at the same speed from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow down the unwinding motion. At the instant considered, the deceleration of the cable attached at A is 4 m/s^2, while that of the cable at B is 1.5 m/s^2. Determine (*a*) the angular acceleration of the beam, (*b*) the acceleration of point C.

15.106 The acceleration of point C is 0.3 m/s^2 downward and the angular acceleration of the beam is 0.8 rad/s^2 clockwise. Knowing that the angular velocity of the beam is zero at the instant considered, determine the acceleration of each cable.

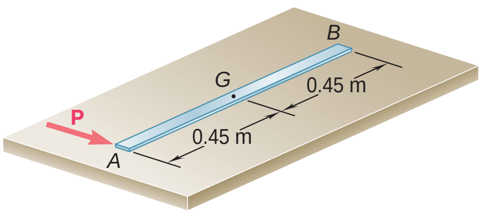


Fig. P15.107 and P15.108

15.107 A 900-mm rod rests on a horizontal table. A force **P** applied as shown produces the following accelerations: $\mathbf{a}_A = 3.6$ m/s^2 to the right, $\alpha = 6$ rad/s^2 counterclockwise as viewed from above. Determine the acceleration (*a*) of point G, (*b*) of point B.

15.108 In Prob. 15.107, determine the point of the rod that (*a*) has no acceleration, (*b*) has an acceleration of 2.4 m/s^2 to the right.

15.109 Knowing that at the instant shown crank BC has a constant angular velocity of 45 rpm clockwise, determine the acceleration (*a*) of point A, (*b*) of point D.

15.110 End A of rod AB moves to the right with a constant velocity of 6 ft/s. For the position shown, determine (*a*) the angular acceleration of rod AB, (*b*) the acceleration of the midpoint G of rod AB.

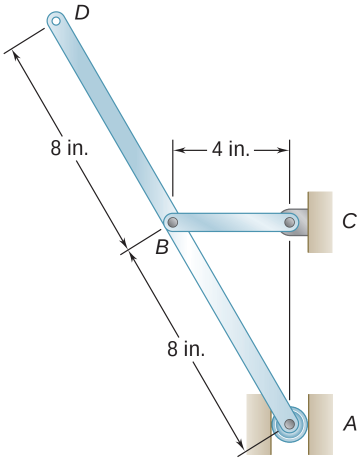


Fig. P15.109

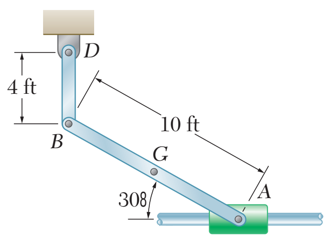


Fig. P15.110

15.111 An automobile travels to the left at a constant speed of 72 km/h. Knowing that the diameter of the wheel is 560 mm, determine the acceleration (*a*) of point *B*, (*b*) of point *C*, (*c*) of point *D*.

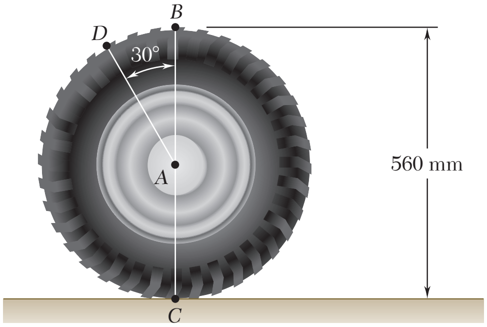


Fig. P15.111

15.112 The 18-in.-radius flywheel is rigidly attached to a 1.5-in.-radius shaft that can roll along parallel rails. Knowing that at the instant shown the center of the shaft has a velocity of 1.2 in./s and an acceleration of 0.5 in./s^2, both directed down to the left, determine the acceleration (*a*) of point *A*, (*b*) of point *B*.

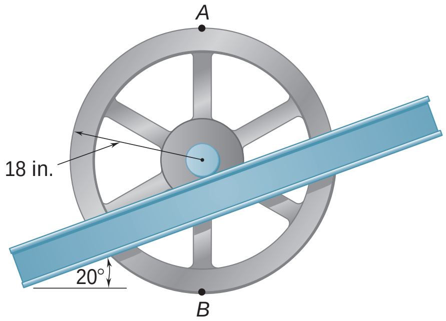


Fig. P15.112

15.113 and 15.114 A 3-in.-radius drum is rigidly attached to a 5-in.-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that at the instant shown end *D* of the cord has a velocity of 8 in./s and an acceleration of 30 in./s^2, both directed to the left, determine the accelerations of points *A*, *B*, and *C* of the drums.

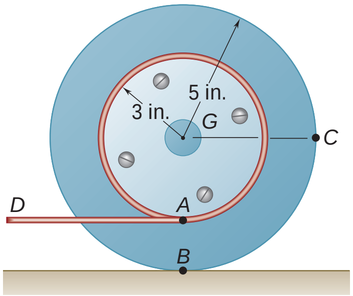


Fig. P15.113

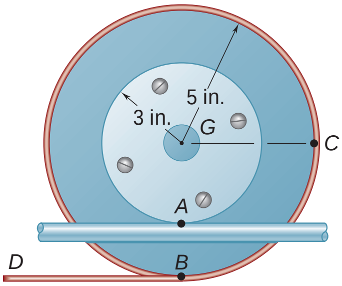


Fig. P15.114

15.115 A carriage *C* is supported by a caster *A* and a cylinder *B*, each of 50-mm diameter. Knowing that at the instant shown the carriage has an acceleration of 2.4 m/s^2 and a velocity of 1.5 m/s, both directed to the left, determine (*a*) the angular accelerations of the caster and of the cylinder, (*b*) the accelerations of the centers of the caster and of the cylinder.

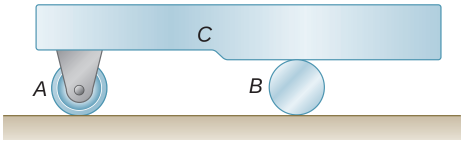


Fig. P15.115

15.116 A wheel rolls without slipping on a fixed cylinder. Knowing that at the instant shown the angular velocity of the wheel is 10 rad/s clockwise and its angular acceleration is 30 rad/s^2 counterclockwise, determine the acceleration of (*a*) point *A*, (*b*) point *B*, (*c*) point *C*.

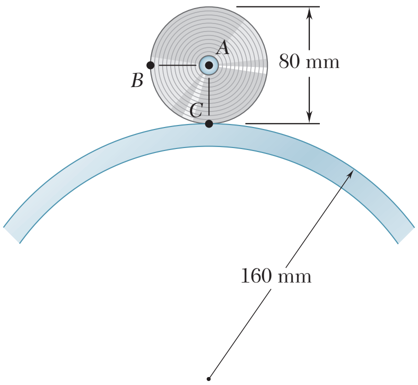


Fig. *P15.116*

15.117 The 100-mm-radius drum rolls without slipping on a portion of a belt which moves downward to the left with a constant velocity of 120 mm/s. Knowing that at a given instant the velocity and acceleration of the center *A* of the drum are as shown, determine the acceleration of point *D*.

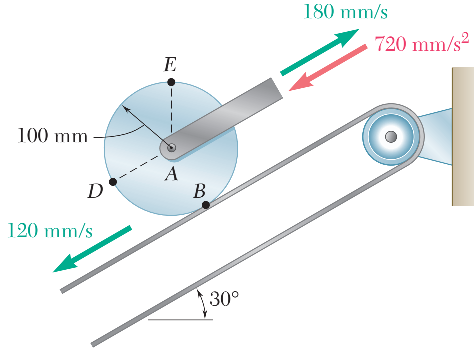


Fig. *P15.117*

15.118 In the planetary gear system shown, the radius of gears *A*, *B*, *C*, and *D* is 3 in. and the radius of the outer gear *E* is 9 in. Knowing that gear *A* has a constant angular velocity of 150 rpm clockwise and that the outer gear *E* is stationary, determine the magnitude of the acceleration of the tooth of gear *D* that is in contact with (*a*) gear *A*, (*b*) gear *E*.

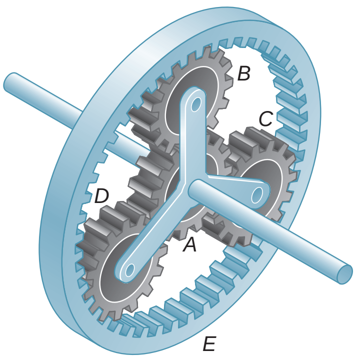


Fig. P15.118

15.119 The 200-mm-radius disk rolls without sliding on the surface shown. Knowing that the distance *BG* is 160 mm and that at the instant shown the disk has an angular velocity of 8 rad/s counterclockwise and an angular acceleration of 2 rad/s^2 clockwise, determine the acceleration of *A*.

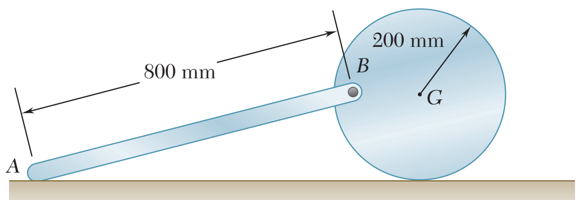


Fig. *P15.119*

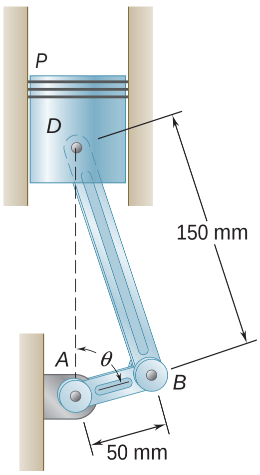


Fig. P15.120 and P15.121

15.120 Knowing that crank *AB* rotates about point *A* with a constant angular velocity of 900 rpm clockwise, determine the acceleration of the piston *P* when u = 60°.

15.121 Knowing that crank *AB* rotates about point *A* with a constant angular velocity of 900 rpm clockwise, determine the acceleration of the piston *P* when u = 120°.

15.122 In the two-cylinder air compressor shown the connecting rods BD and BE are each 190 mm long and crank AB rotates about the fixed point A with a constant angular velocity of 1500 rpm clockwise. Determine the acceleration of each piston when u $= 0$.

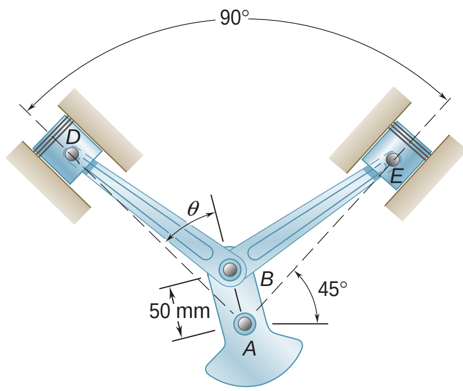


Fig. P15.122

15.123 The disk shown has a constant angular velocity of 500 rpm counterclockwise. Knowing that rod BD is 10 in. long, determine the acceleration of collar D when (a) u $= 90°$, (b) u $= 180°$.

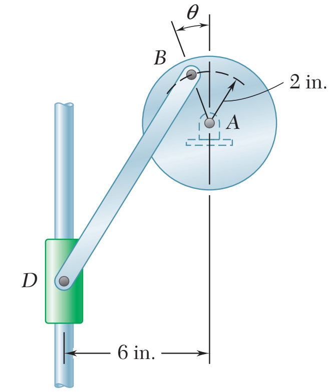


Fig. *P15.123*

15.124 Arm AB has a constant angular velocity of 16 rad/s counterclockwise. At the instant when u $= 90°$, determine the acceleration (a) of collar D, (b) of the midpoint G of bar BD.

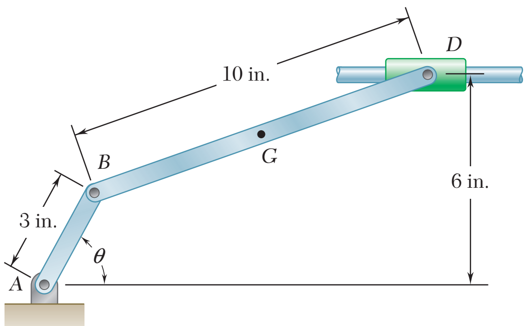


Fig. P15.124 and P15.125

15.125 Arm AB has a constant angular velocity of 16 rad/s counterclockwise. At the instant when u $= 60°$, determine the acceleration of collar D.

15.126 A straight rack rests on a gear of radius $r = 3$ in. and is attached to a block B as shown. Knowing that at the instant shown u $= 20°$, the angular velocity of gear D is 3 rad/s clockwise, and it is speeding up at a rate of 2 rad/s^2, determine (a) the angular acceleration of AB, (b) the acceleration of block B.

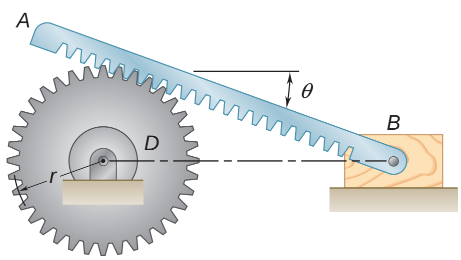


Fig. *P15.126*

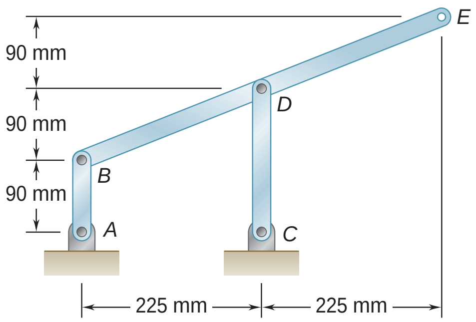


Fig. P15.127 and P15.128

15.127 Knowing that at the instant shown rod *AB* has a constant angular velocity of 6 rad/s clockwise, determine the acceleration of point *D*.

15.128 Knowing that at the instant shown rod *AB* has a constant angular velocity of 6 rad/s clockwise, determine (*a*) the angular acceleration of member *BDE*, (*b*) the acceleration of point *E*.

15.129 Knowing that at the instant shown bar *AB* has a constant angular velocity of 19 rad/s clockwise, determine (*a*) the angular acceleration of bar *BGD*, (*b*) the angular acceleration of bar *DE*.

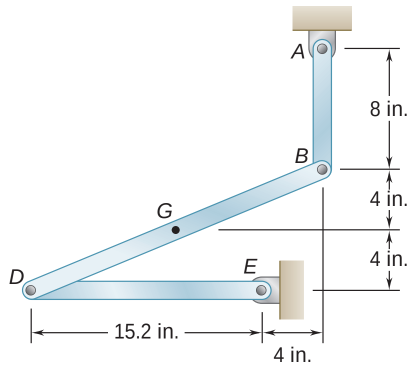


Fig. P15.129 and P15.130

15.130 Knowing that at the instant shown bar *DE* has a constant angular velocity of 18 rad/s clockwise, determine (*a*) the acceleration of point *B*, (*b*) the acceleration of point *G*.

15.131 and 15.132 Knowing that at the instant shown bar *AB* has a constant angular velocity of 4 rad/s clockwise, determine the angular acceleration (*a*) of bar *BD*, (*b*) of bar *DE*.

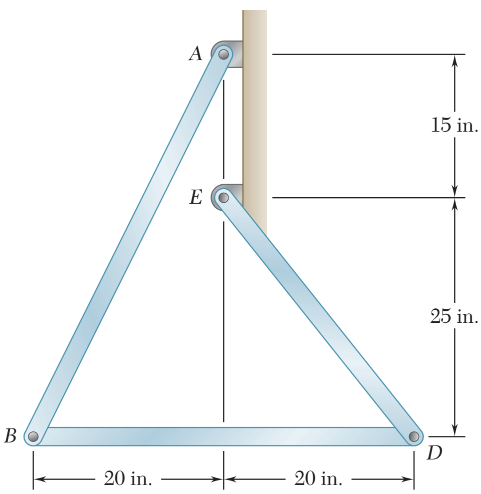


Fig. P15.131 and P15.133

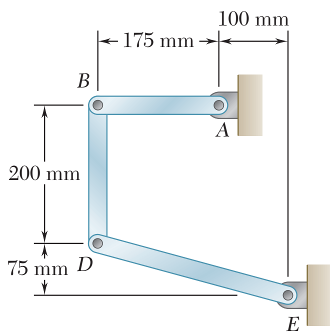


Fig. P15.132 and P15.134

15.133 and 15.134 Knowing that at the instant shown bar *AB* has an angular velocity of 4 rad/s and an angular acceleration of 2 rad/s^2, both clockwise, determine the angular acceleration (*a*) of bar *BD*, (*b*) of bar *DE* by using the vector approach as is done in Sample Prob. 15.8.

15.135 Roberts linkage is named after Richard Roberts (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at point *F*. The distance *AB* is the same as *BF*, *DF*, and *DE*. Knowing that at the instant shown, bar *AB* has a constant angular velocity of 4 rad/s clockwise, determine (*a*) the angular acceleration of bar *DE*, (*b*) the acceleration of point *F*.

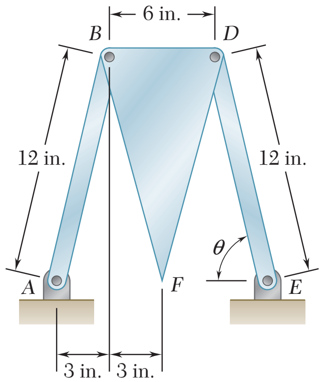


Fig. P15.135

15.136 For the oil pump rig shown, link *AB* causes the beam *BCE* to oscillate as the crank *OA* revolves. Knowing that *OA* has a radius of 0.6 m and a constant clockwise angular velocity of 20 rpm, determine the velocity and acceleration of point *D* at the instant shown.

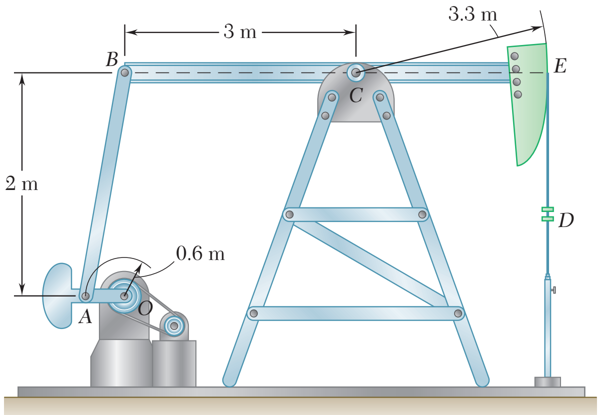


Fig. P15.136

15.137 Denoting by $\mathbf{r}_A$ the position vector of a point *A* of a rigid slab that is in plane motion, show that (*a*) the position vector $\mathbf{r}_C$ of the instantaneous center of rotation is

$$\mathbf{r}_C = \mathbf{r}_A + \frac{\mathrm{V} \times \mathbf{v}_A}{\mathrm{V}^2}$$

where V is the angular velocity of the slab and $\mathbf{v}_A$ is the velocity of point *A*, (*b*) the acceleration of the instantaneous center of rotation is zero if, and only if,

$$\mathbf{a}_A = \frac{\mathrm{a}}{\mathrm{V}}\mathbf{v}_A + \mathrm{V} \times \mathbf{v}_A$$

where A = a**k** is the angular acceleration of the slab.

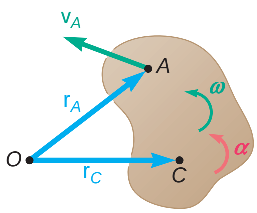


Fig. P15.137

***15.138** The drive disk of the Scotch crosshead mechanism shown has an angular velocity V and an angular acceleration A, both directed counterclockwise. Using the method of Sec. 15.9, derive expressions for the velocity and acceleration of point *B*.

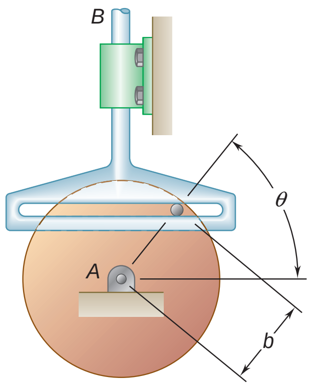


Fig. P15.138

***15.139** The wheels attached to the ends of rod AB roll along the surfaces shown. Using the method of Sec. 15.9, derive an expression for the angular velocity of the rod in terms of v_B, u, l, and b.

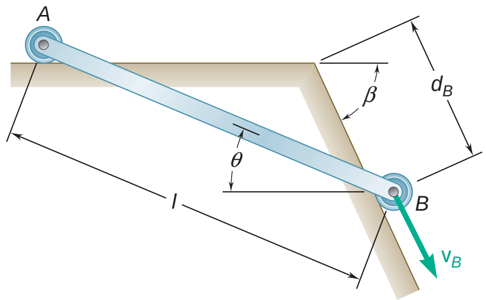


Fig. P15.139 and P15.140

***15.140** The wheels attached to the ends of rod AB roll along the surfaces shown. Using the method of Sec. 15.9 and knowing that the acceleration of wheel B is zero, derive an expression for the angular acceleration of the rod in terms of v_B, u, l, and b.

***15.141** A disk of radius r rolls to the right with a constant velocity **v**. Denoting by P the point of the rim in contact with the ground at $t = 0$, derive expressions for the horizontal and vertical components of the velocity of P at any time t.

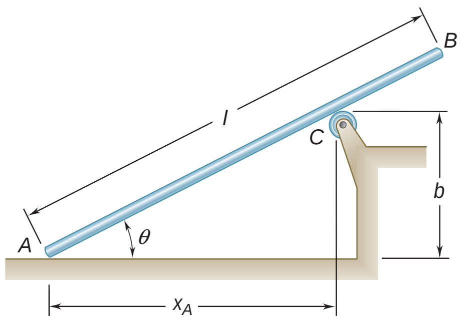


Fig. P15.142 and P15.143

***15.142** Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity $\mathbf{v}_A$. Using the method of Sec. 15.9, derive expressions for the angular velocity and angular acceleration of the rod.

***15.143** Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity $\mathbf{v}_A$. Using the method of Sec. 15.9, derive expressions for the horizontal and vertical components of the velocity of point B.

15.144 Crank AB rotates with a constant clockwise angular velocity $\boldsymbol{\omega}$. Using the method of Sec. 15.9, derive expressions for the angular velocity of rod BD and the velocity of the point on the rod coinciding with point E in terms of u, v, b, and l.

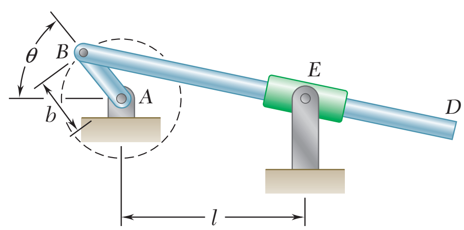


Fig. P15.144 and P15.145

15.145 Crank AB rotates with a constant clockwise angular velocity $\boldsymbol{\omega}$. Using the method of Sec. 15.9, derive an expression for the angular acceleration of rod BD in terms of u, v, b, and l.

15.146 Pin C is attached to rod CD and slides in a slot cut in arm AB. Knowing that rod CD moves vertically upward with a constant velocity $\mathbf{v}_0$, derive an expression for (*a*) the angular velocity of arm AB, (*b*) the components of the velocity of point A, (*c*) an expression for the angular acceleration of arm AB.

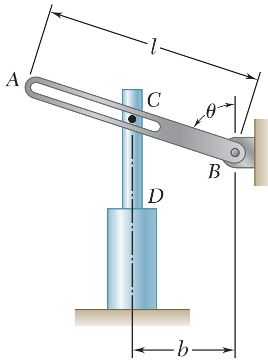


Fig. P15.146

***15.147** The position of rod AB is controlled by a disk of radius r which is attached to yoke CD. Knowing that the yoke moves vertically upward with a constant velocity $\mathbf{v}_0$, derive expressions for the angular velocity and angular acceleration of rod AB.

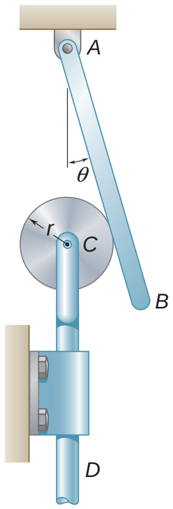


Fig. P15.147

***15.148** A wheel of radius r rolls without slipping along the inside of a fixed cylinder of radius R with a constant angular velocity V. Denoting by P the point of the wheel in contact with the cylinder at $t = 0$, derive expressions for the horizontal and vertical components of the velocity of P at any time t. (The curve described by point P is a *hypocycloid*.)

***15.149** In Prob. 15.148, show that the path of P is a vertical straight line when $r = R/2$. Derive expressions for the corresponding velocity and acceleration of P at any time t.

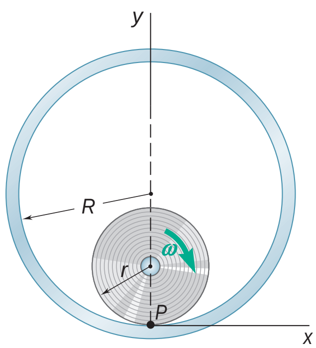


Fig. P15.148

15.10 RATE OF CHANGE OF A VECTOR WITH RESPECT TO A ROTATING FRAME

We saw in Sec. 11.10 that the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation. In this section, the rates of change of a vector $\mathbf{Q}$ with respect to a fixed frame and with respect to a rotating frame of reference will be considered.† You will learn to determine the rate of change of $\mathbf{Q}$ with respect to one frame of reference when $\mathbf{Q}$ is defined by its components in another frame.

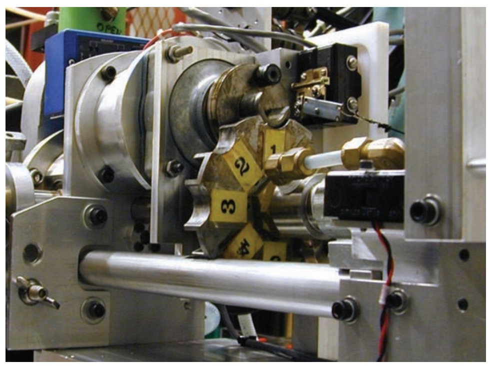


Photo 15.7 A Geneva mechanism is used to convert rotary motion into intermittent motion.

†It is recalled that the selection of a fixed frame of reference is arbitrary. Any frame may be designated as "fixed"; all others will then be considered as moving.

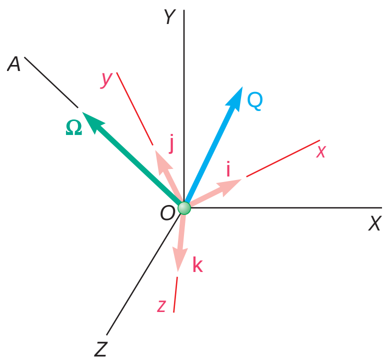


Fig. 15.26

Consider two frames of reference centered at O, a fixed frame $OXYZ$ and a frame $Oxyz$ which rotates about the fixed axis OA; let $\boldsymbol{\Omega}$ denote the angular velocity of the frame $Oxyz$ at a given instant (Fig. 15.26). Consider now a vector function $\mathbf{Q}(t)$ represented by the vector $\mathbf{Q}$ attached at O; as the time t varies, both the direction and the magnitude of $\mathbf{Q}$ change. Since the variation of $\mathbf{Q}$ is viewed differently by an observer using $OXYZ$ as a frame of reference and by an observer using $Oxyz$, we should expect the rate of change of $\mathbf{Q}$ to depend upon the frame of reference which has been selected. Therefore, the rate of change of $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ will be denoted by $(\dot{\mathbf{Q}})_{OXYZ}$, and the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$ will be denoted by $(\dot{\mathbf{Q}})_{Oxyz}$. We propose to determine the relation existing between these two rates of change.

Let us first resolve the vector $\mathbf{Q}$ into components along the x, y, and z axes of the rotating frame. Denoting by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ the corresponding unit vectors, we write

$$\mathbf{Q} = Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k} \tag{15.27}$$

Differentiating (15.27) with respect to t and considering the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as fixed, we obtain the rate of change of $\mathbf{Q}$ *with respect to the rotating frame Oxyz:*

$$(\dot{\mathbf{Q}})_{Oxyz} = \dot{Q}_x\mathbf{i} + \dot{Q}_y\mathbf{j} + \dot{Q}_z\mathbf{k} \tag{15.28}$$

To obtain the rate of change of $\mathbf{Q}$ *with respect to the fixed frame OXYZ,* we must consider the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as variable when differentiating (15.27). We therefore write

$$(\dot{\mathbf{Q}})_{OXYZ} = \dot{Q}_x\mathbf{i} + \dot{Q}_y\mathbf{j} + \dot{Q}_z\mathbf{k} + Q_x\frac{d\mathbf{i}}{dt} + Q_y\frac{d\mathbf{j}}{dt} + Q_z\frac{d\mathbf{k}}{dt} \tag{15.29}$$

Recalling (15.28), we observe that the sum of the first three terms in the right-hand member of (15.29) represents the rate of change $(\dot{\mathbf{Q}})_{Oxyz}$. We note, on the other hand, that the rate of change $(\dot{\mathbf{Q}})_{OXYZ}$ would reduce to the last three terms in (15.29) if the vector $\mathbf{Q}$ were fixed within the frame $Oxyz$, since $(\dot{\mathbf{Q}})_{Oxyz}$ would then be zero. But in that case, $(\dot{\mathbf{Q}})_{OXYZ}$ would represent the velocity of a particle located at the tip of $\mathbf{Q}$ and belonging to a body rigidly attached to the frame $Oxyz$. Thus, the last three terms in (15.29) represent the velocity of that particle; since the frame $Oxyz$ has an angular velocity $\boldsymbol{\Omega}$ with respect to $OXYZ$ at the instant considered, we write, by (15.5),

$$Q_x\frac{d\mathbf{i}}{dt} + Q_y\frac{d\mathbf{j}}{dt} + Q_z\frac{d\mathbf{k}}{dt} = \boldsymbol{\Omega} \times \mathbf{Q} \tag{15.30}$$

Substituting from (15.28) and (15.30) into (15.29), we obtain the fundamental relation

$$\boxed{(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q}} \tag{15.31}$$

We conclude that the rate of change of the vector $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ is made of two parts: The first part represents the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$; the second part, $\boldsymbol{\Omega} \times \mathbf{Q}$, is induced by the rotation of the frame $Oxyz$.

The use of relation (15.31) simplifies the determination of the rate of change of a vector $\mathbf{Q}$ with respect to a fixed frame of reference $OXYZ$ when the vector $\mathbf{Q}$ is defined by its components along the axes of a rotating frame $Oxyz$, since this relation does not require the separate computation of the derivatives of the unit vectors defining the orientation of the rotating frame.

15.11 PLANE MOTION OF A PARTICLE RELATIVE TO A ROTATING FRAME. CORIOLIS ACCELERATION

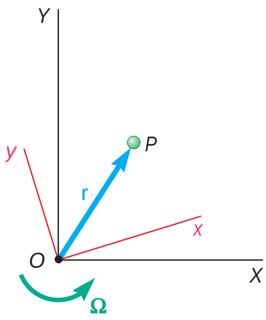


Fig. 15.27

Consider two frames of reference, both centered at O and both in the plane of the figure, a fixed frame OXY and a rotating frame Oxy (Fig. 15.27). Let P be a particle moving in the plane of the figure. The position vector $\mathbf{r}$ of P is the same in both frames, but its rate of change depends upon the frame of reference which has been selected.

The absolute velocity $\mathbf{v}_P$ of the particle is defined as the velocity observed from the fixed frame OXY and is equal to the rate of change $(\dot{\mathbf{r}})_{OXY}$ of $\mathbf{r}$ with respect to that frame. We can, however, express $\mathbf{v}_P$ in terms of the rate of change $(\dot{\mathbf{r}})_{Oxy}$ observed from the rotating frame if we make use of Eq. (15.31). Denoting by $\boldsymbol{\Omega}$ the angular velocity of the frame Oxy with respect to OXY at the instant considered, we write

$$\mathbf{v}_P = (\dot{\mathbf{r}})_{OXY} = \boldsymbol{\Omega} \times \mathbf{r} + (\dot{\mathbf{r}})_{Oxy} \tag{15.32}$$

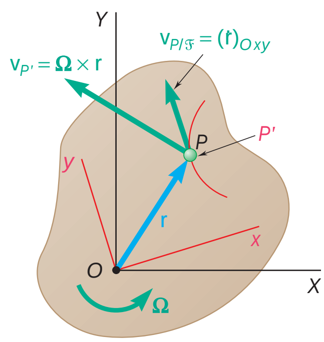


Fig. 15.28

But $(\dot{\mathbf{r}})_{Oxy}$ defines the velocity of the particle P relative to the rotating frame Oxy. Denoting the rotating frame by $\mathcal{F}$ for short, we represent the velocity $(\dot{\mathbf{r}})_{Oxy}$ of P relative to the rotating frame by $\mathbf{v}_{P/\mathcal{F}}$. Let us imagine that a rigid slab has been attached to the rotating frame. Then $v_{P/\mathcal{F}}$ represents the velocity of P along the path that it describes on that slab (Fig. 15.28), and the term $\boldsymbol{\Omega} \times \mathbf{r}$ in (15.32) represents the velocity $\mathbf{v}_{P'}$ of the point P' of the slab—or rotating frame—which coincides with P at the instant considered. Thus, we have

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.33}$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame $\mathcal{F}$

The absolute acceleration $\mathbf{a}_P$ of the particle is defined as the rate of change of $\mathbf{v}_P$ with respect to the fixed frame OXY. Computing the rates of change with respect to OXY of the terms in (15.32), we write

$$\mathbf{a}_P = \dot{\mathbf{v}}_P = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times \dot{\mathbf{r}} + \frac{d}{dt}[(\dot{\mathbf{r}})_{Oxy}] \tag{15.34}$$

where all derivatives are defined with respect to OXY, except where indicated otherwise. Referring to Eq. (15.31), we note that the last term in (15.34) can be expressed as

$$\frac{d}{dt}[(\dot{\mathbf{r}})_{Oxy}] = (\ddot{\mathbf{r}})_{Oxy} + \boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxy}$$

On the other hand, $\dot{\mathbf{r}}$ represents the velocity $\mathbf{v}_P$ and can be replaced by the right-hand member of Eq. (15.32). After completing these two substitutions into (15.34), we write

$$\mathbf{a}_P = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxy} + (\ddot{\mathbf{r}})_{Oxy} \tag{15.35}$$

Referring to the expression (15.8) obtained in Sec. 15.3 for the acceleration of a particle in a rigid body rotating about a fixed axis, we note that the sum of the first two terms represents the acceleration $\mathbf{a}_{P'}$ of the point P' of the rotating frame which coincides with P at the instant considered. On the other hand, the last term defines the acceleration $\mathbf{a}_{P/\text{f}}$ of P relative to the rotating frame. If it were not for the third term, which has not been accounted for, a relation similar to (15.33) could be written for the accelerations, and $\mathbf{a}_P$ could be expressed as the sum of $\mathbf{a}_{P'}$ and $\mathbf{a}_{P/\text{f}}$. However, it is clear that *such a relation would be incorrect* and that we must include the additional term. This term, which will be denoted by $\mathbf{a}_c$, is called the *complementary acceleration*, or *Coriolis acceleration*, after the French mathematician de Coriolis (1792–1843). We write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.36}$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame $\mathcal{F}$
$\mathbf{a}_c = 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxy} = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration†

We note that since point P' moves in a circle about the origin O, its acceleration $\mathbf{a}_{P'}$ has, in general, two components: a component $(\mathbf{a}_{P'})_t$ tangent to the circle, and a component $(\mathbf{a}_{P'})_n$ directed toward O. Similarly, the acceleration $\mathbf{a}_{P/\mathcal{F}}$ generally has two components: a component $(\mathbf{a}_{P/\mathcal{F}})_t$ tangent to the path that P describes on the rotating slab, and a component $(\mathbf{a}_{P/\mathcal{F}})_n$ directed toward the center of curvature of that path. We further note that since the vector $\boldsymbol{\Omega}$ is perpendicular to the plane of motion, and thus to $\mathbf{v}_{P/\mathcal{F}}$, the magnitude of the Coriolis acceleration $\mathbf{a}_c = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$ is equal to $2\Omega v_{P/\mathcal{F}}$, and its direction can be obtained by rotating the vector $\mathbf{v}_{P/\mathcal{F}}$ through 90° in the sense of rotation of the moving frame (Fig. 15.29). The Coriolis acceleration reduces to zero when either $\boldsymbol{\Omega}$ or $\mathbf{v}_{P/\mathcal{F}}$ is zero.

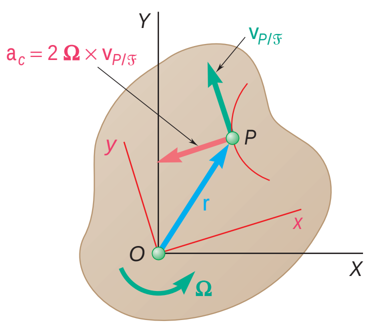


Fig. 15.29

The following example will help in understanding the physical meaning of the Coriolis acceleration. Consider a collar P which is

†It is important to note the difference between Eq. (15.36) and Eq. (15.21) of Sec. 15.8. When we wrote

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{15.21}$$

in Sec. 15.8, we were expressing the absolute acceleration of point B as the sum of its acceleration $\mathbf{a}_{B/A}$ relative to a *frame in translation* and of the acceleration $\mathbf{a}_A$ of a point of that frame. We are now trying to relate the absolute acceleration of point P to its acceleration $\mathbf{a}_{P/\text{f}}$ relative to a *rotating frame* f and to the acceleration $\mathbf{a}_{P'}$ of the point P' of that frame which coincides with P; Eq. (15.36) shows that because the frame is rotating, it is necessary to include an additional term representing the Coriolis acceleration $\mathbf{a}_c$.

made to slide at a constant relative speed u along a rod OB rotating at a constant angular velocity ω about O (Fig. 15.30*a*). According to formula (15.36), the absolute acceleration of P can be obtained by adding vectorially the acceleration $\mathbf{a}_A$ of the point A of the rod coinciding with P, the relative acceleration $\mathbf{a}_{P/OB}$ of P with respect to the rod, and the Coriolis acceleration $\mathbf{a}_c$. Since the angular velocity ω of the rod is constant, $\mathbf{a}_A$ reduces to its normal component $(\mathbf{a}_A)_n$ of magnitude $r\omega^2$; and since u is constant, the relative acceleration $\mathbf{a}_{P/OB}$ is zero. According to the definition given above, the Coriolis acceleration is a vector perpendicular to OB, of magnitude $2\omega u$, and directed as shown in the figure. The acceleration of the collar P consists, therefore, of the two vectors shown in Fig. 15.30*a*. Note that the result obtained can be checked by applying the relation (11.44).

To understand better the significance of the Coriolis acceleration, let us consider the absolute velocity of P at time t and at time $t + \Delta t$ (Fig. 15.30*b*). The velocity at time t can be resolved into its components $\mathbf{u}$ and $\mathbf{v}_A$; the velocity at time $t + \Delta t$ can be resolved into its components $\mathbf{u}'$ and $\mathbf{v}_{A'}$. Drawing these components from the same origin (Fig. 15.30*c*), we note that the change in velocity during the time Δt can be represented by the sum of three vectors, $\overrightarrow{RR'}$, $\overrightarrow{TT''}$, and $\overrightarrow{T''T'}$. The vector $\overrightarrow{TT''}$ measures the change in direction of the velocity $\mathbf{v}_A$, and the quotient $\overrightarrow{TT''}/\Delta t$ represents the acceleration $\mathbf{a}_A$ when Δt approaches zero. We check that the direction of $\overrightarrow{TT''}$ is that of $\mathbf{a}_A$ when Δt approaches zero and that

$$\lim_{\Delta t \to 0} \frac{TT''}{\Delta t} = \lim_{\Delta t \to 0} v_A \frac{\Delta \theta}{\Delta t} = r\omega\omega = r\omega^2 = a_A$$

The vector $\overrightarrow{RR'}$ measures the change in direction of $\mathbf{u}$ due to the rotation of the rod; the vector $\overrightarrow{T''T'}$ measures the change in magnitude of $\mathbf{v}_A$ due to the motion of P on the rod. The vectors $\overrightarrow{RR'}$ and $\overrightarrow{T''T'}$ result from the *combined effect* of the relative motion of P and of the rotation of the rod; they would vanish if *either* of these two motions stopped. It is easily verified that the sum of these two vectors defines the Coriolis acceleration. Their direction is that of $\mathbf{a}_c$ when Δt approaches zero, and since $RR' = u\,\Delta\theta$ and $T''T' = v_{A'} - v_A = (r + \Delta r)\omega - r\omega = \omega\,\Delta r$, we check that a_c is equal to

$$\lim_{\Delta t \to 0}\left(\frac{RR'}{\Delta t} + \frac{T''T'}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(u\frac{\Delta\theta}{\Delta t} + \omega\frac{\Delta r}{\Delta t}\right) = u\omega + \omega u = 2\omega u$$

Formulas (15.33) and (15.36) can be used to analyze the motion of mechanisms which contain parts sliding on each other. They make it possible, for example, to relate the absolute and relative motions of sliding pins and collars (see Sample Probs. 15.9 and 15.10). The concept of Coriolis acceleration is also very useful in the study of long-range projectiles and of other bodies whose motions are appreciably affected by the rotation of the earth. As was pointed out in Sec. 12.2, a system of axes attached to the earth does not truly constitute a newtonian frame of reference; such a system of axes should actually be considered as rotating. The formulas derived in this section will therefore facilitate the study of the motion of bodies with respect to axes attached to the earth.

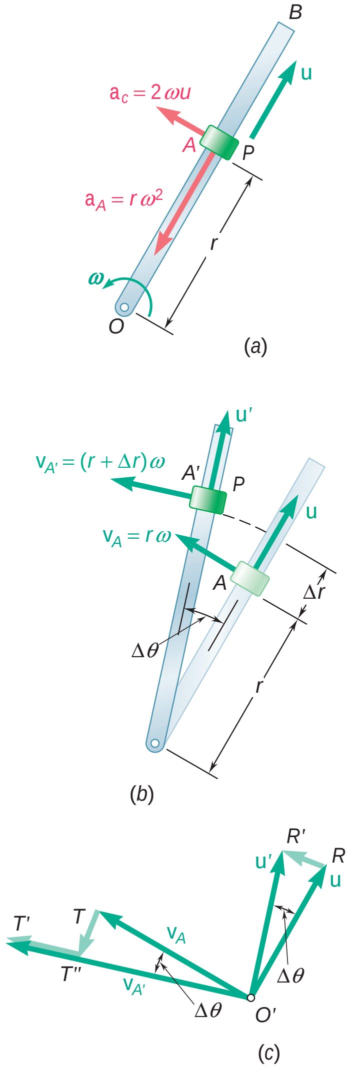
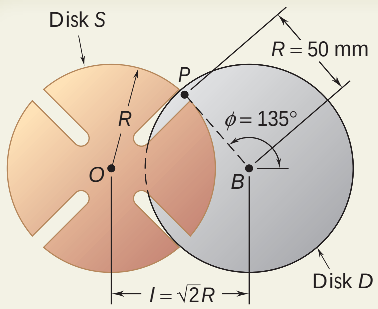


Fig. 15.30

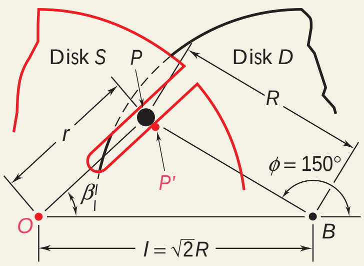


SAMPLE PROBLEM 15.9

The Geneva mechanism shown is used in many counting instruments and in other applications where an intermittent rotary motion is required. Disk D rotates with a constant counterclockwise angular velocity ω_D of 10 rad/s. A pin P is attached to disk D and slides along one of several slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each slot; in the case of four slots, this will occur if the distance between the centers of the disks is $l = \sqrt{2}R$.

At the instant when $\phi = 150°$, determine (*a*) the angular velocity of disk S, (*b*) the velocity of pin P relative to disk S.

SOLUTION



We solve triangle OPB, which corresponds to the position $\phi = 150°$. Using the law of cosines, we write

$$r^2 = R^2 + l^2 - 2Rl \cos 30° = 0.551R^2 \qquad r = 0.742R = 37.1 \text{ mm}$$

From the law of sines,

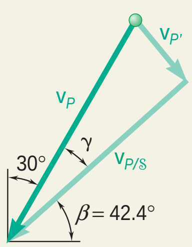
$$\frac{\sin \beta}{R} = \frac{\sin 30°}{r} \qquad \sin \beta = \frac{\sin 30°}{0.742} \qquad \beta = 42.4°$$

Since pin P is attached to disk D, and since disk D rotates about point B, the magnitude of the absolute velocity of P is

$$v_P = R\omega_D = (50 \text{ mm})(10 \text{ rad/s}) = 500 \text{ mm/s}$$
$$\mathbf{v}_P = 500 \text{ mm/s} \measuredangle 60°$$

We consider now the motion of pin P along the slot in disk S. Denoting by P' the point of disk S which coincides with P at the instant considered and selecting a rotating frame $\mathcal{S}$ attached to disk S, we write

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{S}}$$



Noting that $\mathbf{v}_{P'}$ is perpendicular to the radius OP and that $\mathbf{v}_{P/\mathcal{S}}$ is directed along the slot, we draw the velocity triangle corresponding to the equation above. From the triangle, we compute

$$\gamma = 90° - 42.4° - 30° = 17.6°$$
$$v_{P'} = v_P \sin \gamma = (500 \text{ mm/s}) \sin 17.6°$$
$$\mathbf{v}_{P'} = 151.2 \text{ mm/s} \measuredangle 42.4°$$
$$v_{P/\mathcal{S}} = v_P \cos \gamma = (500 \text{ mm/s}) \cos 17.6°$$
$$\mathbf{v}_{P/S} = \mathbf{v}_{P/\mathcal{S}} = 477 \text{ mm/s} \measuredangle 42.4° \quad ◀$$

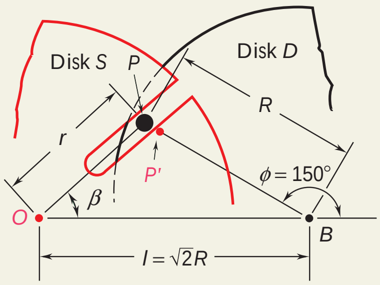
Since $\mathbf{v}_{P'}$ is perpendicular to the radius OP, we write

$$v_{P'} = r\omega_{\mathcal{S}} \qquad 151.2 \text{ mm/s} = (37.1 \text{ mm})\omega_{\mathcal{S}}$$
$$\boldsymbol{\omega}_S = \boldsymbol{\omega}_{\mathcal{S}} = 4.08 \text{ rad/s} \circlearrowright \quad ◀$$

SAMPLE PROBLEM 15.10

In the Geneva mechanism of Sample Prob. 15.9, disk D rotates with a constant counterclockwise angular velocity V_D of 10 rad/s. At the instant when $\mathrm{f} = 150°$, determine the angular acceleration of disk S.

SOLUTION



Referring to Sample Prob. 15.9, we obtain the angular velocity of the frame $\mathcal{S}$ attached to disk S and the velocity of the pin relative to $\mathcal{S}$:

$$\mathrm{V}_{\mathcal{S}} = 4.08 \text{ rad/s i}$$
$$\mathrm{b} = 42.4° \qquad \mathbf{v}_{P/\mathcal{S}} = 477 \text{ mm/s} \;\; \mathrm{d} \;\; 42.4°$$

Since pin P moves with respect to the rotating frame $\mathcal{S}$, we write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{S}} + \mathbf{a}_c \tag{1}$$

Each term of this vector equation is investigated separately.

Absolute Acceleration $\mathbf{a}_P$. Since disk D rotates with a constant angular velocity, the absolute acceleration $\mathbf{a}_P$ is directed toward B. We have

$$a_P = R\mathrm{V}_D^2 = (500 \text{ mm})(10 \text{ rad/s})^2 = 5000 \text{ mm/s}^2$$
$$\mathbf{a}_P = 5000 \text{ mm/s}^2 \;\; \mathrm{c} \;\; 30°$$

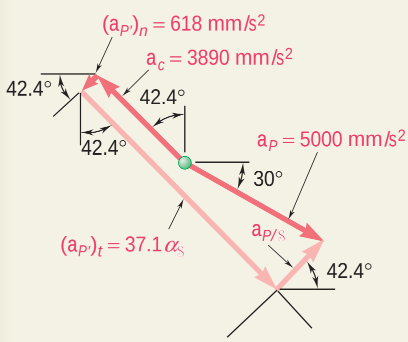
Acceleration $\mathbf{a}_{P'}$ of the Coinciding Point P'. The acceleration $\mathbf{a}_{P'}$ of the point P' of the frame $\mathcal{S}$ which coincides with P at the instant considered is resolved into normal and tangential components. (We recall from Sample Prob. 15.9 that $r = 37.1$ mm.)

$$(a_{P'})_n = r\mathrm{V}_{\mathcal{S}}^2 = (37.1 \text{ mm})(4.08 \text{ rad/s})^2 = 618 \text{ mm/s}^2$$
$$(\mathbf{a}_{P'})_n = 618 \text{ mm/s}^2 \;\; \mathrm{d} \;\; 42.4°$$
$$(a_{P'})_t = r\mathrm{a}_{\mathcal{S}} = 37.1\mathrm{a}_{\mathcal{S}} \qquad (\mathbf{a}_{P'})_t = 37.1\mathrm{a}_{\mathcal{S}} \;\; \mathrm{b} \;\; 42.4°$$

Relative Acceleration $\mathbf{a}_{P/\mathcal{S}}$. Since the pin P moves in a straight slot cut in disk S, the relative acceleration $\mathbf{a}_{P/\mathcal{S}}$ must be parallel to the slot; i.e., its direction must be a 42.4°.

Coriolis Acceleration $\mathbf{a}_c$. Rotating the relative velocity $\mathbf{v}_{P/\mathcal{S}}$ through 90° in the sense of $\mathrm{V}_{\mathcal{S}}$, we obtain the direction of the Coriolis component of the acceleration: h 42.4°. We write

$$a_c = 2\mathrm{V}_{\mathcal{S}} v_{P/\mathcal{S}} = 2(4.08 \text{ rad/s})(477 \text{ mm/s}) = 3890 \text{ mm/s}^2$$
$$\mathbf{a}_c = 3890 \text{ mm/s}^2 \;\; \mathrm{h} \;\; 42.4°$$



We rewrite Eq. (1) and substitute the accelerations found above:

$$\mathbf{a}_P = (\mathbf{a}_{P'})_n + (\mathbf{a}_{P'})_t + \mathbf{a}_{P/\mathcal{S}} + \mathbf{a}_c$$
$$[5000 \;\mathrm{c}\; 30°] = [618 \;\mathrm{d}\; 42.4°] + [37.1\mathrm{a}_{\mathcal{S}} \;\mathrm{b}\; 42.4°] + [a_{P/\mathcal{S}} \;\mathrm{a}\; 42.4°] + [3890 \;\mathrm{h}\; 42.4°]$$

Equating components in a direction perpendicular to the slot,

$$5000 \cos 17.6° = 37.1\mathrm{a}_{\mathcal{S}} - 3890$$
$$\mathrm{A}_S = \mathrm{A}_{\mathcal{S}} = 233 \text{ rad/s}^2 \mathrm{i} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you studied the rate of change of a vector with respect to a rotating frame and then applied your knowledge to the analysis of the plane motion of a particle relative to a rotating frame.

1. Rate of change of a vector with respect to a fixed frame and with respect to a rotating frame. Denoting by $(\dot{\mathbf{Q}})_{OXYZ}$ the rate of change of a vector $\mathbf{Q}$ with respect to a fixed frame $OXYZ$ and by $(\dot{\mathbf{Q}})_{Oxyz}$ its rate of change with respect to a rotating frame $Oxyz$, we obtained the fundamental relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \tag{15.31}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame.

This fundamental relation will now be applied to the solution of two-dimensional problems.

2. Plane motion of a particle relative to a rotating frame. Using the above fundamental relation and designating by $\mathcal{F}$ the rotating frame, we obtained the following expressions for the velocity and the acceleration of a particle P:

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.33}$$

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.36}$$

In these equations:

a. The subscript *P* refers to the absolute motion of the particle P, that is, to its motion with respect to a fixed frame of reference OXY.

b. The subscript *P′* refers to the motion of the point P' of the rotating frame $\mathcal{F}$ which coincides with P at the instant considered.

c. The subscript *P/ℱ* refers to the motion of the particle P relative to the rotating frame $\mathcal{F}$.

d. The term $\mathbf{a}_c$ represents the Coriolis acceleration of point *P*. Its magnitude is $2\Omega v_{P/\mathcal{F}}$, and its direction is found by rotating $\mathbf{v}_{P/\mathcal{F}}$ through 90° in the sense of rotation of the frame $\mathcal{F}$.

You should keep in mind that the Coriolis acceleration should be taken into account whenever a part of the mechanism you are analyzing is moving with respect to another part that is rotating. The problems you will encounter in this lesson involve collars that slide on rotating rods, booms that extend from cranes rotating in a vertical plane, etc.

When solving a problem involving a rotating frame, you will find it convenient to draw vector diagrams representing Eqs. (15.33) and (15.36), respectively, and use these diagrams to obtain either an analytical or a graphical solution.

PROBLEMS

CONCEPT QUESTION

15.CQ8 A person walks radially inward on a platform that is rotating counterclockwise about its center. Knowing that the platform has a constant angular velocity $\boldsymbol{\omega}$ and the person walks with a constant speed **u** relative to the platform, what is the direction of the acceleration of the person at the instant shown?

a. Negative x
b. Negative y
c. Negative x and positive y
d. Positive x and positive y
e. Negative x and negative y

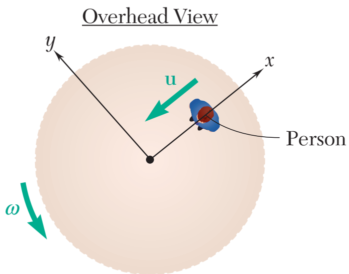


Fig. P15.CQ8

END-OF-SECTION PROBLEMS

15.150 and 15.151 Pin P is attached to the collar shown; the motion of the pin is guided by a slot cut in rod BD and by the collar that slides on rod AE. Knowing that at the instant considered the rods rotate clockwise with constant angular velocities, determine for the given data the velocity of pin P.

15.150 $\omega_{AE} = 8$ rad/s, $\omega_{BD} = 3$ rad/s
15.151 $\omega_{AE} = 7$ rad/s, $\omega_{BD} = 4.8$ rad/s

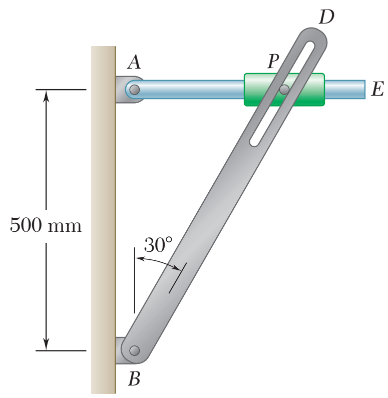


Fig. P15.150 and P15.151

15.152 and 15.153 Two rotating rods are connected by slider block P. The rod attached at A rotates with a constant clockwise angular velocity ω_A. For the given data, determine for the position shown (*a*) the angular velocity of the rod attached at B, (*b*) the relative velocity of slider block P with respect to the rod on which it slides.

15.152 $b = 8$ in., $\omega_A = 6$ rad/s
15.153 $b = 300$ mm, $\omega_A = 10$ rad/s

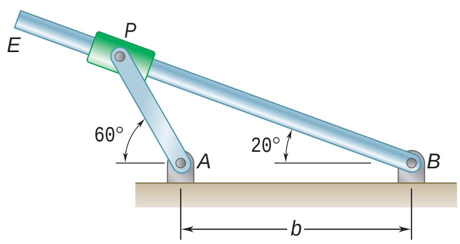


Fig. P15.152

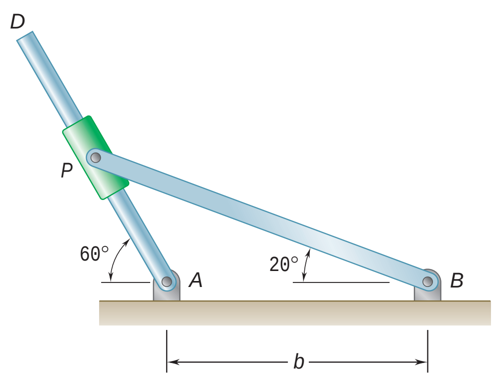


Fig. P15.153

15.154 Pin P is attached to the wheel shown and slides in a slot cut in bar BD. The wheel rolls to the right without slipping with a constant angular velocity of 20 rad/s. Knowing that $x = 480$ mm when u $= 0$, determine the angular velocity of the bar and the relative velocity of pin P with respect to the rod when *(a)* u $= 0$, *(b)* u $= 90°$.

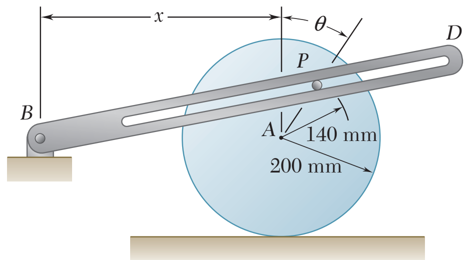


Fig. P15.154

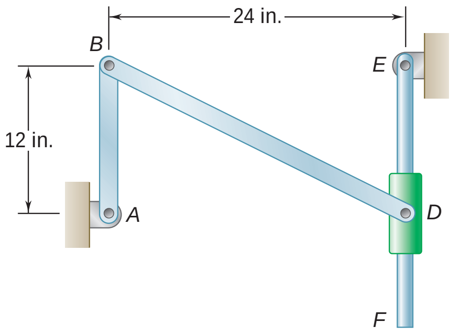


Fig. P15.155 and *P15.156*

15.155 Bar AB rotates clockwise with a constant angular velocity of 8 rad/s and rod EF rotates clockwise with a constant angular velocity of 6 rad/s. Determine at the instant shown (*a*) the angular velocity of bar BD, (*b*) the relative velocity of collar D with respect to rod EF.

15.156 Bar AB rotates clockwise with a constant angular velocity of 4 rad/s. Knowing that the magnitude of the velocity of collar D is 20 ft/s and that the angular velocity of bar BD is counterclockwise at the instant shown, determine (*a*) the angular velocity of bar EF, (*b*) the relative velocity of collar D with respect to rod EF.

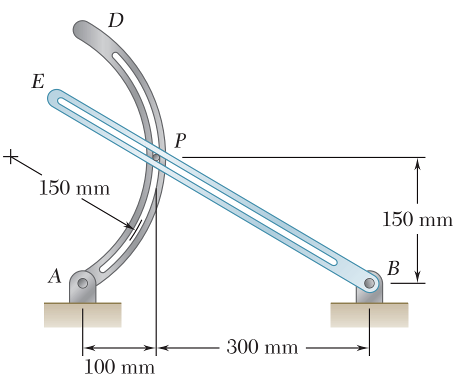


Fig. *P15.157*

15.157 The motion of pin P is guided by slots cut in rods AD and BE. Knowing that bar AD has a constant angular velocity of 4 rad/s clockwise and bar BE has an angular velocity of 5 rad/s counterclockwise and is slowing down at a rate of 2 rad/s^2, determine the velocity of P for the position shown.

15.158 Four pins slide in four separate slots cut in a circular plate as shown. When the plate is at rest, each pin has a velocity directed as shown and of the same constant magnitude u. If each pin maintains the same velocity relative to the plate when the plate rotates about O with a constant counterclockwise angular velocity V, determine the acceleration of each pin.

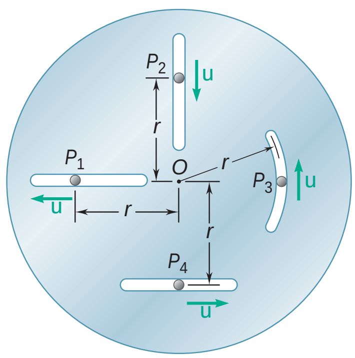


Fig. *P15.158*

15.159 Solve Prob. 15.158, assuming that the plate rotates about O with a constant clockwise angular velocity V.

15.160 Pin P slides in the circular slot cut in the plate shown at a constant relative speed $u = 500$ mm/s. Assuming that at the instant shown the angular velocity of the plate is 6 rad/s and is increasing at the rate of 20 rad/s^2, determine the acceleration of pin P when u = 90°.

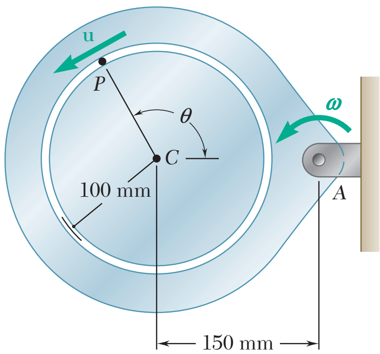


Fig. P15.160

15.161 The cage of a mine elevator moves downward at a constant speed of 40 ft/s. Determine the magnitude and direction of the Coriolis acceleration of the cage if the elevator is located (*a*) at the equator, (*b*) at latitude 40° north, (*c*) at latitude 40° south.

15.162 A rocket sled is tested on a straight track that is built along a meridian. Knowing that the track is located at latitude 40° north, determine the Coriolis acceleration of the sled when it is moving north at a speed of 900 km/h.

15.163 The motion of blade D is controlled by the robot arm ABC. At the instant shown the arm is rotating clockwise at the constant rate V = 1.8 rad/s and the length of portion BC of the arm is being decreased at the constant rate of 250 mm/s. Determine (*a*) the velocity of D, (*b*) the acceleration of D.

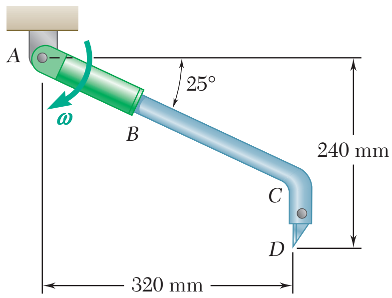


Fig. P15.163

15.164 At the instant shown the length of the boom AB is being *decreased* at the constant rate of 0.2 m/s and the boom is being lowered at the constant rate of 0.08 rad/s. Determine (*a*) the velocity of point B, (*b*) the acceleration of point B.

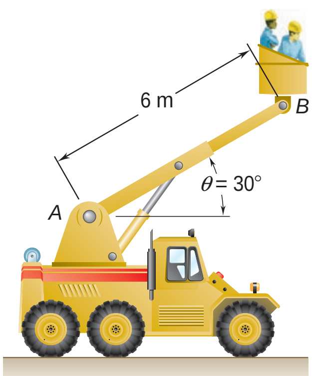


Fig. P15.164 and P15.165

15.165 At the instant shown the length of the boom AB is being *increased* at the constant rate of 0.2 m/s and the boom is being lowered at the constant rate of 0.08 rad/s. Determine (*a*) the velocity of point B, (*b*) the acceleration of point B.

15.166 and 15.167 The sleeve BC is welded to an arm that rotates about A with a constant angular velocity V. In the position shown rod DF is being moved to the left at a constant speed $u = 400$ mm/s relative to the sleeve. For the given angular velocity V, determine the acceleration (*a*) of point D, (*b*) of the point of rod DF that coincides with point E.

15.166 V = (3 rad/s) **i**
15.167 V = (3 rad/s) **j**

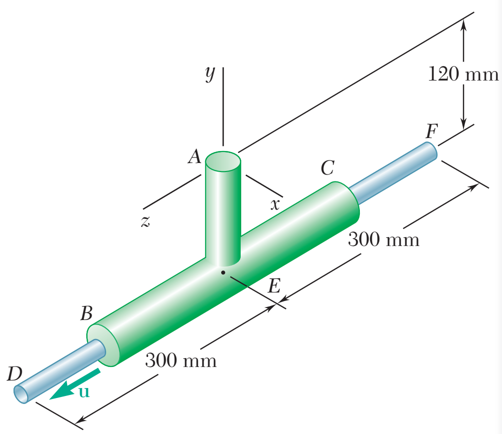


Fig. P15.166 and P15.167

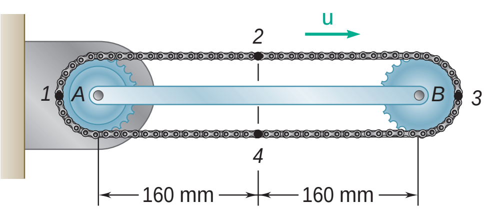


Fig. P15.168 and P15.169

15.168 and 15.169 A chain is looped around two gears of radius 40 mm that can rotate freely with respect to the 320-mm arm AB. The chain moves about arm AB in a clockwise direction at the constant rate of 80 mm/s relative to the arm. Knowing that in the position shown arm AB rotates clockwise about A at the constant rate v = 0.75 rad/s, determine the acceleration of each of the chain links indicated.

15.168 Links 1 and 2
15.169 Links 3 and 4

15.170 A basketball player shoots a free throw in such a way that his shoulder can be considered a pin joint at the moment of release as shown. Knowing that at the instant shown the upper arm SE has a constant angular velocity of 2 rad/s counterclockwise and the forearm EW has a constant clockwise angular velocity of 4 rad/s with respect to SE, determine the velocity and acceleration of the wrist W.

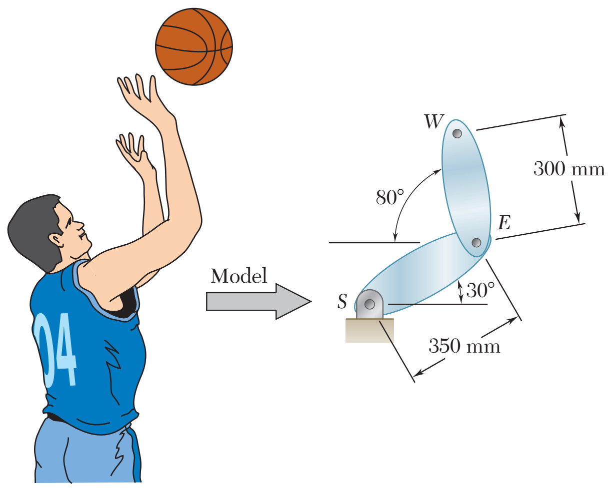


Fig. P15.170

15.171 The human leg can be crudely approximated as two rigid bars (the femur and the tibia) connected with a pin joint. At the instant shown, the velocity of the ankle A is zero, the tibia AK has an angular velocity of 1.5 rad/s counterclockwise and an angular acceleration of 1 rad/s^2 counterclockwise. Determine the relative angular velocity and relative angular acceleration of the femur KH with respect to AK so that the velocity and acceleration of H are both straight up at this instant.

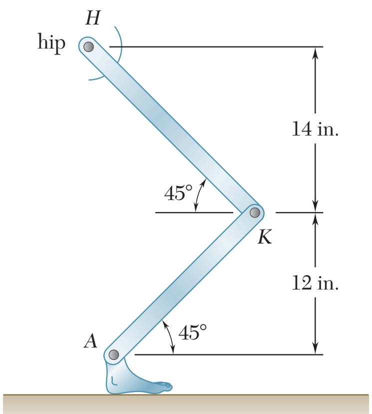


Fig. P15.171

15.172 The collar P slides outward at a constant relative speed u along rod AB, which rotates counterclockwise with a constant angular velocity of 20 rpm. Knowing that r = 250 mm when u = 0 and that the collar reaches B when u = 90°, determine the magnitude of the acceleration of the collar P just as it reaches B.

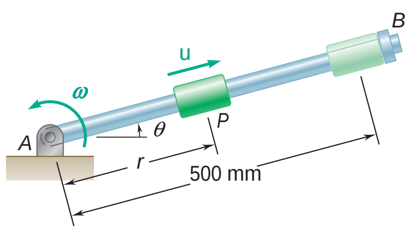


Fig. *P15.172*

15.173 Pin P slides in a circular slot cut in the plate shown at a constant relative speed $u = 90$ mm/s. Knowing that at the instant shown the plate rotates clockwise about A at the constant rate V = 3 rad/s, determine the acceleration of the pin if it is located at (*a*) point A, (*b*) point B, (*c*) point C.

15.174 Pin P slides in a circular slot cut in the plate shown at a constant relative speed $u = 90$ mm/s. Knowing that at the instant shown the angular velocity V of the plate is 3 rad/s clockwise and is decreasing at the rate of 5 rad/s^2, determine the acceleration of the pin if it is located at (*a*) point A, (*b*) point B, (*c*) point C.

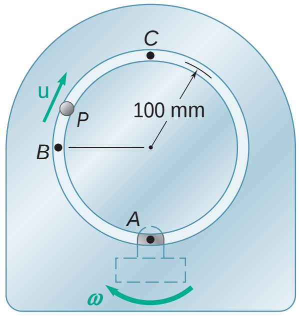


Fig. *P15.173* and P15.174

15.175 Pin P is attached to the wheel shown and slides in a slot cut in bar BD. The wheel rolls to the right without slipping with a constant angular velocity of 20 rad/s. Knowing that $x = 480$ mm when u = 0, determine (*a*) the angular acceleration of the bar, (*b*) the relative acceleration of pin P with respect to the bar when u = 0.

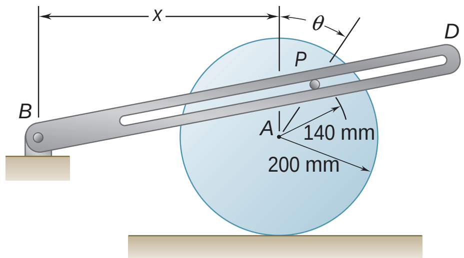


Fig. P15.175

15.176 Knowing that at the instant shown the rod attached at A has an angular velocity of 5 rad/s counterclockwise and an angular acceleration of 2 rad/s^2 clockwise, determine the angular velocity and the angular acceleration of the rod attached at B.

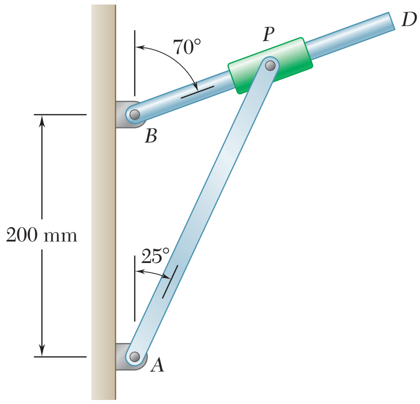


Fig. P15.176

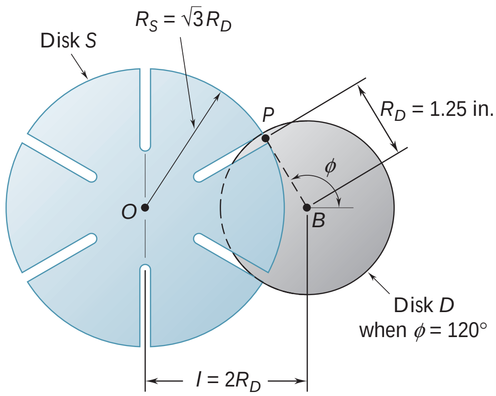


Fig. P15.177

15.177 The Geneva mechanism shown is used to provide an intermittent rotary motion of disk S. Disk D rotates with a constant counterclockwise angular velocity V_D of 8 rad/s. A pin P is attached to disk D and can slide in one of the six equally spaced slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each of the six slots; this will occur if the distance between the centers of the disks and the radii of the disks are related as shown. Determine the angular velocity and angular acceleration of disk S at the instant when f = 150°.

15.178 In Prob. 15.177, determine the angular velocity and angular acceleration of disk S at the instant when f = 135°.

15.179 At the instant shown bar BC has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s^2, both counterclockwise; determine the angular acceleration of the plate.

15.180 At the instant shown bar BC has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s^2, both clockwise; determine the angular acceleration of the plate.

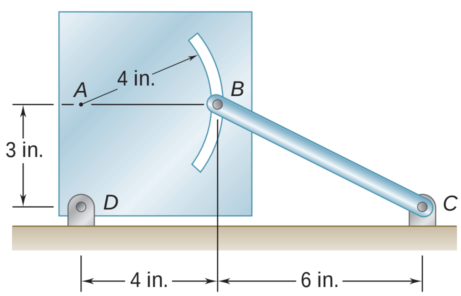


Fig. *P15.179* and *P15.180*

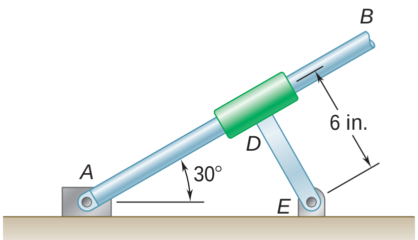


Fig. P15.181

***15.181** Rod AB passes through a collar which is welded to link DE. Knowing that at the instant shown block A moves to the right at a constant speed of 75 in./s, determine (*a*) the angular velocity of rod AB, (*b*) the velocity relative to the collar of the point of the rod in contact with the collar, (*c*) the acceleration of the point of the rod in contact with the collar. (*Hint:* Rod AB and link DE have the same $\boldsymbol{\omega}$ and the same $\boldsymbol{\alpha}$.)

***15.182** Solve Prob. 15.181 assuming block A moves to the left at a constant speed of 75 in./s.

***15.183** In Prob. 15.157, determine the acceleration of pin P.

*15.12 MOTION ABOUT A FIXED POINT

In Sec. 15.3 the motion of a rigid body constrained to rotate about a fixed axis was considered. The more general case of the motion of a rigid body which has a fixed point O will now be examined.

First, it will be proved that *the most general displacement of a rigid body with a fixed point O is equivalent to a rotation of the body about an axis through O.*† Instead of considering the rigid body itself, we can detach a sphere of center O from the body and analyze the motion of that sphere. Clearly, the motion of the sphere completely characterizes the motion of the given body. Since three points define the position of a solid in space, the center O and two points A and B on the surface of the sphere will define the position of the sphere and thus the position of the body. Let A_1 and B_1 characterize the position of the sphere at one instant, and let A_2 and B_2 characterize its position at a later instant (Fig. 15.31*a*). Since the sphere is rigid, the lengths of the arcs of great circle A_1B_1 and A_2B_2 must be equal, but except for this requirement, the positions of A_1, A_2, B_1, and B_2 are arbitrary. We propose to prove that the points A and B can be brought, respectively, from A_1 and B_1 into A_2 and B_2 by a single rotation of the sphere about an axis.

For convenience, and without loss of generality, we select point B so that its initial position coincides with the final position of A; thus, $B_1 = A_2$ (Fig. 15.31*b*). We draw the arcs of great circle A_1A_2, A_2B_2 and the arcs bisecting, respectively, A_1A_2 and A_2B_2. Let C be the point of intersection of these last two arcs; we complete the construction by drawing A_1C, A_2C, and B_2C. As pointed out above, because of the rigidity of the sphere, $A_1B_1 = A_2B_2$. Since C is by construction equidistant from A_1, A_2, and B_2, we also have $A_1C = A_2C = B_2C$. As a result, the spherical triangles A_1CA_2 and B_1CB_2 are congruent and the angles A_1CA_2 and B_1CB_2 are equal. Denoting by θ the common value of these angles, we conclude that the sphere can be brought from its initial position into its final position by a single rotation through θ about the axis OC.

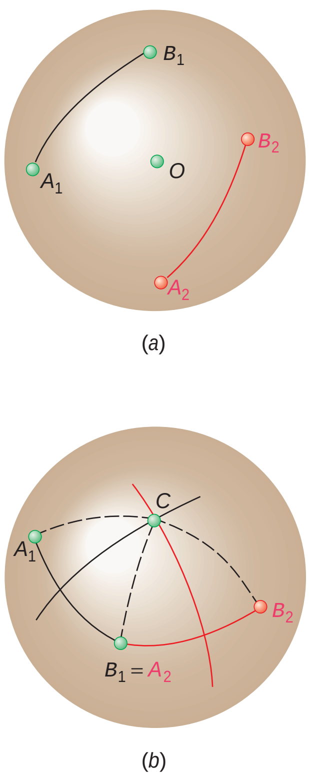


Fig. 15.31

†This is known as *Euler's theorem.*

It follows that the motion during a time interval Δt of a rigid body with a fixed point O can be considered as a rotation through Δu about a certain axis. Drawing along that axis a vector of magnitude Δu$/\Delta t$ and letting Δt approach zero, we obtain at the limit the *instantaneous axis of rotation* and the angular velocity V of the body at the instant considered (Fig. 15.32). The velocity of a particle P of the body can then be obtained, as in Sec. 15.3, by forming the vector product of V and of the position vector $\mathbf{r}$ of the particle:

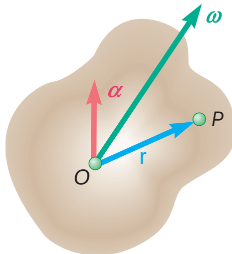


Fig. 15.32

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathrm{V} \times \mathbf{r} \tag{15.37}$$

The acceleration of the particle is obtained by differentiating (15.37) with respect to t. As in Sec. 15.3 we have

$$\mathbf{a} = \mathrm{A} \times \mathbf{r} + \mathrm{V} \times (\mathrm{V} \times \mathbf{r}) \tag{15.38}$$

where the angular acceleration A is defined as the derivative

$$\mathrm{A} = \frac{d\mathrm{V}}{dt} \tag{15.39}$$

of the angular velocity V.

In the case of the motion of a rigid body with a fixed point, the direction of V and of the instantaneous axis of rotation changes from one instant to the next. The angular acceleration A therefore reflects the change in direction of V as well as its change in magnitude and, in general, *is not directed along the instantaneous axis of rotation.* While the particles of the body located on the instantaneous axis of rotation have zero velocity at the instant considered, they do not have zero acceleration. Also, the accelerations of the various particles of the body *cannot* be determined as if the body were rotating permanently about the instantaneous axis.

Recalling the definition of the velocity of a particle with position vector $\mathbf{r}$, we note that the angular acceleration A, as expressed in (15.39), represents the velocity of the tip of the vector V. This property may be useful in the determination of the angular acceleration of a rigid body. For example, it follows that the vector A is tangent to the curve described in space by the tip of the vector V.

We should note that the vector V moves within the body, as well as in space. It thus generates two cones called, respectively, the *body cone* and the *space cone* (Fig. 15.33).† It can be shown that at any given instant, the two cones are tangent along the instantaneous axis of rotation and that as the body moves, the body cone appears to *roll* on the space cone.

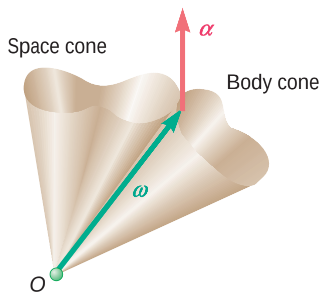


Fig. 15.33

†It is recalled that a *cone* is, by definition, a surface generated by a straight line passing through a fixed point. In general, the cones considered here *will not be circular cones.*

Photo 15.8 When the ladder rotates about its fixed base, its angular velocity can be obtained by adding the angular velocities which correspond to simultaneous rotations about two different axes.

Before concluding our analysis of the motion of a rigid body with a fixed point, we should prove that angular velocities are actually vectors. As indicated in Sec. 2.3, some quantities, such as the *finite rotations* of a rigid body, have magnitude and direction but do not obey the parallelogram law of addition; these quantities cannot be considered as vectors. In contrast, angular velocities (and also *infinitesimal rotations*), as will be demonstrated presently, *do obey* the parallelogram law and thus are truly vector quantities.

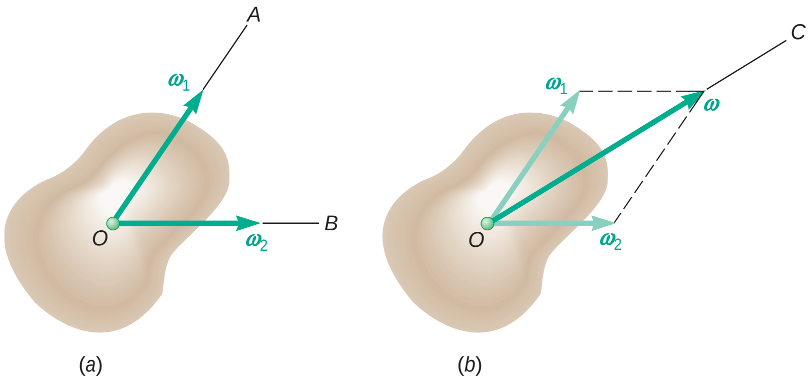


Fig. 15.34

Consider a rigid body with a fixed point O which at a given instant rotates simultaneously about the axes OA and OB with angular velocities V_1 and V_2 (Fig. 15.34*a*). We know that this motion must be equivalent at the instant considered to a single rotation of angular velocity V. We propose to show that

$$\mathsf{V} = \mathsf{V}_1 + \mathsf{V}_2 \tag{15.40}$$

i.e., that the resulting angular velocity can be obtained by adding V_1 and V_2 by the parallelogram law (Fig. 15.34*b*).

Consider a particle P of the body, defined by the position vector $\mathbf{r}$. Denoting, respectively, by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}$ the velocity of P when the body rotates about OA only, about OB only, and about both axes simultaneously, we write

$$\mathbf{v} = \mathsf{V} \times \mathbf{r} \qquad \mathbf{v}_1 = \mathsf{V}_1 \times \mathbf{r} \qquad \mathbf{v}_2 = \mathsf{V}_2 \times \mathbf{r} \tag{15.41}$$

But the vectorial character of *linear* velocities is well established (since they represent the derivatives of position vectors). We therefore have

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

where the plus sign indicates vector addition. Substituting from (15.41), we write

$$\mathsf{V} \times \mathbf{r} = \mathsf{V}_1 \times \mathbf{r} + \mathsf{V}_2 \times \mathbf{r}$$
$$\mathsf{V} \times \mathbf{r} = (\mathsf{V}_1 + \mathsf{V}_2) \times \mathbf{r}$$

where the plus sign still indicates vector addition. Since the relation obtained holds for an arbitrary $\mathbf{r}$, we conclude that (15.40) must be true.

*15.13 GENERAL MOTION

The most general motion of a rigid body in space will now be considered. Let A and B be two particles of the body. We recall from Sec. 11.12 that the velocity of B with respect to the fixed frame of reference $OXYZ$ can be expressed as

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.42}$$

where $\mathbf{v}_{B/A}$ is the velocity of B relative to a frame $AX'Y'Z'$ attached to A and of fixed orientation (Fig. 15.35). Since A is fixed in this frame, the motion of the body relative to $AX'Y'Z'$ is the motion of a body with a fixed point. The relative velocity $\mathbf{v}_{B/A}$ can therefore be obtained from (15.37) after $\mathbf{r}$ has been replaced by the position vector $\mathbf{r}_{B/A}$ of B relative to A. Substituting for $\mathbf{v}_{B/A}$ into (15.42), we write

$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \mathsf{V} \times \mathbf{r}_{B/A}} \tag{15.43}$$

where V is the angular velocity of the body at the instant considered.

The acceleration of B is obtained by a similar reasoning. We first write

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

and, recalling Eq. (15.38),

$$\boxed{\mathbf{a}_B = \mathbf{a}_A + \mathsf{A} \times \mathbf{r}_{B/A} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r}_{B/A})} \tag{15.44}$$

where A is the angular acceleration of the body at the instant considered.

Equations (15.43) and (15.44) show that *the most general motion of a rigid body is equivalent, at any given instant, to the sum of a translation,* in which all the particles of the body have the same velocity and acceleration as a reference particle A, *and of a motion in which particle A is assumed to be fixed.*†

It is easily shown, by solving (15.43) and (15.44) for $\mathbf{v}_A$ and $\mathbf{a}_A$, that the motion of the body with respect to a frame attached to B would be characterized by the same vectors V and A as its motion relative to $AX'Y'Z'$. The angular velocity and angular acceleration of a rigid body at a given instant are thus independent of the choice of reference point. On the other hand, one should keep in mind that whether the moving frame is attached to A or to B, it should maintain a fixed orientation; that is, it should remain parallel to the fixed reference frame $OXYZ$ throughout the motion of the rigid body. In many problems it will be more convenient to use a moving frame which is allowed to rotate as well as to translate. The use of such moving frames will be discussed in Secs. 15.14 and 15.15.

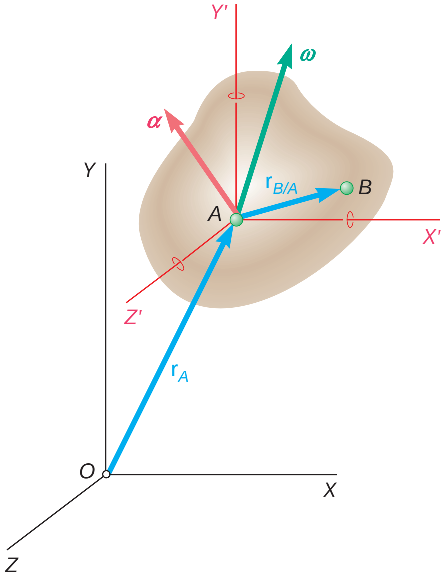
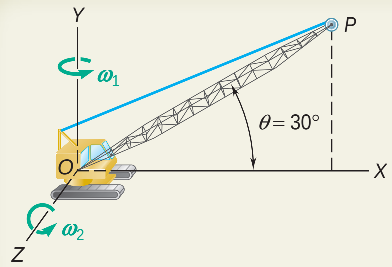


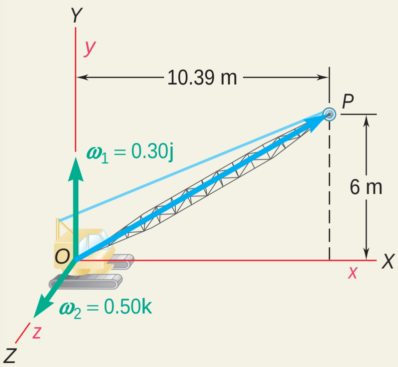
Fig. 15.35

†It is recalled from Sec. 15.12 that, in general, the vectors V and A are not collinear, and that the accelerations of the particles of the body in their motion relative to the frame $AX'Y'Z'$ cannot be determined as if the body were rotating permanently about the instantaneous axis through A.



SAMPLE PROBLEM 15.11

The crane shown rotates with a constant angular velocity V_1 of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity V_2 of 0.50 rad/s relative to the cab. Knowing that the length of the boom OP is $l = 12$ m, determine (*a*) the angular velocity V of the boom, (*b*) the angular acceleration A of the boom, (*c*) the velocity $\mathbf{v}$ of the tip of the boom, (*d*) the acceleration $\mathbf{a}$ of the tip of the boom.



SOLUTION

a. Angular Velocity of Boom. Adding the angular velocity V_1 of the cab and the angular velocity V_2 of the boom relative to the cab, we obtain the angular velocity V of the boom at the instant considered:

$$V = V_1 + V_2 \qquad V = (0.30 \text{ rad/s})\mathbf{j} + (0.50 \text{ rad/s})\mathbf{k} \quad \blacktriangleleft$$

b. Angular Acceleration of Boom. The angular acceleration A of the boom is obtained by differentiating V. Since the vector V_1 is constant in magnitude and direction, we have

$$A = \dot{V} = \dot{V}_1 + \dot{V}_2 = 0 + \dot{V}_2$$

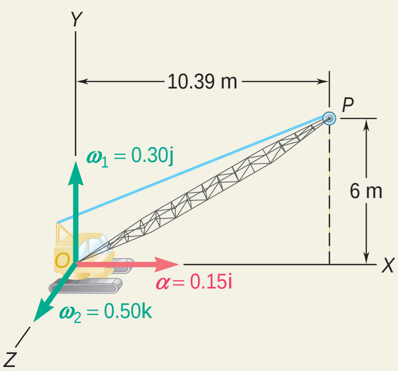
where the rate of change $\dot{V}_2$ is to be computed with respect to the fixed frame $OXYZ$. However, it is more convenient to use a frame $Oxyz$ attached to the cab and rotating with it, since the vector V_2 also rotates with the cab and therefore has zero rate of change with respect to that frame. Using Eq. (15.31) with $\mathbf{Q} = V_2$ and $\boldsymbol{\Omega} = V_1$, we write

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q}$$

$$(\dot{V}_2)_{OXYZ} = (\dot{V}_2)_{Oxyz} + V_1 \times V_2$$

$$A = (\dot{V}_2)_{OXYZ} = 0 + (0.30 \text{ rad/s})\mathbf{j} \times (0.50 \text{ rad/s})\mathbf{k}$$

$$A = (0.15 \text{ rad/s}^2)\mathbf{i} \quad \blacktriangleleft$$



c. Velocity of Tip of Boom. Noting that the position vector of point P is $\mathbf{r} = (10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}$ and using the expression found for V in part *a*, we write

$$\mathbf{v} = V \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.30 \text{ rad/s} & 0.50 \text{ rad/s} \\ 10.39 \text{ m} & 6 \text{ m} & 0 \end{vmatrix}$$

$$\mathbf{v} = -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j} - (3.12 \text{ m/s})\mathbf{k} \quad \blacktriangleleft$$

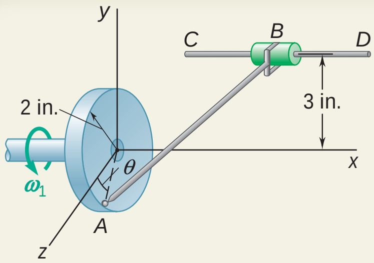
d. Acceleration of Tip of Boom. Recalling that $\mathbf{v} = V \times \mathbf{r}$, we write

$$\mathbf{a} = A \times \mathbf{r} + V \times (V \times \mathbf{r}) = A \times \mathbf{r} + V \times \mathbf{v}$$

$$\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.15 & 0 & 0 \\ 10.39 & 6 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.30 & 0.50 \\ -3 & 5.20 & -3.12 \end{vmatrix}$$

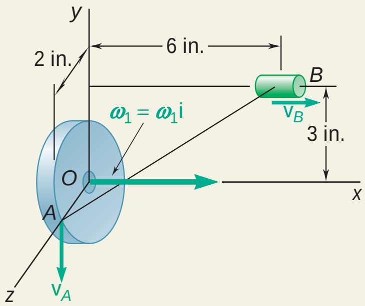
$$= 0.90\mathbf{k} - 0.94\mathbf{i} - 2.60\mathbf{i} - 1.50\mathbf{j} + 0.90\mathbf{k}$$

$$\mathbf{a} = -(3.54 \text{ m/s}^2)\mathbf{i} - (1.50 \text{ m/s}^2)\mathbf{j} + (1.80 \text{ m/s}^2)\mathbf{k} \quad \blacktriangleleft$$

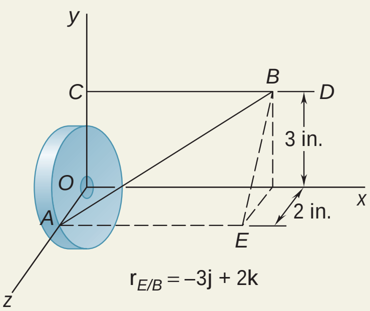


SAMPLE PROBLEM 15.12

The rod AB, of length 7 in., is attached to the disk by a ball-and-socket connection and to the collar B by a clevis. The disk rotates in the yz plane at a constant rate $V_1 = 12$ rad/s, while the collar is free to slide along the horizontal rod CD. For the position $u = 0$, determine (*a*) the velocity of the collar, (*b*) the angular velocity of the rod.



$\omega_1 = 12\,\mathbf{i}$
$\mathbf{r}_A = 2\mathbf{k}$
$\mathbf{r}_B = 6\mathbf{i} + 3\mathbf{j}$
$\mathbf{r}_{B/A} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$



SOLUTION

a. Velocity of Collar. Since point A is attached to the disk and since collar B moves in a direction parallel to the x axis, we have

$$\mathbf{v}_A = V_1 \times \mathbf{r}_A = 12\mathbf{i} \times 2\mathbf{k} = -24\mathbf{j} \qquad \mathbf{v}_B = v_B\mathbf{i}$$

Denoting by V the angular velocity of the rod, we write

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + V \times \mathbf{r}_{B/A}$$

$$v_B\mathbf{i} = -24\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ V_x & V_y & V_z \\ 6 & 3 & -2 \end{vmatrix}$$

$$v_B\mathbf{i} = -24\mathbf{j} + (-2V_y - 3V_z)\mathbf{i} + (6V_z + 2V_x)\mathbf{j} + (3V_x - 6V_y)\mathbf{k}$$

Equating the coefficients of the unit vectors, we obtain

$$v_B = \quad -2V_y \; -3V_z \qquad (1)$$
$$24 = 2V_x \quad +6V_z \qquad (2)$$
$$0 = 3V_x \; -6V_y \qquad (3)$$

Multiplying Eqs. (1), (2), (3), respectively, by 6, 3, −2 and adding, we write

$$6v_B + 72 = 0 \qquad v_B = -12 \qquad \mathbf{v}_B = -(12 \text{ in./s})\mathbf{i} \blacktriangleleft$$

b. Angular Velocity of Rod *AB*. We note that the angular velocity cannot be determined from Eqs. (1), (2), and (3), since the determinant formed by the coefficients of V_x, V_y, and V_z is zero. We must therefore obtain an additional equation by considering the constraint imposed by the clevis at B.

The collar-clevis connection at B permits rotation of AB about the rod CD and also about an axis perpendicular to the plane containing AB and CD. It prevents rotation of AB about the axis EB, which is perpendicular to CD and lies in the plane containing AB and CD. Thus the projection of V on $\mathbf{r}_{E/B}$ must be zero and we write†

$$V \cdot \mathbf{r}_{E/B} = 0 \qquad (V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}) \cdot (-3\mathbf{j} + 2\mathbf{k}) = 0$$
$$-3V_y + 2V_z = 0 \qquad (4)$$

Solving Eqs. (1) through (4) simultaneously, we obtain

$$v_B = -12 \qquad V_x = 3.69 \qquad V_y = 1.846 \qquad V_z = 2.77$$
$$V = (3.69 \text{ rad/s})\mathbf{i} + (1.846 \text{ rad/s})\mathbf{j} + (2.77 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

†We could also note that the direction of EB is that of the vector triple product $\mathbf{r}_{B/C} \times (\mathbf{r}_{B/C} \times \mathbf{r}_{B/A})$ and write $V \cdot [\mathbf{r}_{B/C} \times (\mathbf{r}_{B/C} \times \mathbf{r}_{B/A})] = 0$. This formulation would be particularly useful if the rod CD were skew.

SOLVING PROBLEMS ON YOUR OWN

In this lesson you started the study of the *kinematics of rigid bodies in three dimensions.* You first studied the *motion of a rigid body about a fixed point* and then the *general motion of a rigid body.*

A. Motion of a rigid body about a fixed point. To analyze the motion of a point B of a body rotating about a fixed point O you may have to take some or all of the following steps.

1. Determine the position vector r connecting the fixed point O to point B.

2. Determine the angular velocity V of the body with respect to a fixed frame of reference. The angular velocity V will often be obtained by adding two component angular velocities V_1 and V_2 [Sample Prob. 15.11].

3. Compute the velocity of *B* by using the equation

$$\mathbf{v} = V \times \mathbf{r} \tag{15.37}$$

Your computation will usually be facilitated if you express the vector product as a determinant.

4. Determine the angular acceleration A of the body. The angular acceleration A represents the rate of change $(\dot{V})_{OXYZ}$ of the vector V *with respect to a fixed frame of reference OXYZ* and reflects both *a change in magnitude and a change in direction* of the angular velocity. However, when computing A you may find it convenient to first compute the rate of change $(\dot{V})_{Oxyz}$ of V with respect to a rotating frame of reference *Oxyz* of your choice and use Eq. (15.31) of the preceding lesson to obtain A. You will write

$$A = (\dot{V})_{OXYZ} = (\dot{V})_{Oxyz} + \boldsymbol{\Omega} \times V$$

where $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame *Oxyz* [Sample Prob. 15.11].

5. Compute the acceleration of *B* by using the equation

$$\mathbf{a} = A \times \mathbf{r} + V \times (V \times \mathbf{r}) \tag{15.38}$$

Note that the vector product $(V \times \mathbf{r})$ represents the velocity of point B and was computed in step 3. Also, the computation of the first vector product in (15.38) will be facilitated if you express this product in determinant form. Remember that, as was the case with the plane motion of a rigid body, the instantaneous axis of rotation *cannot* be used to determine accelerations.

B. General motion of a rigid body. The general motion of a rigid body may be considered as *the sum of a translation and a rotation.* Keep the following in mind:

a. In the translation part of the motion, all the points of the body have the *same velocity* $\mathbf{v}_A$ *and the same acceleration* $\mathbf{a}_A$ as the point A of the body that has been selected as the reference point.

b. In the rotation part of the motion, the same reference point A is assumed to be a *fixed point.*

1. To determine the velocity of a point *B* of the rigid body when you know the velocity $\mathbf{v}_A$ of the reference point A and the angular velocity V of the body, you simply add $\mathbf{v}_A$ to the velocity $\mathbf{v}_{B/A} = \mathsf{V} \times \mathbf{r}_{B/A}$ of B in its rotation about A:

$$\mathbf{v}_B = \mathbf{v}_A + \mathsf{V} \times \mathbf{r}_{B/A} \tag{15.43}$$

As indicated earlier, the computation of the vector product will usually be facilitated if you express this product in determinant form.

Equation (15.43) can also be used to determine the magnitude of $\mathbf{v}_B$ when its direction is known, even if V is not known. While the corresponding three scalar equations are linearly dependent and the components of V are indeterminate, these components can be eliminated and $\mathbf{v}_A$ can be found by using an appropriate linear combination of the three equations [Sample Prob. 15.12, part *a*]. Alternatively, you can assign an arbitrary value to one of the components of V and solve the equations for $\mathbf{v}_A$. However, an additional equation must be sought in order to determine the true values of the components of V [Sample Prob. 15.12, part *b*].

2. To determine the acceleration of a point *B* of the rigid body when you know the acceleration $\mathbf{a}_A$ of the reference point A and the angular acceleration A of the body, you simply add $\mathbf{a}_A$ to the acceleration of B in its rotation about A, as expressed by Eq. (15.38):

$$\mathbf{a}_B = \mathbf{a}_A + \mathsf{A} \times \mathbf{r}_{B/A} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r}_{B/A}) \tag{15.44}$$

Note that the vector product $(\mathsf{V} \times \mathbf{r}_{B/A})$ represents the velocity $\mathbf{v}_{B/A}$ of B relative to A and may already have been computed as part of your calculation of $\mathbf{v}_B$. We also remind you that the computation of the other two vector products will be facilitated if you express these products in determinant form.

The three scalar equations associated with Eq. (15.44) can also be used to determine the magnitude of $\mathbf{a}_B$ when its direction is known, even if V and A are not known. While the components of V and A are indeterminate, you can assign arbitrary values to one of the components of V and to one of the components of A and solve the equations for $\mathbf{a}_B$.

PROBLEMS

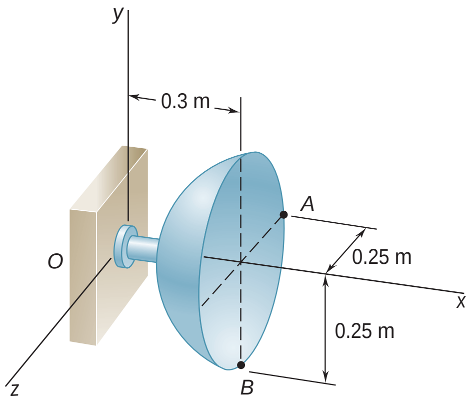


Fig. P15.184 and P15.185

END-OF-SECTION PROBLEMS

15.184 At the instant considered the radar antenna shown rotates about the origin of coordinates with an angular velocity $\mathrm{V} = \mathrm{v}_x\mathbf{i} + \mathrm{v}_y\mathbf{j} + \mathrm{v}_z\mathbf{k}$. Knowing that $(v_A)_y = 300$ mm/s, $(v_B)_y = 180$ mm/s, and $(v_B)_z = 360$ mm/s, determine (*a*) the angular velocity of the antenna, (*b*) the velocity of point *A*.

15.185 At the instant considered the radar antenna shown rotates about the origin of coordinates with an angular velocity $\mathrm{V} = \mathrm{v}_x\mathbf{i} + \mathrm{v}_y\mathbf{j} + \mathrm{v}_z\mathbf{k}$. Knowing that $(v_A)_x = 100$ mm/s, $(v_A)_y = -90$ mm/s, and $(v_B)_z = 120$ mm/s, determine (*a*) the angular velocity of the antenna, (*b*) the velocity of point *A*.

15.186 Plate *ABD* and rod *OB* are rigidly connected and rotate about the ball-and-socket joint *O* with an angular velocity $\mathrm{V} = \mathrm{v}_x\mathbf{i} + \mathrm{v}_y\mathbf{j} + \mathrm{v}_z\mathbf{k}$. Knowing that $\mathbf{v}_A = (80 \text{ mm/s})\mathbf{i} + (360 \text{ mm/s})\mathbf{j} + (v_A)_z\mathbf{k}$ and $\mathrm{v}_x = 1.5$ rad/s, determine (*a*) the angular velocity of the assembly, (*b*) the velocity of point *D*.

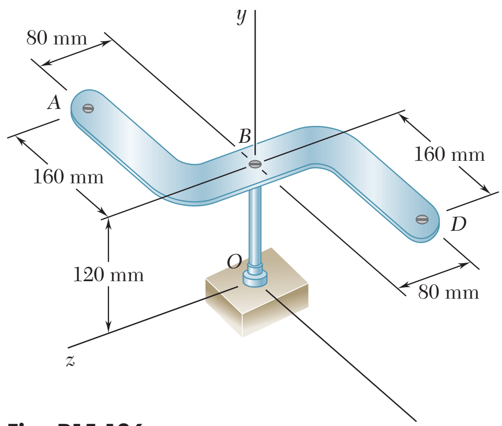


Fig. P15.186

15.187 The bowling ball shown rolls without slipping on the horizontal *xz* plane with an angular velocity $\boldsymbol{\omega} = \mathrm{v}_x\mathbf{i} + \mathrm{v}_y\mathbf{j} + \mathrm{v}_z\mathbf{k}$. Knowing that $\mathbf{v}_A = (14.4 \text{ ft/s})\mathbf{i} - (14.4 \text{ ft/s})\mathbf{j} + (10.8 \text{ ft/s})\mathbf{k}$ and $\mathbf{v}_D = (28.8 \text{ ft/s})\mathbf{i} + (21.6 \text{ ft/s})\mathbf{k}$, determine (*a*) the angular velocity of the bowling ball, (*b*) the velocity of its center *C*.

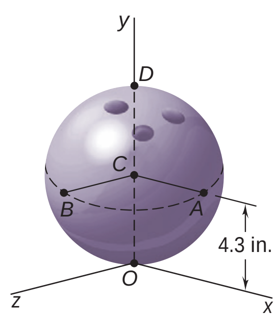


Fig. P15.187

15.188 The rotor of an electric motor rotates at the constant rate $\omega_1 =$ 1800 rpm. Determine the angular acceleration of the rotor as the motor is rotated about the y axis with a constant angular velocity $\boldsymbol{\omega}_2$ of 6 rpm counterclockwise when viewed from the positive y axis.

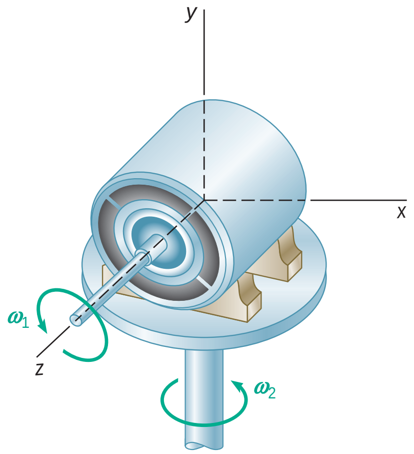


Fig. P15.188

15.189 The disk of a portable sander rotates at the constant rate $\omega_1 =$ 4400 rpm as shown. Determine the angular acceleration of the disk as a worker rotates the sander about the z axis with an angular velocity of 0.5 rad/s and an angular acceleration of 2.5 rad/s^2, both clockwise when viewed from the positive z axis.

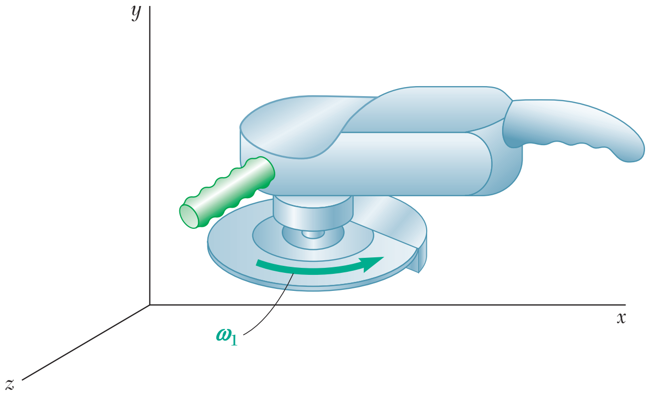


Fig. P15.189

15.190 Knowing that the turbine rotor shown rotates at a constant rate $\omega_1 =$ 9000 rpm, determine the angular acceleration of the rotor if the turbine housing has a constant angular velocity of 2.4 rad/s clockwise as viewed from (*a*) the positive y axis, (*b*) the positive z axis.

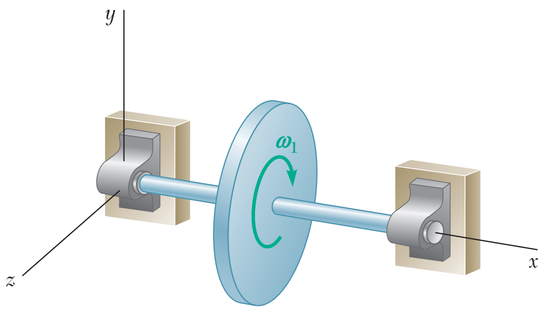


Fig. P15.190

15.191 In the system shown, disk A is free to rotate about the horizontal rod OA. Assuming that disk B is stationary ($\omega_2 = 0$), and that shaft OC rotates with a constant angular velocity $\boldsymbol{\omega}_1$, determine (*a*) the angular velocity of disk A, (*b*) the angular acceleration of disk A.

15.192 In the system shown, disk A is free to rotate about the horizontal rod OA. Assuming that shaft OC and disk B rotate with constant angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, respectively, both counterclockwise, determine (*a*) the angular velocity of disk A, (*b*) the angular acceleration of disk A.

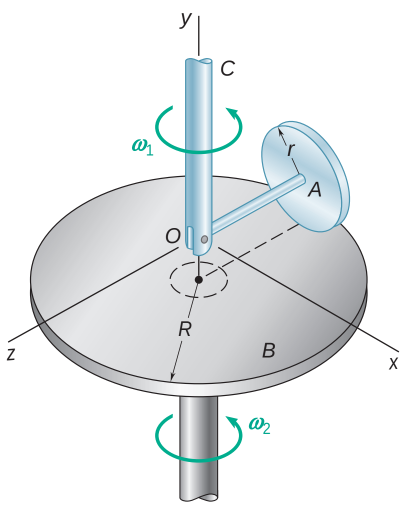


Fig. *P15.191* and *P15.192*

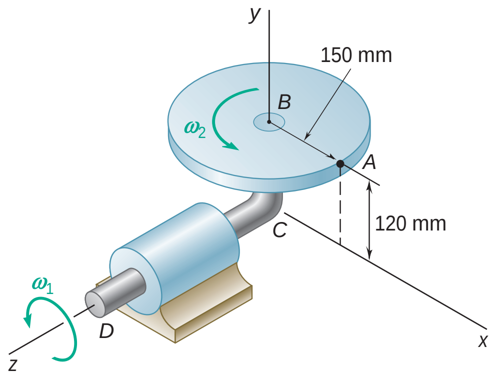


Fig. P15.193

15.193 The L-shaped arm *BCD* rotates about the *z* axis with a constant angular velocity $v_1 = 5$ rad/s. Knowing that the 150-mm-radius disk rotates about *BC* with a constant angular velocity $v_2 = 4$ rad/s, determine (*a*) the velocity of point *A*, (*b*) the acceleration of point *A*.

15.194 A gun barrel of length $OP = 4$ m is mounted on a turret as shown. To keep the gun aimed at a moving target the azimuth angle b is being increased at the rate $d\text{b}/dt = 30°/\text{s}$ and the elevation angle g is being increased at the rate $d\text{g}/dt = 10°/\text{s}$. For the position $\text{b} = 90°$ and $\text{g} = 30°$, determine (*a*) the angular velocity of the barrel, (*b*) the angular acceleration of the barrel, (*c*) the velocity and acceleration of point *P*.

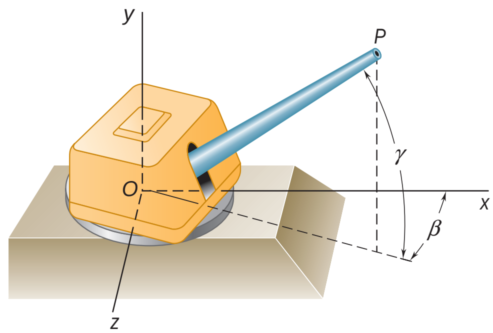


Fig. *P15.194*

15.195 A 3-in.-radius disk spins at the constant rate $v_2 = 4$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $v_1 = 5$ rad/s. For the position shown, determine (*a*) the angular acceleration of the disk, (*b*) the acceleration of point *P* on the rim of the disk if $\text{u} = 0$, (*c*) the acceleration of point *P* on the rim of the disk if $\text{u} = 90°$.

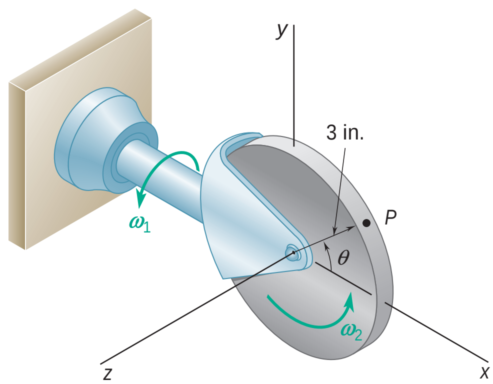


Fig. P15.195 and P15.196

15.196 A 3-in.-radius disk spins at the constant rate $v_2 = 4$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $v_1 = 5$ rad/s. Knowing that $\text{u} = 30°$, determine the acceleration of point *P* on the rim of the disk.

15.197 A 30-mm-radius wheel is mounted on an axle OB of length 100 mm. The wheel rolls without sliding on the horizontal floor, and the axle is perpendicular to the plane of the wheel. Knowing that the system rotates about the y axis at a constant rate ω_1 = 2.4 rad/s, determine (*a*) the angular velocity of the wheel, (*b*) the angular acceleration of the wheel, (*c*) the acceleration of point C located at the highest point on the rim of the wheel.

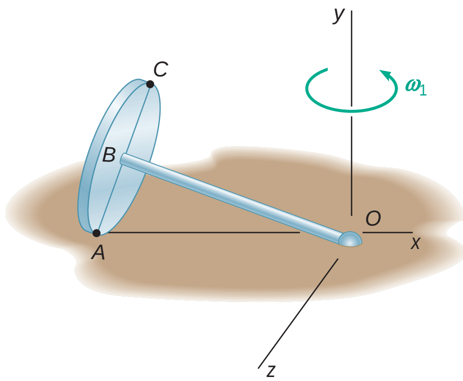


Fig. P15.197

15.198 At the instant shown, the robotic arm ABC is being rotated simultaneously at the constant rate ω_1 = 0.15 rad/s about the y axis, and at the constant rate ω_2 = 0.25 rad/s about the z axis. Knowing that the length of arm ABC is 1 m, determine (*a*) the angular acceleration of the arm, (*b*) the velocity of point C, (*c*) the acceleration of point C.

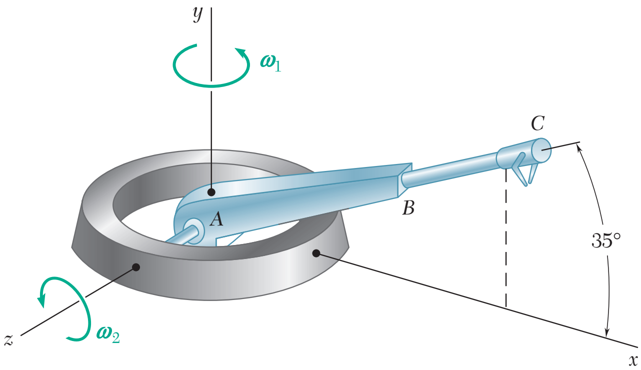


Fig. P15.198

15.199 In the planetary gear system shown, gears A and B are rigidly connected to each other and rotate as a unit about the inclined shaft. Gears C and D rotate with constant angular velocities of 30 rad/s and 20 rad/s, respectively (both counterclockwise when viewed from the right). Choosing the x axis to the right, the y axis upward, and the z axis pointing out of the plane of the figure, determine (*a*) the common angular velocity of gears A and B, (*b*) the angular velocity of shaft FH, which is rigidly attached to the inclined shaft.

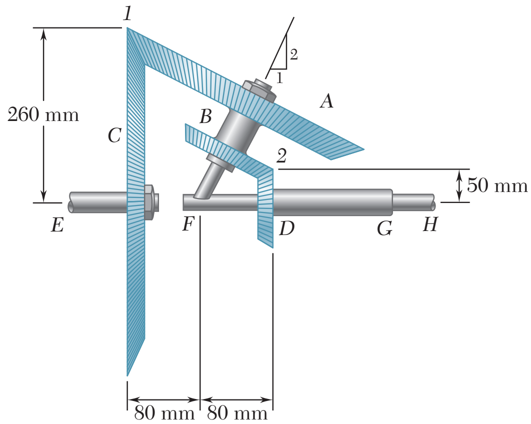


Fig. P15.199

15.200 In Prob. 15.199, determine (*a*) the common angular acceleration of gears A and B, (*b*) the acceleration of the tooth of gear A which is in contact with gear C at point l.

15.201 Several rods are brazed together to form the robotic guide arm shown which is attached to a ball-and-socket joint at O. Rod OA slides in a straight inclined slot while rod OB slides in a slot parallel to the z axis. Knowing that at the instant shown $\mathbf{v}_B = (9 \text{ in./s})\mathbf{k}$, determine (*a*) the angular velocity of the guide arm, (*b*) the velocity of point A, (*c*) the velocity of point C.

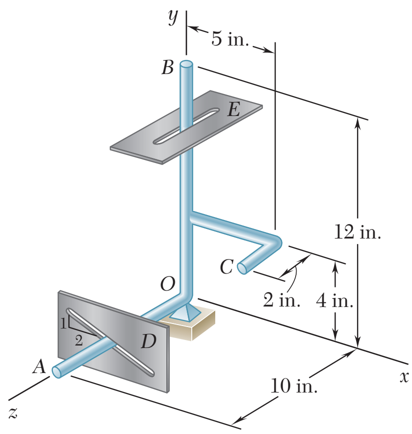


Fig. *P15.201*

15.202 In Prob. 15.201, the speed of point B is known to be constant. For the position shown, determine (*a*) the angular acceleration of the guide arm, (*b*) the acceleration of point C.

15.203 Rod AB of length 25 in. is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point E at a constant speed of 20 in./s, determine the velocity of collar A as collar B passes through point D.

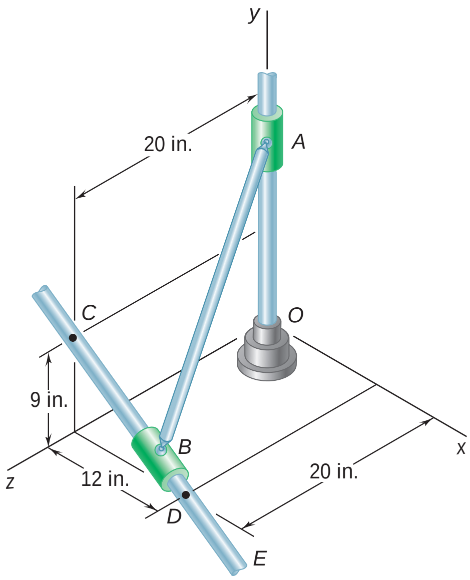


Fig. P15.203

15.204 Rod AB, of length 11 in., is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves downward at a constant speed of 54 in./s, determine the velocity of collar A when c = 2 in.

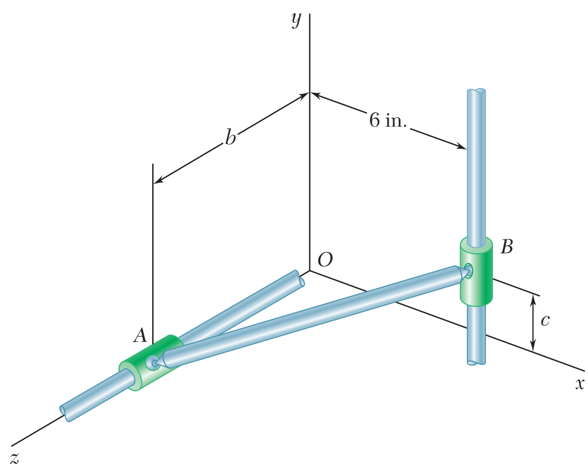


Fig. P15.204

15.205 Rod BC and BD are each 840 mm long and are connected by ball-and-socket joints to collars which may slide on the fixed rods shown. Knowing that collar B moves toward A at a constant speed of 390 mm/s, determine the velocity of collar C for the position shown.

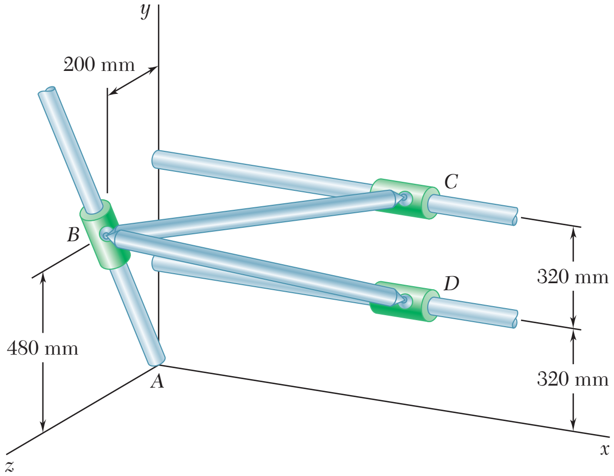


Fig. P15.205

15.206 Rod AB is connected by ball-and-socket joints to collar A and to the 16-in.-diameter disk C. Knowing that disk C rotates counterclockwise at the constant rate $\omega_0 = 3$ rad/s in the zx plane, determine the velocity of collar A for the position shown.

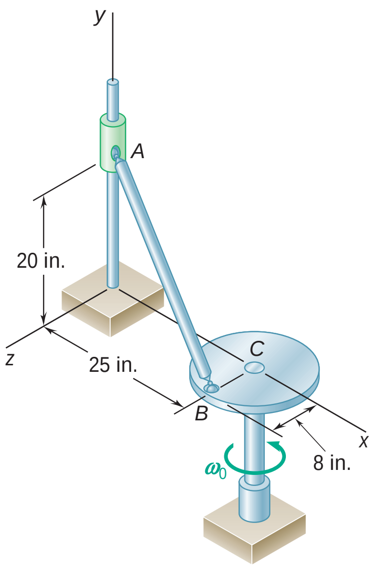


Fig. P15.206

15.207 Rod AB of length 29 in. is connected by ball-and-socket joints to the rotating crank BC and to the collar A. Crank BC is of length 8 in. and rotates in the horizontal xy plane at the constant rate $\omega_0 = 10$ rad/s. At the instant shown, when crank BC is parallel to the z axis, determine the velocity of collar A.

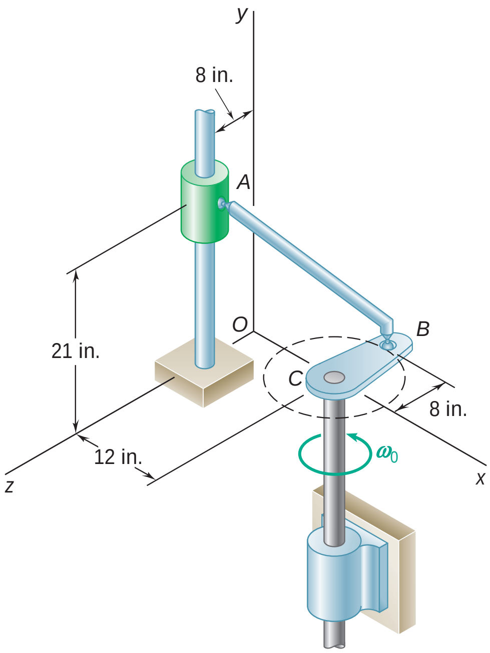


Fig. P15.207

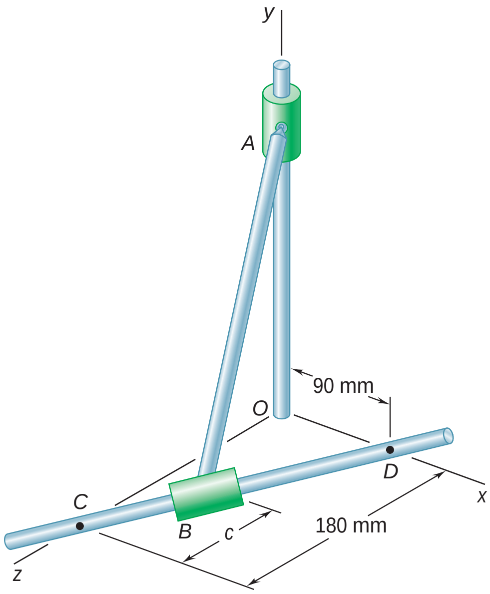


Fig. *P15.208* and *P15.209*

15.208 Rod AB of length 300 mm is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point D at a constant speed of 50 mm/s, determine the velocity of collar A when $c = 80$ mm.

15.209 Rod AB of length 300 mm is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point D at a constant speed of 50 mm/s, determine the velocity of collar A when $c = 120$ mm.

15.210 Two shafts AC and EG, which lie in the vertical yz plane, are connected by a universal joint at D. Shaft AC rotates with a constant angular velocity V_1 as shown. At a time when the arm of the crosspiece attached to shaft AC is vertical, determine the angular velocity of shaft EG.

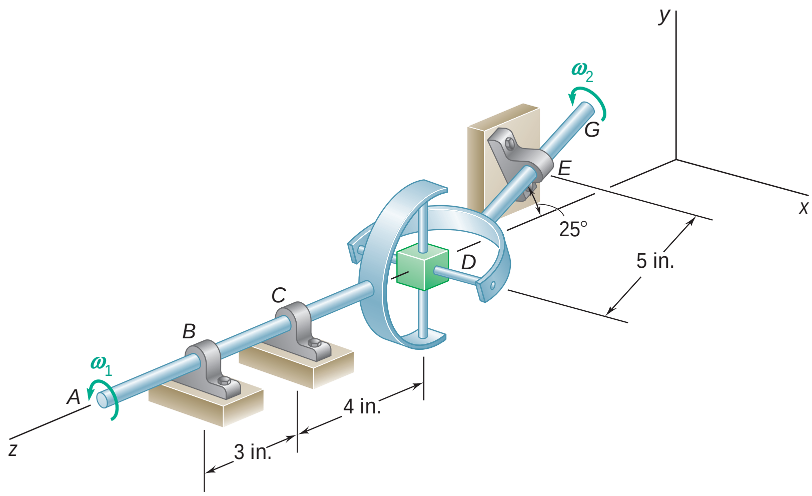


Fig. P15.210

15.211 Solve Prob. 15.210, assuming that the arm of the crosspiece attached to shaft AC is horizontal.

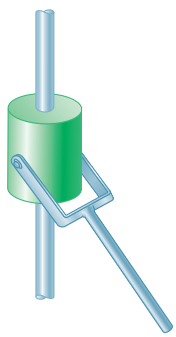


Fig. P15.212

15.212 In Prob. 15.206, the ball-and-socket joint between the rod and collar A is replaced by the clevis shown. Determine (a) the angular velocity of the rod, (b) the velocity of collar A.

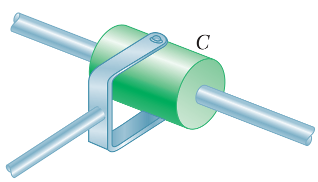


Fig. P15.213

15.213 In Prob. 15.205, the ball-and-socket joint between the rod and collar C is replaced by the clevis connection shown. Determine (a) the angular velocity of the rod, (b) the velocity of collar C.

15.214 In Prob. 15.204, determine the acceleration of collar A when $c = 2$ in.

****15.215*** In Prob. 15.205, determine the acceleration of collar C.

15.216 In Prob. 15.206, determine the acceleration of collar A.

15.217 In Prob. 15.207, determine the acceleration of collar A.

15.218 In Prob. 15.208, determine the acceleration of collar A.

15.219 In Prob. 15.209, determine the acceleration of collar A.

*15.14 THREE-DIMENSIONAL MOTION OF A PARTICLE RELATIVE TO A ROTATING FRAME. CORIOLIS ACCELERATION

We saw in Sec. 15.10 that given a vector function $\mathbf{Q}(t)$ and two frames of reference centered at O—a fixed frame $OXYZ$ and a rotating frame $Oxyz$—the rates of change of $\mathbf{Q}$ with respect to the two frames satisfy the relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \tag{15.31}$$

We had assumed at the time that the frame $Oxyz$ was constrained to rotate about a fixed axis OA. However, the derivation given in Sec. 15.10 remains valid when the frame $Oxyz$ is constrained only to have a fixed point O. Under this more general assumption, the axis OA represents the *instantaneous* axis of rotation of the frame $Oxyz$ (Sec. 15.12) and the vector $\boldsymbol{\Omega}$, its angular velocity at the instant considered (Fig. 15.36).

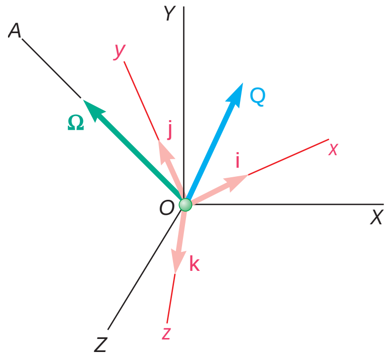


Fig. 15.36

Let us now consider the three-dimensional motion of a particle P relative to a rotating frame $Oxyz$ constrained to have a fixed origin O. Let $\mathbf{r}$ be the position vector of P at a given instant and $\boldsymbol{\Omega}$ be the angular velocity of the frame $Oxyz$ with respect to the fixed frame $OXYZ$ at the same instant (Fig. 15.37). The derivations given in Sec. 15.11 for the two-dimensional motion of a particle can be readily extended to the three-dimensional case, and the absolute velocity $\mathbf{v}_P$ of P (i.e., its velocity with respect to the fixed frame $OXYZ$) can be expressed as

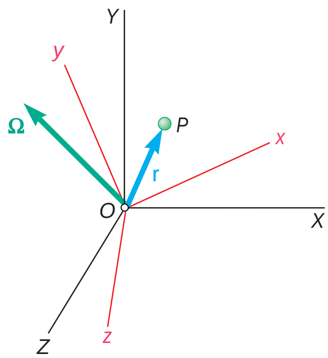


Fig. 15.37

$$\boxed{\mathbf{v}_P = \boldsymbol{\Omega} \times \mathbf{r} + (\dot{\mathbf{r}})_{Oxyz}} \tag{15.45}$$

Denoting by $\mathcal{F}$ the rotating frame $Oxyz$, we write this relation in the alternative form

$$\boxed{\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}}} \tag{15.46}$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame $\mathcal{F}$

The absolute acceleration $\mathbf{a}_P$ of P can be expressed as

$$\boxed{\mathbf{a}_P = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} + (\ddot{\mathbf{r}})_{Oxyz}} \tag{15.47}$$

An alternative form is

$$\boxed{\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c} \tag{15.48}$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame $\mathcal{F}$
$\mathbf{a}_c = 2\mathbf{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} = 2\mathbf{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration†

We note that the Coriolis acceleration is perpendicular to the vectors $\mathbf{\Omega}$ and $\mathbf{v}_{P/\mathcal{F}}$. However, since these vectors are usually not perpendicular to each other, the magnitude of $\mathbf{a}_c$ is in general *not* equal to $2\Omega v_{P/\mathcal{F}}$, as was the case for the plane motion of a particle. We further note that the Coriolis acceleration reduces to zero when the vectors $\mathbf{\Omega}$ and $\mathbf{v}_{P/\mathcal{F}}$ are parallel, or when either of them is zero.

Rotating frames of reference are particularly useful in the study of the three-dimensional motion of rigid bodies. If a rigid body has a fixed point O, as was the case for the crane of Sample Prob. 15.11, we can use a frame $Oxyz$ which is neither fixed nor rigidly attached to the rigid body. Denoting by $\mathbf{\Omega}$ the angular velocity of the frame $Oxyz$, we then resolve the angular velocity V of the body into the components $\mathbf{\Omega}$ and $\mathrm{V}_{B/\mathcal{F}}$, where the second component represents the angular velocity of the body relative to the frame $Oxyz$ (see Sample Prob. 15.14). An appropriate choice of the rotating frame often leads to a simpler analysis of the motion of the rigid body than would be possible with axes of fixed orientation. This is especially true in the case of the general three-dimensional motion of a rigid body, i.e., when the rigid body under consideration has no fixed point (see Sample Prob. 15.15).

*15.15 FRAME OF REFERENCE IN GENERAL MOTION

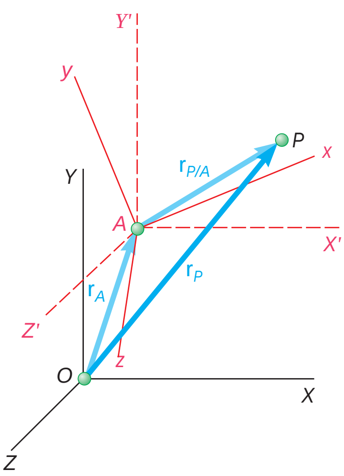


Fig. 15.38

Consider a fixed frame of reference $OXYZ$ and a frame $Axyz$ which moves in a known, but arbitrary, fashion with respect to $OXYZ$ (Fig. 15.38). Let P be a particle moving in space. The position of P is defined at any instant by the vector $\mathbf{r}_P$ in the fixed frame, and by the vector $\mathbf{r}_{P/A}$ in the moving frame. Denoting by $\mathbf{r}_A$ the position vector of A in the fixed frame, we have

$$\mathbf{r}_P = \mathbf{r}_A + \mathbf{r}_{P/A} \tag{15.49}$$

The absolute velocity $\mathbf{v}_P$ of the particle is obtained by writing

$$\mathbf{v}_P = \dot{\mathbf{r}}_P = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{P/A} \tag{15.50}$$

where the derivatives are defined with respect to the fixed frame $OXYZ$. The first term in the right-hand member of (15.50) thus represents the velocity $\mathbf{v}_A$ of the origin A of the moving axes. On the other hand, since the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation (Sec. 11.10), the second term can be regarded as the velocity $\mathbf{v}_{P/A}$ of

†It is important to note the difference between Eq. (15.48) and Eq. (15.21) of Sec. 15.8. See the footnote on page 988.

P relative to the frame $AX'Y'Z'$ of the same orientation as $OXYZ$ and the same origin as $Axyz$. We therefore have

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} \tag{15.51}$$

But the velocity $\mathbf{v}_{P/A}$ of P relative to $AX'Y'Z'$ can be obtained from (15.45) by substituting $\mathbf{r}_{P/A}$ for $\mathbf{r}$ in that equation. We write

$$\mathbf{v}_P = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{P/A} + (\dot{\mathbf{r}}_{P/A})_{Axyz} \tag{15.52}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the frame $Axyz$ at the instant considered.

The absolute acceleration $\mathbf{a}_P$ of the particle is obtained by differentiating (15.51) and writing

$$\mathbf{a}_P = \dot{\mathbf{v}}_P = \dot{\mathbf{v}}_A + \dot{\mathbf{v}}_{P/A} \tag{15.53}$$

where the derivatives are defined with respect to either of the frames $OXYZ$ or $AX'Y'Z'$. Thus, the first term in the right-hand member of (15.53) represents the acceleration $\mathbf{a}_A$ of the origin A of the moving axes and the second term represents the acceleration $\mathbf{a}_{P/A}$ of P relative to the frame $AX'Y'Z'$. This acceleration can be obtained from (15.47) by substituting $\mathbf{r}_{P/A}$ for $\mathbf{r}$. We therefore write

$$\begin{aligned}\mathbf{a}_P = \mathbf{a}_A &+ \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{P/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/A}) \\ &+ 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}}_{P/A})_{Axyz} + (\ddot{\mathbf{r}}_{P/A})_{Axyz}\end{aligned} \tag{15.54}$$

Formulas (15.52) and (15.54) make it possible to determine the velocity and acceleration of a given particle with respect to a fixed frame of reference, when the motion of the particle is known with respect to a moving frame. These formulas become more significant, and considerably easier to remember, if we note that the sum of the first two terms in (15.52) represents the velocity of the point P' of the moving frame which coincides with P at the instant considered, and that the sum of the first three terms in (15.54) represents the acceleration of the same point. Thus, the relations (15.46) and (15.48) of the preceding section are still valid in the case of a reference frame in general motion, and we can write

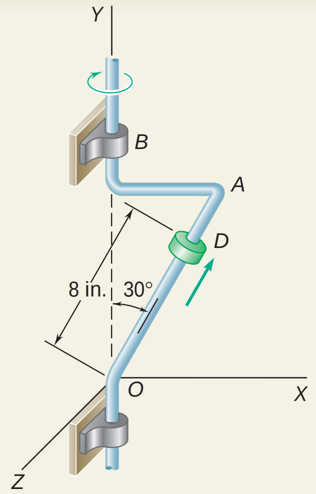
$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.46}$$

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.48}$$

where the various vectors involved have been defined in Sec. 15.14.

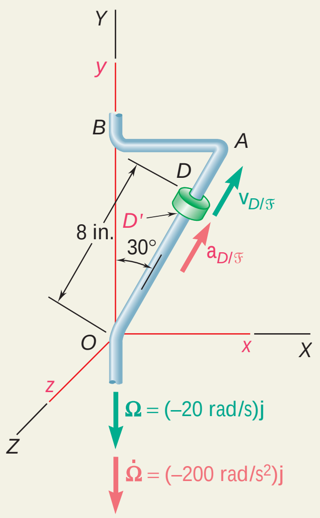
It should be noted that if the moving reference frame $\mathcal{F}$ (or $Axyz$) is in translation, the velocity and acceleration of the point P' of the frame which coincides with P become, respectively, equal to the velocity and acceleration of the origin A of the frame. On the other hand, since the frame maintains a fixed orientation, $\mathbf{a}_c$ is zero, and the relations (15.46) and (15.48) reduce, respectively, to the relations (11.33) and (11.34) derived in Sec. 11.12.

Photo 15.9 The motion of air particles in a hurricane can be considered as motion relative to a frame of reference attached to the Earth and rotating with it.



SAMPLE PROBLEM 15.13

The bent rod *OAB* rotates about the vertical *OB*. At the instant considered, its angular velocity and angular acceleration are, respectively, 20 rad/s and 200 rad/s^2, both clockwise when viewed from the positive *Y* axis. The collar *D* moves along the rod, and at the instant considered, *OD* = 8 in. The velocity and acceleration of the collar relative to the rod are, respectively, 50 in./s and 600 in./s^2, both upward. Determine (*a*) the velocity of the collar, (*b*) the acceleration of the collar.



SOLUTION

Frames of Reference. The frame *OXYZ* is fixed. We attach the rotating frame *Oxyz* to the bent rod. Its angular velocity and angular acceleration relative to *OXYZ* are therefore $\boldsymbol{\Omega} = (-20 \text{ rad/s})\mathbf{j}$ and $\dot{\boldsymbol{\Omega}} = (-200 \text{ rad/s}^2)\mathbf{j}$, respectively. The position vector of *D* is

$$\mathbf{r} = (8 \text{ in.})(\sin 30°\mathbf{i} + \cos 30°\mathbf{j}) = (4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}$$

a. Velocity v_D. Denoting by D' the point of the rod which coincides with *D* and by $\mathcal{F}$ the rotating frame *Oxyz*, we write from Eq. (15.46)

$$\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/\mathcal{F}} \tag{1}$$

where

$$\mathbf{v}_{D'} = \boldsymbol{\Omega} \times \mathbf{r} = (-20 \text{ rad/s})\mathbf{j} \times [(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}] = (80 \text{ in./s})\mathbf{k}$$
$$\mathbf{v}_{D/\mathcal{F}} = (50 \text{ in./s})(\sin 30°\mathbf{i} + \cos 30°\mathbf{j}) = (25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j}$$

Substituting the values obtained for $\mathbf{v}_{D'}$ and $\mathbf{v}_{D/\mathcal{F}}$ into (1), we find

$$\mathbf{v}_D = (25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j} + (80 \text{ in./s})\mathbf{k} \quad ◀$$

b. Acceleration a_D. From Eq. (15.48) we write

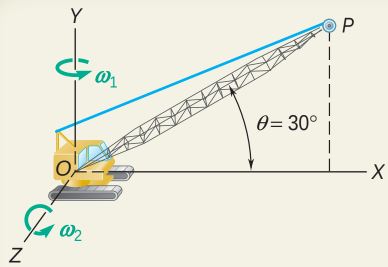
$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/\mathcal{F}} + \mathbf{a}_c \tag{2}$$

where

$$\begin{aligned}
\mathbf{a}_{D'} &= \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \\
&= (-200 \text{ rad/s}^2)\mathbf{j} \times [(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}] - (20 \text{ rad/s})\mathbf{j} \times (80 \text{ in./s})\mathbf{k} \\
&= +(800 \text{ in./s}^2)\mathbf{k} - (1600 \text{ in./s}^2)\mathbf{i} \\
\mathbf{a}_{D/\mathcal{F}} &= (600 \text{ in./s}^2)(\sin 30°\mathbf{i} + \cos 30°\mathbf{j}) = (300 \text{ in./s}^2)\mathbf{i} + (520 \text{ in./s}^2)\mathbf{j} \\
\mathbf{a}_c &= 2\boldsymbol{\Omega} \times \mathbf{v}_{D/\mathcal{F}} \\
&= 2(-20 \text{ rad/s})\mathbf{j} \times [(25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j}] = (1000 \text{ in./s}^2)\mathbf{k}
\end{aligned}$$

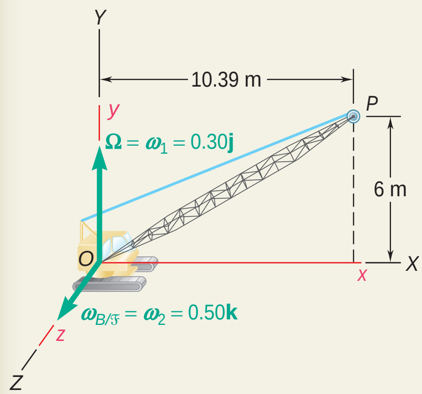
Substituting the values obtained for $\mathbf{a}_{D'}$, $\mathbf{a}_{D/\mathcal{F}}$, and $\mathbf{a}_c$ into (2),

$$\mathbf{a}_D = -(1300 \text{ in./s}^2)\mathbf{i} + (520 \text{ in./s}^2)\mathbf{j} + (1800 \text{ in./s}^2)\mathbf{k} \quad ◀$$



SAMPLE PROBLEM 15.14

The crane shown rotates with a constant angular velocity V_1 of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity V_2 of 0.50 rad/s relative to the cab. Knowing that the length of the boom OP is $l = 12$ m, determine (*a*) the velocity of the tip of the boom, (*b*) the acceleration of the tip of the boom.



SOLUTION

Frames of Reference. The frame $OXYZ$ is fixed. We attach the rotating frame $Oxyz$ to the cab. Its angular velocity with respect to the frame $OXYZ$ is therefore $\boldsymbol{\Omega} = V_1 = (0.30 \text{ rad/s})\mathbf{j}$. The angular velocity of the boom relative to the cab and the rotating frame $Oxyz$ (or $\mathcal{F}$, for short) is $V_{B/\mathcal{F}} = V_2 = (0.50 \text{ rad/s})\mathbf{k}$.

a. Velocity $\mathbf{v}_P$. From Eq. (15.46) we write

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \qquad (1)$$

where $\mathbf{v}_{P'}$ is the velocity of the point P' of the rotating frame which coincides with P:

$$\mathbf{v}_{P'} = \boldsymbol{\Omega} \times \mathbf{r} = (0.30 \text{ rad/s})\mathbf{j} \times [(10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}] = -(3.12 \text{ m/s})\mathbf{k}$$

and where $\mathbf{v}_{P/\mathcal{F}}$ is the velocity of P relative to the rotating frame $Oxyz$. But the angular velocity of the boom relative to $Oxyz$ was found to be $V_{B/\mathcal{F}} = (0.50 \text{ rad/s})\mathbf{k}$. The velocity of its tip P relative to $Oxyz$ is therefore

$$\begin{aligned}\mathbf{v}_{P/\mathcal{F}} = V_{B/\mathcal{F}} \times \mathbf{r} &= (0.50 \text{ rad/s})\mathbf{k} \times [(10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}]\\ &= -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}\end{aligned}$$

Substituting the values obtained for $\mathbf{v}_{P'}$ and $\mathbf{v}_{P/\mathcal{F}}$ into (1), we find

$$\mathbf{v}_P = -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j} - (3.12 \text{ m/s})\mathbf{k} \quad ◀$$

b. Acceleration $\mathbf{a}_P$. From Eq. (15.48) we write

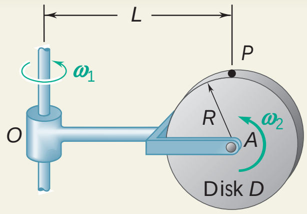
$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \qquad (2)$$

Since $\boldsymbol{\Omega}$ and $V_{B/\mathcal{F}}$ are both constant, we have

$$\begin{aligned}\mathbf{a}_{P'} &= \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = (0.30 \text{ rad/s})\mathbf{j} \times (-3.12 \text{ m/s})\mathbf{k} = -(0.94 \text{ m/s}^2)\mathbf{i}\\ \mathbf{a}_{P/\mathcal{F}} &= V_{B/\mathcal{F}} \times (V_{B/\mathcal{F}} \times \mathbf{r})\\ &= (0.50 \text{ rad/s})\mathbf{k} \times [-(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}]\\ &= -(1.50 \text{ m/s}^2)\mathbf{j} - (2.60 \text{ m/s}^2)\mathbf{i}\\ \mathbf{a}_c &= 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}\\ &= 2(0.30 \text{ rad/s})\mathbf{j} \times [-(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}] = (1.80 \text{ m/s}^2)\mathbf{k}\end{aligned}$$

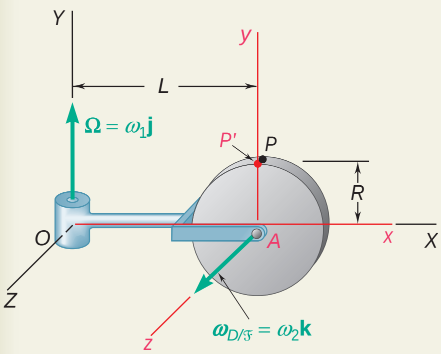
Substituting for $\mathbf{a}_{P'}$, $\mathbf{a}_{P/\mathcal{F}}$, and $\mathbf{a}_c$ into (2), we find

$$\mathbf{a}_P = -(3.54 \text{ m/s}^2)\mathbf{i} - (1.50 \text{ m/s}^2)\mathbf{j} + (1.80 \text{ m/s}^2)\mathbf{k} \quad ◀$$



SAMPLE PROBLEM 15.15

Disk D, of radius R, is pinned to end A of the arm OA of length L located in the plane of the disk. The arm rotates about a vertical axis through O at the constant rate v_1, and the disk rotates about A at the constant rate v_2. Determine (*a*) the velocity of point P located directly above A, (*b*) the acceleration of P, (*c*) the angular velocity and angular acceleration of the disk.



SOLUTION

Frames of Reference. The frame $OXYZ$ is fixed. We attach the moving frame $Axyz$ to the arm OA. Its angular velocity with respect to the frame $OXYZ$ is therefore $\boldsymbol{\Omega} = v_1\mathbf{j}$. The angular velocity of disk D relative to the moving frame $Axyz$ (or $\mathcal{F}$, for short) is $V_{D/\mathcal{F}} = v_2\mathbf{k}$. The position vector of P relative to O is $\mathbf{r} = L\mathbf{i} + R\mathbf{j}$, and its position vector relative to A is $\mathbf{r}_{P/A} = R\mathbf{j}$.

a. Velocity $\mathbf{v}_P$. Denoting by P' the point of the moving frame which coincides with P, we write from Eq. (15.46)

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{1}$$

where $\mathbf{v}_{P'} = \boldsymbol{\Omega} \times \mathbf{r} = v_1\mathbf{j} \times (L\mathbf{i} + R\mathbf{j}) = -v_1L\mathbf{k}$

$$\mathbf{v}_{P/\mathcal{F}} = V_{D/\mathcal{F}} \times \mathbf{r}_{P/A} = v_2\mathbf{k} \times R\mathbf{j} = -v_2R\mathbf{i}$$

Substituting the values obtained for $\mathbf{v}_{P'}$ and $\mathbf{v}_{P/\mathcal{F}}$ into (1), we find

$$\mathbf{v}_P = -v_2R\mathbf{i} - v_1L\mathbf{k} \quad ◀$$

b. Acceleration $\mathbf{a}_P$. From Eq. (15.48) we write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{2}$$

Since $\boldsymbol{\Omega}$ and $V_{D/\mathcal{F}}$ are both constant, we have

$$\begin{aligned}
\mathbf{a}_{P'} &= \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = v_1\mathbf{j} \times (-v_1L\mathbf{k}) = -v_1^2L\mathbf{i} \\
\mathbf{a}_{P/\mathcal{F}} &= V_{D/\mathcal{F}} \times (V_{D/\mathcal{F}} \times \mathbf{r}_{P/A}) = v_2\mathbf{k} \times (-v_2R\mathbf{i}) = -v_2^2R\mathbf{j} \\
\mathbf{a}_c &= 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}} = 2v_1\mathbf{j} \times (-v_2R\mathbf{i}) = 2v_1v_2R\mathbf{k}
\end{aligned}$$

Substituting the values obtained into (2), we find

$$\mathbf{a}_P = -v_1^2L\mathbf{i} - v_2^2R\mathbf{j} + 2v_1v_2R\mathbf{k} \quad ◀$$

c. Angular Velocity and Angular Acceleration of Disk.

$$v = \Omega + v_{D/\mathcal{F}} \qquad V = v_1\mathbf{j} + v_2\mathbf{k} \quad ◀$$

Using Eq. (15.31) with $\mathbf{Q} = V$, we write

$$\begin{aligned}
A = (\dot{V})_{OXYZ} &= (\dot{V})_{Axyz} + \boldsymbol{\Omega} \times V \\
&= 0 + v_1\mathbf{j} \times (v_1\mathbf{j} + v_2\mathbf{k})
\end{aligned}$$

$$A = v_1v_2\mathbf{i} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you concluded your study of the kinematics of rigid bodies by learning how to use an auxiliary frame of reference $\mathcal{F}$ to analyze the three-dimensional motion of a rigid body. This auxiliary frame may be a *rotating frame* with a fixed origin O, or it may be a *frame in general motion.*

A. Using a rotating frame of reference. As you approach a problem involving the use of a rotating frame $\mathcal{F}$ you should take the following steps.

1. Select the rotating frame $\mathcal{F}$ that you wish to use and draw the corresponding coordinate axes x, y, and z from the fixed point O.

2. Determine the angular velocity $\boldsymbol{\Omega}$ of the frame $\mathcal{F}$ with respect to a fixed frame $OXYZ$. In most cases, you will have selected a frame which is attached to some rotating element of the system; $\boldsymbol{\Omega}$ will then be the angular velocity of that element.

3. Designate as P' the point of the rotating frame $\mathcal{F}$ that coincides with the point P of interest at the instant you are considering. Determine the velocity $\mathbf{v}_{P'}$ and the acceleration $\mathbf{a}_{P'}$ of point P'. Since P' is part of $\mathcal{F}$ and has the same position vector $\mathbf{r}$ as P, you will find that

$$\mathbf{v}_{P'} = \boldsymbol{\Omega} \times \mathbf{r} \qquad \text{and} \qquad \mathbf{a}_{P'} = \mathsf{A} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

where A is the angular acceleration of $\mathcal{F}$. However, in many of the problems that you will encounter, the angular velocity of $\mathcal{F}$ is constant in both magnitude and direction, and $\mathsf{A} = 0$.

4. Determine the velocity and acceleration of point P with respect to the frame $\mathcal{F}$. As you are trying to determine $\mathbf{v}_{P/\mathcal{F}}$ and $\mathbf{a}_{P/\mathcal{F}}$ you will find it useful to visualize the motion of P on frame $\mathcal{F}$ when the frame is not rotating. If P is a point of a rigid body $\mathcal{B}$ which has an angular velocity $\mathsf{V}_{\mathcal{B}}$ and an angular acceleration $\mathsf{A}_{\mathcal{B}}$ relative to $\mathcal{F}$ [Sample Prob. 15.14], you will find that

$$\mathbf{v}_{P/\mathcal{F}} = \mathsf{V}_{\mathcal{B}} \times \mathbf{r} \qquad \text{and} \qquad \mathbf{a}_{P/\mathcal{F}} = \mathsf{A}_{\mathcal{B}} \times \mathbf{r} + \mathsf{V}_{\mathcal{B}} \times (\mathsf{V}_{\mathcal{B}} \times \mathbf{r})$$

In many of the problems that you will encounter, the angular velocity of body $\mathcal{B}$ relative to frame $\mathcal{F}$ is constant in both magnitude and direction, and $\mathsf{A}_{\mathcal{B}} = 0$.

5. Determine the Coriolis acceleration. Considering the angular velocity $\boldsymbol{\Omega}$ of frame $\mathcal{F}$ and the velocity $\mathbf{v}_{P/\mathcal{F}}$ of point P relative to that frame, which was computed in the previous step, you write

$$\mathbf{a}_c = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$$

(continued)

6. The velocity and the acceleration of P with respect to the fixed frame $OXYZ$ can now be obtained by adding the expressions you have determined:

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathscr{F}} \tag{15.46}$$
$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathscr{F}} + \mathbf{a}_c \tag{15.48}$$

B. Using a frame of reference in general motion. The steps that you will take differ only slightly from those listed under A. They consist of the following:

1. Select the frame $\mathscr{F}$ that you wish to use and a reference point A in that frame, from which you will draw the coordinate axes, x, y, and z defining that frame. You will consider the motion of the frame as the sum of a *translation with A and a rotation about A.*

2. Determine the velocity $\mathbf{v}_A$ of point A and the angular velocity $\boldsymbol{\Omega}$ of the frame. In most cases, you will have selected a frame which is attached to some element of the system; $\boldsymbol{\Omega}$ will then be the angular velocity of that element.

3. Designate as P' the point of frame $\mathscr{F}$ that coincides with the point P of interest at the instant you are considering, and determine the velocity $\mathbf{v}_{P'}$ and the acceleration $\mathbf{a}_{P'}$ of that point. In some cases, this can be done by visualizing the motion of P if that point were prevented from moving with respect to $\mathscr{F}$ [Sample Prob. 15.15]. A more general approach is to recall that the motion of P' is the sum of a translation with the reference point A and a rotation about A. The velocity $\mathbf{v}_{P'}$ and the acceleration $\mathbf{a}_{P'}$ of P', therefore, can be obtained by adding $\mathbf{v}_A$ and $\mathbf{a}_A$, respectively, to the expressions found in paragraph A3 and replacing the position vector $\mathbf{r}$ by the vector $\mathbf{r}_{P/A}$ drawn from A to P:

$$\mathbf{v}_{P'} = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{P/A} \qquad \mathbf{a}_{P'} = \mathbf{a}_A + \mathsf{A} \times \mathbf{r}_{P/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/A})$$

Steps 4, 5, and 6 are the same as in Part A, except that the vector $\mathbf{r}$ should again be replaced by $\mathbf{r}_{P/A}$. Thus, Eqs. (15.46) and (15.48) can still be used to obtain the velocity and the acceleration of P with respect to the fixed frame of reference $OXYZ$.

PROBLEMS

END-OF-SECTION PROBLEMS

15.220 A square plate of side 18 in. is hinged at A and B to a clevis. The plate rotates at the constant rate $v_2 = 4$ rad/s with respect to the clevis, which itself rotates at the constant rate $v_1 = 3$ rad/s about the Y axis. For the position shown, determine (*a*) the velocity of point C, (*b*) the acceleration of point C.

15.221 A square plate of side 18 in. is hinged at A and B to a clevis. The plate rotates at the constant rate $v_2 = 4$ rad/s with respect to the clevis, which itself rotates at the constant rate $v_1 = 3$ rad/s about the Y axis. For the position shown, determine (*a*) the velocity of corner D, (*b*) the acceleration of corner D.

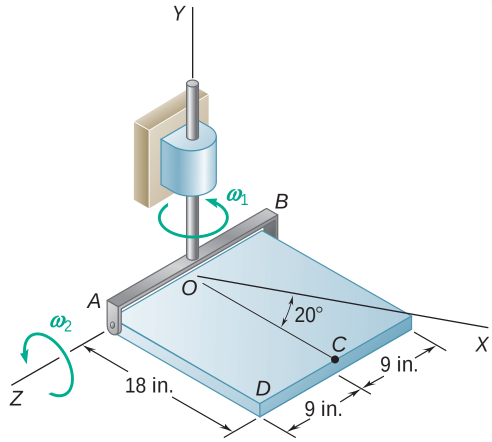


Fig. P15.220 and P15.221

15.222 and 15.223 The rectangular plate shown rotates at the constant rate $v_2 = 12$ rad/s with respect to arm AE, which itself rotates at the constant rate $v_1 = 9$ rad/s about the Z axis. For the position shown, determine the velocity and acceleration of the point of the plate indicated.

15.222 Corner B
15.223 Corner C

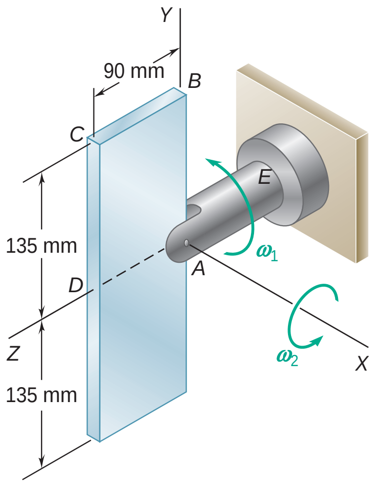


Fig. P15.222 and P15.223

15.224 Rod AB is welded to the 0.3-m-radius plate which rotates at the constant rate $v_1 = 6$ rad/s. Knowing that collar D moves toward end B of the rod at a constant speed $u = 1.3$ m/s, determine, for the position shown, (*a*) the velocity of D, (*b*) the acceleration of D.

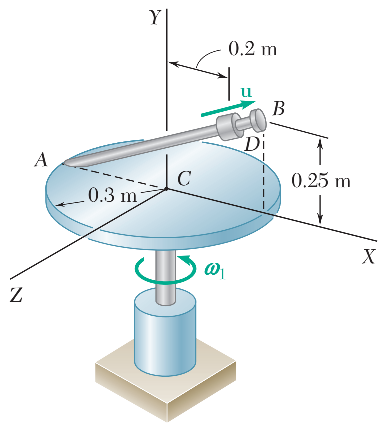


Fig. P15.224

15.225 The bent rod ABC rotates at the constant rate $\omega_1 = 4$ rad/s. Knowing that collar D moves downward along the rod at a constant relative speed $u = 65$ in./s, determine, for the position shown, (*a*) the velocity of D, (*b*) the acceleration of D.

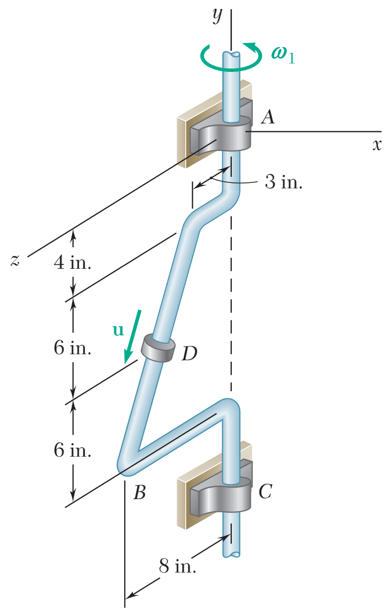


Fig. *P15.225*

15.226 The bent pipe shown rotates at the constant rate $\omega_1 = 10$ rad/s. Knowing that a ball bearing D moves in portion BC of the pipe toward end C at a constant relative speed $u = 2$ ft/s, determine at the instant shown (*a*) the velocity of D, (*b*) the acceleration of D.

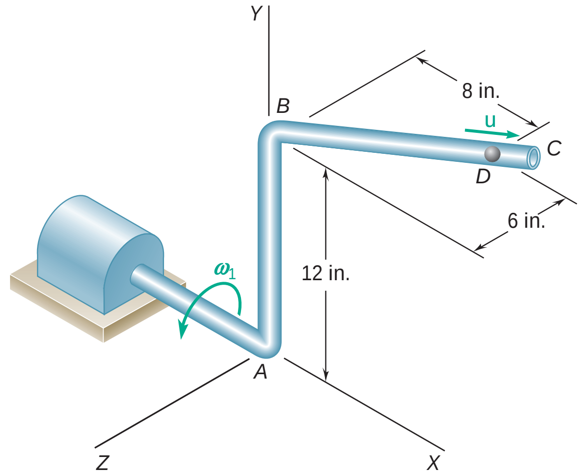


Fig. *P15.226*

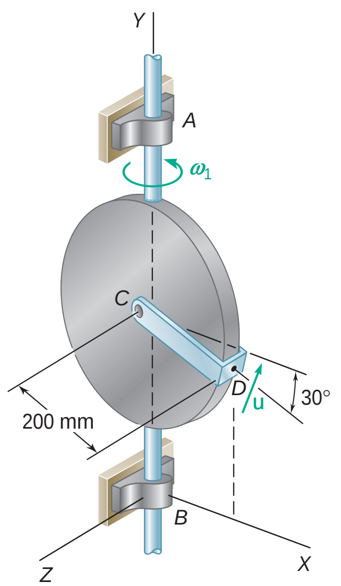


Fig. P15.227

15.227 The circular plate shown rotates about its vertical diameter at the constant rate $\omega_1 = 10$ rad/s. Knowing that in the position shown the disk lies in the XY plane and point D of strap CD moves upward at a constant relative speed $u = 1.5$ m/s, determine (*a*) the velocity of D, (b) the acceleration of D.

15.228 Manufactured items are spray-painted as they pass through the automated work station shown. Knowing that the bent pipe ACE rotates at the constant rate $\omega_1 = 0.4$ rad/s and that at point D the paint moves through the pipe at a constant relative speed $u = 150$ mm/s, determine, for the position shown, (*a*) the velocity of the paint at D, (*b*) the acceleration of the paint at D.

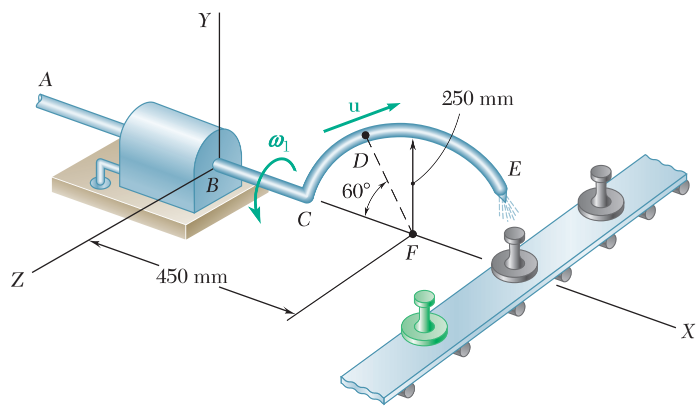


Fig. P15.228

15.229 Solve Prob. 15.227, assuming that at the instant shown the angular velocity ω_1 of the plate is 10 rad/s and is decreasing at the rate of 25 rad/s^2, while the relative speed u of point D of strap CD is 1.5 m/s and is decreasing at the rate of 3 m/s^2.

15.230 Solve Prob. 15.226 assuming that at the instant shown the angular velocity ω_1 of the pipe is 10 rad/s and is decreasing at the rate of 15 rad/s^2, while the relative speed u of the ball bearing is 2 ft/s and is increasing at the rate of 10 ft/s^2.

15.231 Using the method of Sec. 15.14, solve Prob. 15.192.

15.232 Using the method of Sec. 15.14, solve Prob. 15.196.

15.233 Using the method of Sec. 15.14, solve Prob. 15.198.

15.234 A disk of radius 120 mm rotates at the constant rate $\omega_2 = 5$ rad/s with respect to the arm AB, which itself rotates at the constant rate $\omega_1 = 3$ rad/s. For the position shown, determine the velocity and acceleration of point C.

15.235 A disk of radius 120 mm rotates at the constant rate $\omega_2 = 5$ rad/s with respect to the arm AB, which itself rotates at the constant rate $\omega_1 = 3$ rad/s. For the position shown, determine the velocity and acceleration of point D.

15.236 The arm AB of length 16 ft is used to provide an elevated platform for construction workers. In the position shown, arm AB is being raised at the constant rate $d\theta/dt = 0.25$ rad/s; simultaneously, the unit is being rotated about the Y axis at the constant rate $\omega_1 = 0.15$ rad/s. Knowing that $\theta = 20°$, determine the velocity and acceleration of point B.

15.237 The remote manipulator system (RMS) shown is used to deploy payloads from the cargo bay of space shuttles. At the instant shown, the whole RMS is rotating at the constant rate $\omega_1 = 0.03$ rad/s about the axis AB. At the same time, portion BCD rotates as a rigid body at the constant rate $\omega_2 = d\beta/dt = 0.04$ rad/s about an axis through B parallel to the X axis. Knowing that $\beta = 30°$, determine (*a*) the angular acceleration of BCD, (*b*) the velocity of D, (*c*) the acceleration of D.

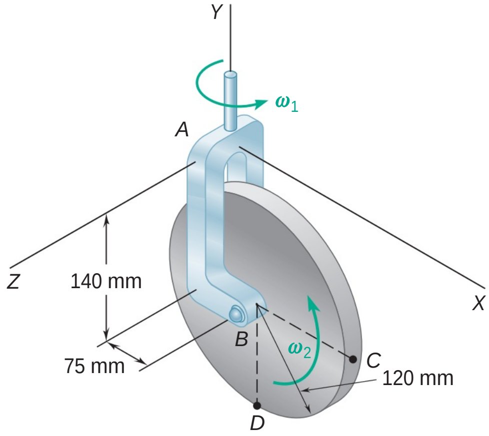


Fig. P15.234 and P15.235

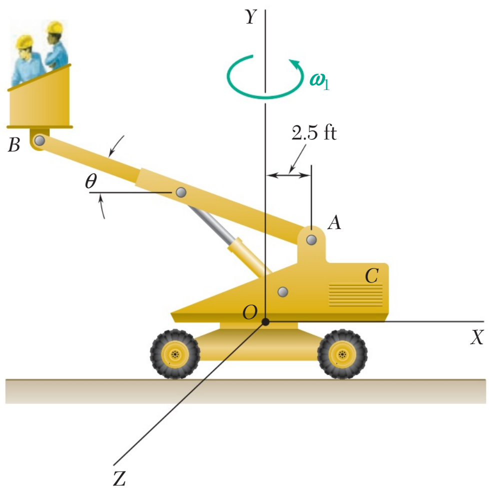


Fig. P15.236

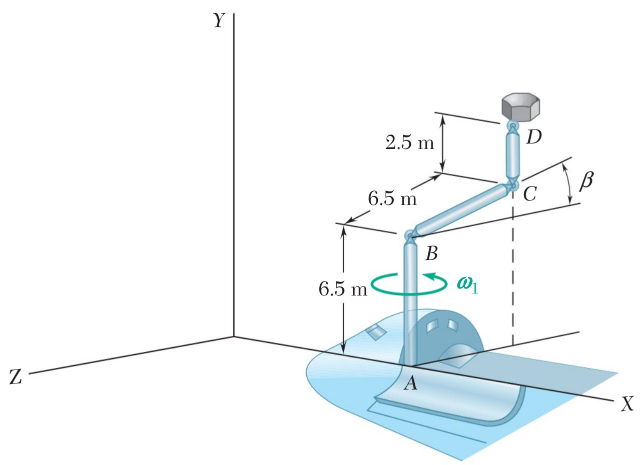


Fig. *P15.237*

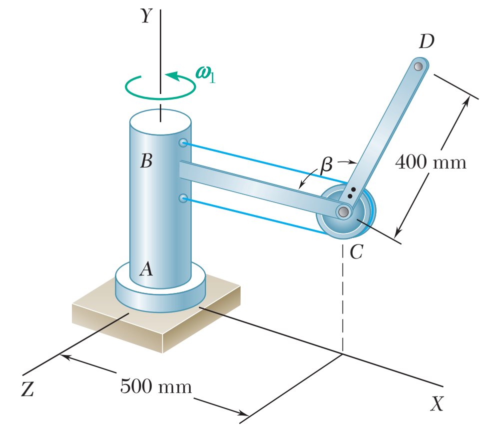


Fig. *P15.238*

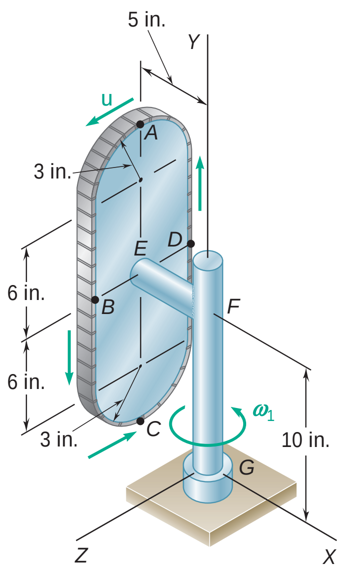


Fig. P15.240 and P15.241

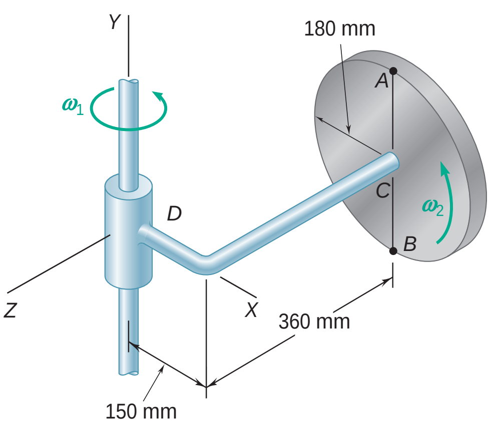


Fig. P15.242 and P15.243

15.238 The body AB and rod BC of the robotic component shown rotate at the constant rate $\omega_1 = 0.60$ rad/s about the Y axis. Simultaneously a wire-and-pulley control causes arm CD to rotate about C at the constant rate $\omega = d\beta/dt = 0.45$ rad/s. Knowing $\beta = 120°$, determine (*a*) the angular acceleration of arm CD, (*b*) the velocity of D, (*c*) the acceleration of D.

15.239 The crane shown rotates at the constant rate $\omega_1 = 0.25$ rad/s; simultaneously, the telescoping boom is being lowered at the constant rate $\omega_2 = 0.40$ rad/s. Knowing that at the instant shown the length of the boom is 20 ft and is increasing at the constant rate $u = 1.5$ ft/s, determine the velocity and acceleration of point B.

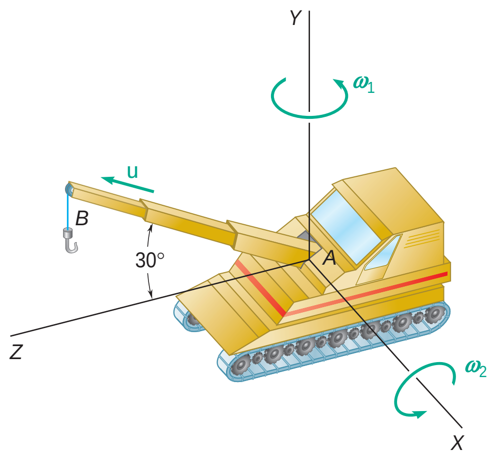


Fig. P15.239

15.240 The vertical plate shown is welded to arm EFG, and the entire unit rotates at the constant rate $\omega_1 = 1.6$ rad/s about the Y axis. At the same time, a continuous link belt moves around the perimeter of the plate at a constant speed $u = 4.5$ in./s. For the position shown, determine the acceleration of the link of the belt located (*a*) at point A, (*b*) at point B.

15.241 The vertical plate shown is welded to arm EFG, and the entire unit rotates at the constant rate $\omega_1 = 1.6$ rad/s about the Y axis. At the same time, a continuous link belt moves around the perimeter of the plate at a constant speed $u = 4.5$ in./s. For the position shown, determine the acceleration of the link of the belt located (*a*) at point C, (*b*) at point D.

15.242 A disk of 180-mm radius rotates at the constant rate $\omega_2 = 12$ rad/s with respect to arm CD, which itself rotates at the constant rate $\omega_1 = 8$ rad/s about the Y axis. Determine at the instant shown the velocity and acceleration of point A on the rim of the disk.

15.243 A disk of 180-mm radius rotates at the constant rate $\omega_2 = 12$ rad/s with respect to arm CD, which itself rotates at the constant rate $\omega_1 = 8$ rad/s about the Y axis. Determine at the instant shown the velocity and acceleration of point B on the rim of the disk.

15.244 A square plate of side $2r$ is welded to a vertical shaft which rotates with a constant angular velocity V_1. At the same time, rod *AB* of length r rotates about the center of the plate with a constant angular velocity V_2 with respect to the plate. For the position of the plate shown, determine the acceleration of end *B* of the rod if (*a*) $u = 0$, (*b*) $u = 90°$, (*c*) $u = 180°$.

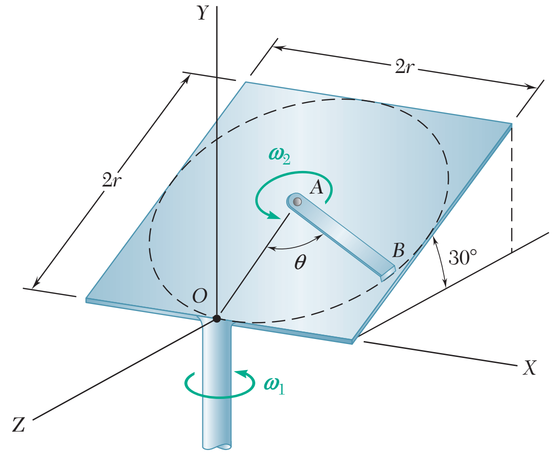


Fig. P15.244

15.245 Two disks, each of 130-mm radius, are welded to the 500-mm rod *CD*. The rod-and-disks unit rotates at the constant rate $v_2 = 3$ rad/s with respect to arm *AB*. Knowing that at the instant shown $v_1 = 4$ rad/s, determine the velocity and acceleration of (*a*) point *E*, (*b*) point *F*.

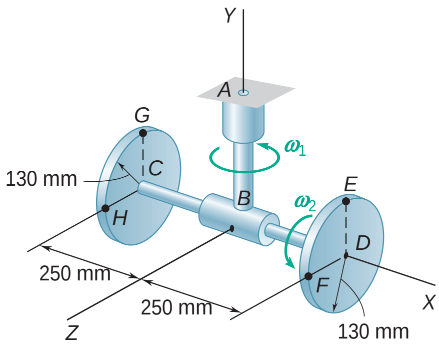


Fig. P15.245

15.246 In Prob. 15.245, determine the velocity and acceleration of (*a*) point *G*, (*b*) point *H*.

15.247 The position of the stylus tip *A* is controlled by the robot shown. In the position shown, the stylus moves at a constant speed $u = 180$ mm/s relative to the solenoid *BC*. At the same time, arm *CD* rotates at the constant rate $v_2 = 1.6$ rad/s with respect to component *DEG*. Knowing that the entire robot rotates about the *X* axis at the constant rate $v_1 = 1.2$ rad/s, determine (*a*) the velocity of *A*, (*b*) the acceleration of *A*.

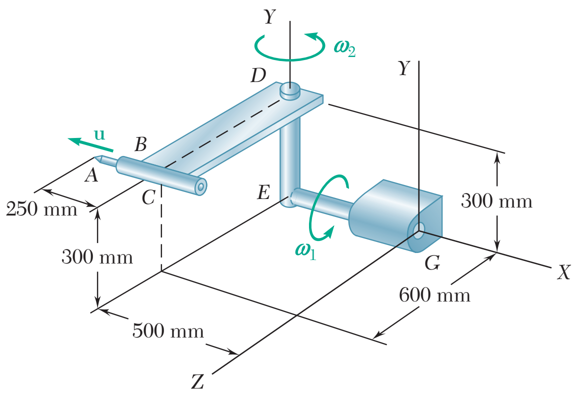


Fig. P15.247

REVIEW AND SUMMARY

This chapter was devoted to the study of the kinematics of rigid bodies.

Rigid body in translation

We first considered the *translation* of a rigid body [Sec. 15.2] and observed that in such a motion, *all points of the body have the same velocity and the same acceleration at any given instant.*

Rigid body in rotation about a fixed axis

We next considered the *rotation* of a rigid body about a fixed axis [Sec. 15.3]. The position of the body is defined by the angle u that the line BP, drawn from the axis of rotation to a point P of the body, forms with a fixed plane (Fig. 15.39). We found that the magnitude of the velocity of P is

$$v = \frac{ds}{dt} = r\dot{u} \sin f \tag{15.4}$$

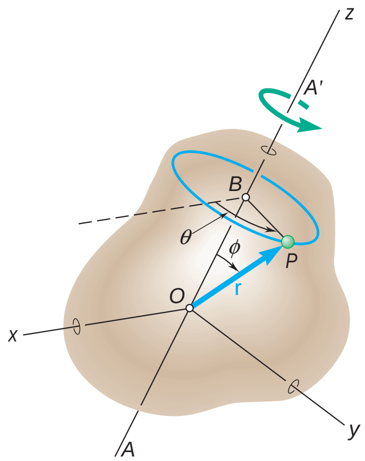


Fig. 15.39

where $\dot{u}$ is the time derivative of u. We then expressed the velocity of P as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathrm{V} \times \mathbf{r} \tag{15.5}$$

where the vector

$$\mathrm{V} = \mathrm{v}\mathbf{k} = \dot{\mathrm{u}}\mathbf{k} \tag{15.6}$$

is directed along the fixed axis of rotation and represents the *angular velocity* of the body.

Denoting by A the derivative $d\mathrm{V}/dt$ of the angular velocity, we expressed the acceleration of P as

$$\mathbf{a} = \mathrm{A} \times \mathbf{r} + \mathrm{V} \times (\mathrm{V} \times \mathbf{r}) \tag{15.8}$$

Differentiating (15.6), and recalling that $\mathbf{k}$ is constant in magnitude and direction, we found that

$$\mathrm{A} = \mathrm{a}\mathbf{k} = \dot{\mathrm{v}}\mathbf{k} = \ddot{\mathrm{u}}\mathbf{k} \tag{15.9}$$

The vector A represents the *angular acceleration* of the body and is directed along the fixed axis of rotation.

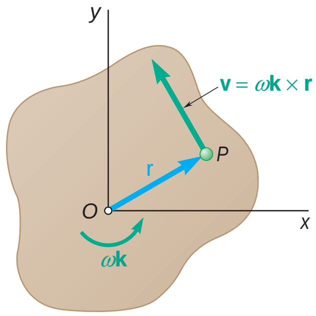


Fig. 15.40

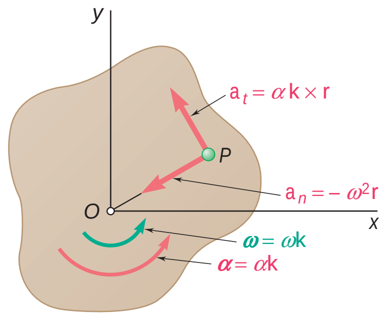


Fig. 15.41

Rotation of a representative slab

Next we considered the motion of a representative slab located in a plane perpendicular to the axis of rotation of the body (Fig. 15.40). Since the angular velocity is perpendicular to the slab, the velocity of a point P of the slab was expressed as

$$\mathbf{v} = \mathsf{v}\mathbf{k} \times \mathbf{r} \qquad (15.10)$$

where $\mathbf{v}$ is contained in the plane of the slab. Substituting $\mathsf{V} = \mathsf{v}\mathbf{k}$ and $\mathsf{A} = \mathsf{a}\mathbf{k}$ into (15.8), we found that the acceleration of P could be resolved into tangential and normal components (Fig. 15.41) respectively equal to

Tangential and normal components

$$\begin{aligned} \mathbf{a}_t &= \mathsf{a}\mathbf{k} \times \mathbf{r} \qquad & a_t &= r\mathsf{a} \\ \mathbf{a}_n &= -\mathsf{v}^2\mathbf{r} \qquad & a_n &= r\mathsf{v}^2 \end{aligned} \qquad (15.11')$$

Angular velocity and angular acceleration of rotating slab

Recalling Eqs. (15.6) and (15.9), we obtained the following expressions for the *angular velocity* and the *angular acceleration* of the slab [Sec. 15.4]:

$$\mathsf{v} = \frac{d\mathsf{u}}{dt} \qquad (15.12)$$

$$\mathsf{a} = \frac{d\mathsf{v}}{dt} = \frac{d^2\mathsf{u}}{dt^2} \qquad (15.13)$$

or

$$\mathsf{a} = \mathsf{v}\frac{d\mathsf{v}}{d\mathsf{u}} \qquad (15.14)$$

We noted that these expressions are similar to those obtained in Chap. 11 for the rectilinear motion of a particle.

Two particular cases of rotation are frequently encountered: *uniform rotation* and *uniformly accelerated rotation.* Problems involving either of these motions can be solved by using equations similar to those used in Secs. 11.4 and 11.5 for the uniform rectilinear motion and the uniformly accelerated rectilinear motion of a particle, but where x, v, and a are replaced by u, v, and a, respectively [Sample Prob. 15.1].

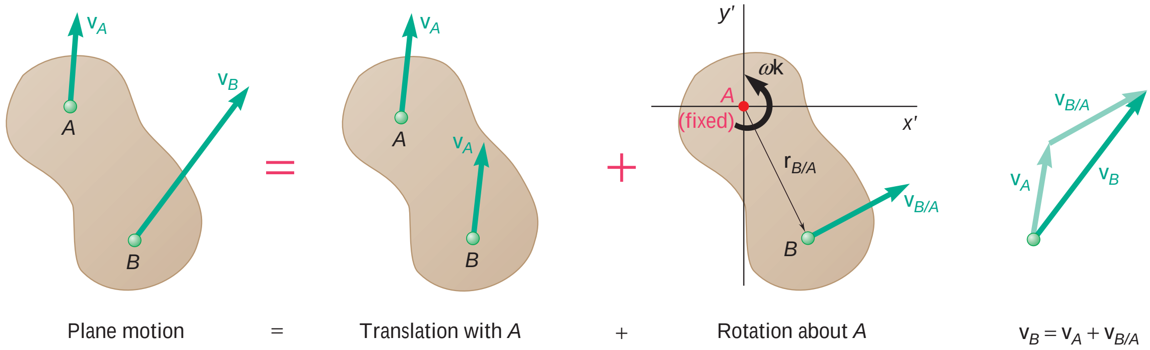


Fig. 15.42

Velocities in plane motion

The *most general plane motion* of a rigid slab can be considered as the *sum of a translation and a rotation* [Sec. 15.5]. For example, the slab shown in Fig. 15.42 can be assumed to translate with point A, while simultaneously rotating about A. It follows [Sec. 15.6] that the velocity of any point B of the slab can be expressed as

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.17}$$

where $\mathbf{v}_A$ is the velocity of A and $\mathbf{v}_{B/A}$ the relative velocity of B with respect to A or, more precisely, with respect to axes $x'y'$ translating with A. Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, we found that

$$\mathbf{v}_{B/A} = \omega\mathbf{k} \times \mathbf{r}_{B/A} \qquad v_{B/A} = r\omega \tag{15.18}$$

The fundamental equation (15.17) relating the absolute velocities of points A and B and the relative velocity of B with respect to A was expressed in the form of a vector diagram and used to solve problems involving the motion of various types of mechanisms [Sample Probs. 15.2 and 15.3].

Instantaneous center of rotation

Another approach to the solution of problems involving the velocities of the points of a rigid slab in plane motion was presented in Sec. 15.7 and used in Sample Probs. 15.4 and 15.5. It is based on the determination of the *instantaneous center of rotation* C of the slab (Fig. 15.43).

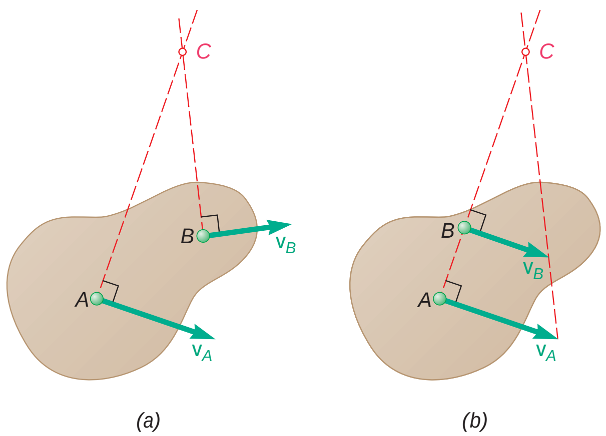


Fig. 15.43

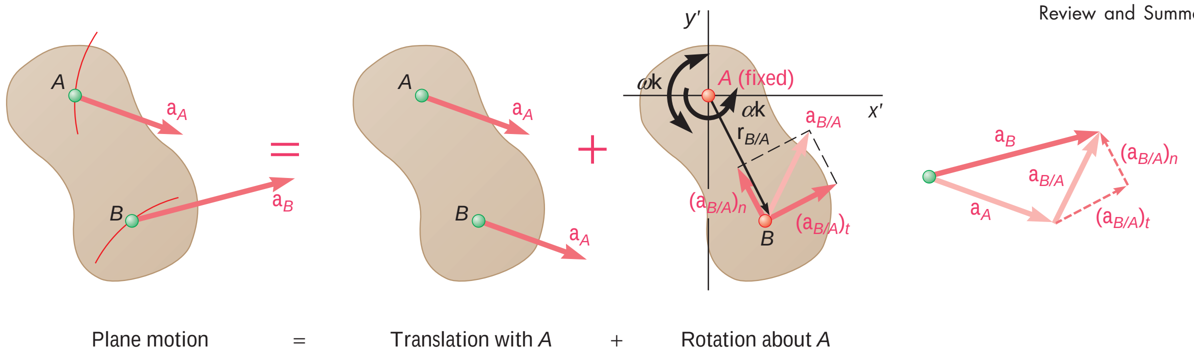


Fig. 15.44

Accelerations in plane motion

The fact that any plane motion of a rigid slab can be considered as the sum of a translation of the slab with a reference point A and a rotation about A was used in Sec. 15.8 to relate the absolute accelerations of any two points A and B of the slab and the relative acceleration of B with respect to A. We had

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{15.21}$$

where $\mathbf{a}_{B/A}$ consisted of a *normal component* $(\mathbf{a}_{B/A})_n$ of magnitude $r\omega^2$ directed toward A, and a *tangential component* $(\mathbf{a}_{B/A})_t$ of magnitude $r\alpha$ perpendicular to the line AB (Fig. 15.44). The fundamental relation (15.21) was expressed in terms of vector diagrams or vector equations and used to determine the accelerations of given points of various mechanisms [Sample Probs. 15.6 through 15.8]. It should be noted that the instantaneous center of rotation C considered in Sec. 15.7 cannot be used for the determination of accelerations, since point C, in general, does *not* have zero acceleration.

Coordinates expressed in terms of a parameter

In the case of certain mechanisms, it is possible to express the coordinates x and y of all significant points of the mechanism by means of simple analytic expressions containing a *single parameter*. The components of the absolute velocity and acceleration of a given point are then obtained by differentiating twice with respect to the time t the coordinates x and y of that point [Sec. 15.9].

Rate of change of a vector with respect to a rotating frame

While the rate of change of a vector is the same with respect to a fixed frame of reference and with respect to a frame in translation, the rate of change of a vector with respect to a rotating frame is different. Therefore, in order to study the motion of a particle relative to a rotating frame we first had to compare the rates of change of a general vector $\mathbf{Q}$ with respect to a fixed frame $OXYZ$ and with respect to a frame $Oxyz$ rotating with an angular velocity $\boldsymbol{\Omega}$ [Sec. 15.10] (Fig. 15.45). We obtained the fundamental relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \tag{15.31}$$

and we concluded that the rate of change of the vector $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ is made of two parts: The first part represents the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$; the second part, $\boldsymbol{\Omega} \times \mathbf{Q}$, is induced by the rotation of the frame $Oxyz$.

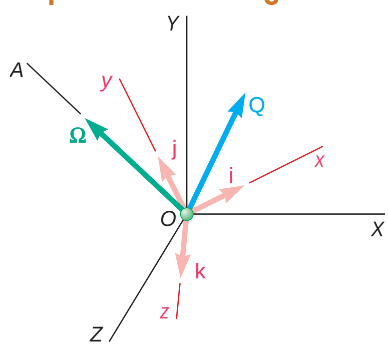


Fig. 15.45

Plane motion of a particle relative to a rotating frame

The next part of the chapter [Sec. 15.11] was devoted to the two-dimensional kinematic analysis of a particle P moving with respect to a frame $\mathcal{F}$ rotating with an angular velocity $\boldsymbol{\Omega}$ about a fixed axis (Fig. 15.46). We found that the absolute velocity of P could be expressed as

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.33}$$

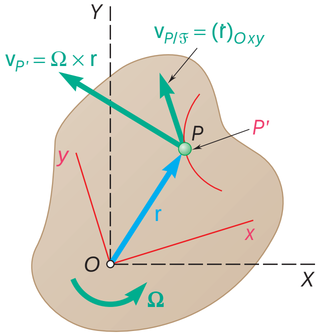


Fig. 15.46

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame $\mathcal{F}$

We noted that the same expression for $\mathbf{v}_P$ is obtained if the frame is in translation rather than in rotation. However, when the frame is in rotation, the expression for the acceleration of P is found to contain an additional term $\mathbf{a}_c$ called the *complementary acceleration* or *Coriolis acceleration*. We wrote

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.36}$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame $\mathcal{F}$
$\mathbf{a}_c = 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxy} = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration

Since $\boldsymbol{\Omega}$ and $\mathbf{v}_{P/\mathcal{F}}$ are perpendicular to each other in the case of plane motion, the Coriolis acceleration was found to have a magnitude $a_c = 2\Omega v_{P/\mathcal{F}}$ and to point in the direction obtained by rotating the vector $\mathbf{v}_{P/\mathcal{F}}$ through 90° in the sense of rotation of the moving frame. Formulas (15.33) and (15.36) can be used to analyze the motion of mechanisms which contain parts sliding on each other [Sample Probs. 15.9 and 15.10].

Motion of a rigid body with a fixed point

The last part of the chapter was devoted to the study of the kinematics of rigid bodies in three dimensions. We first considered the motion of a rigid body with a fixed point [Sec. 15.12]. After proving that the most general displacement of a rigid body with a fixed point O is equivalent to a rotation of the body about an axis through O, we were able to define the angular velocity V and the *instantaneous axis of rotation* of the body at a given instant. The velocity of a point P of the body (Fig. 15.47) could again be expressed as

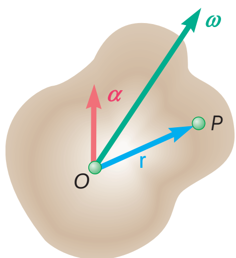


Fig. 15.47

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathsf{V} \times \mathbf{r} \tag{15.37}$$

Differentiating this expression, we also wrote

$$\mathbf{a} = \mathsf{A} \times \mathbf{r} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r}) \tag{15.38}$$

However, since the direction of V changes from one instant to the next, the angular acceleration A is, in general, not directed along the instantaneous axis of rotation [Sample Prob. 15.11].

General motion in space

It was shown in Sec. 15.13 that *the most general motion of a rigid body in space is equivalent, at any given instant, to the sum of a translation and a rotation.* Considering two particles A and B of the body, we found that

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.42}$$

where $\mathbf{v}_{B/A}$ is the velocity of B relative to a frame $AX'Y'Z'$ attached to A and of fixed orientation (Fig. 15.48). Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, we wrote

$$\mathbf{v}_B = \mathbf{v}_A + \mathrm{V} \times \mathbf{r}_{B/A} \tag{15.43}$$

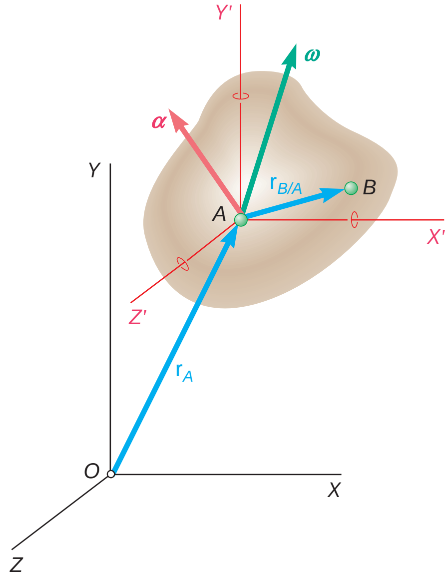


Fig. 15.48

where V is the angular velocity of the body at the instant considered [Sample Prob. 15.12]. The acceleration of B was obtained by a similar reasoning. We first wrote

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

and, recalling Eq. (15.38),

$$\mathbf{a}_B = \mathbf{a}_A + \mathrm{A} \times \mathbf{r}_{B/A} + \mathrm{V} \times (\mathrm{V} \times \mathbf{r}_{B/A}) \tag{15.44}$$

Three-dimensional motion of a particle relative to a rotating frame

In the final two sections of the chapter we considered the three-dimensional motion of a particle P relative to a frame $Oxyz$ rotating with an angular velocity $\boldsymbol{\Omega}$ with respect to a fixed frame $OXYZ$ (Fig. 15.49). In Sec. 15.14 we expressed the absolute velocity $\mathbf{v}_P$ of P as

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.46}$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame $\mathcal{F}$

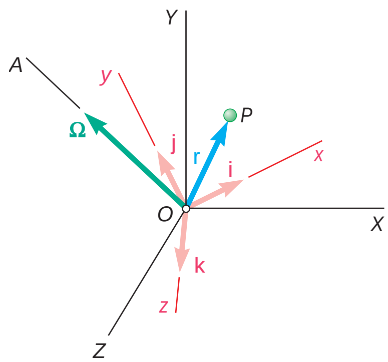


Fig. 15.49

The absolute acceleration $\mathbf{a}_P$ of P was then expressed as

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \qquad (15.48)$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame $\mathcal{F}$
$\mathbf{a}_c = 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration

It was noted that the magnitude a_c of the Coriolis acceleration is not equal to $2\Omega v_{P/\mathcal{F}}$ [Sample Prob. 15.13] except in the special case when $\boldsymbol{\Omega}$ and $\mathbf{v}_{P/\mathcal{F}}$ are perpendicular to each other.

Frame of reference in general motion

We also observed [Sec. 15.15] that Eqs. (15.46) and (15.48) remain valid when the frame $Axyz$ moves in a known, but arbitrary, fashion with respect to the fixed frame $OXYZ$ (Fig. 15.50), provided that the motion of A is included in the terms $\mathbf{v}_{P'}$ and $\mathbf{a}_{P'}$ representing the absolute velocity and acceleration of the coinciding point P'.

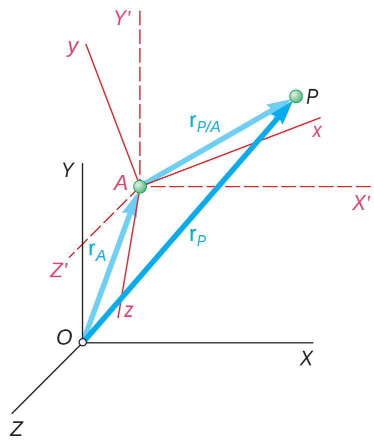


Fig. 15.50

Rotating frames of reference are particularly useful in the study of the three-dimensional motion of rigid bodies. Indeed, there are many cases where an appropriate choice of the rotating frame will lead to a simpler analysis of the motion of the rigid body than would be possible with axes of fixed orientation [Sample Probs. 15.14 and 15.15].

REVIEW PROBLEMS

15.248 The angular acceleration of the 600-mm-radius circular plate shown is defined by the relation $\alpha = \alpha_0 e^{-t}$. Knowing that the plate is at rest when $t = 0$ and that $\alpha_0 = 10$ rad/s^2, determine the magnitude of the total acceleration of point B when (*a*) $t = 0$, (*b*) $t = 0.5$ s, (*c*) $t = \infty$.

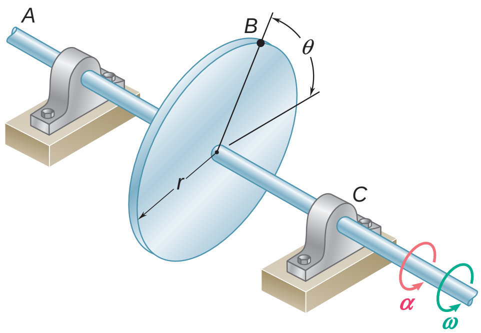


Fig. P15.248

15.249 Cylinder A is moving downward with a velocity of 9 ft/s when the brake is suddenly applied to the drum. Knowing that the cylinder moves 18 ft downward before coming to rest and assuming uniformly accelerated motion, determine (*a*) the angular acceleration of the drum, (*b*) the time required for the cylinder to come to rest.

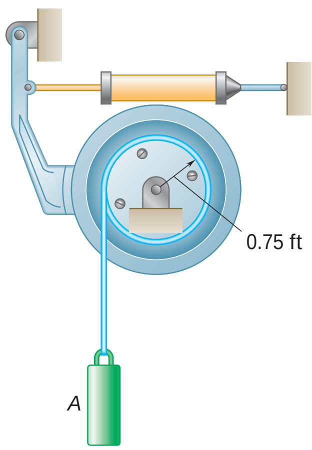


Fig. P15.249

15.250 A baseball pitching machine is designed to deliver a baseball with a ball speed of 70 mph and a ball rotation of 300 rpm clockwise. Knowing that there is no slipping between the wheels and the baseball during the ball launch, determine the angular velocities of wheels A and B.

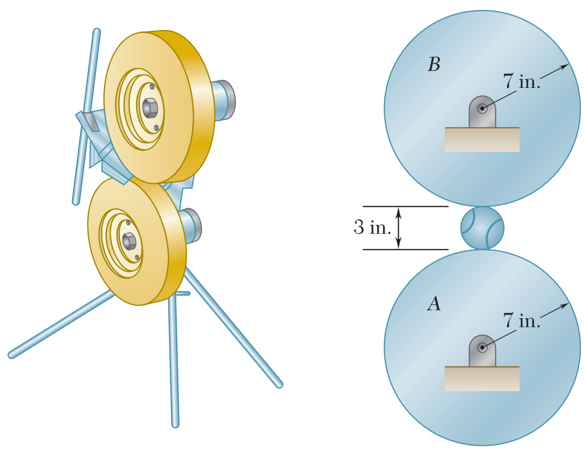


Fig. *P15.250*

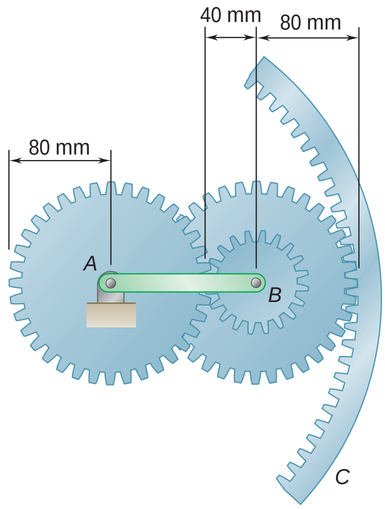


Fig. *P15.251*

15.251 Knowing that inner gear A is stationary and outer gear C starts from rest and has a constant angular acceleration of 4 rad/s^2 clockwise, determine at $t = 5$ s (*a*) the angular velocity of arm AB, (*b*) the angular velocity of gear B, (*c*) the acceleration of the point on gear B that is in contact with gear A.

15.252 Knowing that at the instant shown bar AB has an angular velocity of 10 rad/s clockwise and it is slowing down at a rate of 2 rad/s^2, determine the angular accelerations of bar BD and bar DE.

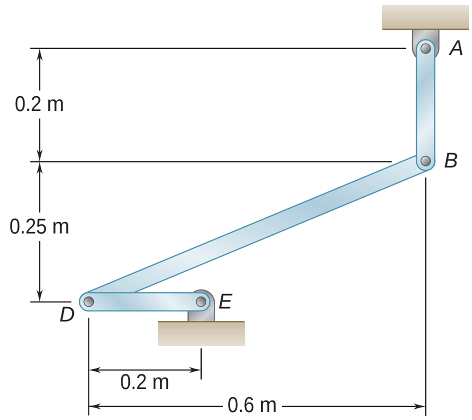


Fig. P15.252

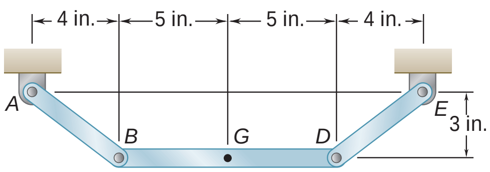


Fig. P15.253

15.253 Knowing that at the instant shown rod AB has zero angular acceleration and an angular velocity of 15 rad/s counterclockwise, determine (*a*) the angular acceleration of arm DE, (*b*) the acceleration of point D.

15.254 Rod AB is attached to a collar at A and is fitted with a wheel at B that has a radius $r = 15$ mm. Knowing that when u = 60° the collar has a velocity of 250 mm/s upward and it is slowing down at a rate of 150 mm/s^2, determine (*a*) the angular acceleration of rod AB, (*b*) the angular acceleration of the wheel.

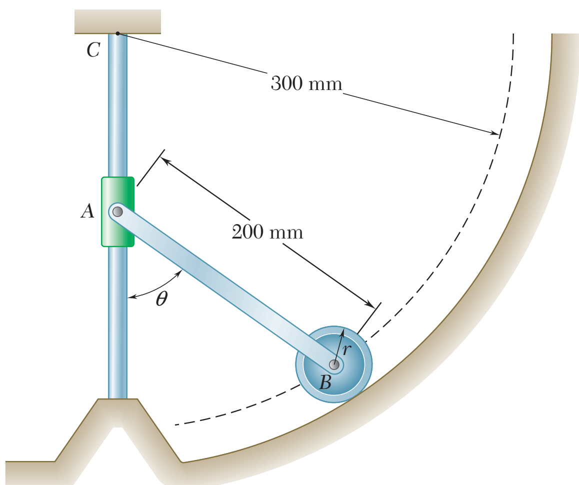


Fig. *P15.254*

15.255 Water flows through a curved pipe AB that rotates with a constant clockwise angular velocity of 90 rpm. If the velocity of the water relative to the pipe is 8 m/s, determine the total acceleration of a particle of water at point P.

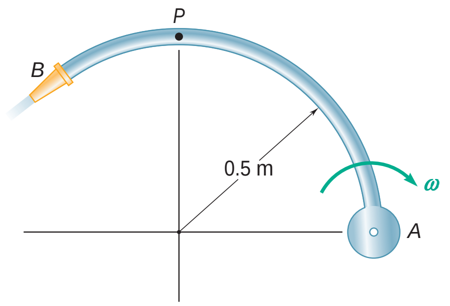


Fig. P15.255

15.256 A disk of 0.15-m radius rotates at the constant rate ω_2 with respect to plate BC, which itself rotates at the constant rate ω_1 about the y axis. Knowing that $\omega_1 = \omega_2 = 3$ rad/s, determine, for the position shown, the velocity and acceleration (*a*) of point D, (*b*) of point F.

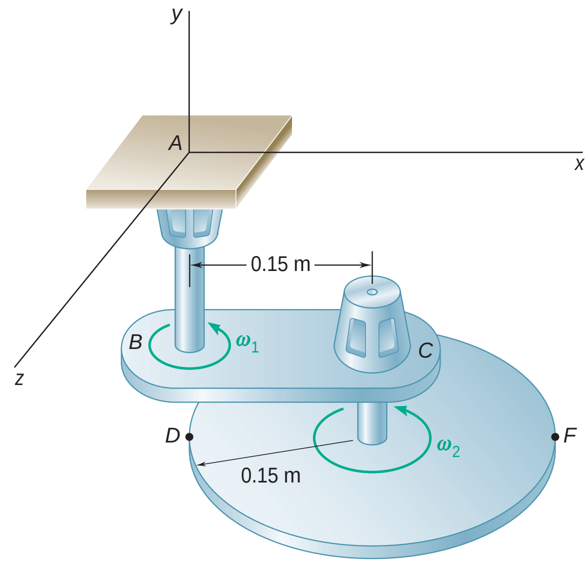


Fig. P15.256

15.257 Two rods AE and BD pass through holes drilled into a hexagonal block. (The holes are drilled in different planes so that the rods will not touch each other.) Knowing that rod AE has an angular velocity of 20 rad/s clockwise and an angular acceleration of 4 rad/s^2 counterclockwise when $\theta = 90°$, determine (*a*) the relative velocity of the block with respect to each rod, (*b*) the relative acceleration of the block with respect to each rod.

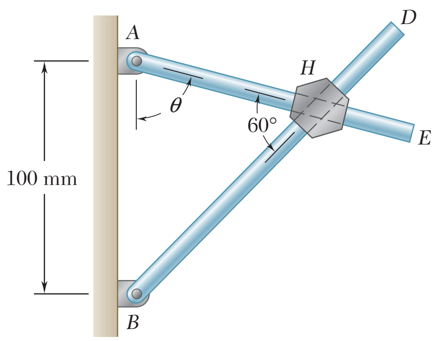


Fig. *P15.257*

15.258 Rod BC of length 24 in. is connected by ball-and-socket joints to a rotating arm AB and to a collar C that slides on the fixed rod DE. Knowing that the length of arm AB is 4 in. and that it rotates at the constant rate $\omega_1 = 10$ rad/s, determine the velocity of collar C when $\theta = 0$.

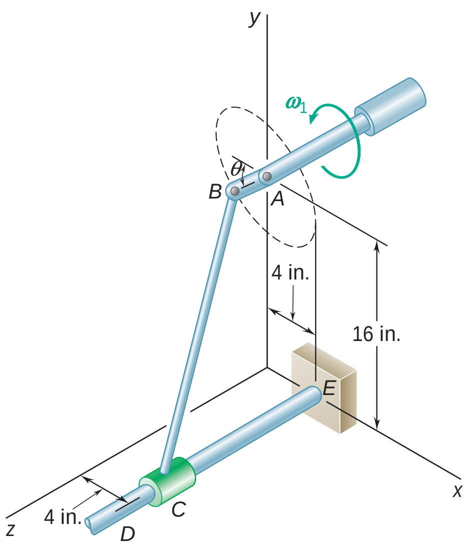


Fig. P15.258

15.259 In the position shown the thin rod moves at a constant speed $u = 3$ in./s out of the tube BC. At the same time tube BC rotates at the constant rate $v_2 = 1.5$ rad/s with respect to arm CD. Knowing that the entire assembly rotates about the X axis at the constant rate $v_1 = 1.2$ rad/s, determine the velocity and acceleration of end A of the rod.

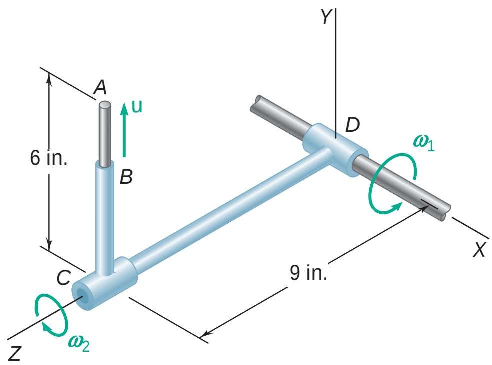


Fig. P15.259

15.C1 The disk shown has a constant angular velocity of 500 rpm counterclockwise. Knowing that rod BD is 250 mm long, use computational software to determine and plot for values of θ from 0 to 360° and using 30° increments, the velocity of collar D and the angular velocity of rod BD. Determine the two values of θ for which the speed of collar D is zero.

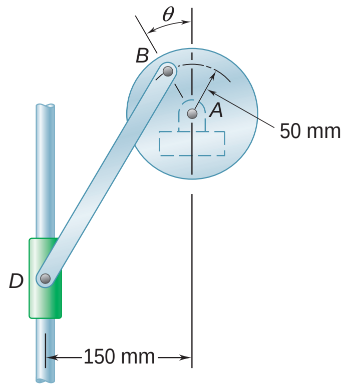


Fig. P15.C1

15.C2 Two rotating rods are connected by a slider block P as shown. Knowing that rod BP rotates with a constant angular velocity of 6 rad/s counterclockwise, use computational software to determine and plot for values of θ from 0 to 180° the angular velocity and angular acceleration of rod AE. Determine the value of θ for which the angular acceleration α_{AE} of rod AE is maximum and the corresponding value of α_{AE}.

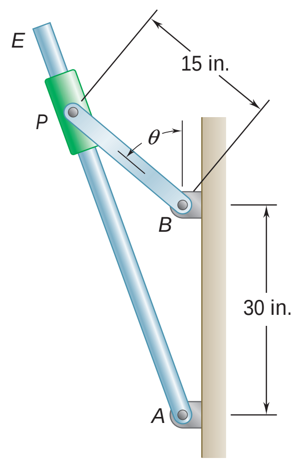


Fig. P15.C2

15.C3 In the engine system shown, l = 160 mm and b = 60 mm. Knowing that crank AB rotates with a constant angular velocity of 1000 rpm clockwise, use computational software to determine and plot for values of u from 0 to 180° and using 10° increments, (*a*) the angular velocity and angular acceleration of rod BD, (*b*) the velocity and acceleration of the piston P.

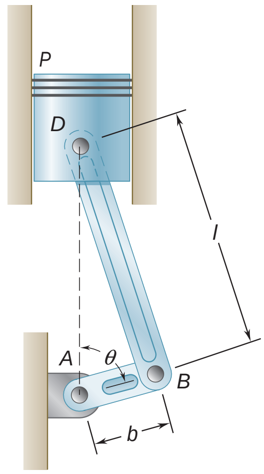


Fig. P15.C3

15.C4 Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity of 180 mm/s. Use computational software to determine and plot for values of u from 20° to 90° and using 5° increments, the velocity of point B and the angular acceleration of the rod. Determine the value of u for which the angular acceleration a of the rod is maximum and the corresponding value of a.

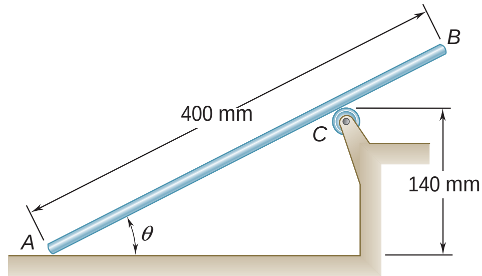


Fig. P15.C4

15.C5 Rod BC of length 24 in. is connected by ball-and-socket joints to the rotating arm AB and to collar C that slides on the fixed rod DE. Arm AB of length 4 in. rotates in the XY plane with a constant angular velocity of 10 rad/s. Use computational software to determine and plot for values of u from 0 to 360° the velocity of collar C. Determine the two values of u for which the velocity of collar C is zero.

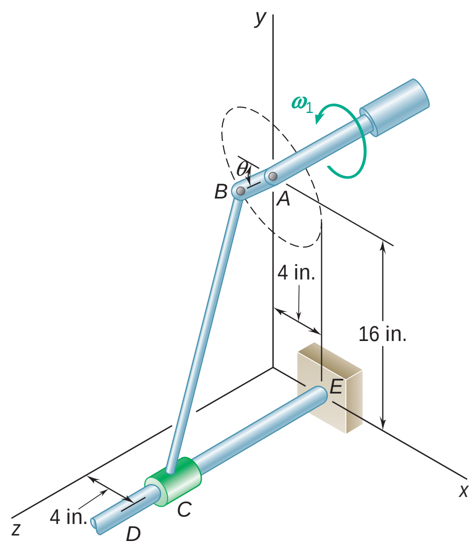


Fig. P15.C5

15.C6 Rod AB of length 25 in. is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Collar B moves toward support E at a constant speed of 20 in./s. Denoting by d the distance from point C to collar B, use computational software to determine and plot the velocity of collar A for values of d from 0 to 15 in.

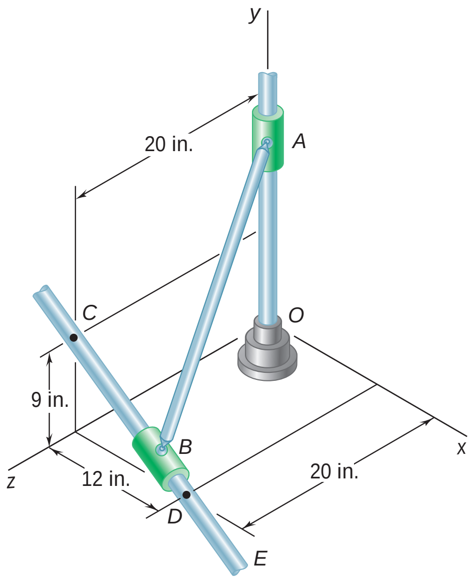
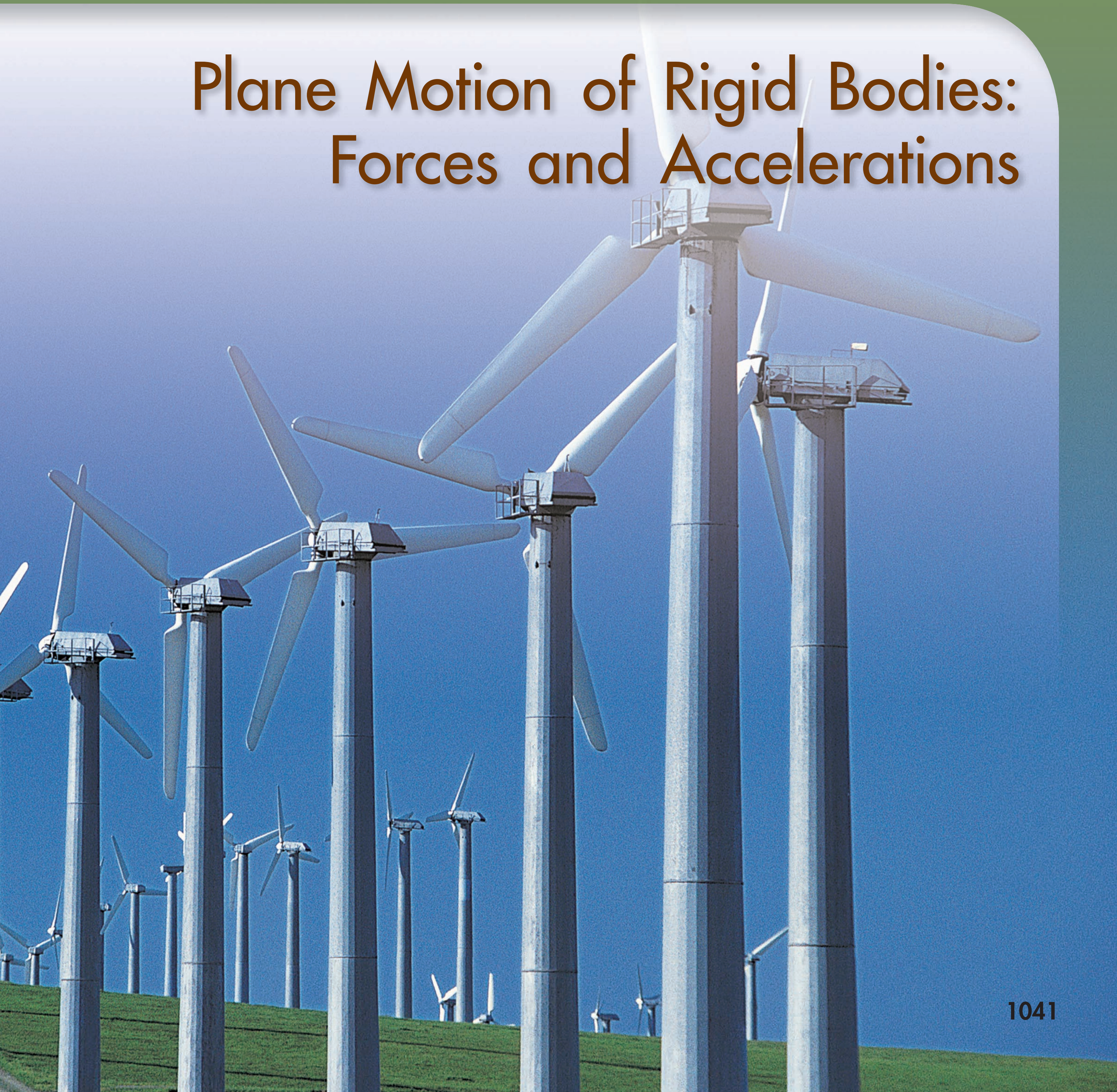


Fig. P15.C6

Three-bladed wind turbines, similar to the ones shown in this picture of a wind farm, are currently the most common design. In this chapter you will learn to analyze the motion of a rigid body by considering the motion of its mass center, the motion relative to its mass center, and the external forces acting on it.

CHAPTER 16

Plane Motion of Rigid Bodies: Forces and Accelerations



16.1 INTRODUCTION

In this chapter and in Chaps. 17 and 18, you will study the *kinetics of rigid bodies,* i.e., the relations existing between the forces acting on a rigid body, the shape and mass of the body, and the motion produced. In Chaps. 12 and 13, you studied similar relations, assuming then that the body could be considered as a particle, i.e., that its mass could be concentrated in one point and that all forces acted at that point. The shape of the body, as well as the exact location of the points of application of the forces, will now be taken into account. You will also be concerned not only with the motion of the body as a whole but also with the motion of the body about its mass center.

Our approach will be to consider rigid bodies as made of large numbers of particles and to use the results obtained in Chap. 14 for the motion of systems of particles. Specifically, two equations from Chap. 14 will be used: Eq. (14.16), $\Sigma\mathbf{F} = m\bar{\mathbf{a}}$, which relates the resultant of the external forces and the acceleration of the mass center G of the system of particles, and Eq. (14.23), $\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G$, which relates the moment resultant of the external forces and the angular momentum of the system of particles about G.

Except for Sec. 16.2, which applies to the most general case of the motion of a rigid body, the results derived in this chapter will be limited in two ways: (1) They will be restricted to the *plane motion* of rigid bodies, i.e., to a motion in which each particle of the body remains at a constant distance from a fixed reference plane. (2) The rigid bodies considered will consist only of plane slabs and of bodies which are symmetrical with respect to the reference plane.† The study of the plane motion of nonsymmetrical three-dimensional bodies and, more generally, the motion of rigid bodies in three-dimensional space will be postponed until Chap. 18.

In Sec. 16.3, we define the angular momentum of a rigid body in plane motion and show that the rate of change of the angular momentum $\dot{\mathbf{H}}_G$ about the mass center is equal to the product $\bar{I}\alpha$ of the centroidal mass moment of inertia $\bar{I}$ and the angular acceleration α of the body. D'Alembert's principle, introduced in Sec. 16.4, is used to prove that the external forces acting on a rigid body are equivalent to a vector $m\bar{\mathbf{a}}$ attached at the mass center and a couple of moment $\bar{I}\alpha$.

In Sec. 16.5, we derive the principle of transmissibility using only the parallelogram law and Newton's laws of motion, allowing us to remove this principle from the list of axioms (Sec. 1.2) required for the study of the statics and dynamics of rigid bodies.

Free-body-diagram equations are introduced in Sec. 16.6 and will be used in the solution of all problems involving the plane motion of rigid bodies.

After considering the plane motion of connected rigid bodies in Sec. 16.7, you will be prepared to solve a variety of problems involving the translation, centroidal rotation, and unconstrained motion of rigid bodies. In Sec. 16.8 and in the remaining part of the chapter, the solution of problems involving noncentroidal rotation, rolling motion, and other partially constrained plane motions of rigid bodies will be considered.

†Or, more generally, bodies which have a principal centroidal axis of inertia perpendicular to the reference plane.

16.2 EQUATIONS OF MOTION FOR A RIGID BODY

Consider a rigid body acted upon by several external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . (Fig. 16.1). We can assume that the body is made of a large number n of particles of mass Δm_i ($i = 1, 2, \ldots, n$) and apply the results obtained in Chap. 14 for a system of particles (Fig. 16.2). Considering first the motion of the mass center G of the body with respect to the newtonian frame of reference $Oxyz$, we recall Eq. (14.16) and write

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{16.1}$$

where m is the mass of the body and $\bar{\mathbf{a}}$ is the acceleration of the mass center G. Turning now to the motion of the body relative to the centroidal frame of reference $Gx'y'z'$, we recall Eq. (14.23) and write

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{16.2}$$

where $\dot{\mathbf{H}}_G$ represents the rate of change of $\mathbf{H}_G$, the angular momentum about G of the system of particles forming the rigid body. In the following, $\mathbf{H}_G$ will simply be referred to as the *angular momentum of the rigid body about its mass center G*. Together Eqs. (16.1) and (16.2) express that *the system of the external forces is equipollent to the system consisting of the vector m$\bar{\mathbf{a}}$ attached at G and the couple of moment* $\dot{\mathbf{H}}_G$ (Fig. 16.3).†

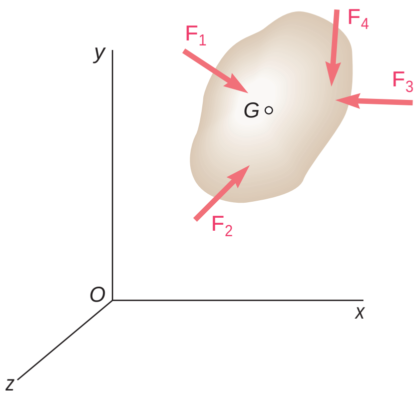


Fig. 16.1

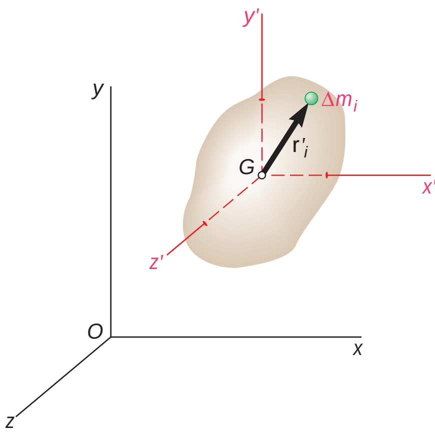


Fig. 16.2

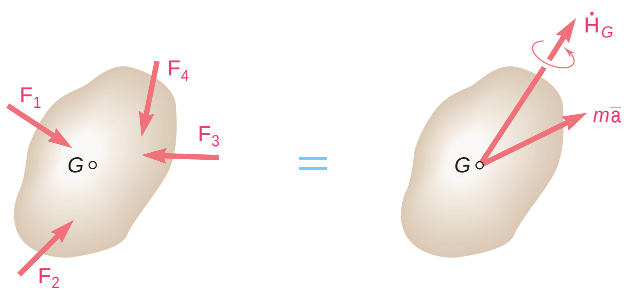


Fig. 16.3

Equations (16.1) and (16.2) apply in the most general case of the motion of a rigid body. In the rest of this chapter, however, our analysis will be limited to the *plane motion* of rigid bodies, i.e., to a motion in which each particle remains at a constant distance from a fixed reference plane, and it will be assumed that the rigid bodies considered consist only of plane slabs and of bodies which are symmetrical with respect to the reference plane. Further study of the plane motion of nonsymmetrical three-dimensional bodies and of the motion of rigid bodies in three-dimensional space will be postponed until Chap. 18.

†Since the systems involved act on a rigid body, we could conclude at this point, by referring to Sec. 3.19, that the two systems are *equivalent* as well as equipollent and use red rather than blue equals signs in Fig. 16.3. However, by postponing this conclusion, we will be able to arrive at it independently (Secs. 16.4 and 18.5), thereby eliminating the necessity of including the principle of transmissibility among the axioms of mechanics (Sec. 16.5).

Photo 16.1 The system of external forces acting on the man and wakeboard includes the weights, the tension in the tow rope, and the forces exerted by the water and the air.

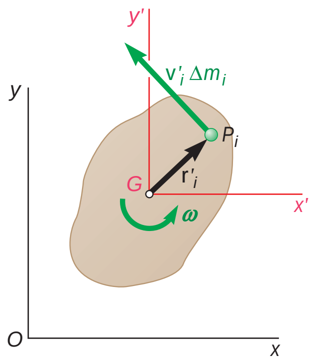


Fig. 16.4

16.3 ANGULAR MOMENTUM OF A RIGID BODY IN PLANE MOTION

Consider a rigid slab in plane motion. Assuming that the slab is made of a large number n of particles P_i of mass Δm_i and recalling Eq. (14.24) of Sec. 14.5, we note that the angular momentum $\mathbf{H}_G$ of the slab about its mass center G can be computed by taking the moments about G of the momenta of the particles of the slab in their motion with respect to either of the frames Oxy or $Gx'y'$ (Fig. 16.4). Choosing the latter course, we write

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times \mathbf{v}'_i \,\Delta m_i) \tag{16.3}$$

where $\mathbf{r}'_i$ and $\mathbf{v}'_i \,\Delta m_i$ denote, respectively, the position vector and the linear momentum of the particle P_i relative to the centroidal frame of reference $Gx'y'$. But since the particle belongs to the slab, we have $\mathbf{v}'_i = \mathsf{V} \times \mathbf{r}'_i$, where V is the angular velocity of the slab at the instant considered. We write

$$\mathbf{H}_G = \sum_{i=1}^{n} [\mathbf{r}'_i \times (\mathsf{V} \times \mathbf{r}'_i)\,\Delta m_i]$$

Referring to Fig. 16.4, we easily verify that the expression obtained represents a vector of the same direction as V (i.e., perpendicular to the slab) and of magnitude equal to $\mathsf{v}\Sigma r_i'^2\,\Delta m_i$. Recalling that the sum $\Sigma r_i'^2\,\Delta m_i$ represents the moment of inertia $\bar{I}$ of the slab about a centroidal axis perpendicular to the slab, we conclude that the angular momentum $\mathbf{H}_G$ of the slab about its mass center is

$$\mathbf{H}_G = \bar{I}\mathsf{V} \tag{16.4}$$

Differentiating both members of Eq. (16.4) we obtain

$$\dot{\mathbf{H}}_G = \bar{I}\dot{\mathsf{V}} = \bar{I}\mathsf{A} \tag{16.5}$$

Thus the rate of change of the angular momentum of the slab is represented by a vector of the same direction as A (i.e., perpendicular to the slab) and of magnitude $\bar{I}\mathsf{a}$.

It should be kept in mind that the results obtained in this section have been derived for a rigid slab in plane motion. As you will see in Chap. 18, they remain valid in the case of the plane motion of rigid bodies which are symmetrical with respect to the reference plane.† However, they do not apply in the case of nonsymmetrical bodies or in the case of three-dimensional motion.

Photo 16.2 The hard disk and pick-up arms of a hard disk computer undergo centroidal rotation.

†Or, more generally, bodies which have a principal centroidal axis of inertia perpendicular to the reference plane.

16.4 PLANE MOTION OF A RIGID BODY. D'ALEMBERT'S PRINCIPLE

Consider a rigid slab of mass m moving under the action of several external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , contained in the plane of the slab (Fig. 16.5). Substituting for $\dot{\mathbf{H}}_G$ from Eq. (16.5) into Eq. (16.2) and writing the fundamental equations of motion (16.1) and (16.2) in scalar form, we have

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma M_G = \bar{I}\alpha \tag{16.6}$$

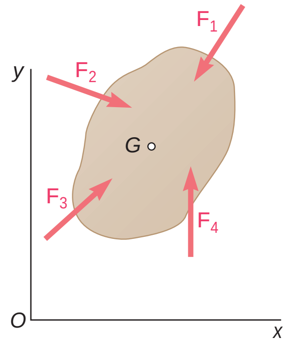


Fig. 16.5

Equations (16.6) show that the acceleration of the mass center G of the slab and its angular acceleration α are easily obtained once the resultant of the external forces acting on the slab and their moment resultant about G have been determined. Given appropriate initial conditions, the coordinates $\bar{x}$ and $\bar{y}$ of the mass center and the angular coordinate θ of the slab can then be obtained by integration at any instant t. Thus *the motion of the slab is completely defined by the resultant and moment resultant about G of the external forces acting on it*.

This property, which will be extended in Chap. 18 to the case of the three-dimensional motion of a rigid body, is characteristic of the motion of a rigid body. Indeed, as we saw in Chap. 14, the motion of a system of particles which are not rigidly connected will in general depend upon the specific external forces acting on the various particles, as well as upon the internal forces.

Since the motion of a rigid body depends only upon the resultant and moment resultant of the external forces acting on it, it follows that *two systems of forces which are equipollent,* i.e., which have the same resultant and the same moment resultant, *are also equivalent;* that is, they have exactly the same effect on a given rigid body.†

Consider in particular the system of the external forces acting on a rigid body (Fig. 16.6*a*) and the system of the effective forces associated with the particles forming the rigid body (Fig. 16.6*b*). It was shown in Sec. 14.2 that the two systems thus defined are equipollent. But since the particles considered now form a rigid body, it follows from the discussion above that the two systems are also equivalent. We can thus state that *the external forces acting on a rigid body are equivalent to the effective forces of the various particles forming the body*. This statement is referred to as *d'Alembert's principle* after the French mathematician Jean le Rond d'Alembert (1717–1783), even though d'Alembert's original statement was written in a somewhat different form.

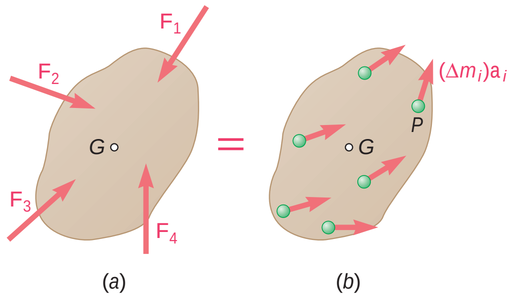


Fig. 16.6

The fact that the system of external forces is *equivalent* to the system of the effective forces has been emphasized by the use of a red equals sign in Fig. 16.6 and also in Fig. 16.7, where using results obtained earlier in this section, we have replaced the effective forces by a vector $m\bar{\mathbf{a}}$ attached at the mass center G of the slab and a couple of moment $\bar{I}\alpha$.

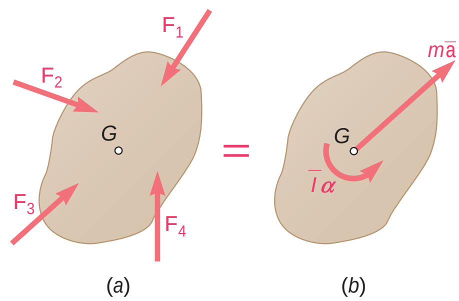


Fig. 16.7

†This result has already been derived in Sec. 3.19 from the principle of transmissibility (Sec. 3.3). The present derivation is independent of that principle, however, and will make possible its elimination from the axioms of mechanics (Sec. 16.5).

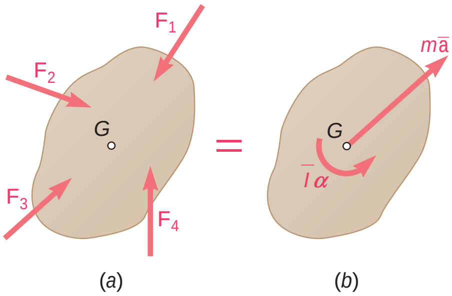


Fig. 16.7 (repeated)

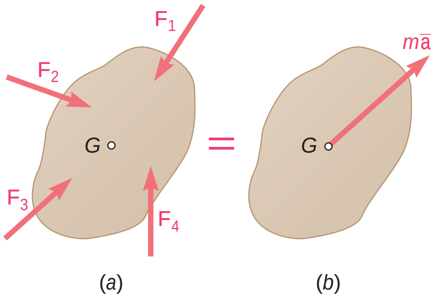


Fig. 16.8 Translation.

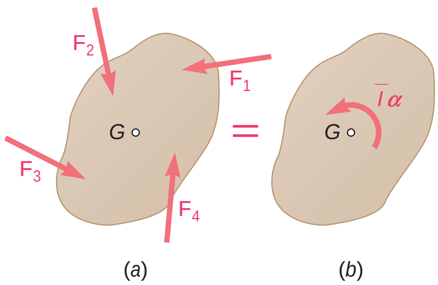


Fig. 16.9 Centroidal rotation.

Translation. In the case of a body in translation, the angular acceleration of the body is identically equal to zero and its effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G (Fig. 16.8). Thus, the resultant of the external forces acting on a rigid body in translation passes through the mass center of the body and is equal to $m\bar{\mathbf{a}}$.

Centroidal Rotation. When a slab, or, more generally, a body symmetrical with respect to the reference plane, rotates about a fixed axis perpendicular to the reference plane and passing through its mass center G, we say that the body is in *centroidal rotation*. Since the acceleration $\bar{\mathbf{a}}$ is identically equal to zero, the effective forces of the body reduce to the couple $\bar{I}\alpha$ (Fig. 16.9). Thus, the external forces acting on a body in centroidal rotation are equivalent to a couple of moment $\bar{I}\alpha$.

General Plane Motion. Comparing Fig. 16.7 with Figs. 16.8 and 16.9, we observe that from the point of view of *kinetics*, the most general plane motion of a rigid body symmetrical with respect to the reference plane can be replaced by the sum of a translation and a centroidal rotation. We should note that this statement is more restrictive than the similar statement made earlier from the point of view of *kinematics* (Sec. 15.5), since we now require that the mass center of the body be selected as the reference point.

Referring to Eqs. (16.6), we observe that the first two equations are identical with the equations of motion of a particle of mass m acted upon by the given forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . We thus check that *the mass center G of a rigid body in plane motion moves as if the entire mass of the body were concentrated at that point, and as if all the external forces acted on it*. We recall that this result has already been obtained in Sec. 14.4 in the general case of a system of particles, the particles being not necessarily rigidly connected. We also note, as we did in Sec. 14.4, that the system of the external forces does not, in general, reduce to a single vector $m\bar{\mathbf{a}}$ attached at G. Therefore, in the general case of the plane motion of a rigid body, *the resultant of the external forces acting on the body does not pass through the mass center of the body*.

Finally, it should be observed that the last of Eqs. (16.6) would still be valid if the rigid body, while subjected to the same applied forces, were constrained to rotate about a fixed axis through G. Thus, *a rigid body in plane motion rotates about its mass center as if this point were fixed*.

*16.5 A REMARK ON THE AXIOMS OF THE MECHANICS OF RIGID BODIES

The fact that two equipollent systems of external forces acting on a rigid body are also equivalent, i.e., have the same effect on that rigid body, has already been established in Sec. 3.19. But there it was derived from the *principle of transmissibility*, one of the axioms used in our study of the statics of rigid bodies. It should be noted that this axiom has not been used in the present chapter because Newton's second and third laws of motion make its use unnecessary in the study of the dynamics of rigid bodies.

In fact, the principle of transmissibility may now be *derived* from the other axioms used in the study of mechanics. This principle stated, without proof (Sec. 3.3), that the conditions of equilibrium or motion of a rigid body remain unchanged if a force **F** acting at a given point of the rigid body is replaced by a force **F′** of the same magnitude and same direction, but acting at a different point, provided that the two forces have the same line of action. But since **F** and **F′** have the same moment about any given point, it is clear that they form two equipollent systems of external forces. Thus, we may now *prove,* as a result of what we established in the preceding section, that **F** and **F′** have the same effect on the rigid body (Fig. 3.3).

The principle of transmissibility can therefore be removed from the list of axioms required for the study of the mechanics of rigid bodies. These axioms are reduced to the parallelogram law of addition of vectors and to Newton's laws of motion.

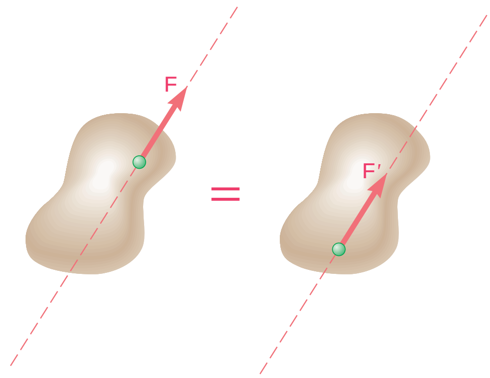


Fig. 3.3 (repeated)

16.6 SOLUTION OF PROBLEMS INVOLVING THE MOTION OF A RIGID BODY

We saw in Sec. 16.4 that when a rigid body is in plane motion, there exists a fundamental relation between the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , acting on the body, the acceleration $\bar{\mathbf{a}}$ of its mass center, and the angular acceleration α of the body. This relation, which is represented in Fig. 16.7 in the form of a *free-body-diagram equation,* can be used to determine the acceleration $\bar{\mathbf{a}}$ and the angular acceleration α produced by a given system of forces acting on a rigid body or, conversely, to determine the forces which produce a given motion of the rigid body.

The three algebraic equations (16.6) can be used to solve problems of plane motion.† However, our experience in statics suggests that the solution of many problems involving rigid bodies could be simplified by an appropriate choice of the point about which the moments of the forces are computed. It is therefore preferable to remember the relation existing between the forces and the accelerations in the pictorial form shown in Fig. 16.7 and to derive from this fundamental relation the component or moment equations which fit best the solution of the problem under consideration.

The fundamental relation shown in Fig. 16.7 can be presented in an alternative form if we add to the external forces an inertia vector $-m\bar{\mathbf{a}}$ of sense opposite to that of $\bar{\mathbf{a}}$, attached at G, and an inertia couple $-\bar{I}\alpha$ of moment equal in magnitude to $\bar{I}\alpha$ and of sense opposite to that of α (Fig. 16.10). The system obtained is equivalent to zero, and the rigid body is said to be in *dynamic equilibrium.*

Whether the principle of equivalence of external and effective forces is directly applied, as in Fig. 16.7, or whether the concept of dynamic equilibrium is introduced, as in Fig. 16.10, the use of free-body-diagram equations showing vectorially the relationship existing between the forces applied on the rigid body and the resulting linear and angular accelerations presents considerable advantages over the blind application of formulas (16.6). These advantages can be summarized as follows:

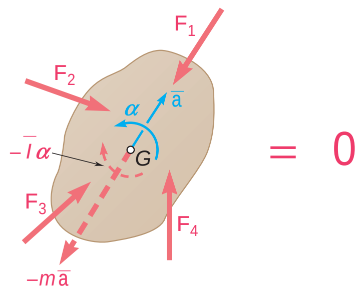


Fig. 16.10

1. The use of a pictorial representation provides a much clearer understanding of the effect of the forces on the motion of the body.

†We recall that the last of Eqs. (16.6) is valid only in the case of the plane motion of a rigid body symmetrical with respect to the reference plane. In all other cases, the methods of Chap. 18 should be used.

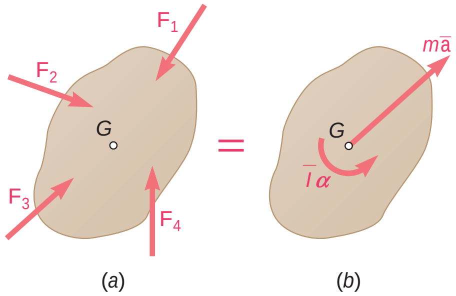


Fig. 16.7 (repeated)

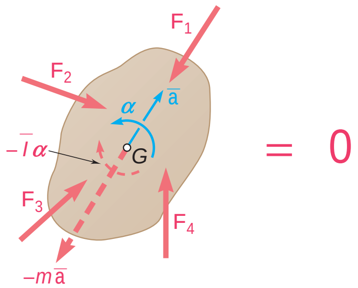


Fig. 16.10 (repeated)

2. This approach makes it possible to divide the solution of a dynamics problem into two parts: In the first part, the analysis of the kinematic and kinetic characteristics of the problem leads to the free-body diagrams of Fig. 16.7 or 16.10; in the second part, the diagram obtained is used to analyze the various forces and vectors involved by the methods of Chap. 3.
3. A unified approach is provided for the analysis of the plane motion of a rigid body, regardless of the particular type of motion involved. While the kinematics of the various motions considered may vary from one case to the other, the approach to the kinetics of the motion is consistently the same. In every case a diagram will be drawn showing the external forces, the vector $m\bar{\mathbf{a}}$ associated with the motion of G, and the couple $\bar{I}$A associated with the rotation of the body about G.
4. The resolution of the plane motion of a rigid body into a translation and a centroidal rotation, which is used here, is a basic concept which can be applied effectively throughout the study of mechanics. It will be used again in Chap. 17 with the method of work and energy and the method of impulse and momentum.
5. As you will see in Chap. 18, this approach can be extended to the study of the general three-dimensional motion of a rigid body. The motion of the body will again be resolved into a translation and a rotation about the mass center, and free-body-diagram equations will be used to indicate the relationship existing between the external forces and the rates of change of the linear and angular momentum of the body.

16.7 SYSTEMS OF RIGID BODIES

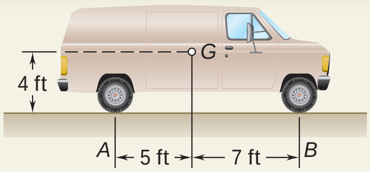
The method described in the preceding section can also be used in problems involving the plane motion of several connected rigid bodies. For each part of the system, a diagram similar to Fig. 16.7 or Fig. 16.10 can be drawn. The equations of motion obtained from these diagrams are solved simultaneously.

In some cases, as in Sample Prob. 16.3, a single diagram can be drawn for the entire system. This diagram should include all the external forces, as well as the vectors $m\bar{\mathbf{a}}$ and the couples $\bar{I}$A associated with the various parts of the system. However, internal forces such as the forces exerted by connecting cables, can be omitted since they occur in pairs of equal and opposite forces and are thus equipollent to zero. The equations obtained by expressing that the system of the external forces is equipollent to the system of the effective forces can be solved for the remaining unknowns.†

It is not possible to use this second approach in problems involving more than three unknowns, since only three equations of motion are available when a single diagram is used. We need not elaborate upon this point, since the discussion involved would be completely similar to that given in Sec. 6.11 in the case of the equilibrium of a system of rigid bodies.

Photo 16.3 The forklift and moving load can be analyzed as a system of two connected rigid bodies in plane motion.

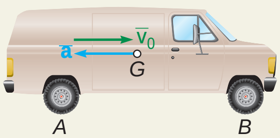
†Note that we cannot speak of *equivalent* systems since we are not dealing with a single rigid body.



SAMPLE PROBLEM 16.1

When the forward speed of the truck shown was 30 ft/s, the brakes were suddenly applied, causing all four wheels to stop rotating. It was observed that the truck skidded to rest in 20 ft. Determine the magnitude of the normal reaction and of the friction force at each wheel as the truck skidded to rest.

SOLUTION

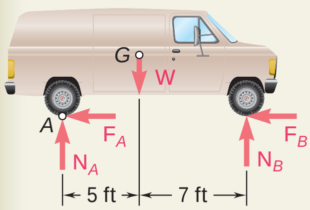


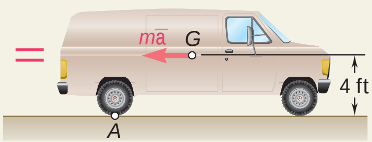
Kinematics of Motion. Choosing the positive sense to the right and using the equations of uniformly accelerated motion, we write

$$\bar{v}_0 = +30 \text{ ft/s} \qquad \bar{v}^2 = \bar{v}_0^2 + 2\bar{a}\bar{x} \qquad 0 = (30)^2 + 2\bar{a}(20)$$

$$\bar{a} = -22.5 \text{ ft/s}^2 \qquad \bar{\mathbf{a}} = 22.5 \text{ ft/s}^2 \leftarrow$$

Equations of Motion. The external forces consist of the weight **W** of the truck and of the normal reactions and friction forces at the wheels. (The vectors $\mathbf{N}_A$ and $\mathbf{F}_A$ represent the sum of the reactions at the rear wheels, while $\mathbf{N}_B$ and $\mathbf{F}_B$ represent the sum of the reactions at the front wheels.) Since the truck is in translation, the effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G. Three equations of motion are obtained by expressing that the system of the external forces is equivalent to the system of the effective forces.





$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N_A + N_B - W = 0$$

Since $F_A = \mu_k N_A$ and $F_B = \mu_k N_B$, where μ_k is the coefficient of kinetic friction, we find that

$$F_A + F_B = \mu_k(N_A + N_B) = \mu_k W$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad -(F_A + F_B) = -m\bar{a}$$

$$-\mu_k W = -\frac{W}{32.2 \text{ ft/s}^2}(22.5 \text{ ft/s}^2)$$

$$\mu_k = 0.699$$

$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \qquad -W(5 \text{ ft}) + N_B(12 \text{ ft}) = m\bar{a}(4 \text{ ft})$$

$$-W(5 \text{ ft}) + N_B(12 \text{ ft}) = \frac{W}{32.2 \text{ ft/s}^2}(22.5 \text{ ft/s}^2)(4 \text{ ft})$$

$$N_B = 0.650W$$

$$F_B = \mu_k N_B = (0.699)(0.650W) \qquad F_B = 0.454W$$

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N_A + N_B - W = 0$$

$$N_A + 0.650W - W = 0$$

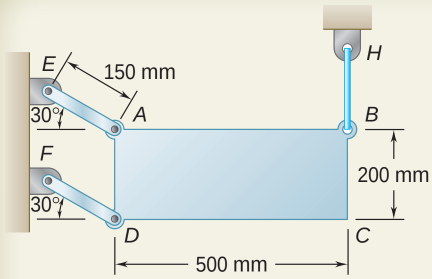
$$N_A = 0.350W$$

$$F_A = \mu_k N_A = (0.699)(0.350W) \qquad F_A = 0.245W$$

Reactions at Each Wheel. Recalling that the values computed above represent the sum of the reactions at the two front wheels or the two rear wheels, we obtain the magnitude of the reactions at each wheel by writing

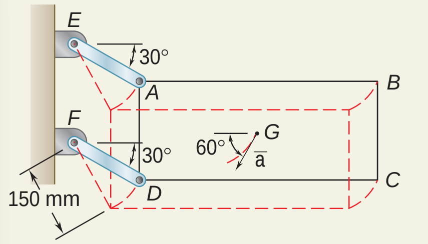
$$N_{\text{front}} = \tfrac{1}{2}N_B = 0.325W \qquad N_{\text{rear}} = \tfrac{1}{2}N_A = 0.175W \blacktriangleleft$$

$$F_{\text{front}} = \tfrac{1}{2}F_B = 0.227W \qquad F_{\text{rear}} = \tfrac{1}{2}F_A = 0.122W \blacktriangleleft$$



SAMPLE PROBLEM 16.2

The thin plate $ABCD$ of mass 8 kg is held in the position shown by the wire BH and two links AE and DF. Neglecting the mass of the links, determine immediately after wire BH has been cut (a) the acceleration of the plate, (b) the force in each link.

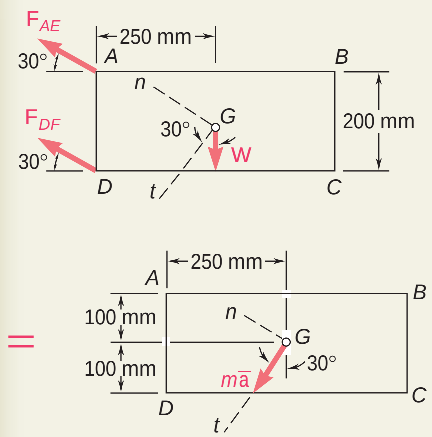


SOLUTION

Kinematics of Motion. After wire BH has been cut, we observe that corners A and D move along parallel circles of radius 150 mm centered, respectively, at E and F. The motion of the plate is thus a curvilinear translation; the particles forming the plate move along parallel circles of radius 150 mm.

At the instant wire BH is cut, the velocity of the plate is zero. Thus the acceleration $\bar{\mathbf{a}}$ of the mass center G of the plate is tangent to the circular path which will be described by G.

Equations of Motion. The external forces consist of the weight $\mathbf{W}$ and the forces $\mathbf{F}_{AE}$ and $\mathbf{F}_{DF}$ exerted by the links. Since the plate is in translation, the effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G and directed along the t axis. A free-body-diagram equation is drawn to show that the system of the external forces is equivalent to the system of the effective forces.



a. Acceleration of the Plate.

$+\swarrow \Sigma F_t = \Sigma (F_t)_{\text{eff}}$:

$$W \cos 30° = m\bar{a}$$
$$mg \cos 30° = m\bar{a}$$
$$\bar{a} = g \cos 30° = (9.81\ \text{m/s}^2) \cos 30° \qquad (1)$$
$$\bar{\mathbf{a}} = 8.50\ \text{m/s}^2 \ \text{d}\ 60°$$

b. Forces in Links AE and DF.

$+\nwarrow \Sigma F_n = \Sigma (F_n)_{\text{eff}}$: $\quad F_{AE} + F_{DF} - W \sin 30° = 0 \qquad (2)$

$+\text{i}\,\Sigma M_G = \Sigma (M_G)_{\text{eff}}$:

$$(F_{AE} \sin 30°)(250\ \text{mm}) - (F_{AE} \cos 30°)(100\ \text{mm})$$
$$+ (F_{DF} \sin 30°)(250\ \text{mm}) + (F_{DF} \cos 30°)(100\ \text{mm}) = 0$$
$$38.4 F_{AE} + 211.6 F_{DF} = 0$$
$$F_{DF} = -0.1815 F_{AE} \qquad (3)$$

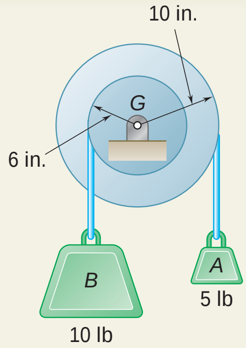
Substituting for F_{DF} from (3) into (2), we write

$$F_{AE} - 0.1815 F_{AE} - W \sin 30° = 0$$
$$F_{AE} = 0.6109 W$$
$$F_{DF} = -0.1815(0.6109 W) = -0.1109 W$$

Noting that $W = mg = (8\ \text{kg})(9.81\ \text{m/s}^2) = 78.48$ N, we have

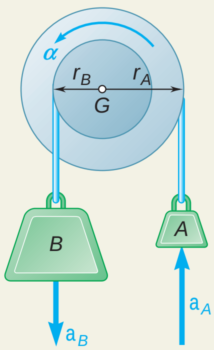
$$F_{AE} = 0.6109(78.48\ \text{N}) \qquad F_{AE} = 47.9\ \text{N}\ T$$
$$F_{DF} = -0.1109(78.48\ \text{N}) \qquad F_{DF} = 8.70\ \text{N}\ C$$

SAMPLE PROBLEM 16.3



A pulley weighing 12 lb and having a radius of gyration of 8 in. is connected to two blocks as shown. Assuming no axle friction, determine the angular acceleration of the pulley and the acceleration of each block.

SOLUTION



Sense of Motion. Although an arbitrary sense of motion can be assumed (since no friction forces are involved) and later checked by the sign of the answer, we may prefer to determine the actual sense of rotation of the pulley first. The weight of block B required to maintain the equilibrium of the pulley when it is acted upon by the 5-lb block A is first determined. We write

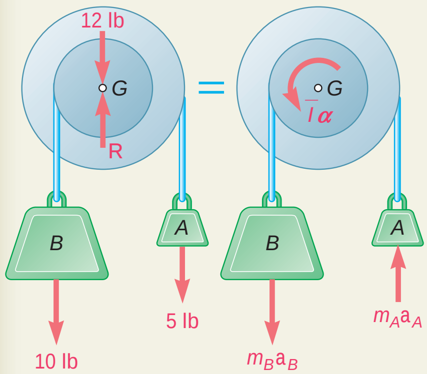
$$+\circlearrowleft \Sigma M_G = 0: \qquad W_B(6 \text{ in.}) - (5 \text{ lb})(10 \text{ in.}) = 0 \qquad W_B = 8.33 \text{ lb}$$

Since block B actually weighs 10 lb, the pulley will rotate counterclockwise.

Kinematics of Motion. Assuming α counterclockwise and noting that $a_A = r_A\alpha$ and $a_B = r_B\alpha$, we obtain

$$\mathbf{a}_A = (\tfrac{10}{12} \text{ ft})\alpha \uparrow \qquad \mathbf{a}_B = (\tfrac{6}{12} \text{ ft})\alpha \downarrow$$

Equations of Motion. A single system consisting of the pulley and the two blocks is considered. Forces external to this system consist of the weights of the pulley and the two blocks and of the reaction at G. (The forces exerted by the cables on the pulley and on the blocks are internal to the system considered and cancel out.) Since the motion of the pulley is a centroidal rotation and the motion of each block is a translation, the effective forces reduce to the couple $\bar{I}\alpha$ and the two vectors $m\mathbf{a}_A$ and $m\mathbf{a}_B$. The centroidal moment of inertia of the pulley is



$$\bar{I} = m\bar{k}^2 = \frac{W}{g}\bar{k}^2 = \frac{12 \text{ lb}}{32.2 \text{ ft/s}^2}(\tfrac{8}{12} \text{ ft})^2 = 0.1656 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Since the system of the external forces is equipollent to the system of the effective forces, we write

$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}:$$

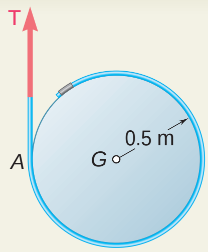
$$(10 \text{ lb})(\tfrac{6}{12} \text{ ft}) - (5 \text{ lb})(\tfrac{10}{12} \text{ ft}) = +\bar{I}\alpha + m_B a_B(\tfrac{6}{12} \text{ ft}) + m_A a_A(\tfrac{10}{12} \text{ ft})$$

$$(10)(\tfrac{6}{12}) - (5)(\tfrac{10}{12}) = 0.1656\alpha + \tfrac{10}{32.2}(\tfrac{6}{12}\alpha)(\tfrac{6}{12}) + \tfrac{5}{32.2}(\tfrac{10}{12}\alpha)(\tfrac{10}{12})$$

$$\alpha = +2.374 \text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 2.37 \text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

$$a_A = r_A\alpha = (\tfrac{10}{12} \text{ ft})(2.374 \text{ rad/s}^2) \qquad \mathbf{a}_A = 1.978 \text{ ft/s}^2 \uparrow \blacktriangleleft$$

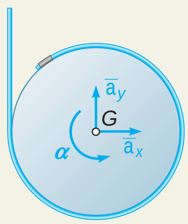
$$a_B = r_B\alpha = (\tfrac{6}{12} \text{ ft})(2.374 \text{ rad/s}^2) \qquad \mathbf{a}_B = 1.187 \text{ ft/s}^2 \downarrow \blacktriangleleft$$



SAMPLE PROBLEM 16.4

A cord is wrapped around a homogeneous disk of radius $r = 0.5$ m and mass $m = 15$ kg. If the cord is pulled upward with a force $\mathbf{T}$ of magnitude 180 N, determine (*a*) the acceleration of the center of the disk, (*b*) the angular acceleration of the disk, (*c*) the acceleration of the cord.

SOLUTION



Equations of Motion. We assume that the components $\bar{\mathbf{a}}_x$ and $\bar{\mathbf{a}}_y$ of the acceleration of the center are directed, respectively, to the right and upward and that the angular acceleration of the disk is counterclockwise. The external forces acting on the disk consist of the weight $\mathbf{W}$ and the force $\mathbf{T}$ exerted by the cord. This system is equivalent to the system of the effective forces, which consists of a vector of components $m\bar{\mathbf{a}}_x$ and $m\bar{\mathbf{a}}_y$ attached at G and a couple $\bar{I}\boldsymbol{\alpha}$. We write

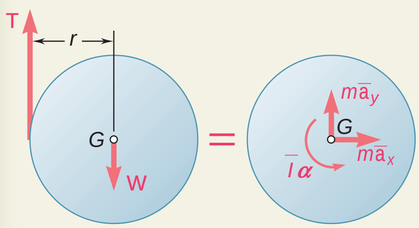
$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad 0 = m\bar{a}_x \qquad \bar{\mathbf{a}}_x = 0 \blacktriangleleft$$

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad T - W = m\bar{a}_y$$

$$\bar{a}_y = \frac{T - W}{m}$$

Since $T = 180$ N, $m = 15$ kg, and $W = (15 \text{ kg})(9.81 \text{ m/s}^2) = 147.1$ N, we have

$$\bar{a}_y = \frac{180 \text{ N} - 147.1 \text{ N}}{15 \text{ kg}} = +2.19 \text{ m/s}^2 \qquad \bar{\mathbf{a}}_y = 2.19 \text{ m/s}^2 \uparrow \blacktriangleleft$$

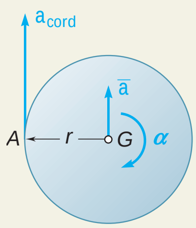


$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \qquad -Tr = \bar{I}\alpha$$

$$-Tr = (\tfrac{1}{2}mr^2)\alpha$$

$$\alpha = -\frac{2T}{mr} = -\frac{2(180 \text{ N})}{(15 \text{ kg})(0.5 \text{ m})} = -48.0 \text{ rad/s}^2$$

$$\boldsymbol{\alpha} = 48.0 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$

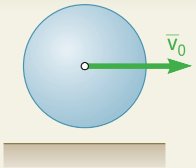


Acceleration of Cord. Since the acceleration of the cord is equal to the tangential component of the acceleration of point A on the disk, we write

$$\mathbf{a}_{\text{cord}} = (\mathbf{a}_A)_t = \bar{\mathbf{a}} + (\mathbf{a}_{A/G})_t$$

$$= [2.19 \text{ m/s}^2 \uparrow] + [(0.5 \text{ m})(48 \text{ rad/s}^2)\uparrow]$$

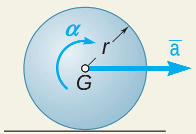
$$\mathbf{a}_{\text{cord}} = 26.2 \text{ m/s}^2 \uparrow \blacktriangleleft$$



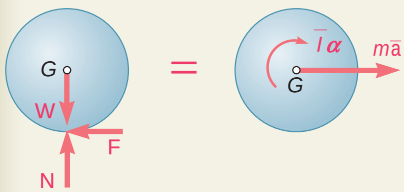
SAMPLE PROBLEM 16.5

A uniform sphere of mass m and radius r is projected along a rough horizontal surface with a linear velocity $\overline{\mathbf{v}}_0$ and no angular velocity. Denoting by μ_k the coefficient of kinetic friction between the sphere and the floor, determine (*a*) the time t_1 at which the sphere will start rolling without sliding, (*b*) the linear velocity and angular velocity of the sphere at time t_1.

SOLUTION



Equations of Motion. The positive sense is chosen to the right for $\overline{\mathbf{a}}$ and clockwise for $\boldsymbol{\alpha}$. The external forces acting on the sphere consist of the weight **W**, the normal reaction **N**, and the friction force **F**. Since the point of the sphere in contact with the surface is sliding to the right, the friction force **F** is directed to the left. While the sphere is sliding, the magnitude of the friction force is $F = \mu_k N$. The effective forces consist of the vector $m\overline{\mathbf{a}}$ attached at G and the couple $\overline{I}\boldsymbol{\alpha}$. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write



$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N - W = 0$$
$$N = W = mg \qquad F = \mu_k N = \mu_k mg$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad -F = m\overline{a} \qquad -\mu_k mg = m\overline{a} \qquad \overline{a} = -\mu_k g$$

$$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \qquad Fr = \overline{I}\alpha$$

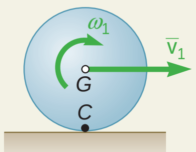
Noting that $\overline{I} = \frac{2}{5}mr^2$ and substituting the value obtained for F, we write

$$(\mu_k mg)r = \tfrac{2}{5}mr^2\alpha \qquad \alpha = \frac{5}{2}\frac{\mu_k g}{r}$$

Kinematics of Motion. As long as the sphere both rotates and slides, its linear and angular motions are uniformly accelerated.

$$t = 0, \overline{v} = \overline{v}_0 \qquad \overline{v} = \overline{v}_0 + \overline{a}t = \overline{v}_0 - \mu_k g t \tag{1}$$

$$t = 0, \omega_0 = 0 \qquad \omega = \omega_0 + \alpha t = 0 + \left(\frac{5}{2}\frac{\mu_k g}{r}\right)t \tag{2}$$



The sphere will start rolling without sliding when the velocity $\mathbf{v}_C$ of the point of contact C is zero. At that time, $t = t_1$, point C becomes the instantaneous center of rotation, and we have

$$\overline{v}_1 = r\omega_1 \tag{3}$$

Substituting in (3) the values obtained for $\overline{v}_1$ and ω_1 by making $t = t_1$ in (1) and (2), respectively, we write

$$\overline{v}_0 - \mu_k g t_1 = r\left(\frac{5}{2}\frac{\mu_k g}{r}t_1\right) \qquad t_1 = \frac{2}{7}\frac{\overline{v}_0}{\mu_k g} \quad ◀$$

Substituting for t_1 into (2), we have

$$\omega_1 = \frac{5}{2}\frac{\mu_k g}{r}t_1 = \frac{5}{2}\frac{\mu_k g}{r}\left(\frac{2}{7}\frac{\overline{v}_0}{\mu_k g}\right) \qquad \omega_1 = \frac{5}{7}\frac{\overline{v}_0}{r} \qquad \boldsymbol{\omega}_1 = \frac{5}{7}\frac{\overline{v}_0}{r}\ \circlearrowright \quad ◀$$

$$\overline{v}_1 = r\omega_1 = r\left(\frac{5}{7}\frac{\overline{v}_0}{r}\right) \qquad \overline{v}_1 = \tfrac{5}{7}\overline{v}_0 \qquad \overline{\mathbf{v}}_1 = \tfrac{5}{7}\overline{v}_0 \rightarrow \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

This chapter deals with the *plane motion* of rigid bodies, and in this first lesson we considered rigid bodies that are free to move under the action of applied forces.

1. Effective forces. We first recalled that a rigid body consists of a large number of particles. The effective forces of the particles forming the body were found to be equivalent to a vector $m\bar{\mathbf{a}}$ attached at the mass center G of the body and a couple of moment $\bar{I}\boldsymbol{\alpha}$ [Fig. 16.7]. Noting that the applied forces are equivalent to the effective forces, we wrote

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma M_G = \bar{I}\alpha \tag{16.5}$$

where $\bar{a}_x$ and $\bar{a}_y$ are the x and y components of the acceleration of the mass center G of the body and α is the angular acceleration of the body. It is important to note that when these equations are used, *the moments of the applied forces must be computed with respect to the mass center of the body*. However, you learned a more efficient method of solution based on the use of a free-body-diagram equation.

2. Free-body-diagram equation. Your first step in the solution of a problem should be to draw a *free-body-diagram equation*.

a. A free-body-diagram equation consists of two diagrams representing two equivalent systems of vectors. *In the first diagram* you should show *the forces exerted on the body*, including the applied forces, the reactions at the supports, and the weight of the body. *In the second diagram* you should show the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$ representing *the effective forces*.

b. Using a free-body-diagram equation allows you to *sum components in any direction and to sum moments about any point.* When writing the three equations of motion needed to solve a given problem, you can therefore select one or more equations involving a single unknown. Solving these equations first and substituting the values obtained for the unknowns into the remaining equation(s) will yield a simpler solution.

3. Plane motion of a rigid body. The problems that you will be asked to solve will fall into one of the following categories.

a. Rigid body in translation. For a body in translation, the angular acceleration is zero. The effective forces reduce to *the vector* $m\bar{\mathbf{a}}$ applied at the mass center [Sample Probs. 16.1 and 16.2].

b. Rigid body in centroidal rotation. For a body in centroidal rotation, the acceleration of the mass center is zero. The effective forces reduce to *the couple* $\bar{I}\mathsf{A}$ [Sample Prob. 16.3].

c. Rigid body in general plane motion. You can consider the general plane motion of a rigid body as the sum of a translation and a centroidal rotation. The effective forces are equivalent to the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\mathsf{A}$ [Sample Probs. 16.4 and 16.5].

4. Plane motion of a system of rigid bodies. You first should draw a free-body-diagram equation that includes all the rigid bodies of the system. A vector $m\bar{\mathbf{a}}$ and a couple $\bar{I}\mathsf{A}$ are attached to each body. However, the forces exerted on each other by the various bodies of the system can be omitted, since they occur in pairs of equal and opposite forces.

a. If no more than three unknowns are involved, you can use this free-body-diagram equation and sum components in any direction and sum moments about any point to obtain equations that can be solved for the desired unknowns [Sample Prob. 16.3].

b. If more than three unknowns are involved, you must draw a separate free-body-diagram equation for each of the rigid bodies of the system. Both internal forces and external forces should be included in each of the free-body-diagram equations, and care should be taken to represent with equal and opposite vectors the forces that two bodies exert on each other.

PROBLEMS

CONCEPT QUESTIONS

16.CQ1 Two pendulums, A and B, with the masses and lengths shown are released from rest. Which system has a larger mass moment of inertia about its pivot point?

a. A

b. B

c. They are the same.

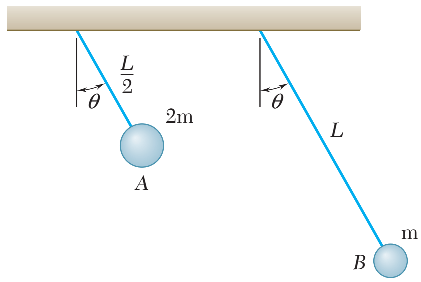


Fig. P16.CQ1 and P16.CQ2

16.CQ2 Two pendulums, A and B, with the masses and lengths shown are released from rest. Which system has a larger angular acceleration immediately after release?

a. A

b. B

c. They are the same.

16.CQ3 Two solid cylinders, A and B, have the same mass m and the radii $2r$ and r, respectively. Each is accelerated from rest with a force applied as shown. In order to impart identical angular accelerations to both cylinders, what is the relationship between F_1 and F_2?

a. $F_1 = 0.5F_2$

b. $F_1 = F_2$

c. $F_1 = 2F_2$

d. $F_1 = 4F_2$

e. $F_1 = 8F_2$

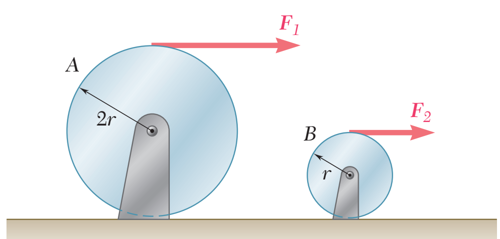


Fig. P16.CQ3

FREE BODY PRACTICE PROBLEMS

16.F1 A 6-ft board is placed in a truck with one end resting against a block secured to the floor and the other leaning against a vertical partition. Draw the FBD and KD necessary to determine the maximum allowable acceleration of the truck if the board is to remain in the position shown.

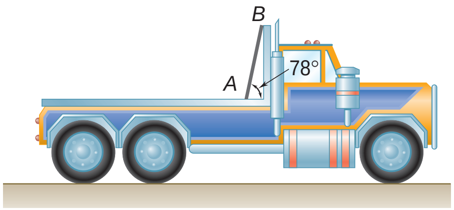


Fig. P16.F1

16.F2 A uniform circular plate of mass 3 kg is attached to two links *AC* and *BD* of the same length. Knowing that the plate is released from rest in the position shown, in which lines joining *G* to *A* and *B* are, respectively, horizontal and vertical, draw the FBD and KD for the plate.

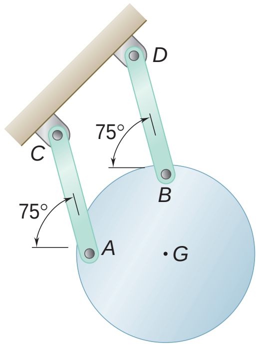


Fig. P16.F2

16.F3 Two uniform disks and two cylinders are assembled as indicated. Disk *A* weighs 20 lb and disk *B* weighs 12 lb. Knowing that the system is released from rest, draw the FBD and KD for the whole system.

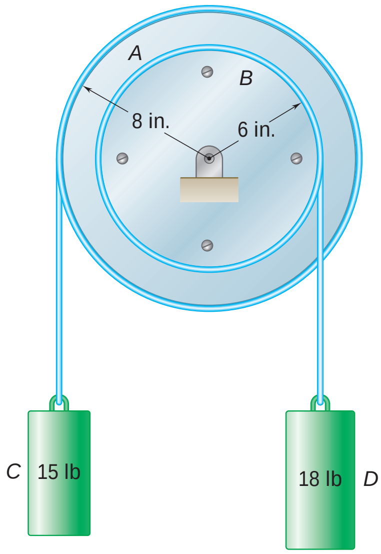


Fig. P16.F3

16.F4 The 400-lb crate shown is lowered by means of two overhead cranes. Knowing the tension in each cable, draw the FBD and KD that can be used to determine the angular acceleration of the crate and the acceleration of the center of gravity.

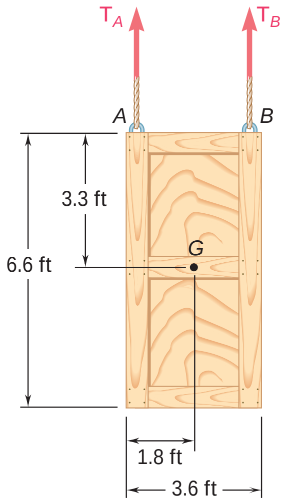


Fig. P16.F4

END-OF-SECTION PROBLEMS

16.1 A conveyor system is fitted with vertical panels, and a 15-in. rod *AB* weighing 5 lb is lodged between two panels as shown. If the rod is to remain in the position shown, determine the maximum allowable acceleration of the system.

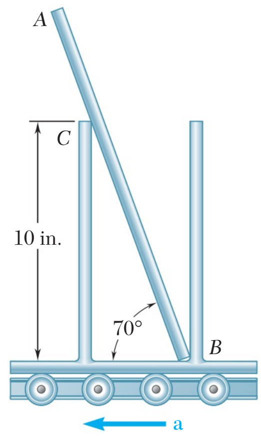


Fig. P16.1 and P16.2

16.2 A conveyor system is fitted with vertical panels, and a 15-in. rod *AB* weighing 5 lb is lodged between two panels as shown. Knowing that the acceleration of the system is 3 ft/s^2 to the left, determine (*a*) the force exerted on the rod at *C*, (*b*) the reaction at *B*.

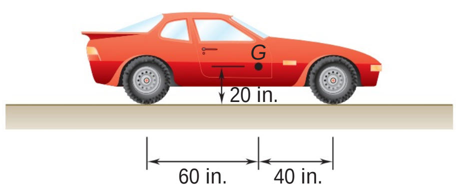


Fig. P16.3

16.3 Knowing that the coefficient of static friction between the tires and the road is 0.80 for the automobile shown, determine the maximum possible acceleration on a level road, assuming (*a*) four-wheel drive, (*b*) rear-wheel drive, (*c*) front-wheel drive.

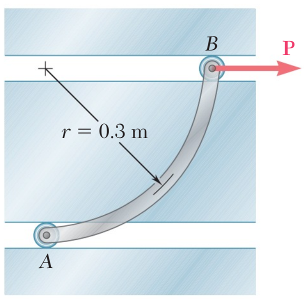


Fig. P16.4

16.4 The motion of the 2.5-kg rod *AB* is guided by two small wheels which roll freely in horizontal slots. If a force **P** of magnitude 8 N is applied at *B*, determine (*a*) the acceleration of the rod, (*b*) the reactions at *A* and *B*.

16.5 A uniform rod *BC* of mass 4 kg is connected to a collar *A* by a 250-mm cord *AB*. Neglecting the mass of the collar and cord, determine (*a*) the smallest constant acceleration $\mathbf{a}_A$ for which the cord and the rod will lie in a straight line, (*b*) the corresponding tension in the cord.

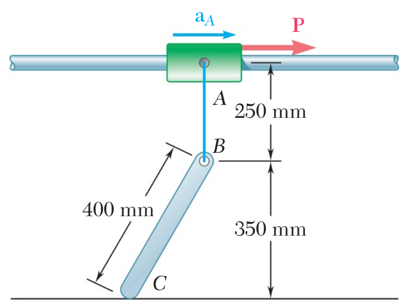


Fig. P16.5

16.6 A 2000-kg truck is being used to lift a 400-kg boulder B that is on a 50-kg pallet A. Knowing the acceleration of the rear-wheel-drive truck is 1 m/s^2, determine (a) the reaction at each of the front wheels, (b) the force between the boulder and the pallet.

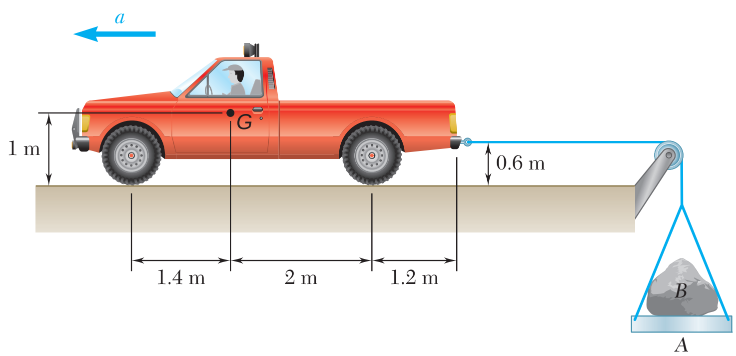


Fig. P16.6

16.7 The support bracket shown is used to transport a cylindrical can from one elevation to another. Knowing that $\mu_s = 0.25$ between the can and the bracket, determine (a) the magnitude of the upward acceleration **a** for which the can will slide on the bracket, (b) the smallest ratio h/d for which the can will tip before it slides.

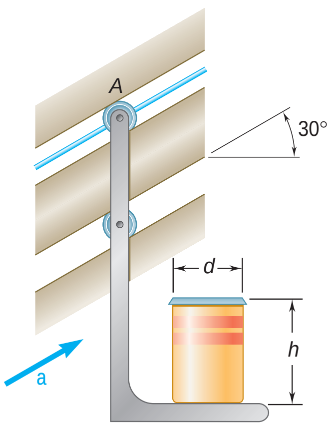


Fig. *P16.7*

16.8 Solve Prob. 16.7, assuming that the acceleration **a** of the bracket is directed downward.

16.9 A 20-kg cabinet is mounted on casters that allow it to move freely ($\mu = 0$) on the floor. If a 100-N force is applied as shown, determine (a) the acceleration of the cabinet, (b) the range of values of h for which the cabinet will not tip.

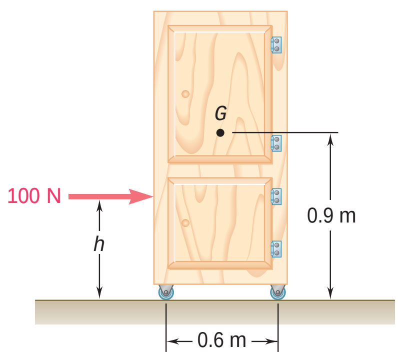


Fig. P16.9

16.10 Solve Prob. 16.9, assuming that the casters are locked and slide on the rough floor ($\mu_k = 0.25$).

16.11 A completely filled barrel and its contents have a combined mass of 90 kg. A cylinder C is connected to the barrel at a height h = 550 mm as shown. Knowing μ_s = 0.40 and μ_k = 0.35, determine the maximum mass of C so the barrel will not tip.

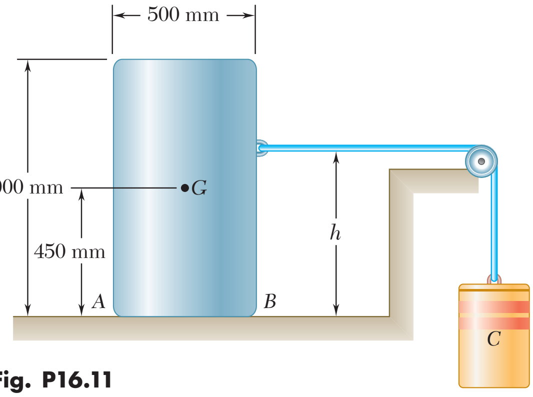


Fig. P16.11

16.12 A 40-kg vase has a 200-mm-diameter base and is being moved using a 100-kg utility cart as shown. The cart moves freely (μ = 0) on the ground. Knowing the coefficient of static friction between the vase and the cart is μ_s = 0.4, determine the maximum force **F** that can be applied if the vase is not to slide or tip.

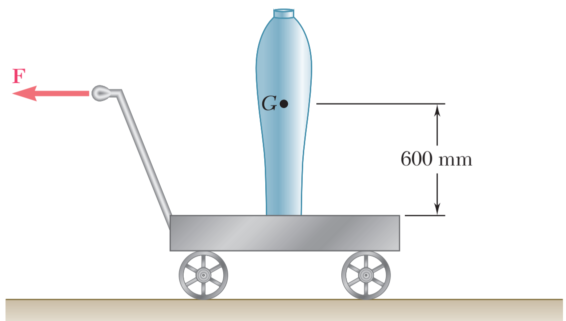


Fig. *P16.12*

16.13 The retractable shelf shown is supported by two identical linkage-and-spring systems; only one of the systems is shown. A 20-kg machine is placed on the shelf so that half of its weight is supported by the system shown. If the springs are removed and the system is released from rest, determine (a) the acceleration of the machine, (b) the tension in link AB. Neglect the weight of the shelf and links.

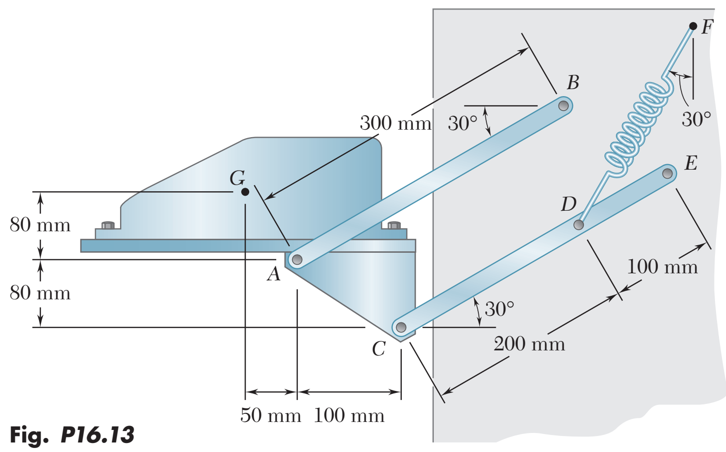


Fig. *P16.13*

16.14 A uniform rectangular plate has a mass of 5 kg and is held in position by three ropes as shown. Knowing that u = 30°, determine, immediately after rope *CF* has been cut, (*a*) the acceleration of the plate, (*b*) the tension in ropes *AD* and *BE*.

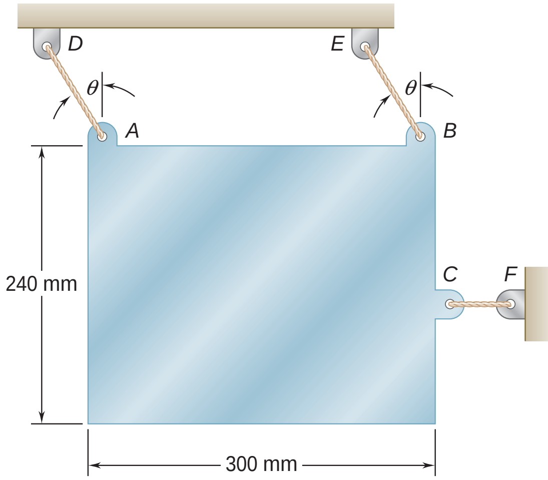


Fig. P16.14 and P16.15

16.15 A uniform rectangular plate has a mass of 5 kg and is held in position by three ropes as shown. Determine the largest value of u for which both ropes *AD* and *BE* remain taut immediately after rope *CF* has been cut.

16.16 Three bars, each of mass 3 kg, are welded together and pin-connected to two links *BE* and *CF*. Neglecting the weight of the links, determine the force in each link immediately after the system is released from rest.

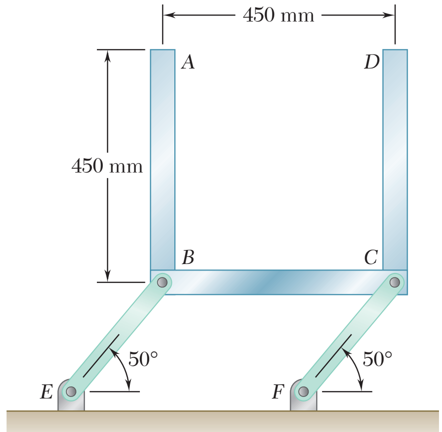


Fig. *P16.16*

16.17 Members *ACE* and *DCB* are each 600 mm long and are connected by a pin at *C*. The mass center of the 10-kg member *AB* is located at *G*. Determine (*a*) the acceleration of *AB* immediately after the system has been released from rest in the position shown, (*b*) the corresponding force exerted by roller *A* on member *AB*. Neglect the weight of members *ACE* and *DCB*.

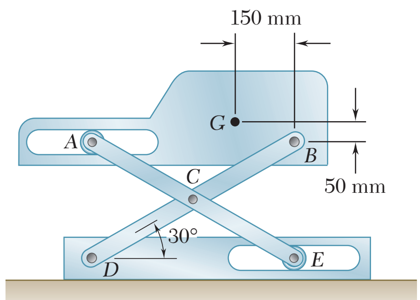


Fig. *P16.17*

16.18 The 15-lb rod *BC* connects a disk centered at *A* to crank *CD*. Knowing that the disk is made to rotate at the constant speed of 180 rpm, determine for the position shown the vertical components of the forces exerted on rod *BC* by pins at *B* and *C*.

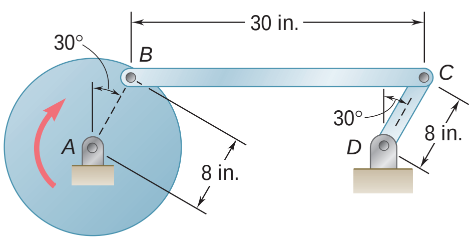


Fig. P16.18

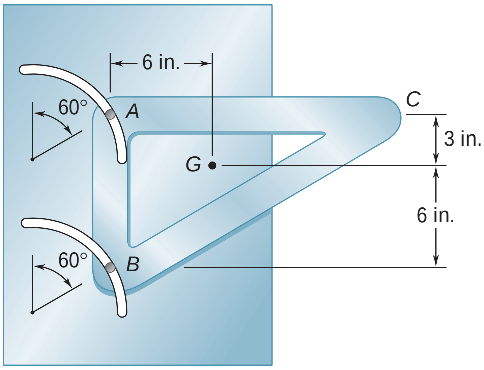


Fig. P16.19

16.19 The triangular weldment ABC is guided by two pins that slide freely in parallel curved slots of radius 6 in. cut in a vertical plate. The weldment weighs 16 lb and its mass center is located at point G. Knowing that at the instant shown the velocity of each pin is 30 in./s downward along the slots, determine (*a*) the acceleration of the weldment, (*b*) the reactions at A and B.

16.20 The coefficients of friction between the 30-lb block and the 5-lb platform BD are $\mu_s = 0.50$ and $\mu_k = 0.40$. Determine the accelerations of the block and of the platform immediately after wire AB has been cut.

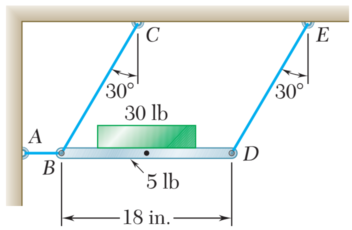


Fig. P16.20

16.21 Draw the shear and bending-moment diagrams for the vertical rod AB of Prob. 16.16.

16.22 Draw the shear and bending-moment diagrams for the connecting rod BC of Prob. 16.18.

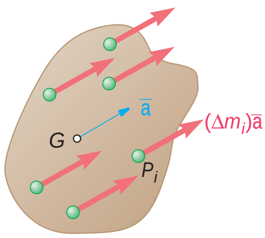


Fig. P16.23

16.23 For a rigid slab in translation, show that the system of the effective forces consists of vectors $(\Delta m_i)\bar{\mathbf{a}}$ attached to the various particles of the slab, where $\bar{\mathbf{a}}$ is the acceleration of the mass center G of the slab. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a single vector $m\bar{\mathbf{a}}$ attached at G.

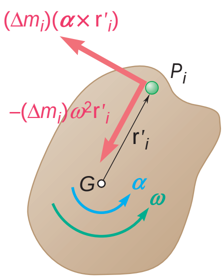


Fig. P16.24

16.24 For a rigid slab in centroidal rotation, show that the system of the effective forces consists of vectors $-(\Delta m_i)\omega^2\mathbf{r}'_i$ and $(\Delta m_i)(\boldsymbol{\alpha}\times\mathbf{r}'_i)$ attached to the various particles P_i of the slab, where $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the angular velocity and angular acceleration of the slab, and where $\mathbf{r}'_i$ denotes the position vector of the particle P_i relative to the mass center G of the slab. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a couple $\bar{I}\boldsymbol{\alpha}$.

16.25 The rotor of an electric motor has an angular velocity of 3600 rpm when the load and power are cut off. The 50-kg rotor, which has a centroidal radius of gyration of 180 mm, then coasts to rest. Knowing that kinetic friction results in a couple of magnitude 3.5 N · m exerted on the rotor, determine the number of revolutions that the rotor executes before coming to rest.

16.26 It takes 10 min for a 6000-lb flywheel to coast to rest from an angular velocity of 300 rpm. Knowing that the radius of gyration of the flywheel is 36 in., determine the average magnitude of the couple due to kinetic friction in the bearings.

16.27 The 8-in.-radius brake drum is attached to a larger flywheel that is not shown. The total mass moment of inertia of the drum and the flywheel is 14 lb · ft · s^2 and the coefficient of kinetic friction between the drum and the brake shoe is 0.35. Knowing that the angular velocity of the flywheel is 360 rpm counterclockwise when a force **P** of magnitude 75 lb is applied to the pedal C, determine the number of revolutions executed by the flywheel before it comes to rest.

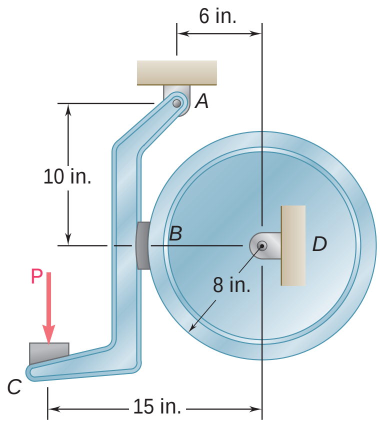


Fig. P16.27

16.28 Solve Prob. 16.27, assuming that the initial angular velocity of the flywheel is 360 rpm clockwise.

16.29 The 100-mm-radius brake drum is attached to a flywheel which is not shown. The drum and flywheel together have a mass of 300 kg and a radius of gyration of 600 mm. The coefficient of kinetic friction between the brake band and the drum is 0.30. Knowing that a force **P** of magnitude 50 N is applied at A when the angular velocity is 180 rpm counterclockwise, determine the time required to stop the flywheel when a = 200 mm and b = 160 mm.

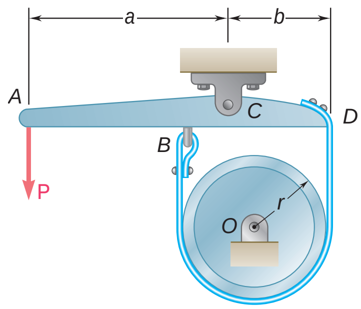


Fig. P16.29

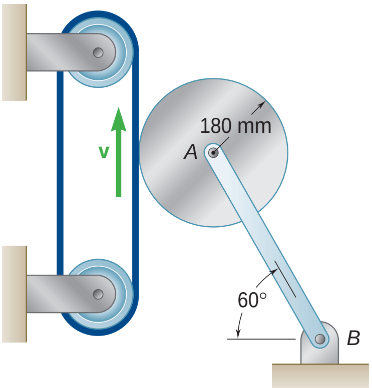


Fig. P16.30

16.30 The 180-mm-radius disk is at rest when it is placed in contact with a belt moving at a constant speed. Neglecting the weight of the link AB and knowing that the coefficient of kinetic friction between the disk and the belt is 0.40, determine the angular acceleration of the disk while slipping occurs.

16.31 Solve Prob. 16.30, assuming that the direction of motion of the belt is reversed.

16.32 In order to determine the mass moment of inertia of a flywheel of radius 600 mm, a 12-kg block is attached to a wire that is wrapped around the flywheel. The block is released and is observed to fall 3 m in 4.6 s. To eliminate bearing friction from the computation, a second block of mass 24 kg is used and is observed to fall 3 m in 3.1 s. Assuming that the moment of the couple due to friction remains constant, determine the mass moment of inertia of the flywheel.

16.33 The flywheel shown has a radius of 20 in., a weight of 250 lb, and a radius of gyration of 15 in. A 30-lb block A is attached to a wire that is wrapped around the flywheel, and the system is released from rest. Neglecting the effect of friction, determine (a) the acceleration of block A, (b) the speed of block A after it has moved 5 ft.

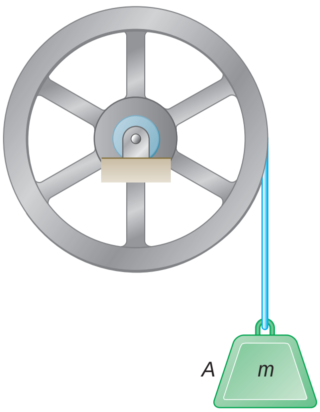


Fig. *P16.32* and P16.33

16.34 Each of the double pulleys shown has a mass moment of inertia of 15 lb · ft · s^2 and is initially at rest. The outside radius is 18 in., and the inner radius is 9 in. Determine (*a*) the angular acceleration of each pulley, (*b*) the angular velocity of each pulley after point *A* on the cord has moved 10 ft.

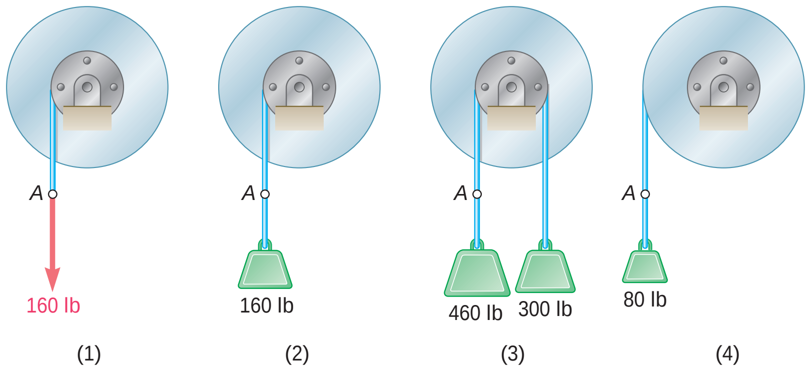


Fig. P16.34

16.35 Each of the gears *A* and *B* has a mass of 9 kg and has a radius of gyration of 200 mm; gear *C* has a mass of 3 kg and has a radius of gyration of 75 mm. If a couple **M** of constant magnitude 5 N-m is applied to gear *C*, determine (*a*) the angular acceleration of gear *A*, (*b*) the tangential force which gear *C* exerts on gear *A*.

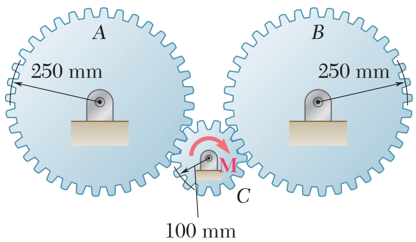


Fig. *P16.35*

16.36 Solve Prob. 16.35, assuming that the couple **M** is applied to disk *A*.

16.37 Gear *A* weighs 1 lb and has a radius of gyration of 1.3 in; gear *B* weighs 6 lb and has a radius of gyration of 3 in.; gear *C* weighs 9 lb and has a radius of gyration of 4.3 in. Knowing a couple **M** of constant magnitude of 40 lb · in is applied to gear *A*, determine (*a*) the angular acceleration of gear *C*, (*b*) the tangential force which gear *B* exerts on gear *C*.

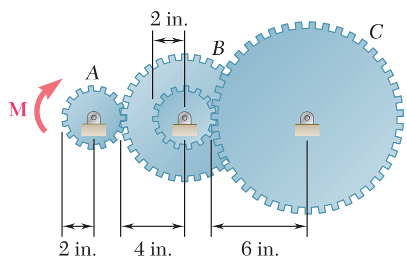


Fig. *P16.37*

16.38 Disks A and B are bolted together, and cylinders D and E are attached to separate cords wrapped on the disks. A single cord passes over disks B and C. Disk A weighs 20 lb and disks B and C each weigh 12 lb. Knowing that the system is released from rest and that no slipping occurs between the cords and the disks, determine the acceleration (*a*) of cylinder D, (*b*) of cylinder E.

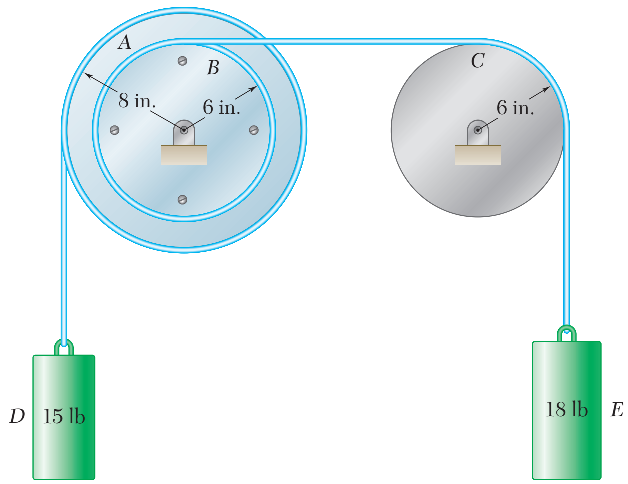


Fig. *P16.38*

16.39 A belt of negligible mass passes between cylinders A and B and is pulled to the right with a force **P**. Cylinders A and B weigh, respectively, 5 and 20 lb. The shaft of cylinder A is free to slide in a vertical slot and the coefficients of friction between the belt and each of the cylinders are $\mu_s = 0.50$ and $\mu_k = 0.40$. For $P = 3.6$ lb, determine (*a*) whether slipping occurs between the belt and either cylinder, (*b*) the angular acceleration of each cylinder.

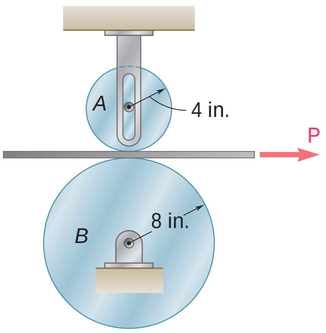


Fig. P16.39

16.40 Solve Prob. 16.39 for $P = 2.00$ lb.

16.41 Disk A has a mass of 6 kg and an initial angular velocity of 360 rpm clockwise; disk B has a mass of 3 kg and is initially at rest. The disks are brought together by applying a horizontal force of magnitude 20 N to the axle of disk A. Knowing that $\mu_k = 0.15$ between the disks and neglecting bearing friction, determine (*a*) the angular acceleration of each disk, (*b*) the final angular velocity of each disk.

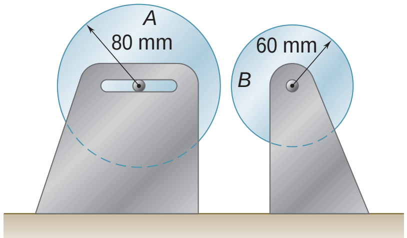


Fig. P16.41

16.42 Solve Prob. 16.41, assuming that initially disk A is at rest and disk B has an angular velocity of 360 rpm clockwise.

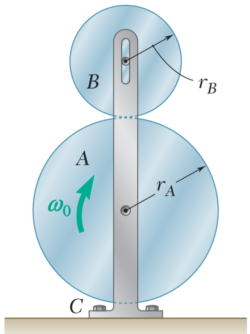


Fig. P16.43 and P16.44

16.43 Disk A has a mass $m_A = 4$ kg, a radius $r_A = 300$ mm, and an initial angular velocity $\omega_0 = 300$ rpm clockwise. Disk B has a mass $m_B = 1.6$ kg, a radius $r_B = 180$ mm, and is at rest when it is brought into contact with disk A. Knowing that $\mu_k = 0.35$ between the disks and neglecting bearing friction, determine (*a*) the angular acceleration of each disk, (*b*) the reaction at the support C.

16.44 Disk B is at rest when it is brought into contact with disk A, which has an initial angular velocity ω_0. (*a*) Show that the final angular velocities of the disks are independent of the coefficient of friction μ_k between the disks as long as $\mu_k \neq 0$. (*b*) Express the final angular velocity of disk A in terms of ω_0 and the ratio of the masses of the two disks m_A/m_B.

16.45 Cylinder A has an initial angular velocity of 720 rpm clockwise, and cylinders B and C are initially at rest. Disks A and B each weigh 5 lb and have radius $r = 4$ in. Disk C weighs 20 lb and has a radius of 8 in. The disks are brought together when C is placed gently onto A and B. Knowing that $\mu_k = 0.25$ between A and C and no slipping occurs between B and C, determine (*a*) the angular acceleration of each disk, (*b*) the final angular velocity of each disk.

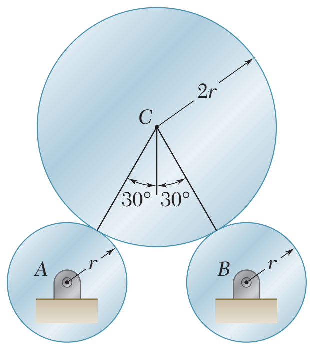


Fig. *P16.45*

16.46 Show that the system of the effective forces for a rigid slab in plane motion reduces to a single vector, and express the distance from the mass center G of the slab to the line of action of this vector in terms of the centroidal radius of gyration $\bar{k}$ of the slab, the magnitude $\bar{a}$ of the acceleration of G, and the angular acceleration α.

16.47 For a rigid slab in plane motion, show that the system of the effective forces consists of vectors $(\Delta m_i)\bar{\mathbf{a}}$, $-(\Delta m_i)\omega^2\mathbf{r}'_i$, and $(\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}'_i)$ attached to the various particles P_i of the slab, where $\bar{\mathbf{a}}$ is the acceleration of the mass center G of the slab, ω is the angular velocity of the slab, $\boldsymbol{\alpha}$ is its angular acceleration, and $\mathbf{r}'_i$ denotes the position vector of the particle P_i, relative to G. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a vector $m\bar{\mathbf{a}}$ attached at G and a couple $\bar{I}\boldsymbol{\alpha}$.

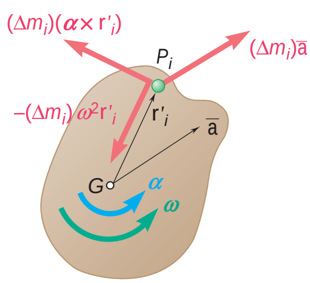


Fig. P16.47

16.48 A uniform slender rod AB rests on a frictionless horizontal surface, and a force **P** of magnitude 0.25 lb is applied at A in a direction perpendicular to the rod. Knowing that the rod weighs 1.75 lb, determine (a) the acceleration of point A, (b) the acceleration of point B, (c) the location of the point on the bar that has zero acceleration.

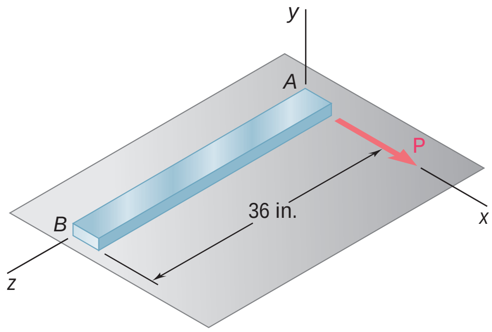


Fig. P16.48

16.49 (a) In Prob. 16.48, determine the point of the rod AB at which the force **P** should be applied if the acceleration of point B is to be zero. (b) Knowing that $P = 0.25$ lb, determine the corresponding acceleration of point A.

16.50 A force **P** of magnitude 3 N is applied to a tape wrapped around a thin hoop of mass 2.4 kg. Knowing that the body rests on a frictionless horizontal surface, determine the acceleration of (a) point A, (b) point B.

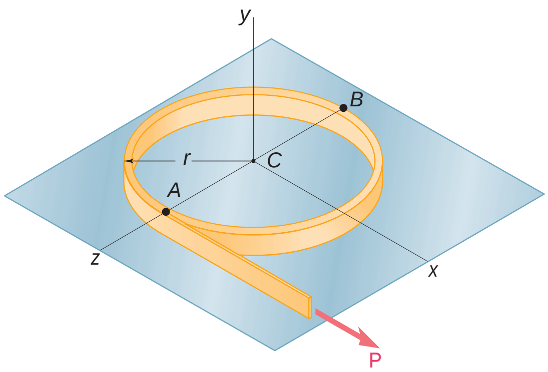


Fig. P16.50

16.51 A force **P** is applied to a tape wrapped around a uniform disk that rests on a frictionless horizontal surface. Show that for each 360° rotation of the disk the center of the disk will move a distance pr.

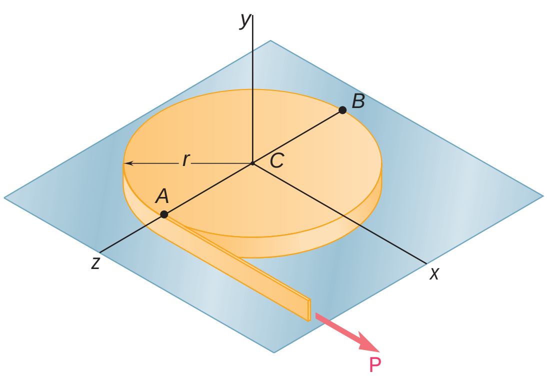


Fig. P16.51

16.52 A 250-lb satellite has a radius of gyration of 24 in. with respect to the y axis and is symmetrical with respect to the zx plane. Its orientation is changed by firing four small rockets A, B, C, and D, each of which produces a 4-lb thrust **T** directed as shown. Determine the angular acceleration of the satellite and the acceleration of its mass center G (a) when all four rockets are fired, (b) when all rockets except D are fired.

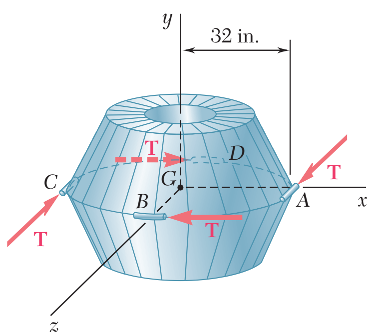


Fig. P16.52

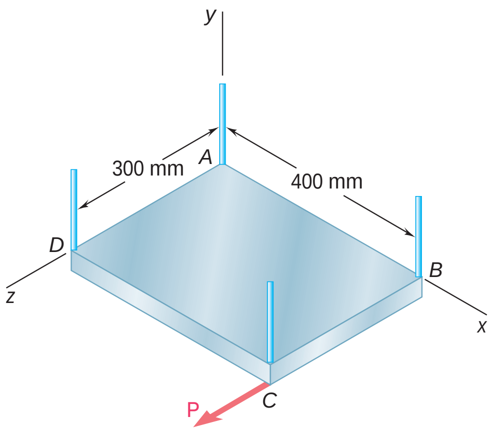


Fig. P16.53

16.53 A rectangular plate of mass 5 kg is suspended from four vertical wires, and a force **P** of magnitude 6 N is applied to corner C as shown. Immediately after **P** is applied, determine the acceleration of (a) the midpoint of edge BC, (b) corner B.

16.54 A uniform slender L-shaped bar ABC is at rest on a horizontal surface when a force **P** of magnitude 4 N is applied at point A. Neglecting friction between the bar and the surface and knowing that the mass of the bar is 2 kg, determine (a) the initial angular acceleration of the bar, (b) the initial acceleration of point B.

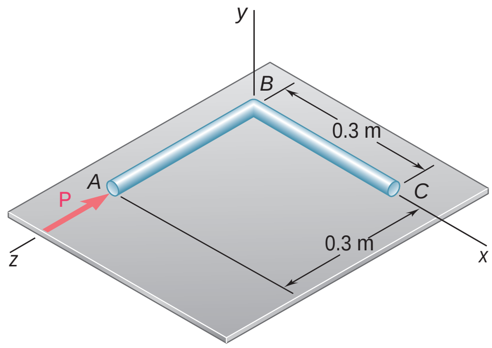


Fig. P16.54

16.55 By pulling on the string of a yo-yo, a person manages to make the yo-yo spin, while remaining at the same elevation above the floor. Denoting the mass of the yo-yo by m, the radius of the inner drum on which the string is wound by r, and the centroidal radius of gyration of the yo-yo by $\bar{k}$, determine the angular acceleration of the yo-yo.

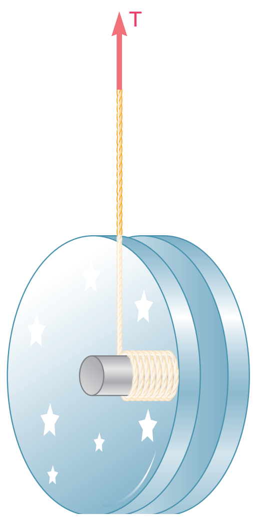


Fig. P16.55 and P16.56

16.56 The 80-g yo-yo shown has a centroidal radius of gyration of 30 mm. The radius of the inner drum on which a string is wound is 6 mm. Knowing that at the instant shown the acceleration of the center of the yo-yo is 1 m/s^2 upward, determine (a) the required tension **T** in the string, (b) the corresponding angular acceleration of the yo-yo.

16.57 A 6-lb sprocket wheel has a centroidal radius of gyration of 2.75 in. and is suspended from a chain as shown. Determine the acceleration of points A and B of the chain, knowing that $T_A = 3$ lb and $T_B = 4$ lb.

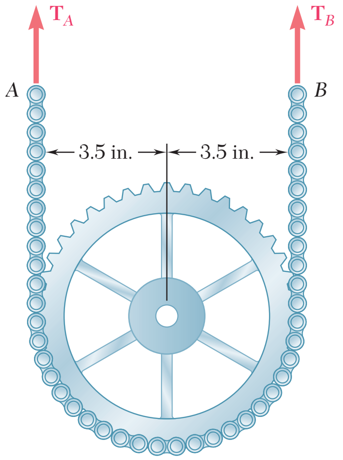


Fig. P16.57

16.58 The steel roll shown has a mass of 1200 kg, a centroidal radius of gyration of 150 mm, and is lifted by two cables looped around its shaft. Knowing that for each cable $T_A = 3100$ N and $T_B = 3300$ N, determine (*a*) the angular acceleration of the roll, (*b*) the acceleration of its mass center.

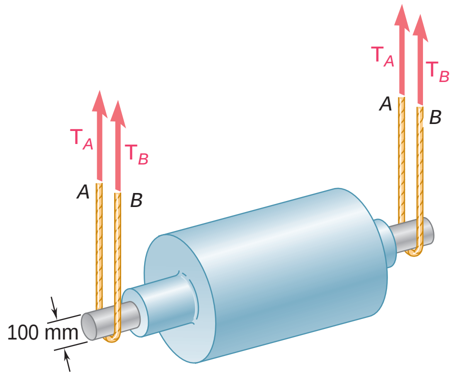


Fig. P16.58 and P16.59

16.59 The steel roll shown has a mass of 1200 kg, has a centroidal radius of gyration of 150 mm, and is lifted by two cables looped around its shaft. Knowing that at the instant shown the acceleration of the roll is 150 mm/s^2 downward and that for each cable $T_A = 3000$ N, determine (*a*) the corresponding tension T_B, (*b*) the angular acceleration of the roll.

16.60 and *16.61* A 15-ft beam weighing 500 lb is lowered by means of two cables unwinding from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow the unwinding motion. Knowing that the deceleration of cable A is 20 ft/s^2 and the deceleration of cable B is 2 ft/s^2, determine the tension in each cable.

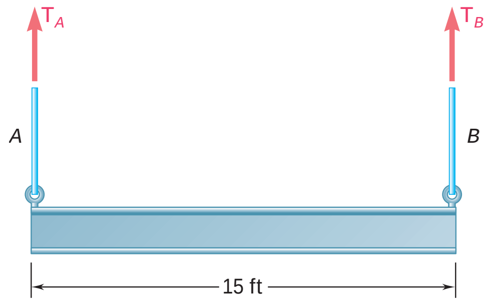


Fig. P16.60

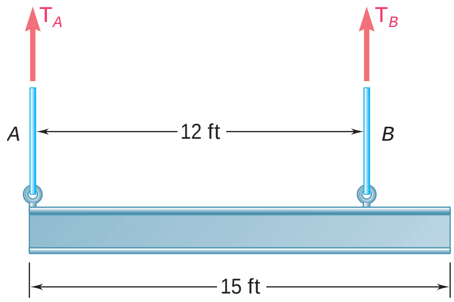


Fig. *P16.61*

16.62 Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a belt as shown. If the system is released from rest, determine (*a*) the angular acceleration of each cylinder, (*b*) the tension in the portion of belt connecting the two cylinders, (*c*) the velocity of the center of the cylinder *A* after it has moved through 3 ft.

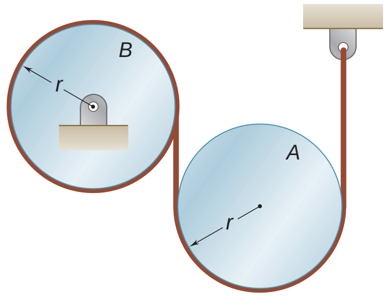


Fig. *P16.62*

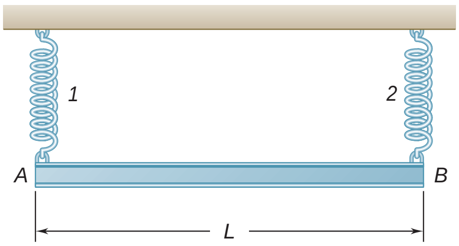


Fig. P16.63

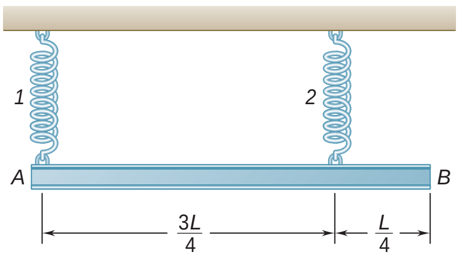


Fig. P16.64

16.63 through 16.65 A beam *AB* of mass *m* and of uniform cross section is suspended from two springs as shown. If spring 2 breaks, determine at that instant (*a*) the angular acceleration of the bar, (*b*) the acceleration of point *A*, (*c*) the acceleration of point *B*.

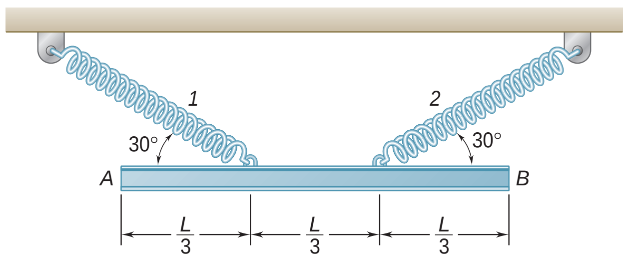


Fig. P16.65

16.66 through *16.68* A thin plate of the shape indicated and of mass *m* is suspended from two springs as shown. If spring 2 breaks, determine the acceleration at that instant (*a*) of point *A*, (*b*) of point *B*.

16.66 A square plate of side *b*
16.67 A circular plate of diameter *b*
16.68 A rectangular plate of height *b* and width *a*

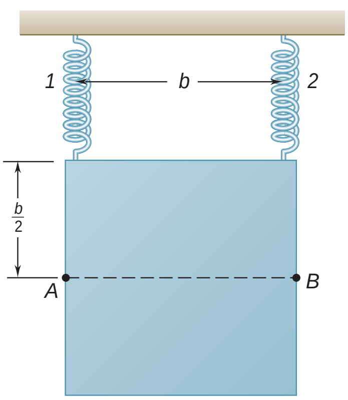


Fig. P16.66

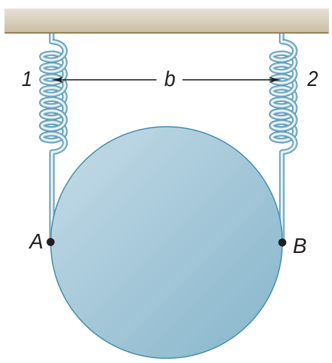


Fig. P16.67

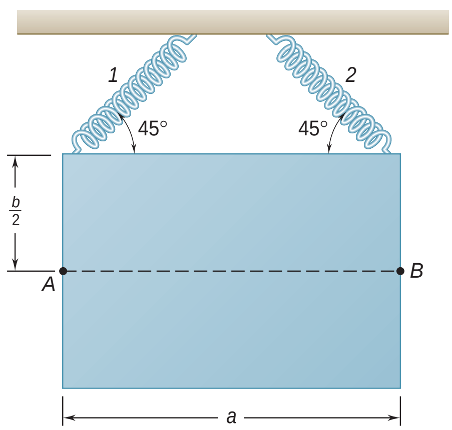


Fig. *P16.68*

16.69 A sphere of radius r and mass m is projected along a rough horizontal surface with the initial velocities indicated. If the final velocity of the sphere is to be zero, express, in terms of v_0, r, and m_k, (*a*) the required magnitude of V_0, (*b*) the time t_1 required for the sphere to come to rest, (*c*) the distance the sphere will move before coming to rest.

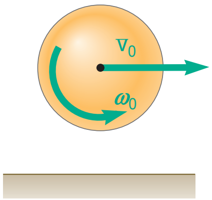


Fig. P16.69

16.70 Solve Prob. 16.69, assuming that the sphere is replaced by a uniform thin hoop of radius r and mass m.

16.71 A bowler projects an 8-in.-diameter ball weighing 12 lb along an alley with a forward velocity $\mathbf{v}_0$ of 15 ft/s and a backspin V_0 of 9 rad/s. Knowing that the coefficient of kinetic friction between the ball and the alley is 0.10, determine (*a*) the time t_1 at which the ball will start rolling without sliding, (*b*) the speed of the ball at time t_1, (*c*) the distance the ball will have traveled at time t_1.

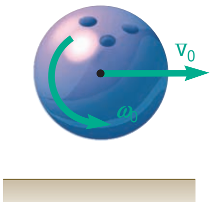


Fig. P16.71

16.72 Solve Prob. 16.71, assuming that the bowler projects the ball with the same forward velocity but with a backspin of 18 rad/s.

16.73 A uniform sphere of radius r and mass m is placed with no initial velocity on a belt that moves to the right with a constant velocity $\mathbf{v}_1$. Denoting by m_k the coefficient of kinetic friction between the sphere and the belt, determine (*a*) the time t_1 at which the sphere will start rolling without sliding, (*b*) the linear and angular velocities of the sphere at time t_1.

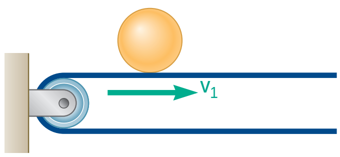


Fig. *P16.73*

16.74 A sphere of radius r and mass m has a linear velocity $\mathbf{v}_0$ directed to the left and no angular velocity as it is placed on a belt moving to the right with a constant velocity $\mathbf{v}_1$. If after first sliding on the belt the sphere is to have no linear velocity relative to the ground as it starts rolling on the belt without sliding, determine in terms of v_1 and the coefficient of kinetic friction m_k between the sphere and the belt (*a*) the required value of v_0, (*b*) the time t_1 at which the sphere will start rolling on the belt, (*c*) the distance the sphere will have moved relative to the ground at time t_1.

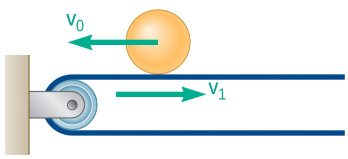


Fig. *P16.74*

16.8 CONSTRAINED PLANE MOTION

Most engineering applications deal with rigid bodies which are moving under given constraints. For example, cranks must rotate about a fixed axis, wheels must roll without sliding, and connecting rods must describe certain prescribed motions. In all such cases, definite relations exist between the components of the acceleration $\bar{\mathbf{a}}$ of the mass center G of the body considered and its angular acceleration A; the corresponding motion is said to be a *constrained motion*.

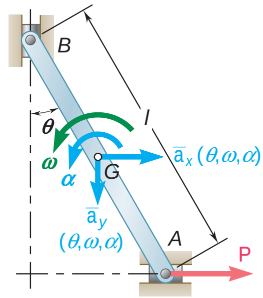


Fig. 16.11

The solution of a problem involving a constrained plane motion calls first for a *kinematic analysis* of the problem. Consider, for example, a slender rod AB of length l and mass m whose extremities are connected to blocks of negligible mass which slide along horizontal and vertical frictionless tracks. The rod is pulled by a force $\mathbf{P}$ applied at A (Fig. 16.11). We know from Sec. 15.8 that the acceleration $\bar{\mathbf{a}}$ of the mass center G of the rod can be determined at any given instant from the position of the rod, its angular velocity, and its angular acceleration at that instant. Suppose, for example, that the values of u, v, and a are known at a given instant and that we wish to determine the corresponding value of the force $\mathbf{P}$, as well as the reactions at A and B. We should first *determine the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of the mass center G* by the method of Sec. 15.8. We next apply d'Alembert's principle (Fig. 16.12), using the expressions obtained for $\bar{a}_x$ and $\bar{a}_y$. The unknown forces $\mathbf{P}$, $\mathbf{N}_A$, and $\mathbf{N}_B$ can then be determined by writing and solving the appropriate equations.

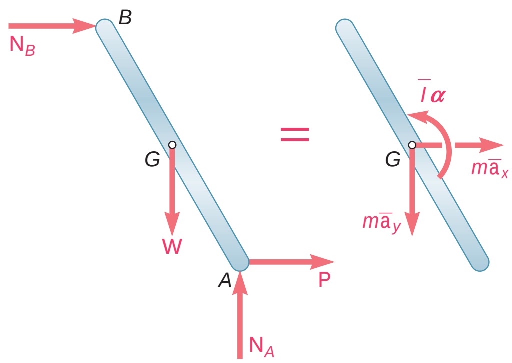


Fig. 16.12

Suppose now that the applied force $\mathbf{P}$, the angle u, and the angular velocity v of the rod are known at a given instant and that we wish to find the angular acceleration a of the rod and the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of its mass center at that instant, as well as the reactions at A and B. The preliminary kinematic study of the problem will have for its object *to express the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of G in terms of the angular acceleration* a *of the rod*. This will be done by first expressing the acceleration of a suitable reference point such as A in terms of the angular acceleration a. The components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of G can then be determined in terms of a, and the expressions obtained carried into Fig. 16.12. Three equations can then be derived in terms of a, N_A, and N_B and solved for the three unknowns (see Sample

Prob. 16.10). Note that the method of dynamic equilibrium can also be used to carry out the solution of the two types of problems we have considered (Fig. 16.13).

When a mechanism consists of *several moving parts*, the approach just described can be used with each part of the mechanism. The procedure required to determine the various unknowns is then similar to the procedure followed in the case of the equilibrium of a system of connected rigid bodies (Sec. 6.11).

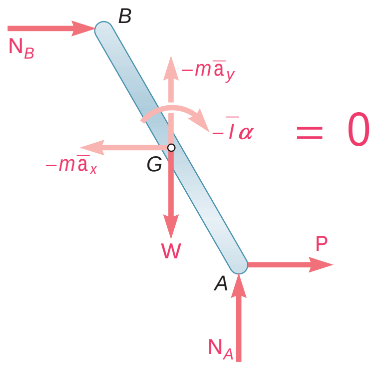


Fig. 16.13

Earlier, we analyzed two particular cases of constrained plane motion: the translation of a rigid body, in which the angular acceleration of the body is constrained to be zero, and the centroidal rotation, in which the acceleration $\bar{\mathbf{a}}$ of the mass center of the body is constrained to be zero. Two other particular cases of constrained plane motion are of special interest: the *noncentroidal rotation* of a rigid body and the *rolling motion* of a disk or wheel. These two cases can be analyzed by one of the general methods described above. However, in view of the range of their applications, they deserve a few special comments.

Noncentroidal Rotation. The motion of a rigid body constrained to rotate about a fixed axis which does not pass through its mass center is called *noncentroidal rotation*. The mass center G of the body moves along a circle of radius $\bar{r}$ centered at the point O, where the axis of rotation intersects the plane of reference (Fig. 16.14). Denoting, respectively, by V and A the angular velocity and the angular acceleration of the line OG, we obtain the following expressions for the tangential and normal components of the acceleration of G:

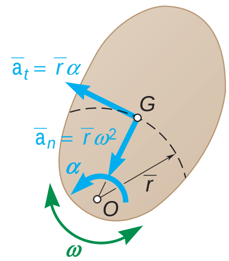
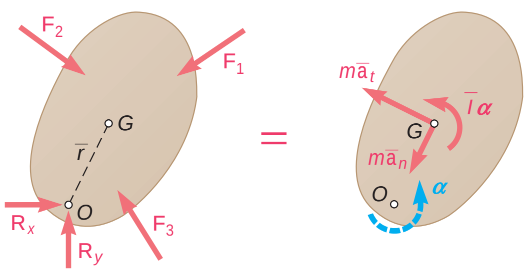


Fig. 16.14

$$\boxed{\bar{a}_t = \bar{r}a \qquad \bar{a}_n = \bar{r}v^2} \tag{16.7}$$

Since line OG belongs to the body, its angular velocity V and its angular acceleration A also represent the angular velocity and the angular acceleration of the body in its motion relative to G. Equations (16.7) define, therefore, the kinematic relation existing between the motion of the mass center G and the motion of the body about G. They should be used to eliminate $\bar{a}_t$ and $\bar{a}_n$ from the equations obtained by applying d'Alembert's principle (Fig. 16.15) or the method of dynamic equilibrium (Fig. 16.16).



(*a*) (*b*)

Fig. 16.15

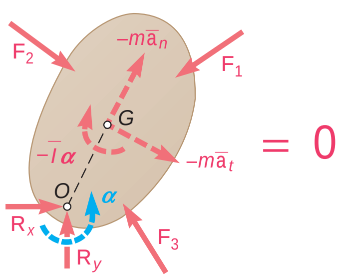


Fig. 16.16

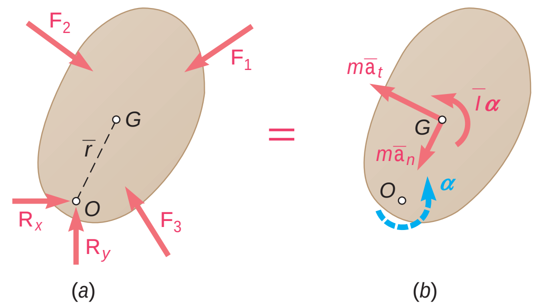


Fig. 16.15 (repeated)

An interesting relation is obtained by equating the moments about the fixed point O of the forces and vectors shown, respectively, in parts a and b of Fig. 16.15. We write

$$+\mathrm{l}\,\Sigma M_O = \bar{I}\mathrm{a} + (m\bar{r}\mathrm{a})\bar{r} = (\bar{I} + m\bar{r}^2)\mathrm{a}$$

But according to the parallel-axis theorem, we have $\bar{I} + m\bar{r}^2 = I_O$, where I_O denotes the moment of inertia of the rigid body about the fixed axis. We therefore write

$$\boxed{\Sigma M_O = I_O\mathrm{a}} \tag{16.8}$$

Although formula (16.8) expresses an important relation between the sum of the moments of the external forces about the fixed point O and the product $I_O\mathrm{a}$, it should be clearly understood that this formula does not mean that the system of the external forces is equivalent to a couple of moment $I_O\mathrm{a}$. The system of the effective forces, and thus the system of the external forces, reduces to a couple only when O coincides with G—that is, *only when the rotation is centroidal* (Sec. 16.4). In the more general case of noncentroidal rotation, the system of the external forces does not reduce to a couple.

A particular case of noncentroidal rotation is of special interest—the case of *uniform rotation*, in which the angular velocity V is constant. Since A is zero, the inertia couple in Fig. 16.16 vanishes and the inertia vector reduces to its normal component. This component (also called *centrifugal force*) represents the tendency of the rigid body to break away from the axis of rotation.

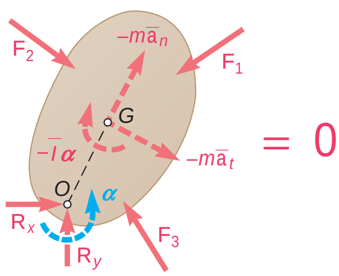


Fig. 16.16 (repeated)

Rolling Motion. Another important case of plane motion is the motion of a disk or wheel rolling on a plane surface. If the disk is constrained to roll without sliding, the acceleration $\bar{\mathbf{a}}$ of its mass center G and its angular acceleration A are not independent. Assuming that the disk is balanced, so that its mass center and its geometric center coincide, we first write that the distance $\bar{x}$ traveled by G during a rotation u of the disk is $\bar{x} = r\mathrm{u}$, where r is the radius of the disk. Differentiating this relation twice, we write

$$\boxed{\bar{a} = r\mathrm{a}} \tag{16.9}$$

Recalling that the system of the effective forces in plane motion reduces to a vector $m\bar{\mathbf{a}}$ and a couple $\bar{I}\boldsymbol{\alpha}$, we find that in the particular case of the rolling motion of a balanced disk, the effective forces reduce to a vector of magnitude $mr\alpha$ attached at G and to a couple of magnitude $\bar{I}\alpha$. We may thus express that the external forces are equivalent to the vector and couple shown in Fig. 16.17.

When a disk *rolls without sliding,* there is no relative motion between the point of the disk in contact with the ground and the ground itself. Thus, as far as the computation of the friction force $\mathbf{F}$ is concerned, a rolling disk can be compared with a block at rest on a surface. The magnitude F of the friction force can have any value, as long as this value does not exceed the maximum value $F_m = \mu_s N$, where μ_s is the coefficient of static friction and N is the magnitude of the normal force. In the case of a rolling disk, the magnitude F of the friction force should therefore be determined independently of N by solving the equation obtained from Fig. 16.17.

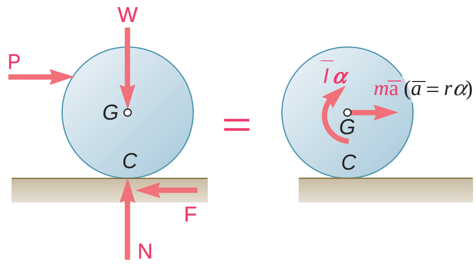


Fig. 16.17

When *sliding is impending,* the friction force reaches its maximum value $F_m = \mu_s N$ and can be obtained from N.

When the disk *rotates and slides* at the same time, a relative motion exists between the point of the disk which is in contact with the ground and the ground itself, and the force of friction has the magnitude $F_k = \mu_k N$, where μ_k is the coefficient of kinetic friction. In this case, however, the motion of the mass center G of the disk and the rotation of the disk about G are independent, and $\bar{a}$ is not equal to $r\alpha$.

These three different cases can be summarized as follows:

Rolling, no sliding:	$F \le \mu_s N$	$\bar{a} = r\alpha$
Rolling, sliding impending:	$F = \mu_s N$	$\bar{a} = r\alpha$
Rotating and sliding:	$F = \mu_k N$	$\bar{a}$ *and* α *independent*

When it is not known whether or not a disk slides, it should first be assumed that the disk rolls without sliding. If F is found smaller than or equal to $\mu_s N$, the assumption is proved correct. If F is found larger than $\mu_s N$, the assumption is incorrect and the problem should be started again, assuming rotating and sliding.

Photo 16.4 As the ball hits the bowling alley, it first spins and slides, then rolls without sliding.

When a disk is *unbalanced,* i.e., when its mass center G does not coincide with its geometric center O, the relation (16.9) does not hold between $\bar{a}$ and α. However, a similar relation holds between the magnitude a_O of the acceleration of the geometric center and the angular acceleration α of an unbalanced disk which rolls without sliding. We have

$$a_O = r\alpha \tag{16.10}$$

To determine $\bar{a}$ in terms of the angular acceleration α and the angular velocity ω of the disk, we can use the relative-acceleration formula

$$\begin{aligned} \bar{\mathbf{a}} = \bar{\mathbf{a}}_G &= \mathbf{a}_O + \mathbf{a}_{G/O} \\ &= \mathbf{a}_O + (\mathbf{a}_{G/O})_t + (\mathbf{a}_{G/O})_n \end{aligned} \tag{16.11}$$

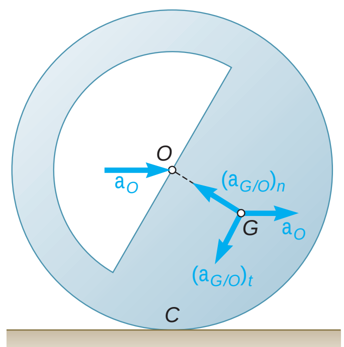
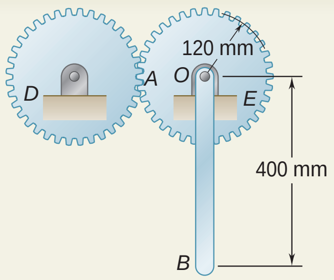


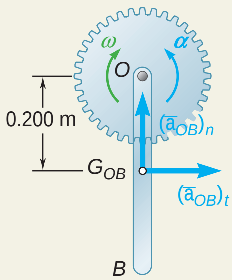
Fig. 16.18

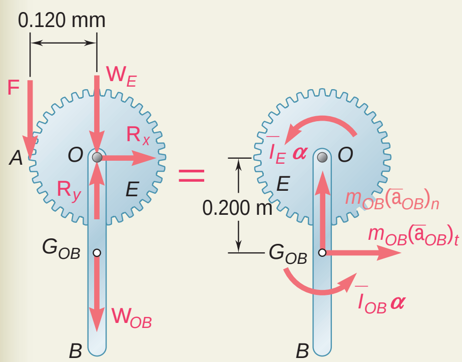
where the three component accelerations obtained have the directions indicated in Fig. 16.18 and the magnitudes $a_O = r\alpha$, $(a_{G/O})_t = (OG)\alpha$, and $(a_{G/O})_n = (OG)\omega^2$.



SAMPLE PROBLEM 16.6

The portion AOB of a mechanism consists of a 400-mm steel rod OB welded to a gear E of radius 120 mm which can rotate about a horizontal shaft O. It is actuated by a gear D and, at the instant shown, has a clockwise angular velocity of 8 rad/s and a counterclockwise angular acceleration of 40 rad/s^2. Knowing that rod OB has a mass of 3 kg and gear E a mass of 4 kg and a radius of gyration of 85 mm, determine (a) the tangential force exerted by gear D on gear E, (b) the components of the reaction at shaft O.





SOLUTION

In determining the effective forces of the rigid body AOB, gear E and rod OB will be considered separately. Therefore, the components of the acceleration of the mass center G_{OB} of the rod will be determined first:

$$(\bar{a}_{OB})_t = \bar{r}\mathbf{a} = (0.200 \text{ m})(40 \text{ rad/s}^2) = 8 \text{ m/s}^2$$
$$(\bar{a}_{OB})_n = \bar{r}\mathbf{v}^2 = (0.200 \text{ m})(8 \text{ rad/s})^2 = 12.8 \text{ m/s}^2$$

Equations of Motion. Two sketches of the rigid body AOB have been drawn. The first shows the external forces consisting of the weight $\mathbf{W}_E$ of gear E, the weight $\mathbf{W}_{OB}$ of the rod OB, the force $\mathbf{F}$ exerted by gear D, and the components $\mathbf{R}_x$ and $\mathbf{R}_y$ of the reaction at O. The magnitudes of the weights are, respectively,

$$W_E = m_E g = (4 \text{ kg})(9.81 \text{ m/s}^2) = 39.2 \text{ N}$$
$$W_{OB} = m_{OB} g = (3 \text{ kg})(9.81 \text{ m/s}^2) = 29.4 \text{ N}$$

The second sketch shows the effective forces, which consist of a couple $\bar{I}_E\mathbf{A}$ (since gear E is in centroidal rotation) and of a couple and two vector components at the mass center of OB. Since the accelerations are known, we compute the magnitudes of these components and couples:

$$\bar{I}_E\mathbf{a} = m_E\bar{k}_E^2\mathbf{a} = (4 \text{ kg})(0.085 \text{ m})^2(40 \text{ rad/s}^2) = 1.156 \text{ N}\cdot\text{m}$$
$$m_{OB}(\bar{a}_{OB})_t = (3 \text{ kg})(8 \text{ m/s}^2) = 24.0 \text{ N}$$
$$m_{OB}(\bar{a}_{OB})_n = (3 \text{ kg})(12.8 \text{ m/s}^2) = 38.4 \text{ N}$$
$$\bar{I}_{OB}\mathbf{a} = (\tfrac{1}{12}m_{OB}L^2)\mathbf{a} = \tfrac{1}{12}(3 \text{ kg})(0.400 \text{ m})^2(40 \text{ rad/s}^2) = 1.600 \text{ N}\cdot\text{m}$$

Expressing that the system of the external forces is equivalent to the system of the effective forces, we write the following equations:

$+\text{l}\,\Sigma M_O = \Sigma(M_O)_{\text{eff}}$:

$$F(0.120 \text{ m}) = \bar{I}_E\mathbf{a} + m_{OB}(\bar{a}_{OB})_t(0.200 \text{ m}) + \bar{I}_{OB}\mathbf{a}$$
$$F(0.120 \text{ m}) = 1.156 \text{ N}\cdot\text{m} + (24.0 \text{ N})(0.200 \text{ m}) + 1.600 \text{ N}\cdot\text{m}$$

$F = 63.0$ N **F** $= 63.0$ N w ◀

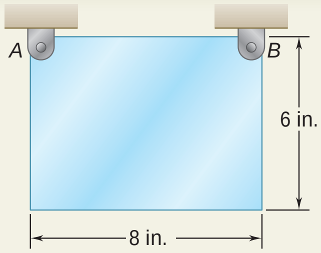
$\overset{+}{\text{y}}\,\Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $R_x = m_{OB}(\bar{a}_{OB})_t$

$R_x = 24.0$ N $\mathbf{R}_x = 24.0$ N y ◀

$+\text{x}\Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $R_y - F - W_E - W_{OB} = m_{OB}(\bar{a}_{OB})_n$

$$R_y - 63.0 \text{ N} - 39.2 \text{ N} - 29.4 \text{ N} = 38.4 \text{ N}$$

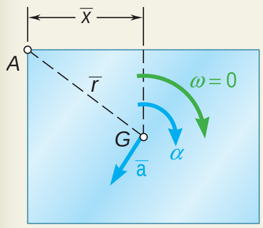
$R_y = 170.0$ N $\mathbf{R}_y = 170.0$ N x ◀



SAMPLE PROBLEM 16.7

A 6 × 8 in. rectangular plate weighing 60 lb is suspended from two pins A and B. If pin B is suddenly removed, determine (a) the angular acceleration of the plate, (b) the components of the reaction at pin A, immediately after pin B has been removed.

SOLUTION



a. Angular Acceleration. We observe that as the plate rotates about point A, its mass center G describes a circle of radius $\bar{r}$ with center at A.

Since the plate is released from rest ($\omega = 0$), the normal component of the acceleration of G is zero. The magnitude of the acceleration $\bar{\mathbf{a}}$ of the mass center G is thus $\bar{a} = \bar{r}\alpha$. We draw the diagram shown to express that the external forces are equivalent to the effective forces:

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \qquad W\bar{x} = (m\bar{a})\bar{r} + \bar{I}\alpha$$

Since $\bar{a} = \bar{r}\alpha$, we have

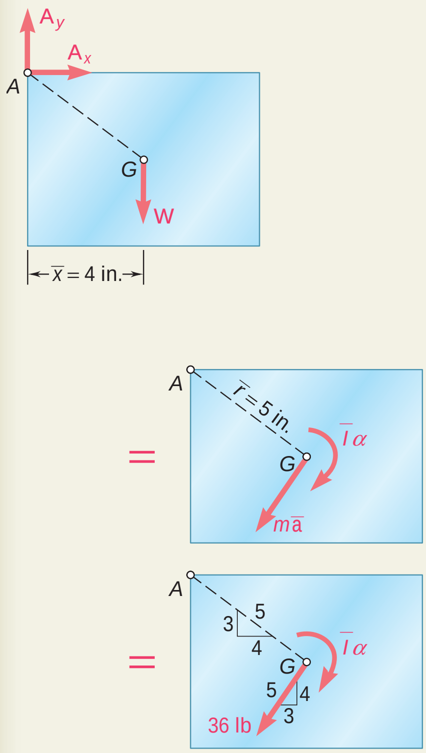
$$W\bar{x} = m(\bar{r}\alpha)\bar{r} + \bar{I}\alpha \qquad \alpha = \frac{W\bar{x}}{\dfrac{W}{g}\bar{r}^2 + \bar{I}} \tag{1}$$

The centroidal moment of inertia of the plate is

$$\bar{I} = \frac{m}{12}(a^2 + b^2) = \frac{60 \text{ lb}}{12(32.2 \text{ ft/s}^2)}[(\tfrac{8}{12}\text{ ft})^2 + (\tfrac{6}{12}\text{ ft})^2]$$
$$= 0.1078 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Substituting this value of $\bar{I}$ together with $W = 60$ lb, $\bar{r} = \frac{5}{12}$ ft, and $\bar{x} = \frac{4}{12}$ ft into Eq. (1), we obtain

$$\alpha = +46.4 \text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 46.4 \text{ rad/s}^2 \circlearrowright \blacktriangleleft$$



b. Reaction at A. Using the computed value of α, we determine the magnitude of the vector $m\bar{\mathbf{a}}$ attached at G.

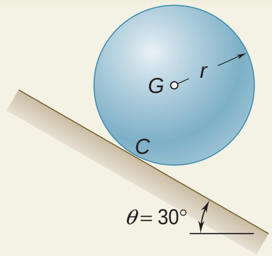
$$m\bar{a} = m\bar{r}\alpha = \frac{60 \text{ lb}}{32.2 \text{ ft/s}^2}(\tfrac{5}{12}\text{ ft})(46.4 \text{ rad/s}^2) = 36.0 \text{ lb}$$

Showing this result on the diagram, we write the equations of motion

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad A_x = -\tfrac{3}{5}(36 \text{ lb})$$
$$= -21.6 \text{ lb} \qquad \mathbf{A}_x = 21.6 \text{ lb} \leftarrow \blacktriangleleft$$

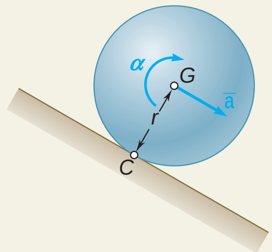
$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad A_y - 60 \text{ lb} = -\tfrac{4}{5}(36 \text{ lb})$$
$$A_y = +31.2 \text{ lb} \qquad \mathbf{A}_y = 31.2 \text{ lb} \uparrow \blacktriangleleft$$

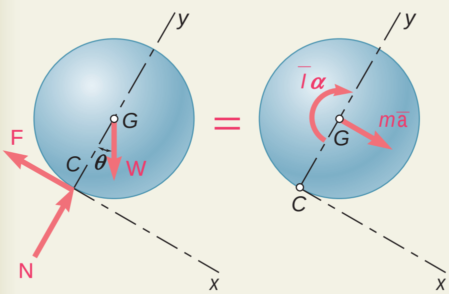
The couple $\bar{I}\boldsymbol{\alpha}$ is not involved in the last two equations; nevertheless, it should be indicated on the diagram.



SAMPLE PROBLEM 16.8

A sphere of radius r and weight W is released with no initial velocity on the incline and rolls without slipping. Determine (*a*) the minimum value of the coefficient of static friction compatible with the rolling motion, (*b*) the velocity of the center G of the sphere after the sphere has rolled 10 ft, (*c*) the velocity of G if the sphere were to move 10 ft down a frictionless 30° incline.





SOLUTION

a. Minimum M$_s$ for Rolling Motion. The external forces **W**, **N**, and **F** form a system equivalent to the system of effective forces represented by the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}$A. Since the sphere rolls without sliding, we have $\bar{a} = r$a.

$$+\text{i}\Sigma M_C = \Sigma(M_C)_{\text{eff}}: \qquad (W \sin \text{u})r = (m\bar{a})r + \bar{I}\text{a}$$
$$(W \sin \text{u})r = (mr\text{a})r + \bar{I}\text{a}$$

Noting that $m = W/g$ and $\bar{I} = \frac{2}{5}mr^2$, we write

$$(W \sin \text{u})r = \left(\frac{W}{g}r\text{a}\right)r + \frac{2}{5}\frac{W}{g}r^2\text{a} \qquad \text{a} = +\frac{5g \sin \text{u}}{7r}$$

$$\bar{a} = r\text{a} = \frac{5g \sin \text{u}}{7} = \frac{5(32.2 \text{ ft/s}^2) \sin 30°}{7} = 11.50 \text{ ft/s}^2$$

$$+\searrow\Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad W \sin \text{u} - F = m\bar{a}$$
$$W \sin \text{u} - F = \frac{W}{g}\frac{5g \sin \text{u}}{7}$$
$$F = +\tfrac{2}{7}W \sin \text{u} = \tfrac{2}{7}W \sin 30° \qquad \mathbf{F} = 0.143W \ \text{b} \ 30°$$

$$+\nearrow\Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N - W \cos \text{u} = 0$$
$$N = W \cos \text{u} = 0.866W \qquad \mathbf{N} = 0.866W \ \text{a} \ 60°$$

$$\text{m}_s = \frac{F}{N} = \frac{0.143W}{0.866W} \qquad \text{m}_s = 0.165 \quad \blacktriangleleft$$

b. Velocity of Rolling Sphere. We have uniformly accelerated motion:

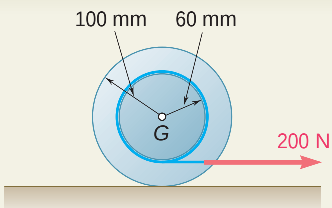
$$\bar{v}_0 = 0 \qquad \bar{a} = 11.50 \text{ ft/s}^2 \qquad \bar{x} = 10 \text{ ft} \qquad \bar{x}_0 = 0$$
$$\bar{v}^2 = \bar{v}_0^2 + 2\bar{a}(\bar{x} - \bar{x}_0) \qquad \bar{v}^2 = 0 + 2(11.50 \text{ ft/s}^2)(10 \text{ ft})$$
$$\bar{v} = 15.17 \text{ ft/s} \qquad \bar{\mathbf{v}} = 15.17 \text{ ft/s} \ \text{c} \ 30° \quad \blacktriangleleft$$

c. Velocity of Sliding Sphere. Assuming now no friction, we have $F = 0$ and obtain

$$+\text{i}\Sigma M_G = \Sigma(M_G)_{\text{eff}}: \qquad 0 = \bar{I}\text{a} \qquad \text{a} = 0$$
$$+\searrow\Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad W \sin 30° = m\bar{a} \qquad 0.50W = \frac{W}{g}\bar{a}$$
$$\bar{a} = +16.1 \text{ ft/s}^2 \qquad \bar{\mathbf{a}} = 16.1 \text{ ft/s}^2 \ \text{c} \ 30°$$

Substituting $\bar{a} = 16.1 \text{ ft/s}^2$ into the equations for uniformly accelerated motion, we obtain

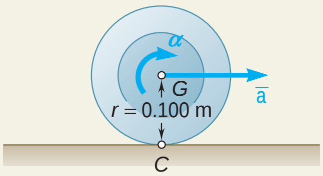
$$\bar{v}^2 = \bar{v}_0^2 + 2\bar{a}(\bar{x} - \bar{x}_0) \qquad \bar{v}^2 = 0 + 2(16.1 \text{ ft/s}^2)(10 \text{ ft})$$
$$\bar{v} = 17.94 \text{ ft/s} \qquad \bar{\mathbf{v}} = 17.94 \text{ ft/s} \ \text{c} \ 30° \quad \blacktriangleleft$$



SAMPLE PROBLEM 16.9

A cord is wrapped around the inner drum of a wheel and pulled horizontally with a force of 200 N. The wheel has a mass of 50 kg and a radius of gyration of 70 mm. Knowing that $\mu_s = 0.20$ and $\mu_k = 0.15$, determine the acceleration of G and the angular acceleration of the wheel.

SOLUTION



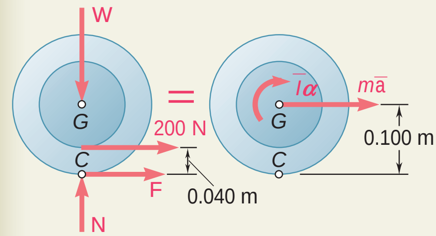
a. Assume Rolling without Sliding. In this case, we have

$$\bar{a} = r\alpha = (0.100\text{ m})\alpha$$

We can determine whether this assumption is justified by comparing the friction force obtained with the maximum available friction force. The moment of inertia of the wheel is

$$\bar{I} = m\bar{k}^2 = (50\text{ kg})(0.070\text{ m})^2 = 0.245\text{ kg}\cdot\text{m}^2$$

Equations of Motion



$$+\circlearrowleft \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \quad (200\text{ N})(0.040\text{ m}) = m\bar{a}(0.100\text{ m}) + \bar{I}\alpha$$
$$8.00\text{ N}\cdot\text{m} = (50\text{ kg})(0.100\text{ m})\alpha(0.100\text{ m}) + (0.245\text{ kg}\cdot\text{m}^2)\alpha$$
$$\alpha = +10.74\text{ rad/s}^2$$
$$\bar{a} = r\alpha = (0.100\text{ m})(10.74\text{ rad/s}^2) = 1.074\text{ m/s}^2$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad F + 200\text{ N} = m\bar{a}$$
$$F + 200\text{ N} = (50\text{ kg})(1.074\text{ m/s}^2)$$
$$F = -146.3\text{ N} \qquad \mathbf{F} = 146.3\text{ N} \leftarrow$$

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}:$$
$$N - W = 0 \qquad N - W = mg = (50\text{ kg})(9.81\text{ m/s}^2) = 490.5\text{ N}$$
$$\mathbf{N} = 490.5\text{ N} \uparrow$$

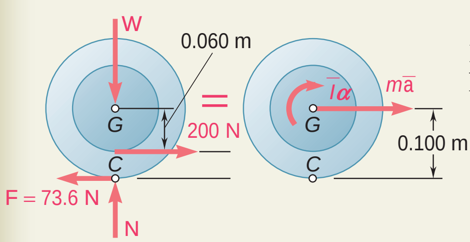
Maximum Available Friction Force

$$F_{\max} = \mu_s N = 0.20(490.5\text{ N}) = 98.1\text{ N}$$

Since $F > F_{\max}$, the assumed motion is impossible.

b. Rotating and Sliding. Since the wheel must rotate and slide at the same time, we draw a new diagram, where $\bar{\mathbf{a}}$ and $\boldsymbol{\alpha}$ are independent and where

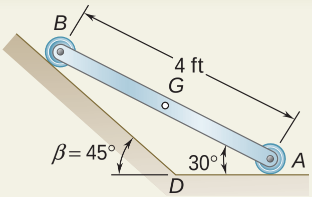
$$F = F_k = \mu_k N = 0.15(490.5\text{ N}) = 73.6\text{ N}$$



From the computation of part a, it appears that **F** should be directed to the left. We write the following equations of motion:

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \quad 200\text{ N} - 73.6\text{ N} = (50\text{ kg})\bar{a}$$
$$\bar{a} = +2.53\text{ m/s}^2 \qquad \bar{\mathbf{a}} = 2.53\text{ m/s}^2 \rightarrow \blacktriangleleft$$

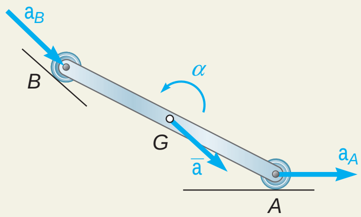
$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}:$$
$$(73.6\text{ N})(0.100\text{ m}) - (200\text{ N})(0.060\text{ m}) = (0.245\text{ kg}\cdot\text{m}^2)\alpha$$
$$\alpha = -18.94\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 18.94\text{ rad/s}^2 \circlearrowright \blacktriangleleft$$



SAMPLE PROBLEM 16.10

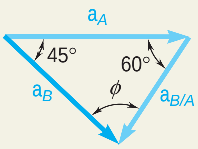
The extremities of a 4-ft rod weighing 50 lb can move freely and with no friction along two straight tracks as shown. If the rod is released with no velocity from the position shown, determine (*a*) the angular acceleration of the rod, (*b*) the reactions at *A* and *B*.

SOLUTION



Kinematics of Motion. Since the motion is constrained, the acceleration of *G* must be related to the angular acceleration A. To obtain this relation, we first determine the magnitude of the acceleration $\mathbf{a}_A$ of point *A* in terms of a. Assuming that A is directed counterclockwise and noting that $a_{B/A} =$ 4a, we write

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$
$$[a_B \text{ c } 45°] = [a_A \text{ y }] + [4\text{a} \text{ d } 60°]$$

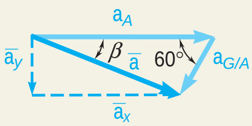


Noting that f = 75° and using the law of sines, we obtain

$$a_A = 5.46\text{a} \qquad a_B = 4.90\text{a}$$

The acceleration of *G* is now obtained by writing

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_A + \mathbf{a}_{G/A}$$
$$\bar{\mathbf{a}} = [5.46\text{a} \text{ y }] + [2\text{a} \text{ d } 60°]$$



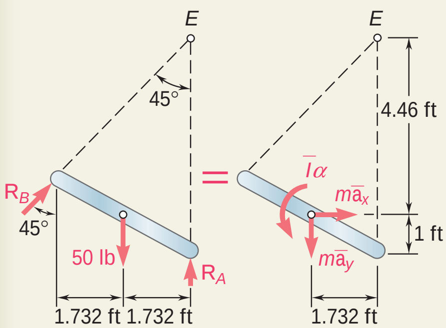
Resolving $\bar{\mathbf{a}}$ into *x* and *y* components, we obtain

$$\bar{a}_x = 5.46\text{a} - 2\text{a} \cos 60° = 4.46\text{a} \qquad \bar{\mathbf{a}}_x = 4.46\text{a} \text{ y}$$
$$\bar{a}_y = -2\text{a} \sin 60° = -1.732\text{a} \qquad \bar{\mathbf{a}}_y = 1.732\text{a w}$$

Kinetics of Motion. We draw a free-body-diagram equation expressing that the system of the external forces is equivalent to the system of the effective forces represented by the vector of components $m\bar{\mathbf{a}}_x$ and $m\bar{\mathbf{a}}_y$ attached at *G* and the couple $\bar{I}$A. We compute the following magnitudes:

$$\bar{I} = \tfrac{1}{12}ml^2 = \frac{1}{12}\frac{50 \text{ lb}}{32.2 \text{ ft/s}^2}(4 \text{ ft})^2 = 2.07 \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \qquad \bar{I}\text{a} = 2.07\text{a}$$

$$m\bar{a}_x = \frac{50}{32.2}(4.46\text{a}) = 6.93\text{a} \qquad m\bar{a}_y = -\frac{50}{32.2}(1.732\text{a}) = -2.69\text{a}$$



Equations of Motion

$+\text{l}\Sigma M_E = \Sigma(M_E)_{\text{eff}}$:
$$(50)(1.732) = (6.93\text{a})(4.46) + (2.69\text{a})(1.732) + 2.07\text{a}$$
$$\text{a} = +2.30 \text{ rad/s}^2 \qquad \text{A} = 2.30 \text{ rad/s}^2 \text{ l} \quad ◀$$

$\overset{+}{\text{y}}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\quad R_B \sin 45° = (6.93)(2.30) = 15.94$
$$R_B = 22.5 \text{ lb} \qquad \mathbf{R}_B = 22.5 \text{ lb a } 45° \quad ◀$$

$+\text{x}\Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $\quad R_A + R_B \cos 45° - 50 = -(2.69)(2.30)$
$$R_A = -6.19 - 15.94 + 50 = 27.9 \text{ lb} \qquad \mathbf{R}_A = 27.9 \text{ lb x} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we considered the *plane motion of rigid bodies under constraints*. We found that the types of constraints involved in engineering problems vary widely. For example, a rigid body may be constrained to rotate about a fixed axis or to roll on a given surface, or it may be pin-connected to collars or to other bodies.

1. Your solution of a problem involving the constrained motion of a rigid body, will, in general, consist of two steps. First, you will consider the *kinematics of the motion,* and then you will solve the *kinetics portion of the problem*.

2. The kinematic analysis of the motion is done by using the methods you learned in Chap. 15. Due to the constraints, linear and angular accelerations will be related. (They will *not* be independent, as they were in the last lesson.) You should establish *relationships among the accelerations* (angular as well as linear), and your goal should be to express all accelerations in terms of a *single unknown acceleration*. This is the first step taken in the solution of each of the sample problems in this lesson.

a. For a body in noncentroidal rotation, the components of the acceleration of the mass center are $\bar{a}_t = \bar{r}\alpha$ and $\bar{a}_n = \bar{r}\omega^2$, where ω will generally be known [Sample Probs. 16.6 and 16.7].

b. For a rolling disk or wheel, the acceleration of the mass center is $\bar{a} = r\alpha$ [Sample Prob. 16.8].

c. For a body in general plane motion, your best course of action, if neither $\bar{a}$ nor α is known or readily obtainable, is to express $\bar{a}$ in terms of α [Sample Prob. 16.10].

3. The kinetic analysis of the motion is carried out as follows.

a. Start by drawing a free-body-diagram equation. This was done in all the sample problems of this lesson. In each case the left-hand diagram shows the external forces, including the applied forces, the reactions, and the weight of the body. The right-hand diagram shows the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$.

b. Next, reduce the number of unknowns in the free-body-diagram equation by using the relationships among the accelerations that you found in your kinematic analysis. You will then be ready to consider equations that can be written by summing components or moments. Choose first an equation that involves a single unknown. After solving for that unknown, substitute the value obtained into the other equations, which you will then solve for the remaining unknowns.

(continued)

4. When solving problems involving rolling disks or wheels, keep in mind the following.

a. If sliding is impending, the friction force exerted on the rolling body has reached its maximum value, $F_m = \mu_s N$, where N is the normal force exerted on the body and μ_s is the coefficient of *static friction* between the surfaces of contact.

b. If sliding is not impending, the friction force F can have *any value* smaller than F_m and should, therefore, be considered as an independent unknown. After you have determined F, be sure to check that it is smaller than F_m; if it is not, *the body does not roll*, but rotates and slides as described in the next paragraph.

c. If the body rotates and slides at the same time, then the body is *not rolling* and the acceleration $\bar{a}$ of the mass center is *independent* of the angular acceleration α of the body: $\bar{a} \neq r\alpha$. On the other hand, the friction force has a well-defined value, $F = \mu_k N$, where μ_k is the coefficient of *kinetic friction* between the surfaces of contact.

d. For an unbalanced rolling disk or wheel, the relation $\bar{a} = r\alpha$ between the acceleration $\bar{a}$ of the mass center G and the angular acceleration α of the disk or wheel *does not hold anymore*. However, a similar relation holds between the acceleration a_O of the *geometric center* O and the angular acceleration α of the disk or wheel: $a_O = r\alpha$. This relation can be used to express $\bar{a}$ in terms of α and ω (Fig. 16.18).

5. For a system of connected rigid bodies, the goal of your *kinematic analysis* should be to determine all the accelerations from the given data, or to express them all in terms of a single unknown. (For systems with several degrees of freedom, you will need to use as many unknowns as there are degrees of freedom.)

Your *kinetic analysis* will generally be carried out by drawing a free-body-diagram equation for the entire system, as well as for one or several of the rigid bodies involved. In the latter case, both internal and external forces should be included, and care should be taken to represent with equal and opposite vectors the forces that two bodies exert on each other.

PROBLEMS

CONCEPT QUESTIONS

16.CQ4 A cord is attached to a spool when a force **P** is applied to the cord as shown. Assuming the spool rolls without slipping, what direction does the spool move for each case?

Case 1: **a.** left **b.** right **c.** It would not move.
Case 2: **a.** left **b.** right **c.** It would not move.
Case 3: **a.** left **b.** right **c.** It would not move.

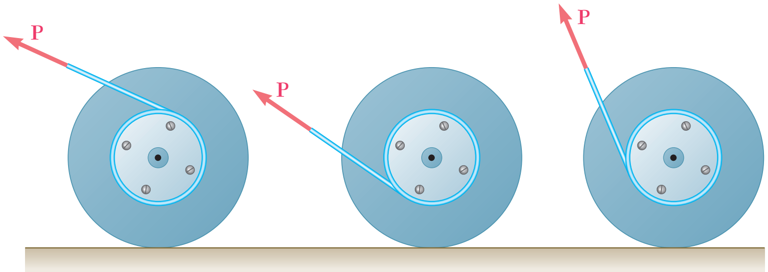


Fig. P16.CQ4 and P16.CQ5

16.CQ5 A cord is attached to a spool when a force **P** is applied to the cord as shown. Assuming the spool rolls without slipping, in what direction does the friction force act for each case?

Case 1: **a.** left **b.** right **c.** The friction force would be zero.
Case 2: **a.** left **b.** right **c.** The friction force would be zero.
Case 3: **a.** left **b.** right **c.** The friction force would be zero.

16.CQ6 A front-wheel-drive car starts from rest and accelerates to the right. Knowing that the tires do not slip on the road, what is the direction of the friction force the road applies to the front tires?

a. left
b. right
c. The friction force is zero.

16.CQ7 A front-wheel-drive car starts from rest and accelerates to the right. Knowing that the tires do not slip on the road, what is the direction of the friction force the road applies to the rear tires?

a. left
b. right
c. The friction force is zero.

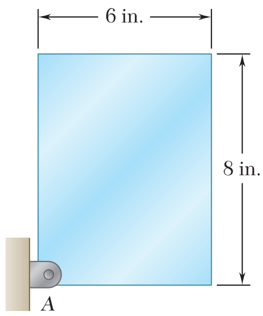


Fig. P16.F5

FREE BODY PRACTICE PROBLEMS

16.F5 A uniform 6 × 8-in. rectangular plate of mass m is pinned at A. Knowing the angular velocity of the plate at the instant shown is V, draw the FBD and KD.

16.F6 Two identical 4-lb slender rods AB and BC are connected by a pin at B and by the cord AC. The assembly rotates in a vertical plane under the combined effect of gravity and a couple **M** applied to rod AB. Knowing that in the position shown the angular velocity of the assembly is V, draw the FBD and KD that can be used to determine the angular acceleration of the assembly.

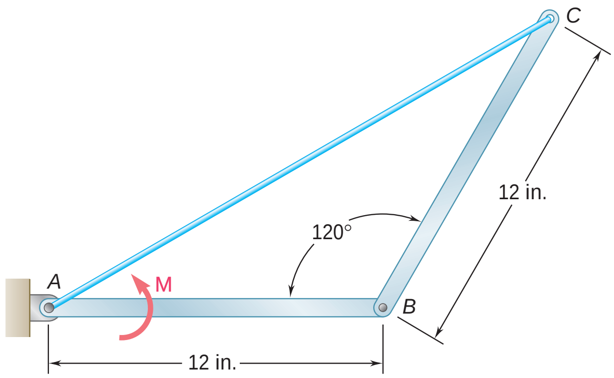


Fig. P16.F6

16.F7 The 4-lb uniform rod AB is attached to collars of negligible mass that slide without friction along the fixed rods shown. Rod AB is at rest in the position u = 25° when a horizontal force **P** is applied to collar A causing it to start moving to the left. Draw the FBD and KD for the rod.

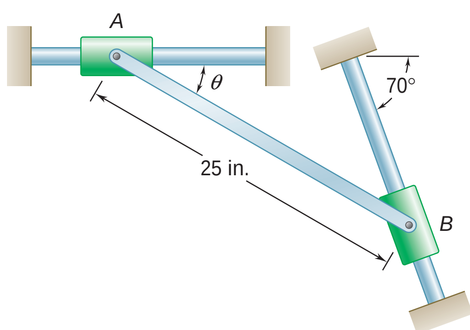


Fig. P16.F7

16.F8 A uniform disk of mass $m = 4$ kg and radius $r = 150$ mm is supported by a belt $ABCD$ that is bolted to the disk at B and C. If the belt suddenly breaks at a point located between A and B, draw the FBD and KD for the disk immediately after the break.

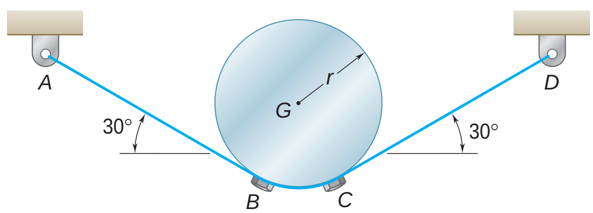


Fig. P16.F8

END-OF-SECTION PROBLEMS

16.75 Show that the couple $\bar{I}\boldsymbol{\alpha}$ of Fig. 16.15 can be eliminated by attaching the vectors $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ at a point P called the *center of percussion,* located on line OG at a distance $GP = \bar{k}^2/\bar{r}$ from the mass center of the body.

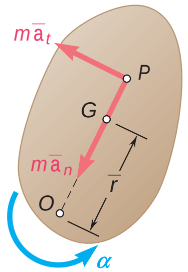


Fig. P16.75

16.76 A uniform slender rod of length $L = 900$ mm and mass $m = 4$ kg is suspended from a hinge at C. A horizontal force $\mathbf{P}$ of magnitude 75 N is applied at end B. Knowing that $\bar{r} = 225$ mm, determine (*a*) the angular acceleration of the rod, (*b*) the components of the reaction at C.

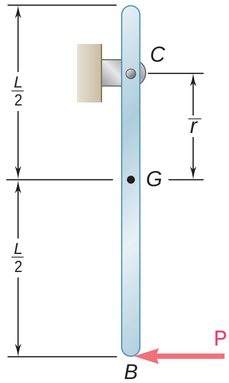


Fig. P16.76

16.77 In Prob. 16.76, determine (*a*) the distance $\bar{r}$ for which the horizontal component of the reaction at C is zero, (*b*) the corresponding angular acceleration of the rod.

16.78 A uniform slender rod of length $L = 36$ in. and weight $W = 4$ lb hangs freely from a hinge at A. If a force $\mathbf{P}$ of magnitude 1.5 lb is applied at B horizontally to the left ($h = L$), determine (*a*) the angular acceleration of the rod, (*b*) the components of the reaction at A.

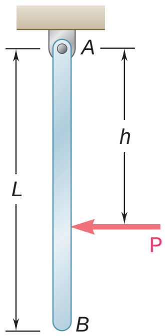


Fig. P16.78

16.79 In Prob. 16.78, determine (*a*) the distance h for which the horizontal component of the reaction at A is zero, (*b*) the corresponding angular acceleration of the rod.

16.80 The uniform slender rod AB is welded to the hub D, and the system rotates about the vertical axis DE with a constant angular velocity $\boldsymbol{\omega}$. (*a*) Denoting by w the mass per unit length of the rod, express the tension in the rod at a distance z from end A in terms of w, l, z, and ω, (*b*) Determine the tension in the rod for $w = 0.3$ kg/m, $l = 400$ mm, $z = 250$ mm, and $\omega = 150$ rpm.

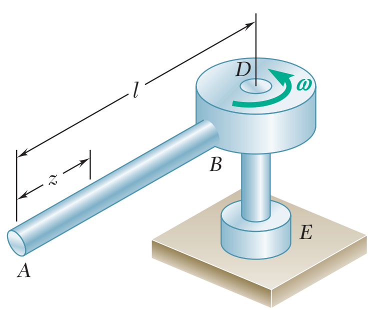


Fig. P16.80

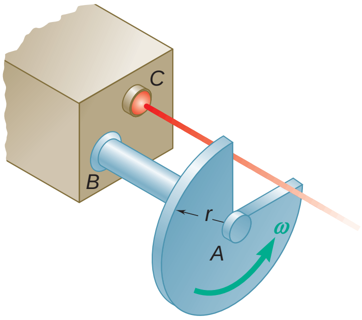


Fig. *P16.81*

16.81 The shutter shown was formed by removing one quarter of a disk of 0.75-in. radius and is used to interrupt a beam of light emanating from a lens at C. Knowing that the shutter weighs 0.125 lb and rotates at the constant rate of 24 cycles per second, determine the magnitude of the force exerted by the shutter on the shaft at A.

16.82 A 6-in.-diameter hole is cut as shown in a thin disk of 15-in. diameter. The disk rotates in a horizontal plane about its geometric center A at the constant rate of 480 rpm. Knowing that the disk has a mass of 60 lb after the hole has been cut, determine the horizontal component of the force exerted by the shaft on the disk at A.

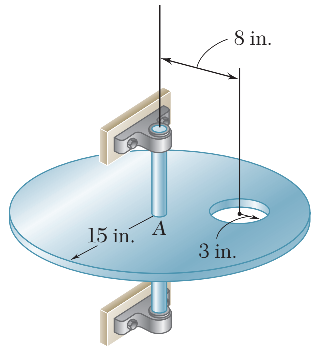


Fig. *P16.82*

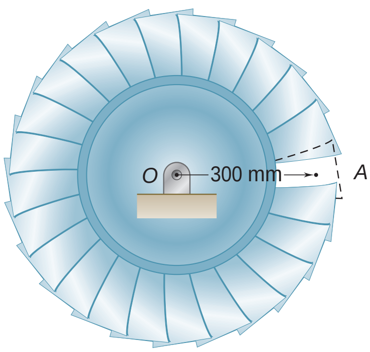


Fig. P16.83

16.83 A turbine disk of mass 26 kg rotates at a constant rate of 9600 rpm. Knowing that the mass center of the disk coincides with the center of rotation O, determine the reaction at O immediately after a single blade at A, of mass 45 g, becomes loose and is thrown off.

16.84 and 16.85 A uniform rod of length L and mass m is supported as shown. If the cable attached at end B suddenly breaks, determine (*a*) the acceleration of end B, (*b*) the reaction at the pin support.

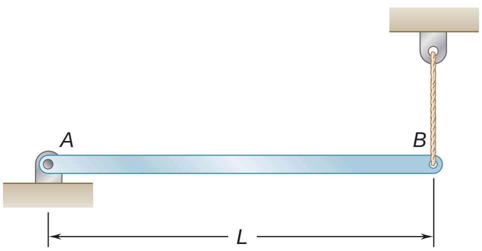


Fig. P16.84

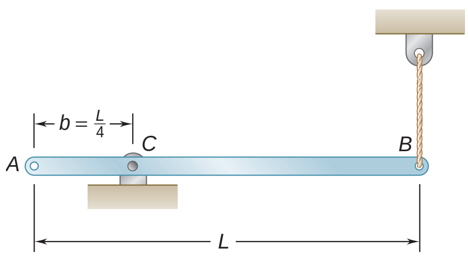


Fig. P16.85

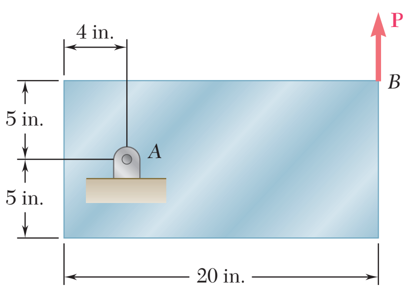


Fig. P16.86

16.86 A 12-lb uniform plate rotates about A in a vertical plane under the combined effect of gravity and of the vertical force **P**. Knowing that at the instant shown the plate has an angular velocity of 20 rad/s and an angular acceleration of 30 rad/s^2 both counterclockwise, determine (*a*) the force **P**, (*b*) the components of the reaction at A.

16.87 A 1.5-kg slender rod is welded to a 5-kg uniform disk as shown. The assembly swings freely about *C* in a vertical plane. Knowing that in the position shown the assembly has an angular velocity of 10 rad/s clockwise, determine (*a*) the angular acceleration of the assembly, (*b*) the components of the reaction at *C*.

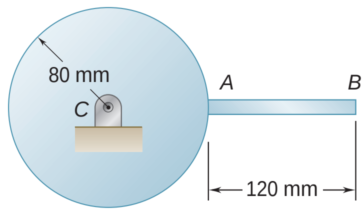


Fig. P16.87

16.88 Two uniform rods, *ABC* of weight 6 lb and *DCE* of weight 8 lb, are connected by a pin at *C* and by two cords *BD* and *BE*. The T-shaped assembly rotates in a vertical plane under the combined effect of gravity and of a couple **M** which is applied to rod *ABC*. Knowing that at the instant shown the tension in cord *BE* is 2 lb and the tension in cord *BD* is 0.5 lb, determine (*a*) the angular acceleration of the assembly, (*b*) the couple **M**.

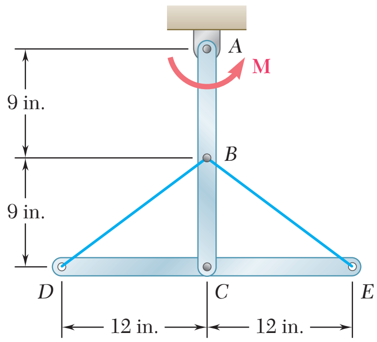


Fig. P16.88

16.89 The object *ABC* consists of two slender rods welded together at point *B*. Rod *AB* has a weight of 2 lb and bar *BC* has a weight of 4 lb. Knowing the magnitude of the angular velocity of *ABC* is 10 rad/s when u = 0°, determine the components of the reaction at point *C* when u = 0°.

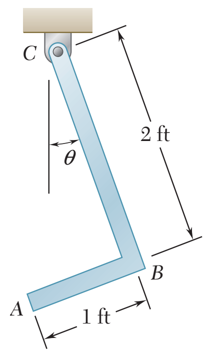


Fig. *P16.89*

16.90 A 3.5-kg slender rod *AB* and a 2-kg slender rod *BC* are connected by a pin at *B* and by the cord *AC*. The assembly can rotate in a vertical plane under the combined effect of gravity and a couple **M** applied to rod *BC*. Knowing that in the position shown the angular velocity of the assembly is zero and the tension in cord *AC* is equal to 25 N, determine (*a*) the angular acceleration of the assembly, (*b*) the magnitude of the couple **M**.

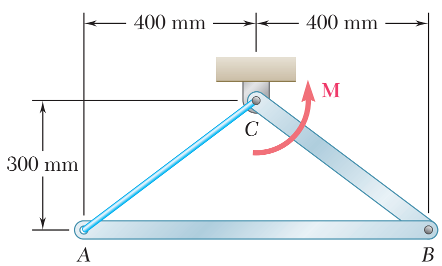


Fig. *P16.90*

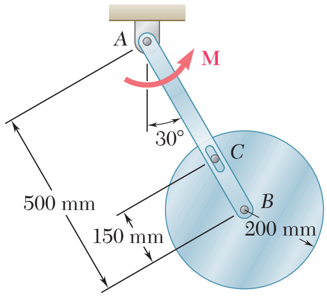


Fig. *P16.91*

16.91 A 9-kg uniform disk is attached to the 5-kg slender rod AB by means of frictionless pins at B and C. The assembly rotates in a vertical plane under the combined effect of gravity and of a couple **M** which is applied to rod AB. Knowing that at the instant shown the assembly has an angular velocity of 6 rad/s and an angular acceleration of 25 rad/s^2, both counterclockwise, determine (a) the couple **M**, (b) the force exerted by pin C on member AB.

16.92 Derive the equation $\Sigma M_C = I_C\alpha$ for the rolling disk of Fig. 16.17, where ΣM_C represents the sum of the moments of the external forces about the instantaneous center C, and I_C is the moment of inertia of the disk about C.

16.93 Show that in the case of an unbalanced disk, the equation derived in Prob. 16.92 is valid only when the mass center G, the geometric center O, and the instantaneous center C happen to lie in a straight line.

16.94 A wheel of radius r and centroidal radius of gyration $\bar{k}$ is released from rest on the incline and rolls without sliding. Derive an expression for the acceleration of the center of the wheel in terms of r, $\bar{k}$, β, and g.

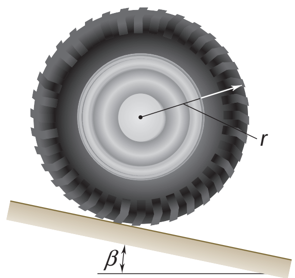


Fig. P16.94

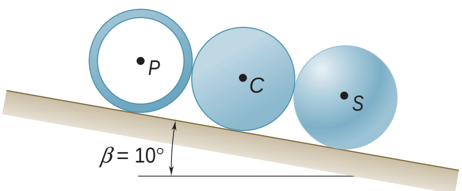


Fig. P16.95

16.95 A homogeneous sphere S, a uniform cylinder C, and a thin pipe P are in contact when they are released from rest on the incline shown. Knowing that all three objects roll without slipping, determine, after 4 s of motion, the clear distance between (a) the pipe and the cylinder, (b) the cylinder and the sphere.

16.96 A 40-kg flywheel of radius $R = 0.5$ m is rigidly attached to a shaft of radius $r = 0.05$ m that can roll along parallel rails. A cord is attached as shown and pulled with a force **P** of magnitude 150 N. Knowing the centroidal radius of gyration is $\bar{k} = 0.4$ m, determine (a) the angular acceleration of the flywheel, (b) the velocity of the center of gravity after 5 s.

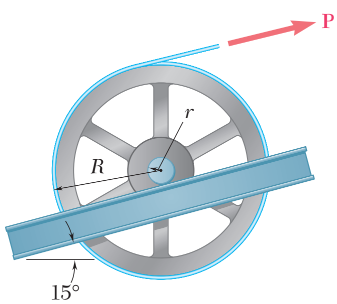


Fig. *P16.96* and *P16.97*

16.97 A 40-kg flywheel of radius $R = 0.5$ m is rigidly attached to a shaft of radius $r = 0.05$ m that can roll along parallel rails. A cord is attached as shown and pulled with a force **P**. Knowing the centroidal radius of gyration is $\bar{k} = 0.4$ m and the coefficient of static friction is $\mu_s = 0.4$, determine the largest magnitude of force **P** for which no slipping will occur.

16.98 through *16.101* A drum of 60-mm radius is attached to a disk of 120-mm radius. The disk and drum have a total mass of 6 kg and a combined radius of gyration of 90 mm. A cord is attached as shown and pulled with a force **P** of magnitude 20 N. Knowing that the disk rolls without sliding, determine (*a*) the angular acceleration of the disk and the acceleration of *G*, (*b*) the minimum value of the coefficient of static friction compatible with this motion.

16.102 through 16.105 A drum of 4-in. radius is attached to a disk of 8-in. radius. The disk and drum have a combined weight of 10 lb and a combined radius of gyration of 6 in. A cord is attached as shown and pulled with a force **P** of magnitude 5 lb. Knowing that the coefficients of static and kinetic friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, respectively, determine (*a*) whether or not the disk slides, (*b*) the angular acceleration of the disk and the acceleration of *G*.

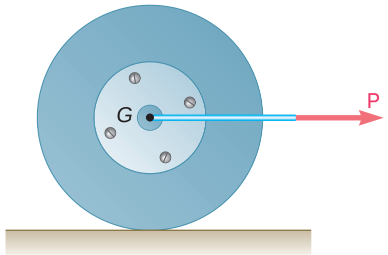


Fig. P16.98 and P16.102

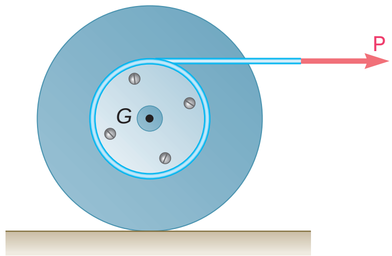


Fig. P16.99 and P16.103

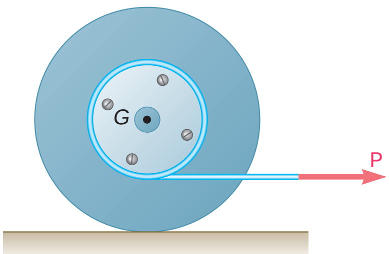


Fig. *P16.100* and P16.104

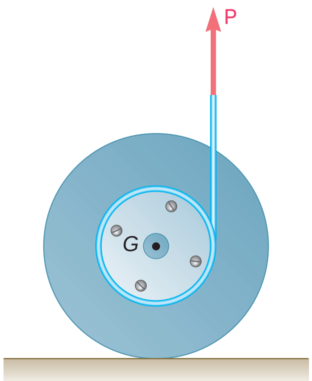


Fig. *P16.101* and P16.105

***16.106* and 16.107** A 12-in.-radius cylinder of weight 16 lb rests on a 6-lb carriage. The system is at rest when a force **P** of magnitude 4 lb is applied. Knowing that the cylinder rolls without sliding on the carriage and neglecting the mass of the wheels of the carriage, determine (*a*) the acceleration of the carriage, (*b*) the acceleration of point *A*, (*c*) the distance the cylinder has rolled with respect to the carriage after 0.5 s.

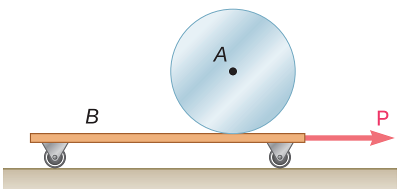


Fig. *P16.106*

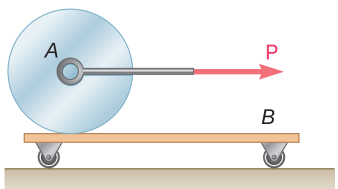


Fig. P16.107

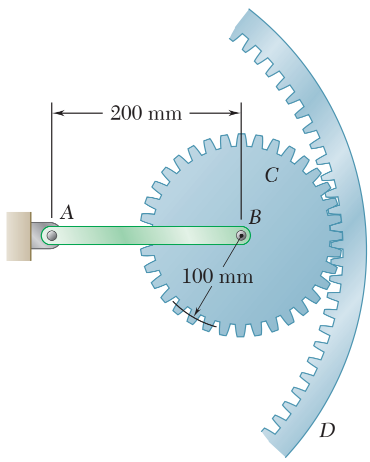


Fig. P16.108

16.108 Gear *C* has a mass of 5 kg and a centroidal radius of gyration of 75 mm. The uniform bar *AB* has a mass of 3 kg and gear *D* is stationary. If the system is released from rest in the position shown, determine (*a*) the angular acceleration of gear *C*, (*b*) the acceleration of point *B*.

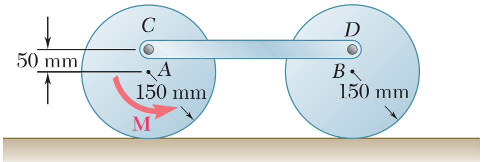


Fig. P16.109

16.109 Two uniform disks A and B, each of mass 2 kg, are connected by a 1.5-kg rod CD as shown. A counterclockwise couple **M** of moment 2.5 N-m is applied to disk A. Knowing that the disks roll without sliding, determine (*a*) the acceleration of the center of each disk, (*b*) the horizontal component of the force exerted on disk B by pin D.

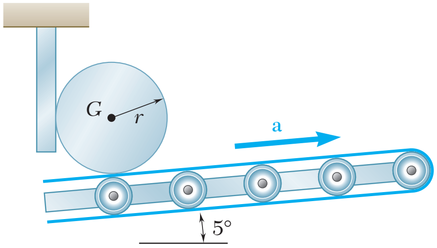


Fig. *P16.110*

16.110 A 10-lb cylinder of radius $r = 4$ in. is resting on a conveyor belt when the belt is suddenly turned on and it experiences an acceleration of magnitude $a = 6\ \text{ft/s}^2$. The smooth vertical bar holds the cylinder in place when the belt is not moving. Knowing the cylinder rolls without slipping and the friction between the vertical bar and the cylinder is negligible, determine (*a*) the angular acceleration of the cylinder, (*b*) the components of the force the conveyor belt applies to the cylinder.

16.111 A hemisphere of weight W and radius r is released from rest in the position shown. Determine (*a*) the minimum value of m_s for which the hemisphere starts to roll without sliding, (*b*) the corresponding acceleration of point B [*Hint:* Note that $OG = \frac{3}{8}r$ and that, by the parallel-axis theorem, $\bar{I} = \frac{2}{5}mr^2 - m(OG)^2$.]

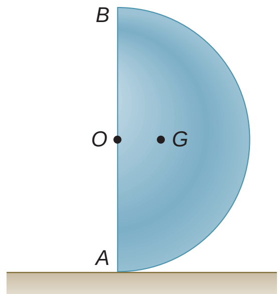


Fig. P16.111

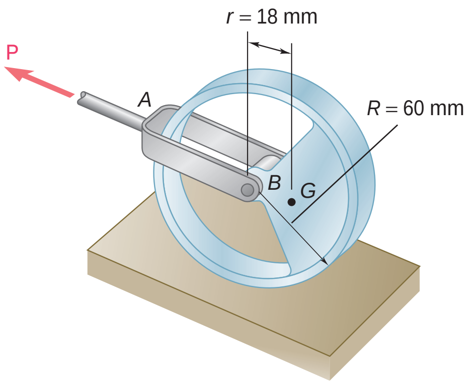


Fig. P16.113

16.112 Solve Prob. 16.111, considering a half cylinder instead of a hemisphere. [*Hint:* Note that $OG = 4r/3p$ and that, by the parallel-axis theorem, $\bar{I} = \frac{1}{2}mr^2 - m(OG)^2$.]

16.113 The center of gravity G of a 1.5-kg unbalanced tracking wheel is located at a distance $r = 18$ mm from its geometric center B. The radius of the wheel is $R = 60$ mm and its centroidal radius of gyration is 44 mm. At the instant shown the center B of the wheel has a velocity of 0.35 m/s and an acceleration of 1.2 m/s^2, both directed to the left. Knowing that the wheel rolls without sliding and neglecting the mass of the driving yoke AB, determine the horizontal force **P** applied to the yoke.

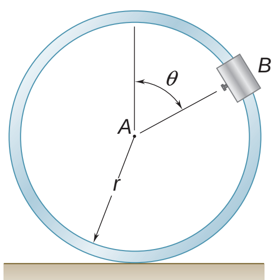


Fig. P16.114 and P16.115

16.114 A small clamp of mass m_B is attached at B to a hoop of mass m_h. The system is released from rest when $u = 90°$ and rolls without sliding. Knowing that $m_h = 3m_B$, determine (*a*) the angular acceleration of the hoop, (*b*) the horizontal and vertical components of the acceleration of B.

16.115 A small clamp of mass m_B is attached at B to a hoop of mass m_h. Knowing that the system is released from rest and rolls without sliding, derive an expression for the angular acceleration of the hoop in terms of m_B, m_h, r, and u.

16.116 A 4-lb bar is attached to a 10-lb uniform cylinder by a square pin, P, as shown. Knowing that $r = 16$ in., $h = 8$ in., $\theta = 20°$, $L = 20$ in., and $\omega = 2$ rad/s at the instant shown, determine the reactions at P at this instant assuming that the cylinder rolls without sliding down the incline.

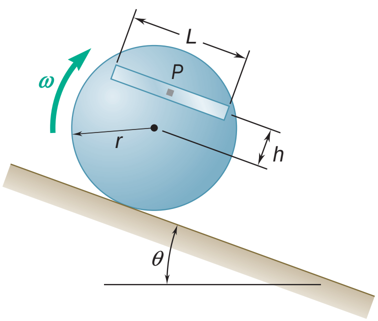


Fig. P16.116

16.117 The ends of the 20-lb uniform rod AB are attached to collars of negligible mass that slide without friction along fixed rods. If the rod is released from rest when $\theta = 25°$, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the reaction at A, (*c*) the reaction at B.

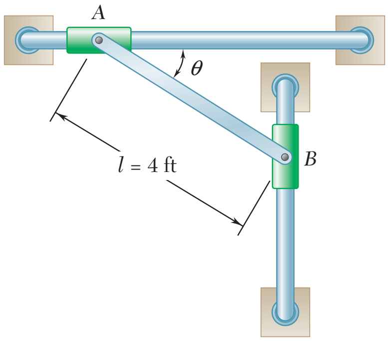


Fig. P16.117 and P16.118

16.118 The ends of the 20-lb uniform rod AB are attached to collars of negligible mass that slide without friction along fixed rods. A vertical force **P** is applied to collar B when $\theta = 25°$, causing the collar to start from rest with an upward acceleration of 40 ft/s^2. Determine (*a*) the force **P**, (*b*) the reaction at A.

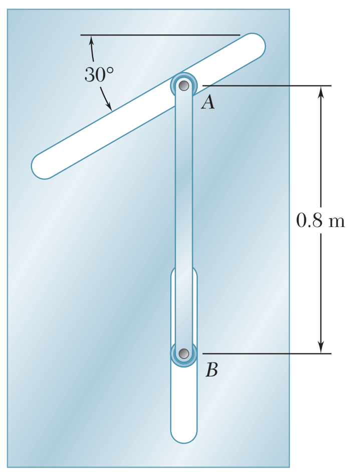


Fig. P16.119

16.119 The motion of the 3-kg uniform rod AB is guided by small wheels of negligible weight that roll along without friction in the slots shown. If the rod is released from rest in the position shown, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the reaction at B.

16.120 A beam AB of length L and mass m is supported by two cables as shown. If cable BD breaks, determine at that instant the tension in the remaining cable as a function of its initial angular orientation θ.

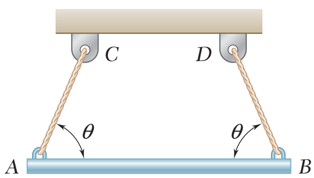


Fig. P16.120

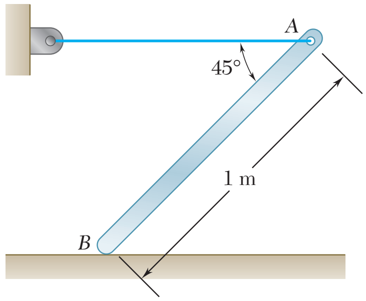


Fig. P16.121

16.121 End A of a uniform 10-kg bar is attached to a horizontal rope and end B contacts a floor with negligible friction. Knowing that the bar is released from rest in the position shown, determine immediately after release (*a*) the angular acceleration of the bar, (*b*) the tension in the rope, (*c*) the reaction at B.

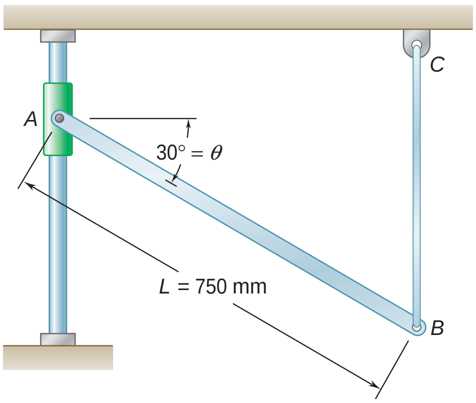


Fig. P16.122

16.122 End A of the 8-kg uniform rod AB is attached to a collar that can slide without friction on a vertical rod. End B of the rod is attached to a vertical cable BC. If the rod is released from rest in the position shown, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the reaction at A.

16.123 A uniform thin plate $ABCD$ has a mass of 8 kg and is held in position by three inextensible cords AE, BF, and CG. If cord AE is cut, determine at that instant (*a*) if the plate is undergoing translation or general plane motion, (*b*) the tension in cords BF and CG.

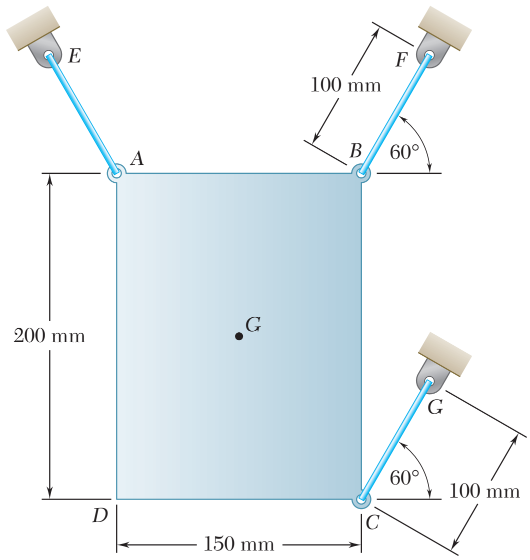


Fig. P16.123

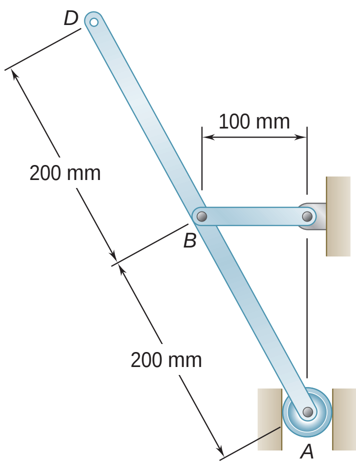


Fig. P16.124

16.124 The 4-kg uniform rod ABD is attached to the crank BC and is fitted with a small wheel that can roll without friction along a vertical slot. Knowing that at the instant shown crank BC rotates with an angular velocity of 6 rad/s clockwise and an angular acceleration of 15 rad/s^2 counterclockwise, determine the reaction at A.

16.125 The 7-lb uniform rod AB is connected to crank BD and to a collar of negligible weight, which can slide freely along rod EF. Knowing that in the position shown crank BD rotates with an angular velocity of 15 rad/s and an angular acceleration of 60 rad/s^2, both clockwise, determine the reaction at A.

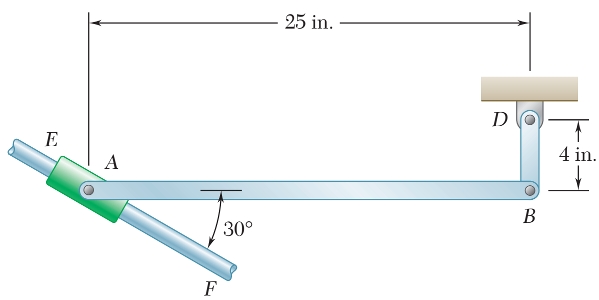


Fig. P16.125

16.126 In Prob. 16.125, determine the reaction at A, knowing that in the position shown crank BD rotates with an angular velocity of 15 rad/s clockwise and an angular acceleration of 60 rad/s^2 counterclockwise.

16.127 The 250-mm uniform rod BD, of mass 5 kg, is connected as shown to disk A and to a collar of negligible mass, that may slide freely along a vertical rod. Knowing that disk A rotates counterclockwise at a constant rate of 500 rpm, determine the reactions at D when $u = 0$.

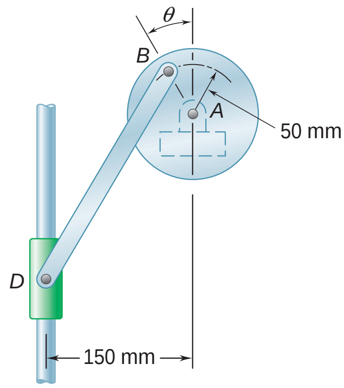


Fig. P16.127

16.128 Solve Prob. 16.127 when $u = 90°$.

16.129 The 4-kg uniform slender bar BD is attached to bar AB and a wheel of negligible mass that rolls on a circular surface. Knowing that at the instant shown bar AB has an angular velocity of 6 rad/s and no angular acceleration, determine the reaction at point D.

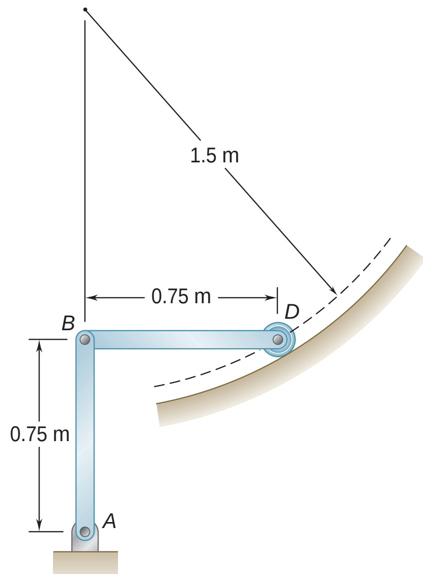


Fig. P16.129

16.130 The motion of the uniform slender rod of length $L = 0.5$ m and mass $m = 3$ kg is guided by pins at A and B that slide freely in frictionless slots, circular and horizontal, cut into a vertical plate as shown. Knowing that at the instant shown the rod has an angular velocity of 3 rad/s counterclockwise and $u = 30°$, determine the reactions at points A and B.

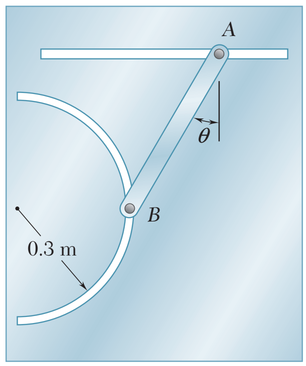


Fig. *P16.130*

16.131 At the instant shown, the 20-ft-long, uniform 100-lb pole ABC has an angular velocity of 1 rad/s counterclockwise and point C is sliding to the right. A 120-lb horizontal force **P** acts at B. Knowing the coefficient of kinetic friction between the pole and the ground is 0.3, determine at this instant (a) the acceleration of the center of gravity, (b) the normal force between the pole and the ground.

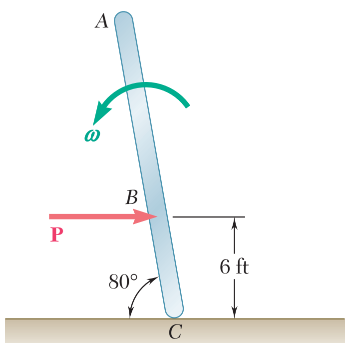


Fig. P16.131

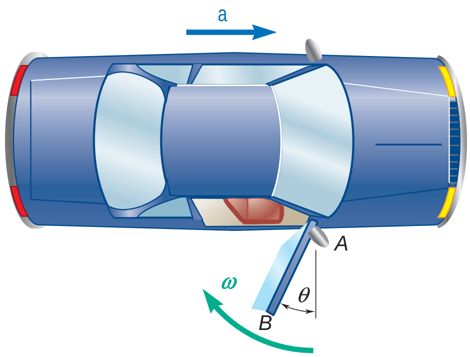


Fig. *P16.132*

16.132 A driver starts his car with the door on the passenger's side wide open (u = 0). The 80-lb door has a centroidal radius of gyration $\bar{k}$ = 12.5 in., and its mass center is located at a distance r = 22 in. from its vertical axis of rotation. Knowing that the driver maintains a constant acceleration of 6 ft/s^2, determine the angular velocity of the door as it slams shut (u = 90°).

16.133 For the car of Prob. 16.132, determine the smallest constant acceleration that the driver can maintain if the door is to close and latch, knowing that as the door hits the frame its angular velocity must be at least 2 rad/s for the latching mechanism to operate.

16.134 Two 8-lb uniform bars are connected to form the linkage shown. Neglecting the effect of friction, determine the reaction at D immediately after the linkage is released from rest in the position shown.

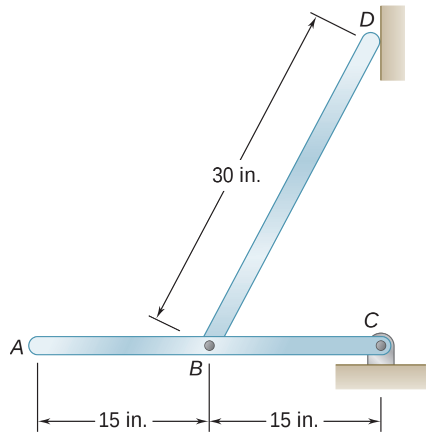


Fig. P16.134

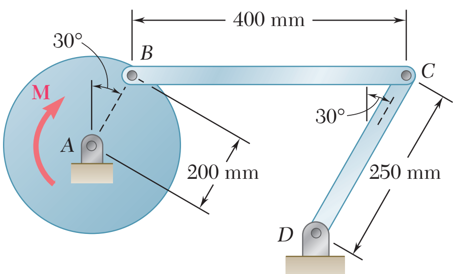


Fig. P16.135 and P16.136

***16.135** The 6-kg rod BC connects a 10-kg disk centered at A to a 5-kg rod CD. The motion of the system is controlled by the couple **M** applied to disk A. Knowing that at the instant shown disk A has an angular velocity of 36 rad/s clockwise and no angular acceleration, determine (a) the couple **M**, (b) the components of the force exerted at C on rod BC.

***16.136** The 6-kg rod BC connects a 10-kg disk centered at A to a 5-kg rod CD. The motion of the system is controlled by the couple **M** applied to disk A. Knowing that at the instant shown disk A has an angular velocity of 36 rad/s clockwise and an angular acceleration of 150 rad/s^2 counterclockwise, determine (a) the couple **M**, (b) the components of the force exerted at C on rod BC.

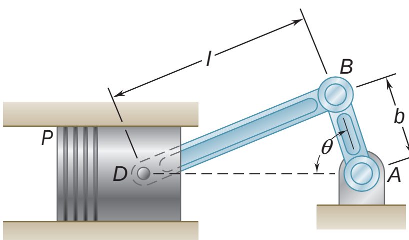


Fig. P16.137

16.137 In the engine system shown l = 250 mm and b = 100 mm. The connecting rod BD is assumed to be a 1.2-kg uniform slender rod and is attached to the 1.8-kg piston P. During a test of the system, crank AB is made to rotate with a constant angular velocity of 600 rpm clockwise with no force applied to the face of the piston. Determine the forces exerted on the connecting rod at B and D when u = 180°. (Neglect the effect of the weight of the rod.)

16.138 Solve Prob. 16.137 when u = 90°.

16.139 The 4-lb rod *AB* and the 6-lb rod *BC* are connected as shown to a disk that is made to rotate in a vertical plane at a constant angular velocity of 6 rad/s clockwise. For the position shown, determine the forces exerted at *A* and *B* on rod *AB*.

16.140 The 4-lb rod *AB* and the 6-lb rod *BC* are connected as shown to a disk that is made to rotate in a vertical plane. Knowing that at the instant shown the disk has an angular acceleration of 18 rad/s^2 clockwise and no angular velocity, determine the components of the forces exerted at *A* and *B* on rod *AB*.

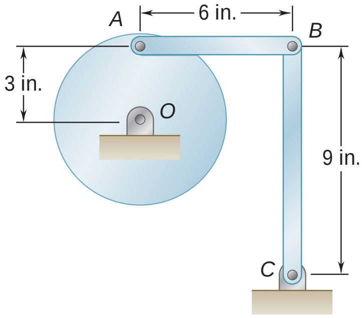


Fig. P16.139 and P16.140

16.141 Two rotating rods in the vertical plane are connected by a slider block *P* of negligible mass. The rod attached at *A* has a weight of 1.6 lb and a length of 8 in. Rod *BP* weighs 2 lb and is 10 in. long and the friction between block *P* and *AE* is negligible. The motion of the system is controlled by a couple **M** applied to rod *BP*. Knowing that rod *BP* has a constant angular velocity of 20 rad/s clockwise, determine (*a*) the couple **M**, (*b*) the components of the force exerted on *AE* by block *P*.

16.142 Two rotating rods in the vertical plane are connected by a slider block *P* of negligible mass. The rod attached at *A* has a mass of 0.8 kg and a length of 160 mm. Rod *BP* has a mass of 1 kg and is 200 mm long and the friction between block *P* and *AE* is negligible. The motion of the system is controlled by a couple **M** applied to bar *BP*. Knowing that at the instant shown rod *BP* has an angular velocity of 20 rad/s clockwise and an angular acceleration of 80 rad/s^2 clockwise, determine (*a*) the couple **M**, (*b*) the components of the force exerted on *AE* by block *P*.

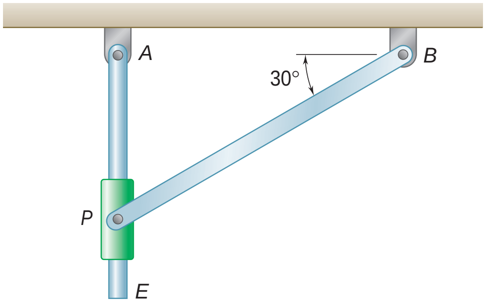


Fig. *P16.141* and *P16.142*

***16.143** Draw the shear and bending-moment diagrams for the rod of Prob. 16.77 immediately after the cable at *B* breaks.

***16.144** A uniform slender bar *AB* of mass *m* is suspended as shown from a uniform disk of the same mass *m*. Neglecting the effect of friction, determine the accelerations of points *A* and *B* immediately after a horizontal force **P** has been applied at *B*.

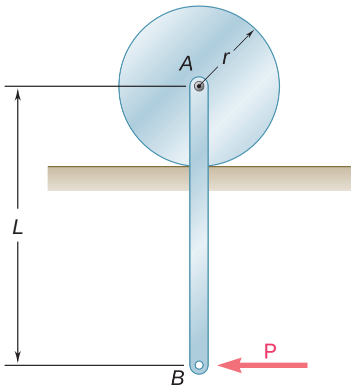


Fig. P16.144

16.145 A uniform rod *AB*, of mass 15 kg and length 1 m, is attached to the 20-kg cart *C*. Neglecting friction, determine immediately after the system has been released from rest, (*a*) the acceleration of the cart, (*b*) the angular acceleration of the rod.

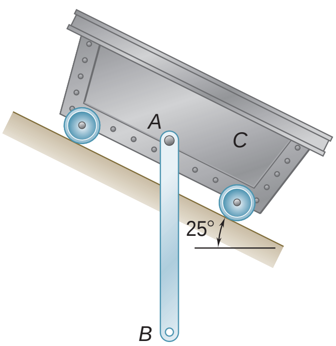


Fig. P16.145

***16.146** The 5-kg slender rod AB is pin-connected to an 8-kg uniform disk as shown. Immediately after the system is released from rest, determine the acceleration of (a) point A, (b) point B.

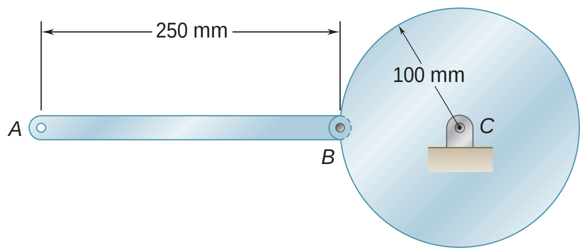


Fig. P16.146

***16.147 and *16.148** The 6-lb cylinder B and the 4-lb wedge A are held at rest in the position shown by cord C. Assuming that the cylinder rolls without sliding on the wedge and neglecting friction between the wedge and the ground, determine, immediately after cord C has been cut, (a) the acceleration of the wedge, (b) the angular acceleration of the cylinder.

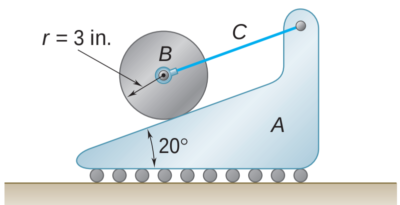


Fig. P16.147

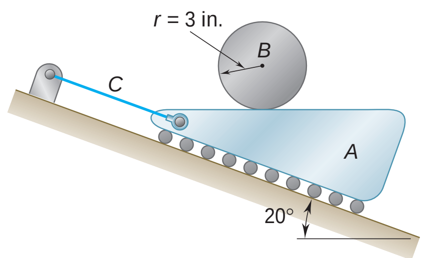


Fig. P16.148

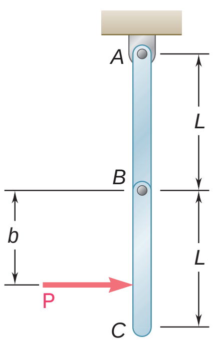


Fig. *P16.149* and *P16.150*

***16.149** Each of the 3-kg bars AB and BC is of length $L = 500$ mm. A horizontal force $\mathbf{P}$ of magnitude 20 N is applied to bar BC as shown. Knowing that $b = L$ ($\mathbf{P}$ is applied at C), determine the angular acceleration of each bar.

***16.150** Each of the 3-kg bars AB and BC is of length $L = 500$ mm. A horizontal force $\mathbf{P}$ of magnitude 20 N is applied to bar BC. For the position shown, determine (a) the distance b for which the bars move as if they formed a single rigid body, (b) the corresponding angular acceleration of the bars.

***16.151** (a) Determine the magnitude and the location of the maximum bending moment in the rod of Prob. 16.78. (b) Show that the answer to part a is independent of the weight of the rod.

***16.152** Draw the shear and bending-moment diagrams for the rod of Prob. 16.84 immediately after the cable at B breaks.

REVIEW AND SUMMARY

In this chapter, we studied the *kinetics of rigid bodies,* i.e., the relations existing between the forces acting on a rigid body, the shape and mass of the body, and the motion produced. Except for the first two sections, which apply to the most general case of the motion of a rigid body, our analysis was restricted to the *plane motion of rigid slabs* and rigid bodies symmetrical with respect to the reference plane. The study of the plane motion of nonsymmetrical rigid bodies and of the motion of rigid bodies in three-dimensional space will be considered in Chap. 18.

Fundamental equations of motion for a rigid body

We first recalled [Sec. 16.2] the two fundamental equations derived in Chap. 14 for the motion of a system of particles and observed that they apply in the most general case of the motion of a rigid body. The first equation defines the motion of the mass center G of the body; we have

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \qquad (16.1)$$

where m is the mass of the body and $\bar{\mathbf{a}}$ the acceleration of G. The second is related to the motion of the body relative to a centroidal frame of reference; we wrote

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \qquad (16.2)$$

where $\dot{\mathbf{H}}_G$ is the rate of change of the angular momentum $\mathbf{H}_G$ of the body about its mass center G. Together, Eqs. (16.1) and (16.2) express that *the system of the external forces is equipollent to the system consisting of the vector* $m\bar{\mathbf{a}}$ *attached at* G *and the couple of moment* $\dot{\mathbf{H}}_G$ (Fig. 16.19).

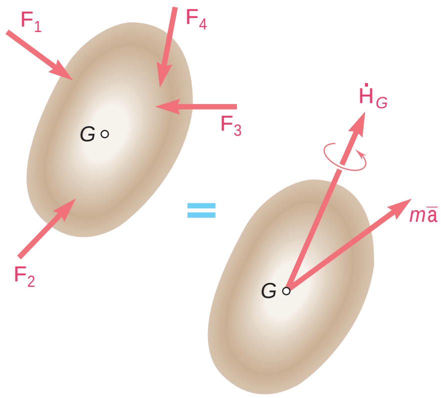


Fig. 16.19

Angular momentum in plane motion

Restricting our analysis at this point and for the rest of the chapter to the plane motion of rigid slabs and rigid bodies symmetrical with respect to the reference plane, we showed [Sec. 16.3] that the angular momentum of the body could be expressed as

$$\mathbf{H}_G = \bar{I}V \qquad (16.4)$$

where $\bar{I}$ is the moment of inertia of the body about a centroidal axis perpendicular to the reference plane and V is the angular velocity of the body. Differentiating both members of Eq. (16.4), we obtained

$$\dot{\mathbf{H}}_G = \bar{I}\dot{V} = \bar{I}A \qquad (16.5)$$

which shows that in the restricted case considered here, the rate of change of the angular momentum of the rigid body can be represented by a vector of the same direction as $\boldsymbol{\alpha}$ (i.e., perpendicular to the plane of reference) and of magnitude $\bar{I}\alpha$.

Equations for the plane motion of a rigid body

It follows from [Sec. 16.4] that the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane is defined by the three scalar equations

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma M_G = \bar{I}\alpha \tag{16.6}$$

D'Alembert's principle

It further follows that *the external forces acting on the rigid body are actually* ***equivalent*** *to the effective forces of the various particles forming the body*. This statement, known as *d'Alembert's principle*, can be expressed in the form of the vector diagram shown in Fig. 16.20, where the effective forces have been represented by a vector $m\bar{\mathbf{a}}$ attached at G and a couple $\bar{I}\boldsymbol{\alpha}$. In the particular case of a slab in *translation*, the effective forces shown in part *b* of this figure reduce to the single vector $m\bar{\mathbf{a}}$, while in the particular case of a slab in *centroidal rotation*, they reduce to the single couple $\bar{I}\boldsymbol{\alpha}$; in any other case of plane motion, both the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$ should be included.

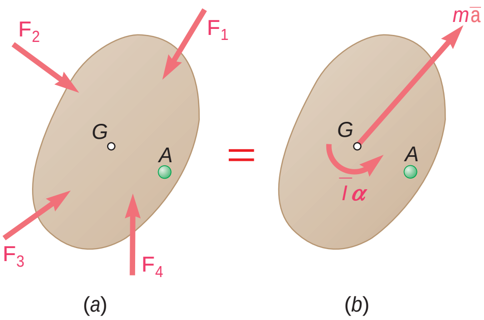


Fig. 16.20

Free-body-diagram equation

Any problem involving the plane motion of a rigid slab may be solved by drawing a *free-body-diagram equation* similar to that of Fig. 16.20 [Sec. 16.6]. Three equations of motion can then be obtained by equating the x components, y components, and moments about an arbitrary point A, of the forces and vectors involved [Sample Probs. 16.1, 16.2, 16.4, and 16.5]. An alternative solution can be obtained by adding to the external forces an *inertia vector* $-m\bar{\mathbf{a}}$ of sense opposite to that of $\bar{\mathbf{a}}$, attached at G, and an *inertia couple* $-\bar{I}\boldsymbol{\alpha}$ of sense opposite to that of $\boldsymbol{\alpha}$. The system obtained in this way is equivalent to zero, and the slab is said to be in *dynamic equilibrium*.

Connected rigid bodies

The method described above can also be used to solve problems involving the plane motion of several connected rigid bodies [Sec. 16.7]. A free-body-diagram equation is drawn for each part of the system and the equations of motion obtained are solved simultaneously. In some cases, however, a single diagram can be drawn for the entire system, including all the external forces as well as the vectors $m\bar{\mathbf{a}}$ and the couples $\bar{I}\boldsymbol{\alpha}$ associated with the various parts of the system [Sample Prob. 16.3].

Constrained plane motion

In the second part of the chapter, we were concerned with rigid bodies *moving under given constraints* [Sec. 16.8]. While the kinetic analysis of the constrained plane motion of a rigid slab is the same as above, it must be supplemented by a *kinematic analysis* which has for its object to express the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of the mass center G of the slab in terms of its angular acceleration α. Problems solved in this way included the *noncentroidal rotation* of rods and plates [Sample Probs. 16.6 and 16.7], the *rolling motion* of spheres and wheels [Sample Probs. 16.8 and 16.9], and the plane motion of *various types of linkages* [Sample Prob. 16.10].

REVIEW PROBLEMS

16.153 A cyclist is riding a bicycle at a speed of 20 mph on a horizontal road. The distance between the axles is 42 in., and the mass center of the cyclist and the bicycle is located 26 in. behind the front axle and 40 in. above the ground. If the cyclist applies the brakes only on the front wheel, determine the shortest distance in which he can stop without being thrown over the front wheel.

16.154 The forklift truck shown weighs 2250 lb and is used to lift a crate of weight $W = 2500$ lb. The truck is moving to the left at a speed of 10 ft/s when the brakes are applied on all four wheels. Knowing that the coefficient of static friction between the crate and the fork lift is 0.30, determine the smallest distance in which the truck can be brought to a stop if the crate is not to slide and if the truck is not to tip forward.

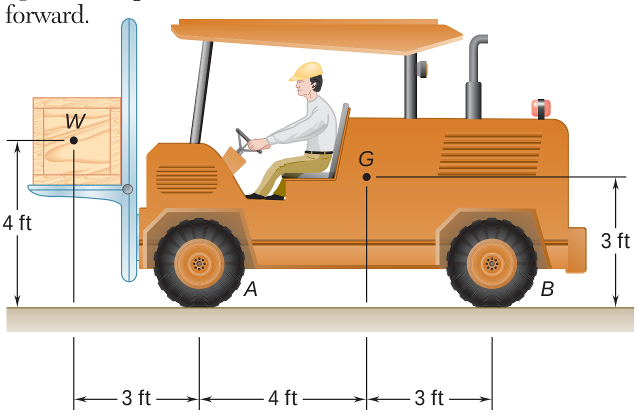


Fig. P16.154

16.155 A 5-kg uniform disk is attached to the 3-kg uniform rod BC by means of a frictionless pin AB. An elastic cord is wound around the edge of the disk and is attached to a ring at E. Both ring E and rod BC can rotate freely about the vertical shaft. Knowing that the system is released from rest when the tension in the elastic cord is 15 N, determine (*a*) the angular acceleration of the disk, (*b*) the acceleration of the center of the disk.

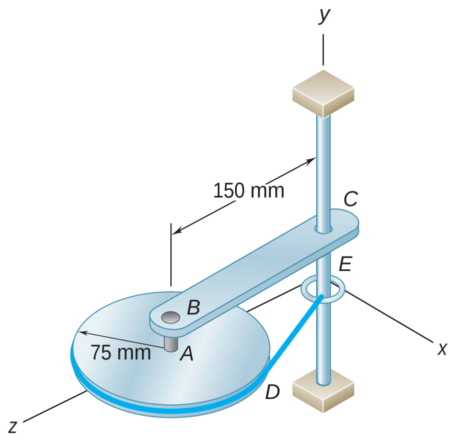


Fig. *P16.155*

16.156 Identical cylinders of mass m and radius r are pushed by a series of moving arms. Assuming the coefficient of friction between all surfaces to be m < 1 and denoting by a the magnitude of the acceleration of the arms, derive an expression for (a) the maximum allowable value of a if each cylinder is to roll without sliding, (b) the minimum allowable value of a if each cylinder is to move to the right without rotating.

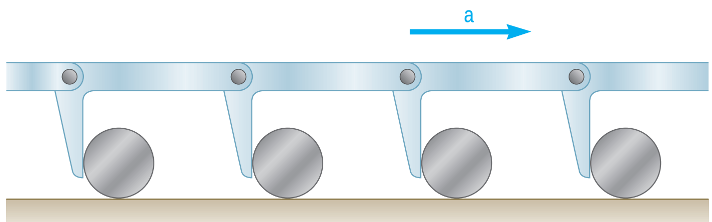


Fig. P16.156

16.157 The uniform rod AB of weight W is released from rest when b $= 70°$. Assuming that the friction force between end A and the surface is large enough to prevent sliding, determine immediately after release (a) the angular acceleration of the rod, (b) the normal reaction at A, (c) the friction force at A.

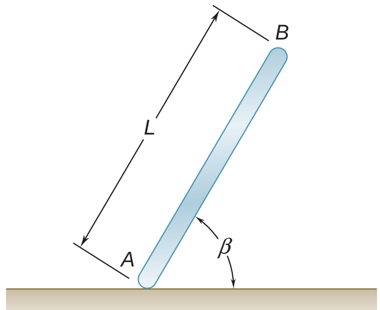


Fig. P16.157 and P16.158

16.158 The uniform rod AB of weight W is released from rest when b $= 70°$. Assuming that the friction force is zero between end A and the surface, determine immediately after release (a) the angular acceleration of the rod, (b) the acceleration of the mass center of the rod, (c) the reaction at A.

16.159 A bar of mass $m = 5$ kg is held as shown between four disks, each of mass $m' = 2$ kg and radius $r = 75$ mm. Knowing that the normal forces on the disks are sufficient to prevent any slipping, for each of the cases shown determine the acceleration of the bar immediately after it has been released from rest.

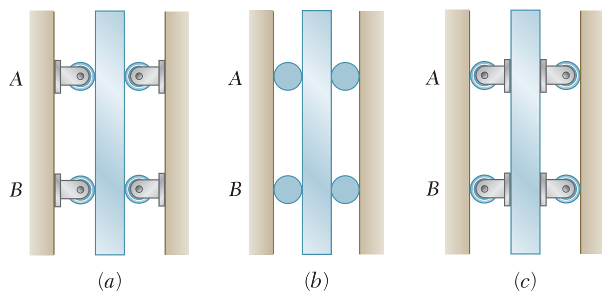


Fig. *P16.159*

16.160 A uniform plate of mass m is suspended in each of the ways shown. For each case determine immediately after the connection B has been released (a) the angular acceleration of the plate, (b) the acceleration of its mass center.

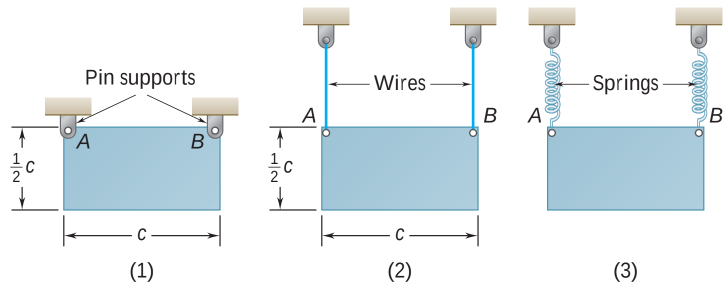


Fig. P16.160

16.161 A cylinder with a circular hole is rolling without slipping on a fixed curved surface as shown. The cylinder would have a weight of 16 lb without the hole, but with the hole it has a weight of 15 lb. Knowing that at the instant shown the disk has an angular velocity of 5 rad/s clockwise, determine (a) the angular acceleration of the disk, (b) the components of the reaction force between the cylinder and the ground at this instant.

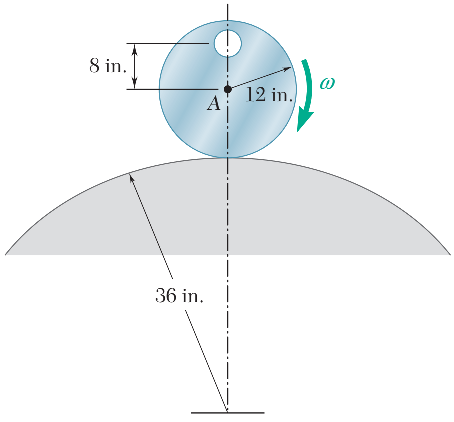


Fig. *P16.161*

16.162 The motion of a square plate of side 150 mm and mass 2.5 kg is guided by pins at corners A and B that slide in slots cut in a vertical wall. Immediately after the plate is released from rest in the position shown, determine (a) the angular acceleration of the plate, (b) the reaction at corner A.

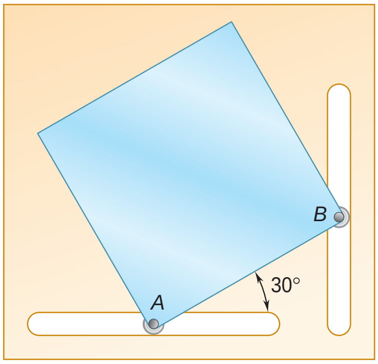


Fig. P16.162

16.163 The motion of a square plate of side 150 mm and mass 2.5 kg is guided by a pin at corner A that slides in a horizontal slot cut in a vertical wall. Immediately after the plate is released from rest in the position shown, determine (*a*) the angular acceleration of the plate, (*b*) the reaction at corner A.

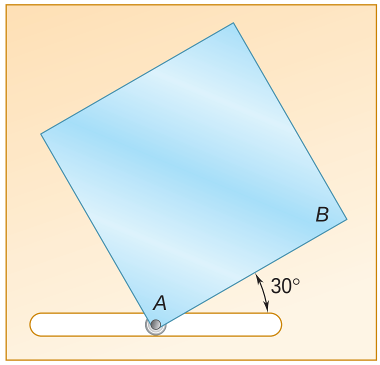


Fig. P16.163

16.164 The Geneva mechanism shown is used to provide an intermittent rotary motion of disk S. Disk D weighs 2 lb and has a radius of gyration of 0.9 in., and disk S weighs 6 lb and has a radius of gyration of 1.5 in. The motion of the system is controlled by a couple **M** applied to disk D. A pin P is attached to disk D and can slide in one of the six equally spaced slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each of the six slots; this will occur if the distance between the centers of the disks and the radii of the disks are related as shown. Knowing disk D rotates with a constant counterclockwise angular velocity of 8 rad/s and the friction between the slot and pin P is negligible, determine when f $= 150°$ (*a*) the couple **M**, (*b*) the magnitude of the force pin P applies to disk S.

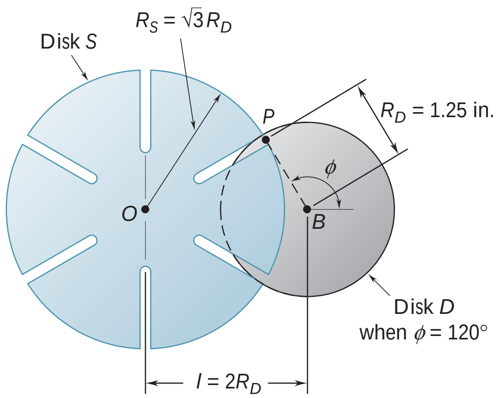


Fig. P16.164

COMPUTER PROBLEMS

16.C1 The 5-lb rod AB is released from rest in the position shown. (*a*) Assuming that the friction force between end A and the surface is large enough to prevent sliding, using software calculate the normal reaction and the friction force at A immediately after release for values of b from 0 to 85°. (*b*) Knowing that the coefficient of static friction between the rod and the floor is actually equal to 0.50, determine the range of values of b for which the rod will slip immediately after being released from rest.

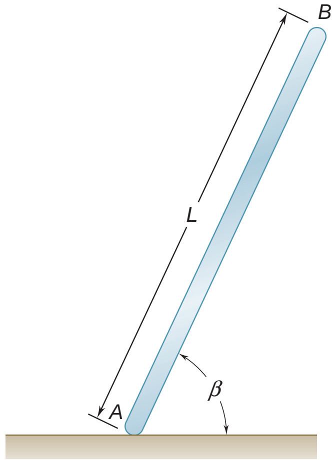


Fig. P16.C1

16.C2 End A of the 5-kg rod AB is moved to the left at a constant speed v_A = 1.5 m/s. Using computational software calculate and plot the normal reactions at ends A and B of the rod for values of u from 0 to 50°. Determine the value of u at which end B of the rod loses contact with the wall.

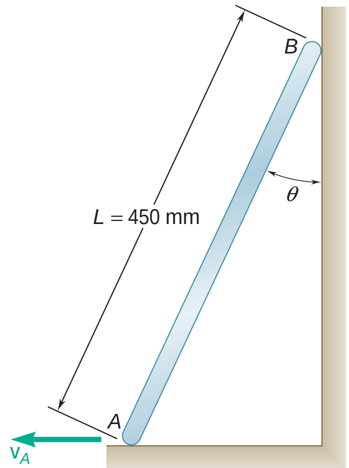


Fig. P16.C2

16.C3 A 30-lb cylinder of diameter b = 8 in. and height h = 6 in. is placed on a 10-lb platform CD that is held in the position shown by three cables. It is desired to determine the minimum value of m_s between the cylinder and the platform for which the cylinder does not slip on the platform, immediately after cable AB is cut. Using computational software calculate and plot the minimum allowable value of m_s for values of u from 0 to 30°. Knowing that the actual value of m_s is 0.60, determine the value of u at which slipping impends.

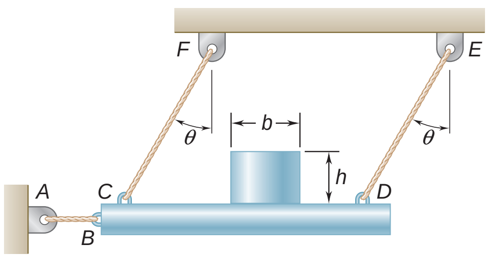


Fig. P16.C3

16.C4 For the engine system of Prob. 15.C3 of Chap. 15, the masses of piston P and the connecting rod BD are 2.5 kg and 3 kg, respectively. Knowing that during a test of the system no force is applied to the face of the piston, use computational software to calculate and plot the horizontal and vertical components of the dynamic reactions exerted on the connecting rod at B and D for values of u from 0 to 180°.

16.C5 A uniform slender bar AB of mass m is suspended from springs AC and BD as shown. Using computational software calculate and plot the accelerations of ends A and B, immediately after spring AC has broken, for values of u from 0 to 90°.

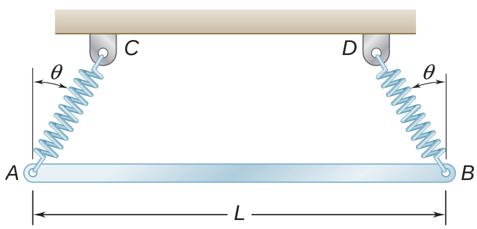
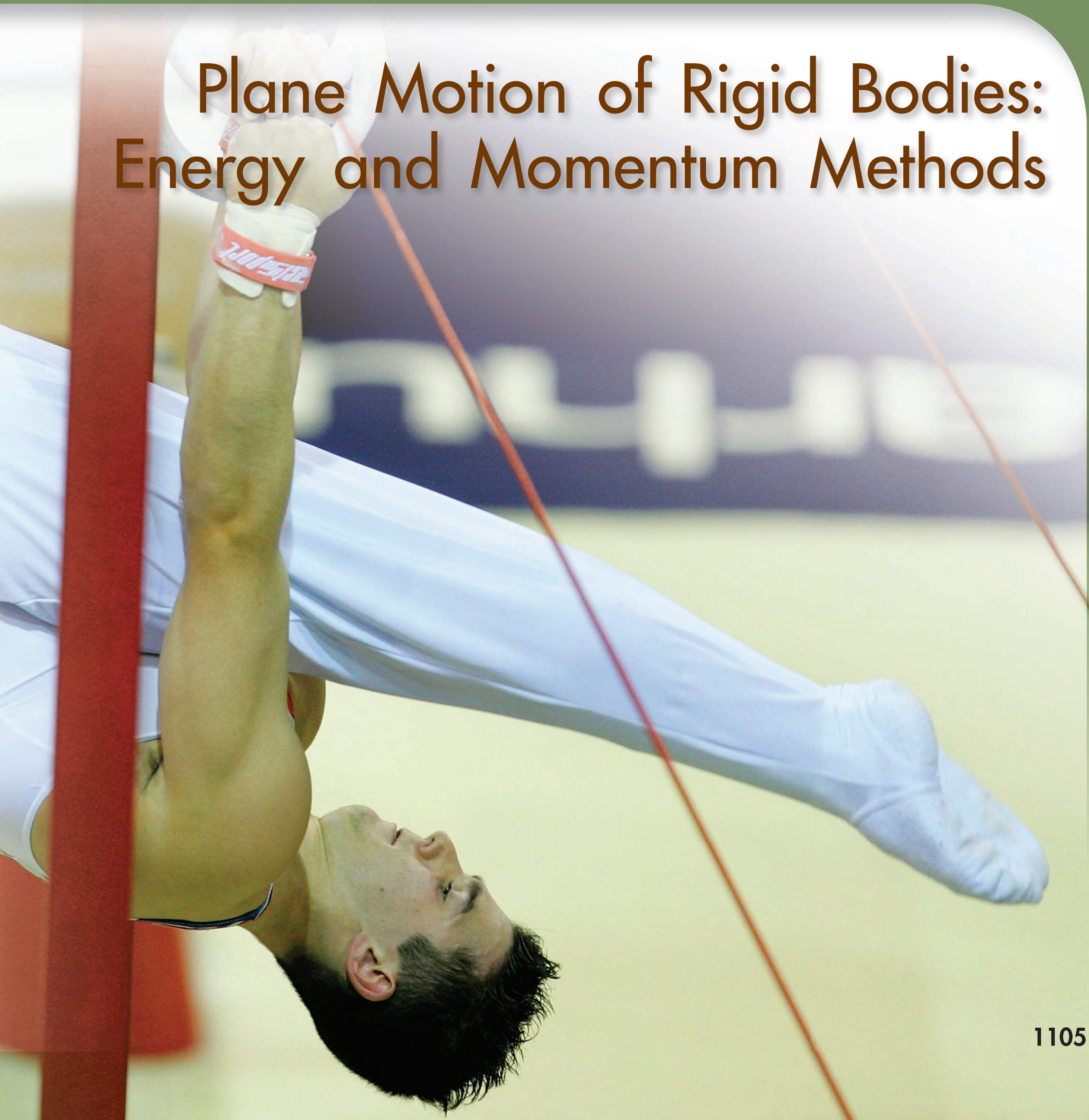


Fig. P16.C5

In this chapter the energy and momentum methods will be added to the tools available for your study of the motion of rigid bodies. For example, by using the principle of conservation of energy and direct application of Newton's second law, the forces exerted on the hands of this gymnast can be determined as he swings from one stationary hold to another.

CHAPTER 17

Plane Motion of Rigid Bodies: Energy and Momentum Methods



Chapter 17 Plane Motion of Rigid Bodies: Energy and Momentum Methods

Photo 17.1 The work done by friction reduces the kinetic energy of the wheel.

17.1 INTRODUCTION

In this chapter the method of work and energy and the method of impulse and momentum will be used to analyze the plane motion of rigid bodies and of systems of rigid bodies.

The method of work and energy will be considered first. In Secs. 17.2 through 17.5, the work of a force and of a couple will be defined, and an expression for the kinetic energy of a rigid body in plane motion will be obtained. The principle of work and energy will then be used to solve problems involving displacements and velocities. In Sec. 17.6, the principle of conservation of energy will be applied to the solution of a variety of engineering problems.

In the second part of the chapter, the principle of impulse and momentum will be applied to the solution of problems involving velocities and time (Secs. 17.8 and 17.9) and the concept of conservation of angular momentum will be introduced and discussed (Sec. 17.10).

In the last part of the chapter (Secs. 17.11 and 17.12), problems involving the eccentric impact of rigid bodies will be considered. As was done in Chap. 13, where we analyzed the impact of particles, the coefficient of restitution between the colliding bodies will be used together with the principle of impulse and momentum in the solution of impact problems. It will also be shown that the method used is applicable not only when the colliding bodies move freely after the impact but also when the bodies are partially constrained in their motion.

17.2 PRINCIPLE OF WORK AND ENERGY FOR A RIGID BODY

The principle of work and energy will now be used to analyze the plane motion of rigid bodies. As was pointed out in Chap. 13, the method of work and energy is particularly well adapted to the solution of problems involving velocities and displacements. Its main advantage resides in the fact that the work of forces and the kinetic energy of particles are scalar quantities.

In order to apply the principle of work and energy to the analysis of the motion of a rigid body, it will again be assumed that the rigid body is made of a large number n of particles of mass Δm_i. Recalling Eq. (14.30) of Sec. 14.8, we write

$$\boxed{T_1 + U_{1 y 2} = T_2} \tag{17.1}$$

where T_1, T_2 = initial and final values of total kinetic energy of particles forming the rigid body

$U_{1 y 2}$ = work of all forces acting on various particles of the body

The total kinetic energy

$$T = \frac{1}{2}\sum_{i=1}^{n} \Delta m_i\, v_i^2 \tag{17.2}$$

is obtained by adding positive scalar quantities and is itself a positive scalar quantity. You will see later how T can be determined for various types of motion of a rigid body.

The expression $U_{1 y 2}$ in (17.1) represents the work of all the forces acting on the various particles of the body, whether these forces are internal or external. However, as you will see presently, the total work of the internal forces holding together the particles of a rigid body is zero. Consider two particles A and B of a rigid body and the two equal and opposite forces **F** and $-$**F** they exert on each other (Fig. 17.1). While, in general, small displacements $d\mathbf{r}$ and $d\mathbf{r}'$ of the two particles are different, the components of these displacements along AB must be equal; otherwise, the particles would not remain at the same distance from each other and the body would not be rigid. Therefore, the work of **F** is equal in magnitude and opposite in sign to the work of $-$**F**, and their sum is zero. Thus, the total work of the internal forces acting on the particles of a rigid body is zero, and *the expression* $U_{1 y 2}$ *in Eq. (17.1) reduces to the work of the external forces* acting on the body during the displacement considered.

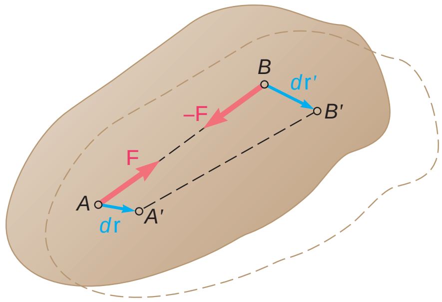


Fig. 17.1

17.3 WORK OF FORCES ACTING ON A RIGID BODY

We saw in Sec. 13.2 that the work of a force **F** during a displacement of its point of application from A_1 to A_2 is

$$U_{1 y 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \tag{17.3}$$

or

$$U_{1 y 2} = \int_{s_1}^{s_2} (F \cos \mathsf{a})\, ds \tag{17.3$'$}$$

where F is the magnitude of the force, a is the angle it forms with the direction of motion of its point of application A, and s is the variable of integration which measures the distance traveled by A along its path.

In computing the work of the external forces acting on a rigid body, it is often convenient to determine the work of a couple without considering separately the work of each of the two forces forming the couple. Consider the two forces **F** and $-$**F** forming a couple of moment **M** and acting on a rigid body (Fig. 17.2). Any small displacement of the rigid body bringing A and B, respectively, into A' and B'' can be divided into two parts: in one part points A and B undergo equal displacements $d\mathbf{r}_1$; in the other part A' remains fixed while B' moves into B'' through a displacement $d\mathbf{r}_2$ of magnitude $ds_2 = r\, d\mathsf{u}$. In the first part of the motion, the work of **F** is equal in magnitude and opposite in sign to the work of $-$**F** and their sum is zero. In the second part of the motion, only force **F** works, and its work is $dU = F\, ds_2 = Fr\, d\mathsf{u}$. But the product Fr is equal to the magnitude M of the moment of the couple. Thus, the work of a couple of moment **M** acting on a rigid body is

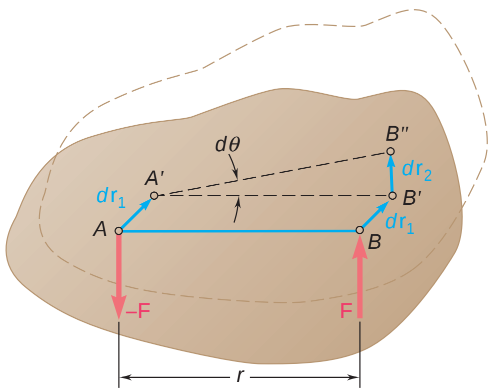


Fig. 17.2

$$dU = M\, d\mathsf{u} \tag{17.4}$$

where $d\mathsf{u}$ is the small angle expressed in radians through which the body rotates. We again note that work should be expressed in units obtained by multiplying units of force by units of length. The work of the couple during a finite rotation of the rigid body is obtained by integrating both members of (17.4) from the initial value u_1 of the angle u to its final value u_2. We write

$$U_{1\,\mathrm{y}\,2} = \int_{\mathrm{u}_1}^{\mathrm{u}_2} M\, d\mathrm{u} \tag{17.5}$$

When the moment **M** *of the couple is constant,* formula (17.5) reduces to

$$U_{1\,\mathrm{y}\,2} = M(\mathrm{u}_2 - \mathrm{u}_1) \tag{17.6}$$

It was pointed out in Sec. 13.2 that a number of forces encountered in problems of kinetics *do no work.* They are forces applied to fixed points or acting in a direction perpendicular to the displacement of their point of application. Among the forces which do no work the following have been listed: the reaction at a frictionless pin when the body supported rotates about the pin, the reaction at a frictionless surface when the body in contact moves along the surface, and the weight of a body when its center of gravity moves horizontally. We can add now that *when a rigid body rolls without sliding on a fixed surface, the friction force* **F** *at the point of contact C does no work.* The velocity $\mathbf{v}_C$ of the point of contact C is zero, and the work of the friction force **F** during a small displacement of the rigid body is

$$dU = F\, ds_C = F(v_C\, dt) = 0$$

17.4 KINETIC ENERGY OF A RIGID BODY IN PLANE MOTION

Consider a rigid body of mass m in plane motion. We recall from Sec. 14.7 that, if the absolute velocity $\mathbf{v}_i$ of each particle P_i of the body is expressed as the sum of the velocity $\bar{\mathbf{v}}$ of the mass center G of the body and of the velocity $\mathbf{v}'_i$ of the particle relative to a frame $Gx'y'$ attached to G and of fixed orientation (Fig. 17.3), the kinetic energy of the system of particles forming the rigid body can be written in the form

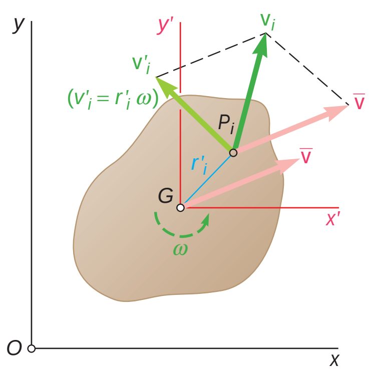


Fig. 17.3

$$T = \tfrac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n} \Delta m_i v_i'^2 \tag{17.7}$$

But the magnitude v'_i of the relative velocity of P_i is equal to the product $r'_i\mathrm{v}$ of the distance r'_i of P_i from the axis through G perpendicular to the plane of motion and of the magnitude v of the angular velocity of the body at the instant considered. Substituting into (17.7), we have

$$T = \tfrac{1}{2}m\bar{v}^2 + \frac{1}{2}\left(\sum_{i=1}^{n} r_i'^2\, \Delta m_i\right) w^2 \tag{17.8}$$

or, since the sum represents the moment of inertia $\bar{I}$ of the body about the axis through G,

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\mathrm{v}^2 \tag{17.9}$$

We note that in the particular case of a body in translation ($\omega = 0$), the expression obtained reduces to $\frac{1}{2}m\bar{v}^2$, while in the case of a centroidal rotation ($\bar{v} = 0$), it reduces to $\frac{1}{2}\bar{I}\omega^2$. We conclude that the kinetic energy of a rigid body in plane motion can be separated into two parts: (1) the kinetic energy $\frac{1}{2}m\bar{v}^2$ associated with the motion of the mass center G of the body, and (2) the kinetic energy $\frac{1}{2}\bar{I}\omega^2$ associated with the rotation of the body about G.

Noncentroidal Rotation. The relation (17.9) is valid for any type of plane motion and can therefore be used to express the kinetic energy of a rigid body rotating with an angular velocity $\boldsymbol{\omega}$ about a fixed axis through O (Fig. 17.4). In that case, however, the kinetic energy of the body can be expressed more directly by noting that the speed v_i of the particle P_i is equal to the product $r_i\omega$ of the distance r_i of P_i from the fixed axis and the magnitude ω of the angular velocity of the body at the instant considered. Substituting into (17.2), we write

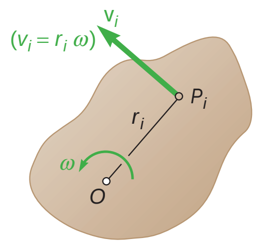


Fig. 17.4

$$T = \frac{1}{2}\sum_{i=1}^{n} \Delta m_i (r_i\omega)^2 = \frac{1}{2}\left(\sum_{i=1}^{n} r_i^2\,\Delta m_i\right)\omega^2$$

or, since the last sum represents the moment of inertia I_O of the body about the fixed axis through O,

$$\boxed{T = \tfrac{1}{2}I_O\omega^2} \tag{17.10}$$

We note that the results obtained are not limited to the motion of plane slabs or to the motion of bodies which are symmetrical with respect to the reference plane, and can be applied to the study of the plane motion of any rigid body, regardless of its shape. However, since Eq. (17.9) is applicable to any plane motion while Eq. (17.10) is applicable only in cases involving noncentroidal rotation, Eq. (17.9) will be used in the solution of all the sample problems.

17.5 SYSTEMS OF RIGID BODIES

When a problem involves several rigid bodies, each rigid body can be considered separately and the principle of work and energy can be applied to each body. Adding the kinetic energies of all the particles and considering the work of all the forces involved, we can also write the equation of work and energy for the entire system. We have

$$T_1 + U_{1\rightarrow 2} = T_2 \tag{17.11}$$

where T represents the arithmetic sum of the kinetic energies of the rigid bodies forming the system (all terms are positive) and $U_{1\rightarrow 2}$ represents the work of all the forces acting on the various bodies, whether these forces are *internal* or *external* from the point of view of the system as a whole.

The method of work and energy is particularly useful in solving problems involving pin-connected members, blocks and pulleys connected by inextensible cords, and meshed gears. In all these cases, the internal forces occur by pairs of equal and opposite forces, and the points of application of the forces in each pair *move through equal distances* during a small displacement of the system. As a result, the work of the internal forces is zero and $U_{1\rightarrow 2}$ reduces to the work of the *forces external to the system.*

17.6 CONSERVATION OF ENERGY

We saw in Sec. 13.6 that the work of conservative forces, such as the weight of a body or the force exerted by a spring, can be expressed as a change in potential energy. When a rigid body, or a system of rigid bodies, moves under the action of conservative forces, the principle of work and energy stated in Sec. 17.2 can be expressed in a modified form. Substituting for $U_{1\rightarrow 2}$ from (13.19′) into (17.1), we write

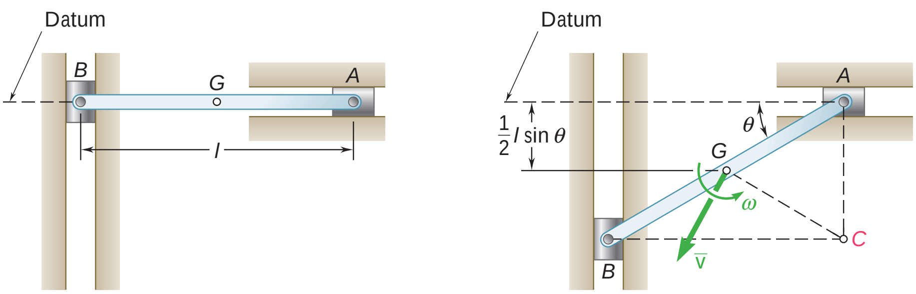
$$\boxed{T_1 + V_1 = T_2 + V_2} \tag{17.12}$$

Formula (17.12) indicates that when a rigid body, or a system of rigid bodies, moves under the action of conservative forces, *the sum of the kinetic energy and of the potential energy of the system remains constant.* It should be noted that in the case of the plane motion of a rigid body, the kinetic energy of the body should include both the *translational* term $\frac{1}{2}m\bar{v}^2$ and the *rotational* term $\frac{1}{2}\bar{I}\omega^2$.

As an example of application of the principle of conservation of energy, let us consider a slender rod AB, of length l and mass m, whose extremities are connected to blocks of negligible mass sliding along horizontal and vertical tracks. We assume that the rod is released with no initial velocity from a horizontal position (Fig. 17.5*a*), and we wish to determine its angular velocity after it has rotated through an angle θ (Fig. 17.5*b*).

Since the initial velocity is zero, we have $T_1 = 0$. Measuring the potential energy from the level of the horizontal track, we write $V_1 = 0$. After the rod has rotated through θ, the center of gravity G of the rod is at a distance $\frac{1}{2}l \sin\theta$ below the reference level and we have

$$V_2 = -\tfrac{1}{2}Wl\sin\theta = -\tfrac{1}{2}mgl\sin\theta$$



(*a*) (*b*)

Fig. 17.5

Observing that in this position the instantaneous center of the rod is located at C and that $CG = \frac{1}{2}l$, we write $\bar{v}_2 = \frac{1}{2}l\mathsf{v}$ and obtain

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\mathsf{v}_2^2 = \tfrac{1}{2}m(\tfrac{1}{2}l\mathsf{v})^2 + \tfrac{1}{2}(\tfrac{1}{12}ml^2)\mathsf{v}^2$$
$$= \frac{1}{2}\frac{ml^2}{3}\mathsf{v}^2$$

Applying the principle of conservation of energy, we write

$$T_1 + V_1 = T_2 + V_2$$
$$0 = \frac{1}{2}\frac{ml^2}{3}\mathsf{v}^2 - \tfrac{1}{2}mgl \sin \mathsf{u}$$
$$\mathsf{v} = \left(\frac{3g}{l}\sin \mathsf{u}\right)^{1/2}$$

The advantages of the method of work and energy, as well as its shortcomings, were indicated in Sec. 13.4. Here we should add that the method of work and energy must be supplemented by the application of d'Alembert's principle when reactions at fixed axles, rollers, or sliding blocks are to be determined. For example, in order to compute the reactions at the extremities A and B of the rod of Fig. 17.5*b*, a diagram should be drawn to express that the system of the external forces applied to the rod is equivalent to the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\mathsf{A}$. The angular velocity V of the rod, however, is determined by the method of work and energy before the equations of motion are solved for the reactions. The complete analysis of the motion of the rod and of the forces exerted on the rod requires, therefore, the combined use of the method of work and energy and of the principle of equivalence of the external and effective forces.

17.7 POWER

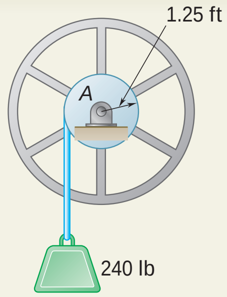
Power was defined in Sec. 13.5 as the time rate at which work is done. In the case of a body acted upon by a force $\mathbf{F}$, and moving with a velocity $\mathbf{v}$, the power was expressed as follows:

$$\text{Power} = \frac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad (13.13)$$

In the case of a rigid body rotating with an angular velocity V and acted upon by a couple of moment $\mathbf{M}$ parallel to the axis of rotation, we have, by (17.4),

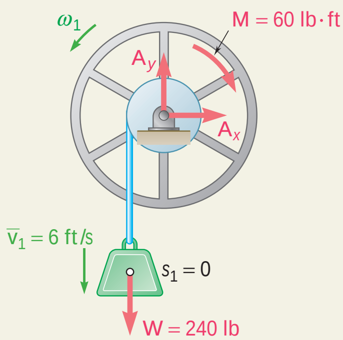
$$\text{Power} = \frac{dU}{dt} = \frac{M\,d\mathsf{u}}{dt} = M\mathsf{v} \qquad (17.13)$$

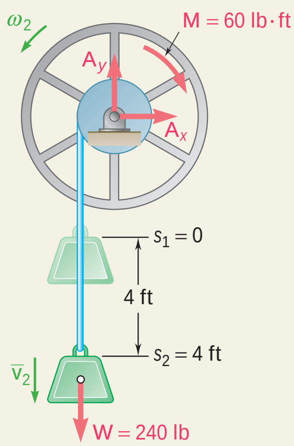
The various units used to measure power, such as the watt and the horsepower, were defined in Sec. 13.5.



SAMPLE PROBLEM 17.1

A 240-lb block is suspended from an inextensible cable which is wrapped around a drum of 1.25-ft radius rigidly attached to a flywheel. The drum and flywheel have a combined centroidal moment of inertia $\bar{I} = 10.5$ lb · ft · s^2. At the instant shown, the velocity of the block is 6 ft/s directed downward. Knowing that the bearing at A is poorly lubricated and that the bearing friction is equivalent to a couple $\mathbf{M}$ of magnitude 60 lb · ft, determine the velocity of the block after it has moved 4 ft downward.





SOLUTION

We consider the system formed by the flywheel and the block. Since the cable is inextensible, the work done by the internal forces exerted by the cable cancels. The initial and final positions of the system and the external forces acting on the system are as shown.

Kinetic Energy. ***Position 1.***

Block: $\bar{v}_1 = 6$ ft/s

Flywheel: $\omega_1 = \dfrac{\bar{v}_1}{r} = \dfrac{6 \text{ ft/s}}{1.25 \text{ ft}} = 4.80$ rad/s

$$
\begin{aligned}
T_1 &= \tfrac{1}{2}m\bar{v}_1^2 + \tfrac{1}{2}\bar{I}\omega_1^2 \\
&= \frac{1}{2}\frac{240 \text{ lb}}{32.2 \text{ ft/s}^2}(6 \text{ ft/s})^2 + \tfrac{1}{2}(10.5 \text{ lb}\cdot\text{ft}\cdot\text{s}^2)(4.80 \text{ rad/s})^2 \\
&= 255 \text{ ft}\cdot\text{lb}
\end{aligned}
$$

Position 2. Noting that $\omega_2 = \bar{v}_2/1.25$, we write

$$
\begin{aligned}
T_2 &= \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 \\
&= \frac{1}{2}\frac{240}{32.2}(\bar{v}_2)^2 + (\tfrac{1}{2})(10.5)\left(\frac{\bar{v}_2}{1.25}\right)^2 = 7.09\bar{v}_2^2
\end{aligned}
$$

Work. During the motion, only the weight $\mathbf{W}$ of the block and the friction couple $\mathbf{M}$ do work. Noting that $\mathbf{W}$ does positive work and that the friction couple $\mathbf{M}$ does negative work, we write

$$
s_1 = 0 \qquad s_2 = 4 \text{ ft}
$$

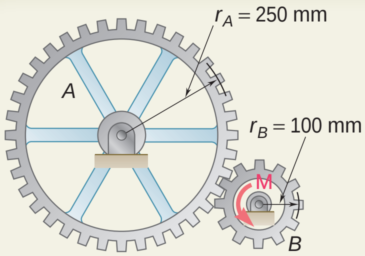
$$
\theta_1 = 0 \qquad \theta_2 = \frac{s_2}{r} = \frac{4 \text{ ft}}{1.25 \text{ ft}} = 3.20 \text{ rad}
$$

$$
\begin{aligned}
U_{1\rightarrow 2} &= W(s_2 - s_1) - M(\theta_2 - \theta_1) \\
&= (240 \text{ lb})(4 \text{ ft}) - (60 \text{ lb}\cdot\text{ft})(3.20 \text{ rad}) \\
&= 768 \text{ ft}\cdot\text{lb}
\end{aligned}
$$

Principle of Work and Energy

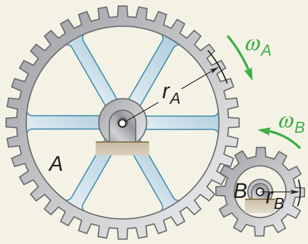
$$
\begin{aligned}
T_1 + U_{1\rightarrow 2} &= T_2 \\
255 \text{ ft}\cdot\text{lb} + 768 \text{ ft}\cdot\text{lb} &= 7.09\bar{v}_2^2 \\
\bar{v}_2 &= 12.01 \text{ ft/s} \qquad \bar{\mathbf{v}}_2 = 12.01 \text{ ft/s} \downarrow
\end{aligned}
$$

SAMPLE PROBLEM 17.2



Gear A has a mass of 10 kg and a radius of gyration of 200 mm; gear B has a mass of 3 kg and a radius of gyration of 80 mm. The system is at rest when a couple **M** of magnitude 6 N · m is applied to gear B. Neglecting friction, determine (*a*) the number of revolutions executed by gear B before its angular velocity reaches 600 rpm, (*b*) the tangential force which gear B exerts on gear A.

SOLUTION



Motion of Entire System. Noting that the peripheral speeds of the gears are equal, we write

$$r_A\omega_A = r_B\omega_B \qquad \omega_A = \omega_B\frac{r_B}{r_A} = \omega_B\frac{100\text{ mm}}{250\text{ mm}} = 0.40\omega_B$$

For $\omega_B = 600$ rpm, we have

$$\omega_B = 62.8\text{ rad/s} \qquad \omega_A = 0.40\omega_B = 25.1\text{ rad/s}$$
$$\bar{I}_A = m_A\bar{k}_A^2 = (10\text{ kg})(0.200\text{ m})^2 = 0.400\text{ kg}\cdot\text{m}^2$$
$$\bar{I}_B = m_B\bar{k}_B^2 = (3\text{ kg})(0.080\text{ m})^2 = 0.0192\text{ kg}\cdot\text{m}^2$$

Kinetic Energy. Since the system is initially at rest, $T_1 = 0$. Adding the kinetic energies of the two gears when $\omega_B = 600$ rpm, we obtain

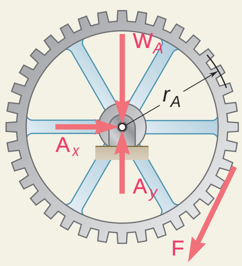
$$\begin{aligned} T_2 &= \tfrac{1}{2}\bar{I}_A\omega_A^2 + \tfrac{1}{2}\bar{I}_B\omega_B^2 \\ &= \tfrac{1}{2}(0.400\text{ kg}\cdot\text{m}^2)(25.1\text{ rad/s})^2 + \tfrac{1}{2}(0.0192\text{ kg}\cdot\text{m}^2)(62.8\text{ rad/s})^2 \\ &= 163.9\text{ J} \end{aligned}$$

Work. Denoting by θ_B the angular displacement of gear B, we have

$$U_{1\to 2} = M\theta_B = (6\text{ N}\cdot\text{m})(\theta_B\text{ rad}) = (6\theta_B)\text{ J}$$

Principle of Work and Energy

$$T_1 + U_{1\to 2} = T_2$$
$$0 + (6\theta_B)\text{ J} = 163.9\text{ J}$$
$$\theta_B = 27.32\text{ rad} \qquad \theta_B = 4.35\text{ rev} \quad ◀$$



Motion of Gear A. ***Kinetic Energy.*** Initially, gear A is at rest, so $T_1 = 0$. When $\omega_B = 600$ rpm, the kinetic energy of gear A is

$$T_2 = \tfrac{1}{2}\bar{I}_A\omega_A^2 = \tfrac{1}{2}(0.400\text{ kg}\cdot\text{m}^2)(25.1\text{ rad/s})^2 = 126.0\text{ J}$$

Work. The forces acting on gear A are as shown. The tangential force **F** does work equal to the product of its magnitude and of the length $\theta_A r_A$ of the arc described by the point of contact. Since $\theta_A r_A = \theta_B r_B$, we have

$$U_{1\to 2} = F(\theta_B r_B) = F(27.3\text{ rad})(0.100\text{ m}) = F(2.73\text{ m})$$

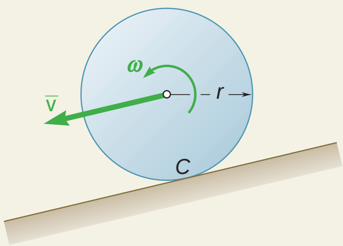
Principle of Work and Energy

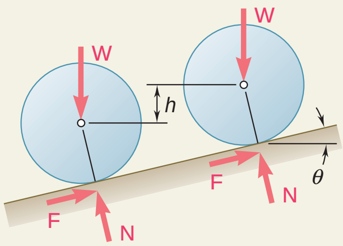
$$T_1 + U_{1\to 2} = T_2$$
$$0 + F(2.73\text{ m}) = 126.0\text{ J}$$
$$F = +46.2\text{ N} \qquad \mathbf{F} = 46.2\text{ N}\swarrow \quad ◀$$

SAMPLE PROBLEM 17.3

A sphere, a cylinder, and a hoop, each having the same mass and the same radius, are released from rest on an incline. Determine the velocity of each body after it has rolled through a distance corresponding to a change in elevation h.

SOLUTION





The problem will first be solved in general terms, and then results for each body will be found. We denote the mass by m, the centroidal moment of inertia by $\bar{I}$, the weight by W, and the radius by r.

Kinematics. Since each body rolls, the instantaneous center of rotation is located at C and we write

$$\omega = \frac{\bar{v}}{r}$$

Kinetic Energy

$$T_1 = 0$$
$$T_2 = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2$$
$$= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\left(\frac{\bar{v}}{r}\right)^2 = \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2$$

Work. Since the friction force **F** in rolling motion does no work,

$$U_{1\rightarrow 2} = Wh$$

Principle of Work and Energy

$$T_1 + U_{1\rightarrow 2} = T_2$$
$$0 + Wh = \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2 \qquad \bar{v}^2 = \frac{2Wh}{m + \bar{I}/r^2}$$

Noting that $W = mg$, we rearrange the result and obtain

$$\bar{v}^2 = \frac{2gh}{1 + \bar{I}/mr^2}$$

Velocities of Sphere, Cylinder, and Hoop. Introducing successively the particular expression for $\bar{I}$, we obtain

Sphere: $\bar{I} = \tfrac{2}{5} mr^2$ $\qquad \bar{v} = 0.845\sqrt{2gh}$ ◀

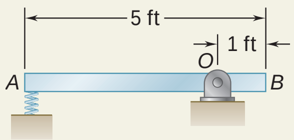
Cylinder: $\bar{I} = \tfrac{1}{2} mr^2$ $\qquad \bar{v} = 0.816\sqrt{2gh}$ ◀

Hoop: $\bar{I} = mr^2$ $\qquad \bar{v} = 0.707\sqrt{2gh}$ ◀

Remark. Let us compare the results with the velocity attained by a frictionless block sliding through the same distance. The solution is identical to the above solution except that $\omega = 0$; we find $\bar{v} = \sqrt{2gh}$.

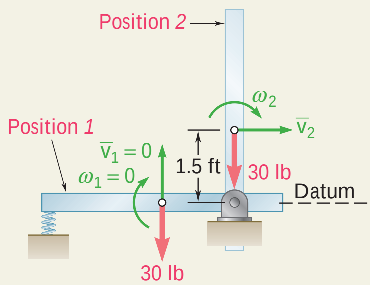
Comparing the results, we note that the velocity of the body is independent of both its mass and radius. However, the velocity does depend upon the quotient $\bar{I}/mr^2 = \bar{k}^2/r^2$, which measures the ratio of the rotational kinetic energy to the translational kinetic energy. Thus the hoop, which has the largest $\bar{k}$ for a given radius r, attains the smallest velocity, while the sliding block, which does not rotate, attains the largest velocity.

SAMPLE PROBLEM 17.4



A 30-lb slender rod AB is 5 ft long and is pivoted about a point O which is 1 ft from end B. The other end is pressed against a spring of constant k = 1800 lb/in. until the spring is compressed 1 in. The rod is then in a horizontal position. If the rod is released from this position, determine its angular velocity and the reaction at the pivot O as the rod passes through a vertical position.

SOLUTION



Position 1. ***Potential Energy.*** Since the spring is compressed 1 in., we have $x_1 = 1$ in.

$$V_e = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(1800 \text{ lb/in.})(1 \text{ in.})^2 = 900 \text{ in} \cdot \text{lb}$$

Choosing the datum as shown, we have $V_g = 0$; therefore,

$$V_1 = V_e + V_g = 900 \text{ in} \cdot \text{lb} = 75 \text{ ft} \cdot \text{lb}$$

Kinetic Energy. Since the velocity in position *1* is zero, we have $T_1 = 0$.

Position 2. ***Potential Energy.*** The elongation of the spring is zero, and we have $V_e = 0$. Since the center of gravity of the rod is now 1.5 ft above the datum,

$$V_g = (30 \text{ lb})(+1.5 \text{ ft}) = 45 \text{ ft} \cdot \text{lb}$$
$$V_2 = V_e + V_g = 45 \text{ ft} \cdot \text{lb}$$

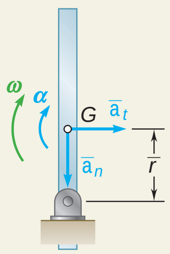
Kinetic Energy. Denoting by ω_2 the angular velocity of the rod in position 2, we note that the rod rotates about O and write $\bar{v}_2 = \bar{r}\omega_2 = 1.5\omega_2$.

$$\bar{I} = \tfrac{1}{12}ml^2 = \frac{1}{12}\frac{30 \text{ lb}}{32.2 \text{ ft/s}^2}(5 \text{ ft})^2 = 1.941 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 = \frac{1}{2}\frac{30}{32.2}(1.5\omega_2)^2 + \tfrac{1}{2}(1.941)\omega_2^2 = 2.019\omega_2^2$$

Conservation of Energy

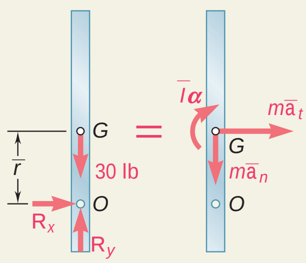
$$T_1 + V_1 = T_2 + V_2$$
$$0 + 75 \text{ ft} \cdot \text{lb} = 2.019\omega_2^2 + 45 \text{ ft} \cdot \text{lb}$$
$$\omega_2 = 3.86 \text{ rad/s} \circlearrowleft \quad \blacktriangleleft$$



Reaction in Position 2. Since $\omega_2 = 3.86$ rad/s, the components of the acceleration of G as the rod passes through position *2* are

$$\bar{a}_n = \bar{r}\omega_2^2 = (1.5 \text{ ft})(3.86 \text{ rad/s})^2 = 22.3 \text{ ft/s}^2 \qquad \bar{\mathbf{a}}_n = 22.3 \text{ ft/s}^2 \downarrow$$
$$\bar{a}_t = \bar{r}\alpha \qquad \bar{\mathbf{a}}_t = \bar{r}\alpha \rightarrow$$

We express that the system of external forces is equivalent to the system of effective forces represented by the vector of components $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ attached at G and the couple $\bar{I}\alpha$.

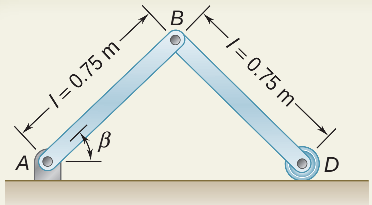


$+\circlearrowleft\Sigma M_O = \Sigma(M_O)_{\text{eff}}$: $\quad 0 = \bar{I}\alpha + m(\bar{r}\alpha)\bar{r} \quad \alpha = 0$

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\quad R_x = m(\bar{r}\alpha) \quad R_x = 0$

$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $\quad R_y - 30 \text{ lb} = -m\bar{a}_n$

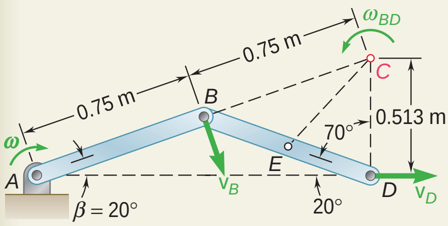
$$R_y - 30 \text{ lb} = -\frac{30 \text{ lb}}{32.2 \text{ ft/s}^2}(22.3 \text{ ft/s}^2)$$
$$R_y = +9.22 \text{ lb} \qquad \mathbf{R} = 9.22 \text{ lb} \uparrow \quad \blacktriangleleft$$



SAMPLE PROBLEM 17.5

Each of the two slender rods shown is 0.75 m long and has a mass of 6 kg. If the system is released from rest with $\beta = 60°$, determine (*a*) the angular velocity of rod AB when $\beta = 20°$, (*b*) the velocity of point D at the same instant.

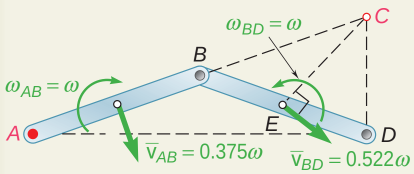
SOLUTION



Kinematics of Motion When B **= 20°.** Since $\mathbf{v}_B$ is perpendicular to the rod AB and $\mathbf{v}_D$ is horizontal, the instantaneous center of rotation of rod BD is located at C. Considering the geometry of the figure, we obtain

$$BC = 0.75 \text{ m} \qquad CD = 2(0.75 \text{ m}) \sin 20° = 0.513 \text{ m}$$

Applying the law of cosines to triangle CDE, where E is located at the mass center of rod BD, we find $EC = 0.522$ m. Denoting by v the angular velocity of rod AB, we have



$$\bar{v}_{AB} = (0.375 \text{ m})\text{v} \qquad \bar{\mathbf{v}}_{AB} = 0.375\text{v} \searrow$$
$$v_B = (0.75 \text{ m})\text{v} \qquad \mathbf{v}_B = 0.75\text{v} \searrow$$

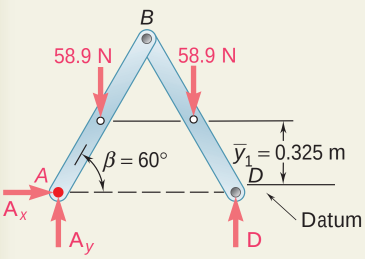
Since rod BD seems to rotate about point C, we write

$$v_B = (BC)\text{v}_{BD} \qquad (0.75 \text{ m})\text{v} = (0.75 \text{ m})\text{v}_{BD} \qquad \text{V}_{BD} = \text{v l}$$
$$\bar{v}_{BD} = (EC)\text{v}_{BD} = (0.522 \text{ m})\text{v} \qquad \bar{\mathbf{v}}_{BD} = 0.522\text{v} \searrow$$

Position 1. ***Potential Energy.*** Choosing the datum as shown, and observing that $W = (6 \text{ kg})(9.81 \text{ m/s}^2) = 58.86$ N, we have

$$V_1 = 2W\bar{y}_1 = 2(58.86 \text{ N})(0.325 \text{ m}) = 38.26 \text{ J}$$

Kinetic Energy. Since the system is at rest, $T_1 = 0$.



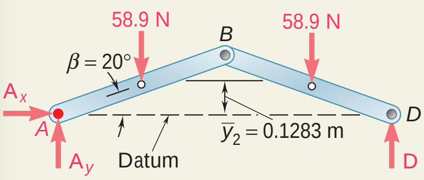
Position *1*

Position 2. ***Potential Energy***

$$V_2 = 2W\bar{y}_2 = 2(58.86 \text{ N})(0.1283 \text{ m}) = 15.10 \text{ J}$$

Kinetic Energy

$$I_{AB} = \bar{I}_{BD} = \tfrac{1}{12}ml^2 = \tfrac{1}{12}(6 \text{ kg})(0.75 \text{ m})^2 = 0.281 \text{ kg} \cdot \text{m}^2$$
$$T_2 = \tfrac{1}{2}m\bar{v}_{AB}^2 + \tfrac{1}{2}\bar{I}_{AB}\text{v}_{AB}^2 + \tfrac{1}{2}m\bar{v}_{BD}^2 + \tfrac{1}{2}\bar{I}_{BD}\text{v}_{BD}^2$$
$$= \tfrac{1}{2}(6)(0.375\text{v})^2 + \tfrac{1}{2}(0.281)\text{v}^2 + \tfrac{1}{2}(6)(0.522\text{v})^2 + \tfrac{1}{2}(0.281)\text{v}^2$$
$$= 1.520\text{v}^2$$



Position *2*

Conservation of Energy

$$T_1 + V_1 = T_2 + V_2$$
$$0 + 38.26 \text{ J} = 1.520\text{v}^2 + 15.10 \text{ J}$$
$$\text{v} = 3.90 \text{ rad/s} \qquad \text{V}_{AB} = 3.90 \text{ rad/s i} \quad ◀$$

Velocity of Point *D*

$$v_D = (CD)\text{v} = (0.513 \text{ m})(3.90 \text{ rad/s}) = 2.00 \text{ m/s}$$
$$\mathbf{v}_D = 2.00 \text{ m/s y} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced energy methods to determine the velocity of rigid bodies for various positions during their motion. As you found out previously in Chap. 13, energy methods should be considered for problems involving displacements and velocities.

1. The method of work and energy, when applied to all of the particles forming a rigid body, yields the equation

$$T_1 + U_{1\,y\,2} = T_2 \tag{17.1}$$

where T_1 and T_2 are, respectively, the initial and final values of the total kinetic energy of the particles forming the body and $U_{1\,y\,2}$ is the *work done by the external forces* exerted on the rigid body.

a. Work of forces and couples. To the expression for the work of a force (Chap. 13), we added the expression for the work of a couple and wrote

$$U_{1\,y\,2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \qquad U_{1\,y\,2} = \int_{\mathrm{u}_1}^{\mathrm{u}_2} M\,d\mathrm{u} \tag{17.3, 17.5}$$

When the moment of a couple is constant, the work of the couple is

$$U_{1\,y\,2} = M(\theta_2 - \theta_1) \tag{17.6}$$

where u_1 and u_2 are expressed in radians [Sample Probs. 17.1 and 17.2].

b. The kinetic energy of a rigid body in plane motion was found by considering the motion of the body as the sum of a translation with its mass center and a rotation about the mass center.

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\mathrm{v}^2 \tag{17.9}$$

where $\bar{v}$ is the velocity of the mass center and v is the angular velocity of the body [Sample Probs. 17.3 and 17.4].

2. For a system of rigid bodies we again used the equation

$$T_1 + U_{1\,y\,2} = T_2 \tag{17.1}$$

where T is the sum of the kinetic energies of the bodies forming the system and U is the work done by *all the forces acting on the bodies,* internal as well as external. Your computations will be simplified if you keep the following in mind.

a. The forces exerted on each other by pin-connected members or by meshed gears are equal and opposite, and, since they have the same point of application, they undergo equal small displacements. Therefore, *their total work is zero* and can be omitted from your calculations [Sample Prob. 17.2].

(*continued*)

b. The forces exerted by an inextensible cord on the two bodies it connects have the same magnitude and their points of application move through equal distances, but the work of one force is positive and the work of the other is negative. Therefore, *their total work is zero* and can again be omitted from your calculations [Sample Prob. 17.1].

c. The forces exerted by a spring on the two bodies it connects also have the same magnitude, but their points of application will generally move through different distances. Therefore, *their total work is usually not zero* and should be taken into account in your calculations.

3. The principle of conservation of energy can be expressed as

$$T_1 + V_1 = T_2 + V_2 \tag{17.12}$$

where V represents the potential energy of the system. This principle can be used when a body or a system of bodies is acted upon by conservative forces, such as the force exerted by a spring or the force of gravity [Sample Probs. 17.4 and 17.5].

4. The last section of this lesson was devoted to power, which is the time rate at which work is done. For a body acted upon by a couple of moment **M**, the power can be expressed as

$$\text{Power} = M\omega \tag{17.13}$$

where ω is the angular velocity of the body expressed in rad/s. As you did in Chap. 13, you should express power either in watts or in horsepower (1 hp = 550 ft · lb/s).

PROBLEMS

CONCEPT QUESTIONS

17.CQ1 A round object of mass m and radius r is released from rest at the top of a curved surface and rolls without slipping until it leaves the surface with a horizontal velocity as shown. Will a solid sphere, a solid cylinder, or a hoop travel the greatest distance x?

a. Solid sphere
b. Solid cylinder
c. Hoop
d. They will all travel the same distance.

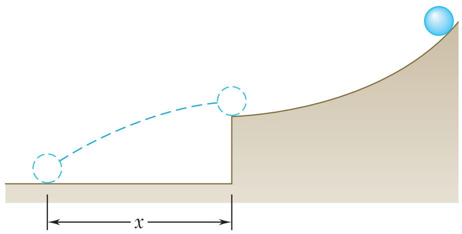


Fig. P17.CQ1

17.CQ2 A solid steel sphere A of radius r and mass m is released from rest and rolls without slipping down an incline as shown. After traveling a distance d, the sphere has a speed v. If a solid steel sphere of radius $2r$ is released from rest on the same incline, what will its speed be after rolling a distance d?

a. $0.25\,v$
b. $0.5\,v$
c. v
d. $2v$
e. $4v$

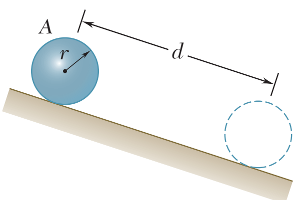


Fig. P17.CQ2

17.CQ3 Slender bar A is rigidly connected to a massless rod BC in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar A is negligible compared to L. In both cases A is released from rest at an angle $u = u_0$. When $u = 0°$, which system will have the larger kinetic energy?

a. Case 1
b. Case 2
c. The kinetic energy will be the same.

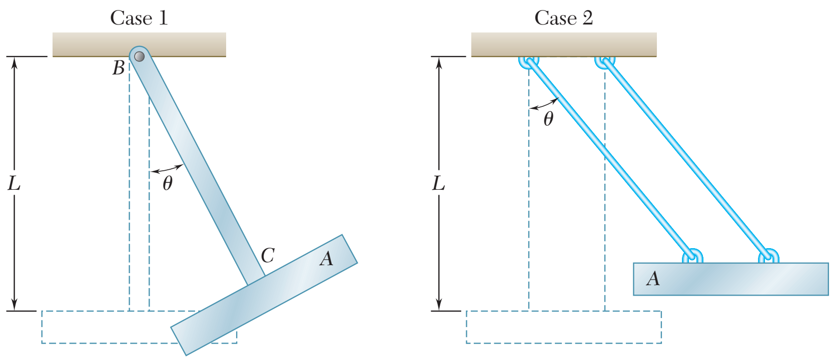


Fig. P17.CQ3 and P17.CQ5

17.CQ4 In Prob. 17.CQ3, how will the speeds of the centers of gravity compare for the two cases when $u = 0°$?

a. Case 1 will be larger.
b. Case 2 will be larger.
c. The speeds will be the same.

17.CQ5 Slender bar A is rigidly connected to a massless rod BC in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar A is not negligible compared to L. In both cases A is released from rest at an angle $u = u_0$. When $u = 0°$, which system will have the largest kinetic energy?

a. Case 1
b. Case 2
c. The kinetic energy will be the same.

END-OF-SECTION PROBLEMS

17.1 The rotor of an electric motor has an angular velocity of 3600 rpm when the load and power are cut off. The 50-kg rotor then coasts to rest after 5000 revolutions. Knowing that the kinetic friction of the rotor produces a couple of magnitude 4 N · m, determine the centroidal radius of gyration of the rotor.

17.2 It is known that 1500 revolutions are required for the 6000-lb flywheel to coast to rest from an angular velocity of 300 rpm. Knowing that the centroidal radius of gyration of the flywheel is 36 in., determine the average magnitude of the couple due to kinetic friction in the bearings.

17.3 Two disks of the same material are attached to a shaft as shown. Disk A has a weight of 30 lb and a radius $r = 5$ in. Disk B is three times as thick as disk A. Knowing that a couple **M** of magnitude 15 lb · ft is to be applied to disk A when the system is at rest, determine the radius nr of disk B if the angular velocity of the system is to be 600 rpm after four revolutions.

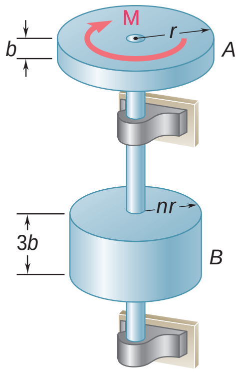


Fig. P17.3 and P17.4

17.4 Two disks of the same material are attached to a shaft as shown. Disk A is of radius r and has a thickness b, while disk B is of radius nr and thickness $3b$. A couple **M** of constant magnitude is applied when the system is at rest and is removed after the system has executed two revolutions. Determine the value of n which results in the largest final speed for a point on the rim of disk B.

17.5 The flywheel of a small punch rotates at 300 rpm. It is known that 1800 ft · lb of work must be done each time a hole is punched. It is desired that the speed of the flywheel after one punching be not less than 90 percent of the original speed of 300 rpm. (*a*) Determine the required moment of inertia of the flywheel. (*b*) If a constant 25-lb · ft couple is applied to the shaft of the flywheel, determine the number of revolutions which must occur between each punching, knowing that the initial velocity is to be 300 rpm at the start of each punching.

17.6 The flywheel of a punching machine has a mass of 300 kg and a radius of gyration of 600 mm. Each punching operation requires 2500 J of work. (*a*) Knowing that the speed of the flywheel is 300 rpm just before a punching, determine the speed immediately after the punching. (*b*) If a constant 25-N · m couple is applied to the shaft of the flywheel, determine the number of revolutions executed before the speed is again 300 rpm.

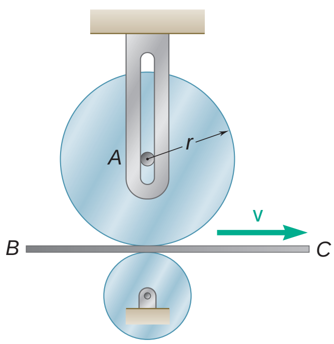


Fig. P17.7 and *P17.8*

17.7 Disk A, of weight 10 lb and radius $r = 6$ in., is at rest when it is placed in contact with belt BC, which moves to the right with a constant speed $v = 40$ ft/s. Knowing that $\mu_k = 0.20$ between the disk and the belt, determine the number of revolutions executed by the disk before it attains a constant angular velocity.

17.8 Disk A is of constant thickness and is at rest when it is placed in contact with belt BC, which moves with a constant velocity **v**. Denoting by μ_k the coefficient of kinetic friction between the disk and the belt, derive an expression for the number of revolutions executed by the disk before it attains a constant angular velocity.

17.9 The 10-in.-radius brake drum is attached to a larger flywheel which is not shown. The total mass moment of inertia of the flywheel and drum is 16 lb · ft · s^2 and the coefficient of kinetic friction between the drum and the brake shoe is 0.40. Knowing that the initial angular velocity is 240 rpm clockwise, determine the force which must be exerted by the hydraulic cylinder if the system is to stop in 75 revolutions.

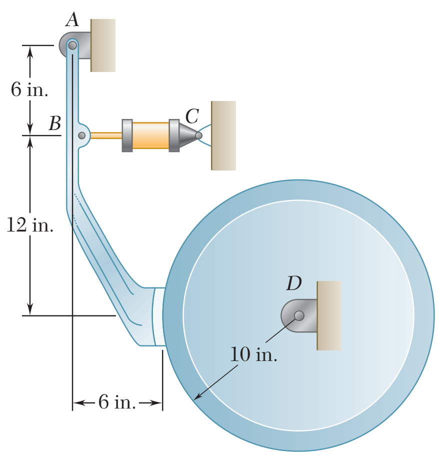


Fig. *P17.9*

17.10 Solve Prob. 17.9, assuming that the initial angular velocity of the flywheel is 240 rpm counterclockwise.

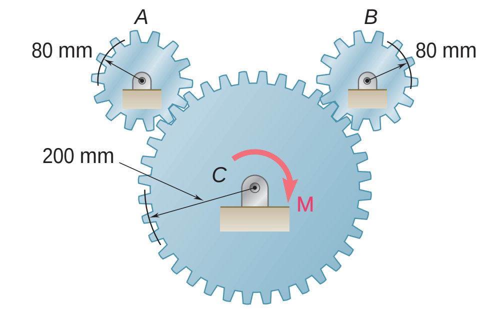


Fig. P17.11

17.11 Each of the gears A and B has a mass of 2.4 kg and a radius of gyration of 60 mm, while gear C has a mass of 12 kg and a radius of gyration of 150 mm. A couple $\mathbf{M}$ of constant magnitude 10 N · m is applied to gear C. Determine (*a*) the number of revolutions of gear C required for its angular velocity to increase from 100 to 450 rpm, (*b*) the corresponding tangential force acting on gear A.

17.12 Solve Prob. 17.11, assuming that the 10-N · m couple is applied to gear B.

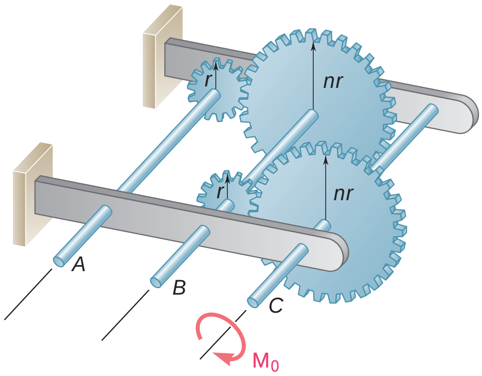


Fig. P17.13

17.13 The gear train shown consists of four gears of the same thickness and of the same material; two gears are of radius r, and the other two are of radius nr. The system is at rest when the couple $\mathbf{M}_0$ is applied to shaft C. Denoting by I_0 the moment of inertia of a gear of radius r, determine the angular velocity of shaft A if the couple $\mathbf{M}_0$ is applied for one revolution of shaft C.

17.14 The double pulley shown has a mass of 15 kg and a centroidal radius of gyration of 160 mm. Cylinder A and block B are attached to cords that are wrapped on the pulleys as shown. The coefficient of kinetic friction between block B and the surface is 0.2. Knowing that the system is at rest in the position shown when a constant force $\mathbf{P} = 200$ N is applied to cylinder A, determine (*a*) the velocity of cylinder A as it strikes the ground, (*b*) the total distance that block B moves before coming to rest.

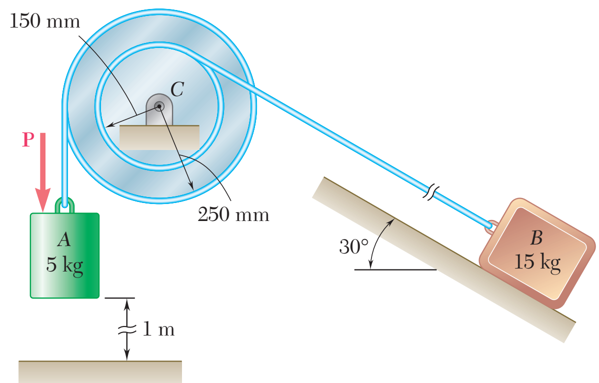


Fig. *P17.14*

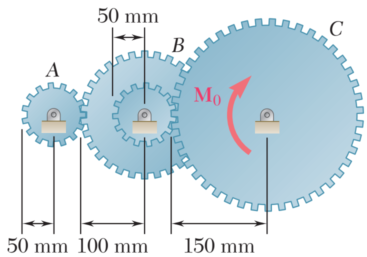


Fig. *P17.15*

17.15 Gear A has a mass of 1 kg and a radius of gyration of 30 mm; gear B has a mass of 4 kg and a radius of gyration of 75 mm; gear C has a mass of 9 kg and a radius of gyration of 100 mm. The system is at rest when a couple $\mathbf{M}_0$ of constant magnitude 4 N · m is applied to gear C. Assuming that no slipping occurs between the gears, determine the number of revolutions required for disk A to reach an angular velocity of 300 rpm.

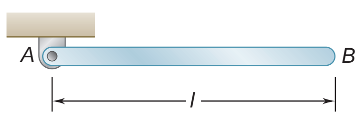


Fig. P17.16

17.16 A slender rod of length l and weight W is pivoted at one end as shown. It is released from rest in a horizontal position and swings freely. (*a*) Determine the angular velocity of the rod as it passes through a vertical position and determine the corresponding reaction at the pivot. (*b*) Solve part *a* for $W = 1.8$ lb and $l = 3$ ft.

17.17 A slender rod of length l is pivoted about a point C located at a distance b from its center G. It is released from rest in a horizontal position and swings freely. Determine (*a*) the distance b for which the angular velocity of the rod as it passes through a vertical position is maximum, (*b*) the corresponding values of its angular velocity and of the reaction at C.

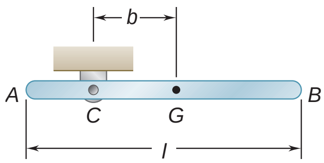


Fig. P17.17

17.18 and 17.19 A slender 9-lb rod can rotate in a vertical plane about a pivot at B. A spring of constant $k = 30$ lb/ft and of unstretched length 6 in. is attached to the rod as shown. Knowing that the rod is released from rest in the position shown, determine its angular velocity after it has rotated through 90°.

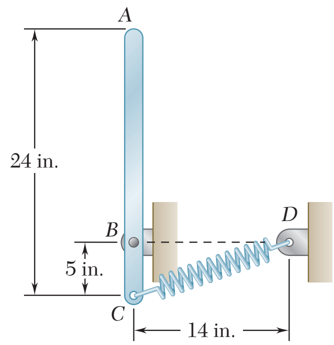


Fig. P17.18

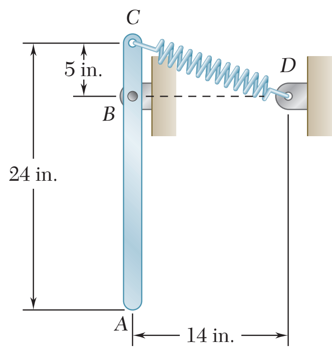


Fig. P17.19

17.20 A 160-lb gymnast is executing a series of full-circle swings on the horizontal bar. In the position shown he has a small and negligible clockwise angular velocity and will maintain his body straight and rigid as he swings downward. Assuming that during the swing the centroidal radius of gyration of his body is 1.5 ft, determine his angular velocity and the force exerted on his hands after he has rotated through (*a*) 90°, (*b*) 180°.

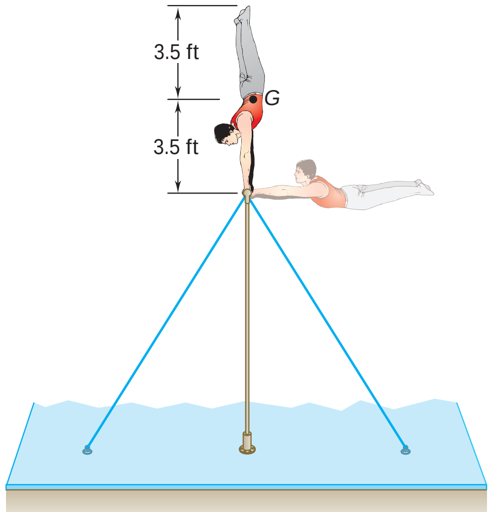


Fig. P17.20

17.21 A collar with a mass of 1 kg is rigidly attached at a distance $d = 300$ mm from the end of a uniform slender rod AB. The rod has a mass of 3 kg and is of length $L = 600$ mm. Knowing that the rod is released from rest in the position shown, determine the angular velocity of the rod after it has rotated through 90°.

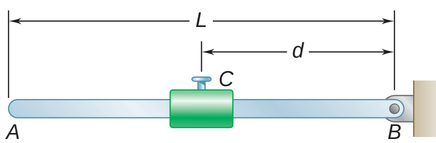


Fig. *P17.21* and *P17.22*

17.22 A collar with a mass of 1 kg is rigidly attached to a slender rod AB of mass 3 kg and length $L = 600$ mm. The rod is released from rest in the position shown. Determine the distance d for which the angular velocity of the rod is maximum after it has rotated through 90°.

17.23 Two identical slender rods AB and BC are welded together to form an L-shaped assembly. The assembly is pressed against a spring at D and released from the position shown. Knowing that the maximum angle of rotation of the assembly in its subsequent motion is 90° counterclockwise, determine the magnitude of the angular velocity of the assembly as it passes through the position where rod AB forms an angle of 30° with the horizontal.

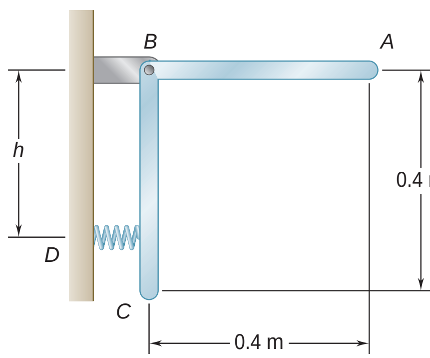


Fig. P17.23

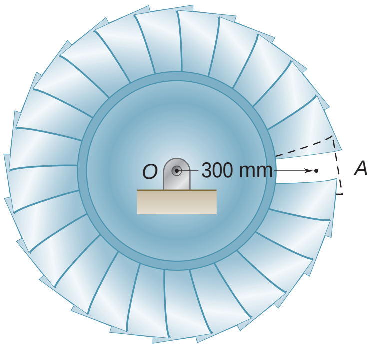


Fig. P17.24

17.24 The 30-kg turbine disk has a centroidal radius of gyration of 175 mm and is rotating clockwise at a constant rate of 60 rpm when a small blade of weight 0.5 N at point A becomes loose and is thrown off. Neglecting friction, determine the change in the angular velocity of the turbine disk after it has rotated through (*a*) 90°, (*b*) 270°.

17.25 A rope is wrapped around a cylinder of radius r and mass m as shown. Knowing that the cylinder is released from rest, determine the velocity of the center of the cylinder after it has moved downward a distance s.

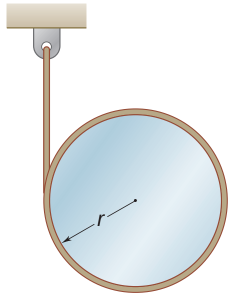


Fig. P17.25

17.26 Solve Prob. 17.25, assuming that the cylinder is replaced by a thin-walled pipe of radius r and mass m.

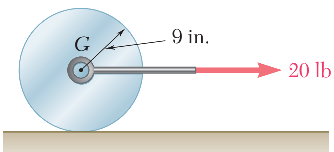


Fig. P17.27

17.27 A 45-lb uniform cylindrical roller, initially at rest, is acted upon by a 20-lb force as shown. Knowing that the body rolls without slipping, determine (*a*) the velocity of its center G after it has moved 5 ft, (*b*) the friction force required to prevent slipping.

17.28 A small sphere of mass m and radius r is released from rest at A and rolls without sliding on the curved surface to point B where it leaves the surface with a horizontal velocity. Knowing that $a = 1.5$ m and $b = 1.2$ m, determine (*a*) the speed of the sphere as it strikes the ground at C, (*b*) the corresponding distance c.

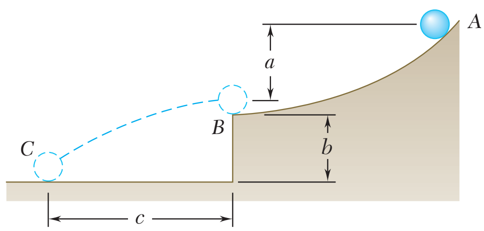


Fig. *P17.28*

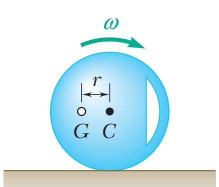


Fig. P17.29

17.29 The mass center G of a 3-kg wheel of radius $R = 180$ mm is located at a distance $r = 60$ mm from its geometric center C. The centroidal radius of gyration of the wheel is $\bar{k} = 90$ mm. As the wheel rolls without sliding, its angular velocity is observed to vary. Knowing that $v = 8$ rad/s in the position shown, determine (*a*) the angular velocity of the wheel when the mass center G is directly above the geometric center C, (*b*) the reaction at the horizontal surface at the same instant.

17.30 A half section of pipe of mass m and radius r is released from rest in the position shown. Knowing that the pipe rolls without sliding, determine (*a*) its angular velocity after it has rolled through 90°, (*b*) the reaction at the horizontal surface at the same instant. [*Hint:* Note that $GO = 2r/\pi$ and that, by the parallel-axis theorem, $\bar{I} = mr^2 - m(GO)^2$.]

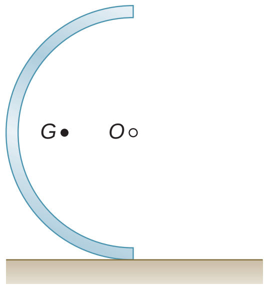


Fig. P17.30

17.31 A sphere of mass m and radius r rolls without slipping inside a curved surface of radius R. Knowing that the sphere is released from rest in the position shown, derive an expression for (*a*) the linear velocity of the sphere as it passes through B, (*b*) the magnitude of the vertical reaction at that instant.

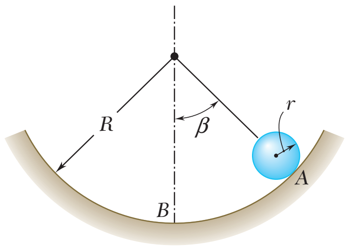


Fig. P17.31

17.32 Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a belt as shown. Knowing that at the instant shown the angular velocity of cylinder B is 30 rad/s clockwise, determine (*a*) the distance through which cylinder A will rise before the angular velocity of cylinder B is reduced to 5 rad/s, (*b*) the tension in the portion of belt connecting the two cylinders.

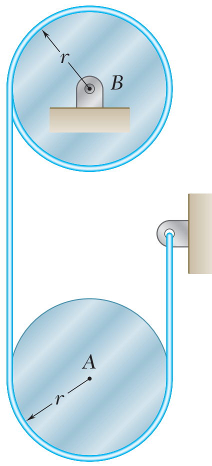


Fig. P17.32 and P17.33

17.33 Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a belt as shown. If the system is released from rest, determine (*a*) the velocity of the center of cylinder A after it has moved through 3 ft, (*b*) the tension in the portion of belt connecting the two cylinders.

17.34 A bar of mass $m = 5$ kg is held as shown between four disks each of mass $m' = 2$ kg and radius $r = 75$ mm. Knowing that the forces exerted on the disks are sufficient to prevent slipping and that the bar is released from rest, for each of the cases shown determine the velocity of the bar after it has moved through the distance h.

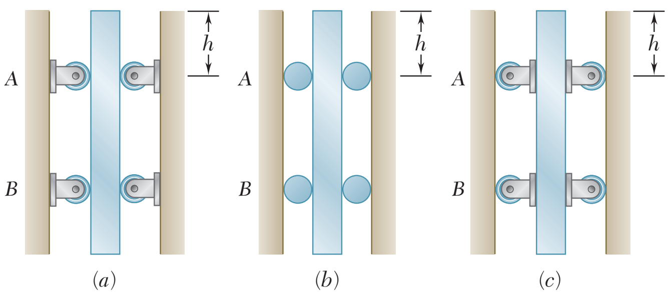


Fig. *P17.34*

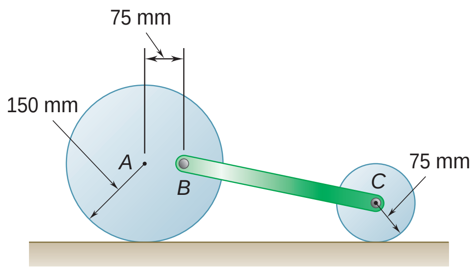


Fig. P17.35

17.35 The 5-kg rod *BC* is attached by pins to two uniform disks as shown. The mass of the 150-mm-radius disk is 6 kg and that of the 75-mm-radius disk is 1.5 kg. Knowing that the system is released from rest in the position shown, determine the velocity of the rod after disk *A* has rotated through 90°.

17.36 The motion of the uniform rod *AB* is guided by small wheels of negligible mass that roll on the surface shown. If the rod is released from rest when u = 0, determine the velocities of *A* and *B* when u = 30°.

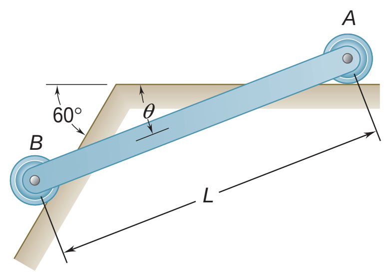


Fig. P17.36

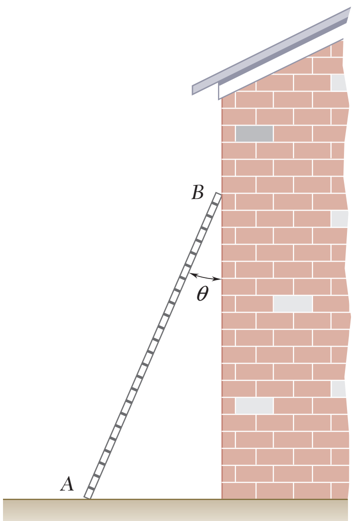


Fig. P17.37 and P17.38

17.37 A 5-m-long ladder has a mass of 15 kg and is placed against a house at an angle u = 20°. Knowing that the ladder is released from rest, determine the angular velocity of the ladder and the velocity of end *A* when u = 45°. Assume the ladder can slide freely on the horizontal ground and on the vertical wall.

17.38 A long ladder of length *l*, mass *m*, and centroidal mass moment of inertia $\bar{I}$ is placed against a house at an angle u = u_0. Knowing that the ladder is released from rest, determine the angular velocity of the ladder when u = u_2. Assume the ladder can slide freely on the horizontal ground and on the vertical wall.

17.39 The ends of a 9-lb rod *AB* are constrained to move along slots cut in a vertical plate as shown. A spring of constant *k* = 3 lb/in. is attached to end *A* in such a way that its tension is zero when u = 0. If the rod is released from rest when u = 50°, determine the angular velocity of the rod and the velocity of end *B* when u = 0.

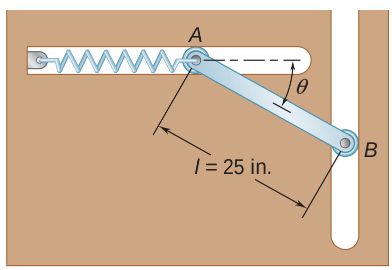


Fig. P17.39 and P17.40

17.40 The ends of a 9-lb rod *AB* are constrained to move along slots cut in a vertical plate as shown. A spring of constant *k* = 3 lb/in. is attached to end *A* in such a way that its tension is zero when u = 0. If the rod is released from rest when u = 0, determine the angular velocity of the rod and the velocity of end *B* when u = 30°.

17.41 The motion of a slender rod of length R is guided by pins at A and B which slide freely in slots cut in a vertical plate as shown. If end B is moved slightly to the left and then released, determine the angular velocity of the rod and the velocity of its mass center (*a*) at the instant when the velocity of end B is zero, (*b*) as end B passes through point D.

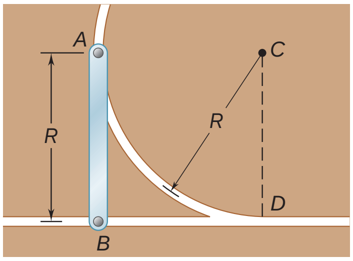


Fig. *P17.41*

17.42 Each of the two rods shown is of length $L = 1$ m and has a mass of 5 kg. Point D is connected to a spring of constant $k = 20$ N/m and is constrained to move along a vertical slot. Knowing that the system is released from rest when rod BD is horizontal and the spring connected to point D is initially unstretched, determine the velocity of point D when it is directly to the right of point A.`

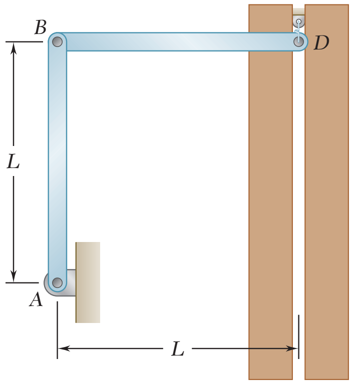


Fig. P17.42

17.43 The 4-kg rod AB is attached to a collar of negligible mass at A and to a flywheel at B. The flywheel has a mass of 16 kg and a radius of gyration of 180 mm. Knowing that in the position shown the angular velocity of the flywheel is 60 rpm clockwise, determine the velocity of the flywheel when point B is directly below C.

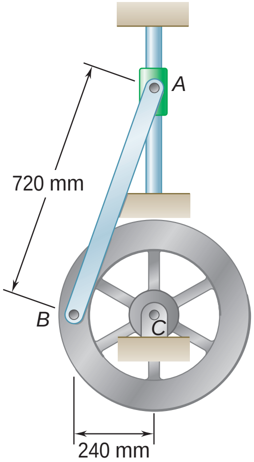


Fig. P17.43 and P17.44

17.44 If in Prob. 17.43 the angular velocity of the flywheel is to be the same in the position shown and when point B is directly above C, determine the required value of its angular velocity in the position shown.

17.45 The uniform rods AB and BC weigh 2.4 kg and 4 kg, respectively, and the small wheel at C is of negligible weight. If the wheel is moved slightly to the right and then released, determine the velocity of pin B after rod AB has rotated through 90°.

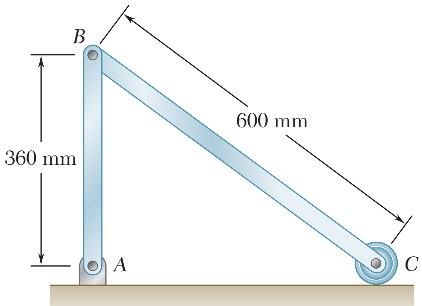


Fig. P17.45 and P17.46

17.46 The uniform rods AB and BC weigh 2.4 kg and 4 kg, respectively, and the small wheel at C is of negligible weight. Knowing that in the position shown the velocity of wheel C is 2 m/s to the right, determine the velocity of pin B after rod AB has rotated through 90°.

17.47 The 80-mm-radius gear shown has a mass of 5 kg and a centroidal radius of gyration of 60 mm. The 4-kg rod *AB* is attached to the center of the gear and to a pin at *B* that slides freely in a vertical slot. Knowing that the system is released from rest when u = 60°, determine the velocity of the center of the gear when u = 20°.

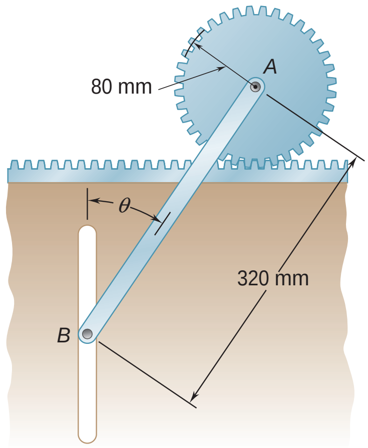


Fig. P17.47

17.48 Knowing that the maximum allowable couple that can be applied to a shaft is 15.5 kip · in., determine the maximum horsepower that can be transmitted by the shaft at (*a*) 180 rpm, (*b*) 480 rpm.

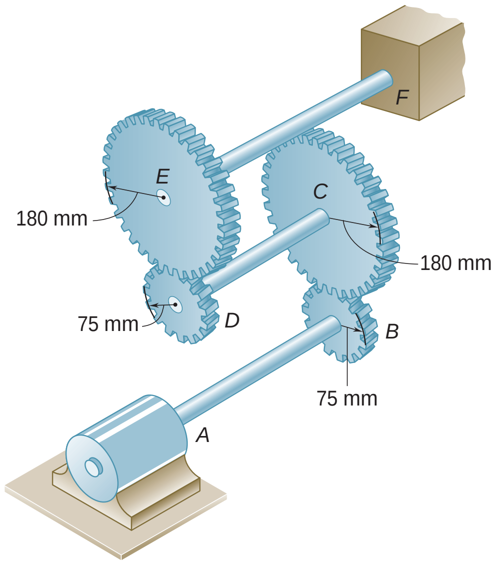


Fig. P17.49

17.49 Three shafts and four gears are used to form a gear train which will transmit 7.5 kW from the motor at *A* to a machine tool at *F*. (Bearings for the shafts are omitted from the sketch.) Knowing that the frequency of the motor is 30 Hz, determine the magnitude of the couple which is applied to shaft (*a*) *AB*, (*b*) *CD*, (*c*) *EF*.

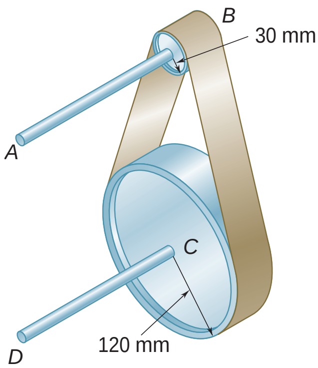


Fig. P17.50

17.50 The shaft-disk-belt arrangement shown is used to transmit 2.4 kW from point *A* to point *D*. Knowing that the maximum allowable couples that can be applied to shafts *AB* and *CD* are 25 N · m and 80 N · m, respectively, determine the required minimum speed of shaft *AB*.

17.51 The experimental setup shown is used to measure the power output of a small turbine. When the turbine is operating at 200 rpm, the readings of the two spring scales are 10 and 22 lb, respectively. Determine the power being developed by the turbine.

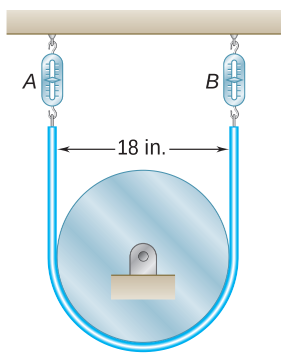


Fig. P17.51

17.8 PRINCIPLE OF IMPULSE AND MOMENTUM FOR THE PLANE MOTION OF A RIGID BODY

The principle of impulse and momentum will now be applied to the analysis of the plane motion of rigid bodies and of systems of rigid bodies. As was pointed out in Chap. 13, the method of impulse and momentum is particularly well adapted to the solution of problems involving time and velocities. Moreover, the principle of impulse and momentum provides the only practicable method for the solution of problems involving impulsive motion or impact (Secs. 17.11 and 17.12).

Considering again a rigid body as made of a large number of particles P_i, we recall from Sec. 14.9 that the system formed by the momenta of the particles at time t_1 and the system of the impulses of the external forces applied from t_1 to t_2 are together equipollent to the system formed by the momenta of the particles at time t_2. Since the vectors associated with a rigid body can be considered as sliding vectors, it follows (Sec. 3.19) that the systems of vectors shown in Fig. 17.6 are not only equipollent but truly *equivalent* in the sense that the vectors on the left-hand side of the equals sign can be transformed into the vectors on the right-hand side through the use of the fundamental operations listed in Sec. 3.13. We therefore write

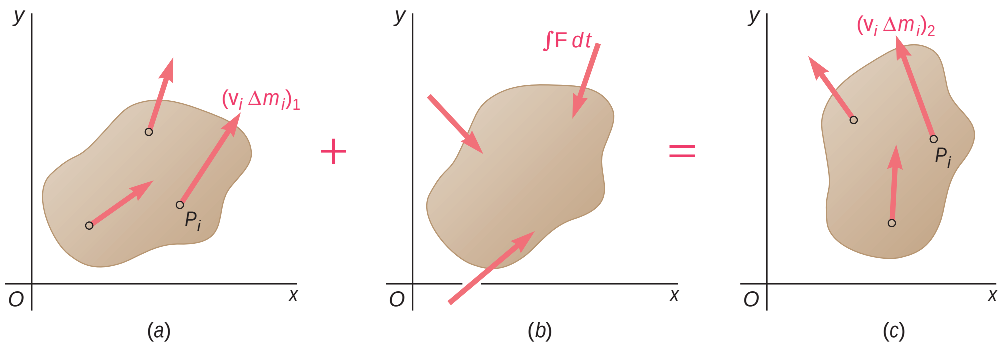


Fig. 17.6

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\,y\,2} = \textbf{Syst Momenta}_2 \tag{17.14}$$

But the momenta $\mathbf{v}_i\,\Delta m_i$ of the particles can be reduced to a vector attached at G, equal to their sum

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{v}_i\,\Delta m_i$$

and a couple of moment equal to the sum of their moments about G

$$\mathbf{H}_G = \sum_{i=1}^{n} \mathbf{r}'_i \times \mathbf{v}_i\,\Delta m_i$$

We recall from Sec. 14.3 that $\mathbf{L}$ and $\mathbf{H}_G$ define, respectively, the linear momentum and the angular momentum about G of the system

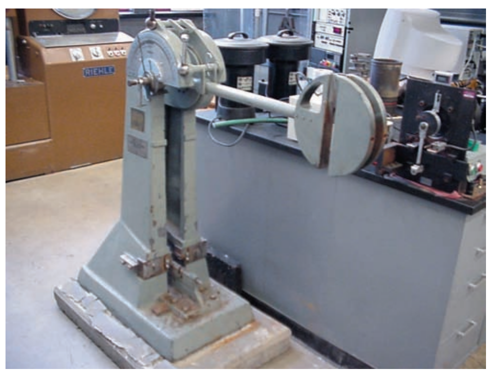


Photo 17.2 A Charpy impact test is used to determine the amount of energy absorbed by a material during impact by subtracting the final gravitation potential energy of the arm from its initial gravitational potential energy.

of particles forming the rigid body. We also note from Eq. (14.14) that $\mathbf{L} = m\bar{\mathbf{v}}$. On the other hand, restricting the present analysis to the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane, we recall from Eq. (16.4) that $\mathbf{H}_G = \bar{I}\boldsymbol{\omega}$. We thus conclude that the system of the momenta $\mathbf{v}_i\,\Delta m_i$ is equivalent to the *linear momentum vector* $m\bar{\mathbf{v}}$ attached at G and to the *angular momentum couple* $\bar{I}\boldsymbol{\omega}$ (Fig. 17.7). Observing that the

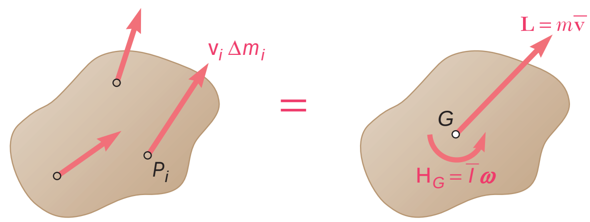


Fig. 17.7

system of momenta reduces to the vector $m\bar{\mathbf{v}}$ in the particular case of a translation ($\boldsymbol{\omega} = 0$) and to the couple $\bar{I}\boldsymbol{\omega}$ in the particular case of a centroidal rotation ($\bar{\mathbf{v}} = \mathbf{0}$), we verify once more that the plane motion of a rigid body symmetrical with respect to the reference plane can be resolved into a translation with the mass center G and a rotation about G.

Replacing the system of momenta in parts *a* and *c* of Fig. 17.6 by the equivalent linear momentum vector and angular momentum couple, we obtain the three diagrams shown in Fig. 17.8. This figure

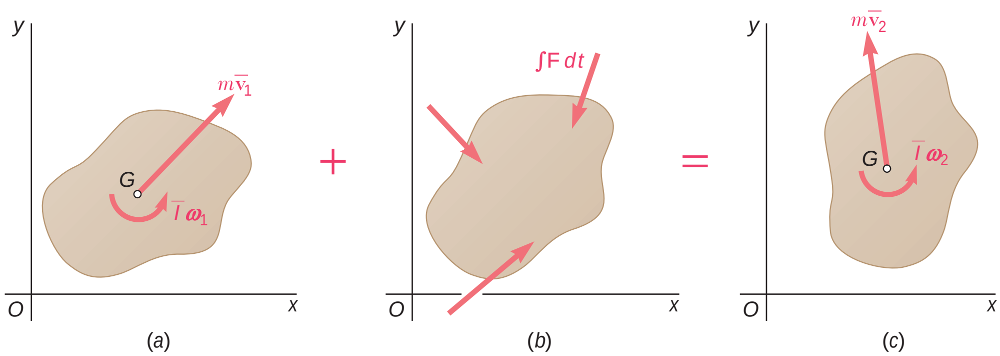


Fig. 17.8

expresses as a free-body-diagram equation the fundamental relation (17.14) in the case of the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane.

Three equations of motion can be derived from Fig. 17.8. Two equations are obtained by summing and equating the x and y *components* of the momenta and impulses, and the third equation is obtained by summing and equating the *moments* of these vectors *about any given point.* The coordinate axes can be chosen fixed in space, or allowed to move with the mass center of the body while maintaining a fixed direction. In either case, the point about which moments are taken should keep the same position relative to the coordinate axes during the interval of time considered.

In deriving the three equations of motion for a rigid body, care should be taken not to add linear and angular momenta indiscriminately. Confusion can be avoided by remembering that $m\bar{v}_x$ and $m\bar{v}_y$ represent the *components of a vector,* namely, the linear momentum vector $m\bar{\mathbf{v}}$, while $\bar{I}\omega$ represents the *magnitude of a couple,* namely, the angular momentum couple $\bar{I}\boldsymbol{\omega}$. Thus the quantity $\bar{I}\omega$ should be added only to the *moment* of the linear momentum $m\bar{\mathbf{v}}$, never to this vector itself nor to its components. All quantities involved will then be expressed in the same units, namely $\mathrm{N \cdot m \cdot s}$ or $\mathrm{lb \cdot ft \cdot s}$.

Noncentroidal Rotation. In this particular case of plane motion, the magnitude of the velocity of the mass center of the body is $\bar{v} = \bar{r}\omega$, where $\bar{r}$ represents the distance from the mass center to the fixed axis of rotation and $\boldsymbol{\omega}$ represents the angular velocity of the body at the instant considered; the magnitude of the momentum vector attached at G is thus $m\bar{v} = m\bar{r}\omega$. Summing the moments about O of the momentum vector and momentum couple (Fig. 17.9)

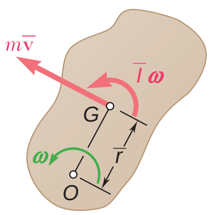


Fig. 17.9

and using the parallel-axis theorem for moments of inertia, we find that the angular momentum $\mathbf{H}_O$ of the body about O has the magnitude†

$$\bar{I}\omega + (m\bar{r}\omega)\bar{r} = (\bar{I} + m\bar{r}^2)\omega = I_O\omega \tag{17.15}$$

Equating the moments about O of the momenta and impulses in (17.14), we write

$$I_O\omega_1 + \sum \int_{t_1}^{t_2} M_O\,dt = I_O\omega_2 \tag{17.16}$$

In the general case of plane motion of a rigid body symmetrical with respect to the reference plane, Eq. (17.16) can be used with respect to the instantaneous axis of rotation under certain conditions. It is recommended, however, that all problems of plane motion be solved by the general method described earlier in this section.

†Note that the sum $\mathbf{H}_A$ of the moments about an arbitrary point A of the momenta of the particles of a rigid slab is, in general, *not* equal to $I_A\omega$. (See Prob. 17.67.)

17.9 SYSTEMS OF RIGID BODIES

The motion of several rigid bodies can be analyzed by applying the principle of impulse and momentum to each body separately (Sample Prob. 17.6). However, in solving problems involving no more than three unknowns (including the impulses of unknown reactions), it is often convenient to apply the principle of impulse and momentum to the system as a whole. The momentum and impulse diagrams are drawn for the entire system of bodies. For each moving part of the system, the diagrams of momenta should include a momentum vector, a momentum couple, or both. Impulses of forces internal to the system can be omitted from the impulse diagram, since they occur in pairs of equal and opposite vectors. Summing and equating successively the x components, y components, and moments of all vectors involved, one obtains three relations which express that the momenta at time t_1 and the impulses of the external forces form a system equipollent to the system of the momenta at time t_2.† Again, care should be taken not to add linear and angular momenta indiscriminately; each equation should be checked to make sure that consistent units have been used. This approach has been used in Sample Prob. 17.8 and, further on, in Sample Probs. 17.9 and 17.10.

17.10 CONSERVATION OF ANGULAR MOMENTUM

When no external force acts on a rigid body or a system of rigid bodies, the impulses of the external forces are zero and the system of the momenta at time t_1 is equipollent to the system of the momenta at time t_2. Summing and equating successively the x components, y components, and moments of the momenta at times t_1 and t_2, we conclude that the total linear momentum of the system is conserved in any direction and that its total angular momentum is conserved about any point.

There are many engineering applications, however, in which *the linear momentum is not conserved* yet *the angular momentum* $\mathbf{H}_O$ of the system about a given point O *is conserved* that is, in which

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \qquad \textbf{(17.17)}$$

Such cases occur when the lines of action of all external forces pass through O or, more generally, when the sum of the angular impulses of the external forces about O is zero.

Problems involving *conservation of angular momentum* about a point O can be solved by the general method of impulse and momentum, i.e., by drawing momentum and impulse diagrams as described in Secs. 17.8 and 17.9. Equation (17.17) is then obtained by summing and equating moments about O (Sample Prob. 17.8). As you will see later in Sample Prob. 17.9, two additional equations can be written by summing and equating x and y components and these equations can be used to determine two unknown linear impulses, such as the impulses of the reaction components at a fixed point.

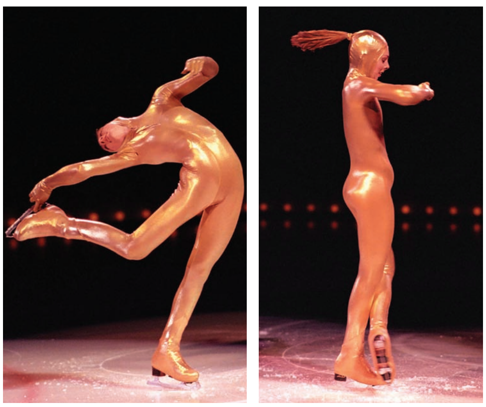
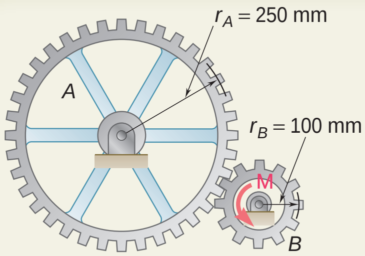


Photo 17.3 A figure skater at the beginning and at the end of a spin. By using the principle of conservation of angular momentum you will find that her angular velocity is much higher at the end of the spin.

†Note that as in Sec. 16.7, we cannot speak of *equivalent* systems since we are not dealing with a single rigid body.



SAMPLE PROBLEM 17.6

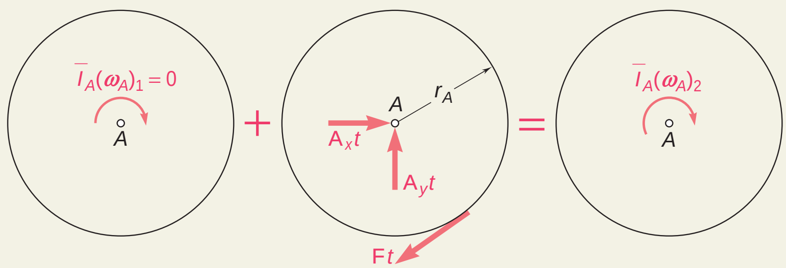
Gear A has a mass of 10 kg and a radius of gyration of 200 mm, and gear B has a mass of 3 kg and a radius of gyration of 80 mm. The system is at rest when a couple **M** of magnitude 6 N · m is applied to gear B. (These gears were considered in Sample Prob. 17.2.) Neglecting friction, determine (*a*) the time required for the angular velocity of gear B to reach 600 rpm, (*b*) the tangential force which gear B exerts on gear A.

SOLUTION

We apply the principle of impulse and momentum to each gear separately. Since all forces and the couple are constant, their impulses are obtained by multiplying them by the unknown time t. We recall from Sample Prob. 17.2 that the centroidal moments of inertia and the final angular velocities are

$$\bar{I}_A = 0.400 \text{ kg}\cdot\text{m}^2 \qquad \bar{I}_B = 0.0192 \text{ kg}\cdot\text{m}^2$$
$$(\omega_A)_2 = 25.1 \text{ rad/s} \qquad (\omega_B)_2 = 62.8 \text{ rad/s}$$

Principle of Impulse and Momentum for Gear *A*. The systems of initial momenta, impulses, and final momenta are shown in three separate sketches.

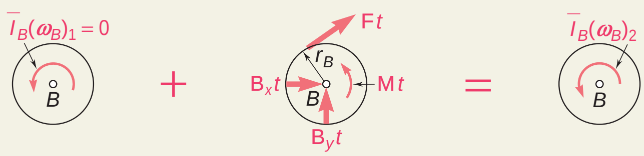


$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to2} = \textbf{Syst Momenta}_2$$

$+\circlearrowleft$ moments about A: $\qquad 0 - Ftr_A = -\bar{I}_A(\omega_A)_2$

$$Ft(0.250 \text{ m}) = (0.400 \text{ kg}\cdot\text{m}^2)(25.1 \text{ rad/s})$$
$$Ft = 40.2 \text{ N}\cdot\text{s}$$

Principle of Impulse and Momentum for Gear *B*.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to2} = \textbf{Syst Momenta}_2$$

$+\circlearrowleft$ moments about B: $\qquad 0 + Mt - Ftr_B = \bar{I}_B(\omega_B)_2$

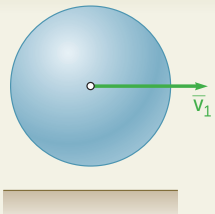
$$+(6 \text{ N}\cdot\text{m})t - (40.2 \text{ N}\cdot\text{s})(0.100 \text{ m}) = (0.0192 \text{ kg}\cdot\text{m}^2)(62.8 \text{ rad/s})$$

$$t = 0.871 \text{ s} \quad ◀$$

Recalling that $Ft = 40.2$ N · s, we write

$$F(0.871 \text{ s}) = 40.2 \text{ N}\cdot\text{s} \qquad F = +46.2 \text{ N}$$

Thus, the force exerted by gear B on gear A is $\qquad \mathbf{F} = 46.2 \text{ N} \swarrow$ ◀



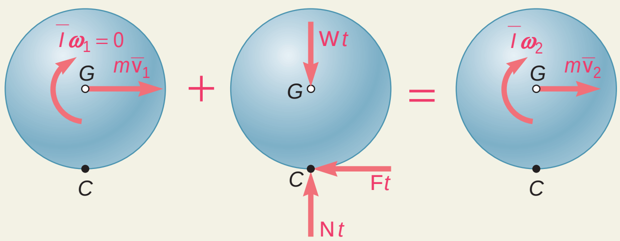
SAMPLE PROBLEM 17.7

A uniform sphere of mass m and radius r is projected along a rough horizontal surface with a linear velocity $\bar{\mathbf{v}}_1$ and no angular velocity. Denoting by m_k the coefficient of kinetic friction between the sphere and the surface, determine (*a*) the time t_2 at which the sphere will start rolling without sliding, (*b*) the linear and angular velocities of the sphere at time t_2.

SOLUTION

While the sphere is sliding relative to the surface, it is acted upon by the normal force **N**, the friction force **F**, and its weight **W** of magnitude $W = mg$.

Principle of Impulse and Momentum. We apply the principle of impulse and momentum to the sphere from the time $t_1 = 0$ when it is placed on the surface until the time $t_2 = t$ when it starts rolling without sliding.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\,y\,2} = \textbf{Syst Momenta}_2$$

+xy components: $\quad Nt - Wt = 0 \qquad (1)$

$\overset{+}{\text{y}}$ x components: $\quad m\bar{v}_1 - Ft = m\bar{v}_2 \qquad (2)$

+i moments about G: $\quad Ftr = \bar{I}\,\mathsf{v}_2 \qquad (3)$

From (1) we obtain $N = W = mg$. During the entire time interval considered, sliding occurs at point C and we have $F = \mathsf{m}_k N = \mathsf{m}_k mg$. Substituting CS for F into (2), we write

$$m\bar{v}_1 - \mathsf{m}_k mgt = m\bar{v}_2 \qquad \bar{v}_2 = \bar{v}_1 - \mathsf{m}_k gt \qquad (4)$$

Substituting $F = \mathsf{m}_k mg$ and $\bar{I} = \frac{2}{5}mr^2$ into (3),

$$\mathsf{m}_k mgtr = \tfrac{2}{5}mr^2\mathsf{v}_2 \qquad \mathsf{v}_2 = \frac{5}{2}\frac{\mathsf{m}_k g}{r}t \qquad (5)$$

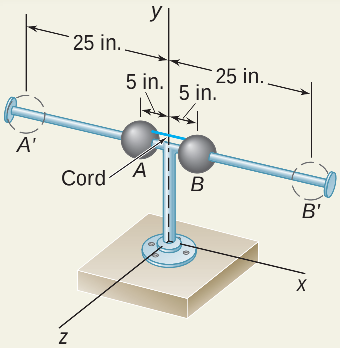
The sphere will start rolling without sliding when the velocity $\mathbf{v}_C$ of the point of contact is zero. At that time, point C becomes the instantaneous center of rotation, and we have $\bar{v}_2 = r\mathsf{v}_2$. Substituting from (4) and (5), we write

$$\bar{v}_2 = r\mathsf{v}_2 \qquad \bar{v}_1 - \mathsf{m}_k gt = r\left(\frac{5}{2}\frac{\mathsf{m}_k g}{r}t\right) \qquad t = \frac{2}{7}\frac{\bar{v}_1}{\mathsf{m}_k g} \quad \blacktriangleleft$$

Substituting this expression for t into (5),

$$\mathsf{v}_2 = \frac{5}{2}\frac{\mathsf{m}_k g}{r}\left(\frac{2}{7}\frac{\bar{v}_1}{\mathsf{m}_k g}\right) \qquad \mathsf{v}_2 = \frac{5}{7}\frac{\bar{v}_1}{r} \qquad \mathsf{V}_2 = \frac{5}{7}\frac{\bar{v}_1}{r}\ \text{i} \quad \blacktriangleleft$$

$$\bar{v}_2 = r\mathsf{v}_2 \qquad \bar{v}_2 = r\left(\frac{5}{7}\frac{v_1}{r}\right) \qquad \bar{\mathbf{v}}_2 = \tfrac{5}{7}\bar{v}_1\ \text{y} \quad \blacktriangleleft$$

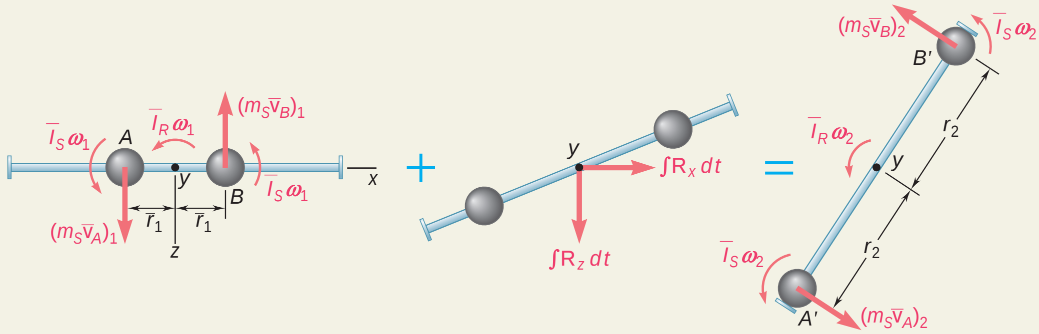


SAMPLE PROBLEM 17.8

Two solid spheres of radius 3 in., weighing 2 lb each, are mounted at A and B on the horizontal rod $A'B'$, which rotates freely about the vertical with a counterclockwise angular velocity of 6 rad/s. The spheres are held in position by a cord which is suddenly cut. Knowing that the centroidal moment of inertia of the rod and pivot is $\bar{I}_R = 0.25 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$, determine (*a*) the angular velocity of the rod after the spheres have moved to positions A' and B', (*b*) the energy lost due to the plastic impact of the spheres and the stops at A' and B'.

SOLUTION

a. Principle of Impulse and Momentum. In order to determine the final angular velocity of the rod, we will express that the initial momenta of the various parts of the system and the impulses of the external forces are together equipollent to the final momenta of the system.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\,y\,2} = \textbf{Syst Momenta}_2$$

Observing that the external forces consist of the weights and the reaction at the pivot, which have no moment about the y axis, and noting that $\bar{v}_A = \bar{v}_B = \bar{r}\mathsf{v}$, we equate moments about the y axis:

$$2(m_S\bar{r}_1\mathsf{v}_1)\bar{r}_1 + 2\bar{I}_S\mathsf{v}_1 + \bar{I}_R\mathsf{v}_1 = 2(m_S\bar{r}_2\mathsf{v}_2)\bar{r}_2 + 2\bar{I}_S\mathsf{v}_2 + \bar{I}_R\mathsf{v}_2$$
$$(2m_S\bar{r}_1^2 + 2\bar{I}_S + \bar{I}_R)\mathsf{v}_1 = (2m_S\bar{r}_2^2 + 2\bar{I}_S + \bar{I}_R)\mathsf{v}_2 \qquad (1)$$

which expresses that *the angular momentum of the system about the y axis is conserved.* We now compute

$$\bar{I}_S = \tfrac{2}{5}m_Sa^2 = \tfrac{2}{5}(2 \text{ lb}/32.2 \text{ ft/s}^2)(\tfrac{3}{12}\text{ ft})^2 = 0.00155 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$
$$m_S\bar{r}_1^2 = (2/32.2)(\tfrac{5}{12})^2 = 0.0108 \qquad m_S\bar{r}_2^2 = (2/32.2)(\tfrac{25}{12})^2 = 0.2696$$

Substituting these values, and $\bar{I}_R = 0.25$ and $\mathsf{v}_1 = 6$ rad/s into (1):

$$0.275(6 \text{ rad/s}) = 0.792\mathsf{v}_2 \qquad \mathsf{V}_2 = 2.08 \text{ rad/s l}$$ ◀

b. Energy Lost. The kinetic energy of the system at any instant is

$$T = 2(\tfrac{1}{2}m_S\bar{v}^2 + \tfrac{1}{2}\bar{I}_S\mathsf{v}^2) + \tfrac{1}{2}\bar{I}_R\mathsf{v}^2 = \tfrac{1}{2}(2m_S\bar{r}^2 + 2\bar{I}_S + \bar{I}_R)\mathsf{v}^2$$

Recalling the numerical values found above, we have

$$T_1 = \tfrac{1}{2}(0.275)(6)^2 = 4.95 \text{ ft} \cdot \text{lb} \qquad T_2 = \tfrac{1}{2}(0.792)(2.08)^2 = 1.713 \text{ ft} \cdot \text{lb}$$
$$\Delta T = T_2 - T_1 = 1.71 - 4.95 \qquad \Delta T = -3.24 \text{ ft} \cdot \text{lb}$$ ◀

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to use the method of impulse and momentum to solve problems involving the plane motion of rigid bodies. As you found out previously in Chap. 13, this method is most effective when used in the solution of problems involving velocities and time.

1. The principle of impulse and momentum for the plane motion of a rigid body is expressed by the following vector equation:

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1 \rightarrow 2} = \textbf{Syst Momenta}_2 \qquad (17.14)$$

where **Syst Momenta** represents the system of the momenta of the particles forming the rigid body, and **Syst Ext Imp** represents the system of all the external impulses exerted during the motion.

a. The system of the momenta of a rigid body is equivalent to a linear momentum vector $m\bar{\mathbf{v}}$ attached at the mass center of the body and an angular momentum couple $\bar{I}\omega$ (Fig. 17.7).

b. You should draw a free-body-diagram equation for the rigid body to express graphically the above vector equation. Your diagram equation will consist of three sketches of the body, representing respectively the initial momenta, the impulses of the external forces, and the final momenta. It will show that the system of the initial momenta and the system of the impulses of the external forces are together equivalent to the system of the final momenta (Fig. 17.8).

c. By using the free-body-diagram equation, you can sum components in any direction and sum moments about any point. When summing moments about a point, remember to include the *angular momentum* $\bar{I}\omega$ of the body, as well as the *moments* of the components of its *linear momentum.* In most cases you will be able to select and solve an equation that involves only one unknown. This was done in all the sample problems of this lesson.

2. In problems involving a system of rigid bodies, you can apply the principle of impulse and momentum to the system as a whole. Since internal forces occur in equal and opposite pairs, they will not be part of your solution [Sample Prob. 17.8].

3. Conservation of angular momentum about a given axis occurs when, for a system of rigid bodies, *the sum of the moments of the external impulses about that axis is zero.* You can indeed easily observe from the free-body-diagram equation that the initial and final angular momenta of the system about that axis are equal and, thus, that *the angular momentum of the system about the given axis is conserved.* You can then sum the angular momenta of the various bodies of the system and the moments of their linear momenta about that axis to obtain an equation which can be solved for one unknown [Sample Prob. 17.8].

PROBLEMS

CONCEPT QUESTIONS

17.CQ6 Slender bar A is rigidly connected to a massless rod BC in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar A is negligible compared to L. If bullet D strikes A with a speed v_0 and becomes embedded in it, how will the speeds of the center of gravity of A immediately after the impact compare for the two cases?

a. Case 1 will be larger.
b. Case 2 will be larger.
c. The speeds will be the same.

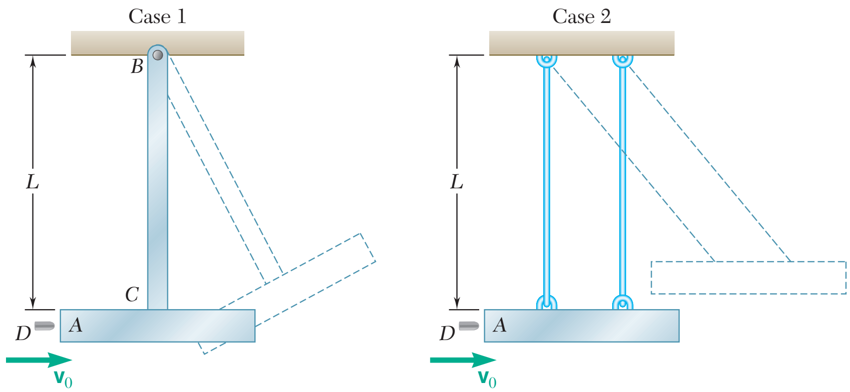


Fig. P17.CQ6

17.CQ7 A 1-m-long uniform slender bar AB has an angular velocity of 12 rad/s and its center of gravity has a velocity of 2 m/s as shown. About which point is the angular momentum of A smallest at this instant?

a. P_1
b. P_2
c. P_3
d. P_4
e. It is the same about all the points.

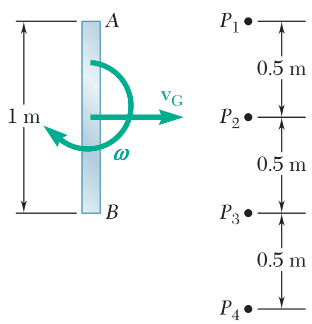


Fig. P17.CQ7

IMPULSE-MOMENTUM PRACTICE PROBLEMS

17.F1 The 350-kg flywheel of a small hoisting engine has a radius of gyration of 600 mm. If the power is cut off when the angular velocity of the flywheel is 100 rpm clockwise, draw an impulse-momentum diagram that can be used to determine the time required for the system to come to rest.

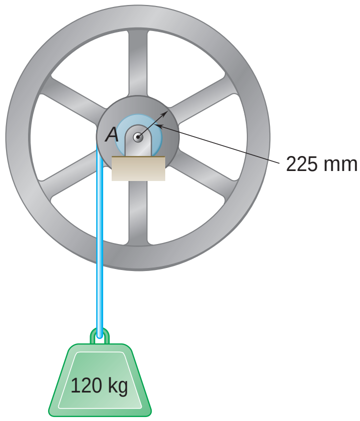


Fig. P17.F1

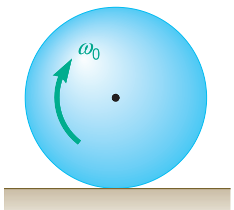


Fig. P17.F2

17.F2 A sphere of radius r and mass m is placed on a horizontal floor with no linear velocity but with a clockwise angular velocity ω_0. Denoting by μ_k the coefficient of kinetic friction between the sphere and the floor, draw the impulse-momentum diagram that can be used to determine the time t_1 at which the sphere will start rolling without sliding.

17.F3 Two panels A and B are attached with hinges to a rectangular plate and held by a wire as shown. The plate and the panels are made of the same material and have the same thickness. The entire assembly is rotating with an angular velocity ω_0 when the wire breaks. Draw the impulse-momentum diagram that is needed to determine the angular velocity of the assembly after the panels have come to rest against the plate.

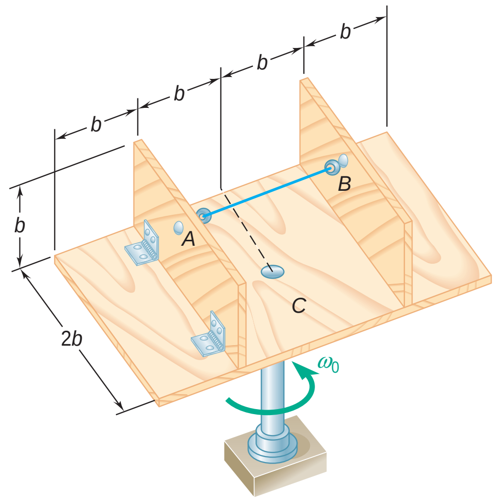


Fig. P17.F3

END-OF-SECTION PROBLEMS

17.52 The rotor of an electric motor has a mass of 25 kg, and it is observed that 4.2 min is required for the rotor to coast to rest from an angular velocity of 3600 rpm. Knowing that kinetic friction produces a couple of magnitude 1.2 N · m, determine the centroidal radius of gyration for the rotor.

17.53 A small grinding wheel is attached to the shaft of an electric motor which has a rated speed of 3600 rpm. When the power is turned off, the unit coasts to rest in 70 s. The grinding wheel and rotor have a combined weight of 6 lb and a combined radius of gyration of 2 in. Determine the average magnitude of the couple due to kinetic friction in the bearings of the motor.

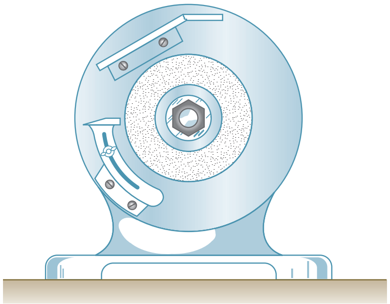


Fig. P17.53

17.54 A bolt located 50 mm from the center of an automobile wheel is tightened by applying the couple shown for 0.10 s. Assuming that the wheel is free to rotate and is initially at rest, determine the resulting angular velocity of the wheel. The wheel has a mass of 19 kg and has a radius of gyration of 250 mm.

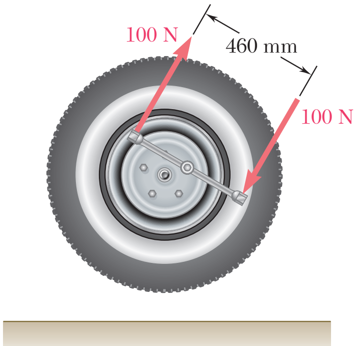


Fig. P17.54

17.55 Two disks of the same thickness and same material are attached to a shaft as shown. The 8-lb disk A has a radius $r_A = 3$ in., and disk B has a radius $r_B = 4.5$ in. Knowing that a couple $\mathbf{M}$ of magnitude 20 lb · in. is applied to disk A when the system is at rest, determine the time required for the angular velocity of the system to reach 960 rpm.

17.56 Two disks of the same thickness and same material are attached to a shaft as shown. The 3-kg disk A has a radius $r_A = 100$ mm, and disk B has a radius $r_B = 125$ mm. Knowing that the angular velocity of the system is to be increased from 200 rpm to 800 rpm during a 3-s interval, determine the magnitude of the couple $\mathbf{M}$ that must be applied to disk A.

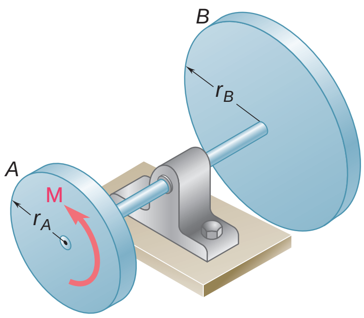


Fig. P17.55 and *P17.56*

17.57 A disk of constant thickness, initially at rest, is placed in contact with a belt that moves with a constant velocity $\mathbf{v}$. Denoting by μ_k the coefficient of kinetic friction between the disk and the belt, derive an expression for the time required for the disk to reach a constant angular velocity.

17.58 Disk A, of weight 5 lb and radius $r = 3$ in., is at rest when it is placed in contact with a belt which moves at a constant speed $v = 50$ ft/s. Knowing that $\mu_k = 0.20$ between the disk and the belt, determine the time required for the disk to reach a constant angular velocity.

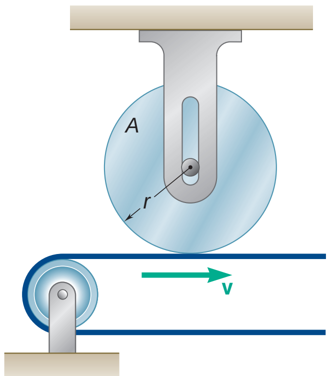


Fig. *P17.57* and P17.58

17.59 A cylinder of radius r and weight W with an initial counterclockwise angular velocity ω_0 is placed in the corner formed by the floor and a vertical wall. Denoting by μ_k the coefficient of kinetic friction between the cylinder and the wall and the floor, derive an expression for the time required for the cylinder to come to rest.

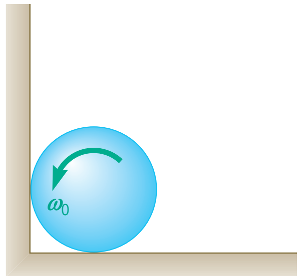


Fig. P17.59

17.60 and 17.61 Two uniform disks and two cylinders are assembled as indicated. Disk A has a mass of 10 kg and disk B has a mass of 6 kg. Knowing that the system is released from rest, determine the time required for cylinder C to have a speed of 0.5 m/s.

17.60 Disks A and B are bolted together and the cylinders are attached to separate cords wrapped on the disks.

17.61 The cylinders are attached to a single cord that passes over the disks. Assume that no slipping occurs between the cord and the disks.

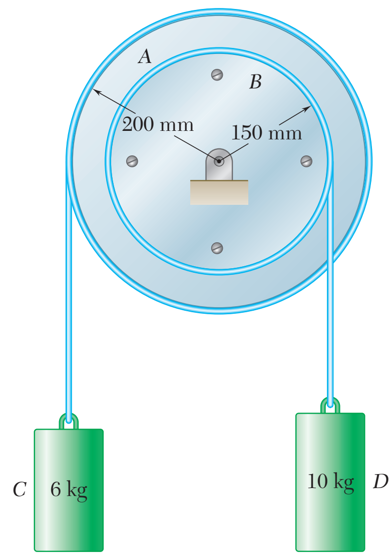


Fig. P17.60

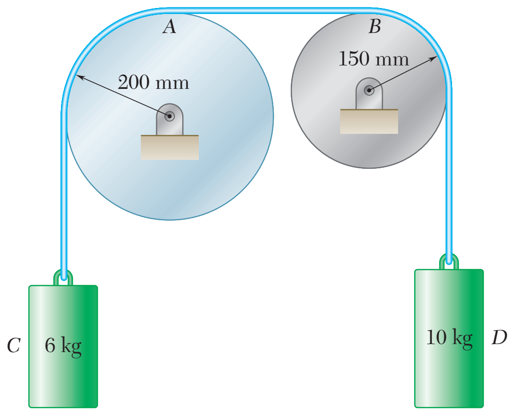


Fig. P17.61

17.62 Disk B has an initial angular velocity ω_0 when it is brought into contact with disk A which is at rest. Show that the final angular velocity of disk B depends only on ω_0 and the ratio of the masses m_A and m_B of the two disks.

17.63 The 7.5-lb disk A has a radius $r_A = 6$ in. and is initially at rest. The 10-lb disk B has a radius $r_B = 8$ in. and an angular velocity ω_0 of 900 rpm when it is brought into contact with disk A. Neglecting friction in the bearings, determine (*a*) the final angular velocity of each disk, (*b*) the total impulse of the friction force exerted on disk A.

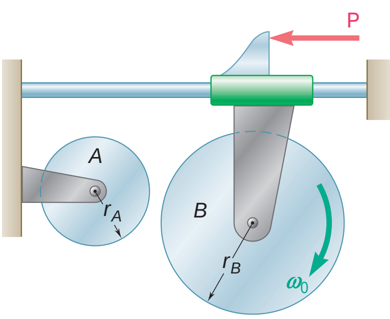


Fig. P17.62 and P17.63

17.64 A tape moves over the two drums shown. Drum A weighs 1.4 lb and has a radius of gyration of 0.75 in., while drum B weighs 3.5 lb and has a radius of gyration of 1.25 in. In the lower portion of the tape the tension is constant and equal to $T_A = 0.75$ lb. Knowing that the tape is initially at rest, determine (*a*) the required constant tension T_B if the velocity of the tape is to be $v = 10$ ft/s after 0.24 s, (*b*) the corresponding tension in the portion of the tape between the drums.

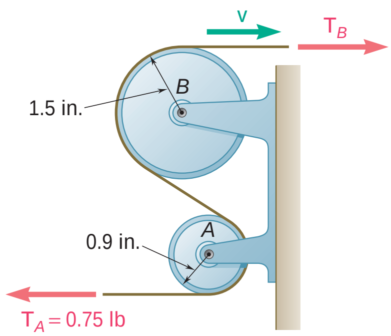


Fig. P17.64

17.65 Show that the system of momenta for a rigid slab in plane motion reduces to a single vector, and express the distance from the mass center G to the line of action of this vector in terms of the centroidal radius of gyration $\bar{k}$ of the slab, the magnitude $\bar{v}$ of the velocity of G, and the angular velocity $\boldsymbol{\omega}$.

17.66 Show that, when a rigid slab rotates about a fixed axis through O perpendicular to the slab, the system of the momenta of its particles is equivalent to a single vector of magnitude $m\bar{r}\omega$, perpendicular to the line OG, and applied to a point P on this line, called the *center of percussion*, at a distance $GP = \bar{k}^2/\bar{r}$ from the mass center of the slab.

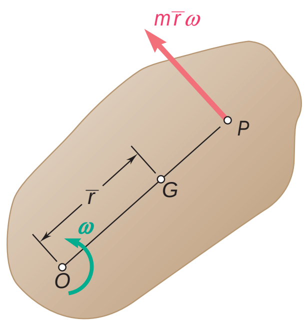


Fig. P17.66

17.67 Show that the sum $\mathbf{H}_A$ of the moments about a point A of the momenta of the particles of a rigid slab in plane motion is equal to $I_A\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the angular velocity of the slab at the instant considered and I_A the moment of inertia of the slab about A, if and only if one of the following conditions is satisfied: (*a*) A is the mass center of the slab, (*b*) A is the instantaneous center of rotation, (*c*) the velocity of A is directed along a line joining point A and the mass center G.

17.68 Consider a rigid slab initially at rest and subjected to an impulsive force $\mathbf{F}$ contained in the plane of the slab. We define the *center of percussion* P as the point of intersection of the line of action of $\mathbf{F}$ with the perpendicular drawn from G. (*a*) Show that the instantaneous center of rotation C of the slab is located on line GP at a distance $GC = \bar{k}^2/GP$ on the opposite side of G. (*b*) Show that if the center of percussion were located at C the instantaneous center of rotation would be located at P.

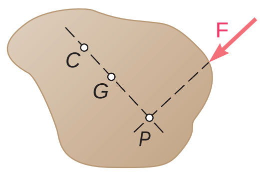


Fig. P17.68

17.69 A flywheel is rigidly attached to a 1.5-in.-radius shaft that rolls without sliding along parallel rails. Knowing that after being released from rest the system attains a speed of 6 in./s in 30 s, determine the centroidal radius of gyration of the system.

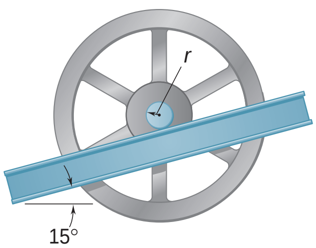


Fig. P17.69

17.70 A wheel of radius r and centroidal radius of gyration $\bar{k}$ is released from rest on the incline shown at time $t = 0$. Assuming that the wheel rolls without sliding, determine (*a*) the velocity of its center at time t, (*b*) the coefficient of static friction required to prevent slipping.

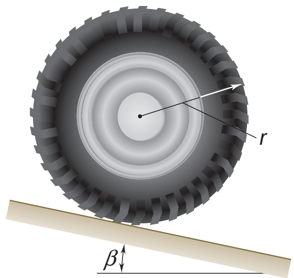


Fig. P17.70

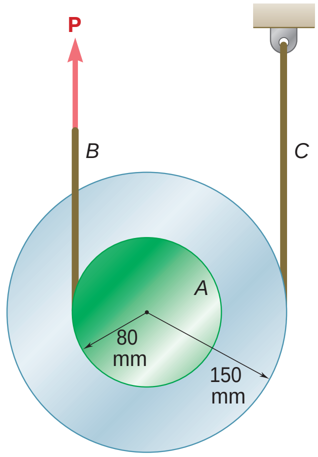


Fig. P17.71

17.71 The double pulley shown has a mass of 3 kg and a radius of gyration of 100 mm. Knowing that when the pulley is at rest, a force **P** of magnitude 24 N is applied to cord B, determine (*a*) the velocity of the center of the pulley after 1.5 s, (*b*) the tension in cord C.

17.72 and *17.73* A 9-in.-radius cylinder of weight 18 lb rests on a 6-lb carriage. The system is at rest when a force **P** of magnitude 2.5 lb is applied as shown for 1.2 s. Knowing that the cylinder rolls without sliding on the carriage and neglecting the mass of the wheels of the carriage, determine the resulting velocity of (*a*) the carriage, (*b*) the center of the cylinder.

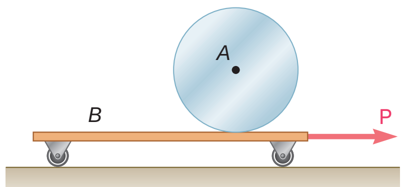


Fig. P17.72

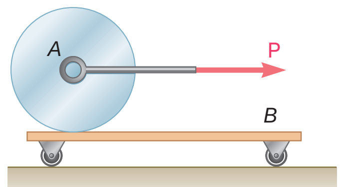


Fig. *P17.73*

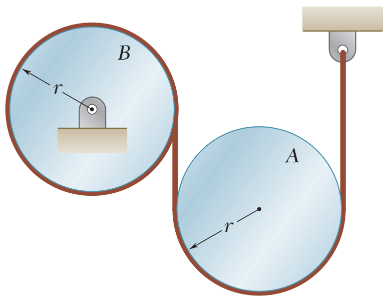


Fig. P17.74 and P17.75

17.74 Two uniform cylinders, each of mass $m = 6$ kg and radius $r = 125$ mm, are connected by a belt as shown. If the system is released from rest when $t = 0$, determine (*a*) the velocity of the center of cylinder B at $t = 3$ s, (*b*) the tension in the portion of belt connecting the two cylinders.

17.75 Two uniform cylinders, each of mass $m = 6$ kg and radius $r = 125$ mm, are connected by a belt as shown. Knowing that at the instant shown the angular velocity of cylinder A is 30 rad/s counterclockwise, determine (*a*) the time required for the angular velocity of cylinder A to be reduced to 5 rad/s, (*b*) the tension in the portion of belt connecting the two cylinders.

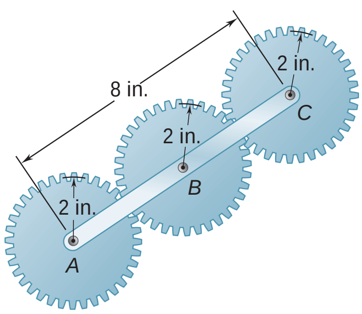


Fig. *P17.76*

17.76 In the gear arrangement shown, gears A and C are attached to rod ABC, which is free to rotate about B, while the inner gear B is fixed. Knowing that the system is at rest, determine the magnitude of the couple **M** which must be applied to rod ABC, if 2.5 s later the angular velocity of the rod is to be 240 rpm clockwise. Gears A and C weigh 2.5 lb each and may be considered as disks of radius 2 in.; rod ABC weighs 4 lb.

17.77 A sphere of radius r and mass m is projected along a rough horizontal surface with the initial velocities shown. If the final velocity of the sphere is to be zero, express (*a*) the required magnitude of V_0 in terms of v_0 and r, (*b*) the time required for the sphere to come to rest in terms of v_0 and the coefficient of kinetic friction m_k.

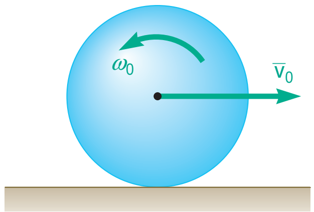


Fig. P17.77

17.78 A bowler projects an 8.5-in.-diameter ball weighing 16 lb along an alley with a forward velocity $\mathbf{v}_0$ of 25 ft/s and a backspin V_0 of 9 rad/s. Knowing that the coefficient of kinetic friction between the ball and the alley is 0.10, determine (*a*) the time t_1 at which the ball will start rolling without sliding, (*b*) the speed of the ball at time t_1.

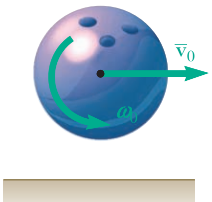


Fig. P17.78

17.79 Four rectangular panels, each of length b and height $\frac{1}{2}b$, are attached with hinges to a circular plate of diameter $1\overline{2}\,b$ and held by a wire loop in the position shown. The plate and the panels are made of the same material and have the same thickness. The entire assembly is rotating with an angular velocity V_0 when the wire breaks. Determine the angular velocity of the assembly after the panels have come to rest in a horizontal position.

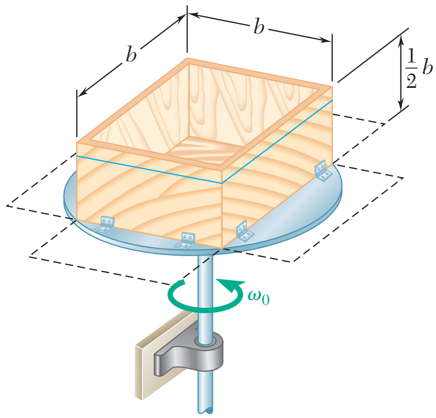


Fig. P17.79

17.80 A 2.5-lb disk of radius 4 in. is attached to the yoke BCD by means of short shafts fitted in bearings at B and D. The 1.5-lb yoke has a radius of gyration of 3 in. about the x axis. Initially the assembly is rotating at 120 rpm with the disk in the plane of the yoke ($\mathsf{u} = 0$). If the disk is slightly disturbed and rotates with respect to the yoke until $\mathsf{u} = 90°$, where it is stopped by a small bar at D, determine the final angular velocity of the assembly.

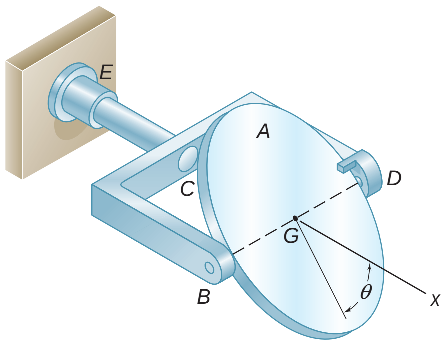


Fig. P17.80

17.81 Two 10-lb disks and a small motor are mounted on a 15-lb rectangular platform which is free to rotate about a central vertical spindle. The normal operating speed of the motor is 180 rpm. If the motor is started when the system is at rest, determine the angular velocity of all elements of the system after the motor has attained its normal operating speed. Neglect the mass of the motor and of the belt.

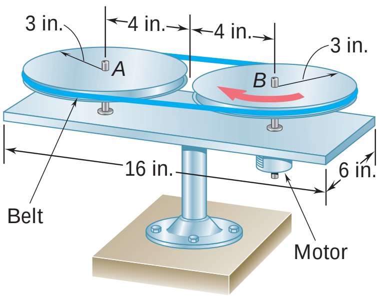


Fig. P17.81

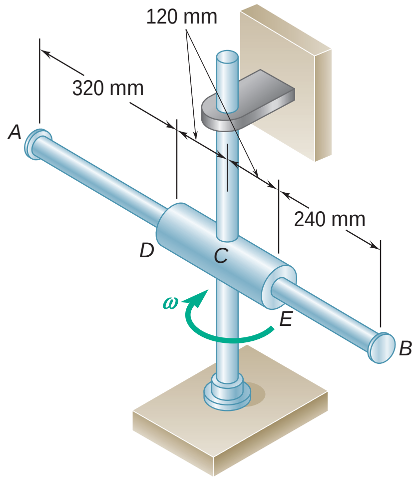


Fig. P17.82

17.82 A 3-kg rod of length 800 mm can slide freely in the 240-mm cylinder DE, which in turn can rotate freely in a horizontal plane. In the position shown the assembly is rotating with an angular velocity of magnitude $\omega = 40$ rad/s and end B of the rod is moving toward the cylinder at a speed of 75 mm/s relative to the cylinder. Knowing that the centroidal mass moment of inertia of the cylinder about a vertical axis is 0.025 kg · m^2 and neglecting the effect of friction, determine the angular velocity of the assembly as end B of the rod strikes end E of the cylinder.

17.83 A 1.6-kg tube AB can slide freely on rod DE which in turn can rotate freely in a horizontal plane. Initially the assembly is rotating with an angular velocity $\omega = 5$ rad/s and the tube is held in position by a cord. The moment of inertia of the rod and bracket about the vertical axis of rotation is 0.30 kg · m^2 and the centroidal moment of inertia of the tube about a vertical axis is 0.0025 kg · m^2. If the cord suddenly breaks, determine (*a*) the angular velocity of the assembly after the tube has moved to end E, (*b*) the energy lost during the plastic impact at E.

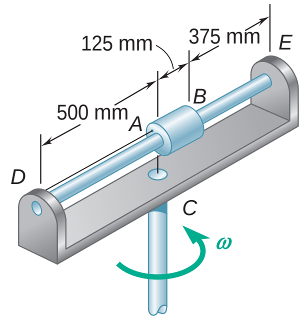


Fig. P17.83

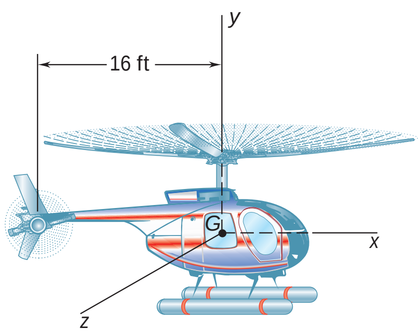


Fig. *P17.84*

17.84 In the helicopter shown, a vertical tail propeller is used to prevent rotation of the cab as the speed of the main blades is changed. Assuming that the tail propeller is not operating, determine the final angular velocity of the cab after the speed of the main blades has been changed from 180 to 240 rpm. (The speed of the main blades is measured relative to the cab, and the cab has a centroidal moment of inertia of 650 lb · ft · s^2. Each of the four main blades is assumed to be a slender 14-ft rod weighing 55 lb.)

17.85 Assuming that the tail propeller in Prob. 17.84 is operating and that the angular velocity of the cab remains zero, determine the final horizontal velocity of the cab when the speed of the main blades is changed from 180 to 240 rpm. The cab weighs 1250 lb and is initially at rest. Also determine the force exerted by the tail propeller if the change in speed takes place uniformly in 12 s.

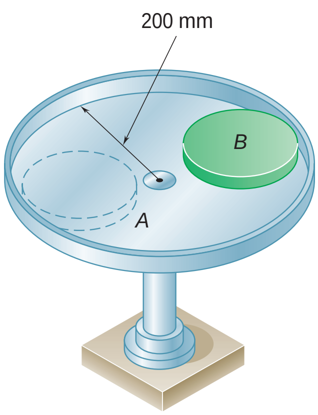


Fig. P17.86

17.86 The circular platform A is fitted with a rim of 200-mm inner radius and can rotate freely about the vertical shaft. It is known that the platform-rim unit has a mass of 5 kg and a radius of gyration of 175 mm with respect to the shaft. At a time when the platform is rotating with an angular velocity of 50 rpm, a 3-kg disk B of radius 80 mm is placed on the platform with no velocity. Knowing that disk B then slides until it comes to rest relative to the platform against the rim, determine the final angular velocity of the platform.

17.87 Two 4-kg disks and a small motor are mounted on a 6-kg rectangular platform which is free to rotate about a central vertical spindle. The normal operating speed of the motor is 240 rpm. If the motor is started when the system is at rest, determine the angular velocity of all elements of the system after the motor has attained its normal operating speed. Neglect the mass of the motor and of the belt.

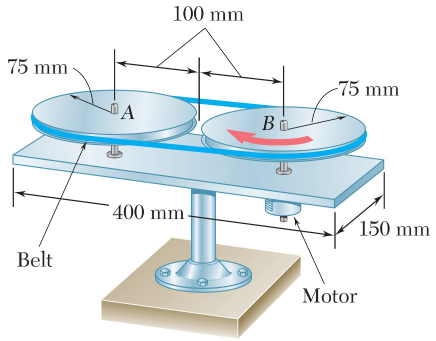


Fig. P17.87

17.88 The 4-kg rod AB can slide freely inside the 6-kg tube CD. The rod was entirely within the tube ($x = 0$) and released with no initial velocity relative to the tube when the angular velocity of the assembly was 5 rad/s. Neglecting the effect of friction, determine the speed of the rod relative to the tube when $x = 400$ mm.

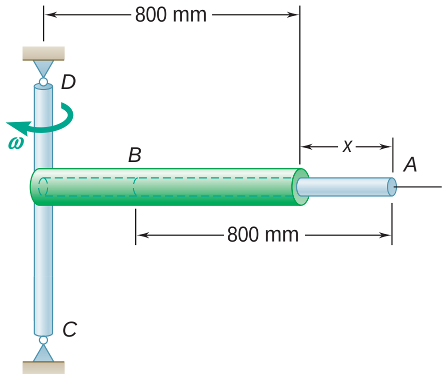


Fig. P17.88

17.89 A 1.8-kg collar A and a 0.7-kg collar B can slide without friction on a frame, consisting of the horizontal rod OE and the vertical rod CD, which is free to rotate about its vertical axis of symmetry. The two collars are connected by a cord running over a pulley that is attached to the frame at O. At the instant shown, the velocity v_A of collar A has a magnitude of 2.1 m/s and a stop prevents collar B from moving. The stop is suddenly removed and collar A moves toward E. As it reaches a distance of 0.12 m from O, the magnitude of its velocity is observed to be 2.5 m/s. Determine at that instant the magnitude of the angular velocity of the frame and the moment of inertia of the frame and pulley system about CD.

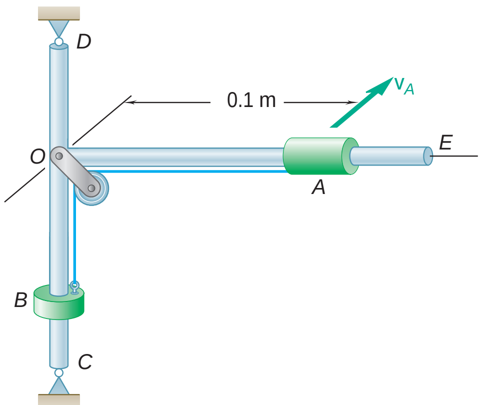


Fig. P17.89

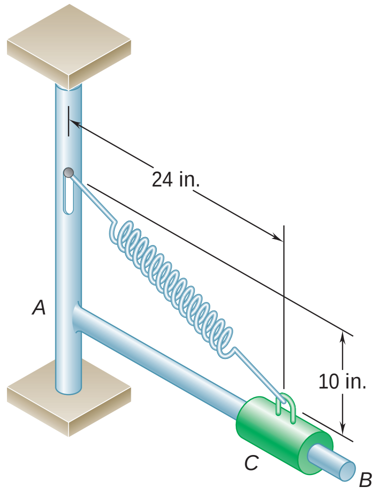


Fig. P17.90

17.90 A 6-lb collar *C* is attached to a spring and can slide on rod *AB*, which in turn can rotate in a horizontal plane. The mass moment of inertia of rod *AB* with respect to end *A* is 0.35 lb · ft · s^2. The spring has a constant k = 15 lb/in. and an undeformed length of 10 in. At the instant shown the velocity of the collar relative to the rod is zero and the assembly is rotating with an angular velocity of 12 rad/s. Neglecting the effect of friction, determine (*a*) the angular velocity of the assembly as the collar passes through a point located 7.5 in. from end *A* of the rod, (*b*) the corresponding velocity of the collar relative to the rod.

17.91 A small 4-lb collar *C* can slide freely on a thin ring of weight 6 lb and radius 10 in. The ring is welded to a short vertical shaft, which can rotate freely in a fixed bearing. Initially the ring has an angular velocity of 35 rad/s and the collar is at the top of the ring (u = 0) when it is given a slight nudge. Neglecting the effect of friction, determine (*a*) the angular velocity of the ring as the collar passes through the position u = 90°, (*b*) the corresponding velocity of the collar relative to the ring.

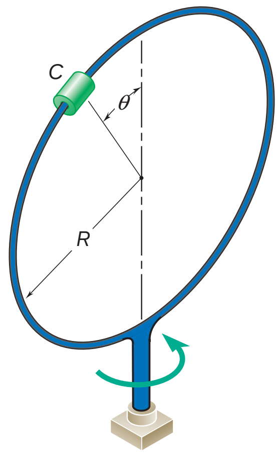


Fig. P17.91

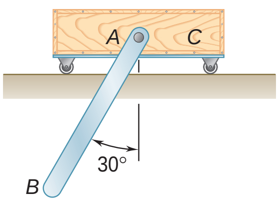


Fig. P17.92

17.92 A uniform rod *AB*, of mass 7 kg and length 1.2 m, is attached to the 11-kg cart *C*. Knowing that the system is released from rest in the position shown and neglecting friction, determine (*a*) the velocity of point *B* as rod *AB* passes through a vertical position, (*b*) the corresponding velocity of cart *C*.

17.93 In Prob. 17.82, determine the velocity of rod *AB* relative to cylinder *DE* as end *B* of the rod strikes end *E* of the cylinder.

17.94 In Prob. 17.83, determine the velocity of the tube relative to the rod as the tube strikes end *E* of the assembly.

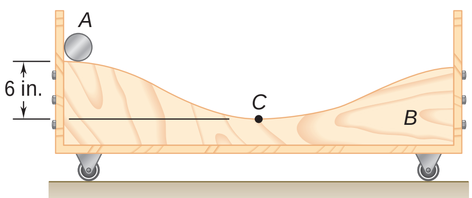


Fig. P17.95

17.95 The 6-lb steel cylinder *A* and the 10-lb wooden cart *B* are at rest in the position shown when the cylinder is given a slight nudge, causing it to roll without sliding along the top surface of the cart. Neglecting friction between the cart and the ground, determine the velocity of the cart as the cylinder passes through the lowest point of the surface at *C*.

17.11 IMPULSIVE MOTION

You saw in Chap. 13 that the method of impulse and momentum is the only practicable method for the solution of problems involving the impulsive motion of a particle. Now you will find that problems involving the impulsive motion of a rigid body are particularly well suited to a solution by the method of impulse and momentum. Since the time interval considered in the computation of linear impulses and angular impulses is very short, the bodies involved can be assumed to occupy the same position during that time interval, making the computation quite simple.

17.12 ECCENTRIC IMPACT

In Secs. 13.13 and 13.14, you learned to solve problems of *central impact,* i.e., problems in which the mass centers of the two colliding bodies are located on the line of impact. You will now analyze the *eccentric impact* of two rigid bodies. Consider two bodies which collide, and denote by $\mathbf{v}_A$ and $\mathbf{v}_B$ the velocities before impact *of the two points of contact* A and B (Fig. 17.10*a*). Under the impact, the two

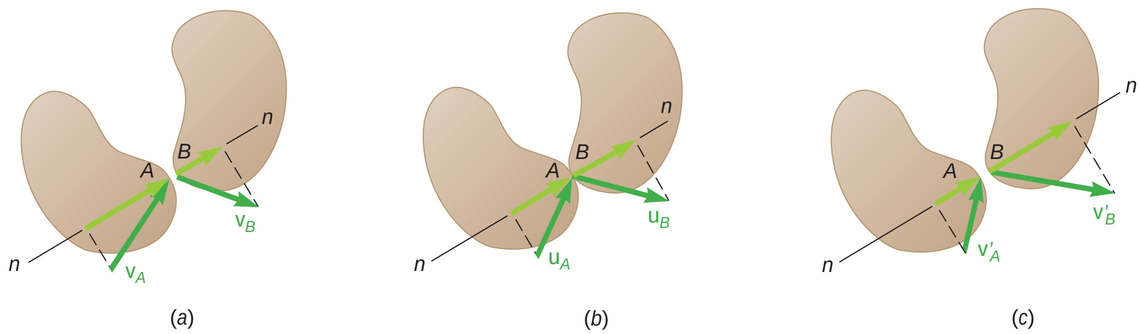


Fig. 17.10

bodies will *deform,* and at the end of the period of deformation, the velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ of A and B will have equal components along the line of impact nn (Fig. 17.10*b*). A period of *restitution* will then take place, at the end of which A and B will have velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ (Fig. 17.10*c*). Assuming that the bodies are frictionless, we find that the forces they exert on each other are directed along the line of impact. Denoting the magnitude of the impulse of one of these forces during the period of deformation by $\int P\,dt$ and the magnitude of its impulse during the period of restitution by $\int R\,dt$, we recall that the coefficient of restitution e is defined as the ratio

$$e = \frac{\int R\,dt}{\int P\,dt} \tag{17.18}$$

We propose to show that the relation established in Sec. 13.13 between the relative velocities of two particles before and after impact also holds between the components along the line of impact

Photo 17.4 When the rotating bat contacts the ball it applies an impulsive force to the ball requiring the method of impulse and momentum to be used to determine the final velocities of the ball and bat.

of the relative velocities of the two points of contact A and B. We propose to show, therefore, that

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{17.19}$$

It will first be assumed that the motion of each of the two colliding bodies of Fig. 17.10 is unconstrained. Thus the only impulsive forces exerted on the bodies during the impact are applied at A and B, respectively. Consider the body to which point A belongs and draw the three momentum and impulse diagrams corresponding to the period of deformation (Fig. 17.11). We denote by $\bar{\mathbf{v}}$ and $\bar{\mathbf{u}}$,

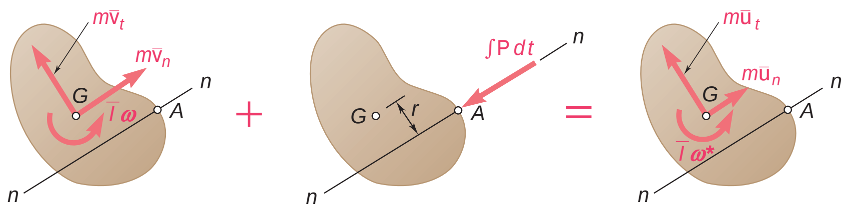


Fig. 17.11

respectively, the velocity of the mass center at the beginning and at the end of the period of deformation, and we denote by V and V* the angular velocity of the body at the same instants. Summing and equating the components of the momenta and impulses along the line of impact nn, we write

$$m\bar{v}_n - \int P\,dt = m\bar{u}_n \tag{17.20}$$

Summing and equating the moments about G of the momenta and impulses, we also write

$$\bar{I}\mathrm{V} - r\int P\,dt = \bar{I}\mathrm{V}^* \tag{17.21}$$

where r represents the perpendicular distance from G to the line of impact. Considering now the period of restitution, we obtain in a similar way

$$m\bar{u}_n - \int R\,dt = m\bar{v}'_n \tag{17.22}$$

$$\bar{I}\mathrm{V}^* - r\int R\,dt = \bar{I}\mathrm{V}' \tag{17.23}$$

where $\bar{\mathbf{v}}'$ and V′ represent, respectively, the velocity of the mass center and the angular velocity of the body after impact. Solving (17.20) and (17.22) for the two impulses and substituting into (17.18), and then solving (17.21) and (17.23) for the same two impulses and substituting again into (17.18), we obtain the following two alternative expressions for the coefficient of restitution:

$$e = \frac{\bar{u}_n - \bar{v}'_n}{\bar{v}_n - \bar{u}_n} \qquad e = \frac{\mathrm{V}^* - \mathrm{V}'}{\mathrm{V} - \mathrm{V}^*} \tag{17.24}$$

Multiplying by r the numerator and denominator of the second expression obtained for e, and adding respectively to the numerator and denominator of the first expression, we have

$$e = \frac{\bar{u}_n + r\omega^* - (\bar{v}'_n + r\omega')}{\bar{v}_n + r\omega - (\bar{u}_n + r\omega^*)} \tag{17.25}$$

Observing that $\bar{v}_n + r\omega$ represents the component $(v_A)_n$ along nn of the velocity of the point of contact A and that, similarly, $\bar{u}_n + r\omega^*$ and $\bar{v}'_n + r\omega'$ represent, respectively, the components $(u_A)_n$ and $(v'_A)_n$, we write

$$e = \frac{(u_A)_n - (v'_A)_n}{(v_A)_n - (u_A)_n} \tag{17.26}$$

The analysis of the motion of the second body leads to a similar expression for e in terms of the components along nn of the successive velocities of point B. Recalling that $(u_A)_n = (u_B)_n$, and eliminating these two velocity components by a manipulation similar to the one used in Sec. 13.13, we obtain relation (17.19).

If one or both of the colliding bodies is constrained to rotate about a fixed point O, as in the case of a compound pendulum (Fig. 17.12*a*), an impulsive reaction will be exerted at O (Fig. 17.12*b*).

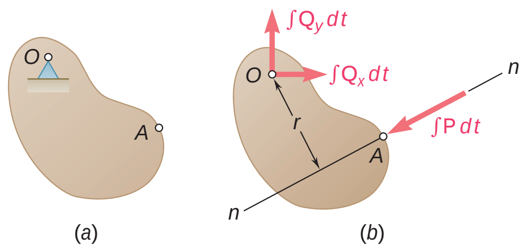


Fig. 17.12

Let us verify that while their derivation must be modified, Eqs. (17.26) and (17.19) remain valid. Applying formula (17.16) to the period of deformation and to the period of restitution, we write

$$I_O\omega - r\int P\,dt = I_O\omega^* \tag{17.27}$$
$$I_O\omega^* - r\int R\,dt = I_O\omega' \tag{17.28}$$

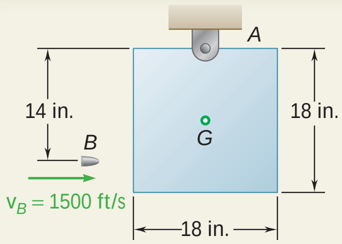
where r represents the perpendicular distance from the fixed point O to the line of impact. Solving (17.27) and (17.28) for the two impulses and substituting into (17.18), and then observing that $r\omega$, $r\omega^*$, and $r\omega'$ represent the components along nn of the successive velocities of point A, we write

$$e = \frac{\omega^* - \omega'}{\omega - \omega^*} = \frac{r\omega^* - r\omega'}{r\omega - r\omega^*} = \frac{(u_A)_n - (v'_A)_n}{(v_A)_n - (u_A)_n}$$

and check that Eq. (17.26) still holds. Thus Eq. (17.19) remains valid when one or both of the colliding bodies is constrained to rotate about a fixed point O.

In order to determine the velocities of the two colliding bodies after impact, relation (17.19) should be used in conjunction with one or several other equations obtained by applying the principle of impulse and momentum (Sample Prob. 17.10).

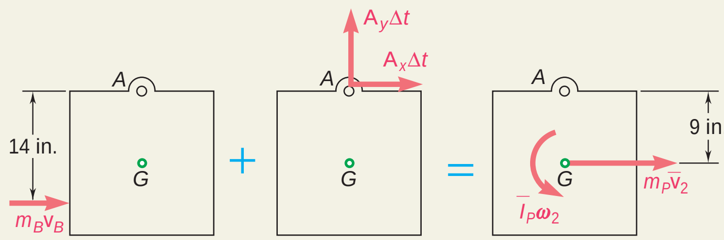
SAMPLE PROBLEM 17.9



A 0.05-lb bullet B is fired with a horizontal velocity of 1500 ft/s into the side of a 20-lb square panel suspended from a hinge at A. Knowing that the panel is initially at rest, determine (a) the angular velocity of the panel immediately after the bullet becomes embedded, (b) the impulsive reaction at A, assuming that the bullet becomes embedded in 0.0006 s.

SOLUTION

Principle of Impulse and Momentum. We consider the bullet and the panel as a single system and express that the initial momenta of the bullet and panel and the impulses of the external forces are together equipollent to the final momenta of the system. Since the time interval $\Delta t = 0.0006$ s is very short, we neglect all nonimpulsive forces and consider only the external impulses $\mathbf{A}_x\,\Delta t$ and $\mathbf{A}_y\,\Delta t$.



Syst Momenta$_1$ + Syst Ext Imp$_{1\to 2}$ = Syst Momenta$_2$

+↺ moments about A: $\quad m_B v_B(\frac{14}{12}\text{ ft}) + 0 = m_P\bar{v}_2(\frac{9}{12}\text{ ft}) + \bar{I}_P\omega_2 \qquad (1)$

$\xrightarrow{+}$ x components: $\quad m_B v_B + A_x\,\Delta t = m_P\bar{v}_2 \qquad (2)$

+↑y components: $\quad 0 + A_y\,\Delta t = 0 \qquad (3)$

The centroidal mass moment of inertia of the square panel is

$$\bar{I}_P = \tfrac{1}{6}m_P b^2 = \frac{1}{6}\left(\frac{20\text{ lb}}{32.2}\right)(\tfrac{18}{12}\text{ ft})^2 = 0.2329\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

Substituting this value as well as the given data into (1) and noting that

$$\bar{v}_2 = (\tfrac{9}{12}\text{ ft})\omega_2$$

we write

$$\left(\frac{0.05}{32.2}\right)(1500)(\tfrac{14}{12}) = 0.2329\omega_2 + \left(\frac{20}{32.2}\right)(\tfrac{9}{12}\omega_2)(\tfrac{9}{12})$$

$$\omega_2 = 4.67\text{ rad/s} \qquad \boldsymbol{\omega}_2 = 4.67\text{ rad/s}↺ \blacktriangleleft$$

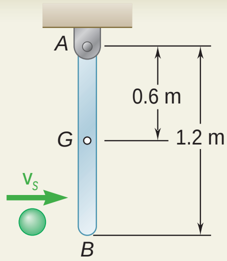
$$\bar{v}_2 = (\tfrac{9}{12}\text{ ft})\omega_2 = (\tfrac{9}{12}\text{ ft})(4.67\text{ rad/s}) = 3.50\text{ ft/s}$$

Substituting $\bar{v}_2 = 3.50$ ft/s, $\Delta t = 0.0006$ s, and the given data into Eq. (2), we have

$$\left(\frac{0.05}{32.2}\right)(1500) + A_x(0.0006) = \left(\frac{20}{32.2}\right)(3.50)$$

$$A_x = -259\text{ lb} \qquad \mathbf{A}_x = 259\text{ lb} \leftarrow \blacktriangleleft$$

From Eq. (3), we find $\quad A_y = 0 \qquad \mathbf{A}_y = 0 \blacktriangleleft$

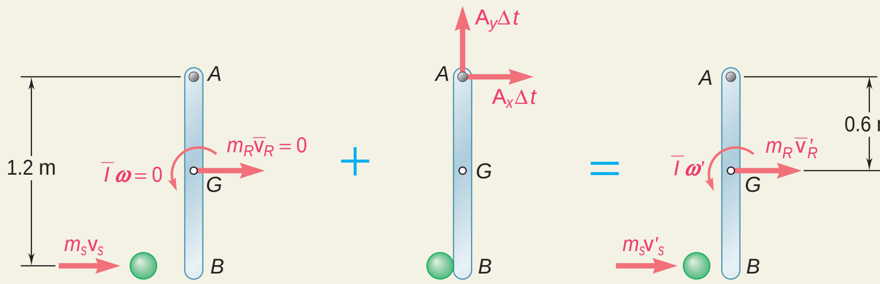


SAMPLE PROBLEM 17.10

A 2-kg sphere moving horizontally to the right with an initial velocity of 5 m/s strikes the lower end of an 8-kg rigid rod *AB*. The rod is suspended from a hinge at *A* and is initially at rest. Knowing that the coefficient of restitution between the rod and the sphere is 0.80, determine the angular velocity of the rod and the velocity of the sphere immediately after the impact.

SOLUTION

Principle of Impulse and Momentum. We consider the rod and sphere as a single system and express that the initial momenta of the rod and sphere and the impulses of the external forces are together equipollent to the final momenta of the system. We note that the only impulsive force external to the system is the impulsive reaction at *A*.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to2} = \textbf{Syst Momenta}_2$$

$+\circlearrowleft$ moments about *A*:

$$m_s v_s(1.2\text{ m}) = m_s v'_s(1.2\text{ m}) + m_R\bar{v}'_R(0.6\text{ m}) + \bar{I}\omega' \qquad (1)$$

Since the rod rotates about *A*, we have $\bar{v}'_R = \bar{r}\omega' = (0.6\text{ m})\omega'$. Also,

$$\bar{I} = \tfrac{1}{12}mL^2 = \tfrac{1}{12}(8\text{ kg})(1.2\text{ m})^2 = 0.96\text{ kg}\cdot\text{m}^2$$

Substituting these values and the given data into Eq. (1), we have

$$(2\text{ kg})(5\text{ m/s})(1.2\text{ m}) = (2\text{ kg})v'_s(1.2\text{ m}) + (8\text{ kg})(0.6\text{ m})\omega'(0.6\text{ m}) + (0.96\text{ kg}\cdot\text{m}^2)\omega'$$

$$12 = 2.4v'_s + 3.84\omega' \qquad (2)$$

Relative Velocities. Choosing positive to the right, we write

$$v'_B - v'_s = e(v_s - v_B)$$

Substituting $v_s = 5$ m/s, $v_B = 0$, and $e = 0.80$, we obtain

$$v'_B - v'_s = 0.80(5\text{ m/s}) \qquad (3)$$

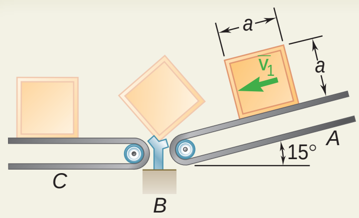
Again noting that the rod rotates about *A*, we write

$$v'_B = (1.2\text{ m})\omega' \qquad (4)$$

Solving Eqs. (2) to (4) simultaneously, we obtain

$$\omega' = 3.21\text{ rad/s} \qquad \boldsymbol{\omega}' = 3.21\text{ rad/s} \circlearrowleft \blacktriangleleft$$

$$v'_s = -0.143\text{ m/s} \qquad \mathbf{v}'_s = -0.143\text{ m/s} \rightarrow \blacktriangleleft$$

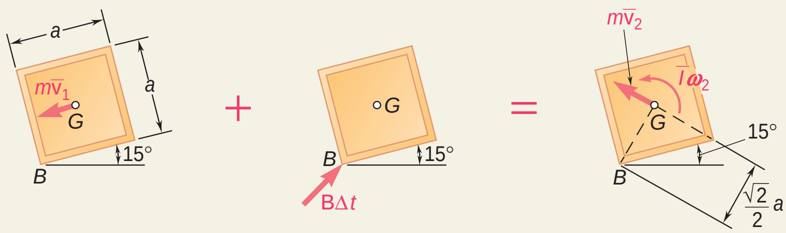


SAMPLE PROBLEM 17.11

A square package of side a and mass m moves down a conveyor belt A with a constant velocity $\bar{\mathbf{v}}_1$. At the end of the conveyor belt, the corner of the package strikes a rigid support at B. Assuming that the impact at B is perfectly plastic, derive an expression for the smallest magnitude of the velocity $\bar{\mathbf{v}}_1$ for which the package will rotate about B and reach conveyor belt C.

SOLUTION

Principle of Impulse and Momentum. Since the impact between the package and the support is perfectly plastic, the package rotates about B during the impact. We apply the principle of impulse and momentum to the package and note that the only impulsive force external to the package is the impulsive reaction at B.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1 \to 2} = \textbf{Syst Momenta}_2$$

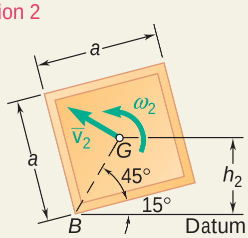
$+\circlearrowleft$ moments about B: $\quad (m\bar{v}_1)(\tfrac{1}{2}a) + 0 = (m\bar{v}_2)(\tfrac{1}{2}\sqrt{2}a) + \bar{I}\omega_2 \qquad (1)$

Since the package rotates about B, we have $\bar{v}_2 = (GB)\omega_2 = \frac{1}{2}\sqrt{2}a\omega_2$. We substitute this expression, together with $\bar{I} = \frac{1}{6}ma^2$, into Eq. (1):

$$(m\bar{v}_1)(\tfrac{1}{2}a) = m(\tfrac{1}{2}\sqrt{2}a\omega_2)(\tfrac{1}{2}\sqrt{2}a) + \tfrac{1}{6}ma^2\omega_2 \qquad \bar{v}_1 = \tfrac{4}{3}a\omega_2 \qquad (2)$$

Principle of Conservation of Energy. We apply the principle of conservation of energy between position 2 and position 3.

Position 2



$GB = \frac{1}{2}\sqrt{2}a = 0.707a$

$h_2 = GB \sin(45° + 15°)$

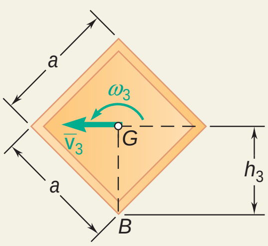
$= 0.612a$

Position 2. $V_2 = Wh_2$. Recalling that $\bar{v}_2 = \frac{1}{2}\sqrt{2}a\omega_2$, we write

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 = \tfrac{1}{2}m(\tfrac{1}{2}\sqrt{2}a\omega_2)^2 + \tfrac{1}{2}(\tfrac{1}{6}ma^2)\omega_2^2 = \tfrac{1}{3}ma^2\omega_2^2$$

Position 3. Since the package must reach conveyor belt C, it must pass through position 3 where G is directly above B. Also, since we wish to determine the smallest velocity for which the package will reach this position, we choose $\bar{v}_3 = \omega_3 = 0$. Therefore $T_3 = 0$ and $V_3 = Wh_3$.

Position 3



$h_3 = GB = 0.707a$

Conservation of Energy

$$T_2 + V_2 = T_3 + V_3$$

$$\tfrac{1}{3}ma^2\omega_2^2 + Wh_2 = 0 + Wh_3$$

$$\omega_2^2 = \frac{3W}{ma^2}(h_3 - h_2) = \frac{3g}{a^2}(h_3 - h_2) \qquad (3)$$

Substituting the computed values of h_2 and h_3 into Eq. (3), we obtain

$$\omega_2^2 = \frac{3g}{a^2}(0.707a - 0.612a) = \frac{3g}{a^2}(0.095a) \qquad \omega_2 = \sqrt{0.285g/a}$$

$$\bar{v}_1 = \tfrac{4}{3}a\omega_2 = \tfrac{4}{3}a\sqrt{0.285g/a} \qquad \bar{v}_1 = 0.712\sqrt{ga} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

This lesson was devoted to the *impulsive motion* and to the *eccentric impact of rigid bodies.*

1. Impulsive motion occurs when a rigid body is subjected to a very large force $\mathbf{F}$ for a very short interval of time Δt; the resulting impulse $\mathbf{F}\,\Delta t$ is both finite and different from zero. Such forces are referred to as *impulsive forces* and are encountered whenever there is an impact between two rigid bodies. Forces for which the impulse is zero are referred to as *nonimpulsive forces.* As you saw in Chap. 13, the following forces can be assumed to be nonimpulsive: the *weight* of a body, the force exerted by a *spring*, and any other force which is *known* to be small by comparison with the impulsive forces. Unknown reactions, however, *cannot be assumed* to be nonimpulsive.

2. Eccentric impact of rigid bodies. You saw that when two bodies collide, the velocity components along the line of impact of the *points of contact A and B* before and after impact satisfy the following equation:

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{17.19}$$

where the left-hand member is the *relative velocity after the impact,* and the right-hand member is the product of the coefficient of restitution and the *relative velocity before the impact.*

This equation expresses the same relation between the velocity components of the points of contact before and after an impact that you used for particles in Chap. 13.

3. To solve a problem involving an impact you should use the *method of impulse and momentum* and take the following steps.

a. Draw a free-body-diagram equation of the body that will express that the system consisting of the momenta immediately before impact and of the impulses of the external forces is equivalent to the system of the momenta immediately after impact.

b. The free-body-diagram equation will relate the velocities before and after impact and the impulsive forces and reactions. In some cases, you will be able to determine the unknown velocities and impulsive reactions by solving equations obtained by summing components and moments [Sample Prob. 17.9].

c. In the case of an impact in which $e > 0$, the number of unknowns will be greater than the number of equations that you can write by summing components and moments, and you should supplement the equations obtained from the free-body-diagram equation with Eq. (17.19), which relates the relative velocities of the points of contact before and after impact [Sample Prob. 17.10].

d. During an impact you must use the method of impulse and momentum. However, *before and after the impact* you can, if necessary, use some of the other methods of solution that you have learned, such as the method of work and energy [Sample Prob. 17.11].

PROBLEMS

IMPULSE-MOMENTUM PRACTICE PROBLEMS

17.F4 A uniform slender rod *AB* of mass m is at rest on a frictionless horizontal surface when hook *C* engages a small pin at *A*. Knowing that the hook is pulled upward with a constant velocity $\mathbf{v}_0$, draw the impulse-momentum diagram that is needed to determine the impulse exerted on the rod at *A* and *B*. Assume that the velocity of the hook is unchanged and that the impact is perfectly plastic.

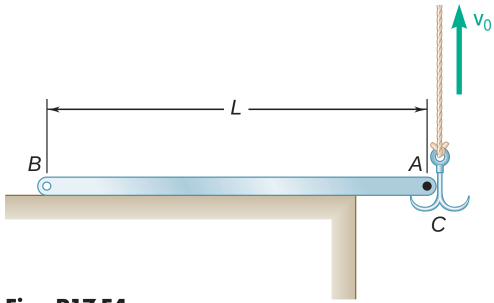


Fig. P17.F4

17.F5 A uniform slender rod *AB* of length L is falling freely with a velocity $\mathbf{v}_0$ when cord *AC* suddenly becomes taut. Assuming that the impact is perfectly plastic, draw the impulse-momentum diagram that is needed to determine the angular velocity of the rod and the velocity of its mass center immediately after the cord becomes taut.

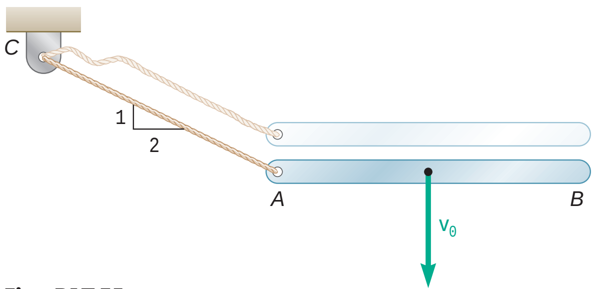


Fig. P17.F5

17.F6 A slender rod *CDE* of length L and mass m is attached to a pin support at its midpoint *D*. A second and identical rod *AB* is rotating about a pin support at *A* with an angular velocity ω_1 when its end *B* strikes end *C* of rod *CDE*. The coefficient of restitution between the rods is e. Draw the impulse-momentum diagrams that are needed to determine the angular velocity of each rod immediately after the impact.

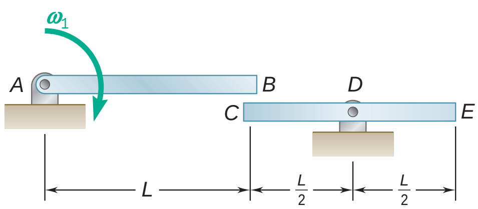


Fig. P17.F6

END-OF-SECTION PROBLEMS

17.96 At what height h above its center G should a billiard ball of radius r be struck horizontally by a cue if the ball is to start rolling without sliding?

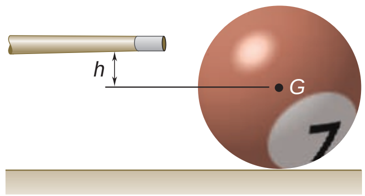


Fig. P17.96

17.97 A bullet weighing 0.08 lb is fired with a horizontal velocity of 1800 ft/s into the lower end of a slender 15-lb bar of length $L = 30$ in. Knowing that $h = 12$ in. and that the bar is initially at rest, determine (*a*) the angular velocity of the bar immediately after the bullet becomes embedded, (*b*) the impulsive reaction at C, assuming that the bullet becomes embedded in 0.001 s.

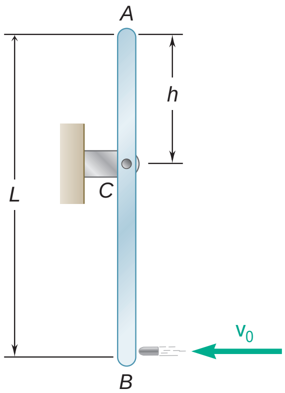


Fig. P17.97

17.98 In Prob. 17.97, determine (*a*) the required distance h if the impulsive reaction at C is to be zero, (*b*) the corresponding angular velocity of the bar immediately after the bullet becomes embedded.

17.99 An 16-lb wooden panel is suspended from a pin support at A and is initially at rest. A 4-lb metal sphere is released from rest at B and falls into a hemispherical cup C attached to the panel at a point located on its top edge. Assuming that the impact is perfectly plastic, determine the velocity of the mass center G of the panel immediately after the impact.

17.100 A 16-lb wooden panel is suspended from a pin support at A and is initially at rest. A 4-lb metal sphere is released from rest at B' and falls into a hemispherical cup C' attached to the panel at the same level as the mass center G. Assuming that the impact is perfectly plastic, determine the velocity of the mass center G of the panel immediately after the impact.

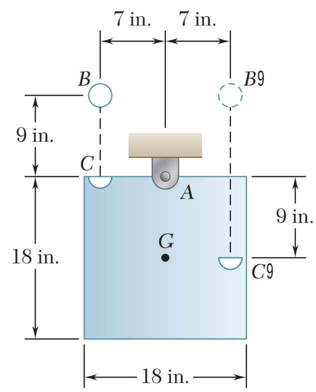


Fig. *P17.99* and *P17.100*

17.101 A 45-g bullet is fired with a velocity of 400 m/s at $u = 30°$ into a 9-kg square panel of side $b = 200$ mm. Knowing that $h = 150$ mm and that the panel is initially at rest, determine (*a*) the velocity of the center of the panel immediately after the bullet becomes embedded, (*b*) the impulsive reaction at A, assuming that the bullet becomes embedded in 2 ms.

17.102 A 45-g bullet is fired with a velocity of 400 m/s at $u = 5°$ into a 9-kg panel of side $b = 200$ mm. Knowing that the panel is initially at rest, determine (*a*) the required distance h if the horizontal component of the impulsive reaction at A is to be zero, (*b*) the corresponding velocity of the center of the panel immediately after the bullet becomes embedded.

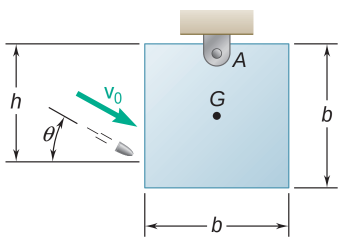


Fig. P17.101 and P17.102

17.103 Two uniform rods, each of mass m, form the L-shaped rigid body ABC which is initially at rest on the frictionless horizontal surface when hook D of the carriage E engages a small pin at C. Knowing that the carriage is pulled to the right with a constant velocity $\mathbf{v}_0$, determine immediately after the impact (*a*) the angular velocity of the body, (*b*) the velocity of corner B. Assume that the velocity of the carriage is unchanged and that the impact is perfectly plastic.

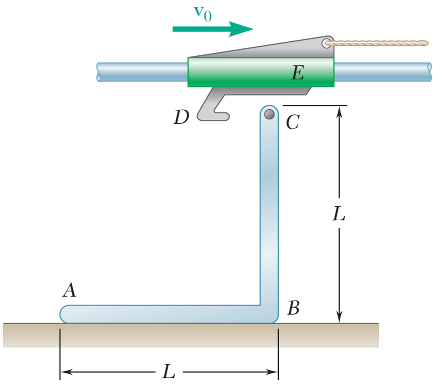


Fig. P17.103

17.104 The uniform slender rod AB of weight 5 lb and length 30 in. forms an angle b = 30° with the vertical as it strikes the smooth corner shown with a vertical velocity $\mathbf{v}_1$ of magnitude 8 ft/s and no angular velocity. Assuming that the impact is perfectly plastic, determine the angular velocity of the rod immediately after the impact.

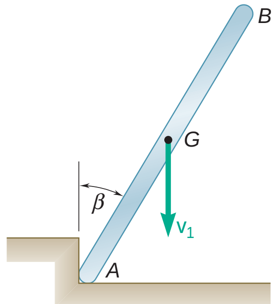


Fig. P17.104

17.105 A bullet weighing 0.08 lb is fired with a horizontal velocity of 1800 ft/s into the 15-lb wooden rod AB of length L = 30 in. The rod, which is initially at rest, is suspended by a cord of length L = 30 in. Determine the distance h for which, immediately after the bullet becomes embedded, the instantaneous center of rotation of the rod is point C.

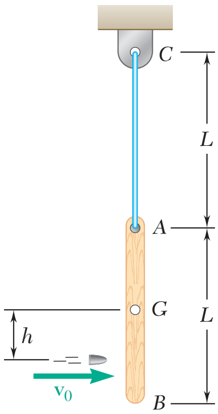


Fig. P17.105

17.106 A uniform sphere of radius r rolls down the incline shown without slipping. It hits a horizontal surface and, after slipping for a while, it starts rolling again. Assuming that the sphere does not bounce as it hits the horizontal surface, determine its angular velocity and the velocity of its mass center after it has resumed rolling.

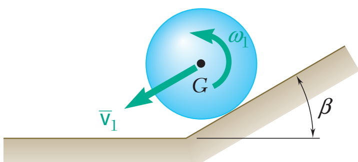


Fig. P17.106

17.107 A uniformly loaded rectangular crate is released from rest in the position shown. Assuming that the floor is sufficiently rough to prevent slipping and that the impact at B is perfectly plastic, determine the smallest value of the ratio a/b for which corner A will remain in contact with the floor.

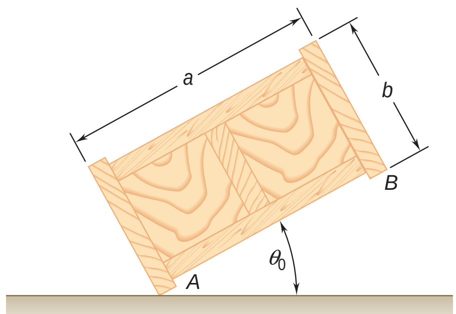


Fig. P17.107

17.108 A bullet of mass m is fired with a horizontal velocity $\mathbf{v}_0$ and at a height $h = \frac{1}{2}R$ into a wooden disk of much larger mass M and radius R. The disk rests on a horizontal plane and the coefficient of friction between the disk and the plane is finite. (*a*) Determine the linear velocity $\bar{v}_1$ and the angular velocity v_1 of the disk immediately after the bullet has penetrated the disk. (*b*) Describe the ensuing motion of the disk and determine its linear velocity after the motion has become uniform.

17.109 Determine the height h at which the bullet of Prob. 17.108 should be fired (*a*) if the disk is to roll without sliding immediately after impact, (*b*) if the disk is to slide without rolling immediately after impact.

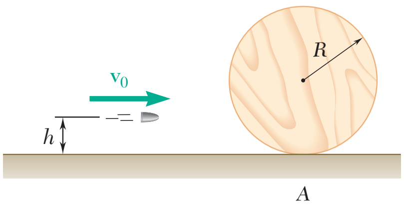


Fig. P17.108 and P17.109

17.110 A uniform slender bar of length $L = 200$ mm and mass $m = 0.5$ kg is supported by a frictionless horizontal table. Initially the bar is spinning about its mass center G with a constant angular speed $v_1 = 6$ rad/s. Suddenly latch D is moved to the right and is struck by end A of the bar. Knowing that the coefficient of restitution between A and D is $e = 0.6$, determine the angular velocity of the bar and the velocity of its mass center immediately after the impact.

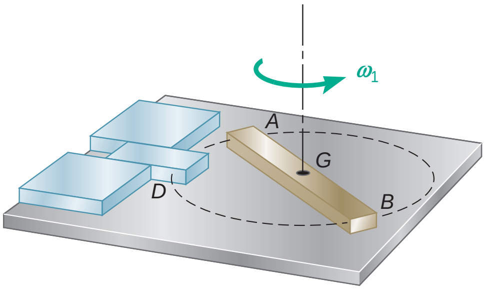


Fig. *P17.110*

17.111 A uniform slender rod of length L is dropped onto rigid supports at A and B. Since support B is slightly lower than support A, the rod strikes A with a velocity $\bar{\mathbf{v}}_1$ before it strikes B. Assuming perfectly elastic impact at both A and B, determine the angular velocity of the rod and the velocity of its mass center immediately after the rod (*a*) strikes support A, (*b*) strikes support B, (*c*) again strikes support A.

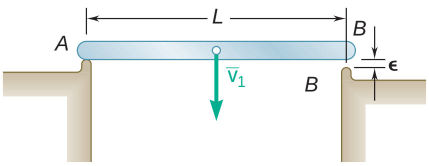


Fig. *P17.111*

17.112 The slender rod AB of length L forms an angle b with the vertical as it strikes the frictionless surface shown with a vertical velocity $\bar{\mathbf{v}}_1$ and no angular velocity. Assuming that the impact is perfectly plastic, derive an expression for the angular velocity of the rod immediately after the impact.

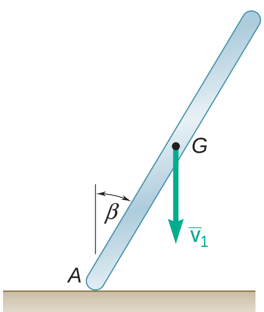


Fig. P17.112

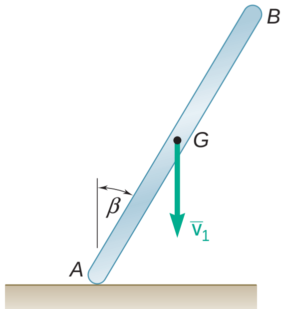


Fig. *P17.113*

17.113 The slender rod AB of length $L = 1$ m forms an angle $b = 30°$ with the vertical as it strikes the frictionless surface shown with a vertical velocity $\bar{\mathbf{v}}_1 = 2$ m/s and no angular velocity. Knowing that the coefficient of restitution between the rod and the ground is $e = 0.8$, determine the angular velocity of the rod immediately after the impact.

17.114 The trapeze/lanyard air drop (t/LAD) launch is a proposed innovative method for airborne launch of a payload-carrying rocket. The release sequence involves several steps as shown in (1) where the payload rocket is shown at various instances during the launch. To investigate the first step of this process, where the rocket body drops freely from the carrier aircraft until the 2-m lanyard stops the vertical motion of B, a trial rocket is tested as shown in (2). The rocket can be considered a uniform 1×7-m rectangle with a mass of 4000 kg. Knowing that the rocket is released from rest and falls vertically 2 m before the lanyard becomes taut, determine the angular velocity of the rocket immediately after the lanyard is taut.

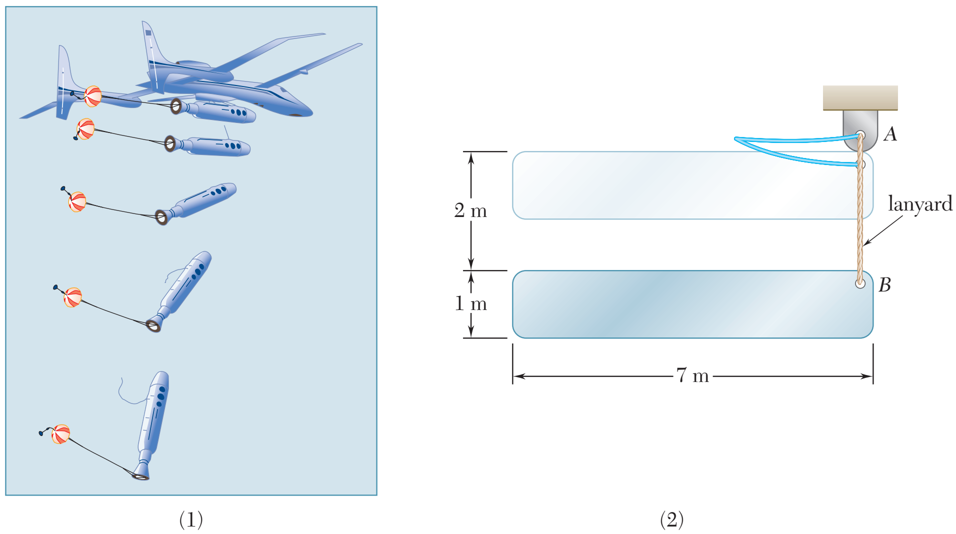


Fig. *P17.114*

17.115 The uniform rectangular block shown is moving along a frictionless surface with a velocity $\bar{\mathbf{v}}_1$ when it strikes a small obstruction at B. Assuming that the impact between corner A and obstruction B is perfectly plastic, determine the magnitude of the velocity $\bar{\mathbf{v}}_1$ for which the maximum angle u through which the block will rotate will be 30°.

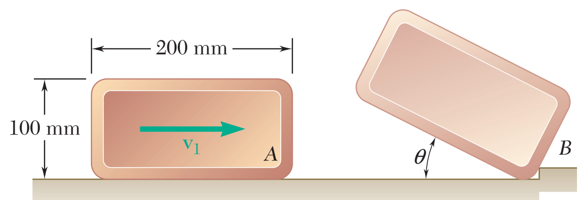


Fig. P17.115

17.116 A slender rod of length L and mass m is released from rest in the position shown. It is observed that after the rod strikes the vertical surface it rebounds to form an angle of 30° with the vertical. (*a*) Determine the coefficient of restitution between knob K and the surface. (*b*) Show that the same rebound can be expected for any position of knob K.

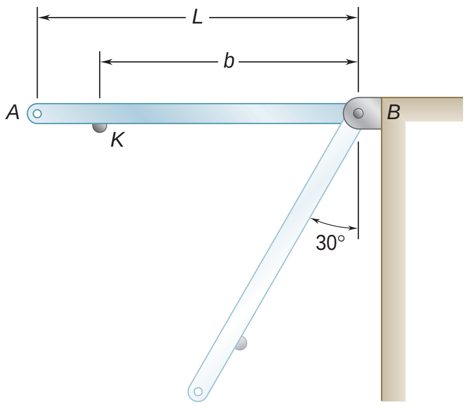


Fig. P17.116

17.117 A slender rod of mass m and length L is released from rest in the position shown and hits edge D. Assuming perfectly plastic impact at D, determine for $b = 0.6L$, (*a*) the angular velocity of the rod immediately after the impact, (*b*) the maximum angle through which the rod will rotate after the impact.

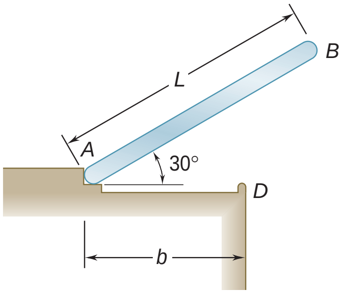


Fig. P17.117

17.118 A uniformly loaded square crate is released from rest with its corner D directly above A; it rotates about A until its corner B strikes the floor, and then rotates about B. The floor is sufficiently rough to prevent slipping and the impact at B is perfectly plastic. Denoting by V_0 the angular velocity of the crate immediately before B strikes the floor, determine (*a*) the angular velocity of the crate immediately after B strikes the floor, (*b*) the fraction of the kinetic energy of the crate lost during the impact, (*c*) the angle u through which the crate will rotate after B strikes the floor.

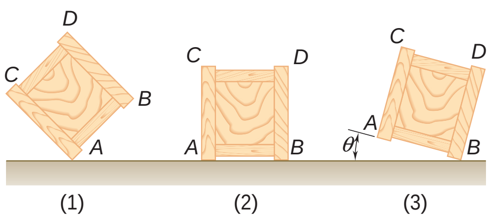


Fig. P17.118

17.119 A 1-oz bullet is fired with a horizontal velocity of 750 mi/h into the 18-lb wooden beam AB. The beam is suspended from a collar of negligible mass that can slide along a horizontal rod. Neglecting friction between the collar and the rod, determine the maximum angle of rotation of the beam during its subsequent motion.

17.120 For the beam of Prob. 17.119, determine the velocity of the 1-oz bullet for which the maximum angle of rotation of the beam will be 90°.

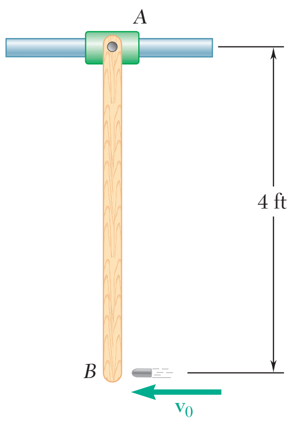


Fig. P17.119

17.121 The plank *CDE* has a mass of 15 kg and rests on a small pivot at *D*. The 55-kg gymnast *A* is standing on the plank at *C* when the 70-kg gymnast *B* jumps from a height of 2.5 m and strikes the plank at *E*. Assuming perfectly plastic impact and that gymnast *A* is standing absolutely straight, determine the height to which gymnast *A* will rise.

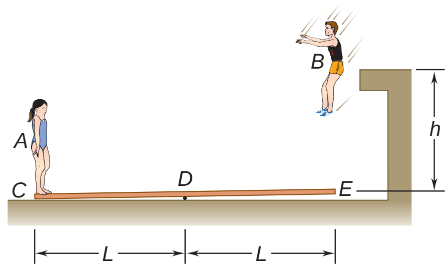


Fig. P17.121

17.122 Solve Prob. 17.121, assuming that the gymnasts change places so that gymnast *A* jumps onto the plank while gymnast *B* stands at *C*.

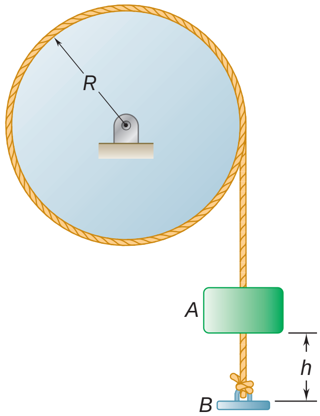


Fig. P17.123

17.123 A small plate *B* is attached to a cord that is wrapped around a uniform 8-lb disk of radius $R = 9$ in. A 3-lb collar *A* is released from rest and falls through a distance $h = 15$ in. before hitting plate *B*. Assuming that the impact is perfectly plastic and neglecting the weight of the plate, determine immediately after the impact (*a*) the velocity of the collar, (*b*) the angular velocity of the disk.

17.124 Solve Prob. 17.123, assuming that the coefficient of restitution between *A* and *B* is 0.8.

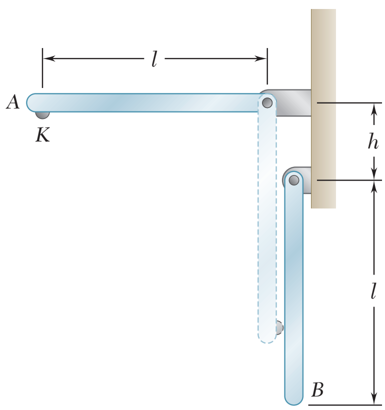


Fig. *P17.125*

17.125 Two identical slender rods may swing freely from the pivots shown. Rod *A* is released from rest in a horizontal position and swings to a vertical position, at which time the small knob *K* strikes rod *B* which was at rest. If $h = \frac{1}{2}l$ and $e = \frac{1}{2}$, determine (*a*) the angle through which rod *B* will swing, (*b*) the angle through which rod *A* will rebound.

17.126 A 2-kg solid sphere of radius $r = 40$ mm is dropped from a height $h = 200$ mm and lands on a uniform slender plank *AB* of mass 4 kg and length $L = 500$ mm which is held by two inextensible cords. Knowing that the impact is perfectly plastic and that the sphere remains attached to the plank at a distance $a = 40$ mm from the left end, determine the velocity of the sphere immediately after impact. Neglect the thickness of the plank.

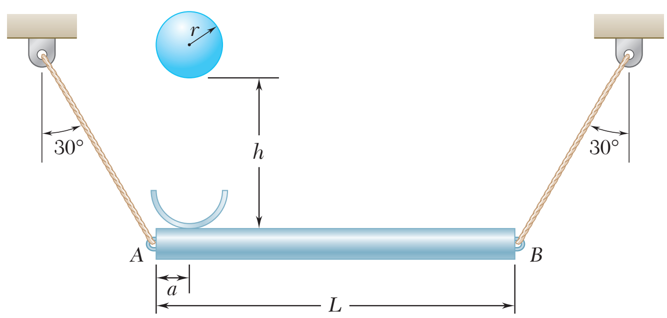


Fig. *P17.126*

17.127 and 17.128 Member ABC has a mass of 2.4 kg and is attached to a pin support at B. An 800-g sphere D strikes the end of member ABC with a vertical velocity $\mathbf{v}_1$ of 3 m/s. Knowing that $L = 750$ mm and that the coefficient of restitution between the sphere and member ABC is 0.5, determine immediately after the impact (*a*) the angular velocity of member ABC, (*b*) the velocity of the sphere.

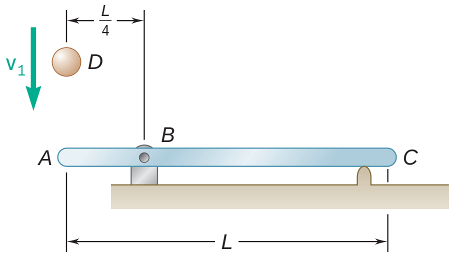


Fig. P17.127

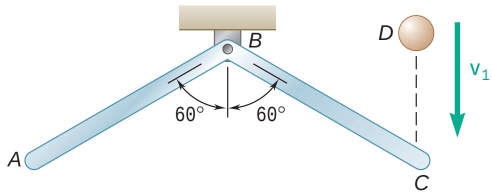


Fig. P17.128

17.129 Sphere A of mass $m_A = 2$ kg and radius $r = 40$ mm rolls without slipping with a velocity $\overline{\mathbf{v}}_1 = 2$ m/s on a horizontal surface when it hits squarely a uniform slender bar B of mass is $m_B = 0.5$ kg and length $L = 100$ mm that is standing on end and is at rest. Denoting by μ_k the coefficient of kinetic friction between the sphere and the horizontal surface, neglecting friction between the sphere and the bar, and knowing the coefficient of restitution between A and B is 0.1, determine the angular velocities of the sphere and the bar immediately after the impact.

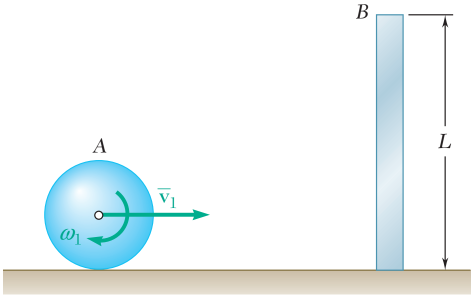


Fig. *P17.129*

17.130 A large 3-lb sphere with a radius $r = 3$ in. is thrown into a light basket at the end of a thin, uniform rod weighing 2 lb and length $L = 10$ in. as shown. Immediately before the impact the angular velocity of the rod is 3 rad/s counterclockwise and the velocity of the sphere is 2 ft/s down. Assume the sphere sticks in the basket. Determine after the impact (*a*) the angular velocity of the bar and sphere, (*b*) the components of the reactions at A.

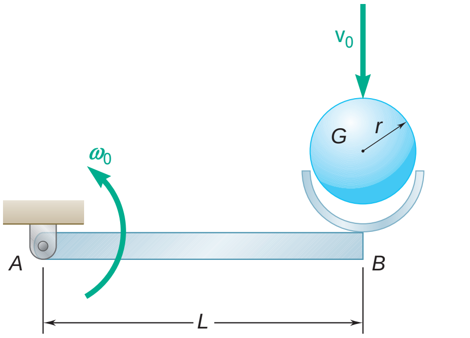


Fig. *P17.130*

17.131 A small rubber ball of radius r is thrown against a rough floor with a velocity $\overline{\mathbf{v}}_A$ of magnitude v_0 and a backspin ω_A of magnitude ω_0. It is observed that the ball bounces from A to B, then from B to A, then from A to B, etc. Assuming perfectly elastic impact, determine the required magnitude ω_0 of the backspin in terms of $\overline{v}_0$ and r.

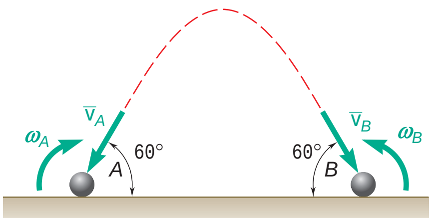


Fig. P17.131

17.132 Sphere A of mass m and radius r rolls without slipping with a velocity $\bar{\mathbf{v}}_1$ on a horizontal surface when it hits squarely an identical sphere B that is at rest. Denoting by μ_k the coefficient of kinetic friction between the spheres and the surface, neglecting friction between the spheres, and assuming perfectly elastic impact, determine (*a*) the linear and angular velocities of each sphere immediately after the impact, (*b*) the velocity of each sphere after it has started rolling uniformly.

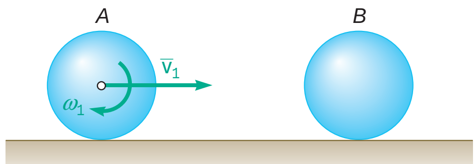


Fig. P17.132

17.133 In a game of pool, ball A is rolling without slipping with a velocity $\bar{\mathbf{v}}_0$ as it hits obliquely ball B, which is at rest. Denoting by r the radius of each ball and by μ_k the coefficient of kinetic friction between a ball and the table, and assuming perfectly elastic impact, determine (*a*) the linear and angular velocity of each ball immediately after the impact, (*b*) the velocity of ball B after it has started rolling uniformly.

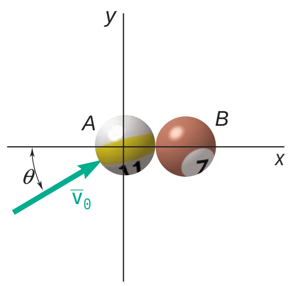


Fig. P17.133

17.134 Each of the bars AB and BC is of length $L = 400$ mm and mass $m = 1.2$ kg. Determine the angular velocity of each bar immediately after the impulse $\mathbf{Q}\Delta t = (1.5\text{ N}\cdot\text{s})\mathbf{i}$ is applied at C.

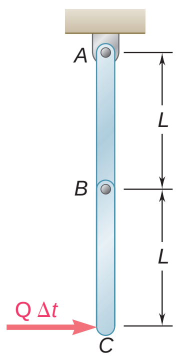


Fig. P17.134

REVIEW AND SUMMARY

In this chapter we again considered the method of work and energy and the method of impulse and momentum. In the first part of the chapter we studied the method of work and energy and its application to the analysis of the motion of rigid bodies and systems of rigid bodies.

Principle of work and energy for a rigid body

In Sec. 17.2, we first expressed the principle of work and energy for a rigid body in the form

$$T_1 + U_{1 \to 2} = T_2 \tag{17.1}$$

where T_1 and T_2 represent the initial and final values of the kinetic energy of the rigid body and $U_{1 \to 2}$ represents the work of the *external forces* acting on the rigid body.

Work of a force or a couple

In Sec. 17.3, we recalled the expression found in Chap. 13 for the work of a force **F** applied at a point A, namely

$$U_{1 \to 2} = \int_{s_1}^{s_2} (F \cos \alpha)\, ds \tag{17.3'}$$

where F was the magnitude of the force, α the angle it formed with the direction of motion of A, and s the variable of integration measuring the distance traveled by A along its path. We also derived the expression for the *work of a couple of moment* **M** applied to a rigid body during a rotation in θ of the rigid body:

$$U_{1 \to 2} = \int_{\theta_1}^{\theta_2} M\, d\theta \tag{17.5}$$

Kinetic energy in plane motion

We then derived an expression for the kinetic energy of a rigid body in plane motion [Sec. 17.4]. We wrote

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 \tag{17.9}$$

where $\bar{v}$ is the velocity of the mass center G of the body, ω is the angular velocity of the body, and $\bar{I}$ is its moment of inertia about an axis through G perpendicular to the plane of reference (Fig. 17.13) [Sample Prob. 17.3]. We noted that the kinetic energy of a rigid body in plane motion can be separated into two parts: (1) the kinetic energy $\frac{1}{2}m\bar{v}^2$ associated with the motion of the mass center G of the body, and (2) the kinetic energy $\frac{1}{2}\bar{I}\omega^2$ associated with the rotation of the body about G.

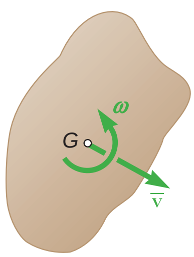


Fig. 17.13

For a rigid body rotating about a fixed axis through O with an angular velocity ω, we had

Kinetic energy in rotation

$$T = \tfrac{1}{2} I_O \omega^2 \qquad (17.10)$$

where I_O was the moment of inertia of the body about the fixed axis. We noted that the result obtained is not limited to the rotation of plane slabs or of bodies symmetrical with respect to the reference plane, but is valid regardless of the shape of the body or of the location of the axis of rotation.

Systems of rigid bodies

Equation (17.1) can be applied to the motion of systems of rigid bodies [Sec. 17.5] as long as all the forces acting on the various bodies involved—internal as well as external to the system—are included in the computation of $U_{1 \rightarrow 2}$. However, in the case of systems consisting of pin-connected members, or blocks and pulleys connected by inextensible cords, or meshed gears, the points of application of the internal forces move through equal distances and the work of these forces cancels out [Sample Probs. 17.1 and 17.2].

Conservation of energy

When a rigid body, or a system of rigid bodies, moves under the action of conservative forces, the principle of work and energy can be expressed in the form

$$T_1 + V_1 = T_2 + V_2 \qquad (17.12)$$

which is referred to as the *principle of conservation of energy* [Sec. 17.6]. This principle can be used to solve problems involving conservative forces such as the force of gravity or the force exerted by a spring [Sample Probs. 17.4 and 17.5]. However, when a reaction is to be determined, the principle of conservation of energy must be supplemented by the application of d'Alembert's principle [Sample Prob. 17.4].

Power

In Sec. 17.7, we extended the concept of power to a rotating body subjected to a couple, writing

$$\text{Power} = \frac{dU}{dt} = \frac{M\,d\theta}{dt} = M\omega \qquad (17.13)$$

where M is the magnitude of the couple and ω the angular velocity of the body.

The middle part of the chapter was devoted to the method of impulse and momentum and its application to the solution of various types of problems involving the plane motion of rigid slabs and rigid bodies symmetrical with respect to the reference plane.

Principle of impulse and momentum for a rigid body

We first recalled the *principle of impulse and momentum* as it was derived in Sec. 14.9 for a system of particles and applied it to the *motion of a rigid body* [Sec. 17.8]. We wrote

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1 \rightarrow 2} = \textbf{Syst Momenta}_2 \qquad (17.14)$$

Next we showed that for a rigid slab or a rigid body symmetrical with respect to the reference plane, the system of the momenta of the particles forming the body is equivalent to a vector $m\bar{\mathbf{v}}$ attached at the mass center G of the body and a couple $\bar{I}\boldsymbol{\omega}$ (Fig. 17.14). The vector

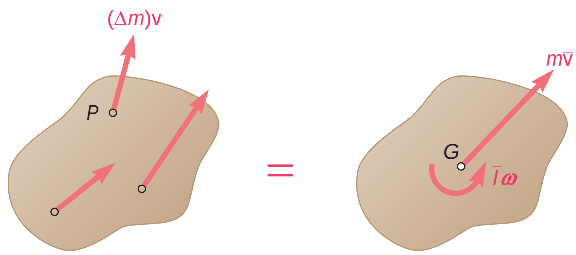


Fig. 17.14

$m\bar{\mathbf{v}}$ is associated with the translation of the body with G and represents the *linear momentum* of the body, while the couple $\bar{I}\boldsymbol{\omega}$ corresponds to the rotation of the body about G and represents the *angular momentum* of the body about an axis through G.

Equation (17.14) can be expressed graphically as shown in Fig. 17.15 by drawing three diagrams representing respectively the system of the initial momenta of the body, the impulses of the external forces acting on the body, and the system of the final momenta of the body.

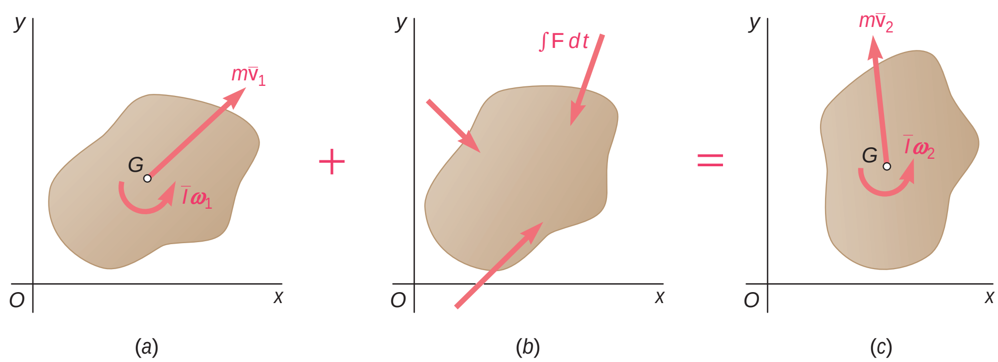


Fig. 17.15

Summing and equating respectively the *x components*, the *y components*, and the *moments about any given point* of the vectors shown in that figure, we obtain three equations of motion which can be solved for the desired unknowns [Sample Probs. 17.6 and 17.7].

In problems dealing with several connected rigid bodies [Sec. 17.9], each body can be considered separately [Sample Prob. 17.6], or, if no more than three unknowns are involved, the principle of impulse and momentum can be applied to the entire system, considering the impulses of the external forces only [Sample Prob. 17.8].

Conservation of angular momentum

When the lines of action of all the external forces acting on a system of rigid bodies pass through a given point O, the angular momentum of the system about O is conserved [Sec. 17.10]. It was suggested that problems involving conservation of angular momentum be solved by the general method described above [Sample Prob. 17.8].

Impulsive motion

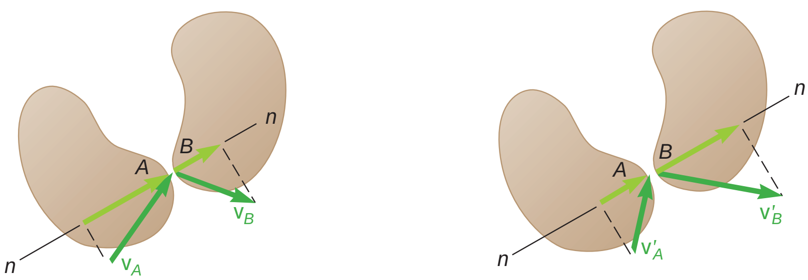
The last part of the chapter was devoted to the *impulsive motion* and the *eccentric impact* of rigid bodies. In Sec. 17.11, we recalled that the method of impulse and momentum is the only practicable method for the solution of problems involving impulsive motion and that the computation of impulses in such problems is particularly simple [Sample Prob. 17.9].

Eccentric impact

In Sec. 17.12, we recalled that the eccentric impact of two rigid bodies is defined as an impact in which the mass centers of the colliding bodies are *not* located on the line of impact. It was shown that in such a situation a relation similar to that derived in Chap. 13 for the central impact of two particles and involving the coefficient of restitution e still holds, but that *the velocities of points A and B where contact occurs during the impact should be used.* We have

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{17.19}$$

where $(v_A)_n$ and $(v_B)_n$ are the components along the line of impact of the velocities of A and B before the impact, and $(v'_A)_n$ and $(v'_B)_n$ are their components after the impact (Fig. 17.16). Equation (17.19)



(*a*) Before impact

(*b*) After impact

Fig. 17.16

is applicable not only when the colliding bodies move freely after the impact but also when the bodies are partially constrained in their motion. It should be used in conjunction with one or several other equations obtained by applying the principle of impulse and momentum [Sample Prob. 17.10]. We also considered problems where the method of impulse and momentum and the method of work and energy can be combined [Sample Prob. 17.11].

17.135 A uniform disk of constant thickness and initially at rest is placed in contact with the belt shown, which moves at a constant speed v = 80 ft/s. Knowing that the coefficient of kinetic friction between the disk and the belt is 0.15, determine (*a*) the number of revolutions executed by the disk before it reaches a constant angular velocity, (*b*) the time required for the disk to reach that constant angular velocity.

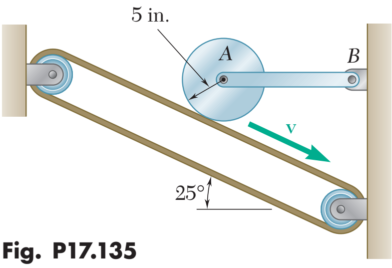


Fig. P17.135

17.136 The 8-in.-radius brake drum is attached to a larger flywheel that is not shown. The total mass moment of inertia of the flywheel and drum is 14 lb · ft · s^2 and the coefficient of kinetic friction between the drum and the brake shoe is 0.35. Knowing that the initial angular velocity of the flywheel is 360 rpm counterclockwise, determine the vertical force **P** that must be applied to the pedal C if the system is to stop in 100 revolutions.

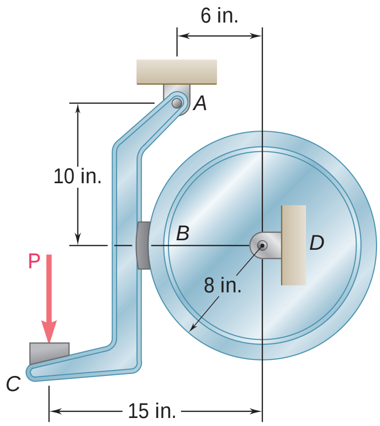


Fig. P17.136

17.137 A 6 × 8-in. rectangular plate is suspended by pins at A and B. The pin at B is removed and the plate swings freely about pin A. Determine (*a*) the angular velocity of the plate after it has rotated through 90°, (*b*) the maximum angular velocity attained by the plate as it swings freely.

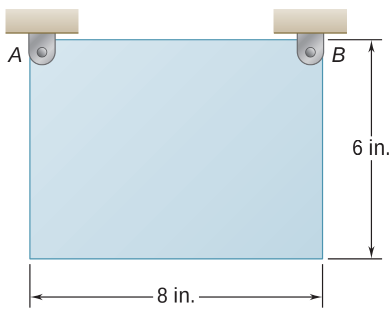


Fig. P17.137

17.138 The gear shown has a radius R = 150 mm and a radius of gyration $\bar{k}$ = 125 mm. The gear is rolling without sliding with a velocity $\bar{\mathbf{v}}_1$ of magnitude 3 m/s when it strikes a step of height h = 75 mm. Because the edge of the step engages the gear teeth, no slipping occurs between the gear and the step. Assuming perfectly plastic impact, determine (*a*) the angular velocity of the gear immediately after the impact, (*b*) the angular velocity of the gear after it has rotated to the top of the step.

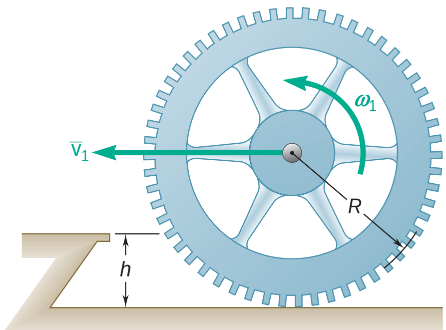


Fig. *P17.138*

17.139 A uniform slender rod is placed at corner *B* and is given a slight clockwise motion. Assuming that the corner is sharp and becomes slightly embedded in the end of the rod, so that the coefficient of static friction at *B* is very large, determine (*a*) the angle β through which the rod will have rotated when it loses contact with the corner, (*b*) the corresponding velocity of end *A*.

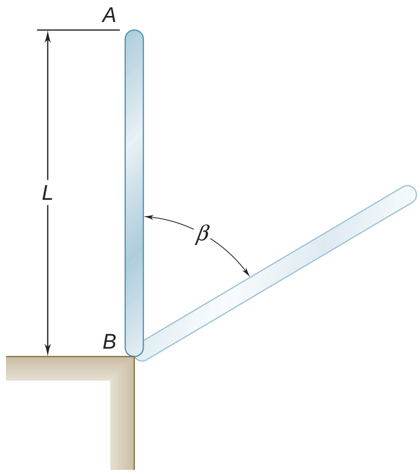


Fig. P17.139

17.140 The motion of the slender 250-mm rod *AB* is guided by pins at *A* and *B* that slide freely in slots cut in a vertical plate as shown. Knowing that the rod has a mass of 2 kg and is released from rest when $\theta = 0$, determine the reactions at *A* and *B* when $\theta = 90°$.

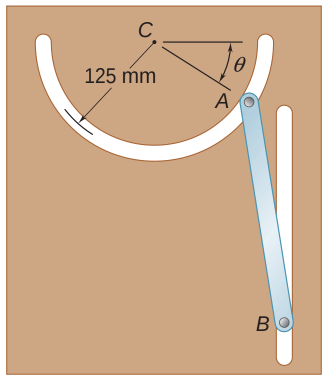


Fig. P17.140

17.141 A 35-g bullet *B* is fired horizontally with a velocity of 400 m/s into the side of a 3-kg square panel suspended from a pin at *A*. Knowing that the panel is initially at rest, determine the components of the reaction at *A* after the panel has rotated 45°.

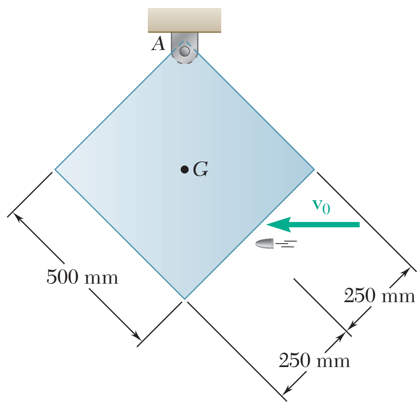


Fig. *P17.141*

17.142 Two panels *A* and *B* are attached with hinges to a rectangular plate and held by a wire as shown. The plate and the panels are made of the same material and have the same thickness. The entire assembly is rotating with an angular velocity ω_0 when the wire breaks. Determine the angular velocity of the assembly after the panels have come to rest against the plate.

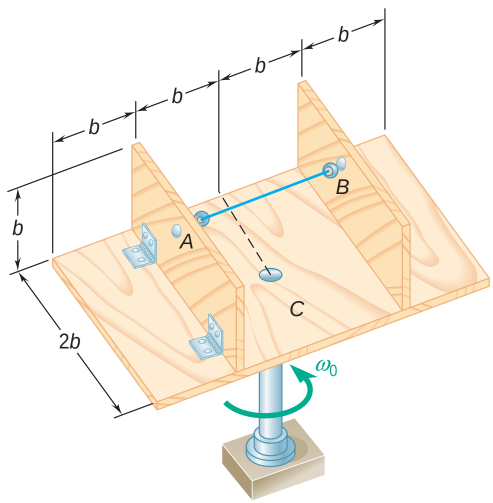


Fig. P17.142

17.143 Disks A and B are made of the same material and are of the same thickness; they can rotate freely about the vertical shaft. Disk B is at rest when it is dropped onto disk A, which is rotating with an angular velocity of 500 rpm. Knowing that disk A has a mass of 8 kg, determine (*a*) the final angular velocity of the disks, (*b*) the change in kinetic energy of the system.

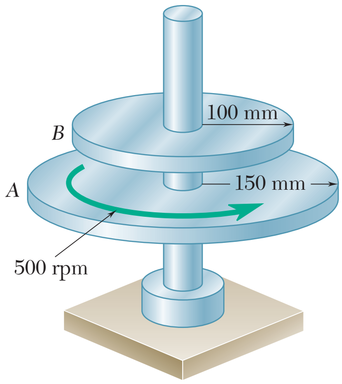


Fig. P17.143

17.144 A square block of mass m is falling with a velocity $\overline{\mathbf{v}}_1$ when it strikes a small obstruction at B. Knowing that the coefficient of restitution for the impact between corner A and the obstruction B is $e = 0.5$, determine immediately after the impact (*a*) the angular velocity of the block, (*b*) the velocity of its mass center G.

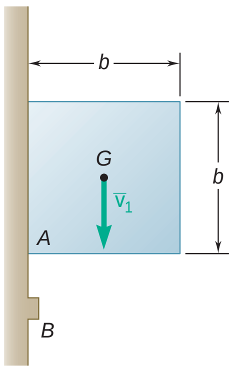


Fig. *P17.144*

17.145 A 3-kg bar AB is attached by a pin at D to a 4-kg square plate, which can rotate freely about a vertical axis. Knowing that the angular velocity of the plate is 120 rpm when the bar is vertical, determine (*a*) the angular velocity of the plate after the bar has swung into a horizontal position and has come to rest against pin C, (*b*) the energy lost during the plastic impact at C.

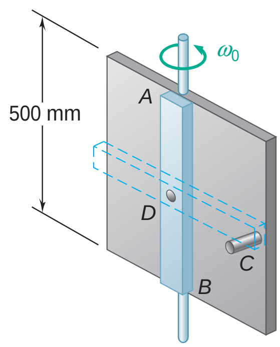


Fig. P17.145

17.146 A 1.8-lb javelin DE impacts a 10-lb slender rod ABC with a horizontal velocity of $\mathbf{v}_0 = 30$ ft/s as shown. Knowing that the javelin becomes embedded into the end of the rod at point C and does not penetrate very far into it, determine immediately after the impact (*a*) the angular velocity of the rod ABC, (*b*) the components of the reaction at B. Assume the javelin and the rod move as a single rigid body after the impact.

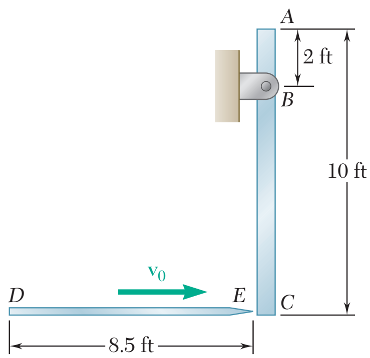


Fig. *P17.146*

COMPUTER PROBLEMS

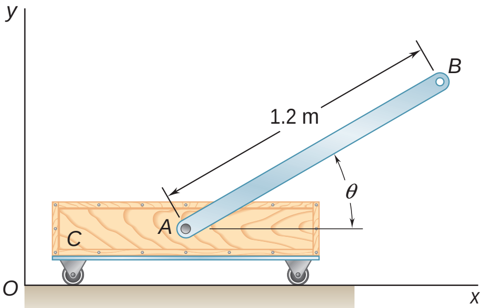


Fig. P17.C1

17.C1 Rod AB has a mass of 3 kg and is attached at A to a 5-kg cart C. Knowing that the system is released from rest when u = 30° and neglecting friction, use computational software to determine the velocity of the cart and the velocity of end B of the rod for values of u from +30° to −90°. Determine the value of u for which the velocity of the cart to the left is maximum and the corresponding value of the velocity.

17.C2 The uniform slender rod AB of length L = 800 mm and mass 5 kg rests on a small wheel at D and is attached to a collar of negligible mass that can slide freely on the vertical rod EF. Knowing that a = 200 mm and that the rod is released from rest when u = 0, use computational software to calculate and plot the angular velocity of the rod and the velocity of end A for values of u from 0 to 50°. Determine the maximum angular velocity of the rod and the corresponding value of u.

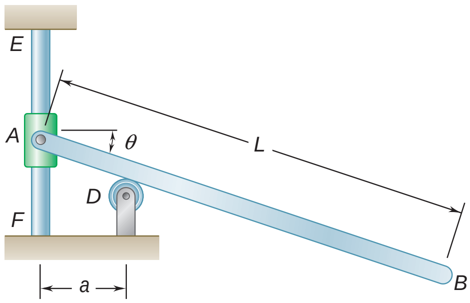


Fig. P17.C2

17.C3 A uniform 10-in.-radius sphere rolls over a series of parallel horizontal bars equally spaced at a distance d. As it rotates without slipping about a given bar, the sphere strikes the next bar and starts rotating about that bar without slipping, until it strikes the next bar, and so on. Assuming perfectly plastic impact and knowing that the sphere has an angular velocity V_0 of 1.5 rad/s as its mass center G is directly above bar A, use computational software to calculate values of the spacing d from 1 to 6 in. (a) the angular velocity V_1 of the sphere as G passes directly above bar B, (b) the number of bars over which the sphere will roll after leaving bar A.

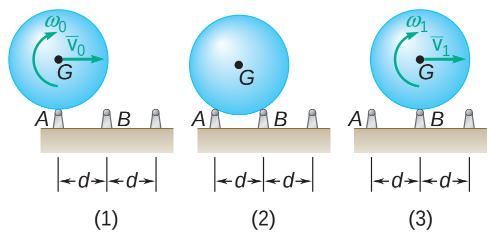


Fig. P17.C3

17.C4 Collar C has a mass of 2.5 kg and can slide without friction on rod AB. A spring of constant 750 N/m and an unstretched length r_0 = 500 mm is attached as shown to the collar and to the hub B. The total mass moment of inertia of the rod, hub, and spring is known to be 0.3 kg · m^2 about B. Initially the collar is held at a distance of 500 mm from the axis of rotation by a small pin protruding from the rod. The pin is suddenly removed as the assembly is rotating in a horizontal plane with an angular velocity V_0 of 10 rad/s. Denoting by r the distance of the collar from the axis of rotation, use computational software to calculate and plot the angular velocity of the assembly and the velocity of the collar relative to the rod for values of r from 500 to 700 mm. Determine the maximum value of r in the ensuing motion.

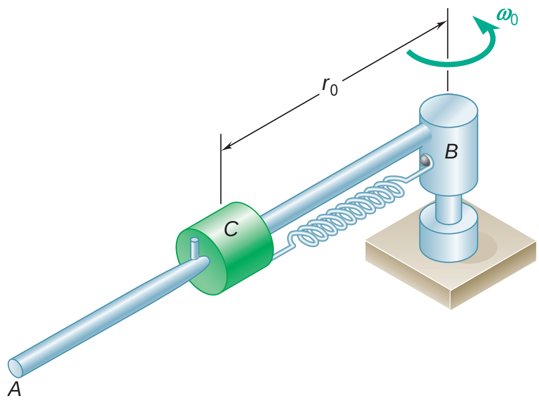


Fig. P17.C4

17.C5 Each of the two identical slender bars shown has a length L = 30 in. Knowing that the system is released from rest when the bars are horizontal, use computational software to calculate and plot the angular velocity of rod AB and the velocity of point D for values of u from 0 to 90°.

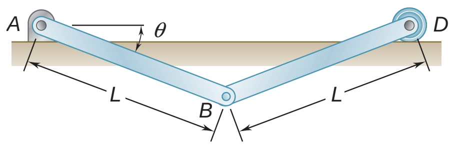
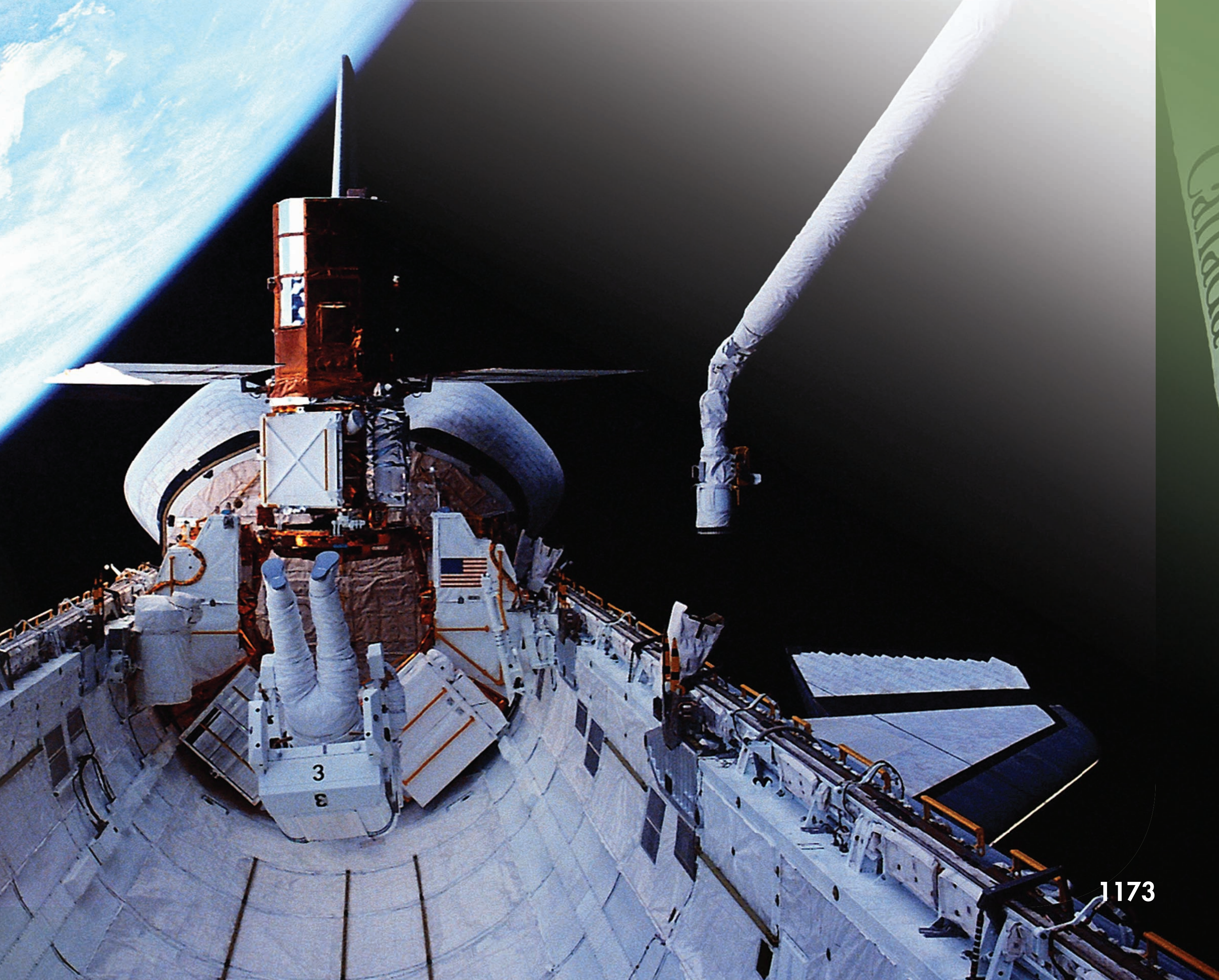


Fig. P17.C5

While the general principles that you learned in earlier chapters can be used again to solve problems involving the three-dimensional motion of rigid bodies, the solution of these problems requires a new approach and is considerably more involved than the solution of two-dimensional problems. One example is the determination of the forces acting on the space shuttle's robotic arm.

CHAPTER 18

Kinetics of Rigid Bodies in Three Dimensions



Chapter 18 Kinetics of Rigid Bodies in Three Dimensions

*18.1 INTRODUCTION

In Chaps. 16 and 17 we were concerned with the plane motion of rigid bodies and of systems of rigid bodies. In Chap. 16 and in the second half of Chap. 17 (momentum method), our study was further restricted to that of plane slabs and of bodies symmetrical with respect to the reference plane. However, many of the fundamental results obtained in these two chapters remain valid in the case of the motion of a rigid body in three dimensions.

For example, the two fundamental equations

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{18.1}$$

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{18.2}$$

on which the analysis of the plane motion of a rigid body was based, remain valid in the most general case of motion of a rigid body. As indicated in Sec. 16.2, these equations express that the system of the external forces is equipollent to the system consisting of the vector $m\bar{\mathbf{a}}$ attached at G and the couple of moment $\dot{\mathbf{H}}_G$ (Fig. 18.1). However,

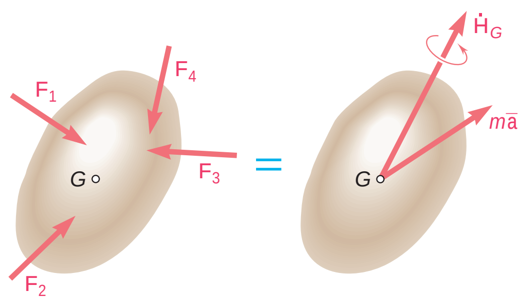


Fig. 18.1

the relation $\mathbf{H}_G = \bar{I}\boldsymbol{\omega}$, which enabled us to determine the angular momentum of a rigid slab and which played an important part in the solution of problems involving the plane motion of slabs and bodies symmetrical with respect to the reference plane, ceases to be valid in the case of nonsymmetrical bodies or three-dimensional motion. Thus in the first part of the chapter, in Sec. 18.2, a more general method for computing the angular momentum $\mathbf{H}_G$ of a rigid body in three dimensions will be developed.

Similarly, although the main feature of the impulse-momentum method discussed in Sec. 17.7, namely, the reduction of the momenta of the particles of a rigid body to a linear momentum vector $m\bar{\mathbf{v}}$ attached at the mass center G of the body and an angular momentum couple $\mathbf{H}_G$, remains valid, the relation $\mathbf{H}_G = \bar{I}\boldsymbol{\omega}$ must be discarded and replaced by the more general relation developed in Sec. 18.2 before this method can be applied to the three-dimensional motion of a rigid body (Sec. 18.3).

We also note that the work-energy principle (Sec. 17.2) and the principle of conservation of energy (Sec. 17.6) still apply in the case of the motion of a rigid body in three dimensions. However, the expression obtained in Sec. 17.4 for the kinetic energy of a rigid body in plane motion will be replaced by a new expression developed in Sec. 18.4 for a rigid body in three-dimensional motion.

In the second part of the chapter, you will first learn to determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of a three-dimensional rigid body, using a rotating frame of reference with respect to which the moments and products of inertia of the body remain constant (Sec. 18.5). Equations (18.1) and (18.2) will then be expressed in the form of free-body-diagram equations, which can be used to solve various problems involving the three-dimensional motion of rigid bodies (Secs. 18.6 through 18.8).

The last part of the chapter (Secs. 18.9 through 18.11) is devoted to the study of the motion of the gyroscope or, more generally, of an axisymmetrical body with a fixed point located on its axis of symmetry. In Sec. 18.10, the particular case of the steady precession of a gyroscope will be considered, and, in Sec. 18.11, the motion of an axisymmetrical body subjected to no force, except its own weight, will be analyzed.

*18.2 ANGULAR MOMENTUM OF A RIGID BODY IN THREE DIMENSIONS

In this section you will see how the angular momentum $\mathbf{H}_G$ of a body about its mass center G can be determined from the angular velocity V of the body in the case of three-dimensional motion.

According to Eq. (14.24), the angular momentum of the body about G can be expressed as

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times \mathbf{v}'_i \, \Delta m_i) \tag{18.3}$$

where $\mathbf{r}'_i$ and $\mathbf{v}'_i$ denote, respectively, the position vector and the velocity of the particle P_i, of mass Δm_i, relative to the centroidal frame $Gxyz$ (Fig. 18.2). But $\mathbf{v}'_i = \text{V} \times \mathbf{r}'_i$, where V is the angular

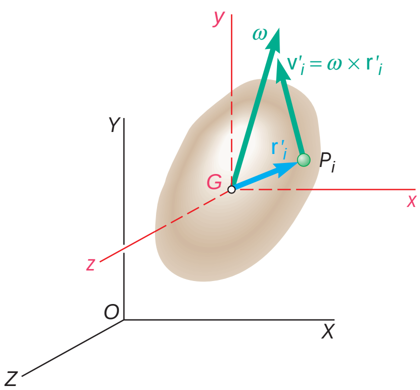


Fig. 18.2

velocity of the body at the instant considered. Substituting into (18.3), we have

$$\mathbf{H}_G = \sum_{i=1}^{n} [\mathbf{r}'_i \times (\boldsymbol{\omega} \times \mathbf{r}'_i)\, \Delta m_i]$$

Recalling the rule for determining the rectangular components of a vector product (Sec. 3.5), we obtain the following expression for the x component of the angular momentum:

$$\begin{aligned} H_x &= \sum_{i=1}^{n} [y_i(\boldsymbol{\omega} \times \mathbf{r}'_i)_z - z_i(\boldsymbol{\omega} \times \mathbf{r}'_i)_y]\, \Delta m_i \\ &= \sum_{i=1}^{n} [y_i(\omega_x y_i - \omega_y x_i) - z_i(\omega_z x_i - \omega_x z_i)]\, \Delta m_i \\ &= \omega_x \sum_i (y_i^2 + z_i^2)\, \Delta m_i - \omega_y \sum_i x_i y_i\, \Delta m_i - \omega_z \sum_i z_i x_i\, \Delta m_i \end{aligned}$$

Replacing the sums by integrals in this expression and in the two similar expressions which are obtained for H_y and H_z, we have

$$\begin{aligned} H_x &= \omega_x \int (y^2 + z^2)\, dm - \omega_y \int xy\, dm - \omega_z \int zx\, dm \\ H_y &= -\omega_x \int xy\, dm + \omega_y \int (z^2 + x^2)\, dm - \omega_z \int yz\, dm \\ H_z &= -\omega_x \int zx\, dm - \omega_y \int yz\, dm + \omega_z \int (x^2 + y^2)\, dm \end{aligned} \tag{18.4}$$

We note that the integrals containing squares represent the *centroidal mass moments of inertia* of the body about the x, y, and z axes, respectively (Sec. 9.11); we have

$$\bar{I}_x = \int (y^2 + z^2)\, dm \qquad \bar{I}_y = \int (z^2 + x^2)\, dm \qquad \bar{I}_z = \int (x^2 + y^2)\, dm \tag{18.5}$$

Similarly, the integrals containing products of coordinates represent the *centroidal mass products of inertia* of the body (Sec. 9.16); we have

$$\bar{I}_{xy} = \int xy\, dm \qquad \bar{I}_{yz} = \int yz\, dm \qquad \bar{I}_{zx} = \int zx\, dm \tag{18.6}$$

Substituting from (18.5) and (18.6) into (18.4), we obtain the components of the angular momentum $\mathbf{H}_G$ of the body about its mass center:

$$\begin{aligned} H_x &= +\bar{I}_x\, \omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z \\ H_y &= -\bar{I}_{yx}\omega_x + \bar{I}_y\, \omega_y - \bar{I}_{yz}\omega_z \\ H_z &= -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\, \omega_z \end{aligned} \tag{18.7}$$

The relations (18.7) show that the operation which transforms the vector $\boldsymbol{\omega}$ into the vector $\mathbf{H}_G$ (Fig. 18.3) is characterized by the array of moments and products of inertia

$$\begin{pmatrix} \bar{I}_x & -\bar{I}_{xy} & -\bar{I}_{xz} \\ -\bar{I}_{yx} & \bar{I}_y & -\bar{I}_{yz} \\ -\bar{I}_{zx} & -\bar{I}_{zy} & \bar{I}_z \end{pmatrix} \tag{18.8}$$

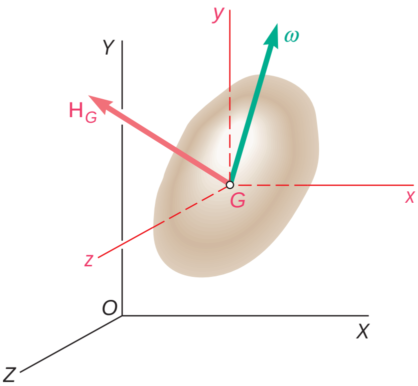


Fig. 18.3

The array (18.8) defines the *inertia tensor* of the body at its mass center G.† A new array of moments and products of inertia would be obtained if a different system of axes were used. The transformation characterized by this new array, however, would still be the same. Clearly, the angular momentum $\mathbf{H}_G$ corresponding to a given angular velocity $\boldsymbol{\omega}$ is independent of the choice of the coordinate axes. As was shown in Secs. 9.17 and 9.18, it is always possible to select a system of axes $Gx'y'z'$, called *principal axes of inertia,* with respect to which all the products of inertia of a given body are zero. The array (18.8) takes then the diagonalized form

$$\begin{pmatrix} \bar{I}_{x'} & 0 & 0 \\ 0 & \bar{I}_{y'} & 0 \\ 0 & 0 & \bar{I}_{z'} \end{pmatrix} \tag{18.9}$$

where $\bar{I}_{x'}$, $\bar{I}_{y'}$, $\bar{I}_{z'}$ represent the *principal centroidal moments of inertia* of the body, and the relations (18.7) reduce to

$$H_{x'} = \bar{I}_{x'}\omega_{x'} \qquad H_{y'} = \bar{I}_{y'}\omega_{y'} \qquad H_{z'} = \bar{I}_{z'}\omega_{z'} \tag{18.10}$$

We note that if the three principal centroidal moments of inertia $\bar{I}_{x'}$, $\bar{I}_{y'}$, $\bar{I}_{z'}$ are equal, the components $H_{x'}$, $H_{y'}$, $H_{z'}$ of the angular momentum about G are proportional to the components $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ of the angular velocity, and the vectors $\mathbf{H}_G$ and $\boldsymbol{\omega}$ are collinear. In general, however, the principal moments of inertia will be different, and the vectors $\mathbf{H}_G$ and $\boldsymbol{\omega}$ *will have different directions,* except when two of the three components of $\boldsymbol{\omega}$ happen to be zero, i.e., when $\boldsymbol{\omega}$ is directed along one of the coordinate axes. Thus, *the angular momentum* $\mathbf{H}_G$ *of a rigid body and its angular velocity* $\boldsymbol{\omega}$ *have the same direction if, and only if,* $\boldsymbol{\omega}$ *is directed along a principal axis of inertia.*‡

†Setting $\bar{I}_x = I_{11}$, $\bar{I}_y = I_{22}$, $\bar{I}_z = I_{33}$, and $-\bar{I}_{xy} = I_{12}$, $-\bar{I}_{xz} = I_{13}$, etc., we may write the inertia tensor (18.8) in the standard form

$$\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

Denoting by H_1, H_2, H_3 the components of the angular momentum $\mathbf{H}_G$ and by ω_1, ω_2, ω_3 the components of the angular velocity $\boldsymbol{\omega}$, we can write the relations (18.7) in the form

$$H_i = \sum_j I_{ij}\omega_j$$

where i and j take the values 1, 2, 3. The quantities I_{ij} are said to be the *components* of the inertia tensor. Since $I_{ij} = I_{ji}$, the inertia tensor is a *symmetric tensor of the second order.*

‡In the particular case when $\bar{I}_{x'} = \bar{I}_{y'} = \bar{I}_{z'}$, any line through G can be considered as a principal axis of inertia, and the vectors $\mathbf{H}_G$ and $\boldsymbol{\omega}$ are always collinear.

Photo 18.1 The design of a robotic welder for an automobile assembly line requires a three-dimensional study of both kinematics and kinetics.

Since this condition is satisfied in the case of the plane motion of a rigid body symmetrical with respect to the reference plane, we were able in Secs. 16.3 and 17.8 to represent the angular momentum $\mathbf{H}_G$ of such a body by the vector $\bar{I}$V. We must realize, however, that this result cannot be extended to the case of the plane motion of a nonsymmetrical body, or to the case of the three-dimensional motion of a rigid body. Except when V happens to be directed along a principal axis of inertia, the angular momentum and angular velocity of a rigid body have different directions, and the relation (18.7) or (18.10) must be used to determine $\mathbf{H}_G$ from V.

Reduction of the Momenta of the Particles of a Rigid Body to a Momentum Vector and a Couple at *G*. We saw in Sec. 17.8 that the system formed by the momenta of the various particles of a rigid body can be reduced to a vector $\mathbf{L}$ attached at the mass center G of the body, representing the linear momentum of the body, and to a couple $\mathbf{H}_G$, representing the angular momentum of the body about G (Fig. 18.4). We are now in a position to determine the vector $\mathbf{L}$

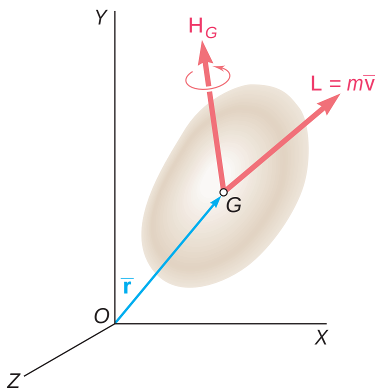


Fig. 18.4

and the couple $\mathbf{H}_G$ in the most general case of three-dimensional motion of a rigid body. As in the case of the two-dimensional motion considered in Sec. 17.8, the linear momentum $\mathbf{L}$ of the body is equal to the product $m\bar{\mathbf{v}}$ of its mass m and the velocity $\bar{\mathbf{v}}$ of its mass center G. The angular momentum $\mathbf{H}_G$, however, can no longer be obtained by simply multiplying the angular velocity V of the body by the scalar $\bar{I}$; it must now be obtained from the components of V and from the centroidal moments and products of inertia of the body through the use of Eq. (18.7) or (18.10).

We should also note that once the linear momentum $m\bar{\mathbf{v}}$ and the angular momentum $\mathbf{H}_G$ of a rigid body have been determined, its angular momentum $\mathbf{H}_O$ about any given point O can be obtained by adding the moments about O of the vector $m\bar{\mathbf{v}}$ and of the couple $\mathbf{H}_G$. We write

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G \tag{18.11}$$

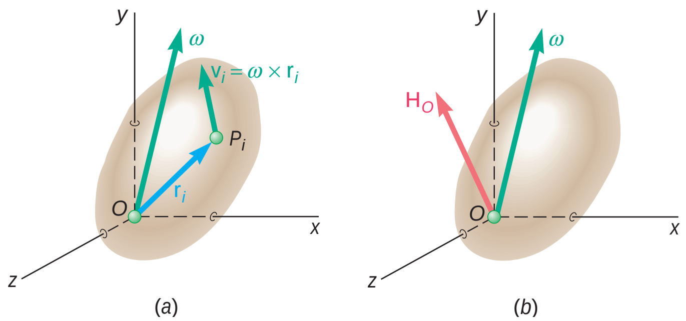


Fig. 18.5

Angular Momentum of a Rigid Body Constrained to Rotate about a Fixed Point. In the particular case of a rigid body constrained to rotate in three-dimensional space about a fixed point O (Fig. 18.5*a*), it is sometimes convenient to determine the angular momentum $\mathbf{H}_O$ of the body about the fixed point O. While $\mathbf{H}_O$ could be obtained by first computing $\mathbf{H}_G$ as indicated above and then using Eq. (18.11), it is often advantageous to determine $\mathbf{H}_O$ directly from the angular velocity V of the body and its moments and products of inertia with respect to a frame $Oxyz$ centered at the fixed point O. Recalling Eq. (14.7), we write

$$\mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times \mathbf{v}_i \, \Delta m_i) \tag{18.12}$$

where $\mathbf{r}_i$ and $\mathbf{v}_i$ denote, respectively, the position vector and the velocity of the particle P_i with respect to the fixed frame $Oxyz$. Substituting $\mathbf{v}_i = \mathrm{V} \times \mathbf{r}_i$, and after manipulations similar to those used in the earlier part of this section, we find that the components of the angular momentum $\mathbf{H}_O$ (Fig. 18.5*b*) are given by the relations

$$\begin{aligned} H_x &= +I_x \mathrm{V}_x - I_{xy}\mathrm{V}_y - I_{xz}\mathrm{V}_z \\ H_y &= -I_{yx}\mathrm{V}_x + I_y \mathrm{V}_y - I_{yz}\mathrm{V}_z \\ H_z &= -I_{zx}\mathrm{V}_x - I_{zy}\mathrm{V}_y + I_z \mathrm{V}_z \end{aligned} \tag{18.13}$$

where the moments of inertia I_x, I_y, I_z and the products of inertia I_{xy}, I_{yz}, I_{zx} are computed with respect to the frame $Oxyz$ centered at the fixed point O.

*18.3 APPLICATION OF THE PRINCIPLE OF IMPULSE AND MOMENTUM TO THE THREE-DIMENSIONAL MOTION OF A RIGID BODY

Before we can apply the fundamental equation (18.2) to the solution of problems involving the three-dimensional motion of a rigid body, we must learn to compute the derivative of the vector $\mathbf{H}_G$. This will be done in Sec. 18.5. The results obtained in the preceding section can, however, be used right away to solve problems by the impulse-momentum method.

Recalling that the system formed by the momenta of the particles of a rigid body reduces to a linear momentum vector $m\overline{\mathbf{v}}$

Photo 18.2 As a result of the impulsive force applied by the bowling ball, a pin acquires both linear momentum and angular momentum.

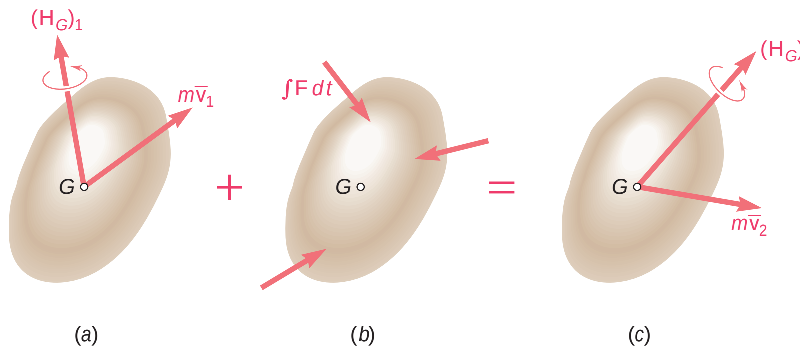


Fig. 18.6

attached at the mass center G of the body and an angular momentum couple $\mathbf{H}_G$, we represent graphically the fundamental relation

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\,y\,2} = \textbf{Syst Momenta}_2 \qquad (17.14)$$

by means of the three sketches shown in Fig. 18.6. To solve a given problem, we can use these sketches to write appropriate component and moment equations, keeping in mind that the components of the angular momentum $\mathbf{H}_G$ are related to the components of the angular velocity V by Eqs. (18.7) of the preceding section.

In solving problems dealing with the motion of a body rotating about a fixed point O, it will be convenient to eliminate the impulse of the reaction at O by writing an equation involving the moments of the momenta and impulses about O. We recall that the angular momentum $\mathbf{H}_O$ of the body about the fixed point O can be obtained either directly from Eqs. (18.13) or by first computing its linear momentum $m\bar{\mathbf{v}}$ and its angular momentum $\mathbf{H}_G$ and then using Eq. (18.11).

*18.4 KINETIC ENERGY OF A RIGID BODY IN THREE DIMENSIONS

Consider a rigid body of mass m in three-dimensional motion. We recall from Sec. 14.6 that if the absolute velocity $\mathbf{v}_i$ of each particle P_i of the body is expressed as the sum of the velocity $\bar{\mathbf{v}}$ of the mass center G of the body and of the velocity $\mathbf{v}'_i$ of the particle relative to a frame $Gxyz$ attached to G and of fixed orientation (Fig. 18.7), the kinetic energy of the system of particles forming the rigid body can be written in the form

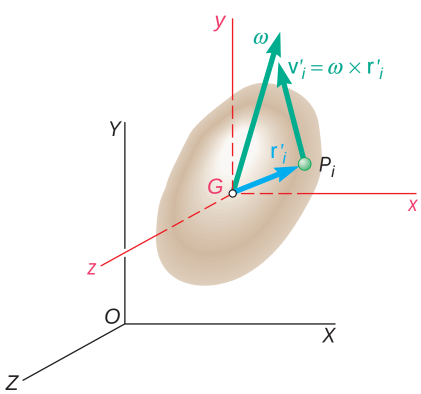


Fig. 18.7

$$T = \tfrac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n}\Delta m_i {v'_i}^2 \qquad (18.14)$$

where the last term represents the kinetic energy T' of the body relative to the centroidal frame $Gxyz$. Since $v'_i = |\mathbf{v}'_i| = |\text{V} \times \mathbf{r}'_i|$, we write

$$T' = \frac{1}{2}\sum_{i=1}^{n}\Delta m_i {v'_i}^2 = \frac{1}{2}\sum_{i=1}^{n}|\text{V} \times \mathbf{r}'_i|^2\,\Delta m_i$$

Expressing the square in terms of the rectangular components of the vector product, and replacing the sums by integrals, we have

$$
\begin{aligned}
T' &= \tfrac{1}{2}\int[(\mathsf{v}_x y - \mathsf{v}_y x)^2 + (\mathsf{v}_y z - \mathsf{v}_z y)^2 + (\mathsf{v}_z x - \mathsf{v}_x z)^2]\,dm \\
&= \tfrac{1}{2}[\mathsf{v}_x^2\int(y^2 + z^2)\,dm + \mathsf{v}_y^2\int(z^2 + x^2)\,dm + \mathsf{v}_z^2\int(x^2 + y^2)\,dm \\
&\qquad - 2\mathsf{v}_x\mathsf{v}_y\int xy\,dm - 2\mathsf{v}_y\mathsf{v}_z\int yz\,dm - 2\mathsf{v}_z\mathsf{v}_x\int zx\,dm]
\end{aligned}
$$

or, recalling the relations (18.5) and (18.6),

$$
T' = \tfrac{1}{2}(\bar{I}_x\mathsf{v}_x^2 + \bar{I}_y\mathsf{v}_y^2 + \bar{I}_z\mathsf{v}_z^2 - 2\bar{I}_{xy}\mathsf{v}_x\mathsf{v}_y - 2\bar{I}_{yz}\mathsf{v}_y\mathsf{v}_z - 2\bar{I}_{zx}\mathsf{v}_z\mathsf{v}_x) \tag{18.15}
$$

Substituting into (18.14) the expression (18.15) we have just obtained for the kinetic energy of the body relative to centroidal axes, we write

$$
\begin{aligned}
T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_x\mathsf{v}_x^2 + \bar{I}_y\mathsf{v}_y^2 + \bar{I}_z\mathsf{v}_z^2 - 2\bar{I}_{xy}\mathsf{v}_x\mathsf{v}_y \\
- 2\bar{I}_{yz}\mathsf{v}_y\mathsf{v}_z - 2\bar{I}_{zx}\mathsf{v}_z\mathsf{v}_x)
\end{aligned} \tag{18.16}
$$

If the axes of coordinates are chosen so that they coincide at the instant considered with the principal axes x', y', z' of the body, the relation obtained reduces to

$$
T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_{x'}\mathsf{v}_{x'}^2 + \bar{I}_{y'}\mathsf{v}_{y'}^2 + \bar{I}_{z'}\mathsf{v}_{z'}^2) \tag{18.17}
$$

where $\bar{\mathbf{v}}$ = velocity of mass center
V = angular velocity
m = mass of rigid body
$\bar{I}_{x'}, \bar{I}_{y'}, \bar{I}_{z'}$ = principal centroidal moments of inertia

The results we have obtained enable us to apply to the three-dimensional motion of a rigid body the principles of work and energy (Sec. 17.2) and conservation of energy (Sec. 17.6).

Kinetic Energy of a Rigid Body with a Fixed Point. In the particular case of a rigid body rotating in three-dimensional space about a fixed point O, the kinetic energy of the body can be expressed in terms of its moments and products of inertia with respect to axes attached at O (Fig. 18.8). Recalling the definition of the kinetic energy of a system of particles, and substituting $v_i = |\mathbf{v}_i| = |\mathsf{V} \times \mathbf{r}_i|$, we write

$$
T = \frac{1}{2}\sum_{i=1}^{n}\Delta m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{n}|\mathsf{V} \times \mathbf{r}_i|^2\,\Delta m_i \tag{18.18}
$$

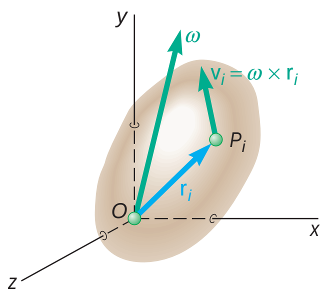


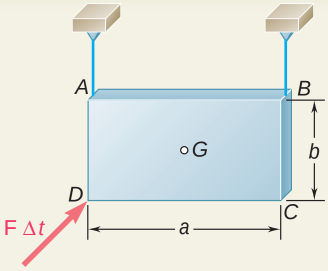
Fig. 18.8

Manipulations similar to those used to derive Eq. (18.15) yield

$$
T = \tfrac{1}{2}(I_x\mathsf{v}_x^2 + I_y\mathsf{v}_y^2 + I_z\mathsf{v}_z^2 - 2I_{xy}\mathsf{v}_x\mathsf{v}_y - 2I_{yz}\mathsf{v}_y\mathsf{v}_z - 2I_{zx}\mathsf{v}_z\mathsf{v}_x) \tag{18.19}
$$

or, if the principal axes x', y', z' of the body at the origin O are chosen as coordinate axes,

$$
T = \tfrac{1}{2}(I_{x'}\mathsf{v}_{x'}^2 + I_{y'}\mathsf{v}_{y'}^2 + I_{z'}\mathsf{v}_{z'}^2) \tag{18.20}
$$



SAMPLE PROBLEM 18.1

A rectangular plate of mass m suspended from two wires at A and B is hit at D in a direction perpendicular to the plate. Denoting by $\mathbf{F}\,\Delta t$ the impulse applied at D, determine immediately after the impact (*a*) the velocity of the mass center G, (*b*) the angular velocity of the plate.

SOLUTION

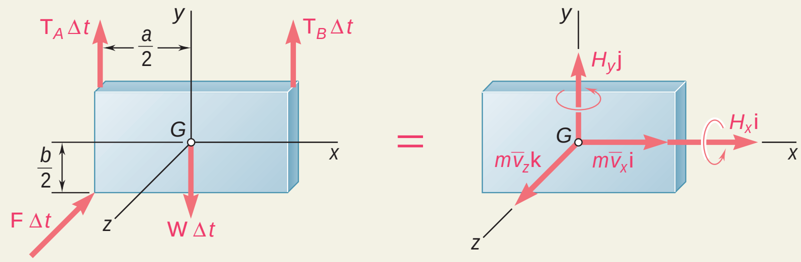
Assuming that the wires remain taut and thus that the components $\bar{v}_y$ of $\bar{\mathbf{v}}$ and v_z of V are zero after the impact, we have

$$\bar{\mathbf{v}} = \bar{v}_x\mathbf{i} + \bar{v}_z\mathbf{k} \qquad \mathrm{V} = \mathrm{v}_x\mathbf{i} + \mathrm{v}_y\mathbf{j}$$

and since the x, y, z axes are principal axes of inertia,

$$\mathbf{H}_G = \bar{I}_x\mathrm{v}_x\mathbf{i} + \bar{I}_y\mathrm{v}_y\mathbf{j} \qquad \mathbf{H}_G = \tfrac{1}{12}mb^2\mathrm{v}_x\mathbf{i} + \tfrac{1}{12}ma^2\mathrm{v}_y\mathbf{j} \tag{1}$$

Principle of Impulse and Momentum. Since the initial momenta are zero, the system of the impulses must be equivalent to the system of the final momenta:



a. Velocity of Mass Center. Equating the components of the impulses and momenta in the x and z directions:

x components: $\quad 0 = m\bar{v}_x \qquad \bar{v}_x = 0$

z components: $\quad -F\,\Delta t = m\bar{v}_z \qquad \bar{v}_z = -F\,\Delta t/m$

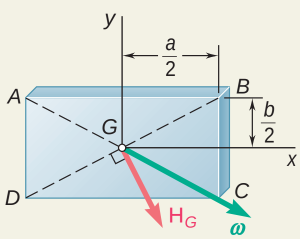
$$\bar{\mathbf{v}} = \bar{v}_x\mathbf{i} + \bar{v}_z\mathbf{k} \qquad \bar{\mathbf{v}} = -(F\,\Delta t/m)\mathbf{k} \quad ◀$$

b. Angular Velocity. Equating the moments of the impulses and momenta about the x and y axes:

About x axis: $\quad \tfrac{1}{2}bF\,\Delta t = H_x$

About y axis: $\quad -\tfrac{1}{2}aF\,\Delta t = H_y$

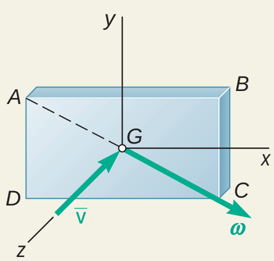
$$\mathbf{H}_G = H_x\mathbf{i} + H_y\mathbf{j} \qquad \mathbf{H}_G = \tfrac{1}{2}bF\,\Delta t\mathbf{i} - \tfrac{1}{2}aF\,\Delta t\mathbf{j} \tag{2}$$



Comparing Eqs. (1) and (2), we conclude that

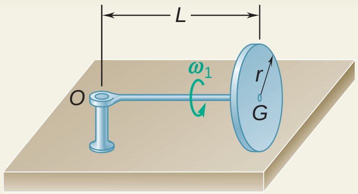
$$\mathrm{v}_x = 6F\,\Delta t/mb \qquad \mathrm{v}_y = -6F\,\Delta t/ma$$

$$\mathrm{V} = \mathrm{v}_x\mathbf{i} + \mathrm{v}_y\mathbf{j} \qquad \mathrm{V} = (6F\,\Delta t/mab)(a\mathbf{i} - b\mathbf{j}) \quad ◀$$



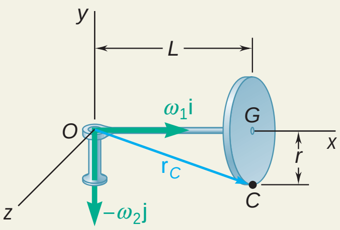
We note that V is directed along the diagonal AC.

Remark: Equating the y components of the impulses and momenta, and their moments about the z axis, we obtain two additional equations which yield $T_A = T_B = \tfrac{1}{2}W$. We thus verify that the wires remain taut and that our assumption was correct.



SAMPLE PROBLEM 18.2

A homogeneous disk of radius r and mass m is mounted on an axle OG of length L and negligible mass. The axle is pivoted at the fixed point O, and the disk is constrained to roll on a horizontal floor. Knowing that the disk rotates counterclockwise at the rate ω_1 about the axle OG, determine (*a*) the angular velocity of the disk, (*b*) its angular momentum about O, (*c*) its kinetic energy, (*d*) the vector and couple at G equivalent to the momenta of the particles of the disk.



SOLUTION

a. Angular Velocity. As the disk rotates about the axle OG it also rotates with the axle about the y axis at a rate ω_2 clockwise. The total angular velocity of the disk is therefore

$$\boldsymbol{\omega} = \omega_1\mathbf{i} - \omega_2\mathbf{j} \tag{1}$$

To determine ω_2 we write that the velocity of C is zero:

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_C = 0$$
$$(\omega_1\mathbf{i} - \omega_2\mathbf{j}) \times (L\mathbf{i} - r\mathbf{j}) = 0$$
$$(L\omega_2 - r\omega_1)\mathbf{k} = 0 \qquad \omega_2 = r\omega_1/L$$

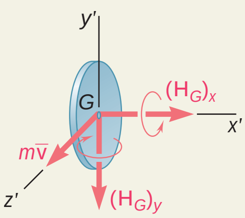
Substituting into (1) for ω_2: $\quad \boldsymbol{\omega} = \omega_1\mathbf{i} - (r\omega_1/L)\mathbf{j}$ ◀

b. Angular Momentum about O. Assuming the axle to be part of the disk, we can consider the disk to have a fixed point at O. Since the x, y, and z axes are principal axes of inertia for the disk,

$$H_x = I_x\omega_x = (\tfrac{1}{2}mr^2)\omega_1$$
$$H_y = I_y\omega_y = (mL^2 + \tfrac{1}{4}mr^2)(-r\omega_1/L)$$
$$H_z = I_z\omega_z = (mL^2 + \tfrac{1}{4}mr^2)0 = 0$$
$$\mathbf{H}_O = \tfrac{1}{2}mr^2\omega_1\mathbf{i} - m(L^2 + \tfrac{1}{4}r^2)(r\omega_1/L)\mathbf{j} \quad ◀$$

c. Kinetic Energy. Using the values obtained for the moments of inertia and the components of $\boldsymbol{\omega}$, we have

$$T = \tfrac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) = \tfrac{1}{2}[\tfrac{1}{2}mr^2\omega_1^2 + m(L^2 + \tfrac{1}{4}r^2)(-r\omega_1/L)^2]$$
$$T = \tfrac{1}{8}mr^2\left(6 + \frac{r^2}{L^2}\right)\omega_1^2 \quad ◀$$



d. Momentum Vector and Couple at G. The linear momentum vector $m\bar{\mathbf{v}}$ and the angular momentum couple $\mathbf{H}_G$ are

$$m\bar{\mathbf{v}} = mr\omega_1\mathbf{k} \quad ◀$$

and

$$\mathbf{H}_G = \bar{I}_{x'}\omega_x\mathbf{i} + \bar{I}_{y'}\omega_y\mathbf{j} + \bar{I}_{z'}\omega_z\mathbf{k} = \tfrac{1}{2}mr^2\omega_1\mathbf{i} + \tfrac{1}{4}mr^2(-r\omega_1/L)\mathbf{j}$$
$$\mathbf{H}_G = \tfrac{1}{2}mr^2\omega_1\left(\mathbf{i} - \frac{r}{2L}\mathbf{j}\right) \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to compute the *angular momentum of a rigid body in three dimensions* and to apply the principle of impulse and momentum to the three-dimensional motion of a rigid body. You also learned to compute the *kinetic energy of a rigid body in three dimensions.* It is important for you to keep in mind that, except for very special situations, the angular momentum of a rigid body in three dimensions *cannot* be expressed as the product $\bar{I}V$ and, therefore, *will not have the same direction as the angular velocity* V (Fig. 18.3).

1. To compute the angular momentum $\mathbf{H}_G$ of a rigid body about its mass center G, you must first determine the angular velocity V of the body with respect to a system of axes *centered at G and of fixed orientation.* Since you will be asked in this lesson to determine the angular momentum of the body *at a given instant only,* select the system of axes which will be most convenient for your computations.

a. If the principal axes of inertia of the body at G are known, use these axes as coordinate axes x', y', and z', since the corresponding products of inertia of the body will be equal to zero. Resolve V into components $V_{x'}$, $V_{y'}$, and $V_{z'}$ along these axes and compute the principal moments of inertia $\bar{I}_{x'}$, $\bar{I}_{y'}$, and $\bar{I}_{z'}$. The corresponding components of the angular momentum $\mathbf{H}_G$ are

$$H_{x'} = \bar{I}_{x'}V_{x'} \qquad H_{y'} = \bar{I}_{y'}V_{y'} \qquad H_{z'} = \bar{I}_{z'}V_{z'} \tag{18.10}$$

b. If the principal axes of inertia of the body at G are not known, you must use Eqs. (18.7) to determine the components of the angular momentum $\mathbf{H}_G$. These equations require prior computation of the *products of inertia* of the body as well as prior computation of its moments of inertia with respect to the selected axes.

c. The magnitude and direction cosines of $\mathbf{H}_G$ are obtained from formulas similar to those used in Statics [Sec. 2.12]. We have

$$H_G = 2\overline{H_x^2 + H_y^2 + H_z^2}$$

$$\cos u_x = \frac{H_x}{H_G} \qquad \cos u_y = \frac{H_y}{H_G} \qquad \cos u_z = \frac{H_z}{H_G}$$

d. Once $\mathbf{H}_G$ has been determined, you can obtain the angular momentum of the body *about any given point O* by observing from Fig. (18.4) that

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G \tag{18.11}$$

where $\bar{\mathbf{r}}$ is the position vector of G relative to O, and $m\bar{\mathbf{v}}$ is the linear momentum of the body.

2. To compute the angular momentum $\mathbf{H}_O$ of a rigid body with a fixed point O, follow the procedure described in paragraph 1, except that you should now use axes centered at the fixed point O.

a. If the principal axes of inertia of the body at O are known, resolve V into components along these axes [Sample Prob. 18.2]. The corresponding components of the angular momentum $\mathbf{H}_G$ are obtained from equations similar to Eqs. (18.10).

b. If the principal axes of inertia of the body at O are not known, you must compute the products as well as the moments of inertia of the body with respect to the axes that you have selected and use Eqs. (18.13) to determine the components of the angular momentum $\mathbf{H}_O$.

3. To apply the principle of impulse and momentum to the solution of a problem involving the three-dimensional motion of a rigid body, you will use the same vector equation that you used for plane motion in Chap. 17,

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1 \text{ y } 2} = \textbf{Syst Momenta}_2 \qquad (17.14)$$

where the initial and final systems of momenta are each represented by a *linear-momentum vector* $m\bar{\mathbf{v}}$ and an *angular-momentum couple* $\mathbf{H}_G$. Now, however, these vector-and-couple systems should be represented in three dimensions as shown in Fig. 18.6, and $\mathbf{H}_G$ should be determined as explained in paragraph 1.

a. In problems involving the application of a known impulse to a rigid body, draw the free-body-diagram equation corresponding to Eq. (17.14). Equating the components of the vectors involved, you will determine the final linear momentum $m\bar{\mathbf{v}}$ of the body and, thus, the corresponding velocity $\bar{\mathbf{v}}$ of its mass center. Equating moments about G, you will determine the final angular momentum $\mathbf{H}_G$ of the body. You will then substitute the values obtained for the components of $\mathbf{H}_G$ into Eqs. (18.10) or (18.7) and solve these equations for the corresponding values of the components of the angular velocity V of the body [Sample Prob. 18.1].

b. In problems involving unknown impulses, draw the free-body-diagram equation corresponding to Eq. (17.4) and write equations which do not involve the unknown impulses. Such equations can be obtained by equating moments about the point or line of impact.

4. To compute the kinetic energy of a rigid body with a fixed point O, resolve the angular velocity V into components along axes of your choice and compute the moments and products of inertia of the body with respect to these axes. As was the case for the computation of the angular momentum, use the principal axes of inertia x', y', and z' if you can easily determine them. The products of inertia will then be zero [Sample Prob. 18.2], and the expression for the kinetic energy will reduce to

$$T = \tfrac{1}{2}(I_{x'}\mathrm{V}_{x'}^2 + I_{y'}\mathrm{V}_{y'}^2 + I_{z'}\mathrm{V}_{x'}^2) \qquad (18.20)$$

If you must use axes other than the principal axes of inertia, the kinetic energy of the body should be expressed as shown in Eq. (18.19).

5. To compute the kinetic energy of a rigid body in general motion, consider the motion as the sum of a *translation with the mass center G and a rotation about G.* The kinetic energy associated with the translation is $\frac{1}{2}m\bar{v}^2$. If principal axes of inertia can be used, the kinetic energy associated with the rotation about G can be expressed in the form used in Eq. (18.20). The total kinetic energy of the rigid body is then

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_{x'}\mathrm{V}_{x'}^2 + \bar{I}_{y'}\mathrm{V}_{y'}^2 + \bar{I}_{z'}\mathrm{V}_{z'}^2) \qquad (18.17)$$

If you must use axes other than the principal axes of inertia to determine the kinetic energy associated with the rotation about G, the total kinetic energy of the body should be expressed as shown in Eq. (18.16).

PROBLEMS

18.1 A thin, homogeneous disk of mass m and radius r spins at the constant rate ω_1 about an axle held by a fork-ended vertical rod which rotates at the constant rate ω_2. Determine the angular momentum $\mathbf{H}_G$ of the disk about its mass center G.

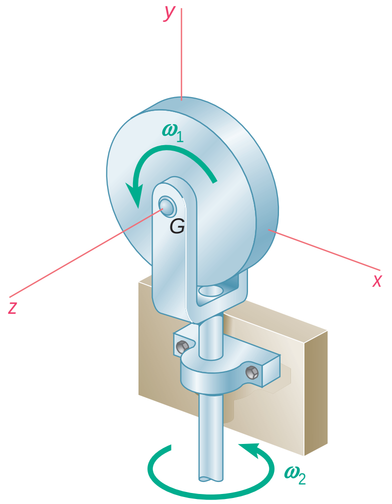


Fig. P18.1

18.2 A thin rectangular plate of weight 15 lb rotates about its vertical diagonal AB with an angular velocity ω. Knowing that the z axis is perpendicular to the plate and that ω is constant and equal to 5 rad/s, determine the angular momentum of the plate about its mass center G.

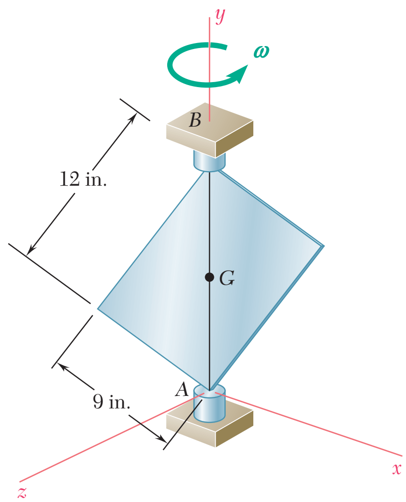


Fig. P18.2

18.3 Two uniform rods AB and CE, each of weight 3 lb and length 2 ft, are welded to each other at their midpoints. Knowing that this assembly has an angular velocity of constant magnitude $\omega = 12$ rad/s, determine the magnitude and direction of the angular momentum $\mathbf{H}_D$ of the assembly about D.

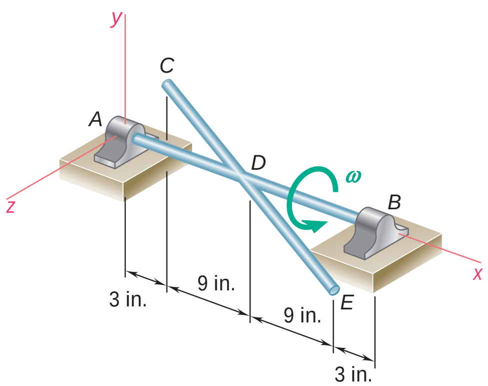


Fig. P18.3

18.4 A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm ABC, which is welded to a shaft DCE rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center A.

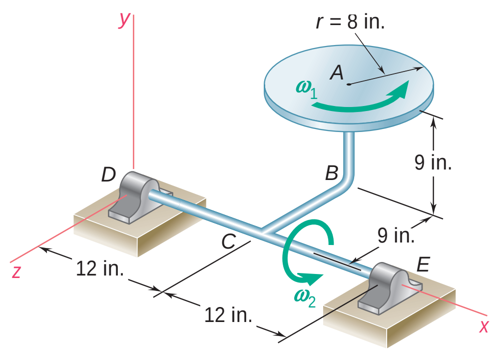


Fig. *P18.4*

18.5 A thin disk of mass $m = 4$ kg rotates at the constant rate $v_2 = 15$ rad/s with respect to arm ABC, which itself rotates at the constant rate $v_1 = 5$ rad/s about the y axis. Determine the angular momentum of the disk about its center C.

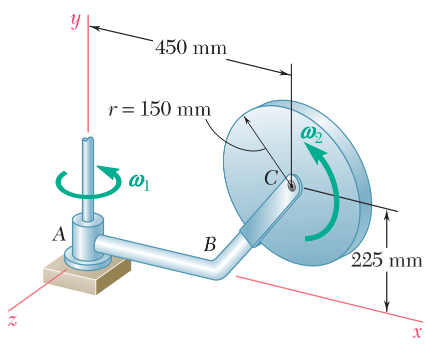


Fig. *P18.5*

18.6 A solid rectangular parallelepiped of mass m has a square base of side a and a length $2a$. Knowing that it rotates at the constant rate v about its diagonal AC' and that its rotation is observed from A as counterclockwise, determine (*a*) the magnitude of the angular momentum $\mathbf{H}_G$ of the parallelepiped about its mass center G, (*b*) the angle that $\mathbf{H}_G$ forms with the diagonal AC'.

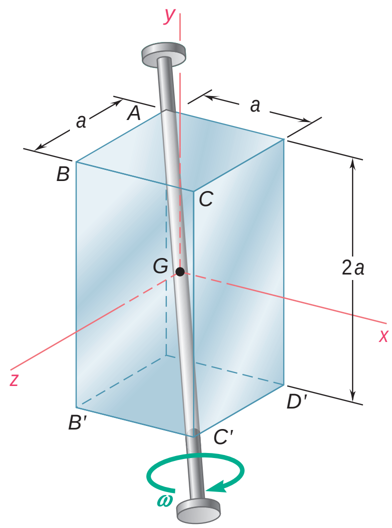


Fig. *P18.6*

18.7 Solve Prob. 18.6, assuming that the solid rectangular parallelepiped has been replaced by a hollow one consisting of six thin metal plates welded together.

18.8 A homogeneous disk of mass m and radius r is mounted on the vertical shaft AB. The normal to the disk at G forms an angle b $= 25°$ with the shaft. Knowing that the shaft has a constant angular velocity V, determine the angle u formed by the shaft AB and the angular momentum $\mathbf{H}_G$ of the disk about its mass center G.

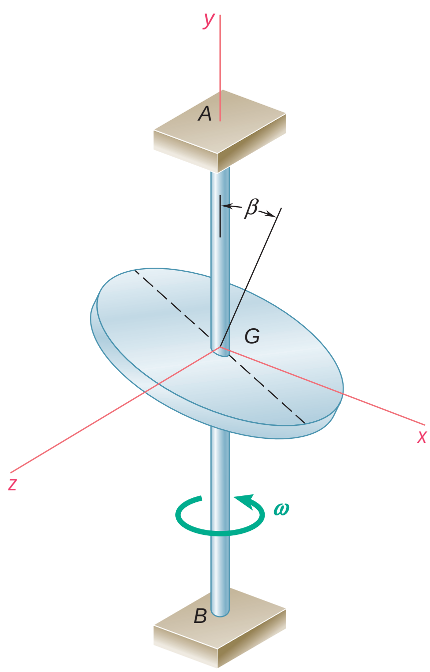


Fig. P18.8

18.9 Determine the angular momentum $\mathbf{H}_D$ of the disk of Prob. 18.4 about point D.

18.10 Determine the angular momentum of the disk of Prob. 18.5 about point A.

18.11 Determine the angular momentum $\mathbf{H}_O$ of the disk of Sample Prob. 18.2 from the expressions obtained for its linear momentum $m\bar{\mathbf{v}}$ and its angular momentum $\mathbf{H}_G$, using Eqs. (18.11). Verify that the result obtained is the same as that obtained by direct computation.

18.12 The 100-kg projectile shown has a radius of gyration of 100 mm about its axis of symmetry Gx and a radius of gyration of 250 mm about the transverse axis Gy. Its angular velocity V can be resolved into two components; one component, directed along Gx, measures the *rate of spin* of the projectile, while the other component, directed along GD, measures its *rate of precession*. Knowing that u = 6° and that the angular momentum of the projectile about its mass center G is $\mathbf{H}_G = (500 \text{ g} \cdot \text{m}^2/\text{s})\mathbf{i} - (10 \text{ g} \cdot \text{m}^2/\text{s})\mathbf{j}$, determine (*a*) the rate of spin, (*b*) the rate of precession.

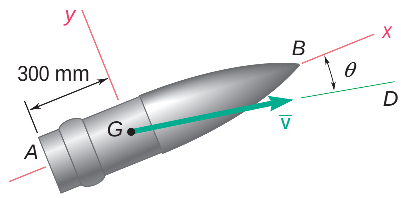


Fig. P18.12

18.13 Determine the angular momentum $\mathbf{H}_A$ of the projectile of Prob. 18.12 about the center A of its base, knowing that its mass center G has a velocity $\bar{\mathbf{v}}$ of 750 m/s. Give your answer in terms of components respectively parallel to the x and y axes shown and to a third axis z pointing toward you.

18.14 (*a*) Show that the angular momentum $\mathbf{H}_B$ of a rigid body about point B can be obtained by adding to the angular momentum $\mathbf{H}_A$ of that body about point A the vector product of the vector $\mathbf{r}_{A/B}$ drawn from B to A and the linear momentum $m\bar{\mathbf{v}}$ of the body:

$$\mathbf{H}_B = \mathbf{H}_A + \mathbf{r}_{A/B} \times m\bar{\mathbf{v}}$$

(*b*) Further show that when a rigid body rotates about a fixed axis, its angular momentum is the same about any two points A and B located on the fixed axis ($\mathbf{H}_A = \mathbf{H}_B$) if, and only if, the mass center G of the body is located on the fixed axis.

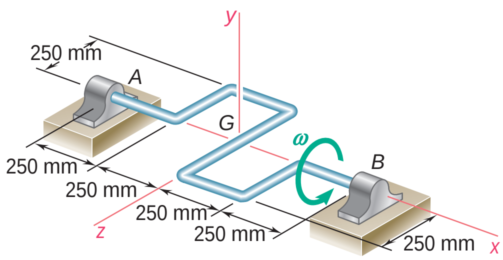


Fig. P18.15

18.15 A 5-kg rod of uniform cross section is used to form the shaft shown. Knowing that the shaft rotates with a constant angular velocity V of magnitude 12 rad/s, determine (*a*) the angular momentum $\mathbf{H}_G$ of the shaft about its mass center G, (*b*) the angle formed by $\mathbf{H}_G$ and the axis AB.

18.16 Determine the angular momentum of the shaft of Prob. 18.15 about (*a*) point A, (*b*) point B.

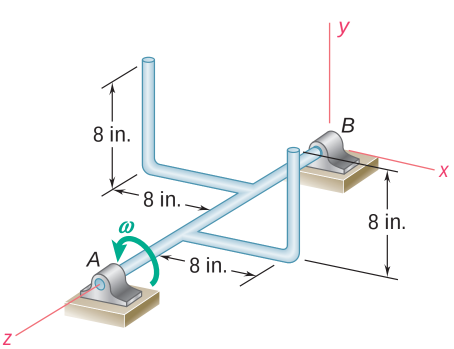


Fig. P18.17

18.17 Two L-shaped arms, each weighing 4 lb, are welded at the third points of the 2-ft shaft AB. Knowing that shaft AB rotates at the constant rate v = 240 rpm, determine (*a*) the angular momentum of the body about A, (*b*) the angle formed by the angular momentum and shaft AB.

18.18 For the body of Prob. 18.17, determine (*a*) the angular momentum about B, (*b*) the angle formed by the angular momentum about shaft BA.

18.19 The triangular plate shown has a mass of 7.5 kg and is welded to a vertical shaft *AB*. Knowing that the plate rotates at the constant rate ω = 12 rad/s, determine its angular momentum about (*a*) point *C*, (*b*) point *A*. (*Hint:* To solve part *b*, find $\bar{\mathbf{v}}$ and use the property indicated in part *a* of Prob. 18.14.)

18.20 The triangular plate shown has a mass of 7.5 kg and is welded to a vertical shaft *AB*. Knowing that the plate rotates at the constant rate ω = 12 rad/s, determine its angular momentum about (*a*) point *C*, (*b*) point *B*. (See hint of Prob. 18.19.)

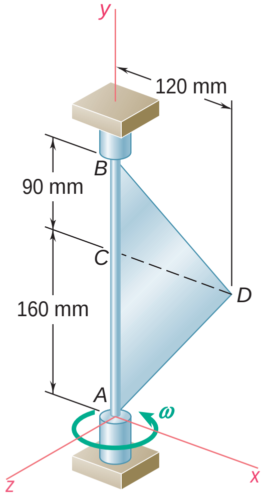


Fig. *P18.19* and *P18.20*

18.21 One of the sculptures displayed on a university campus consists of a hollow cube made of six aluminum sheets, each 1.5 × 1.5 m, welded together and reinforced with internal braces of negligible weight. The cube is mounted on a fixed base at *A* and can rotate freely about its vertical diagonal *AB*. As she passes by this display on the way to a class in mechanics, an engineering student grabs corner *C* of the cube and pushes it for 1.2 s in a direction perpendicular to the plane *ABC* with an average force of 50 N. Having observed that it takes 5 s for the cube to complete one full revolution, she flips out her calculator and proceeds to determine the mass of the cube. What is the result of her calculation? (*Hint:* The perpendicular distance from the diagonal joining two vertices of a cube to any of its other six vertices can be obtained by multiplying the side of the cube by $\sqrt{2/3}$.)

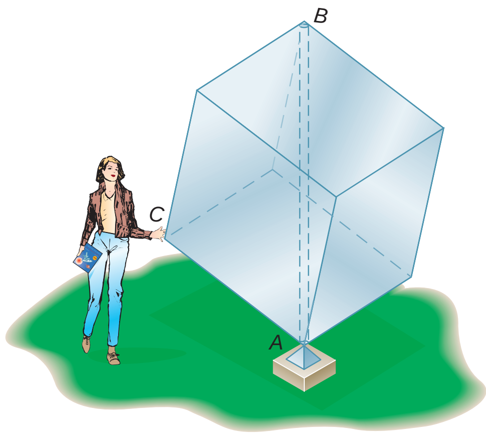


Fig. P18.21

18.22 If the aluminum cube of Prob. 18.21 were replaced by a cube of the same size, made of six plywood sheets with mass 8 kg each, how long would it take for that cube to complete one full revolution if the student pushed its corner *C* in the same way that she pushed the corner of the aluminum cube?

18.23 A uniform rod of total mass *m* is bent into the shape shown and is suspended by a wire attached at *B*. The bent rod is hit at *D* in a direction perpendicular to the plane containing the rod (in the negative *z* direction). Denoting the corresponding impulse by $\mathbf{F}\Delta t$, determine (*a*) the velocity of the mass center of the rod, (*b*) the angular velocity of the rod.

18.24 Solve Prob. 18.23, assuming that the bent rod is hit at *C*.

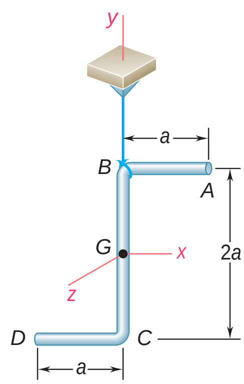


Fig. *P18.23*

18.25 Three slender rods, each of mass m and length $2a$, are welded together to form the assembly shown. The assembly is hit at A in a vertical downward direction. Denoting the corresponding impulse by $\mathbf{F}\,\Delta t$, determine immediately after the impact (*a*) the velocity of the mass center G, (*b*) the angular velocity of the rod.

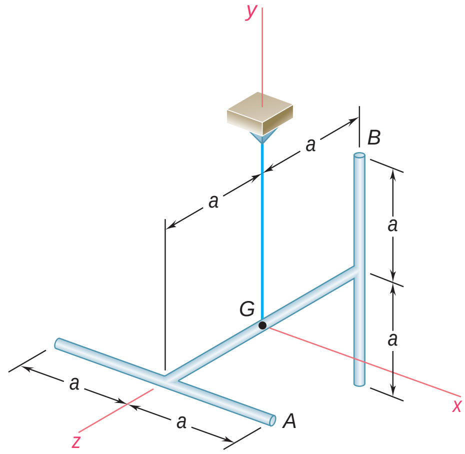


Fig. P18.25

18.26 Solve Prob. 18.25, assuming that the assembly is hit at B in a direction opposite to that of the x axis.

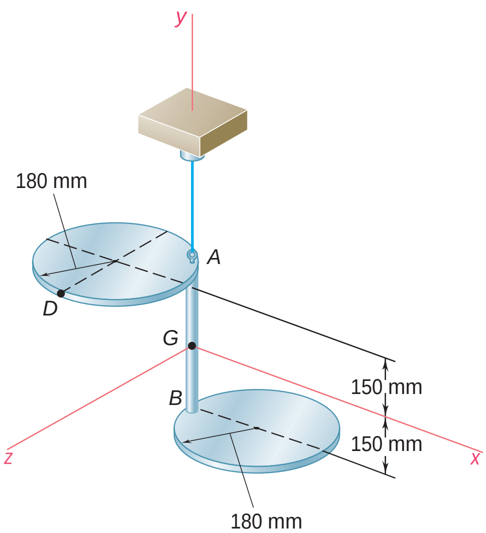


Fig. P18.27 and P18.28

18.27 Two circular plates, each of mass 4 kg, are rigidly connected by a rod AB of negligible mass and are suspended from point A as shown. Knowing that an impulse $\mathbf{F}\,\Delta t = -(2.4\text{ N}\cdot\text{s})\mathbf{k}$ is applied at point D, determine (*a*) the velocity of the mass center G of the assembly, (*b*) the angular velocity of the assembly.

18.28 Two circular plates, each of mass 4 kg, are rigidly connected by a rod AB of negligible mass and are suspended from point A as shown. Knowing that an impulse $\mathbf{F}\,\Delta t = (2.4\text{ N}\cdot\text{s})\mathbf{j}$ is applied at point D, determine (*a*) the velocity of the mass center G of the assembly, (*b*) the angular velocity of the assembly.

18.29 A circular plate of mass m is falling with a velocity $\bar{\mathbf{v}}_0$ and no angular velocity when its edge C strikes an obstruction. A line passing the origin and parallel to the line CG makes a 45° angle with the x-axis. Assuming the impact to be perfectly plastic ($e = 0$), determine the angular velocity of the plate immediately after the impact.

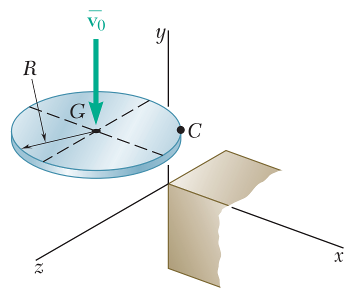


Fig. *P18.29*

18.30 For the plate of Prob. 18.29, determine (*a*) the velocity of its mass center G immediately after the impact, (*b*) the impulse exerted on the plate by the obstruction during the impact.

18.31 A square plate of side a and mass m supported by a ball-and-socket joint at A is rotating about the y axis with a constant angular velocity $V = v_0\mathbf{j}$ when an obstruction is suddenly introduced at B in the xy plane. Assuming the impact at B to be perfectly plastic ($e = 0$), determine immediately after the impact (a) the angular velocity of the plate, (b) the velocity of its mass center G.

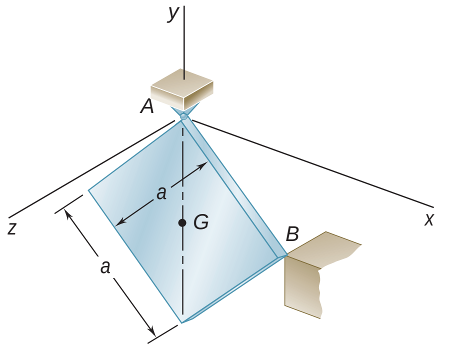


Fig. P18.31

18.32 Determine the impulse exerted on the plate of Prob. 18.31 during the impact by (a) the obstruction at B, (b) the support at A.

18.33 The coordinate axes shown represent the principal centroidal axes of inertia of a 3000-lb space probe whose radii of gyration are $k_x = 1.375$ ft, $k_y = 1.425$ ft, and $k_z = 1.250$ ft. The probe has no angular velocity when a 5-oz meteorite strikes one of its solar panels at point A with a velocity $\mathbf{v}_0 = (2400 \text{ ft/s})\mathbf{i} - (3000 \text{ ft/s})\mathbf{j} + (3200 \text{ ft/s})\mathbf{k}$ relative to the probe. Knowing that the meteorite emerges on the other side of the panel with no change in the direction of its velocity, but with a speed reduced by 20 percent, determine the final angular velocity of the probe.

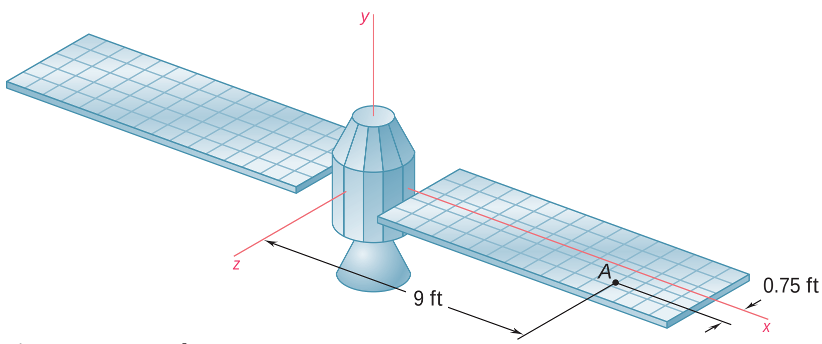


Fig. P18.33 and P18.34

18.34 The coordinate axes shown represent the principal centroidal axes of inertia of a 3000-lb space probe whose radii of gyration are $k_x = 1.375$ ft, $k_y = 1.425$ ft, and $k_z = 1.250$ ft. The probe has no angular velocity when a 5-oz meteorite strikes one of its solar panels at point A and emerges on the other side of the panel with no change in the direction of its velocity, but with a speed reduced by 25 percent. Knowing that the final angular velocity of the probe is $V = (0.05 \text{ rad/s})\mathbf{i} - (0.12 \text{ rad/s})\mathbf{j} + v_z\mathbf{k}$ and that the x component of the resulting change in the velocity of the mass center of the probe is -0.675 in./s, determine (a) the component v_z of the final angular velocity of the probe, (b) the relative velocity $\mathbf{v}_0$ with which the meteorite strikes the panel.

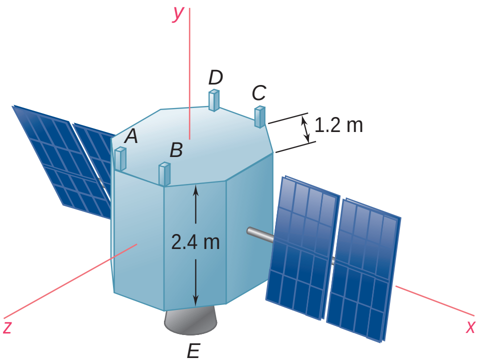


Fig. P18.35

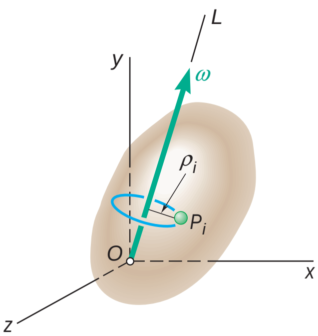


Fig. P18.38

18.35 A 2500-kg probe in orbit about the moon is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the probe, and its radii of gyration are $k_x = 0.98$ m, $k_y = 1.06$ m, and $k_z = 1.02$ m. The probe is equipped with a main 500-N thruster E and with four 20-N thrusters A, B, C, and D which can expel fuel in the positive y direction. The probe has an angular velocity V $= (0.040 \text{ rad/s})\mathbf{i} + (0.060 \text{ rad/s})\mathbf{k}$ when two of the 20-N thrusters are used to reduce the angular velocity to zero. Determine (*a*) which of the thrusters should be used, (*b*) the operating time of each of these thrusters, (*c*) for how long the main thruster E should be activated if the velocity of the mass center of the probe is to remain unchanged.

18.36 Solve Prob. 18.35, assuming that the angular velocity of the probe is V $= (0.060 \text{ rad/s})\mathbf{i} - (0.040 \text{ rad/s})\mathbf{k}$.

18.37 Denoting, respectively, by V, $\mathbf{H}_O$, and T the angular velocity, the angular momentum, and the kinetic energy of a rigid body with a fixed point O, (*a*) prove that $\mathbf{H}_O \cdot$ V $= 2T$; (*b*) show that the angle u between v and $\mathbf{H}_O$ will always be acute.

18.38 Show that the kinetic energy of a rigid body with a fixed point O can be expressed as $T = \frac{1}{2} I_{OL}$v^2, where V is the instantaneous angular velocity of the body and I_{OL} is its moment of inertia about the line of action OL of V. Derive this expression (*a*) from Eqs. (9.46) and (18.19), (*b*) by considering T as the sum of the kinetic energies of particles P_i describing circles of radius r$_i$ about line OL.

18.39 Determine the kinetic energy of the disk of Prob. 18.1.

18.40 Determine the kinetic energy of the plate of Prob. 18.2.

18.41 Determine the kinetic energy of the assembly of Prob. 18.3.

18.42 Determine the kinetic energy of the disk of Prob. 18.4.

18.43 Determine the kinetic energy of the disk of Prob. 18.5.

18.44 Determine the kinetic energy of the solid parallelepiped of Prob. 18.6.

18.45 Determine the kinetic energy of the hollow parallelepiped of Prob. 18.7.

18.46 Determine the kinetic energy of the disk of Prob. 18.8.

18.47 Determine the kinetic energy of the shaft of Prob. 18.15.

18.48 Determine the kinetic energy of the body of Prob. 18.17.

18.49 Determine the kinetic energy of the triangular plate of Prob. 18.19.

18.50 Determine the kinetic energy imparted to the cube of Prob. 18.21.

18.51 Determine the kinetic energy lost when edge C of the plate of Prob. 18.29 hits the obstruction.

18.52 Determine the kinetic energy lost when the plate of Prob. 18.31 hits the obstruction at B.

18.53 Determine the kinetic energy of the space probe of Prob. 18.33 in its motion about its mass center after its collision with the meteorite.

18.54 Determine the kinetic energy of the space probe of Prob. 18.34 in its motion about its mass center after its collision with the meteorite.

*18.5 MOTION OF A RIGID BODY IN THREE DIMENSIONS

As indicated in Sec. 18.2, the fundamental equations

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \qquad (18.1)$$
$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \qquad (18.2)$$

remain valid in the most general case of the motion of a rigid body. Before Eq. (18.2) could be applied to the three-dimensional motion of a rigid body, however, it was necessary to derive Eqs. (18.7), which relate the components of the angular momentum $\mathbf{H}_G$ and those of the angular velocity V. It still remains for us to find an effective and convenient way for computing the components of the derivative $\dot{\mathbf{H}}_G$ of the angular momentum.

Since $\mathbf{H}_G$ represents the angular momentum of the body in its motion relative to centroidal axes $GX'Y'Z'$ of fixed orientation (Fig. 18.9), and since $\dot{\mathbf{H}}_G$ represents the rate of change of $\mathbf{H}_G$ with respect to the same axes, it would seem natural to use components of V and $\mathbf{H}_G$ along the axes X', Y', Z' in writing the relations (18.7). But since the body rotates, its moments and products of inertia would change continually, and it would be necessary to determine their values as functions of the time. It is therefore more convenient to use axes x, y, z attached to the body, ensuring that its moments and products of inertia will maintain the same values during the motion. This is permissible since, as indicated earlier, the transformation of V into $\mathbf{H}_G$ is independent of the system of coordinate axes selected. The angular velocity V, however, should still be *defined* with respect to the frame $GX'Y'Z'$ of fixed orientation. The vector V may then be *resolved* into components along the rotating x, y, and z axes. Applying the relations (18.7), we obtain the *components* of the vector $\mathbf{H}_G$ along the rotating axes. The vector $\mathbf{H}_G$, however, represents the angular momentum about G of the body *in its motion relative to the frame $GX'Y'Z'$*.

Differentiating with respect to t the components of the angular momentum in (18.7), we define the rate of change of the vector $\mathbf{H}_G$ with respect to the rotating frame $Gxyz$:

$$(\dot{\mathbf{H}}_G)_{Gxyz} = \dot{H}_x\mathbf{i} + \dot{H}_y\mathbf{j} + \dot{H}_z\mathbf{k} \qquad (18.21)$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the rotating axes. Recalling from Sec. 15.10 that the rate of change $\dot{\mathbf{H}}_G$ of the vector $\mathbf{H}_G$ with respect to the frame $GX'Y'Z'$ is found by adding to $(\dot{\mathbf{H}}_G)_{Gxyz}$ the vector product $\mathbf{\Omega} \times \mathbf{H}_G$, where $\mathbf{\Omega}$ denotes the angular velocity of the rotating frame, we write

$$\boxed{\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G} \qquad (18.22)$$

where $\mathbf{H}_G$ = angular momentum of body with respect to frame $GX'Y'Z'$ of fixed orientation
$(\dot{\mathbf{H}}_G)_{Gxyz}$ = rate of change of $\mathbf{H}_G$ with respect to rotating frame $Gxyz$, to be computed from the relations (18.7) and (18.21)
$\mathbf{\Omega}$ = angular velocity of rotating frame $Gxyz$

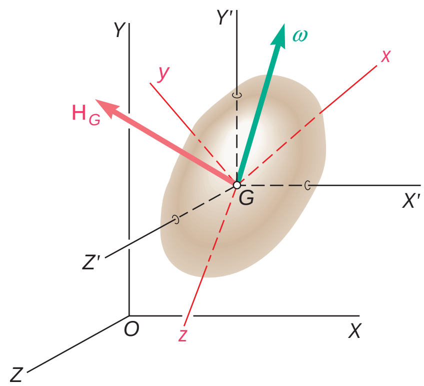


Fig. 18.9

Substituting for $\dot{\mathbf{H}}_G$ from (18.22) into (18.2), we have

$$\Sigma \mathbf{M}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \tag{18.23}$$

If the rotating frame is attached to the body, as has been assumed in this discussion, its angular velocity $\boldsymbol{\Omega}$ is identically equal to the angular velocity $\boldsymbol{\omega}$ of the body. There are many applications, however, where it is advantageous to use a frame of reference which is not actually attached to the body but rotates in an independent manner. For example, if the body considered is axisymmetrical, as in Sample Prob. 18.5 or Sec. 18.9, it is possible to select a frame of reference with respect to which the moments and products of inertia of the body remain constant, but which rotates less than the body itself.† As a result, it is possible to obtain simpler expressions for the angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{H}_G$ of the body than could have been obtained if the frame of reference had actually been attached to the body. It is clear that in such cases the angular velocity $\boldsymbol{\Omega}$ of the rotating frame and the angular velocity $\boldsymbol{\omega}$ of the body are different.

*18.6 EULER'S EQUATIONS OF MOTION. EXTENSION OF D'ALEMBERT'S PRINCIPLE TO THE MOTION OF A RIGID BODY IN THREE DIMENSIONS

If the x, y, and z axes are chosen to coincide with the principal axes of inertia of the body, the simplified relations (18.10) can be used to determine the components of the angular momentum $\mathbf{H}_G$. Omitting the primes from the subscripts, we write

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} \tag{18.24}$$

where $\bar{I}_x$, $\bar{I}_y$, and $\bar{I}_z$ denote the principal centroidal moments of inertia of the body. Substituting for $\mathbf{H}_G$ from (18.24) into (18.23) and setting $\boldsymbol{\Omega} = \boldsymbol{\omega}$, we obtain the three scalar equations

$$\begin{aligned} \Sigma M_x &= \bar{I}_x \dot{\omega}_x - (\bar{I}_y - \bar{I}_z)\omega_y \omega_z \\ \Sigma M_y &= \bar{I}_y \dot{\omega}_y - (\bar{I}_z - \bar{I}_x)\omega_z \omega_x \\ \Sigma M_z &= \bar{I}_z \dot{\omega}_z - (\bar{I}_x - \bar{I}_y)\omega_x \omega_y \end{aligned} \tag{18.25}$$

These equations, called *Euler's equations of motion* after the Swiss mathematician Leonhard Euler (1707–1783), can be used to analyze the motion of a rigid body about its mass center. In the following sections, however, Eq. (18.23) will be used in preference to Eqs. (18.25), since the former is more general and the compact vectorial form in which it is expressed is easier to remember.

Writing Eq. (18.1) in scalar form, we obtain the three additional equations

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma F_z = m\bar{a}_z \tag{18.26}$$

which, together with Euler's equations, form a system of six differential equations. Given appropriate initial conditions, these differential

†More specifically, the frame of reference will have no spin (see Sec. 18.9).

equations have a unique solution. Thus, the motion of a rigid body in three dimensions is completely defined by the resultant and the moment resultant of the external forces acting on it. This result will be recognized as a generalization of a similar result obtained in Sec. 16.4 in the case of the plane motion of a rigid slab. It follows that in three as well as two dimensions, two systems of forces which are equipollent are also equivalent; that is, they have the same effect on a given rigid body.

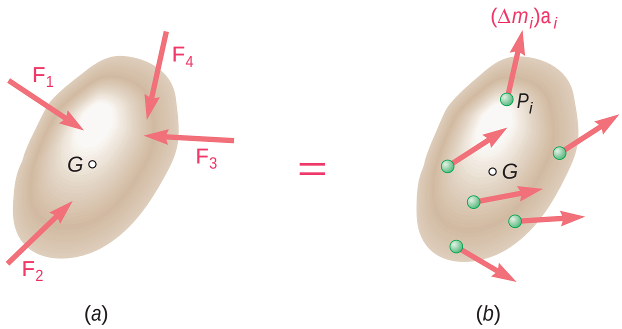


Fig. 18.10

Considering in particular the system of the external forces acting on a rigid body (Fig. 18.10*a*) and the system of the effective forces associated with the particles forming the rigid body (Fig. 18.10*b*), we can state that the two systems—which were shown in Sec. 14.2 to be equipollent—are also equivalent. This is the extension of d'Alembert's principle to the three-dimensional motion of a rigid body. Replacing the effective forces in Fig. 18.10*b* by an equivalent force-couple system, we verify that the system of the external forces acting on a rigid body in three-dimensional motion is equivalent to the system consisting of the vector $m\bar{\mathbf{a}}$ attached at the mass center G of the body and the couple of moment $\dot{\mathbf{H}}_G$ (Fig. 18.11), where $\dot{\mathbf{H}}_G$ is obtained from the relations (18.7) and (18.22). Note that the equivalence of the systems of vectors shown in Fig. 18.10 and in Fig. 18.11 has been indicated by *red* equals signs. Problems involving the three-dimensional motion of a rigid body can be solved by considering the free-body-diagram equation represented in Fig. 18.11 and writing appropriate scalar equations relating the components or moments of the external and effective forces (see Sample Prob. 18.3).

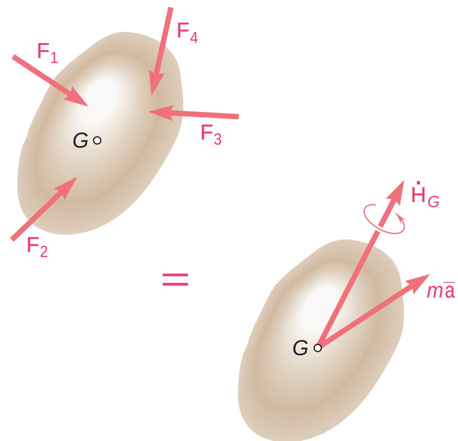


Fig. 18.11

*18.7 MOTION OF A RIGID BODY ABOUT A FIXED POINT

When a rigid body is constrained to rotate about a fixed point O, it is desirable to write an equation involving the moments about O of the external and effective forces, since this equation will not contain the unknown reaction at O. While such an equation can be obtained from Fig. 18.11, it may be more convenient to write it by considering the rate of change of the angular momentum $\mathbf{H}_O$ of the body about the fixed point O (Fig. 18.12). Recalling Eq. (14.11), we write

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \qquad (18.27)$$

where $\dot{\mathbf{H}}_O$ denotes the rate of change of the vector $\mathbf{H}_O$ with respect to the fixed frame $OXYZ$. A derivation similar to that used in Sec. 18.5

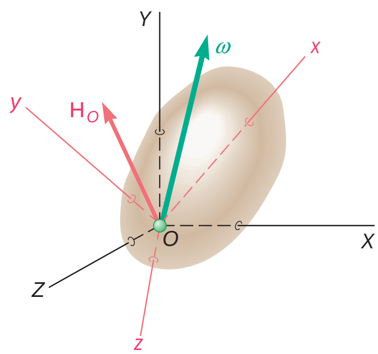


Fig. 18.12

Photo 18.3 The revolving radio telescope is an example of a structure constrained to rotate about a fixed point.

enables us to relate $\dot{\mathbf{H}}_O$ to the rate of change $(\dot{\mathbf{H}}_O)_{Oxyz}$ of $\mathbf{H}_O$ with respect to the rotating frame $Oxyz$. Substitution into (18.27) leads to the equation

$$\Sigma \mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \tag{18.28}$$

where $\Sigma \mathbf{M}_O$ = sum of moments about O of forces applied to rigid body
$\mathbf{H}_O$ = angular momentum of body with respect to fixed frame $OXYZ$
$(\dot{\mathbf{H}}_O)_{Oxyz}$ = rate of change of $\mathbf{H}_O$ with respect to rotating frame $Oxyz$, to be computed from relations (18.13)
$\boldsymbol{\Omega}$ = angular velocity of rotating frame $Oxyz$

If the rotating frame is attached to the body, its angular velocity $\boldsymbol{\Omega}$ is identically equal to the angular velocity $\boldsymbol{\omega}$ of the body. However, as indicated in the last paragraph of Sec. 18.5, there are many applications where it is advantageous to use a frame of reference which is not actually attached to the body but rotates in an independent manner.

*18.8 ROTATION OF A RIGID BODY ABOUT A FIXED AXIS

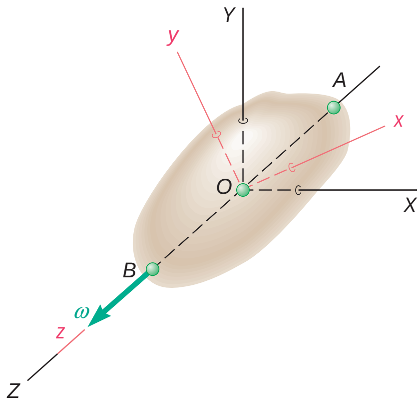


Fig. 18.13

Equation (18.28), which was derived in the preceding section, will be used to analyze the motion of a rigid body constrained to rotate about a fixed axis AB (Fig. 18.13). First, we note that the angular velocity of the body with respect to the fixed frame $OXYZ$ is represented by the vector $\boldsymbol{\omega}$ directed along the axis of rotation. Attaching the moving frame of reference $Oxyz$ to the body, with the z axis along AB, we have $\boldsymbol{\omega} = \omega \mathbf{k}$. Substituting $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$ into the relations (18.13), we obtain the components along the rotating axes of the angular momentum $\mathbf{H}_O$ of the body about O:

$$H_x = -I_{xz}\omega \qquad H_y = -I_{yz}\omega \qquad H_z = I_z\omega$$

Since the frame $Oxyz$ is attached to the body, we have $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and Eq. (18.28) yields

$$\begin{aligned}\Sigma \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\omega} \times \mathbf{H}_O \\ &= (-I_{xz}\mathbf{i} - I_{yz}\mathbf{j} + I_z\mathbf{k})\dot{\omega} + \omega\mathbf{k} \times (-I_{xz}\mathbf{i} - I_{yz}\mathbf{j} + I_z\mathbf{k})\omega \\ &= (-I_{xz}\mathbf{i} - I_{yz}\mathbf{j} + I_z\mathbf{k})\alpha + (-I_{xz}\mathbf{j} + I_{yz}\mathbf{i})\omega^2\end{aligned}$$

The result obtained can be expressed by the three scalar equations

$$\begin{aligned}\Sigma M_x &= -I_{xz}\alpha + I_{yz}\omega^2 \\ \Sigma M_y &= -I_{yz}\alpha - I_{xz}\omega^2 \\ \Sigma M_z &= I_z\alpha\end{aligned} \tag{18.29}$$

When the forces applied to the body are known, the angular acceleration α can be obtained from the last of Eqs. (18.29). The angular velocity ω is then determined by integration and the values obtained for α and ω substituted into the first two equations (18.29). These equations plus the three equations (18.26) which define the motion of the mass center of the body can then be used to determine the reactions at the bearings A and B.

It is possible to select axes other than the ones shown in Fig. 18.13 to analyze the rotation of a rigid body about a fixed axis. In many cases, the principal axes of inertia of the body will be found more advantageous. It is therefore wise to revert to Eq. (18.28) and to select the system of axes which best fits the problem under consideration.

If the rotating body is symmetrical with respect to the xy plane, the products of inertia I_{xz} and I_{yz} are equal to zero and Eqs. (18.29) reduce to

$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = I_z\alpha \tag{18.30}$$

which is in accord with the results obtained in Chap. 16. If, on the other hand, the products of inertia I_{xz} and I_{yz} are different from zero, the sum of the moments of the external forces about the x and y axes will also be different from zero, even when the body rotates at a constant rate ω. Indeed, in the latter case, Eqs. (18.29) yield

$$\Sigma M_x = I_{yz}\omega^2 \qquad \Sigma M_y = -I_{xz}\omega^2 \qquad \Sigma M_z = 0 \tag{18.31}$$

Photo 18.4 The forces exerted by a rotating automobile crankshaft on its bearings are the static and dynamic reactions. The crankshaft can be designed to be dynamically as well as statically balanced.

This last observation leads us to discuss the *balancing of rotating shafts*. Consider, for instance, the crankshaft shown in Fig. 18.14*a*, which is symmetrical about its mass center G. We first observe that when the crankshaft is at rest, it exerts no lateral thrust on its supports, since its center of gravity G is located directly above A. The shaft is said to be *statically balanced*. The reaction at A, often referred to as a *static reaction*, is vertical and its magnitude is equal to the weight W of the shaft. Let us now assume that the shaft rotates with a constant angular velocity $\boldsymbol{\omega}$. Attaching our frame of reference to the shaft, with its origin at G, the z axis along AB, and the y axis in the plane of symmetry of the shaft (Fig. 18.14*b*), we note that I_{xz} is zero and that I_{yz} is positive. According to Eqs. (18.31), the external forces include a couple of moment $I_{yz}\omega^2\mathbf{i}$. Since this couple is formed by the reaction at B and the horizontal component of the reaction at A, we have

$$\mathbf{A}_y = \frac{I_{yz}\omega^2}{l}\mathbf{j} \qquad \mathbf{B} = -\frac{I_{yz}\omega^2}{l}\mathbf{j} \tag{18.32}$$

Since the bearing reactions are proportional to ω^2, the shaft will have a tendency to tear away from its bearings when rotating at high speeds. Moreover, since the bearing reactions $\mathbf{A}_y$ and $\mathbf{B}$, called *dynamic reactions,* are contained in the yz plane, they rotate with the shaft and cause the structure supporting it to vibrate. These undesirable effects will be avoided if, by rearranging the distribution of mass around the shaft or by adding corrective masses, we let I_{yz} become equal to zero. The dynamic reactions $\mathbf{A}_y$ and $\mathbf{B}$ will vanish and the reactions at the bearings will reduce to the static reaction $\mathbf{A}_z$, the direction of which is fixed. The shaft will then be *dynamically as well as statically balanced*.

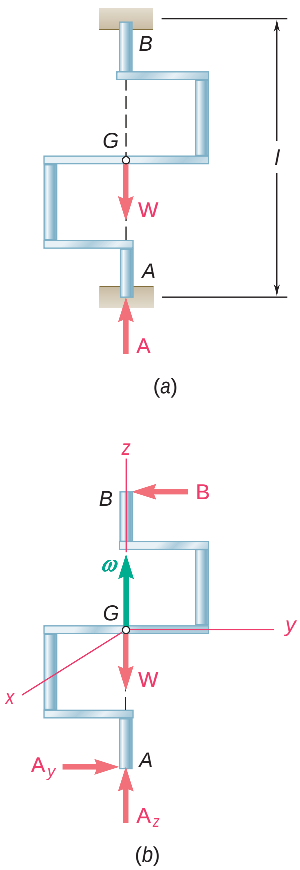
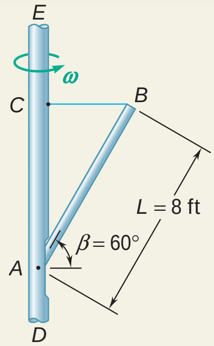


Fig. 18.14



SAMPLE PROBLEM 18.3

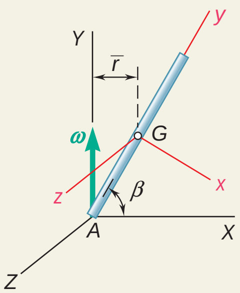
A slender rod AB of length $L = 8$ ft and weight $W = 40$ lb is pinned at A to a vertical axle DE which rotates with a constant angular velocity ω of 15 rad/s. The rod is maintained in position by means of a horizontal wire BC attached to the axle and to the end B of the rod. Determine the tension in the wire and the reaction at A.

SOLUTION

The effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G and the couple $\dot{\mathbf{H}}_G$. Since G describes a horizontal circle of radius $\bar{r} = \frac{1}{2}L \cos \beta$ at the constant rate ω, we have

$$\bar{\mathbf{a}} = \mathbf{a}_n = -\bar{r}\omega^2\mathbf{I} = -(\tfrac{1}{2}L \cos \beta)\omega^2\mathbf{I} = -(450 \text{ ft/s}^2)\mathbf{I}$$

$$m\bar{\mathbf{a}} = \frac{40}{g}(-450\mathbf{I}) = -(559 \text{ lb})\mathbf{I}$$

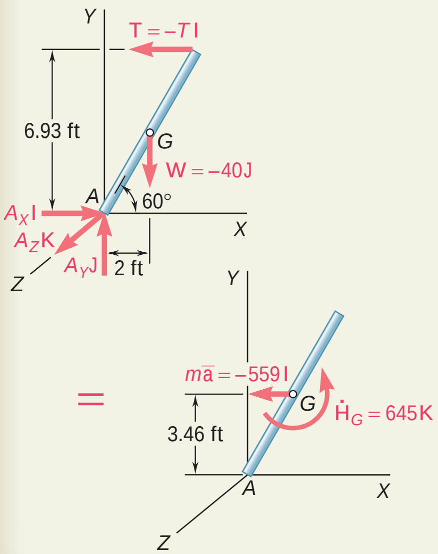


Determination of $\dot{\mathbf{H}}_G$. We first compute the angular momentum $\mathbf{H}_G$. Using the principal centroidal axes of inertia x, y, z, we write

$$\bar{I}_x = \tfrac{1}{12}mL^2 \qquad \bar{I}_y = 0 \qquad \bar{I}_z = \tfrac{1}{12}mL^2$$
$$\omega_x = -\omega \cos \beta \qquad \omega_y = \omega \sin \beta \qquad \omega_z = 0$$
$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$
$$\mathbf{H}_G = -\tfrac{1}{12}mL^2\omega \cos \beta\, \mathbf{i}$$

The rate of change $\dot{\mathbf{H}}_G$ of $\mathbf{H}_G$ with respect to axes of fixed orientation is obtained from Eq. (18.22). Observing that the rate of change $(\dot{\mathbf{H}}_G)_{Gxyz}$ of $\mathbf{H}_G$ with respect to the rotating frame $Gxyz$ is zero, and that the angular velocity $\boldsymbol{\Omega}$ of that frame is equal to the angular velocity $\boldsymbol{\omega}$ of the rod, we have

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\omega} \times \mathbf{H}_G$$
$$\dot{\mathbf{H}}_G = 0 + (-\omega \cos \beta\, \mathbf{i} + \omega \sin \beta\, \mathbf{j}) \times (-\tfrac{1}{12}mL^2\omega \cos \beta\, \mathbf{i})$$
$$\dot{\mathbf{H}}_G = \tfrac{1}{12}mL^2\omega^2 \sin \beta \cos \beta\, \mathbf{k} = (645 \text{ lb} \cdot \text{ft})\,\mathbf{k}$$



Equations of Motion. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write

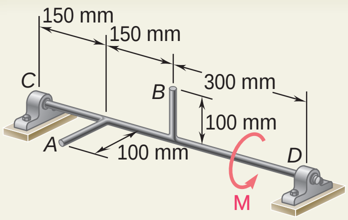
$\Sigma\mathbf{M}_A = \Sigma(\mathbf{M}_A)_{\text{eff}}$:

$$6.93\mathbf{J} \times (-T\mathbf{I}) + 2\mathbf{I} \times (-40\mathbf{J}) = 3.46\mathbf{J} \times (-559\mathbf{I}) + 645\mathbf{K}$$
$$(6.93T - 80)\mathbf{K} = (1934 + 645)\mathbf{K} \qquad T = 384 \text{ lb} \;\blacktriangleleft$$

$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}$: $\quad A_X\mathbf{I} + A_Y\mathbf{J} + A_Z\mathbf{K} - 384\mathbf{I} - 40\mathbf{J} = -559\mathbf{I}$

$$\mathbf{A} = -(175 \text{ lb})\mathbf{I} + (40 \text{ lb})\mathbf{J} \;\blacktriangleleft$$

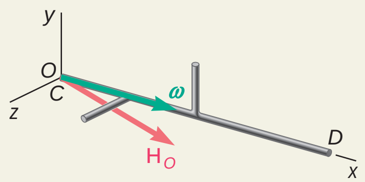
Remark. The value of T could have been obtained from $\mathbf{H}_A$ and Eq. (18.28). However, the method used here also yields the reaction at A. Moreover, it draws attention to the effect of the asymmetry of the rod on the solution of the problem by clearly showing that both the vector $m\bar{\mathbf{a}}$ and the couple $\dot{\mathbf{H}}_G$ must be used to represent the effective forces.



SAMPLE PROBLEM 18.4

Two 100-mm rods A and B, each of mass 300 g, are welded to shaft CD which is supported by bearings at C and D. If a couple $\mathbf{M}$ of magnitude equal to $6\text{ N}\cdot\text{m}$ is applied to the shaft, determine the components of the dynamic reactions at C and D at the instant when the shaft has reached an angular velocity at 1200 rpm. Neglect the moment of inertia of the shaft itself.

SOLUTION

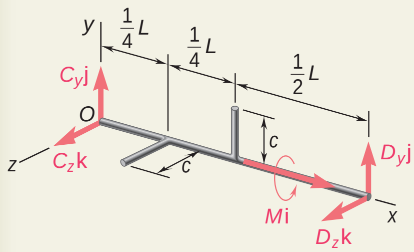


Angular Momentum About O. We attach to the body the frame of reference $Oxyz$ and note that the axes chosen are not principal axes of inertia for the body. Since the body rotates about the x axis, we have $\omega_x = \omega$ and $\omega_y = \omega_z = 0$. Substituting into Eqs. (18.13),

$$H_x = I_x\omega \qquad H_y = -I_{xy}\omega \qquad H_z = -I_{xz}\omega$$
$$\mathbf{H}_O = (I_x\mathbf{i} - I_{xy}\mathbf{j} - I_{xz}\mathbf{k})\omega$$

Moments of the External Forces About O. Since the frame of reference rotates with the angular velocity $\boldsymbol{\omega}$, Eq. (18.28) yields

$$\begin{aligned}\Sigma\mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\omega} \times \mathbf{H}_O \\ &= (I_x\mathbf{i} - I_{xy}\mathbf{j} - I_{xz}\mathbf{k})\alpha + \omega\mathbf{i} \times (I_x\mathbf{i} - I_{xy}\mathbf{j} - I_{xz}\mathbf{k})\omega \\ &= I_x\alpha\mathbf{i} - (I_{xy}\alpha - I_{xz}\omega^2)\mathbf{j} - (I_{xz}\alpha + I_{xy}\omega^2)\mathbf{k} \qquad (1)\end{aligned}$$



Dynamic Reaction at D. The external forces consist of the weights of the shaft and rods, the couple $\mathbf{M}$, the static reactions at C and D, and the dynamic reactions at C and D. Since the weights and static reactions are balanced, the external forces reduce to the couple $\mathbf{M}$ and the dynamic reactions $\mathbf{C}$ and $\mathbf{D}$ as shown in the figure. Taking moments about O, we have

$$\Sigma\mathbf{M}_O = L\mathbf{i} \times (D_y\mathbf{j} + D_z\mathbf{k}) + M\mathbf{i} = M\mathbf{i} - D_zL\mathbf{j} + D_yL\mathbf{k} \qquad (2)$$

Equating the coefficients of the unit vector $\mathbf{i}$ in (1) and (2),

$$M = I_x\alpha \qquad M = 2(\tfrac{1}{3}mc^2)\alpha \qquad \alpha = 3M/2mc^2$$

Equating the coefficients of $\mathbf{k}$ and $\mathbf{j}$ in (1) and (2):

$$D_y = -(I_{xz}\alpha + I_{xy}\omega^2)/L \qquad D_z = (I_{xy}\alpha - I_{xz}\omega^2)/L \qquad (3)$$

Using the parallel-axis theorem, and noting that the product of inertia of each rod is zero with respect to centroidal axes, we have

$$I_{xy} = \Sigma m\bar{x}\bar{y} = m(\tfrac{1}{2}L)(\tfrac{1}{2}c) = \tfrac{1}{4}mLc$$
$$I_{xz} = \Sigma m\bar{x}\bar{z} = m(\tfrac{1}{4}L)(\tfrac{1}{2}c) = \tfrac{1}{8}mLc$$

Substituting into (3) the values found for I_{xy}, I_{xz}, and α:

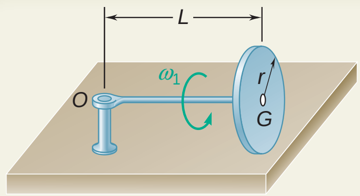
$$D_y = -\tfrac{3}{16}(M/c) - \tfrac{1}{4}mc\omega^2 \qquad D_z = \tfrac{3}{8}(M/c) - \tfrac{1}{8}mc\omega^2$$

Substituting $\omega = 1200\text{ rpm} = 125.7\text{ rad/s}$, $c = 0.100\text{ m}$, $M = 6\text{ N}\cdot\text{m}$, and $m = 0.300\text{ kg}$, we have

$$D_y = -129.8\text{ N} \qquad D_z = -36.8\text{ N} \quad ◀$$

Dynamic Reaction at C. Using a frame of reference attached at D, we obtain equations similar to Eqs. (3), which yield

$$C_y = -152.2\text{ N} \qquad C_z = -155.2\text{ N} \quad ◀$$



SAMPLE PROBLEM 18.5

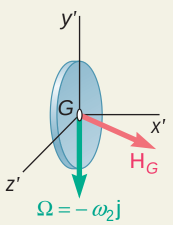
A homogeneous disk of radius r and mass m is mounted on an axle OG of length L and negligible mass. The axle is pivoted at the fixed point O and the disk is constrained to roll on a horizontal floor. Knowing that the disk rotates counterclockwise at the constant rate ω_1 about the axle, determine (*a*) the force (assumed vertical) exerted by the floor on the disk, (*b*) the reaction at the pivot O.

SOLUTION

The effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G and the couple $\dot{\mathbf{H}}_G$. Recalling from Sample Prob. 18.2 that the axle rotates about the y axis at the rate $\omega_2 = r\omega_1/L$, we write

$$m\bar{\mathbf{a}} = -mL\omega_2^2\mathbf{i} = -mL(r\omega_1/L)^2\mathbf{i} = -(mr^2\omega_1^2/L)\mathbf{i} \qquad (1)$$

Determination of $\dot{\mathbf{H}}_G$. We recall from Sample Prob. 18.2 that the angular momentum of the disk about G is



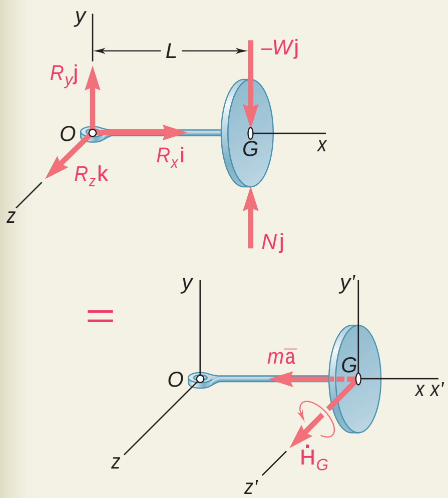
$$\mathbf{H}_G = \tfrac{1}{2}mr^2\omega_1\left(\mathbf{i} - \frac{r}{2L}\mathbf{j}\right)$$

where $\mathbf{H}_G$ is resolved into components along the rotating axes x', y', z', with x' along OG and y' vertical. The rate of change $\dot{\mathbf{H}}_G$ of $\mathbf{H}_G$ with respect to axes of fixed orientation is obtained from Eq. (18.22). Noting that the rate of change $(\dot{\mathbf{H}}_G)_{Gx'y'z'}$ of $\mathbf{H}_G$ with respect to the rotating frame is zero, and that the angular velocity $\boldsymbol{\Omega}$ of that frame is

$$\boldsymbol{\Omega} = -\omega_2\mathbf{j} = -\frac{r\omega_1}{L}\mathbf{j}$$

we have

$$\begin{aligned}
\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gx'y'z'} + \boldsymbol{\Omega} \times \mathbf{H}_G \\
&= 0 - \frac{r\omega_1}{L}\mathbf{j} \times \tfrac{1}{2}mr^2\omega_1\left(\mathbf{i} - \frac{r}{2L}\mathbf{j}\right) \\
&= \tfrac{1}{2}mr^2(r/L)\omega_1^2\mathbf{k} && (2)
\end{aligned}$$



Equations of Motion. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write

$\Sigma\mathbf{M}_O = \Sigma(\mathbf{M}_O)_{\text{eff}}$: $\qquad L\mathbf{i} \times (N\mathbf{j} - W\mathbf{j}) = \dot{\mathbf{H}}_G$

$$(N - W)L\mathbf{k} = \tfrac{1}{2}mr^2(r/L)\omega_1^2\mathbf{k}$$

$$N = W + \tfrac{1}{2}mr(r/L)^2\omega_1^2 \qquad \mathbf{N} = [W + \tfrac{1}{2}mr(r/L)^2\omega_1^2]\mathbf{j} \quad (3) \ \blacktriangleleft$$

$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}$: $\qquad \mathbf{R} + N\mathbf{j} - W\mathbf{j} = m\bar{\mathbf{a}}$

Substituting for N from (3), for $m\bar{\mathbf{a}}$ from (1), and solving for $\mathbf{R}$, we have

$$\mathbf{R} = -(mr^2\omega_1^2/L)\mathbf{i} - \tfrac{1}{2}mr(r/L)^2\omega_1^2\mathbf{j}$$

$$\mathbf{R} = -\frac{mr^2\omega_1^2}{L}\left(\mathbf{i} + \frac{r}{2L}\mathbf{j}\right) \ \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you will be asked to solve problems involving the *three-dimensional motion of rigid bodies*. The method you will use is basically the same that you used in Chap. 16 in your study of the plane motion of rigid bodies. You will draw a free-body-diagram equation showing that the system of the external forces is equivalent to the system of the effective forces, and you will equate sums of components and sums of moments on both sides of this equation. Now, however, the system of the effective forces will be represented by the vector $m\bar{\mathbf{a}}$ and a couple vector $\dot{\mathbf{H}}_G$, the determination of which will be explained in paragraphs 1 and 2 below.

To solve a problem involving the three-dimensional motion of a rigid body, you should take the following steps:

1. Determine the angular momentum $\mathbf{H}_G$ of the body about its mass center G from its angular velocity V with respect to a frame of reference $GX'Y'Z'$ of fixed orientation. This is an operation you learned to perform in the preceding lesson. However, since the configuration of the body will be changing with time, it will now be necessary for you to use an auxiliary system of axes $Gx'y'z'$ (Fig. 18.9) to compute the components of V and the moments and products of inertia of the body. These axes may be rigidly attached to the body, in which case their angular velocity is equal to V [Sample Probs. 18.3 and 18.4], or they may have an angular velocity $\boldsymbol{\Omega}$ of their own [Sample Prob. 18.5].

Recall the following from the preceding lesson:

a. If the principal axes of inertia of the body at G are known, use these axes as coordinate axes x', y', and z', since the corresponding products of inertia of the body will be equal to zero. (Note that if the body is axisymmetric, these axes do not need to be rigidly attached to the body.) Resolve V into components $\mathsf{V}_{x'}$, $\mathsf{V}_{y'}$, and $\mathsf{V}_{z'}$ along these axes and compute the principal moments of inertia $\bar{I}_{x'}$, $\bar{I}_{y'}$, and $\bar{I}_{z'}$. The corresponding components of the angular momentum $\mathbf{H}_G$ are

$$H_{x'} = \bar{I}_{x'}\mathsf{V}_{x'} \qquad H_{y'} = \bar{I}_{y'}\mathsf{V}_{y'} \qquad H_{z'} = \bar{I}_{z'}\mathsf{V}_{z'} \tag{18.10}$$

b. If the principal axes of inertia of the body at G are not known, you must use Eqs. (18.7) to determine the components of the angular momentum $\mathbf{H}_G$. These equations require your prior computation of the *products of inertia* of the body, as well as of its moments of inertia, with respect to the selected axes.

(continued)

2. Compute the rate of change $\dot{\mathrm{H}}_G$ of the angular momentum H_G with respect to the frame *GX′Y′Z′*. Note that this frame has a *fixed orientation,* while the frame $Gx'y'z'$ you used when you calculated the components of the vector V was a *rotating frame*. We refer you to our discussion in Sec. 15.10 of the rate of change of a vector with respect to a rotating frame. Recalling Eq. (15.31), you will express the rate of change $\dot{\mathbf{H}}_G$ as follows:

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gx'y'z'} + \mathbf{\Omega} \times \mathbf{H}_G \tag{18.22}$$

The first term in the right-hand member of Eq. (18.22) represents the rate of change of $\mathbf{H}_G$ with respect to the rotating frame $Gx'y'z'$. This term will drop out if V—and, thus, $\mathbf{H}_G$—remain constant in both magnitude and direction when viewed from that frame. On the other hand, if any of the time derivatives $\dot{\mathsf{v}}_{x'}$, $\dot{\mathsf{v}}_{y'}$, and $\dot{\mathsf{v}}_{z'}$ is different from zero, $(\dot{\mathbf{H}}_G)_{Gx'y'z'}$ will also be different from zero, and its components should be determined by differentiating Eqs. (18.10) with respect to t. Finally, we remind you that if the rotating frame is rigidly attached to the body, its angular velocity will be the same as that of the body, and $\mathbf{\Omega}$ can be replaced by V.

3. Draw the free-body-diagram equation for the rigid body, showing that the system of the external forces exerted on the body is equivalent to the vector $m\bar{\mathbf{a}}$ applied at G and the couple vector $\dot{\mathbf{H}}_G$ (Fig. 18.11). By equating components in any direction and moments about any point, you can write as many as six independent scalar equations of motion [Sample Probs. 18.3 and 18.5].

4. When solving problems involving the motion of a rigid body about a fixed point *O*, you may find it convenient to use the following equation, derived in Sec. 18.7, which eliminates the components of the reaction at the support O,

$$\Sigma\mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_O \tag{18.28}$$

where the first term in the right-hand member represents the rate of change of $\mathbf{H}_O$ with respect to the rotating frame $Oxyz$, and where $\mathbf{\Omega}$ is the angular velocity of that frame.

5. When determining the reactions at the bearings of a rotating shaft, use Eq. (18.28) and take the following steps:

a. Place the fixed point *O* at one of the two bearings supporting the shaft and attach the rotating frame $Oxyz$ to the shaft, with one of the axes directed along it. Assuming, for instance, that the x axis has been aligned with the shaft, you will have $\mathbf{\Omega} = \mathsf{V} = \mathsf{v}\mathbf{i}$ [Sample Prob. 18.4].

b. Since the selected axes, usually, will not be the principal axes of inertia at *O*, you must compute the *products of inertia* of the shaft, as well as its moments of inertia, with respect to these axes, and use Eqs. (18.13) to determine $\mathbf{H}_O$. Assuming again that the x axis has been aligned with the shaft, Eqs. (18.13) reduce to

$$H_x = I_x\omega \qquad H_y = -I_{yx}\omega \qquad H_z = -I_{zx}\omega \tag{18.13$'$}$$

which shows that $\mathbf{H}_O$ *will not be directed along the shaft*.

c. To obtain $\dot{\mathbf{H}}_O$, substitute the expressions obtained into Eq. (18.28), and let $\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$. If the angular velocity of the shaft is constant, the first term in the right-hand member of the equation will drop out. However, if the shaft has an angular acceleration $\boldsymbol{\alpha} = \alpha\mathbf{i}$, the first term will not be zero and must be determined by differentiating with respect to t the expressions in (18.13′). The result will be equations similar to Eqs. (18.13′), with ω replaced by α.

d. Since point *O* coincides with one of the bearings, the three scalar equations corresponding to Eq. (18.28) can be solved for the components of the dynamic reaction at the other bearing. If the mass center G of the shaft is located on the line joining the two bearings, the effective force $m\bar{\mathbf{a}}$ will be zero. Drawing the free-body-diagram equation of the shaft, you will then observe that the components of the dynamic reaction at the first bearing must be equal and opposite to those you have just determined. If G is not located on the line joining the two bearings, you can determine the reaction at the first bearing by placing the fixed point O at the second bearing and repeating the earlier procedure [Sample Prob. 18.4]; or you can obtain additional equations of motion from the free-body-diagram equation of the shaft, making sure to first determine and include the effective force $m\bar{\mathbf{a}}$ applied at G.

e. Most problems call for the determination of the "dynamic reactions" at the bearings, that is, for the *additional forces* exerted by the bearings on the shaft when the shaft is rotating. When determining dynamic reactions, ignore the effect of static loads, such as the weight of the shaft.

PROBLEMS

18.55 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.1.

18.56 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the plate of Prob. 18.2.

18.57 Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Prob. 18.3.

18.58 Determine the rate of change $\dot{\mathbf{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the disk of Prob. 18.4.

18.59 Determine the rate of change $\dot{\mathbf{H}}_C$ of the angular momentum $\mathbf{H}_C$ of the disk of Prob. 18.5.

18.60 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.8.

18.61 Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Prob. 18.3, assuming that at the instant considered the assembly has an angular velocity V = (12 rad/s)$\mathbf{i}$ and an angular acceleration A = −(96 rad/s^2)$\mathbf{i}$.

18.62 Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Prob. 18.3, assuming that at the instant considered the assembly has an angular velocity V = (12 rad/s)$\mathbf{i}$ and an angular acceleration A = (96 rad/s^2)$\mathbf{i}$.

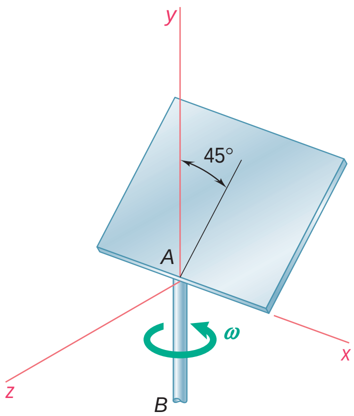


Fig. *P18.63*

18.63 A thin, homogeneous square of mass m and side a is welded to a vertical shaft AB with which it forms an angle of 45°. Knowing that the shaft rotates with an angular velocity V = v$\mathbf{j}$ and an angular acceleration A = a$\mathbf{j}$, determine the rate of change $\dot{\mathbf{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the plate assembly.

18.64 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.8, assuming that at the instant considered the assembly has an angular velocity V = v$\mathbf{j}$ and an angular acceleration A = a$\mathbf{j}$.

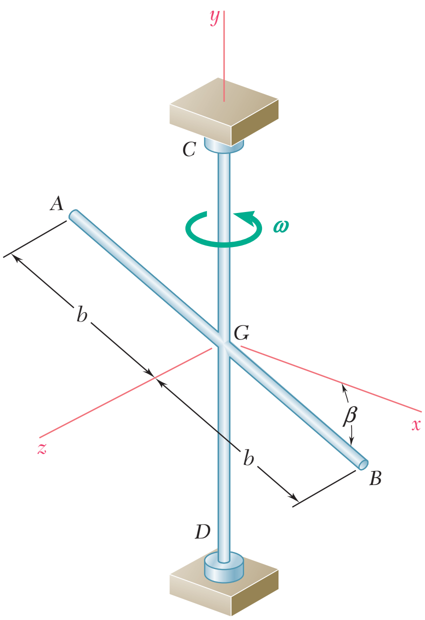


Fig. P18.65

18.65 A slender, uniform rod AB of mass m and a vertical shaft CD, each of length $2b$, are welded together at their midpoints G. Knowing that the shaft rotates at the constant rate v, determine the dynamic reactions at C and D.

18.66 A thin, homogeneous triangular plate of weight 10 lb is welded to a light, vertical axle supported by bearings at A and B. Knowing that the plate rotates at the constant rate $\omega = 8$ rad/s, determine the dynamic reactions at A and B.

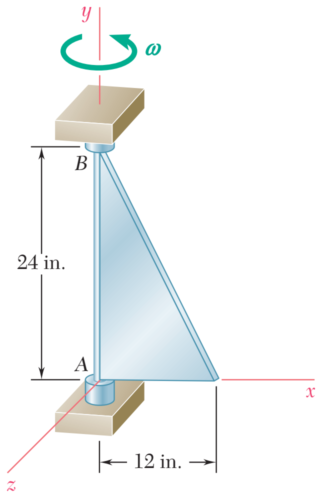


Fig. P18.66

18.67 The assembly shown consists of pieces of sheet aluminum of uniform thickness and of total weight 2.7 lb welded to a light axle supported by bearings at A and B. Knowing that the assembly rotates at the constant rate $\omega = 240$ rpm, determine the dynamic reactions at A and B.

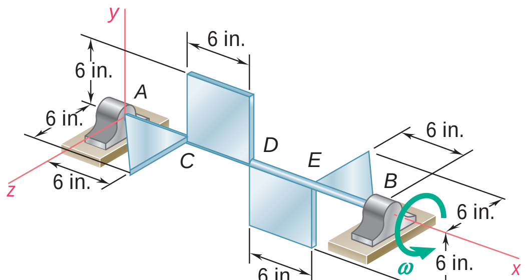


Fig. P18.67

18.68 The 8-kg shaft shown has a uniform cross section. Knowing that the shaft rotates at the constant rate $\omega = 12$ rad/s, determine the dynamic reactions at A and B.

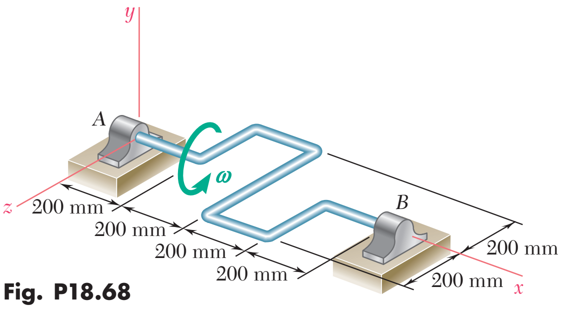


Fig. P18.68

18.69 After attaching the 18-kg wheel shown to a balancing machine and making it spin at the rate of 15 rev/s, a mechanic has found that to balance the wheel both statically and dynamically, he should use two corrective masses, a 170-g mass placed at B and a 56-g mass placed at D. Using a right-handed frame of reference rotating with the wheel (with the z axis perpendicular to the plane of the figure), determine before the corrective masses have been attached (a) the distance from the axis of rotation to the mass center of the wheel and the products of inertia I_{xy} and I_{zx}, (b) the force-couple system at C equivalent to the forces exerted by the wheel on the machine.

18.70 When the 18-kg wheel shown is attached to a balancing machine and made to spin at a rate of 12.5 rev/s, it is found that the forces exerted by the wheel on the machine are equivalent to a force-couple system consisting of a force $\mathbf{F} = (160 \text{ N})\mathbf{j}$ applied at C and a couple $\mathbf{M}_C = (14.7 \text{ N} \cdot \text{m})\mathbf{k}$, where the unit vectors form a triad which rotates with the wheel. (a) Determine the distance from the axis of rotation to the mass center of the wheel and the products of inertia I_{xy} and I_{zx}. (b) If only two corrective masses are to be used to balance the wheel statically and dynamically, what should these masses be and at which of the points A, B, D, or E should they be placed?

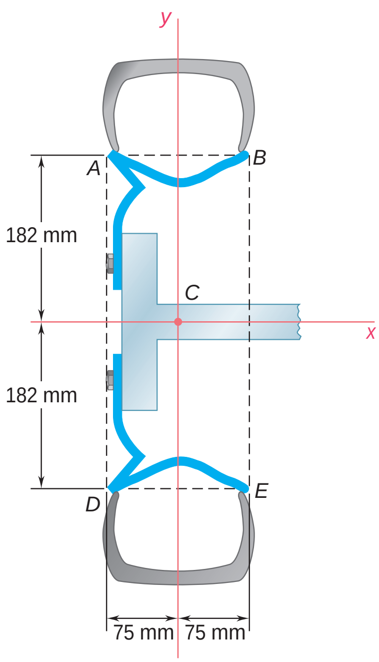


Fig. *P18.69* and *P18.70*

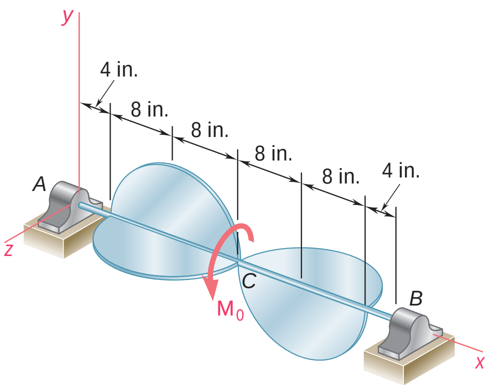


Fig. P18.75

18.71 Knowing that the assembly of Prob. 18.65 is initially at rest ($\mathsf{v} = 0$) when a couple of moment $\mathbf{M}_0 = M_0\mathbf{j}$ is applied to shaft CD, determine (*a*) the resulting angular acceleration of the assembly, (*b*) the dynamic reactions at C and D immediately after the couple is applied.

18.72 Knowing that the plate of Prob. 18.66 is initially at rest ($\mathsf{v} = 0$) when a couple of moment $\mathbf{M}_0 = (0.75 \text{ ft} \cdot \text{lb})\mathbf{j}$ is applied to it, determine (*a*) the resulting angular acceleration of the plate, (*b*) the dynamic reactions A and B immediately after the couple has been applied.

18.73 The assembly of Prob. 18.67 is initially at rest ($\mathsf{v} = 0$) when a couple $\mathbf{M}_0$ is applied to axle AB. Knowing that the resulting angular acceleration of the assembly is $\mathsf{A} = (150 \text{ rad/s}^2)\mathbf{i}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at A and B immediately after the couple is applied.

18.74 The shaft of Prob. 18.68 is initially at rest ($\mathsf{v} = 0$) when a couple $\mathbf{M}_0$ is applied to it. Knowing that the resulting angular acceleration of the shaft is $\mathsf{A} = (20 \text{ rad/s}^2)\mathbf{i}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at A and B immediately after the couple is applied.

18.75 The assembly shown weighs 12 lb and consists of 4 thin 16-in.-diameter semicircular aluminum plates welded to a light 40-in.-long shaft AB. The assembly is at rest ($\mathsf{v} = 0$) at time $t = 0$ when a couple $\mathbf{M}_0$ is applied to it as shown, causing the assembly to complete one full revolution in 2 s. Determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at A and B at $t = 0$.

18.76 For the assembly of Prob. 18.75, determine the dynamic reactions at A and B at $t = 2$ s.

18.77 The sheet-metal component shown is of uniform thickness and has a mass of 600 g. It is attached to a light axle supported by bearings at A and B located 150 mm apart. The component is at rest when it is subjected to a couple $\mathbf{M}_0$ as shown. If the resulting angular acceleration is $\mathsf{A} = (12 \text{ rad/s}^2)\mathbf{k}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions A and B immediately after the couple has been applied.

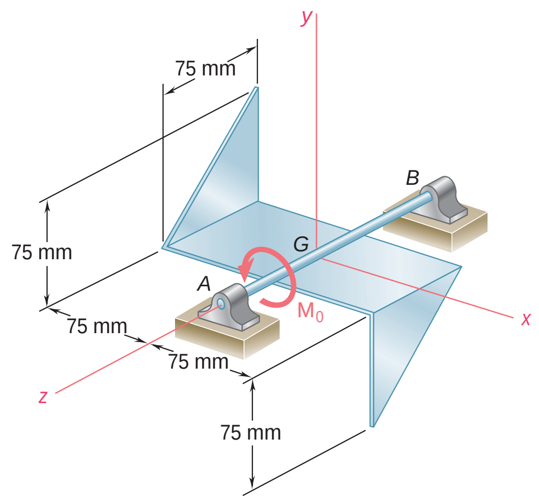


Fig. *P18.77*

18.78 For the sheet-metal component of Prob. 18.77, determine (*a*) the angular velocity of the component 0.6 s after the couple $\mathbf{M}_0$ has been applied to it, (*b*) the magnitude of the dynamic reactions at A and B at that time.

18.79 The blade of an oscillating fan and the rotor of its motor have a total mass of 300 g and a combined radius of gyration of 75 mm. They are supported by bearings at A and B, 125 mm apart, and rotate at the rate $\omega_1 = 1800$ rpm. Determine the dynamic reactions at A and B when the motor casing has an angular velocity $\boldsymbol{\omega}_2 = (0.6 \text{ rad/s})\mathbf{j}$.

18.80 The blade of a portable saw and the rotor of its motor have a total weight of 2.5 lb and a combined radius of gyration of 1.5 in. Knowing that the blade rotates as shown at the rate $\omega_1 = 1500$ rpm, determine the magnitude and direction of the couple **M** that a worker must exert on the handle of the saw to rotate it with a constant angular velocity $\boldsymbol{\omega}_2 = -(2.4 \text{ rad/s})\mathbf{j}$.

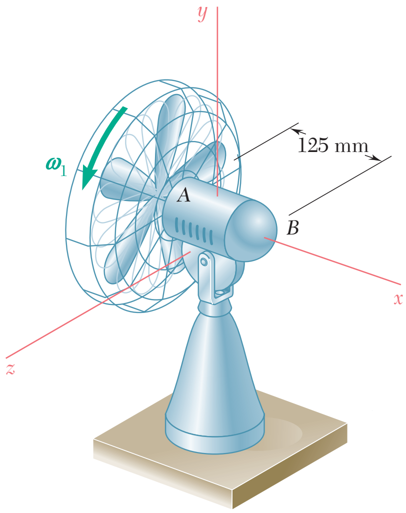


Fig. P18.79

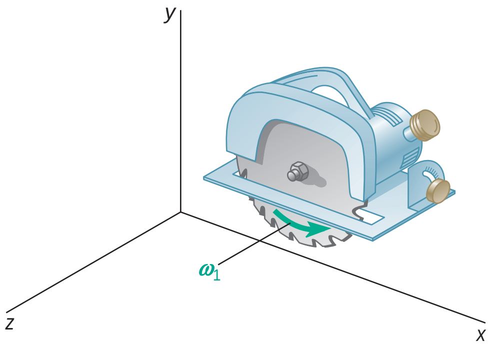


Fig. *P18.80*

18.81 The flywheel of an automobile engine, which is rigidly attached to the crankshaft, is equivalent to a 400-mm-diameter, 15-mm-thick steel plate. Determine the magnitude of the couple exerted by the flywheel on the horizontal crankshaft as the automobile travels around an unbanked curve of 200-m radius at a speed of 90 km/h, with the flywheel rotating at 2700 rpm. Assume the automobile to have (*a*) a rear-wheel drive with the engine mounted longitudinally, (*b*) a front-wheel drive with the engine mounted transversely. (Density of steel = 7860 kg/m^3.)

18.82 Each wheel of an automobile has a mass of 22 kg, a diameter of 575 mm, and a radius of gyration of 225 mm. The automobile travels around an unbanked curve of radius 150 m at a speed of 95 km/h. Knowing that the transverse distance between the wheels is 1.5 m, determine the additional normal force exerted by the ground on each outside wheel due to the motion of the car.

18.83 The uniform, thin 5-lb disk spins at a constant rate $\omega_2 = 6$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 3$ rad/s. Determine the couple which represents the dynamic reaction at the support A.

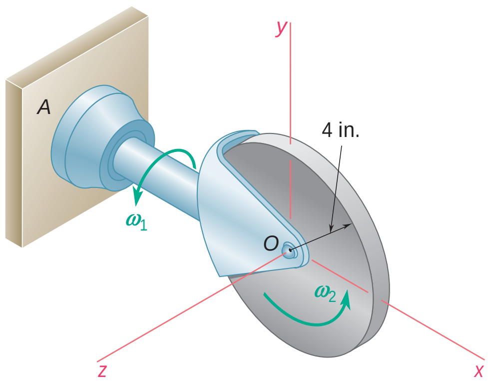


Fig. *P18.83*

18.84 The essential structure of a certain type of aircraft turn indicator is shown. Each spring has a constant of 500 N/m, and the 200-g uniform disk of 40-mm radius spins at the rate of 10 000 rpm. The springs are stretched and exert equal vertical forces on yoke AB when the airplane is traveling in a straight path. Determine the angle through which the yoke will rotate when the pilot executes a horizontal turn of 750-m radius to the right at a speed of 800 km/h. Indicate whether point A will move up or down.

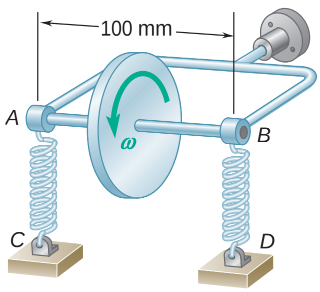


Fig. P18.84

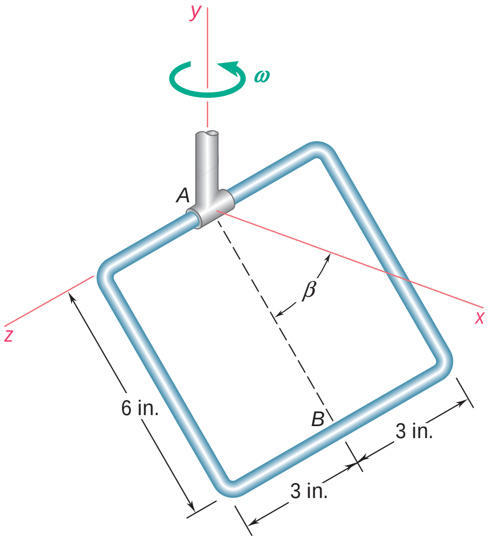


Fig. P18.85

18.85 A slender rod is bent to form a square frame of side 6 in. The frame is attached by a collar at A to a vertical shaft which rotates with a constant angular velocity V. Determine the value of v for which line AB forms an angle b = 48° with the horizontal x axis.

18.86 A uniform semicircular plate of radius 120 mm is hinged at A and B to a clevis which rotates with a constant angular velocity V about a vertical axis. Determine (a) the angle b that the plate forms with the horizontal x axis when v = 15 rad/s, (b) the largest value of v for which the plate remains vertical (b = 90°).

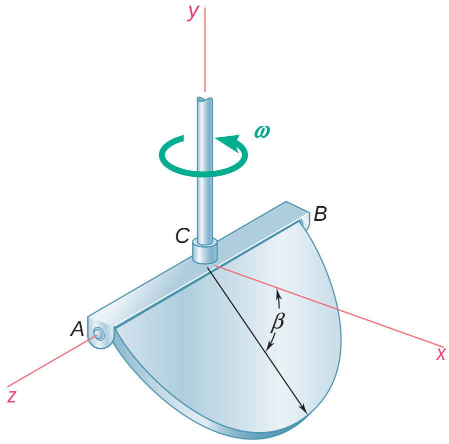


Fig. P18.86 and P18.87

18.87 A uniform semicircular plate of radius 120 mm is hinged at A and B to a clevis which rotates with a constant angular velocity V about a vertical axis. Determine the value of v for which the plate forms an angle b = 50° with the horizontal x axis.

18.88 and *18.89* The slender rod AB is attached by a clevis to arm BCD which rotates with a constant angular velocity V about the centerline of its vertical portion CD. Determine the magnitude of the angular velocity V.

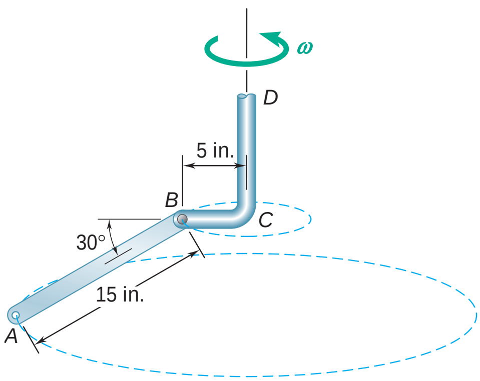


Fig. P18.88

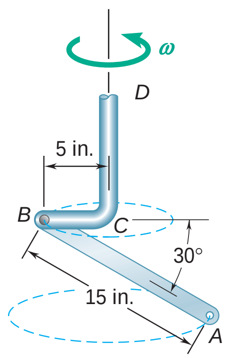


Fig. *P18.89*

18.90 The 950-g gear A is constrained to roll on the fixed gear B, but is free to rotate about axle AD. Axle AD, of length 400 mm and negligible mass, is connected by a clevis to the vertical shaft DE which rotates as shown with a constant angular velocity V_1. Assuming that gear A can be approximated by a thin disk of radius 80 mm, determine the largest allowable value of v_1 if gear A is not to lose contact with gear B.

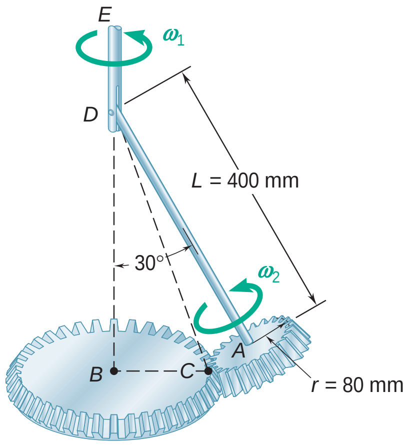


Fig. P18.90

18.91 Determine the force **F** exerted by gear B on gear A of Prob. 18.90 when shaft DE rotates with the constant angular velocity $\mathrm{V}_1 =$ 4 rad/s. (*Hint:* The force **F** must be perpendicular to the line drawn from D to C.)

18.92 The essential structure of a certain type of aircraft turn indicator is shown. Springs AC and BD are initially stretched and exert equal vertical forces at A and B when the airplane is traveling in a straight path. Each spring has a constant of 600 N/m and the uniform disk has a mass of 250 g and spins at the rate of 12 000 rpm. Determine the angle through which the yoke will rotate when the pilot executes a horizontal turn of 800-m radius to the right at a speed of 720 km/h. Indicate whether point A will move up or down.

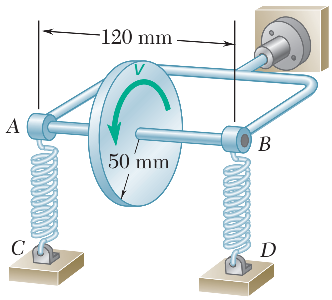


Fig. *P18.92*

18.93 The 10-oz disk shown spins at the rate $\mathrm{v}_1 = 750$ rpm, while axle AB rotates as shown with an angular velocity V_2 of 6 rad/s. Determine the dynamic reactions at A and B.

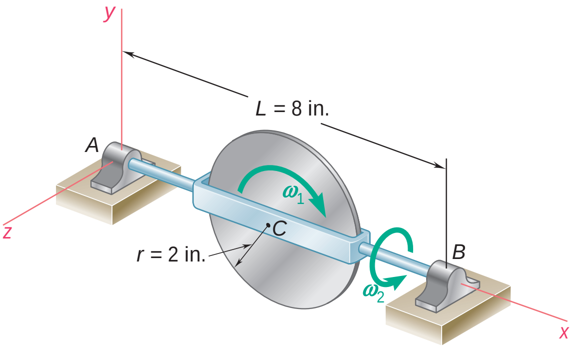


Fig. P18.93 and P18.94

18.94 The 10-oz disk shown spins at the rate $\mathrm{v}_1 = 750$ rpm, while axle AB rotates as shown with an angular velocity V_2. Determine the maximum allowable magnitude of V_2 if the dynamic reactions at A and B are not to exceed 0.25 lb each.

18.95 Two disks, each of mass 5 kg and radius 100 mm, spin as shown at the rate $\mathrm{v}_1 = 1500$ rpm about a rod AB of negligible mass which rotates about a vertical axis at the rate $\mathrm{v}_2 = 45$ rpm. (*a*) Determine the dynamic reactions at C and D. (*b*) Solve part *a* assuming that the direction of spin of disk B is reversed.

18.96 Two disks, each of mass 5 kg and radius 100 mm, spin as shown at the rate $\mathrm{v}_1 = 1500$ rpm about a rod AB of negligible mass which rotates about a vertical axis at a rate v_2. Determine the maximum allowable value of v_2 if the dynamic reactions at C and D are not to exceed 250 N each.

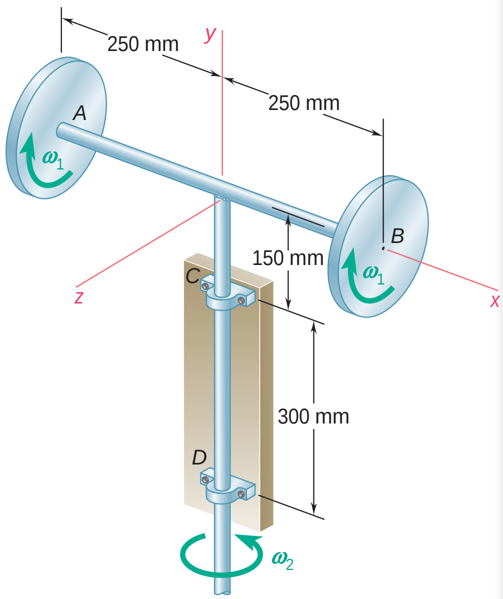


Fig. P18.95 and P18.96

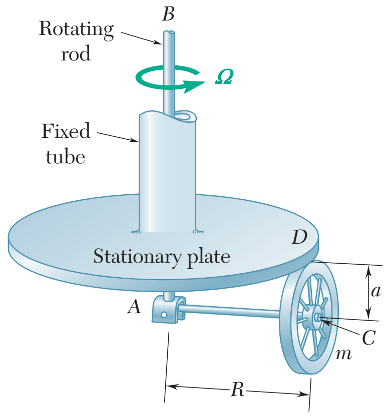


Fig. *P18.97*

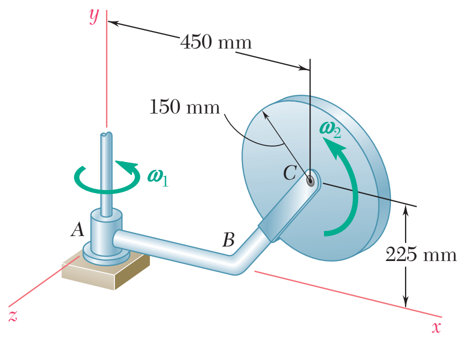


Fig. P18.99

18.97 A stationary horizontal plate is attached to the ceiling by means of a fixed vertical tube. A wheel of radius a and mass m is mounted on a light axle AC which is attached by means of a clevis at A to a rod AB fitted inside the vertical tube. The rod AB is made to rotate with a constant angular velocity Ω causing the wheel to roll on the lower face of the stationary plate. Determine the minimum angular velocity Ω for which contact is maintained between the wheel and the plate. Consider the particular cases (a) when the mass of the wheel is concentrated in the rim, (b) when the wheel is equivalent to a thin disk of radius a.

18.98 Assuming that the wheel of Prob. 18.97 weighs 8 lb, has a radius a = 4 in., and a radius of gyration of 3 in., and that R = 20 in., determine the force exerted by the plate on the wheel when Ω = 25 rad/s.

18.99 A thin disk of mass m = 4 kg rotates with an angular velocity V_2 with respect to arm ABC, which itself rotates with an angular velocity V_1 about the y axis. Knowing that v_1 = 5 rad/s and v_2 = 15 rad/s and that both are constant, determine the force-couple system representing the dynamic reaction at the support at A.

18.100 An experimental Fresnel-lens solar-energy concentrator can rotate about the horizontal axis AB which passes through its mass center G. It is supported at A and B by a steel framework which can rotate about the vertical y axis. The concentrator has a mass of 30 Mg, a radius of gyration of 12 m about its axis of symmetry CD, and a radius of gyration of 10 m about any transverse axis through G. Knowing that the angular velocities V_1 and V_2 have constant magnitudes equal to 0.20 rad/s and 0.25 rad/s, respectively, determine for the position u = 60° (a) the forces exerted on the concentrator at A and B, (b) the couple $M_2\mathbf{k}$ applied to the concentrator at that instant.

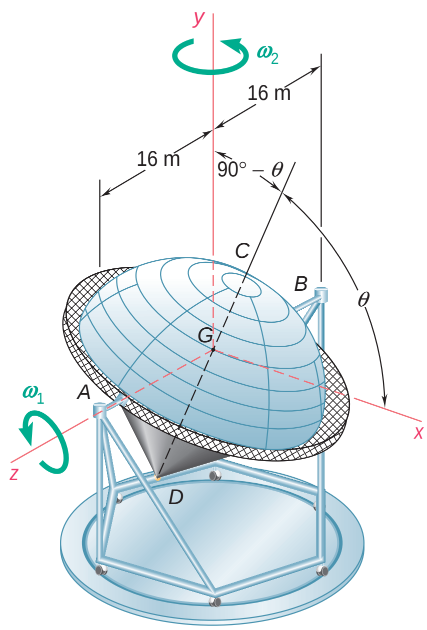


Fig. P18.100

18.101 A 6-lb homogeneous disk of radius 3 in. spins as shown at the constant rate $v_1 = 60$ rad/s. The disk is supported by the fork-ended rod *AB*, which is welded to the vertical shaft *CBD*. The system is at rest when a couple $\mathbf{M}_0 = (0.25 \text{ ft} \cdot \text{lb})\mathbf{j}$ is applied to the shaft for 2 s and then removed. Determine the dynamic reactions at *C* and *D* after the couple has been removed.

18.102 A 6-lb homogeneous disk of radius 3 in. spins as shown at the constant rate $v_1 = 60$ rad/s. The disk is supported by the fork-ended rod *AB*, which is welded to the vertical shaft *CBD*. The system is at rest when a couple $\mathbf{M}_0$ is applied as shown to the shaft for 3 s and then removed. Knowing that the maximum angular velocity reached by the shaft is 18 rad/s, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at *C* and *D* after the couple has been removed.

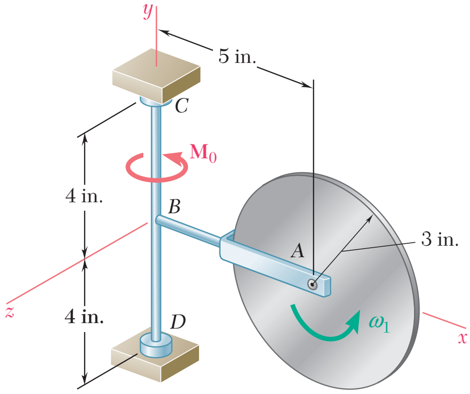


Fig. P18.101 and P18.102

18.103 A 2.5-kg homogeneous disk of radius 80 mm rotates with an angular velocity V_1 with respect to arm *ABC*, which is welded to a shaft *DCE* rotating as shown at the constant rate $v_2 = 12$ rad/s. Friction in the bearing at *A* causes v_1 to decrease at the rate of 15 rad/s^2. Determine the dynamic reactions at *D* and *E* at a time when v_1 has decreased to 50 rad/s.

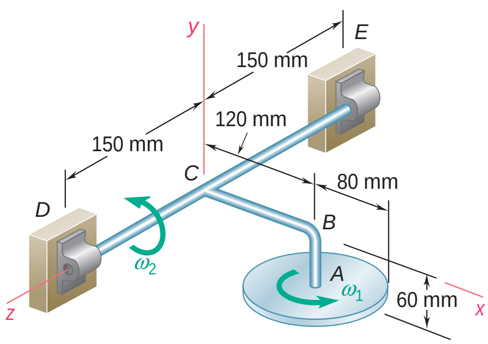


Fig. P18.103 and P18.104

18.104 A 2.5-kg homogeneous disk of radius 80 mm rotates at the constant rate $v_1 = 50$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE*. Knowing that at the instant shown, shaft *DCE* has an angular velocity $V_2 = (12 \text{ rad/s})\mathbf{k}$ and an angular acceleration $A_2 = (8 \text{ rad/s}^2)\mathbf{k}$, determine (*a*) the couple which must be applied to shaft *DCE* to produce that acceleration, (*b*) the corresponding dynamic reactions at *D* and *E*.

18.105 For the disk of Prob. 18.99, determine (*a*) the couple $M_1\mathbf{j}$ which should be applied to arm *ABC* to give it an angular acceleration $A_1 = -(7.5 \text{ rad/s}^2)\mathbf{j}$ when $v_1 = 5$ rad/s, knowing that the disk rotates at the constant rate $v_2 = 15$ rad/s, (*b*) the force-couple system representing the dynamic reaction at *A* at that instant. Assume that *ABC* has a negligible mass.

****18.106*** A slender homogeneous rod *AB* of mass *m* and length *L* is made to rotate at a constant rate v_2 about the horizontal *z* axis, while frame *CD* is made to rotate at the constant rate v_1 about the *y* axis. Express as a function of the angle u (*a*) the couple $\mathbf{M}_1$ required to maintain the rotation of the frame, (*b*) the couple $\mathbf{M}_2$ required to maintain the rotation of the rod, (*c*) the dynamic reactions at the supports *C* and *D*.

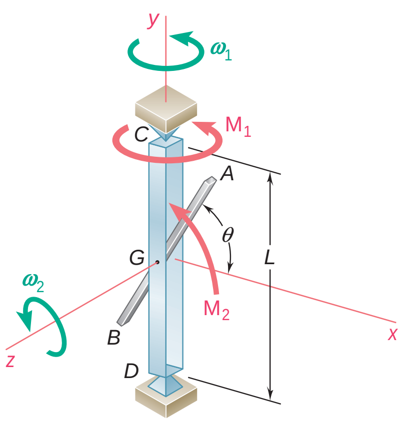


Fig. *P18.106*

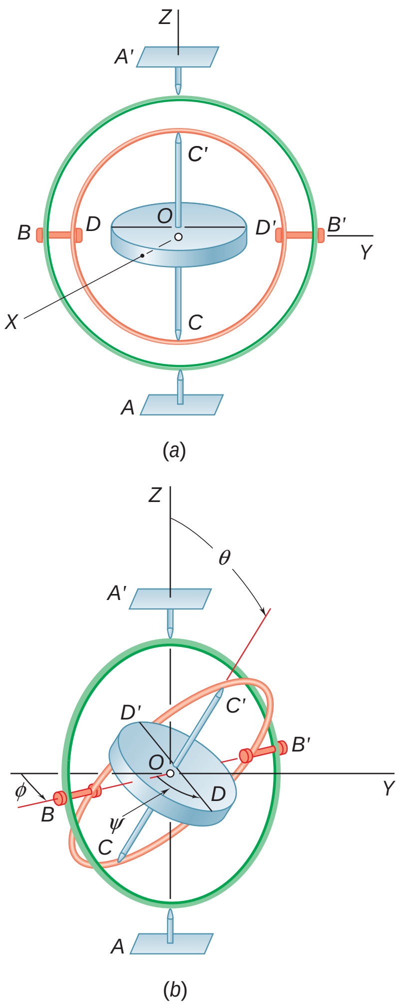


Fig. 18.15

*18.9 MOTION OF A GYROSCOPE. EULERIAN ANGLES

A *gyroscope* consists essentially of a rotor which can spin freely about its geometric axis. When mounted in a Cardan's suspension (Fig. 18.15), a gyroscope can assume any orientation, but its mass center must remain fixed in space. In order to define the position of a gyroscope at a given instant, let us select a fixed frame of reference $OXYZ$, with the origin O located at the mass center of the gyroscope and the Z axis directed along the line defined by the bearings A and A' of the outer gimbal. We will consider a reference position of the gyroscope in which the two gimbals and a given diameter DD' of the rotor are located in the fixed YZ plane (Fig. 18.15*a*). The gyroscope can be brought from this reference position into any arbitrary position (Fig. 18.15*b*) by means of the following steps: (1) a rotation of the outer gimbal through an angle ϕ about the axis AA', (2) a rotation of the inner gimbal through θ about BB', and (3) a rotation of the rotor through ψ about CC'. The angles ϕ, θ, and ψ are called the *Eulerian angles;* they completely characterize the position of the gyroscope at any given instant. Their derivatives $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ define, respectively, the rate of *precession,* the rate of *nutation,* and the rate of *spin* of the gyroscope at the instant considered.

In order to compute the components of the angular velocity and of the angular momentum of the gyroscope, we will use a rotating system of axes $Oxyz$ *attached to the inner gimbal,* with the y axis along BB' and the z axis along CC' (Fig. 18.16). These axes are principal axes of inertia for the gyroscope. While they follow it in its precession and nutation, however, they do not spin; for that reason, they are more convenient to use than axes actually attached to the gyroscope. The angular velocity $\boldsymbol{\omega}$ of the gyroscope with respect to the fixed frame of reference $OXYZ$ will now be expressed as the sum of three partial angular velocities corresponding, respectively, to the precession, the nutation, and the spin of the gyroscope. Denoting by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ the unit vectors along the rotating axes, and by $\mathbf{K}$ the unit vector along the fixed Z axis, we have

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{K} + \dot{\theta}\mathbf{j} + \dot{\psi}\mathbf{k} \tag{18.33}$$

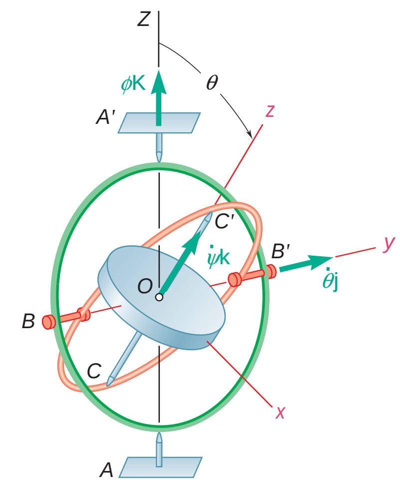


Fig. 18.16

Since the vector components obtained for $\boldsymbol{\omega}$ in (18.33) are not orthogonal (Fig. 18.16), the unit vector $\mathbf{K}$ will be resolved into components along the x and z axes; we write

$$\mathbf{K} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \tag{18.34}$$

and, substituting for $\mathbf{K}$ into (18.33),

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\theta}\mathbf{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k} \tag{18.35}$$

Since the coordinate axes are principal axes of inertia, the components of the angular momentum $\mathbf{H}_O$ can be obtained by multiplying the components of $\boldsymbol{\Omega}$ by the moments of inertia of the rotor about the x, y, and z axes, respectively. Denoting by I the moment of inertia of the rotor about its spin axis, by I' its moment of inertia about a transverse axis through O, and neglecting the mass of the gimbals, we write

$$\mathbf{H}_O = -I'\dot{\phi}\sin\theta\,\mathbf{i} + I'\dot{\theta}\mathbf{j} + I(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k} \tag{18.36}$$

Recalling that the rotating axes are attached to the inner gimbal and thus do not spin, we express their angular velocity as the sum

$$\boldsymbol{\Omega} = \dot{\phi}\mathbf{K} + \dot{\theta}\mathbf{j} \tag{18.37}$$

or, substituting for $\mathbf{K}$ from (18.34),

$$\boldsymbol{\Omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi}\cos\theta\,\mathbf{k} \tag{18.38}$$

Substituting for $\mathbf{H}_O$ and $\boldsymbol{\Omega}$ from (18.36) and (18.38) into the equation

$$\Sigma\mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \tag{18.28}$$

we obtain the three differential equations

$$\begin{aligned}
\Sigma M_x &= -I'(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta) + I\dot{\theta}(\dot{\psi} + \dot{\phi}\cos\theta)\\
\Sigma M_y &= I'(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta) + I\dot{\phi}\sin\theta(\dot{\psi} + \dot{\phi}\cos\theta)\\
\Sigma M_z &= I\frac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta)
\end{aligned} \tag{18.39}$$

The equations (18.39) define the motion of a gyroscope subjected to a given system of forces when the mass of its gimbals is neglected. They can also be used to define the motion of an *axisymmetrical body* (or body of revolution) attached at a point on its axis of symmetry, and to define the motion of an axisymmetrical body about its mass center. While the gimbals of the gyroscope helped us visualize the Eulerian angles, it is clear that these angles can be used to define the position of any rigid body with respect to axes centered at a point of the body, regardless of the way in which the body is actually supported.

Since the equations (18.39) are nonlinear, it will not be possible, in general, to express the Eulerian angles ϕ, θ, and ψ as analytical functions of the time t, and numerical methods of solution may have to be used. However, as you will see in the following sections, there are several particular cases of interest which can be analyzed easily.

Photo 18.5 A gyroscope can be used for measuring orientation and is capable of maintaining the same absolute direction in space.

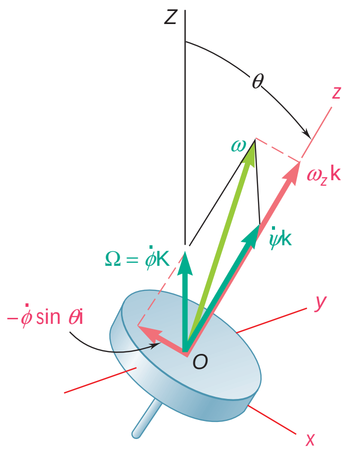


Fig. 18.17

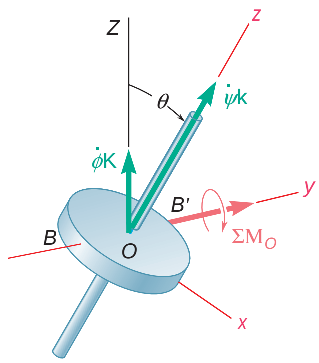


Fig. 18.18

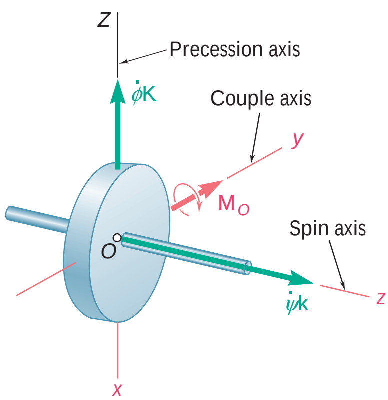


Fig. 18.19

*18.10 STEADY PRECESSION OF A GYROSCOPE

Let us now investigate the particular case of gyroscopic motion in which the angle u, the rate of precession $\dot{f}$, and the rate of spin $\dot{c}$ remain constant. We propose to determine the forces which must be applied to the gyroscope to maintain this motion, known as the *steady precession* of a gyroscope.

Instead of applying the general equations (18.39), we will determine the sum of the moments of the required forces by computing the rate of change of the angular momentum of the gyroscope in the particular case considered. We first note that the angular velocity V of the gyroscope, its angular momentum $\mathbf{H}_O$, and the angular velocity $\mathbf{\Omega}$ of the rotating frame of reference (Fig. 18.17) reduce, respectively, to

$$V = -\dot{f} \sin u \, \mathbf{i} + v_z \mathbf{k} \tag{18.40}$$

$$\mathbf{H}_O = -I' \dot{f} \sin u \, \mathbf{i} + I v_z \mathbf{k} \tag{18.41}$$

$$\mathbf{\Omega} = -\dot{f} \sin u \, \mathbf{i} + \dot{f} \cos u \, \mathbf{k} \tag{18.42}$$

where $v_z = \dot{c} + \dot{f} \cos u$ = rectangular component along spin axis of total angular velocity of gyroscope

Since u, $\dot{f}$, and $\dot{c}$ are constant, the vector $\mathbf{H}_O$ is constant in magnitude and direction with respect to the rotating frame of reference and its rate of change $(\dot{\mathbf{H}}_O)_{Oxyz}$ with respect to that frame is zero. Thus Eq. (18.28) reduces to

$$\Sigma \mathbf{M}_O = \mathbf{\Omega} \times \mathbf{H}_O \tag{18.43}$$

which yields, after substitutions from (18.41) and (18.42),

$$\boxed{\Sigma \mathbf{M}_O = (I v_z - I' \dot{f} \cos u) \dot{f} \sin u \, \mathbf{j}} \tag{18.44}$$

Since the mass center of the gyroscope is fixed in space, we have, by (18.1), $\Sigma \mathbf{F} = 0$; thus, the forces which must be applied to the gyroscope to maintain its steady precession reduce to a couple of moment equal to the right-hand member of Eq. (18.44). We note that *this couple should be applied about an axis perpendicular to the precession axis and to the spin axis of the gyroscope* (Fig. 18.18).

In the particular case when the precession axis and the spin axis are at a right angle to each other, we have $u = 90°$ and Eq. (18.44) reduces to

$$\Sigma \mathbf{M}_O = I \dot{c} \dot{f} \, \mathbf{j} \tag{18.45}$$

Thus, if we apply to the gyroscope a couple $\mathbf{M}_O$ about an axis perpendicular to its axis of spin, the gyroscope will precess about an axis perpendicular to both the spin axis and the couple axis, in a sense such that the vectors representing the spin, the couple, and the precession, respectively, form a right-handed triad (Fig. 18.19).

Because of the relatively large couples required to change the orientation of their axles, gyroscopes are used as stabilizers in torpedoes and ships. Spinning bullets and shells remain tangent to their trajectory because of gyroscopic action. And a bicycle is easier to keep balanced at high speeds because of the stabilizing effect of its spinning wheels. However, gyroscopic action is not always welcome and must be taken into account in the design of bearings supporting rotating shafts subjected to forced precession. The reactions exerted by its propellers on an airplane which changes its direction of flight must also be taken into consideration and compensated for whenever possible.

*18.11 MOTION OF AN AXISYMMETRICAL BODY UNDER NO FORCE

In this section you will analyze the motion about its mass center of an axisymmetrical body under no force except its own weight. Examples of such a motion are furnished by projectiles, if air resistance is neglected, and by artificial satellites and space vehicles after burnout of their launching rockets.

Since the sum of the moments of the external forces about the mass center G of the body is zero, Eq. (18.2) yields $\dot{\mathbf{H}}_G = 0$. It follows that the angular momentum $\mathbf{H}_G$ of the body about G is constant. Thus, the direction of $\mathbf{H}_G$ is fixed in space and can be used to define the Z axis, or axis of precession (Fig. 18.20). Selecting a rotating system of axes $Gxyz$ with the z axis along the axis of symmetry of the body, the x axis in the plane defined by the Z and z axes, and the y axis pointing away from you, we have

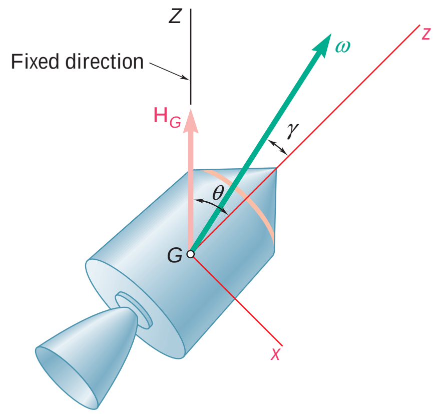


Fig. 18.20

$$H_x = -H_G \sin \mathrm{u} \qquad H_y = 0 \qquad H_z = H_G \cos \mathrm{u} \tag{18.46}$$

where u represents the angle formed by the Z and z axes, and H_G denotes the constant magnitude of the angular momentum of the body about G. Since the x, y, and z axes are principal axes of inertia for the body considered, we can write

$$H_x = I'\mathrm{v}_x \qquad H_y = I'\mathrm{v}_y \qquad H_z = I\mathrm{v}_z \tag{18.47}$$

where I denotes the moment of inertia of the body about its axis of symmetry and I' denotes its moment of inertia about a transverse axis through G. It follows from Eqs. (18.46) and (18.47) that

$$\mathrm{v}_x = -\frac{H_G \sin \mathrm{u}}{I'} \qquad \mathrm{v}_y = 0 \qquad \mathrm{v}_2 = \frac{H_G \cos \mathrm{u}}{I} \tag{18.48}$$

The second of the relations obtained shows that the angular velocity V has no component along the y axis, i.e., along an axis perpendicular to the Zz plane. Thus, the angle u formed by the Z and z axes remains constant and *the body is in steady precession about the Z axis*.

Dividing the first and third of the relations (18.48) member by member, and observing from Fig. 18.21 that $-\mathrm{v}_x/\mathrm{v}_z = \tan \mathrm{g}$, we obtain the following relation between the angles g and u that the

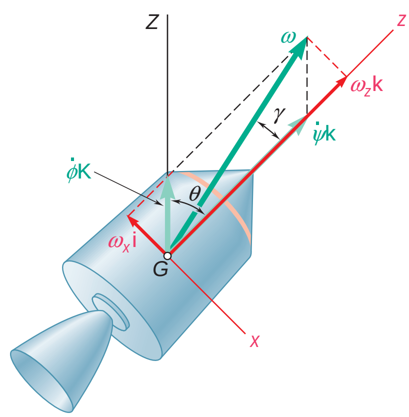
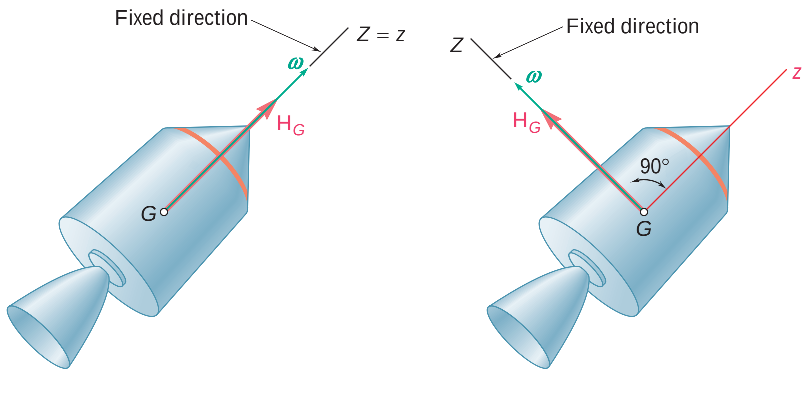


Fig. 18.21



(a) (b)

Fig. 18.22

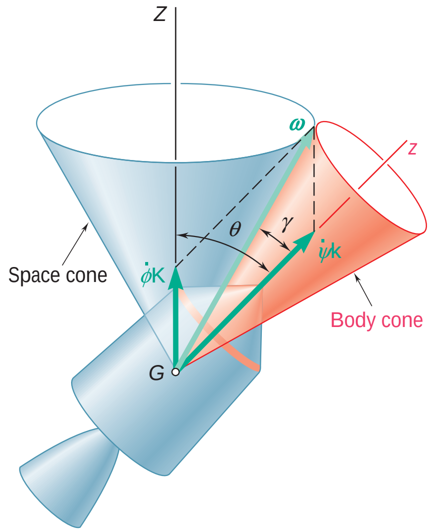


Fig. 18.23

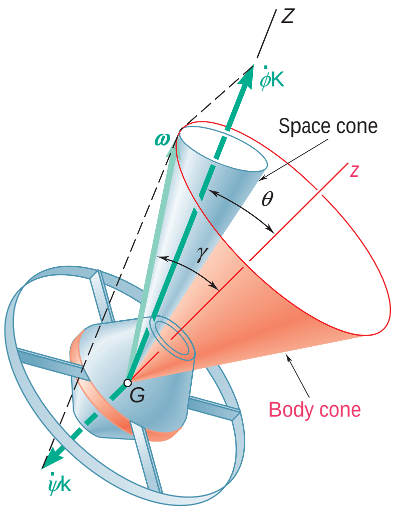


Fig. 18.24

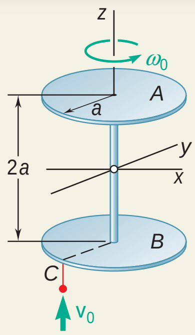
vectors V and $\mathbf{H}_G$, respectively, form with the axis of symmetry of the body:

$$\tan g = \frac{I}{I'} \tan u \tag{18.49}$$

There are two particular cases of motion of an axisymmetrical body under no force which involve no precession: (1) If the body is set to spin about its axis of symmetry, we have $v_x = 0$ and, by (18.47), $H_x = 0$; the vectors V and $\mathbf{H}_G$ have the same orientation and the body keeps spinning about its axis of symmetry (Fig. 18.22*a*). (2) If the body is set to spin about a transverse axis, we have $v_z = 0$ and, by (18.47), $H_z = 0$; again V and $\mathbf{H}_G$ have the same orientation and the body keeps spinning about the given transverse axis (Fig. 18.22*b*).

Considering now the general case represented in Fig. 18.21, we recall from Sec. 15.12 that the motion of a body about a fixed point—or about its mass center—can be represented by the motion of a body cone rolling on a space cone. In the case of steady precession, the two cones are circular, since the angles g and u − g that the angular velocity V forms, respectively, with the axis of symmetry of the body and with the precession axis are constant. Two cases should be distinguished:

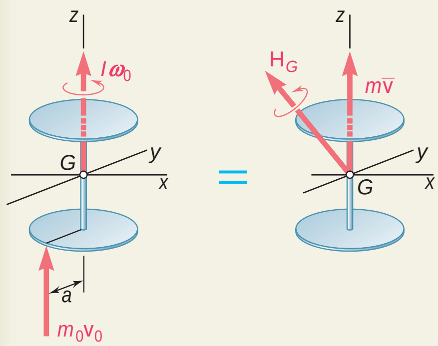
1. $I < I'$. This is the case of an elongated body, such as the space vehicle of Fig. 18.23. By (18.49) we have $g < u$; the vector V lies inside the angle ZGz; the space cone and the body cone are tangent externally; the spin and the precession are both observed as counterclockwise from the positive z axis. The precession is said to be *direct*.
2. $I > I'$. This is the case of a flattened body, such as the satellite of Fig. 18.24. By (18.49) we have $g > u$; since the vector V must lie outside the angle ZGz, the vector $\dot{c}\mathbf{k}$ has a sense opposite to that of the z axis; the space cone is inside the body cone; the precession and the spin have opposite senses. The precession is said to be *retrograde*.



SAMPLE PROBLEM 18.6

A space satellite of mass m is known to be dynamically equivalent to two thin disks of equal mass. The disks are of radius $a = 800$ mm and are rigidly connected by a light rod of length $2a$. Initially the satellite is spinning freely about its axis of symmetry at the rate $\omega_0 = 60$ rpm. A meteorite, of mass $m_0 = m/1000$ and traveling with a velocity $\mathbf{v}_0$ of 2000 m/s relative to the satellite, strikes the satellite and becomes embedded at C. Determine (*a*) the angular velocity of the satellite immediately after impact, (*b*) the precession axis of the ensuing motion, (*c*) the rates of precession and spin of the ensuing motion.

SOLUTION



Moments of Inertia. We note that the axes shown are principal axes of inertia for the satellite and write

$$I = I_z = \tfrac{1}{2}ma^2 \qquad I' = I_x = I_y = 2[\tfrac{1}{4}(\tfrac{1}{2}m)a^2 + (\tfrac{1}{2}m)a^2] = \tfrac{5}{4}ma^2$$

Principle of Impulse and Momentum. We consider the satellite and the meteorite as a single system. Since no external force acts on this system, the momenta before and after impact are equipollent. Taking moments about G, we write

$$-a\mathbf{j} \times m_0v_0\mathbf{k} + I\omega_0\mathbf{k} = \mathbf{H}_G$$
$$\mathbf{H}_G = -m_0v_0a\mathbf{i} + I\omega_0\mathbf{k} \qquad (1)$$

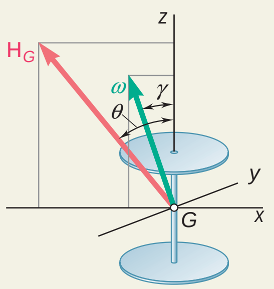
Angular Velocity After Impact. Substituting the values obtained for the components of $\mathbf{H}_G$ and for the moments of inertia into

$$H_x = I_x\omega_x \qquad H_y = I_y\omega_y \qquad H_z = I_z\omega_z$$

we write

$$-m_0v_0a = I'\omega_x = \tfrac{5}{4}ma^2\omega_x \qquad 0 = I'\omega_y \qquad I\omega_0 = I\omega_z$$

$$\omega_x = -\frac{4}{5}\frac{m_0v_0}{ma} \qquad \omega_y = 0 \qquad \omega_z = \omega_0 \qquad (2)$$



For the satellite considered we have $\omega_0 = 60$ rpm $= 6.283$ rad/s, $m_0/m = \frac{1}{1000}$, $a = 0.800$ m, and $v_0 = 2000$ m/s; we find

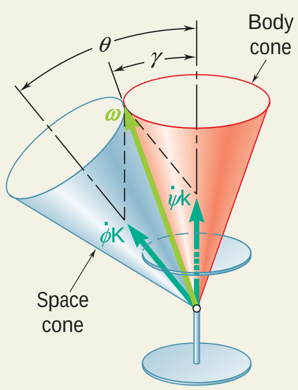
$$\omega_x = -2 \text{ rad/s} \qquad \omega_y = 0 \qquad \omega_z = 6.283 \text{ rad/s}$$

$$\omega = \sqrt{\omega_x^2 + \omega_z^2} = 6.594 \text{ rad/s} \qquad \tan\gamma = \frac{-\omega_x}{\omega_z} = +0.3183$$

$$\omega = 63.0 \text{ rpm} \qquad \gamma = 17.7° \quad ◀$$

Precession Axis. Since in free motion the direction of the angular momentum $\mathbf{H}_G$ is fixed in space, the satellite will precess about this direction. The angle θ formed by the precession axis and the z axis is

$$\tan\theta = \frac{-H_x}{H_z} = \frac{m_0v_0a}{I\omega_0} = \frac{2m_0v_0}{ma\omega_0} = 0.796 \qquad \theta = 38.5° \quad ◀$$



Rates of Precession and Spin. We sketch the space and body cones for the free motion of the satellite. Using the law of sines, we compute the rates of precession and spin.

$$\frac{\omega}{\sin\theta} = \frac{\dot{\phi}}{\sin\gamma} = \frac{\dot{\psi}}{\sin(\theta - \gamma)}$$

$$\dot{\phi} = 30.8 \text{ rpm} \qquad \dot{\psi} = 35.9 \text{ rpm} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we analyzed the motion of *gyroscopes* and of other *axisymmetrical bodies* with a fixed point O. In order to define the position of these bodies at any given instant, we introduced the three *Eulerian angles* f, u, and c (Fig. 18.15), and noted that their time derivatives define, respectively, the rate of *precession*, the rate of *nutation*, and the rate of *spin* (Fig. 18.16). The problems you will encounter fall into one of the following categories.

1. Steady precession. This is the motion of a gyroscope or other axisymmetrical body with a fixed point located on its axis of symmetry, in which the angle u, the rate of precession $\dot{f}$, and the rate of spin $\dot{c}$ all remain constant.

a. Using the rotating frame of reference *Oxyz* shown in Fig. 18.17, which *precesses* with the body, *but does not spin* with it, we obtained the following expressions for the angular velocity V of the body, its angular momentum $\mathbf{H}_O$, and the angular velocity $\boldsymbol{\Omega}$ of the frame $Oxyz$:

$$\mathrm{V} = -\dot{\mathrm{f}} \sin \mathrm{u}\, \mathbf{i} + \mathrm{v}_z \mathbf{k} \tag{18.40}$$
$$\mathbf{H}_O = -I'\dot{\mathrm{f}} \sin \mathrm{u}\, \mathbf{i} + I\, \mathrm{v}_z \mathbf{k} \tag{18.41}$$
$$\boldsymbol{\Omega} = -\dot{\mathrm{f}} \sin \mathrm{u}\, \mathbf{i} + \dot{\mathrm{f}} \cos \mathrm{u}\, \mathbf{k} \tag{18.42}$$

where I = moment of inertia of body about its axis of symmetry
I' = moment of inertia of body about a transverse axis through O
v_z = *rectangular* component of V along z axis = $\dot{\mathrm{c}} + \dot{\mathrm{f}} \cos \mathrm{u}$

b. The sum of the moments about *O* of the forces applied to the body is equal to the rate of change of its angular momentum, as expressed by Eq. (18.28). But, since u and the rates of change $\dot{f}$ and $\dot{c}$ are constant, it follows from Eq. (18.41) that $\mathbf{H}_O$ remains constant in magnitude and direction when viewed from the frame $Oxyz$. Thus, its rate of change is zero with respect to that frame and you can write

$$\Sigma \mathbf{M}_O = \boldsymbol{\Omega} \times \mathbf{H}_O \tag{18.43}$$

where $\boldsymbol{\Omega}$ and $\mathbf{H}_O$ are defined, respectively, by Eq. (18.42) and Eq. (18.41). The equation obtained shows that the moment resultant at O of the forces applied to the body is perpendicular to both the axis of precession and the axis of spin (Fig. 18.18).

c. Keep in mind that the method described applies, not only to gyroscopes, where the fixed point O coincides with the mass center G, but also *to any axisymmetrical body with a fixed point O located on its axis of symmetry*. This method, therefore, can be used to analyze the *steady precession of a top* on a rough floor.

d. When an axisymmetrical body has no fixed point, but is in steady precession about its mass center *G*, you should draw a *free-body-diagram equation* showing that the system of the external forces exerted on the body (including the body's weight) is equivalent to the vector $m\bar{\mathbf{a}}$ applied at G and the couple vector

$\dot{\mathbf{H}}_G$. You can use Eqs. (18.40) through (18.42), replacing $\mathbf{H}_O$ with $\mathbf{H}_G$, and express the moment of the couple as

$$\dot{\mathbf{H}}_G = \boldsymbol{\Omega} \times \mathbf{H}_G$$

You can then use the free-body-diagram equation to write as many as six independent scalar equations.

2. Motion of an axisymmetrical body under no force, except its own weight. We have $\Sigma \mathbf{M}_G = 0$ and, thus, $\dot{\mathbf{H}}_G = 0$; it follows that *the angular momentum* $\mathbf{H}_G$ is constant in magnitude and direction (Sec. 18.11). The body is in *steady precession* with the precession axis GZ directed along $\mathbf{H}_G$ (Fig. 18.20). Using the rotating frame $Gxyz$ and denoting by g the angle that V forms with the spin axis Gz (Fig. 18.21), we obtained the following relation between g and the angle u formed by the precession and spin axes:

$$\tan g = \frac{I}{I'} \tan u \qquad (18.49)$$

The precession is said to be *direct* if $I < I'$ (Fig. 18.23) and *retrograde* if $I > I'$ (Fig. 18.24).

a. In many of the problems dealing with the motion of an axisymmetrical body under no force, you will be asked to determine the *precession axis* and the *rates of precession and spin* of the body, knowing the magnitude of its *angular velocity* V and the angle g that it forms with the axis of symmetry Gz (Fig. 18.21). From Eq. (18.49) you will determine the angle u that the precession axis GZ forms with Gz and resolve V into its two *oblique components* $\dot{f}\mathbf{K}$ and $\dot{c}\mathbf{k}$. Using the law of sines, you will then determine the rate of precession $\dot{f}$ and the rate of spin $\dot{c}$.

b. In other problems, the body will be subjected to *a given impulse* and you will first determine the resulting *angular momentum* $\mathbf{H}_G$. Using Eqs. (18.10), you will calculate the rectangular components of the angular velocity V, its magnitude v, and the angle g that it forms with the axis of symmetry. You will then determine the *precession axis* and the *rates of precession and spin* as described above [Sample Prob. 18.6].

3. General motion of an axisymmetric body with a fixed point O located on its axis of symmetry, and subjected only to its own weight. This is a motion in which the angle u is allowed to vary. At any given instant you should take into account the rate of precession $\dot{f}$, the rate of spin $\dot{c}$, *and the rate of nutation* $\dot{u}$, none of which will remain constant. An example of such a motion is the motion of a top, which is discussed in Probs. 18.137 and 18.138. The rotating frame of reference $Oxyz$ that you will use is still the one shown in Fig. 18.18, but this frame

(*continued*)

will now rotate about the y axis at the rate $\dot{\theta}$. Equations (18.40), (18.41), and (18.42), therefore, should be replaced by the following equations:

$$\boldsymbol{\omega} = -\dot{\phi} \sin \theta \, \mathbf{i} + \dot{\theta} \mathbf{j} + (\dot{\psi} + \dot{\phi} \cos \theta) \, \mathbf{k} \qquad (18.40')$$

$$\mathbf{H}_O = -I' \dot{\phi} \sin \theta \, \mathbf{i} + I' \dot{\theta} \mathbf{j} + I(\dot{\psi} + \dot{\phi} \cos \theta) \, \mathbf{k} \qquad (18.41')$$

$$\boldsymbol{\Omega} = -\dot{\phi} \sin \theta \, \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\phi} \cos \theta \, \mathbf{k} \qquad (18.42')$$

Since substituting these expressions into Eq. (18.44) would lead to nonlinear differential equations, it is preferable, whenever feasible, to apply the following conservation principles.

a. Conservation of energy. Denoting by c the distance between the fixed point O and the mass center G of the body, and by E the total energy, you will write

$$T + V = E: \qquad \tfrac{1}{2}(I' \omega_x^2 + I' \omega_y^2 + I \omega_z^2) + mgc \cos \theta = E$$

and substitute for the components of $\boldsymbol{\omega}$ the expressions obtained in Eq. (18.40′). Note that c will be positive or negative, depending upon the position of G relative to O. Also, $c = 0$ if G coincides with O; the *kinetic energy* is then conserved.

b. Conservation of the angular momentum about the axis of precession. Since the support at O is located on the Z axis, and since the weight of the body and the Z axis are both vertical and, thus, parallel to each other, it follows that $\Sigma M_Z = 0$ and, thus, that H_Z remains constant. This can be expressed by writing that the scalar product $\mathbf{K} \cdot \mathbf{H}_O$ is constant, where $\mathbf{K}$ is the unit vector along the Z axis.

c. Conservation of the angular momentum about the axis of spin. Since the support at O and the center of gravity G are both located on the z axis, it follows that $\Sigma M_z = 0$ and, thus, that H_z remains constant. This is expressed by writing that the coefficient of the unit vector $\mathbf{k}$ in Eq. (18.41′) is constant. Note that this last conservation principle cannot be applied when the body is restrained from spinning about its axis of symmetry, but in that case the only variables are θ and ϕ.

PROBLEMS

18.107 A solid cone of height 9 in. with a circular base of radius 3 in. is supported by a ball-and-socket joint at *A*. Knowing that the cone is observed to precess about the vertical axis *AC* at the constant rate of 40 rpm in the sense indicated and that its axis of symmetry *AB* forms an angle $\beta = 40°$ with *AC*, determine the rate at which the cone spins about the axis *AB*.

18.108 A solid cone of height 9 in. with a circular base of radius 3 in. is supported by a ball-and-socket joint at *A*. Knowing that the cone is spinning about its axis of symmetry *AB* at the rate of 3000 rpm and that *AB* forms an angle $\beta = 60°$ with the vertical axis *AC*, determine the two possible rates of steady precession of the cone about the axis *AC*.

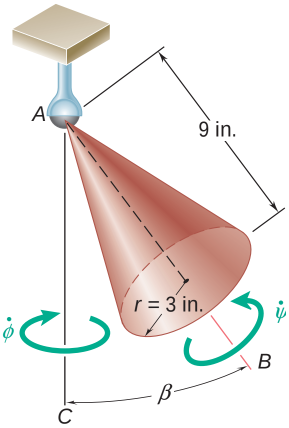


Fig. P18.107 and *P18.108*

18.109 The 85-g top shown is supported at the fixed point *O*. The radii of gyration of the top with respect to its axis of symmetry and with respect to a transverse axis through *O* are 21 mm and 45 mm, respectively. Knowing that $c = 37.5$ mm and that the rate of spin of the top about its axis of symmetry is 1800 rpm, determine the two possible rates of steady precession corresponding to $\theta = 30°$.

18.110 The top shown is supported at the fixed point *O* and its moments of inertia about its axis of symmetry and about a transverse axis through *O* are denoted, respectively, by *I* and *I′*. (*a*) Show that the condition for steady precession of the top is

$$(I\omega_z - I'\dot{\phi}\cos\theta)\dot{\phi} = Wc$$

where $\dot{\phi}$ is the rate of precession and ω_z is the rectangular component of the angular velocity along the axis of symmetry of the top. (*b*) Show that if the rate of spin $\dot{\psi}$ of the top is very large compared with its rate of precession $\dot{\phi}$, the condition for steady precession is $I\dot{\psi}\dot{\phi} \approx Wc$. (*c*) Determine the percentage error introduced when this last relation is used to approximate the slower of the two rates of precession obtained for the top of Prob. 18.109.

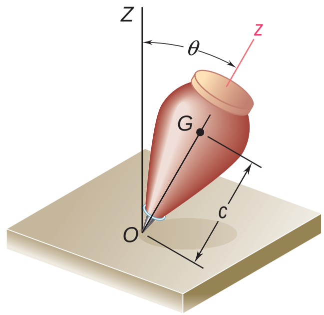


Fig. P18.109 and *P18.110*

18.111 A solid aluminum sphere of radius 4 in. is welded to the end of a 10-in.-long rod *AB* of negligible mass which is supported by a ball-and-socket joint at *A*. Knowing that the sphere is observed to precess about a vertical axis at the constant rate of 60 rpm in the sense indicated and that rod *AB* forms an angle $\beta = 20°$ with the vertical, determine the rate of spin of the sphere about line *AB*.

18.112 A solid aluminum sphere of radius 4 in. is welded to the end of a 10-in.-long rod *AB* of negligible mass which is supported by a ball-and-socket joint at *A*. Knowing that the sphere spins as shown about line *AB* at the rate of 600 rpm, determine the angle β for which the sphere will precess about a vertical axis at the constant rate of 60 rpm in the sense indicated.

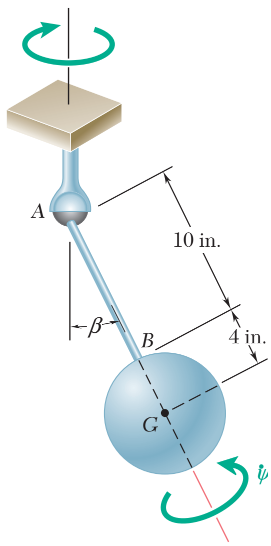


Fig. P18.111 and P18.112

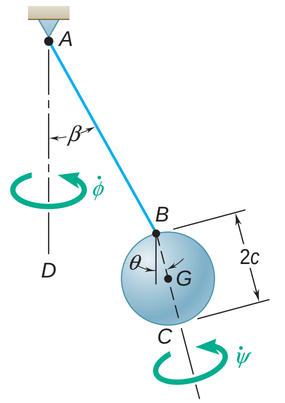


Fig. P18.113 and P18.114

18.113 A solid sphere of radius $c = 3$ in. is attached as shown to cord AB. The sphere is observed to precess at the constant rate $\dot{\phi} = 6$ rad/s about the vertical axis AD. Knowing that $\beta = 40°$, determine the angle θ that the diameter BC forms with the vertical when the sphere (*a*) has no spin, (*b*) spins about its diameter BC at the rate $\dot{\psi} = 50$ rad/s, (*c*) spins about BC at the rate $\dot{\psi} = -50$ rad/s.

18.114 A solid sphere of radius $c = 3$ in. is attached as shown to a cord AB of length 15 in. The sphere spins about its diameter BC and precesses about the vertical axis AD. Knowing that $\theta = 20°$ and $\beta = 35°$, determine (*a*) the rate of spin of the sphere, (*b*) its rate of precession.

18.115 A solid cube of side $c = 80$ mm is attached as shown to cord AB. It is observed to spin at the rate $\dot{\psi} = 40$ rad/s about its diagonal BC and to precess at the constant rate $\dot{\phi} = 5$ rad/s about the vertical axis AD. Knowing that $\beta = 30°$, determine the angle θ that the diagonal BC forms with the vertical. (*Hint:* The moment of inertia of a cube about an axis through its center is independent of the orientation of that axis.)

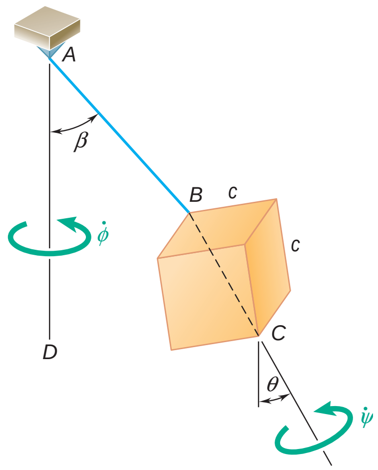


Fig. P18.115 and P18.116

18.116 A solid cube of side $c = 120$ mm is attached as shown to a cord AB of length 240 mm. The cube spins about its diagonal BC and precesses about the vertical axis AD. Knowing that $\theta = 25°$ and $\beta = 40°$, determine (*a*) the rate of spin of the cube, (*b*) its rate of precession. (See hint of Prob. 18.115.)

18.117 A high-speed photographic record shows that a certain projectile was fired with a horizontal velocity $\bar{\mathbf{v}}$ of 2000 ft/s and with its axis of symmetry forming an angle $\beta = 3°$ with the horizontal. The rate of spin $\dot{\psi}$ of the projectile was 6000 rpm, and the atmospheric drag was equivalent to a force $\mathbf{D}$ of 25 lb acting at the center of pressure C_P located at a distance $c = 6$ in. from G. (*a*) Knowing that the projectile has a weight of 45 lb and a radius of gyration of 2 in. with respect to its axis of symmetry, determine its approximate rate of steady precession. (*b*) If it is further known that the radius of gyration of the projectile with respect to a transverse axis through G is 8 in., determine the exact values of the two possible rates of precession.

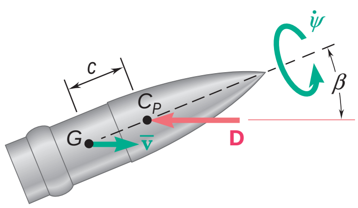


Fig. P18.117

18.118 If the earth were a sphere, the gravitational attraction of the sun, moon, and planets would at all times be equivalent to a single force **R** acting at the mass center of the earth. However, the earth is actually an oblate spheroid and the gravitational system acting on the earth is equivalent to a force **R** and a couple **M**. Knowing that the effect of the couple **M** is to cause the axis of the earth to precess about the axis GA at the rate of one revolution in 25 800 years, determine the average magnitude of the couple **M** applied to the earth. Assume that the average density of the earth is 5.51 g/cm^3, that the average radius of the earth is 6370 km, and that $\bar{I} = \frac{2}{5}mR^2$. (*Note:* This forced precession is known as the precession of the equinoxes and is not to be confused with the free precession discussed in Prob. 18.123.)

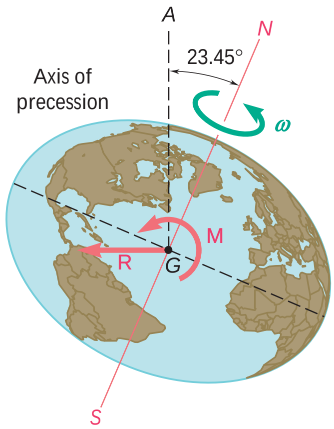


Fig. *P18.118*

18.119 Show that for an axisymmetrical body under no force, the rates of precession and spin can be expressed, respectively, as

$$\dot{f} = \frac{H_G}{I'}$$

and

$$\dot{c} = \frac{H_G \cos u(I' - I)}{II'}$$

where H_G is the constant value of the angular momentum of the body.

18.120 (*a*) Show that for an axisymmetrical body under no force, the rate of precession can be expressed as

$$\dot{f} = \frac{I v_z}{I' \cos u}$$

where v_2 is the rectangular component of V along the axis of symmetry of the body. (*b*) Use this result to check that the condition (18.44) for steady precession is satisfied by an axisymmetrical body under no force.

18.121 Show that the angular velocity vector V of an axisymmetrical body under no force is observed from the body itself to rotate about the axis of symmetry at the constant rate

$$n = \frac{I' - I}{I'} v_z$$

where v_2 is the rectangular component of V along the axis of symmetry of the body.

18.122 For an axisymmetrical body under no force, prove (*a*) that the rate of retrograde precession can never be less than twice the rate of spin of the body about its axis of symmetry, (*b*) that in Fig. 18.24 the axis of symmetry of the body can never lie within the space cone.

18.123 Using the relation given in Prob. 18.121, determine the period of precession of the north pole of the earth about the axis of symmetry of the earth. The earth may be approximated by an oblate spheroid of axial moment of inertia I and of transverse moment of inertia $I' = 0.9967I$. (*Note:* Actual observations show a period of precession of the north pole of about 432.5 mean solar days; the difference between the observed and computed periods is due to the fact that the earth is not a perfectly rigid body. The free precession considered here should not be confused with the much slower precession of the equinoxes, which is a forced precession. See Prob. 18.118.)

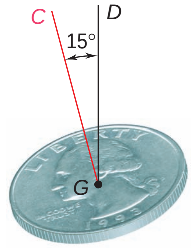


Fig. P18.124

18.124 A coin is tossed into the air. It is observed to spin at the rate of 600 rpm about an axis *GC* perpendicular to the coin and to precess about the vertical direction *GD*. Knowing that *GC* forms an angle of 15° with *GD*, determine (*a*) the angle that the angular velocity V of the coin forms with *GD*, (*b*) the rate of precession of the coin about *GD*.

18.125 The angular velocity vector of a football which has just been kicked is horizontal, and its axis of symmetry *OC* is oriented as shown. Knowing that the magnitude of the angular velocity is 200 rpm and that the ratio of the axis and transverse moments of inertia is $I/I' = \frac{1}{3}$, determine (*a*) the orientation of the axis of precession *OA*, (*b*) the rates of precession and spin.

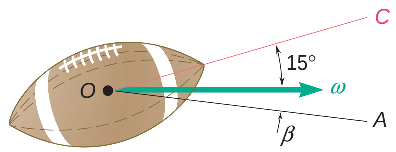


Fig. P18.125

18.126 The space capsule has no angular velocity when the jet at *A* is activated for 1 s in a direction parallel to the *x* axis. Knowing that the capsule has a mass of 1000 kg, that its radii of gyration are $\bar{k}_z = \bar{k}_y = 1.00$ m and $\bar{k}_z = 1.25$ m, and that the jet at *A* produces a thrust of 50 N, determine the axis of precession and the rates of precession and spin after the jet has stopped.

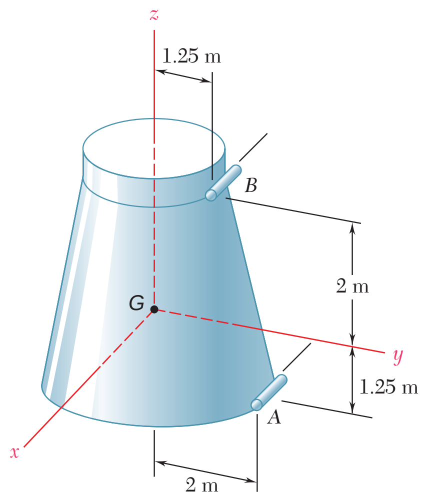


Fig. P18.126 and P18.127

18.127 The space capsule has an angular velocity V = (0.02 rad/s)**j** + (0.10 rad/s)**k** when the jet at *B* is activated for 1 s in a direction parallel to the *x* axis. Knowing that the capsule has a mass of 1000 kg, that its radii of gyration are $\bar{k}_x = \bar{k}_y = 1.00$ m and $\bar{k}_z = 1.25$ m, and that the jet at *B* produces a thrust of 50 N, determine the axis of precession and the rates of precession and spin after the jet has stopped.

18.128 Solve Sample Prob. 18.6, assuming that the meteorite strikes the satellite at *C* with a velocity $\mathbf{v}_0 = (2000 \text{ m/s})\mathbf{i}$.

18.129 An 800-lb geostationary satellite is spinning with an angular velocity V_0 = (1.5 rad/s)**j** when it is hit at B by a 6-oz meteorite traveling with a velocity $\mathbf{v}_0$ = −(1600 ft/s)**i** + (1300 ft/s)**j** + (4000 ft/s)**k** relative to the satellite. Knowing that b = 20 in. and that the radii of gyration of the satellite are $\bar{k}_x = \bar{k}_z$ = 28.8 in. and $\bar{k}_y$ = 32.4 in., determine the precession axis and the rates of precession and spin of the satellite after the impact.

18.130 Solve Prob. 18.129, assuming that the meteorite hits the satellite at A instead of B.

18.131 A homogeneous rectangular plate of mass m and sides c and $2c$ is held at A and B by a fork-ended shaft of negligible mass which is supported by a bearing at C. The plate is free to rotate about AB, and the shaft is free to rotate about a horizontal axis through C. Knowing that, initially, u_0 = 40°, $\dot{u}_0$ = 0, and $\dot{f}_0$ = 10 rad/s, determine for the ensuing motion (*a*) the range of values of u, (*b*) the minimum value of $\dot{f}$, (*c*) the maximum value of $\dot{u}$.

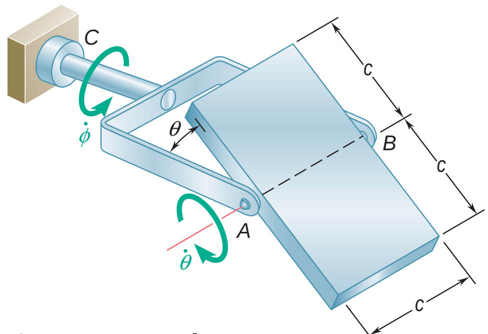


Fig. P18.131 and P18.132

18.132 A homogeneous rectangular plate of mass m and sides c and 2c is held at A and B by a fork-ended shaft of negligible mass which is supported by a bearing at C. The plate is free to rotate about AB, and the shaft is free to rotate about a horizontal axis through C. Initially the plate lies in the plane of the fork (u_0 = 0) and the shaft has an angular velocity $\dot{f}_0$ = 10 rad/s. If the plate is slightly disturbed, determine for the ensuing motion (*a*) the minimum value of $\dot{f}$, (*b*) the maximum value of u.

18.133 A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is connected by a clevis to a vertical shaft AB. The rod and disk can rotate freely about a horizontal axis AC, and shaft AB can rotate freely about a vertical axis. Initially rod AG is horizontal (u_0 = 90°) and has no angular velocity about AC. Knowing that the maximum value $\dot{f}_m$ of the angular velocity of shaft AB in the ensuing motion is twice its initial value $\dot{f}_0$, determine (*a*) the minimum value of u, (*b*) the initial angular velocity $\dot{f}_0$ of shaft AB.

18.134 A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is connected by a clevis to a vertical shaft AB. The rod and disk can rotate freely about a horizontal axis AC, and shaft AB can rotate freely about a vertical axis. Initially rod AG is horizontal (u_0 = 90°) and has no angular velocity about AC. Knowing that the smallest value of u in the ensuing motion is 30°, determine (*a*) the initial angular velocity of shaft AB, (*b*) its maximum angular velocity.

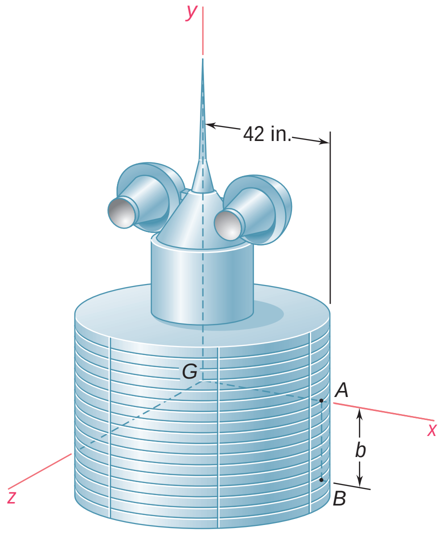


Fig. *P18.129*

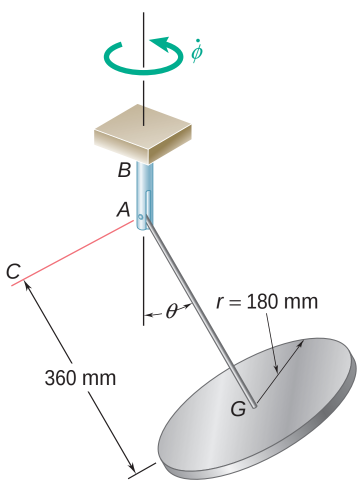


Fig. *P18.133* and *P18.134*

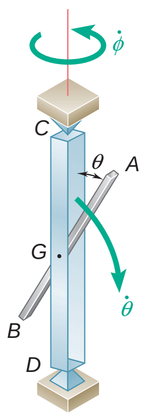


Fig. P18.135

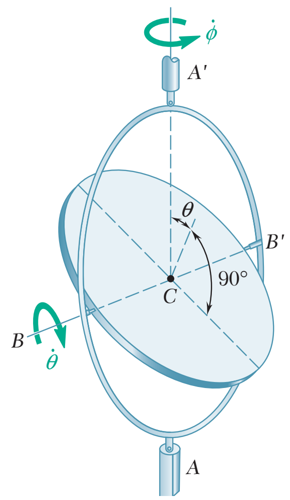


Fig. P18.136

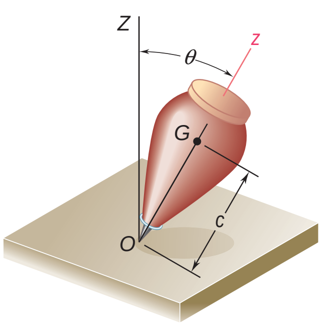


Fig. P18.137 and P18.138

18.135 The slender homogeneous rod AB of mass m and length L is free to rotate about a horizontal axle through its mass center G. The axle is supported by a frame of negligible mass which is free to rotate about the vertical CD. Knowing that, initially, $\theta = \theta_0$, $\dot{\theta} = 0$, and $\dot{\phi} = \dot{\phi}_0$, show that the rod will oscillate about the horizontal axle and determine (*a*) the range of values of angle θ during this motion, (*b*) the maximum value of $\dot{\theta}$, (*c*) the minimum value of $\dot{\phi}$.

18.136 The gimbal $ABA'B'$ is of negligible mass and may rotate freely about the vertical AA'. The uniform disk of radius a and mass m may rotate freely about its diameter BB', which is also the horizontal diameter of the gimbal. (*a*) Applying the principle of conservation of energy, and observing that, since $\Sigma M_{AA'} = 0$, the component of the angular momentum of the disk along the fixed axis AA' must be constant, write two first-order differential equations defining the motion of the disk. (*b*) Given the initial conditions $\theta_0 \neq 0$, $\dot{\phi}_0 \neq 0$, and $\dot{\theta}_0 = 0$, express the rate of nutation $\dot{\theta}$ as a function of θ. (*c*) Show that the angle θ will never be larger than θ_0 during the ensuing motion.

***18.137** The top shown is supported at the fixed point O. Denoting by ϕ, θ, and ψ the Eulerian angles defining the position of the top with respect to a fixed frame of reference, consider the general motion of the top in which all Eulerian angles vary.

(*a*) Observing that $\Sigma M_Z = 0$ and $\Sigma M_z = 0$, and denoting by I and I', respectively, the moments of inertia of the top about its axis of symmetry and about a transverse axis through O, derive the two first-order differential equations of motion

$$I'\dot{\phi}\sin^2\theta + I(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \alpha \tag{1}$$
$$I(\dot{\psi} + \dot{\phi}\cos\theta) = \beta \tag{2}$$

where α and β are constants depending upon the initial conditions. These equations express that the angular momentum of the top is conserved about both the Z and z axes, i.e., that the rectangular component of $\mathbf{H}_O$ along each of these axes is constant.

(*b*) Use Eqs. (1) and (2) to show that the rectangular component ω_z of the angular velocity of the top is constant and that the rate of precession $\dot{\phi}$ depends upon the value of the angle of nutation θ.

***18.138** (*a*) Applying the principle of conservation of energy, derive a third differential equation for the general motion of the top of Prob. 18.137.

(*b*) Eliminating the derivatives $\dot{\phi}$ and $\dot{\psi}$ from the equation obtained and from the two equations of Prob. 18.137, show that the rate of nutation $\dot{\theta}$ is defined by the differential equation $\dot{\theta}^2 = f(\theta)$, where

$$f(\theta) = \frac{1}{I'}\left(2E - \frac{\beta^2}{I} - 2mgc\cos\theta\right) - \left(\frac{\alpha - \beta\cos\theta}{I'\sin\theta}\right)^2 \tag{1}$$

(*c*) Further show, by introducing the auxiliary variable $x = \cos\theta$, that the maximum and minimum values of θ can be obtained by solving for x the cubic equation

$$\left(2E - \frac{\beta^2}{I} - 2mgcx\right)(1 - x^2) - \frac{1}{I'}(\alpha - \beta x)^2 = 0 \tag{2}$$

***18.139** A solid cone of height 180 mm with a circular base of radius 60 mm is supported by a ball and socket at A. The cone is released from the position $\theta_0 = 30°$ with a rate of spin $\dot{\psi}_0 = 300$ rad/s, a rate of precession $\dot{\phi}_0 = 20$ rad/s, and a zero rate of nutation. Determine (*a*) the maximum value of θ in the ensuing motion, (*b*) the corresponding values of the rates of spin and precession. [*Hint:* Use Eq. (2) of Prob. 18.138; you can either solve this equation numerically or reduce it to a quadratic equation, since one of its roots is known.]

***18.140** A solid cone of height 180 mm with a circular base of radius 60 mm is supported by a ball and socket at A. The cone is released from the position $\theta_0 = 30°$ with a rate of spin $\dot{\psi}_0 = 300$ rad/s, a rate of precession $\dot{\phi}_0 = -4$ rad/s, and a zero rate of nutation. Determine (*a*) the maximum value of θ in the ensuing motion, (*b*) the corresponding values of the rates of spin and precession, (*c*) the value of θ for which the sense of the precession is reversed. (See hint of Prob. 18.139.)

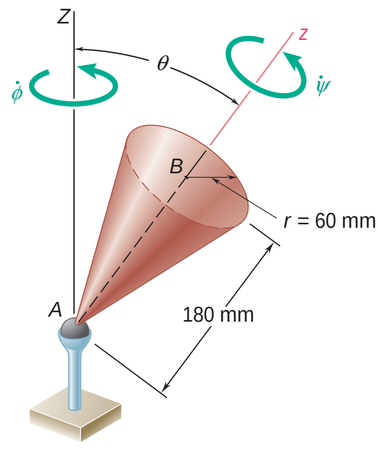


Fig. *P18.139* and *P18.140*

***18.141** A homogeneous sphere of mass m and radius a is welded to a rod AB of negligible mass, which is held by a ball-and-socket support at A. The sphere is released in the position $\beta = 0$ with a rate of precession $\dot{\phi}_0 = \sqrt{17g/11a}$ with no spin or nutation. Determine the largest value of β in the ensuing motion.

***18.142** A homogeneous sphere of mass m and radius a is welded to a rod AB of negligible mass, which is held by a ball-and-socket support at A. The sphere is released in the position $\beta = 0$ with a rate of precession $\dot{\phi} = \dot{\phi}_0$ with no spin or nutation. Knowing that the largest value of β in the ensuing motion is 30°, determine (*a*) the rate of precession $\dot{\phi}_0$ of the sphere in its initial position, (*b*) the rates of precession and spin when $\beta = 30°$.

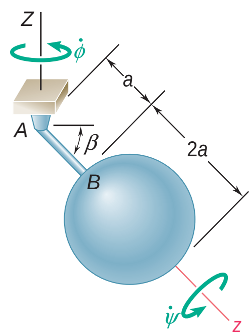


Fig. *P18.141* and *P18.142*

***18.143** Consider a rigid body of arbitrary shape which is attached at its mass center O and subjected to no force other than its weight and the reaction of the support at O.

(*a*) Prove that the angular momentum $\mathbf{H}_O$ of the body about the fixed point O is constant in magnitude and direction, that the kinetic energy T of the body is constant, and that the projection along $\mathbf{H}_O$ of the angular velocity $\boldsymbol{\omega}$ of the body is constant.

(*b*) Show that the tip of the vector $\boldsymbol{\omega}$ describes a curve on a fixed plane in space (called the *invariable plane*), which is perpendicular to $\mathbf{H}_O$ and at a distance $2T/H_O$ from O.

(*c*) Show that with respect to a frame of reference attached to the body and coinciding with its principal axes of inertia, the tip of the vector $\boldsymbol{\omega}$ appears to describe a curve on an ellipsoid of equation

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant}$$

The ellipsoid (called the *Poinsot ellipsoid*) is rigidly attached to the body and is of the same shape as the ellipsoid of inertia, but of a different size.

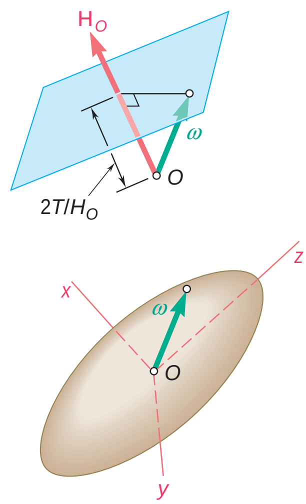


Fig. P18.143

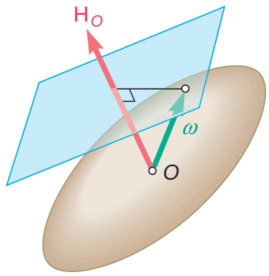


Fig. P18.144

***18.144** Referring to Prob. 18.143, (*a*) prove that the Poinsot ellipsoid is tangent to the invariable plane, (*b*) show that the motion of the rigid body must be such that the Poinsot ellipsoid appears to roll on the invariable plane. [*Hint:* In part *a*, show that the normal to the Poinsot ellipsoid at the tip of V is parallel to $\mathbf{H}_O$. It is recalled that the direction of the normal to a surface of equation $F(x, y, z) =$ constant at a point P is the same as that of **grad** F at point P.]

***18.145** Using the results obtained in Probs. 18.143 and 18.144, show that for an axisymmetrical body attached at its mass center O and under no force other than its weight and the reaction at O, the Poinsot ellipsoid is an ellipsoid of revolution and the space and body cones are both circular and are tangent to each other. Further show that (*a*) the two cones are tangent externally, and the precession is direct, when $I < I'$, where I and I' denote, respectively, the axial and transverse moment of inertia of the body, (*b*) the space cone is inside the body cone, and the precession is retrograde, when $I > I'$.

***18.146** Refer to Probs. 18.143 and 18.144.

(*a*) Show that the curve (called *polhode*) described by the tip of the vector V with respect to a frame of reference coinciding with the principal axes of inertia of the rigid body is defined by the equations

$$I_x \mathsf{v}_x^2 + I_y \mathsf{v}_y^2 + I_z \mathsf{v}_z^2 = 2T = \text{constant} \tag{1}$$
$$I_x^2 \mathsf{v}_x^2 + I_y^2 \mathsf{v}_y^2 + I_z^2 \mathsf{v}_z^2 = H_O^2 = \text{constant} \tag{2}$$

and that this curve can, therefore, be obtained by intersecting the Poinsot ellipsoid with the ellipsoid defined by Eq. (2).

(*b*) Further show, assuming $I_x > I_y > I_z$, that the polhodes obtained for various values of H_O have the shapes indicated in the figure.

(*c*) Using the result obtained in part *b*, show that a rigid body under no force can rotate about a fixed centroidal axis if, and only if, that axis coincides with one of the principal axes of inertia of the body, and that the motion will be stable if the axis of rotation coincides with the major or minor axis of the Poinsot ellipsoid (z or x axis in the figure) and unstable if it coincides with the intermediate axis (y axis).

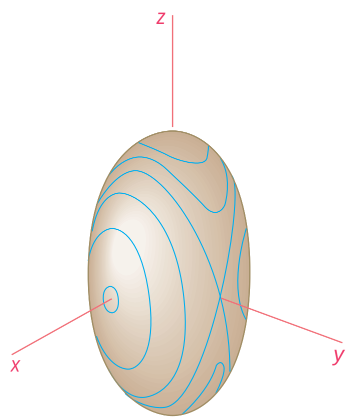


Fig. P18.146

REVIEW AND SUMMARY

This chapter was devoted to the kinetic analysis of the motion of rigid bodies in three dimensions.

Fundamental equations of motion for a rigid body

We first noted [Sec. 18.1] that the two fundamental equations derived in Chap. 14 for the motion of a system of particles

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{18.1}$$
$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{18.2}$$

provide the foundation of our analysis, just as they did in Chap. 16 in the case of the plane motion of rigid bodies. The computation of the angular momentum $\mathbf{H}_G$ of the body and of its derivative $\dot{\mathbf{H}}_G$, however, are now considerably more involved.

Angular momentum of a rigid body in three dimensions

In Sec. 18.2, we saw that the rectangular components of the angular momentum $\mathbf{H}_G$ of a rigid body can be expressed as follows in terms of the components of its angular velocity $\boldsymbol{\omega}$ and of its centroidal moments and products of inertia:

$$\begin{aligned} H_x &= +\bar{I}_x\,\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z \\ H_y &= -\bar{I}_{yx}\omega_x + \bar{I}_y\,\omega_y - \bar{I}_{yz}\omega_z \\ H_z &= -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\,\omega_z \end{aligned} \tag{18.7}$$

If *principal axes of inertia $Gx'y'z'$* are used, these relations reduce to

$$H_{x'} = \bar{I}_{x'}\omega_{x'} \qquad H_{y'} = \bar{I}_{y'}\omega_{y'} \qquad H_{z'} = \bar{I}_{z'}\omega_{z'} \tag{18.10}$$

We observed that, in general, *the angular momentum $\mathbf{H}_G$ and the angular velocity $\boldsymbol{\omega}$ do not have the same direction* (Fig. 18.25). They will, however, have the same direction if $\boldsymbol{\omega}$ is directed along one of the principal axes of inertia of the body.

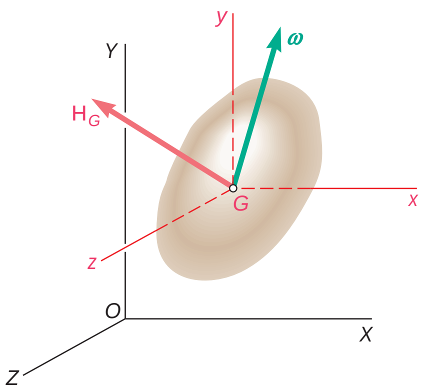


Fig. 18.25

Angular momentum about a given point

Recalling that the system of the momenta of the particles forming a rigid body can be reduced to the vector $m\bar{\mathbf{v}}$ attached at G and the couple $\mathbf{H}_G$ (Fig. 18.26), we noted that, once the linear momentum $m\bar{\mathbf{v}}$ and the angular momentum $\mathbf{H}_G$ of a rigid body have been determined, the angular momentum $\mathbf{H}_O$ of the body about any given point O can be obtained by writing

$$\mathbf{H}_O = \bar{\mathrm{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G \qquad (18.11)$$

Rigid body with a fixed point

In the particular case of a rigid body *constrained to rotate about a fixed point O*, the components of the angular momentum $\mathbf{H}_O$ of the body about O can be obtained directly from the components of its angular velocity and from its moments and products of inertia with respect to axes through O. We wrote

$$\begin{aligned} H_x &= +I_x\,\mathsf{V}_x - I_{xy}\mathsf{V}_y - I_{xz}\mathsf{V}_z \\ H_y &= -I_{yx}\mathsf{V}_x + I_y\,\mathsf{V}_y - I_{yz}\mathsf{V}_z \\ H_z &= -I_{zx}\mathsf{V}_x - I_{zy}\mathsf{V}_y + I_z\,\mathsf{V}_z \end{aligned} \qquad (18.13)$$

Principle of impulse and momentum

The *principle of impulse and momentum* for a rigid body in three-dimensional motion [Sec. 18.3] is expressed by the same fundamental formula that was used in Chap. 17 for a rigid body in plane motion,

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\ \mathrm{y}\ 2} = \textbf{Syst Momenta}_2 \qquad (17.4)$$

but the systems of the initial and final momenta should now be represented as shown in Fig. 18.26, and $\mathbf{H}_G$ should be computed from the relations (18.7) or (18.10) [Sample Probs. 18.1 and 18.2].

Kinetic energy of a rigid body in three dimensions

The *kinetic energy* of a rigid body in three-dimensional motion can be divided into two parts [Sec. 18.4], one associated with the motion of its mass center G and the other with its motion about G. Using principal centroidal axes x', y', z', we wrote

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_{x'}\mathsf{V}_{x'}^2 + \bar{I}_{y'}\mathsf{V}_{y'}^2 + \bar{I}_{z'}\mathsf{V}_{z'}^2) \qquad (18.17)$$

where $\bar{\mathbf{v}}$ = velocity of mass center
V = angular velocity
m = mass of rigid body
$\bar{I}_{x'}, \bar{I}_{y'}, \bar{I}_{z'}$ = principal centroidal moments of inertia

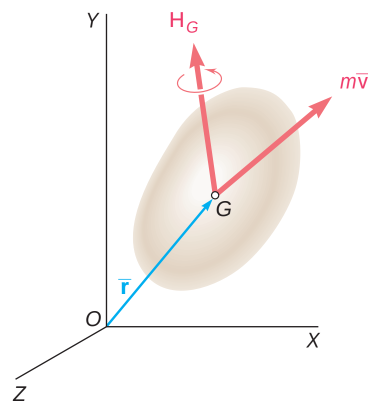


Fig. 18.26

We also noted that, in the case of a rigid body *constrained to rotate about a fixed point* O, the kinetic energy of the body can be expressed as

$$T = \tfrac{1}{2}(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2) \tag{18.20}$$

where the x', y', and z' axes are the principal axes of inertia of the body at O. The results obtained in Sec. 18.4 make it possible to extend to the three-dimensional motion of a rigid body the application of the *principle of work and energy* and of the *principle of conservation of energy*.

Using a rotating frame to write the equations of motion of a rigid body in space

The second part of the chapter was devoted to the application of the fundamental equations

$$\Sigma\mathbf{F} = m\bar{\mathbf{a}} \tag{18.1}$$

$$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G \tag{18.2}$$

to the motion of a rigid body in three dimensions. We first recalled [Sec. 18.5] that $\mathbf{H}_G$ represents the angular momentum of the body relative to a centroidal frame $GX'Y'Z'$ of fixed orientation (Fig. 18.27)

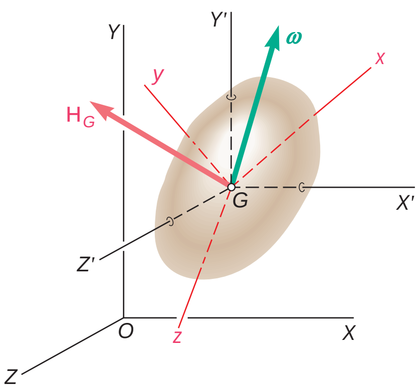


Fig. 18.27

and that $\dot{\mathbf{H}}_G$ in Eq. (18.2) represents the rate of change of $\mathbf{H}_G$ with respect to that frame. We noted that, as the body rotates, its moments and products of inertia with respect to the frame $GX'Y'Z'$ change continually. Therefore, it is more convenient to use a rotating frame $Gxyz$ when resolving ω into components and computing the moments and products of inertia that will be used to determine $\mathbf{H}_G$ from Eqs. (18.7) or (18.10). However, since $\dot{\mathbf{H}}_G$ in Eq. (18.2) represents the rate of change of $\mathbf{H}_G$ with respect to the frame $GX'Y'Z'$ of fixed orientation, we must use the method of Sec. 15.10 to determine its value. Recalling Eq. (15.31), we wrote

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \tag{18.22}$$

where $\mathbf{H}_G$ = angular momentum of body with respect to frame $GX'Y'Z'$ of fixed orientation

$(\dot{\mathbf{H}}_G)_{Gxyz}$ = rate of change of $\mathbf{H}_G$ with respect to rotating frame $Gxyz$, to be computed from relations (18.7)

$\boldsymbol{\Omega}$ = angular velocity of the rotating frame $Gxyz$

Substituting for $\dot{\mathbf{H}}_G$ from (18.22) into (18.2), we obtained

$$\Sigma \mathbf{M}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \qquad (18.23)$$

If the rotating frame is actually attached to the body, its angular velocity $\boldsymbol{\Omega}$ is identically equal to the angular velocity V of the body. There are many applications, however, where it is advantageous to use a frame of reference which is not attached to the body but rotates in an independent manner [Sample Prob. 18.5].

Euler's equations of motion. D'Alembert's principle

Setting $\boldsymbol{\Omega}$ = V in Eq. (18.23), using principal axes, and writing this equation in scalar form, we obtained *Euler's equations of motion* [Sec. 18.6]. A discussion of the solution of these equations and of the scalar equations corresponding to Eq. (18.1) led us to extend d'Alembert's principle to the three-dimensional motion of a rigid body and to conclude that the system of the external forces acting on the rigid body is not only equipollent, but actually *equivalent* to the effective forces of the body represented by the vector $m\bar{\mathbf{a}}$ and the couple $\dot{\mathbf{H}}_G$ (Fig. 18.28). Problems involving the three-dimensional motion of a rigid body can be solved by considering the free-body-diagram equation represented in Fig. 18.28 and writing appropriate scalar equations relating the components or moments of the external and effective forces [Sample Probs. 18.3 and 18.5].

Free-body-diagram equation

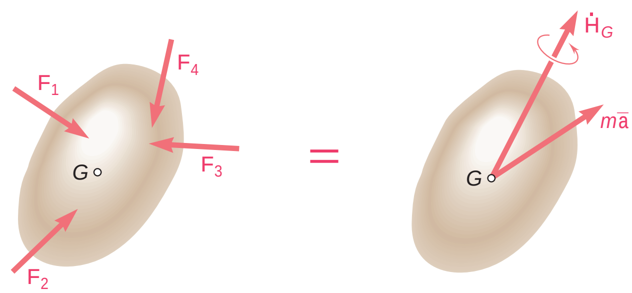


Fig. 18.28

Rigid body with a fixed point

In the case of a rigid body *constrained to rotate about a fixed point O*, an alternative method of solution, involving the moments of the forces and the rate of change of the angular momentum about point O, can be used. We wrote [Sec. 18.7]:

$$\Sigma \mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \qquad (18.28)$$

where $\Sigma \mathbf{M}_O$ = sum of moments about O of forces applied to rigid body
$\mathbf{H}_O$ = angular momentum of body with respect to fixed frame $OXYZ$
$(\dot{\mathbf{H}}_O)_{Oxyz}$ = rate of change of $\mathbf{H}_O$ with respect to rotating frame $Oxyz$, to be computed from relations (18.13)
$\boldsymbol{\Omega}$ = angular velocity of rotating frame $Oxyz$

This approach can be used to solve certain problems involving the rotation of a rigid body about a fixed axis [Sec. 18.8], for example, an unbalanced rotating shaft [Sample Prob. 18.4].

Motion of a gyroscope

In the last part of the chapter, we considered the motion of *gyroscopes* and other *axisymmetrical bodies*. Introducing the *Eulerian angles* f, u, and c to define the position of a gyroscope (Fig. 18.29), we observed that their derivatives $\dot{f}$, $\dot{u}$, and $\dot{c}$ represent, respectively, the rates of *precession, nutation,* and *spin* of the gyroscope [Sec. 18.9]. Expressing the angular velocity V in terms of these derivatives, we wrote

$$V = -\dot{f}\ \sin u\,\mathbf{i} + \dot{u}\mathbf{j} + (\dot{c} + \dot{f}\ \cos u)\mathbf{k} \qquad (18.35)$$

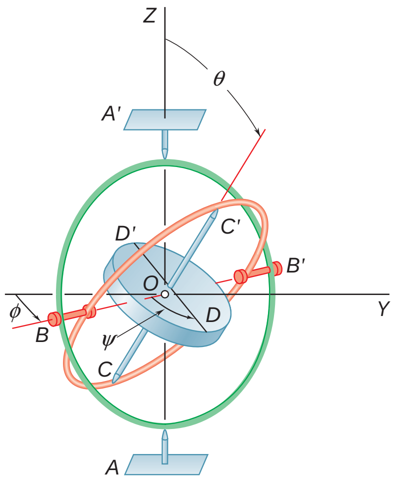


Fig. 18.29

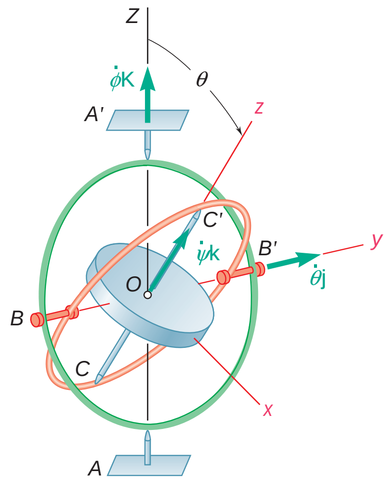


Fig. 18.30

where the unit vectors are associated with a frame *Oxyz* attached to the inner gimbal of the gyroscope (Fig. 18.30) and rotate, therefore, with the angular velocity

$$\boldsymbol{\Omega} = -\dot{f}\ \sin u\,\mathbf{i} + \dot{u}\mathbf{j} + \dot{f}\ \cos u\,\mathbf{k} \qquad (18.38)$$

Denoting by I the moment of inertia of the gyroscope with respect to its spin axis z and by I' its moment of inertia with respect to a transverse axis through O, we wrote

$$\mathbf{H}_O = -I'\dot{f}\ \sin u\,\mathbf{i} + I'\dot{u}\mathbf{j} + I(\dot{c} + \dot{f}\ \cos u)\mathbf{k} \quad (18.36)$$

Substituting for $\mathbf{H}_O$ and $\boldsymbol{\Omega}$ into Eq. (18.28) led us to the differential equations defining the motion of the gyroscope.

Steady precession

In the particular case of the *steady precession* of a gyroscope [Sec. 18.10], the angle u, the rate of precession $\dot{f}$, and the rate of spin $\dot{c}$ remain constant. We saw that such a motion is possible only if the moments of the external forces about O satisfy the relation

$$\Sigma\mathbf{M}_O = (IV_z - I'\dot{f}\ \cos u)\dot{f}\ \sin u\mathbf{j} \qquad (18.44)$$

i.e., if the external forces reduce to a couple of moment equal to the right-hand member of Eq. (18.44) and applied *about an axis perpendicular to the precession axis and to the spin axis* (Fig. 18.31). The chapter ended with a discussion of the motion of an axisymmetrical body spinning and precessing *under no force* [Sec. 18.11; Sample Prob. 18.6].

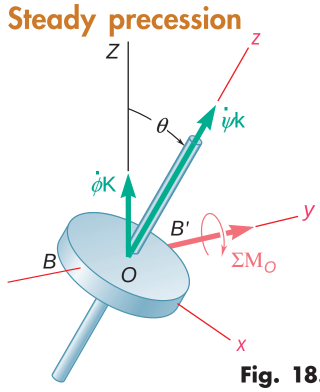


Fig. 18.31

REVIEW PROBLEMS

18.147 Three 25-lb rotor disks are attached to a shaft which rotates at 720 rpm. Disk *A* is attached eccentrically so that its mass center is $\frac{1}{4}$ in. from the axis of rotation, while disks *B* and *C* are attached so that their mass centers coincide with the axis of rotation. Where should 2-lb weights be bolted to disks *B* and *C* to balance the system dynamically?

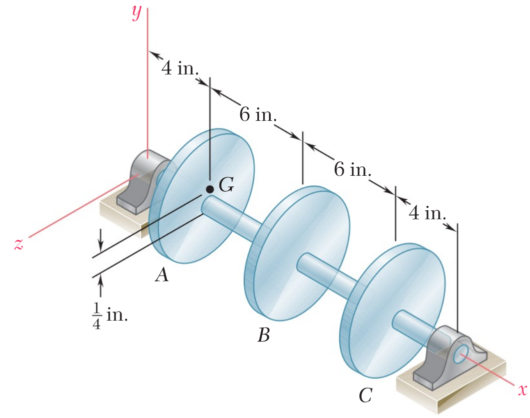


Fig. P18.147

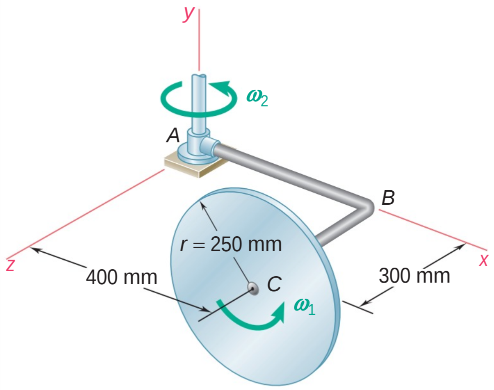


Fig. P18.148

18.148 A homogeneous disk of mass $m = 5$ kg rotates at the constant rate $\omega_1 = 8$ rad/s with respect to the bent axle *ABC*, which itself rotates at the constant rate $\omega_2 = 3$ rad/s about the *y* axis. Determine the angular momentum $\mathbf{H}_C$ of the disk about its center *C*.

18.149 A rod of uniform cross section is used to form the shaft shown. Denoting by *m* the total mass of the shaft and knowing that the shaft rotates with a constant angular velocity ω, determine (*a*) the angular momentum $\mathbf{H}_G$ of the shaft about its mass center *G*, (*b*) the angle formed by $\mathbf{H}_G$ and the axis *AB*, (*c*) the angular momentum of the shaft about point *A*.

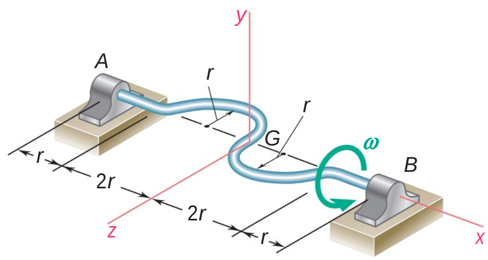


Fig. P18.149

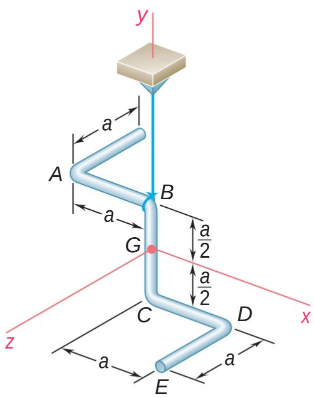


Fig. P18.150

18.150 A uniform rod of mass *m* and length 5*a* is bent into the shape shown and is suspended from a wire attached at point *B*. Knowing that the rod is hit at point *A* in the negative *y* direction and denoting the corresponding impulse by $-(F\,\Delta t)\mathbf{j}$, determine immediately after the impact (*a*) the velocity of the mass center *G*, (*b*) the angular velocity of the rod.

18.151 A four-bladed airplane propeller has a mass of 160 kg and a radius of gyration of 800 mm. Knowing that the propeller rotates at 1600 rpm as the airplane is traveling in a circular path of 600-m radius at 540 km/h, determine the magnitude of the couple exerted by the propeller on its shaft due to the rotation of the airplane.

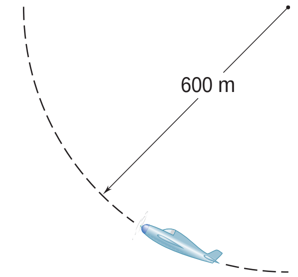


Fig. P18.151

18.152 A 2.4-kg piece of sheet steel with dimensions 160 × 640 mm was bent to form the component shown. The component is at rest ($\omega = 0$) when a couple $\mathbf{M}_0 = (0.8 \text{ N} \cdot \text{m})\mathbf{k}$ is applied to it. Determine (*a*) the angular acceleration of the component, (*b*) the dynamic reactions at *A* and *B* immediately after the couple is applied.

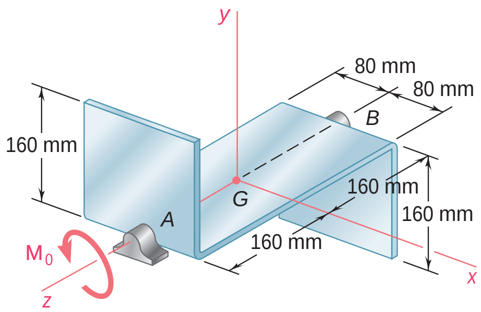


Fig. *P18.152*

18.153 A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE* rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the dynamic reactions at *D* and *E*.

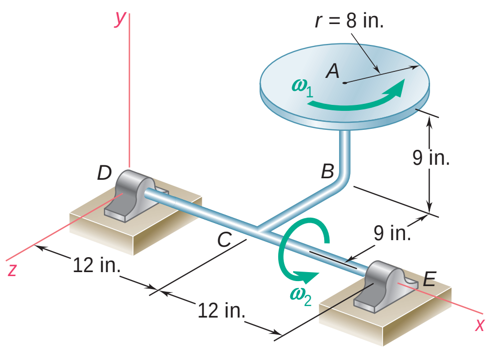


Fig. P18.153

18.154 A 48-kg advertising panel of length $2a = 2.4$ m and width $2b = 1.6$ m is kept rotating at a constant rate ω_1 about its horizontal axis by a small electric motor attached at *A* to frame *ACB*. This frame itself is kept rotating at a constant rate ω_2 about a vertical axis by a second motor attached at *C* to the column *CD*. Knowing that the panel and the frame complete a full revolution in 6 s and 12 s, respectively, express, as a function of the angle θ, the dynamic reaction exerted on column *CD* by its support at *D*.

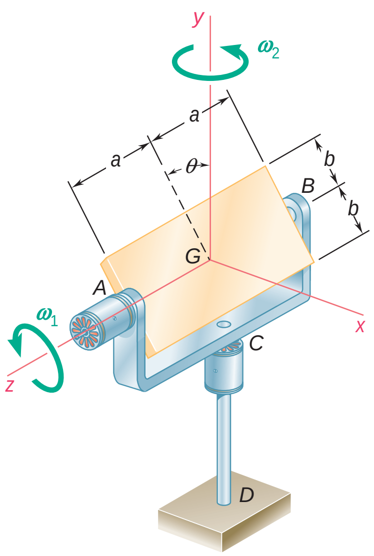


Fig. P18.154

18.155 A 2500-kg satellite is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the satellite, and its radii of gyration are $k_x = k_z = 0.90$ m and $k_y = 0.98$ m. The satellite is equipped with a main 500-N thruster E and four 20-N thrusters A, B, C, and D which can expel fuel in the positive y direction. The satellite is spinning at the rate of 36 rev/h about its axis of symmetry Gy, which maintains a fixed direction in space, when thrusters A and B are activated for 2 s. Determine (*a*) the precession axis of the satellite, (*b*) its rate of precession, (*c*) its rate of spin.

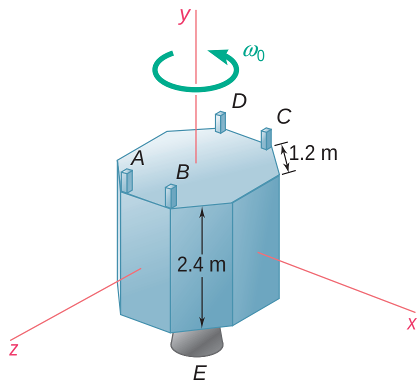


Fig. P18.155

18.156 A thin disk of weight $W = 8$ lb rotates with an angular velocity v_2 with respect to arm OA, which itself rotates with an angular velocity v_1 about the y axis. Determine (*a*) the couple $M_1\mathbf{j}$ which should be applied to arm OA to give it an angular acceleration $A_1 = (6\text{ rad/s}^2)\mathbf{j}$ with $v_1 = 4$ rad/s, knowing that the disk rotates at the constant rate $v_2 = 12$ rad/s, (*b*) the force-couple system representing the dynamic reaction at O at that instant. Assume that arm OA has negligible mass.

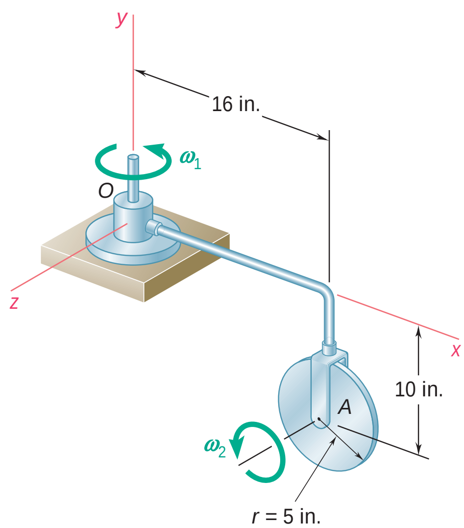


Fig. P18.156

18.157 A homogeneous disk of mass m is connected at A and B to a fork-ended shaft of negligible mass which is supported by a bearing at C. The disk is free to rotate about its horizontal diameter AB and the shaft is free to rotate about a vertical axis through C. Initially the disk lies in a vertical plane ($\theta_0 = 90°$) and the shaft has an angular velocity $\dot{\phi}_0 = 8$ rad/s. If the disk is slightly disturbed, determine for the ensuing motion (*a*) the minimum value of $\dot{\phi}$, (*b*) the maximum value of $\dot{\theta}$.

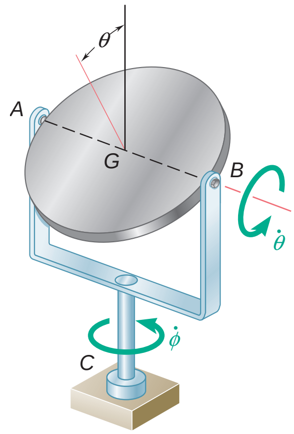


Fig. P18.157

18.158 The essential features of the gyrocompass are shown. The rotor spins at the rate $\dot{\psi}$ about an axis mounted in a single gimbal, which may rotate freely about the vertical axis AB. The angle formed by the axis of the rotor and the plane of the meridian is denoted by θ, and the latitude of the position on the earth is denoted by λ. We note that line OC is parallel to the axis of the earth, and we denote by ω_e the angular velocity of the earth about its axis.

(*a*) Show that the equations of motion of the gyrocompass are

$$I'\ddot{\theta} + I\omega_z\omega_e \cos\lambda \sin\theta - I'\omega_e^2 \cos^2\lambda \sin\theta \cos\theta = 0$$

$$I\dot{\omega}_z = 0$$

where ω_z is the rectangular component of the total angular velocity $\boldsymbol{\omega}$ along the axis of the rotor, and I and I' are the moments of inertia of the rotor with respect to its axis of symmetry and a transverse axis through O, respectively.

(*b*) Neglecting the term containing ω_e^2, show that for small values of θ, we have

$$\ddot{\theta} + \frac{I\omega_z\omega_e \cos\lambda}{I'}\theta = 0$$

and that the axis of the gyrocompass oscillates about the north-south direction.

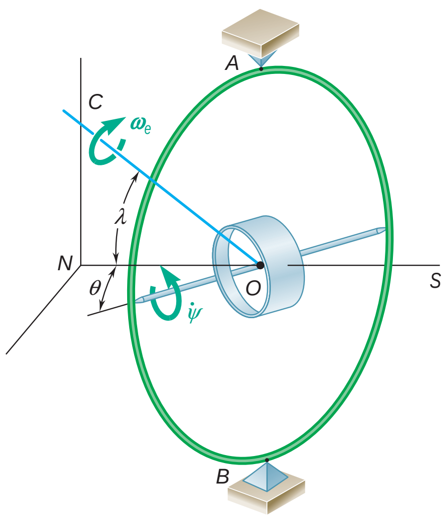


Fig. *P18.158*

COMPUTER PROBLEMS

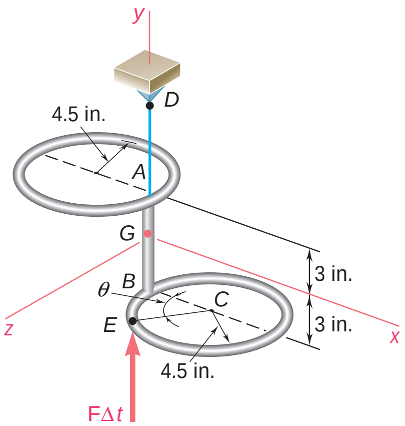


Fig. P18.C1

18.C1 A wire of uniform cross section weighing $\frac{5}{8}$ oz/ft is used to form the wire figure shown, which is suspended from cord AD. An impulse $\mathbf{F}\,\Delta t = (0.5 \text{ lb} \cdot \text{s})\mathbf{j}$ is applied to the wire figure at point E. Use computational software to calculate and plot immediately after the impact, for values of u from 0 to 180°, (*a*) the velocity of the mass center of the wire figure, (*b*) the angular velocity of the figure.

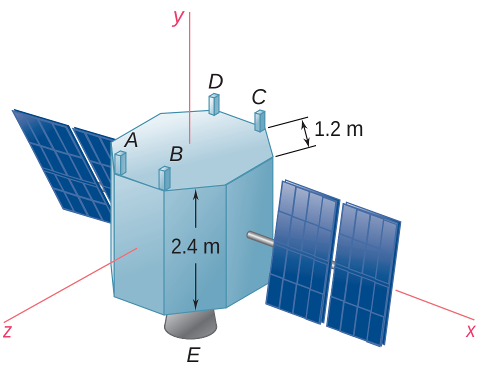


Fig. P18.C2

18.C2 A 2500-kg probe in orbit about the moon is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the probe, and its radii of gyration are $k_x = 0.98$ m, $k_y = 1.06$ m, and $k_z = 1.02$ m. The probe is equipped with a main 500-N thruster E and four 20-N thrusters A, B, C, and D that can expel fuel in the positive y direction. The probe has an angular velocity $V = v_x\,\mathbf{i} + V_z\,\mathbf{k}$ when two of the 20-N thrusters are used to reduce the angular velocity to zero. Use computational software to determine for any pair of values of v_x and v_z less than or equal to 0.06 rad/s, which of the thrusters should be used and for how long each of them should be activated. Apply this program assuming v to be (*a*) $V = (0.040 \text{ rad/s})\mathbf{i} + (0.060 \text{ rad/s})\mathbf{k}$, (*b*) $V = (0.060 \text{ rad/s})\mathbf{i} - (0.040 \text{ rad/s})\mathbf{k}$, (*c*) $V = (0.06 \text{ rad/s})\mathbf{i} + (0.02 \text{ rad/s})\mathbf{k}$, (*d*) $V = -(0.06 \text{ rad/s})\mathbf{i} - (0.02 \text{ rad/s})\mathbf{k}$.

18.C3 A couple $\mathbf{M}_0 = (0.03 \text{ lb} \cdot \text{ft})\mathbf{i}$ is applied to an assembly consisting of pieces of sheet aluminum of uniform thickness and of total weight 2.7 lb, which are welded to a light axle supported by bearings at A and B. Use computational software to determine the dynamic reactions exerted by the bearings on the axle at any time t after the couple has been applied. Resolve these reactions into components directed along y and z axes rotating with the assembly. (*a*) Calculate and plot the components of the reactions from $t = 0$ to $t = 2$ s at 0.1-s intervals. (*b*) Determine the time at which the z components of the reactions at A and B are equal to zero.

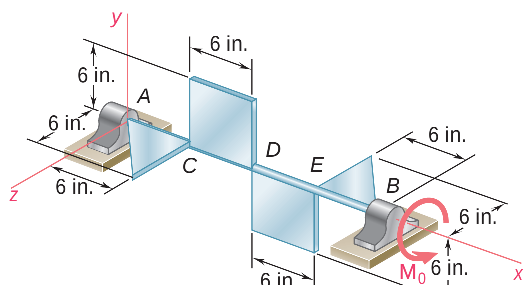


Fig. P18.C3

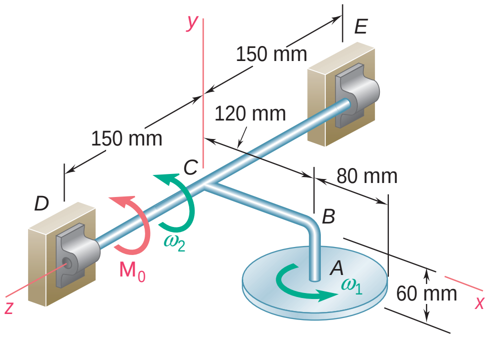


Fig. P18.C4

18.C4 A 2.5-kg homogeneous disk of radius 80 mm can rotate with respect to arm ABC, which is welded to a shaft DCE supported by bearings at D and E. Both the arm and the shaft are of negligible mass. At time $t = 0$ a couple $\mathbf{M}_0 = (0.5 \text{ N} \cdot \text{m})\mathbf{k}$ is applied to shaft DCE. Knowing that at $t = 0$ the angular velocity of the disk is $V_1 = (60 \text{ rad/s})\mathbf{j}$ and that friction in the bearing at A causes the magnitude of V_1 to decrease at the rate of 15 rad/s^2, determine the dynamic reactions exerted on the shaft by the bearings at D and E at any time t. Resolve these reactions into components directed along x and y axes rotating with the shaft. Use computational software (*a*) to calculate the components of the reactions from $t = 0$ to $t = 4$ s, (*b*) to determine the times t_1 and t_2 at which the x and y components of the reaction at E are respectively equal to zero.

18.C5 A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is connected by a clevis to a vertical shaft AB. The rod and disk can rotate freely about a horizontal axis AC, and shaft AB can rotate freely about a vertical axis. Initially rod AG forms a given angle θ_0 with the downward vertical and its angular velocity $\dot{\theta}_0$ about AC is zero. Shaft AB is then given an angular velocity $\dot{\phi}_0$ about the vertical. Use computational software (a) to calculate the minimum value θ_m of the angle θ in the ensuing motion and the period of oscillation in θ, that is, the time required for θ to regain its initial value θ_0, (b) to compute the angular velocity $\dot{\phi}$ of shaft AB for values of θ from θ_0 to θ_m. Apply this program with the initial conditions (i) $\theta_0 = 90°$, $\dot{\phi}_0 = 5$ rad/s, (ii) $\theta_0 = 90°$, $\dot{\phi}_0 = 10$ rad/s, (iii) $\theta_0 = 60°$, $\dot{\phi}_0 = 5$ rad/s. [*Hint:* Use the principle of conservation of energy and the fact that the angular momentum of the body about the vertical through A is conserved to obtain an equation of the form $\dot{\theta}^2 = f(\theta)$. This equation can be integrated by a numerical method.]

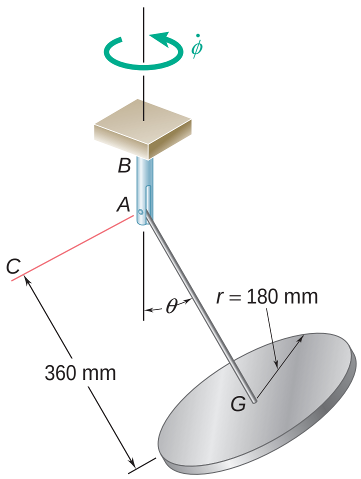


Fig. P18.C5

18.C6 A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is supported by a ball-and-socket joint at A. The disk is released in the position $\theta = \theta_0$ with a rate of spin $\dot{\psi}_0$, a rate of precession $\dot{\phi}_0$, and a zero rate of nutation. Use computational software (a) to calculate the minimum value θ_m of the angle θ in the ensuing motion and the period of oscillation in θ, that is, the time required for θ to regain its initial value θ_0, (b) to compute the rate of spin $\dot{\psi}$ and the rate of precession $\dot{\phi}$ for values of θ from θ_0 to θ_m, using 2° decrements. Apply this program with the initial conditions (i) $\theta_0 = 90°$, $\dot{\psi}_0 = 50$ rad/s, $\dot{\phi}_0 = 0$, (ii) $\theta_0 = 90°$, $\dot{\psi}_0 = 0$, $\dot{\phi}_0 = 5$ rad/s, (iii) $\theta_0 = 90°$, $\dot{\psi}_0 = 50$ rad/s, $\dot{\phi}_0 = 5$ rad/s, (iv) $\theta_0 = 90°$, $\dot{\psi}_0 = 10$ rad/s, $\dot{\phi}_0 = 5$ rad/s, (v) $\theta_0 = 60°$, $\dot{\psi}_0 = 50$ rad/s, $\dot{\phi}_0 = 5$ rad/s. [*Hint:* Use the principle of conservation of energy and the fact that the angular momentum of the body is conserved about both the Z and z axes to obtain an equation of the form $\dot{\theta}^2 = f(\theta)$. This equation can be integrated by a numerical method.]

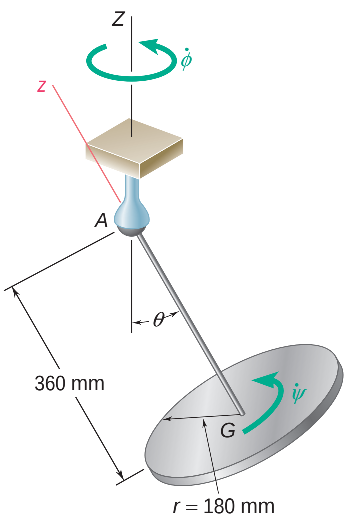
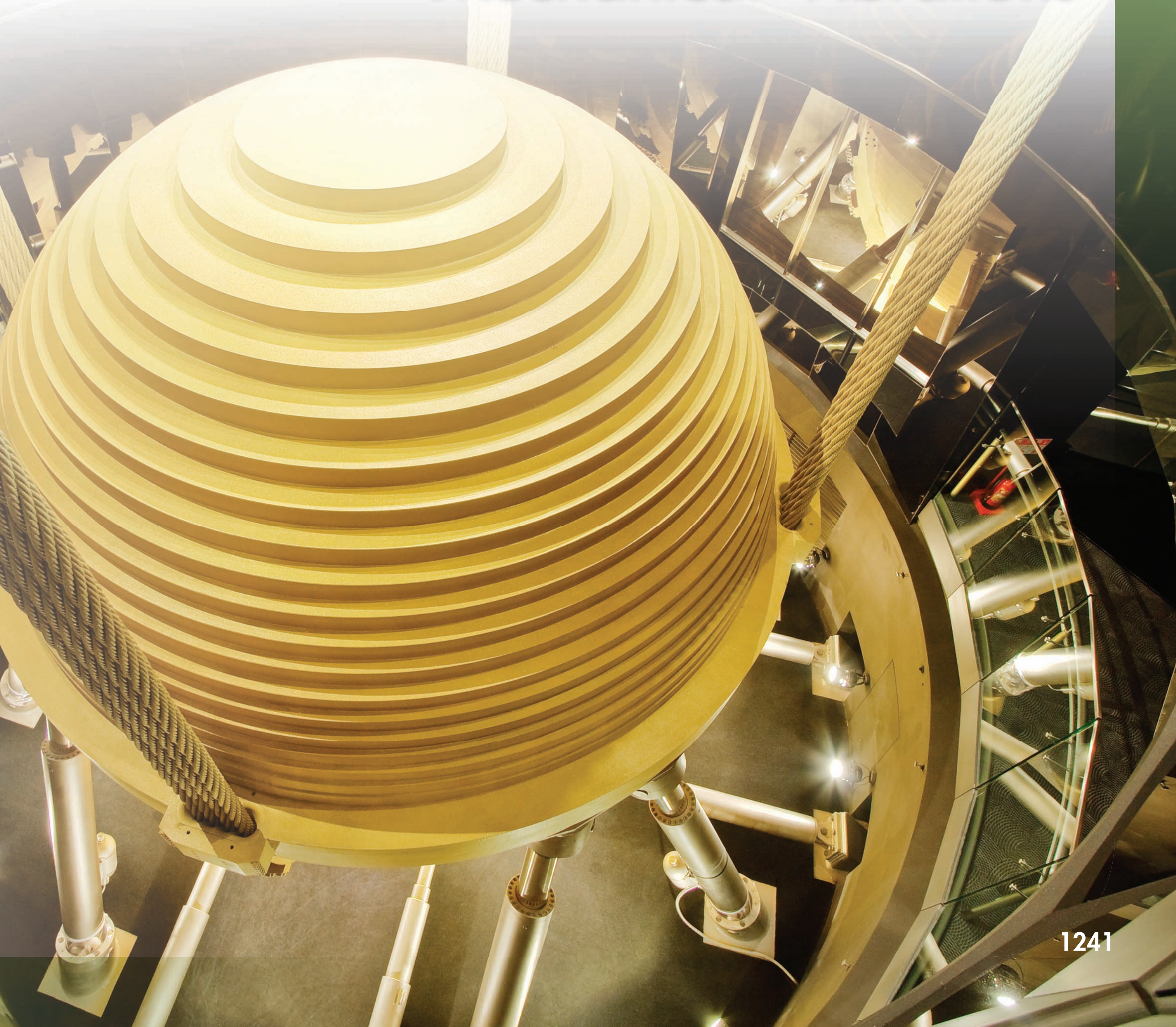


Fig. P18.C6

The Wind Damper inside of Taipei 101 helps protect against typhoons and earthquakes by reducing the effects of wind and vibrations on the building. Mechanical systems may undergo *free vibrations* or they may be subject to *forced vibrations.* The vibrations are *damped* when there is energy dissipation and *undamped* otherwise. This chapter is an introduction to many fundamental concepts in vibration analysis.

CHAPTER 19

Mechanical Vibrations



Chapter 19 Mechanical Vibrations

19.1 INTRODUCTION

A *mechanical vibration* is the motion of a particle or a body which oscillates about a position of equilibrium. Most vibrations in machines and structures are undesirable because of the increased stresses and energy losses which accompany them. They should therefore be eliminated or reduced as much as possible by appropriate design. The analysis of vibrations has become increasingly important in recent years owing to the current trend toward higher-speed machines and lighter structures. There is every reason to expect that this trend will continue and that an even greater need for vibration analysis will develop in the future.

The analysis of vibrations is a very extensive subject to which entire texts have been devoted. Our present study will therefore be limited to the simpler types of vibrations, namely, the vibrations of a body or a system of bodies with one degree of freedom.

A mechanical vibration generally results when a system is displaced from a position of stable equilibrium. The system tends to return to this position under the action of restoring forces (either elastic forces, as in the case of a mass attached to a spring, or gravitational forces, as in the case of a pendulum). But the system generally reaches its original position with a certain acquired velocity which carries it beyond that position. Since the process can be repeated indefinitely, the system keeps moving back and forth across its position of equilibrium. The time interval required for the system to complete a full cycle of motion is called the *period* of the vibration. The number of cycles per unit time defines the *frequency,* and the maximum displacement of the system from its position of equilibrium is called the *amplitude* of the vibration.

When the motion is maintained by the restoring forces only, the vibration is said to be a *free vibration* (Secs. 19.2 to 19.6). When a periodic force is applied to the system, the resulting motion is described as a *forced vibration* (Sec. 19.7). When the effects of friction can be neglected, the vibrations are said to be *undamped*. However, all vibrations are actually *damped* to some degree. If a free vibration is only slightly damped, its amplitude slowly decreases until, after a certain time, the motion comes to a stop. But if damping is large enough to prevent any true vibration, the system then slowly regains its original position (Sec. 19.8). A damped forced vibration is maintained as long as the periodic force which produces the vibration is applied. The amplitude of the vibration, however, is affected by the magnitude of the damping forces (Sec. 19.9).

VIBRATIONS WITHOUT DAMPING

19.2 FREE VIBRATIONS OF PARTICLES. SIMPLE HARMONIC MOTION

Consider a body of mass m attached to a spring of constant k (Fig. 19.1*a*). Since at the present time we are concerned only with the motion of its mass center, we will refer to this body as a particle. When the particle is in static equilibrium, the forces acting on it are its weight $\mathbf{W}$ and the force $\mathbf{T}$ exerted by the spring, of magnitude

$T = k\delta_{st}$, where δ_{st} denotes the elongation of the spring. We have, therefore,

$$W = k\delta_{st}$$

Suppose now that the particle is displaced through a distance x_m from its equilibrium position and released with no initial velocity. If x_m has been chosen smaller than δ_{st}, the particle will move back and forth through its equilibrium position; a vibration of amplitude x_m has been generated. Note that the vibration can also be produced by imparting a certain initial velocity to the particle when it is in its equilibrium position $x = 0$ or, more generally, by starting the particle from any given position $x = x_0$ with a given initial velocity $\mathbf{v}_0$.

To analyze the vibration, let us consider the particle in a position P at some arbitrary time t (Fig. 19.1*b*). Denoting by x the displacement OP measured from the equilibrium position O (positive downward), we note that the forces acting on the particle are its weight $\mathbf{W}$ and the force $\mathbf{T}$ exerted by the spring which, in this position, has a magnitude $T = k(\delta_{st} + x)$. Recalling that $W = k\delta_{st}$, we find that the magnitude of the resultant $\mathbf{F}$ of the two forces (positive downward) is

$$F = W - k(\delta_{st} + x) = -kx \tag{19.1}$$

Thus the *resultant* of the forces exerted on the particle is proportional to the displacement *OP measured from the equilibrium position*. Recalling the sign convention, we note that $\mathbf{F}$ is always directed *toward* the equilibrium position O. Substituting for F into the fundamental equation $F = ma$ and recalling that a is the second derivative $\ddot{x}$ of x with respect to t, we write

$$m\ddot{x} + kx = 0 \tag{19.2}$$

Note that the same sign convention should be used for the acceleration $\ddot{x}$ and for the displacement x, namely, positive downward.

The motion defined by Eq. (19.2) is called a *simple harmonic motion*. It is characterized by the fact that *the acceleration is proportional to the displacement and of opposite direction*. We can verify that each of the functions $x_1 = \sin(\sqrt{k/m}\,t)$ and $x_2 = \cos(\sqrt{k/m}\,t)$ satisfies Eq. (19.2). These functions, therefore, constitute two *particular solutions* of the differential equation (19.2). The *general solution* of Eq. (19.2) is obtained by multiplying each of the particular solutions by an arbitrary constant and adding. Thus, the general solution is expressed as

$$x = C_1x_1 + C_2x_2 = C_1 \sin\left(\sqrt{\frac{k}{m}}\,t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}\,t\right) \tag{19.3}$$

We note that x is a *periodic function* of the time t and does, therefore, represent a vibration of the particle P. The coefficient of t in the expression we have obtained is referred to as the *natural circular frequency* of the vibration and is denoted by ω_n. We have

$$\text{Natural circular frequency} = \omega_n = \sqrt{\frac{k}{m}} \tag{19.4}$$

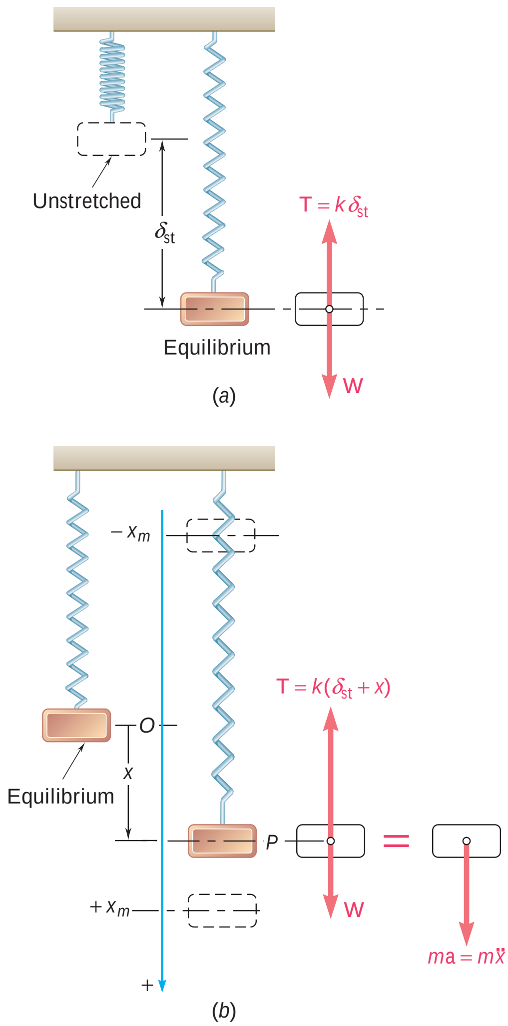


Fig. 19.1

Substituting for $\sqrt{k/m}$ into Eq. (19.3), we write

$$x = C_1 \sin \omega_n t + C_2 \cos \omega_n t \tag{19.5}$$

This is the general solution of the differential equation

$$\ddot{x} + \omega_n^2 x = 0 \tag{19.6}$$

which can be obtained from Eq. (19.2) by dividing both terms by m and observing that $k/m = \omega_n^2$. Differentiating twice both members of Eq. (19.5) with respect to t, we obtain the following expressions for the velocity and the acceleration at time t:

$$v = \dot{x} = C_1\omega_n \cos \omega_n t - C_2\omega_n \sin \omega_n t \tag{19.7}$$
$$a = \ddot{x} = -C_1\omega_n^2 \sin \omega_n t - C_2\omega_n^2 \cos \omega_n t \tag{19.8}$$

The values of the constants C_1 and C_2 depend upon the *initial conditions* of the motion. For example, we have $C_1 = 0$ if the particle is displaced from its equilibrium position and released at $t = 0$ with no initial velocity, and we have $C_2 = 0$ if the particle is started from O at $t = 0$ with a certain initial velocity. In general, substituting $t = 0$ and the initial values x_0 and v_0 of the displacement and the velocity into Eqs. (19.5) and (19.7), we find that $C_1 = v_0/\omega_n$ and $C_2 = x_0$.

The expressions obtained for the displacement, velocity, and acceleration of a particle can be written in a more compact form if we observe that Eq. (19.5) expresses that the displacement $x = OP$ is the sum of the x components of two vectors $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively, of magnitude C_1 and C_2, directed as shown in Fig. 19.2*a*. As t varies, both vectors rotate clockwise; we also note that the magnitude of their resultant $\overrightarrow{OQ}$ is equal to the maximum displacement x_m. The simple harmonic motion of P along the x axis can thus be obtained by projecting on this axis the motion of a point Q describing an *auxiliary circle* of radius x_m *with a constant angular velocity* ω_n (which explains the name of natural *circular* frequency given to ω_n). Denoting by ϕ the angle formed by the vectors $\overrightarrow{OQ}$ and $\mathbf{C}_1$, we write

$$OP = OQ \sin (\omega_n t + \phi) \tag{19.9}$$

which leads to new expressions for the displacement, velocity, and acceleration of P:

$$x = x_m \sin (\omega_n t + \phi) \tag{19.10}$$

$$v = \dot{x} = x_m\omega_n \cos (\omega_n t + \phi) \tag{19.11}$$
$$a = \ddot{x} = -x_m\omega_n^2 \sin (\omega_n t + \phi) \tag{19.12}$$

The displacement-time curve is represented by a sine curve (Fig. 19.2*b*); the maximum value x_m of the displacement is called the *amplitude* of the vibration, and the angle ϕ which defines the initial position of Q on the circle is called the *phase angle*. We note from Fig. 19.2 that a full *cycle* is described as the angle $\omega_n t$ increases by 2π rad. The corresponding value of t, denoted by τ_n, is called the *period* of the free vibration and is measured in seconds. We have

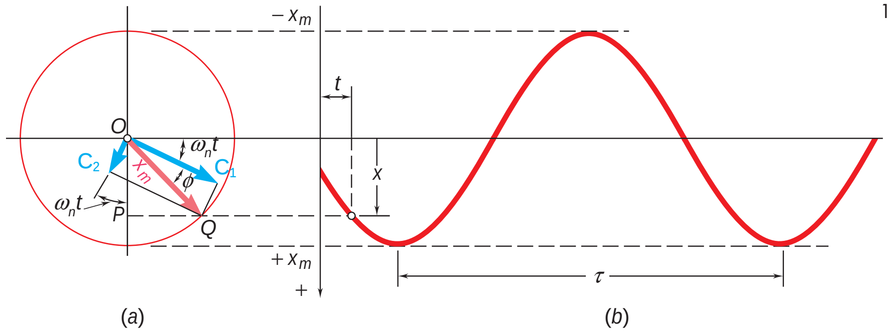


Fig. 19.2

$$\text{period} = \tau_n = \frac{2\pi}{\omega_n} \tag{19.13}$$

The number of cycles described per unit of time is denoted by f_n and is known as the *natural frequency* of the vibration. We write

$$\text{Natural frequency} = f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} \tag{19.14}$$

The unit of frequency is a frequency of 1 cycle per second, corresponding to a period of 1 s. In terms of base units the unit of frequency is thus 1/s or s^{-1}. It is called a *hertz* (Hz) in the SI system of units. It also follows from Eq. (19.14) that a frequency of 1 s^{-1} or 1 Hz corresponds to a circular frequency of 2π rad/s. In problems involving angular velocities expressed in revolutions per minute (rpm), we have 1 rpm $= \frac{1}{60}$ s^{-1} $= \frac{1}{60}$ Hz, or 1 rpm $= (2\pi/60)$ rad/s.

Recalling that ω_n was defined in (19.4) in terms of the constant k of the spring and the mass m of the particle, we observe that the period and the frequency are independent of the initial conditions and of the amplitude of the vibration. Note that τ_n and f_n depend on the *mass* rather than on the *weight* of the particle and thus are independent of the value of g.

The velocity-time and acceleration-time curves can be represented by sine curves of the same period as the displacement-time curve, but with different phase angles. From Eqs. (19.11) and (19.12), we note that the maximum values of the magnitudes of the velocity and acceleration are

$$v_m = x_m \omega_n \qquad a_m = x_m \omega_n^2 \tag{19.15}$$

Since the point Q describes the auxiliary circle, of radius x_m, at the constant angular velocity ω_n, its velocity and acceleration are equal, respectively, to the expressions (19.15). Recalling Eqs. (19.11) and (19.12), we find, therefore, that the velocity and acceleration of P can be obtained at any instant by projecting on the x axis vectors of magnitudes $v_m = x_m \omega_n$ and $a_m = x_m \omega_n^2$ representing, respectively, the velocity and acceleration of Q at the same instant (Fig. 19.3).

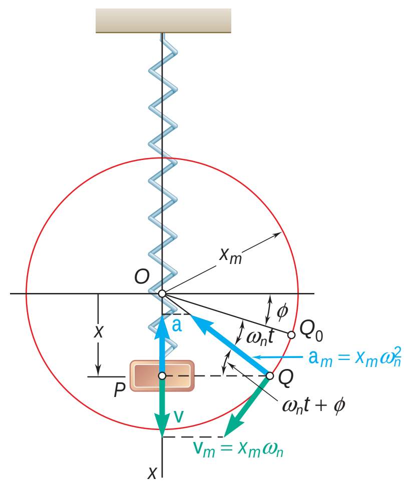


Fig. 19.3

The results obtained are not limited to the solution of the problem of a mass attached to a spring. They can be used to analyze the rectilinear motion of a particle *whenever the resultant* **F** *of the forces acting on the particle is proportional to the displacement* x *and directed toward* O. The fundamental equation of motion $F = ma$ can then be written in the form of Eq. (19.6), which is characteristic of a simple harmonic motion. Observing that the coefficient of x must be equal to ω_n^2, we can easily determine the natural circular frequency ω_n of the motion. Substituting the value obtained for ω_n into Eqs. (19.13) and (19.14), we then obtain the period τ_n and the natural frequency f_n of the motion.

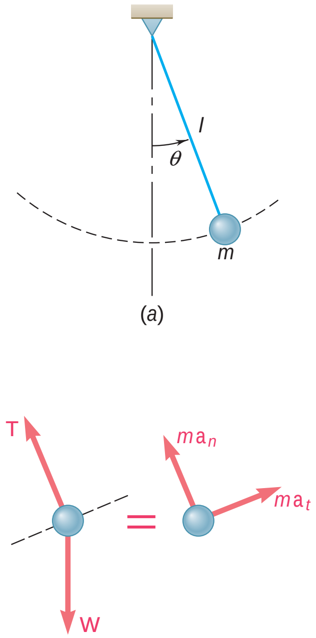


Fig. 19.4

19.3 SIMPLE PENDULUM (APPROXIMATE SOLUTION)

Most of the vibrations encountered in engineering applications can be represented by a simple harmonic motion. Many others, although of a different type, can be *approximated* by a simple harmonic motion, provided that their amplitude remains small. Consider, for example, a *simple pendulum,* consisting of a bob of mass m attached to a cord of length l, which can oscillate in a vertical plane (Fig. 19.4*a*). At a given time t, the cord forms an angle θ with the vertical. The forces acting on the bob are its weight **W** and the force **T** exerted by the cord (Fig. 19.4*b*). Resolving the vector $m\mathbf{a}$ into tangential and normal components, with $m\mathbf{a}_t$ directed to the right, i.e., in the direction corresponding to increasing values of θ, and observing that $a_t = l\alpha = l\ddot{\theta}$, we write

$$\Sigma F_t = ma_t: \qquad -W \sin \theta = ml\ddot{\theta}$$

Noting that $W = mg$ and dividing through by ml, we obtain

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \tag{19.16}$$

For oscillations of small amplitude, we can replace $\sin \theta$ by θ, expressed in radians, and write

$$\ddot{\theta} + \frac{g}{l} \theta = 0 \tag{19.17}$$

Comparison with Eq. (19.6) shows that the differential equation (19.17) is that of a simple harmonic motion with a natural circular frequency ω_n equal to $(g/l)^{1/2}$. The general solution of Eq. (19.17) can, therefore, be expressed as

$$\theta = \theta_m \sin (\omega_n t + \phi)$$

where θ_m is the amplitude of the oscillations and ϕ is a phase angle. Substituting into Eq. (19.13) the value obtained for ω_n, we get the following expression for the period of the small oscillations of a pendulum of length l:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{l}{g}} \tag{19.18}$$

*19.4 SIMPLE PENDULUM (EXACT SOLUTION)

Formula (19.18) is only approximate. To obtain an exact expression for the period of the oscillations of a simple pendulum, we must return to Eq. (19.16). Multiplying both terms by $2\dot{\theta}$ and integrating from an initial position corresponding to the maximum deflection, that is, $\theta = \theta_m$ and $\dot{\theta} = 0$, we write

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\theta_m)$$

Replacing $\cos\theta$ by $1 - 2\sin^2(\theta/2)$ and $\cos\theta_m$ by a similar expression, solving for dt, and integrating over a quarter period from $t = 0$, $\theta = 0$ to $t = \tau_n/4$, $\theta = \theta_m$, we have

$$\tau_n = 2\sqrt{\frac{l}{g}}\int_0^{\theta_m}\frac{d\theta}{\sqrt{\sin^2(\theta_m/2) - \sin^2(\theta/2)}}$$

The integral in the right-hand member is known as an *elliptic integral;* it cannot be expressed in terms of the usual algebraic or trigonometric functions. However, setting

$$\sin(\theta/2) = \sin(\theta_m/2)\sin\phi$$

we can write

$$\tau_n = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - \sin^2(\theta_m/2)\sin^2\phi}} \tag{19.19}$$

where the integral obtained, commonly denoted by K, can be calculated by using a numerical method of integration. It can also be found in *tables of elliptic integrals* for various values of $\theta_m/2$.† In order to compare the result just obtained with that of the preceding section, we write Eq. (19.19) in the form

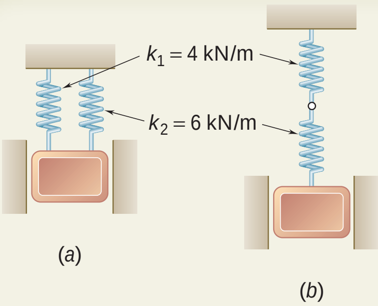
$$\tau_n = \frac{2K}{\pi}\left(2\pi\sqrt{\frac{l}{g}}\right) \tag{19.20}$$

Formula (19.20) shows that the actual value of the period of a simple pendulum can be obtained by multiplying the approximate value given in Eq. (19.18) by the correction factor $2K/\pi$. Values of the correction factor are given in Table 19.1 for various values of the amplitude θ_m. We note that for ordinary engineering computations the correction factor can be omitted as long as the amplitude does not exceed 10°.

TABLE 19.1 Correction Factor for the Period of a Simple Pendulum

θ_m	0°	10°	20°	30°	60°	90°	120°	150°	180°
K	1.571	1.574	1.583	1.598	1.686	1.854	2.157	2.768	∞
$2K/\pi$	1.000	1.002	1.008	1.017	1.073	1.180	1.373	1.762	∞

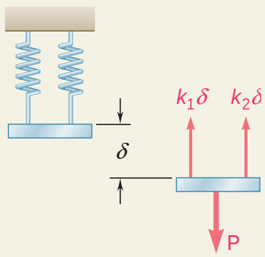
†See, for example, *Standard Mathematical Tables*, Chemical Rubber Publishing Company, Cleveland, Ohio.



SAMPLE PROBLEM 19.1

A 50-kg block moves between vertical guides as shown. The block is pulled 40 mm down from its equilibrium position and released. For each spring arrangement, determine the period of the vibration, the maximum velocity of the block, and the maximum acceleration of the block.

SOLUTION



a. Springs Attached in Parallel. We first determine the constant k of a single spring equivalent to the two springs *by finding the magnitude of the force* **P** required to cause a given deflection δ. Since for a deflection δ the magnitudes of the forces exerted by the springs are, respectively, $k_1\delta$ and $k_2\delta$, we have

$$P = k_1\delta + k_2\delta = (k_1 + k_2)\delta$$

The constant k of the single equivalent spring is

$$k = \frac{P}{\delta} = k_1 + k_2 = 4\text{ kN/m} + 6\text{ kN/m} = 10\text{ kN/m} = 10^4\text{ N/m}$$

Period of Vibration: Since $m = 50$ kg, Eq. (19.4) yields

$$\omega_n^2 = \frac{k}{m} = \frac{10^4\text{ N/m}}{50\text{ kg}} \qquad \omega_n = 14.14\text{ rad/s}$$

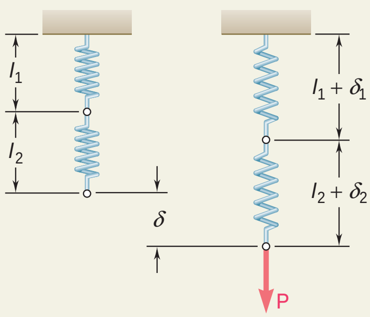
$$\tau_n = 2\pi/\omega_n \qquad \tau_n = 0.444\text{ s} \blacktriangleleft$$

Maximum Velocity: $v_m = x_m\omega_n = (0.040\text{ m})(14.14\text{ rad/s})$

$$v_m = 0.566\text{ m/s} \qquad \mathbf{v}_m = 0.566\text{ m/s} \updownarrow \blacktriangleleft$$

Maximum Acceleration: $a_m = x_m\omega_n^2 = (0.040\text{ m})(14.14\text{ rad/s})^2$

$$a_m = 8.00\text{ m/s}^2 \qquad \mathbf{a}_m = 8.00\text{ m/s}^2 \updownarrow \blacktriangleleft$$



b. Springs Attached in Series. We first determine the constant k of a single spring equivalent to the two springs *by finding the total elongation* δ of the springs under a given static load **P**. To facilitate the computation, a static load of magnitude $P = 12$ kN is used.

$$\delta = \delta_1 + \delta_2 = \frac{P}{k_1} + \frac{P}{k_2} = \frac{12\text{ kN}}{4\text{ kN/m}} + \frac{12\text{ kN}}{6\text{ kN/m}} = 5\text{ m}$$

$$k = \frac{P}{\delta} = \frac{12\text{ kN}}{5\text{ m}} = 2.4\text{ kN/m} = 2400\text{ N/m}$$

Period of Vibration: $\omega_n^2 = \dfrac{k}{m} = \dfrac{2400\text{ N/m}}{50\text{ kg}} \qquad \omega_n = 6.93\text{ rad/s}$

$$\tau_n = \frac{2\pi}{\omega_n} \qquad \tau_n = 0.907\text{ s} \blacktriangleleft$$

Maximum Velocity: $v_m = x_m\omega_n = (0.040\text{ m})(6.93\text{ rad/s})$

$$v_m = 0.277\text{ m/s} \qquad \mathbf{v}_m = 0.277\text{ m/s} \updownarrow \blacktriangleleft$$

Maximum Acceleration: $a_m = x_m\omega_n^2 = (0.040\text{ m})(6.93\text{ rad/s})^2$

$$a_m = 1.920\text{ m/s}^2 \qquad \mathbf{a}_m = 1.920\text{ m/s}^2 \updownarrow \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

This chapter deals with *mechanical vibrations,* i.e., with the motion of a particle or body oscillating about a position of equilibrium.

In this first lesson, we saw that a *free vibration* of a particle occurs when the particle is subjected to a force proportional to its displacement and of opposite direction, such as the force exerted by a spring (Fig. 19.1). The resulting motion, called a *simple harmonic motion,* is characterized by the differential equation

$$m\ddot{x} + kx = 0 \tag{19.2}$$

where x is the displacement of the particle, $\ddot{x}$ is its acceleration, m is its mass, and k is the constant of the spring. The solution of this differential equation was found to be

$$x = x_m \sin(\omega_n t + \phi) \tag{19.10}$$

where x_m = amplitude of the vibration
$\omega_n = \sqrt{k/m}$ = natural circular frequency (rad/s)
ϕ = phase angle (rad)

We also defined the *period* of the vibration as the time $\tau_n = 2\pi/\omega_n$ needed for the particle to complete one cycle, and the *natural frequency* as the number of cycles per second, $f_n = 1/\tau_n = \omega_n/2\pi$, expressed in Hz or s^{-1}. Differentiating Eq. (19.10) twice yields the velocity and the acceleration of the particle at any time. The maximum values of the velocity and acceleration were found to be

$$v_m = x_m\omega_n \qquad a_m = x_m\omega_n^2 \tag{19.15}$$

To determine the parameters in Eq. (19.10) you can follow these steps.

1. Draw a free-body diagram showing the forces exerted on the particle when the particle is at a distance x from its position of equilibrium. The resultant of these forces will be proportional to x and its direction will be opposite to the positive direction of x [Eq. (19.1)].

2. Write the differential equation of motion by equating to $m\ddot{x}$ the resultant of the forces found in step 1. Note that once a positive direction for x has been chosen, the same sign convention should be used for the acceleration $\ddot{x}$. After transposition, you will obtain an equation of the form of Eq. (19.2).

(*continued*)

3. Determine the natural circular frequency ω_n by dividing the coefficient of x by the coefficient of $\ddot{x}$ in this equation and taking the square root of the result obtained. Make sure that ω_n is expressed in rad/s.

4. Determine the amplitude x_m and the phase angle ϕ by substituting the value obtained for ω_n and the initial values of x and $\dot{x}$ into Eq. (19.10) and the equation obtained by differentiating Eq. (19.10) with respect to t.

Equation (19.10) and the two equations obtained by differentiating Eq. (19.10) twice with respect to t can now be used to find the displacement, velocity, and acceleration of the particle at any time. Equations (19.15) yield the maximum velocity v_m and the maximum acceleration a_m.

5. You also saw that for the small oscillations of a simple pendulum, the angle θ that the cord of the pendulum forms with the vertical satisfies the differential equation

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{19.17}$$

where l is the length of the cord and where θ is expressed in radians [Sec. 19.3]. This equation defines again a *simple harmonic motion*, and its solution is of the same form as Eq. (19.10),

$$\theta = \theta_m \sin(\omega_n t + \phi)$$

where the natural circular frequency $\omega_n = \sqrt{g/l}$ is expressed in rad/s. The determination of the various constants in this expression is carried out in a manner similar to that described above. Remember that the velocity of the bob is tangent to the path and that its magnitude is $v = l\dot{\theta}$, while the acceleration of the bob has a tangential component $\mathbf{a}_t$, of magnitude $a_t = l\ddot{\theta}$, and a normal component $\mathbf{a}_n$ directed toward the center of the path and of magnitude $a_n = l\dot{\theta}^2$.

PROBLEMS

19.1 Determine the maximum velocity and maximum acceleration of a particle which moves in simple harmonic motion with an amplitude of 3 mm and a frequency of 20 Hz.

19.2 A particle moves in simple harmonic motion. Knowing that the amplitude is 15 in. and the maximum acceleration is 15 ft/s^2, determine the maximum velocity of the particle and the frequency of its motion.

19.3 Determine the amplitude and maximum velocity of a particle which moves in simple harmonic motion with a maximum acceleration of 15 ft/s^2 and a frequency of 8 Hz.

19.4 A 32-kg block is attached to a spring and can move without friction in a slot as shown. The block is in its equilibrium position when it is struck by a hammer which imparts to the block an initial velocity of 250 mm/s. Determine (*a*) the period and frequency of the resulting motion, (*b*) the amplitude of the motion and the maximum acceleration of the block.

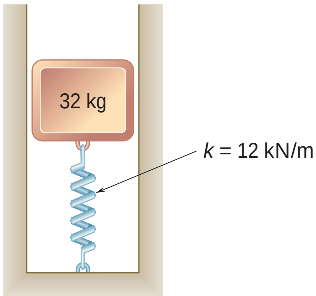


Fig. P19.4

19.5 A 13-kg block is supported by the spring shown. If the block is moved vertically downward from its equilibrium position and released, determine (*a*) the period and frequency of the resulting motion, (*b*) the maximum velocity and acceleration of the block if the amplitude of its motion is 50 mm.

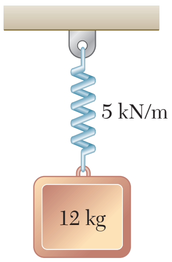


Fig. P19.5

19.6 An instrument package A is bolted to a shaker table as shown. The table moves vertically in simple harmonic motion at the same frequency as the variable-speed motor which drives it. The package is to be tested at a peak acceleration of 150 ft/s^2. Knowing that the amplitude of the shaker table is 2.3 in., determine (*a*) the required speed of the motor in rpm, (*b*) the maximum velocity of the table.

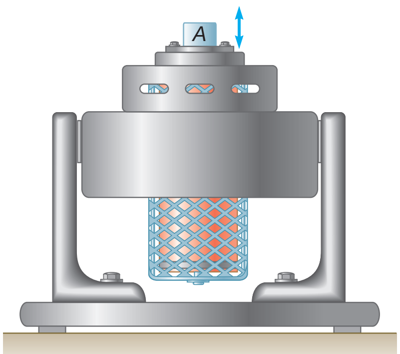


Fig. P19.6

19.7 A simple pendulum consisting of a bob attached to a cord oscillates in a vertical plane with a period of 1.3 s. Assuming simple harmonic motion and knowing that the maximum velocity of the bob is 0.4 m/s, determine (*a*) the amplitude of the motion in degrees, (*b*) the maximum tangential acceleration of the bob.

19.8 A simple pendulum consisting of a bob attached to a cord of length l = 800 mm oscillates in a vertical plane. Assuming simple harmonic motion and knowing that the bob is released from rest when u = 6°, determine (*a*) the frequency of oscillation, (*b*) the maximum velocity of the bob.

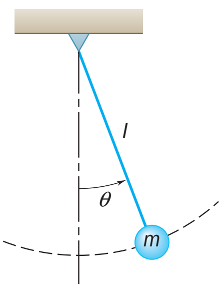


Fig. P19.7 and P19.8

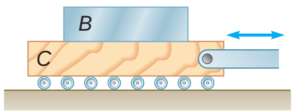


Fig. *P19.9*

19.9 An instrument package B is placed on the shaking table C as shown. The table is made to move horizontally in simple harmonic motion with a frequency of 3 Hz. Knowing that the coefficient of static friction is $m_s = 0.40$ between the package and the table, determine the largest allowable amplitude of the motion if the package is not to slip on the table. Give the answers in both SI and U.S. customary units.

19.10 A 5-kg fragile glass vase is surrounded by packing material in a cardboard box of negligible weight. The packing material has negligible damping and a force-deflection relationship as shown. Knowing that the box is dropped from a height of 1 m and the impact with the ground is perfectly plastic, determine (*a*) the amplitude of vibration for the vase, (*b*) the maximum acceleration the vase experiences in g's.

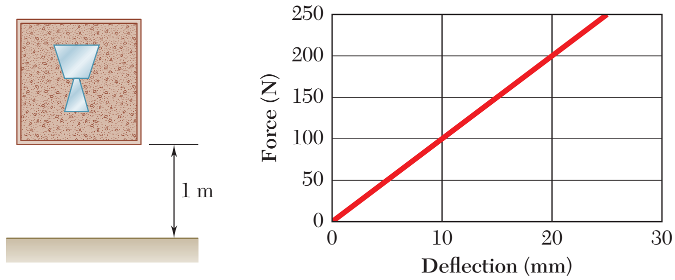


Fig. *P19.10*

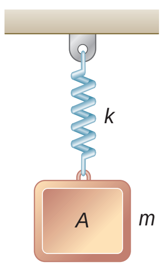


Fig. P19.11

19.11 A 3-lb block is supported as shown by a spring of constant k = 2 lb/in. which can act in tension or compression. The block is in its equilibrium position when it is struck from below by a hammer which imparts to the block an upward velocity of 90 in./s. Determine (*a*) the time required for the block to move 3 in. upward, (*b*) the corresponding velocity and acceleration of the block.

19.12 In Prob. 19.11, determine the position, velocity, and acceleration of the block 0.90 s after it has been struck by the hammer.

19.13 The bob of a simple pendulum of length l = 40 in. is released from rest when u = +5°. Assuming simple harmonic motion, determine 1.6 s after release (*a*) the angle u, (*b*) the magnitudes of the velocity and acceleration of the bob.

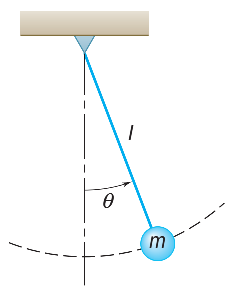


Fig. P19.13

19.14 A 150-kg electromagnet is at rest and is holding 100 kg of scrap steel when the current is turned off and the steel is dropped. Knowing that the cable and the supporting crane have a total stiffness equivalent to a spring of constant 200 kN/m, determine (*a*) the frequency, the amplitude, and the maximum velocity of the resulting motion, (*b*) the minimum tension which will occur in the cable during the motion, (*c*) the velocity of the magnet 0.03 s after the current is turned off.

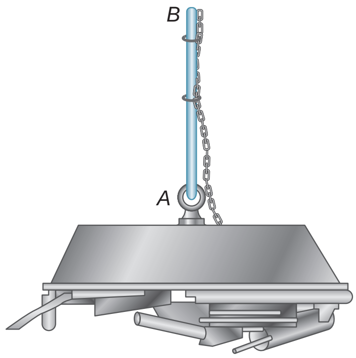


Fig. P19.14

19.15 A variable-speed motor is rigidly attached to beam *BC*. The rotor is slightly unbalanced and causes the beam to vibrate with a frequency equal to the motor speed. When the speed of the motor is less than 600 rpm or more than 1200 rpm, a small object placed at *A* is observed to remain in contact with the beam. For speeds between 600 rpm and 1200 rpm the object is observed to "dance" and actually to lose contact with the beam. Determine the amplitude of the motion of *A* when the speed of the motor is (*a*) 600 rpm, (*b*) 1200 rpm. Give answers in both SI and U.S. customary units.

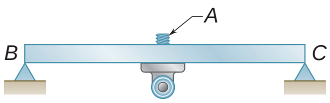


Fig. *P19.15*

19.16 A small bob is attached to a cord of length 1.2 m and is released from rest when $u_A = 5°$. Knowing that $d = 0.6$ m, determine (*a*) the time required for the bob to return to point *A*, (*b*) the amplitude u_C.

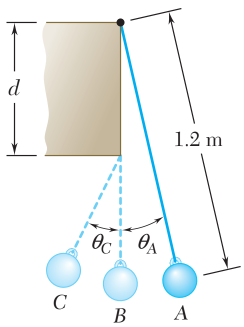


Fig. *P19.16*

19.17 A 5-kg block, attached to the lower end of a spring whose upper end is fixed, vibrates with a period of 6.8 s. Knowing that the constant *k* of a spring is inversely proportional to its length, determine the period of a 3-kg block which is attached to the center of the same spring if the upper and lower ends of the spring are fixed.

19.18 and 19.19 A 75-lb block is supported by the spring arrangement shown. The block is moved vertically downward from its equilibrium position and released. Knowing that the amplitude of the resulting motion is 2 in., determine (*a*) the period and frequency of the motion, (*b*) the maximum velocity and maximum acceleration of the block.

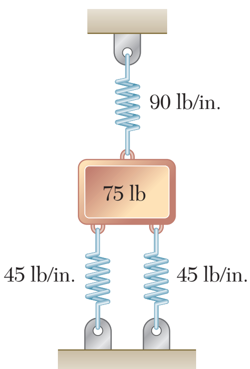


Fig. P19.18

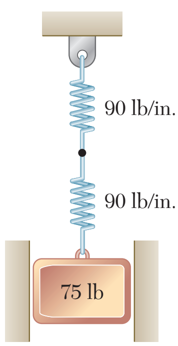


Fig. P19.19

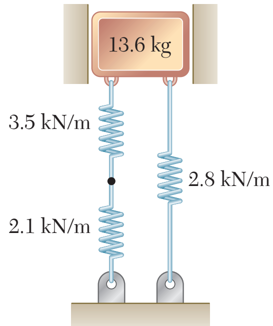


Fig. P19.20

19.20 A 13.6-kg block is supported by the spring arrangement shown. If the block is moved from its equilibrium position 44 mm vertically downward and released, determine (*a*) the period and frequency of the resulting motion, (*b*) the maximum velocity and acceleration of the block.

19.21 An 11-lb block, attached to the lower end of a spring whose upper end is fixed, vibrates with a period of 7.2 s. Knowing that the constant k of a spring is inversely proportional to its length (e.g., if you cut a 10-lb/in. spring in half, the remaining two springs each have a spring constant of 20 lb/in.), determine the period of a 7-lb block which is attached to the center of the same spring if the upper and lower ends of the spring are fixed.

19.22 Block A of mass m is supported by the spring arrangement as shown. Knowing that the mass of the pulley is negligible and that the block is moved vertically downward from its equilibrium position and released, determine the frequency of the motion.

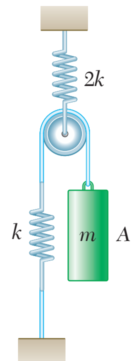


Fig. *P19.22*

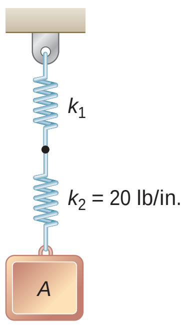


Fig. P19.23

19.23 The period of vibration of the system shown is observed to be 0.2 s. After the spring of constant $k_2 = 20$ lb/in. is removed and block A is connected to the spring of constant k_1, the period is observed to be 0.12 s. Determine (*a*) the constant k_1 of the remaining spring, (*b*) the weight of block A.

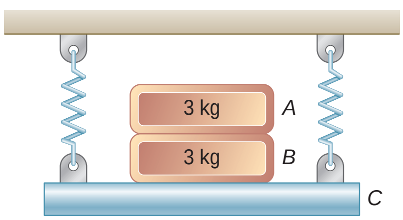


Fig. P19.24

19.24 The period of vibration of the system shown is observed to be 0.8 s. If block A is removed, the period is observed to be 0.7 s. Determine (*a*) the mass of block C, (*b*) the period of vibration when both blocks A and B have been removed.

19.25 The 100-lb platform A is attached to springs B and D, each of which has a constant $k = 120$ lb/ft. Knowing that the frequency of vibration of the platform is to remain unchanged when an 80-lb block is placed on it and a third spring C is added between springs B and D, determine the required constant of spring C.

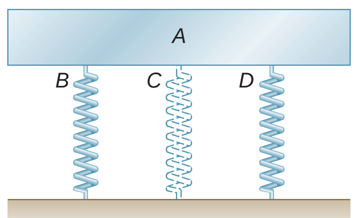


Fig. P19.25

19.26 The period of vibration for a barrel floating in salt water is found to be 0.58 s when the barrel is empty and 1.8 s when it is filled with 55 gallons of crude oil. Knowing that the density of the oil is 900 kg/m^3, determine (*a*) the mass of the empty barrel, (*b*) the density of the salt water, ρ_{sw}. [*Hint*: The force of the water on the bottom of the barrel can be modeled as a spring with constant $k = \rho_{sw}gA$.]

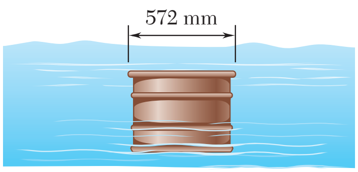


Fig. P19.26

19.27 From mechanics of materials it is known that for a cantilever beam of constant cross section a static load **P** applied at end *B* will cause a deflection $\delta_B = PL^3/3EI$, where *L* is the length of the beam, *E* is the modulus of elasticity, and *I* is the moment of inertia of the cross-sectional area of the beam. Knowing that $L = 10$ ft, $E = 29 \times 10^6$ lb/in^2, and $I = 12.4$ in^4, determine (*a*) the equivalent spring constant of the beam, (*b*) the frequency of vibration of a 520-lb block attached to end *B* of the same beam.

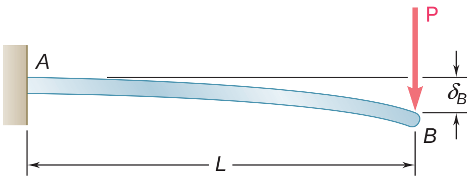
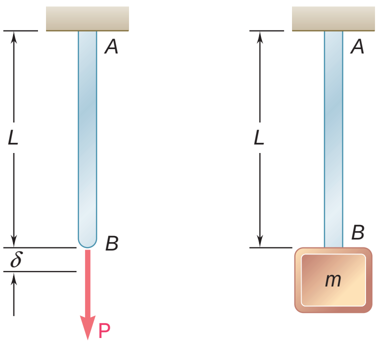


Fig. *P19.27*

19.28 From mechanics of materials it is known that when a static load **P** is applied at the end *B* of a uniform metal rod fixed at end *A*, the length of the rod will increase by an amount $\delta = PL/AE$, where *L* is the length of the undeformed rod, *A* is its cross-sectional area, and *E* is the modulus of elasticity of the metal. Knowing that $L = 450$ mm and $E = 200$ GPa and that the diameter of the rod is 8 mm, and neglecting the mass of the rod, determine (*a*) the equivalent spring constant of the rod, (*b*) the frequency of the vertical vibrations of a block of mass $m = 8$ kg attached to end *B* of the same rod.



(*a*) (*b*)

Fig. P19.28

19.29 Denoting by δ_{st} the static deflection of a beam under a given load, show that the frequency of vibration of the load is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

Neglect the mass of the beam, and assume that the load remains in contact with the beam.

19.30 A 40-mm deflection of the second floor of a building is measured directly under a newly installed 3500-kg piece of rotating machinery which has a slightly unbalanced rotor. Assuming that the deflection of the floor is proportional to the load it supports, determine (*a*) the equivalent spring constant of the floor system, (*b*) the speed in rpm of the rotating machinery that should be avoided if it is not to coincide with the natural frequency of the floor-machinery system.

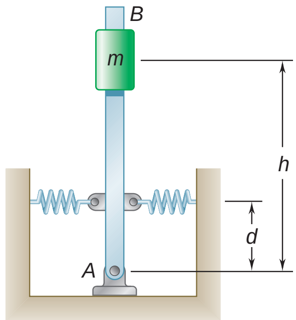


Fig. P19.31

19.31 If $h = 700$ mm and $d = 500$ mm and each spring has a constant $k = 600$ N/m, determine the mass m for which the period of small oscillations is (*a*) 0.50 s, (*b*) infinite. Neglect the mass of the rod and assume that each spring can act in either tension or compression.

19.32 The force-deflection equation for a nonlinear spring fixed at one end is $F = 1.5x^{1/2}$ where F is the force, expressed in newtons, applied at the other end and x is the deflection expressed in meters. (*a*) Determine the deflection x_0 if a 4-oz block is suspended from the spring and is at rest. (*b*) Assuming that the slope of the force-deflection curve at the point corresponding to this loading can be used as an equivalent spring constant, determine the frequency of vibration of the block if it is given a very small downward displacement from its equilibrium position and released.

***19.33** Expanding the integrand in Eq. (19.19) of Sec. 19.4 into a series of even powers of $\sin \phi$ and integrating, show that the period of a simple pendulum of length l may be approximated by the formula

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left(1 + \tfrac{1}{4}\sin^2\frac{\theta_m}{2}\right)$$

where θ_m is the amplitude of the oscillations.

***19.34** Using the formula given in Prob. 19.33, determine the amplitude θ_m for which the period of a simple pendulum is $\frac{1}{2}$ percent longer than the period of the same pendulum for small oscillations.

***19.35** Using the data of Table 19.1, determine the period of a simple pendulum of length $l = 750$ mm (*a*) for small oscillations, (*b*) for oscillations of amplitude $\theta_m = 60°$, (*c*) for oscillations of amplitude $\theta_m = 90°$.

***19.36** Using the data of Table 19.1, determine the length in inches of a simple pendulum which oscillates with a period of 2 s and an amplitude of 90°.

19.5 FREE VIBRATIONS OF RIGID BODIES

The analysis of the vibrations of a rigid body or of a system of rigid bodies possessing a single degree of freedom is similar to the analysis of the vibrations of a particle. An appropriate variable, such as a distance x or an angle θ, is chosen to define the position of the body or system of bodies, and an equation relating this variable and its second derivative with respect to t is written. If the equation obtained is of the same form as (19.6), i.e., if we have

$$\ddot{x} + \omega_n^2 x = 0 \qquad \text{or} \qquad \ddot{\theta} + \omega_n^2 \theta = 0 \tag{19.21}$$

the vibration considered is a simple harmonic motion. The period and natural frequency of the vibration can then be obtained by identifying ω_n and substituting its value into Eqs. (19.13) and (19.14).

In general, a simple way to obtain one of Eqs. (19.21) is to express that the system of the external forces is equivalent to the system of the effective forces by drawing a free-body-diagram equation for an arbitrary value of the variable and writing the appropriate equation of motion. We recall that our goal should be *the determination* *of the coefficient* of the variable x or θ, *not* the determination of the variable itself or of the derivative $\ddot{x}$ or $\ddot{\theta}$. Setting this coefficient equal to ω_n^2, we obtain the natural circular frequency ω_n, from which τ_n and f_n can be determined.

The method we have outlined can be used to analyze vibrations which are truly represented by a simple harmonic motion, or vibrations of small amplitude which can be *approximated* by a simple harmonic motion. As an example, let us determine the period of the small oscillations of a square plate of side $2b$ which is suspended from the midpoint O of one of its sides (Fig. 19.5*a*). We consider the plate in an arbitrary position defined by the angle θ that the line OG forms with the vertical and draw a free-body-diagram equation to express that the weight $\mathbf{W}$ of the plate and the components $\mathbf{R}_x$ and $\mathbf{R}_y$ of the reaction at O are equivalent to the vectors $m\mathbf{a}_t$ and $m\mathbf{a}_n$ and to the couple $\bar{I}\alpha$ (Fig. 19.5*b*). Since the angular velocity and angular acceleration of the plate are equal, respectively, to $\dot{\theta}$ and $\ddot{\theta}$, the magnitudes of the two vectors are, respectively, $mb\ddot{\theta}$ and $mb\dot{\theta}^2$, while the moment of the couple is $\bar{I}\ddot{\theta}$. In previous applications of this method (Chap. 16), we tried whenever possible to assume the correct sense for the acceleration. Here, however, we must assume the same positive sense for θ and $\ddot{\theta}$ in order to obtain an equation of the form (19.21). Consequently, the angular acceleration $\ddot{\theta}$ will be assumed positive counterclockwise, even though this assumption is obviously unrealistic. Equating moments about O, we write

$$+\circlearrowleft \qquad -W(b \sin \theta) = (mb\ddot{\theta})b + \bar{I}\ddot{\theta}$$

Noting that $\bar{I} = \frac{1}{12}m[(2b)^2 + (2b)^2] = \frac{2}{3}mb^2$ and $W = mg$, we obtain

$$\ddot{\theta} + \frac{3}{5}\frac{g}{b}\sin\theta = 0 \tag{19.22}$$

For oscillations of small amplitude, we can replace $\sin\theta$ by θ, expressed in radians, and write

$$\ddot{\theta} + \frac{3}{5}\frac{g}{b}\theta = 0 \tag{19.23}$$

Comparison with (19.21) shows that the equation obtained is that of a simple harmonic motion and that the natural circular frequency ω_n of the oscillations is equal to $(3g/5b)^{1/2}$. Substituting into (19.13), we find that the period of the oscillations is

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{5b}{3g}} \tag{19.24}$$

The result obtained is valid only for oscillations of small amplitude. A more accurate description of the motion of the plate is obtained by comparing Eqs. (19.16) and (19.22). We note that the two equations are identical if we choose l equal to $5b/3$. This means that the plate will oscillate as a simple pendulum of length $l = 5b/3$ and the results of Sec. 19.4 can be used to correct the value of the period given in (19.24). The point A of the plate located on line OG at a distance $l = 5b/3$ from O is defined as the *center of oscillation* corresponding to O (Fig. 19.5*a*).

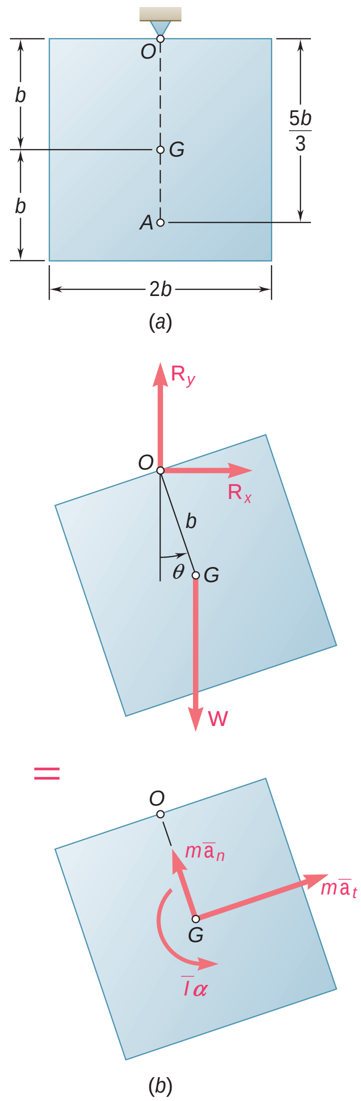
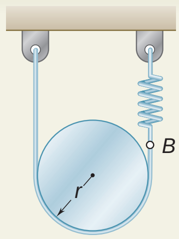
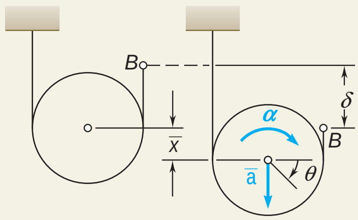


Fig. 19.5



SAMPLE PROBLEM 19.2

A cylinder of weight W and radius r is suspended from a looped cord as shown. One end of the cord is attached directly to a rigid support, while the other end is attached to a spring of constant k. Determine the period and natural frequency of the vibrations of the cylinder.

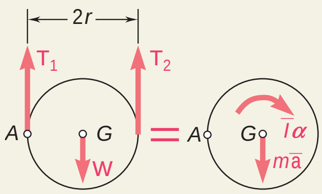


SOLUTION

Kinematics of Motion. We express the linear displacement and the acceleration of the cylinder in terms of the angular displacement θ. Choosing the positive sense clockwise and measuring the displacements from the equilibrium position, we write

$$\bar{x} = r\theta \qquad \delta = 2\bar{x} = 2r\theta$$

$$\boldsymbol{\alpha} = \ddot{\theta} \circlearrowright \qquad \bar{a} = r\alpha = r\ddot{\theta} \qquad \bar{\mathbf{a}} = r\ddot{\theta}\downarrow \tag{1}$$



Equations of Motion. The system of external forces acting on the cylinder consists of the weight $\mathbf{W}$ and of the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ exerted by the cord. We express that this system is equivalent to the system of effective forces represented by the vector $m\bar{\mathbf{a}}$ attached at G and the couple $\bar{I}\boldsymbol{\alpha}$.

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \qquad Wr - T_2(2r) = m\bar{a}r + \bar{I}\alpha \tag{2}$$

When the cylinder is in its position of equilibrium, the tension in the cord is $T_0 = \frac{1}{2}W$. We note that for an angular displacement θ, the magnitude of $\mathbf{T}_2$ is

$$T_2 = T_0 + k\delta = \tfrac{1}{2}W + k\delta = \tfrac{1}{2}W + k(2r\theta) \tag{3}$$

Substituting from (1) and (3) into (2), and recalling that $\bar{I} = \frac{1}{2}mr^2$, we write

$$Wr - (\tfrac{1}{2}W + 2kr\theta)(2r) = m(r\ddot{\theta})r + \tfrac{1}{2}mr^2\ddot{\theta}$$

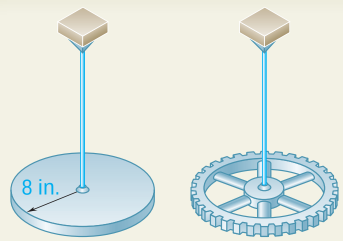
$$\ddot{\theta} + \frac{8}{3}\frac{k}{m}\theta = 0$$

The motion is seen to be simple harmonic, and we have

$$\omega_n^2 = \frac{8}{3}\frac{k}{m} \qquad \omega_n = \sqrt{\frac{8}{3}\frac{k}{m}}$$

$$\tau_n = \frac{2\pi}{\omega_n} \qquad \tau_n = 2\pi\sqrt{\frac{3}{8}\frac{m}{k}} \quad ◀$$

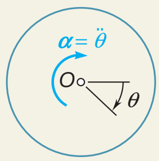
$$f_n = \frac{\omega_n}{2\pi} \qquad f_n = \frac{1}{2\pi}\sqrt{\frac{8}{3}\frac{k}{m}} \quad ◀$$



SAMPLE PROBLEM 19.3

A circular disk, weighing 20 lb and of radius 8 in., is suspended from a wire as shown. The disk is rotated (thus twisting the wire) and then released; the period of the torsional vibration is observed to be 1.13 s. A gear is then suspended from the same wire, and the period of torsional vibration for the gear is observed to be 1.93 s. Assuming that the moment of the couple exerted by the wire is proportional to the angle of twist, determine (*a*) the torsional spring constant of the wire, (*b*) the centroidal moment of inertia of the gear, (*c*) the maximum angular velocity reached by the gear if it is rotated through 90° and released.

SOLUTION



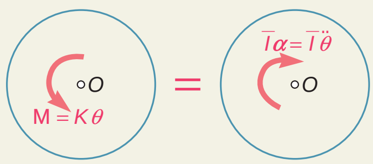
a. Vibration of Disk. Denoting by θ the angular displacement of the disk, we express that the magnitude of the couple exerted by the wire is $M = K\theta$, where K is the torsional spring constant of the wire. Since this couple must be equivalent to the couple $\bar{I}\alpha$ representing the effective forces of the disk, we write

$$+\circlearrowleft \Sigma M_O = \Sigma(M_O)_{\text{eff}}: \qquad +K\theta = -\bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

The motion is seen to be simple harmonic, and we have

$$\omega_n^2 = \frac{K}{\bar{I}} \qquad \tau_n = \frac{2\pi}{\omega_n} \qquad \tau_n = 2\pi\sqrt{\frac{\bar{I}}{K}} \tag{1}$$



For the disk, we have

$$\tau_n = 1.13 \text{ s} \qquad \bar{I} = \tfrac{1}{2}mr^2 = \frac{1}{2}\left(\frac{20 \text{ lb}}{32.2 \text{ ft/s}^2}\right)\left(\frac{8}{12}\text{ ft}\right)^2 = 0.138 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Substituting into (1), we obtain

$$1.13 = 2\pi\sqrt{\frac{0.138}{K}} \qquad K = 4.27 \text{ lb} \cdot \text{ft/rad} \blacktriangleleft$$

b. Vibration of Gear. Since the period of vibration of the gear is 1.93 s and $K = 4.27$ lb · ft/rad, Eq. (1) yields

$$1.93 = 2\pi\sqrt{\frac{\bar{I}}{4.27}} \qquad \bar{I}_{\text{gear}} = 0.403 \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \blacktriangleleft$$

c. Maximum Angular Velocity of Gear. Since the motion is simple harmonic, we have

$$\theta = \theta_m \sin \omega_n t \qquad \omega = \theta_m\omega_n \cos \omega_n t \qquad \omega_m = \theta_m\omega_n$$

Recalling that $\theta_m = 90° = 1.571$ rad and $\tau = 1.93$ s, we write

$$\omega_m = \theta_m\omega_n = \theta_m\left(\frac{2\pi}{\tau}\right) = (1.571 \text{ rad})\left(\frac{2\pi}{1.93 \text{ s}}\right)$$

$$\omega_m = 5.11 \text{ rad/s} \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you saw that a rigid body, or a system of rigid bodies, whose position can be defined by a single coordinate x or θ, will execute a simple harmonic motion if the differential equation obtained by applying Newton's second law is of the form

$$\ddot{x} + \omega_n^2 x = 0 \qquad \text{or} \qquad \ddot{\theta} + \omega_n^2 \theta = 0 \tag{19.21}$$

Your goal should be to determine ω_n, from which you can obtain the period τ_n and the natural frequency f_n. Taking into account the initial conditions, you can then write an equation of the form

$$x = x_m \sin(\omega_n t + \phi) \tag{19.10}$$

where x should be replaced by θ if a rotation is involved. To solve the problems in this lesson, you will follow these steps:

1. Choose a coordinate which will measure the displacement of the body from its equilibrium position. You will find that many of the problems in this lesson involve the rotation of a body about a fixed axis and that the angle measuring the rotation of the body from its equilibrium position is the most convenient coordinate to use. In problems involving the general plane motion of a body, where a coordinate x (and possibly a coordinate y) is used to define the position of the mass center G of the body, and a coordinate θ is used to measure its rotation about G, find kinematic relations which will allow you to express x (and y) in terms of θ [Sample Prob. 19.2].

2. Draw a free-body-diagram equation to express that the system of the external forces is equivalent to the system of the effective forces, which consists of the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$, where $\bar{a} = \ddot{x}$ and $\alpha = \ddot{\theta}$. Be sure that each applied force or couple is drawn in a direction consistent with the assumed displacement and that the senses of $\bar{\mathbf{a}}$ and $\boldsymbol{\alpha}$ are, respectively, those in which the coordinates x and θ are increasing.

3. Write the differential equations of motion by equating the sums of the components of the external and effective forces in the x and y directions and the sums of their moments about a given point. If necessary, use the kinematic relations developed in step 1 to obtain equations involving only the coordinate u. If u is a small angle, replace sin u by u and cos u by 1, if these functions appear in your equations. Eliminating any unknown reactions, you will obtain an equation of the type of Eqs. (19.21). Note that in problems involving a body rotating about a fixed axis, you can immediately obtain such an equation by equating the moments of the external and effective forces about the fixed axis.

4. Comparing the equation you have obtained with one of Eqs. (19.21), you can identify v_n^2 and, thus, determine the natural circular frequency v_n. Remember that the object of your analysis is *not to solve* the differential equation you have obtained, *but to identify* v_n^2.

5. Determine the amplitude and the phase angle F by substituting the value obtained for v_n and the initial values of the coordinate and its first derivative into Eq. (19.10) and the equation obtained by differentiating (19.10) with respect to t. From Eq. (19.10) and the two equations obtained by differentiating (19.10) twice with respect to t, and using the kinematic relations developed in step 1, you will be able to determine the position, velocity, and acceleration of any point of the body at any given time.

6. In problems involving torsional vibrations, the torsional spring constant K is expressed in N · m/rad or lb · ft/rad. The product of K and the angle of twist u, expressed in radians, yields the moment of the restoring couple, which should be equated to the sum of the moments of the effective forces or couples about the axis of rotation [Sample Prob. 19.3].

PROBLEMS

19.37 The uniform rod shown has mass 6 kg and is attached to a spring of constant $k = 700$ N/m. If end B of the rod is depressed 10 mm and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of end B.

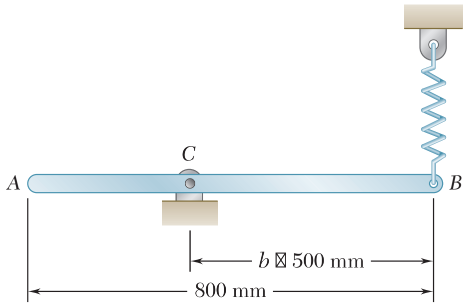


Fig. P19.37

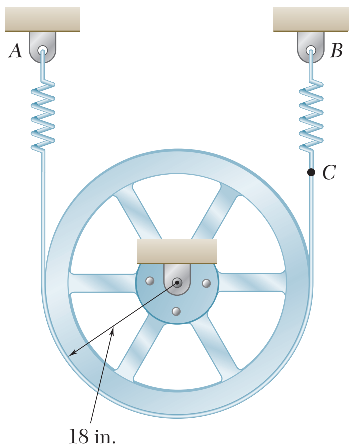


Fig. P19.38

19.38 A belt is placed around the rim of a 500-lb flywheel and attached as shown to two springs, each of constant $k = 85$ lb/in. If end C of the belt is pulled 1.5 in. down and released, the period of vibration of the flywheel is observed to be 0.5 s. Knowing that the initial tension in the belt is sufficient to prevent slipping, determine (*a*) the maximum angular velocity of the flywheel, (*b*) the centroidal radius of gyration of the flywheel.

19.39 An 8-kg uniform rod AB is hinged to a fixed support at A and is attached by means of pins B and C to a 12-kg disk of radius 400 mm. A spring attached at D holds the rod at rest in the position shown. If point B is moved down 25 mm and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of point B.

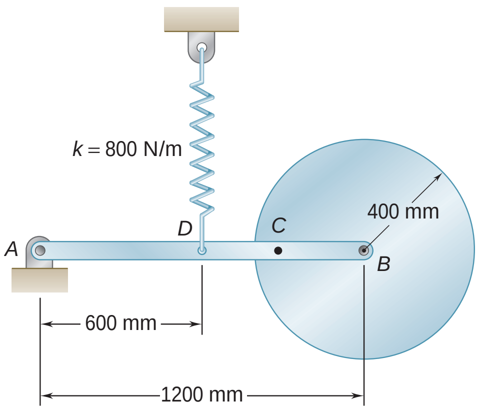


Fig. P19.39

19.40 Solve Prob. 19.39, assuming that pin C is removed and that the disk can rotate freely about pin B.

19.41 A 15-lb slender rod AB is riveted to a 12-lb uniform disk as shown. A belt is attached to the rim of the disk and to a spring which holds the rod at rest in the position shown. If end A of the rod is moved 0.75 in. down and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of end A.

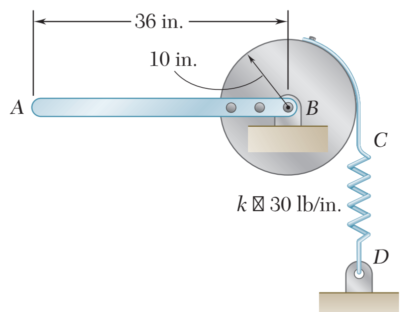


Fig. P19.41

19.42 A 30-lb uniform cylinder can roll without sliding on a 15° incline. A belt is attached to the rim of the cylinder, and a spring holds the cylinder at rest in the position shown. If the center of the cylinder is moved 2 in. down the incline and released, determine (*a*) the period of vibration, (*b*) the maximum acceleration of the center of the cylinder.

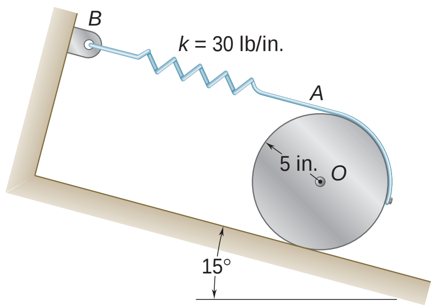


Fig. P19.42

19.43 A square plate of mass m is held by eight springs, each of constant k. Knowing that each spring can act in either tension or compression, determine the frequency of the resulting vibration if (*a*) the plate is given a small vertical displacement and released, (*b*) the plate is rotated through a small angle about G and released.

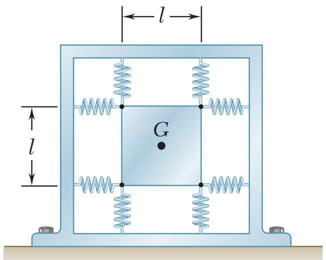


Fig. *P19.43*

19.44 Two small weights w are attached at A and B to the rim of a uniform disk of radius r and weight W. Denoting by τ_0 the period of small oscillations when $\beta = 0$, determine the angle β for which the period of small oscillations is $2\tau_0$.

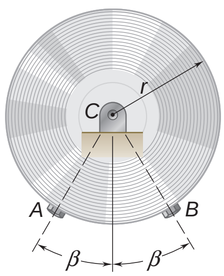


Fig. P19.44 and P19.45

19.45 Two 40-g weights are attached at A and B to the rim of a 1.5-kg uniform disk of radius r = 100 mm. Determine the frequency of small oscillations when β = 60°.

19.46 A three-blade wind turbine used for research is supported on a shaft so that it is free to rotate about O. One technique to determine the centroidal mass moment of inertia of an object is to place a known weight at a known distance from the axis of rotation and to measure the frequency of oscillations after releasing it from rest with a small initial angle. In this case, a weight of W_{add} = 50 lb is attached to one of the blades at a distance R = 20 ft from the axis of rotation. Knowing that when the blade with the added weight is displaced slightly from the vertical axis, and the system is found to have a period of 7.6 s, determine the centroidal mass moment of inertia of the three-blade rotor.

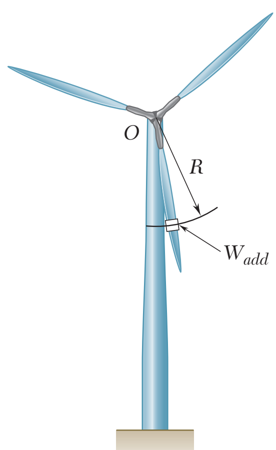


Fig. *P19.46*

19.47 A connecting rod is supported by a knife-edge at point A; the period of its small oscillations is observed to be 0.87 s. The rod is then inverted and supported by a knife edge at point B and the period of its small oscillations is observed to be 0.78 s. Knowing that $r_a + r_b = 10$ in., determine (*a*) the location of the mass center G, (*b*) the centroidal radius of gyration $\bar{k}$.

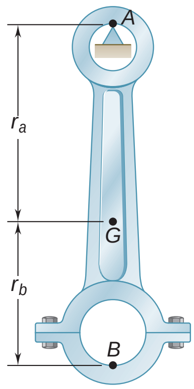


Fig. *P19.47*

19.48 A 75-mm-radius hole is cut in a 200-mm-radius uniform disk which is attached to a frictionless pin at its geometric center O. Determine (*a*) the period of small oscillations of the disk, (*b*) the length of a simple pendulum which has the same period.

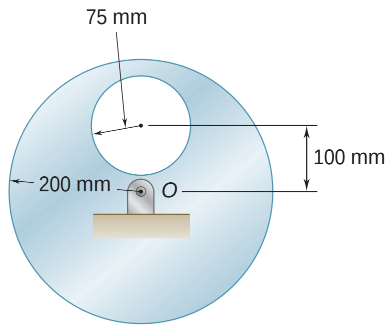


Fig. P19.48

19.49 A uniform disk of radius $r = 250$ mm is attached at A to a 650-mm rod AB of negligible mass which can rotate freely in a vertical plane about B. Determine the period of small oscillations (*a*) if the disk is free to rotate in a bearing at A, (*b*) if the rod is riveted to the disk at A.

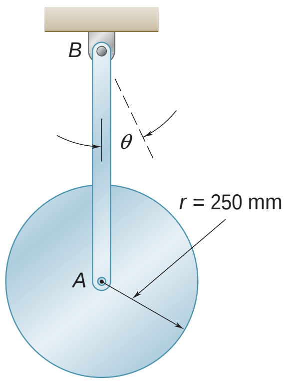


Fig. P19.49

19.50 A small collar of mass 1 kg is rigidly attached to a 3-kg uniform rod of length $L = 750$ mm. Determine (*a*) the distance d to maximize the frequency of oscillation when the rod is given a small initial displacement, (*b*) the corresponding period of oscillation.

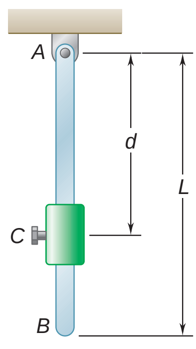


Fig. P19.50

19.51 For the uniform square plate of side $b = 12$ in., determine (*a*) the period of small oscillations if the plate is suspended as shown, (*b*) the distance c from O to a point A from which the plate should be suspended for the period to be a minimum.

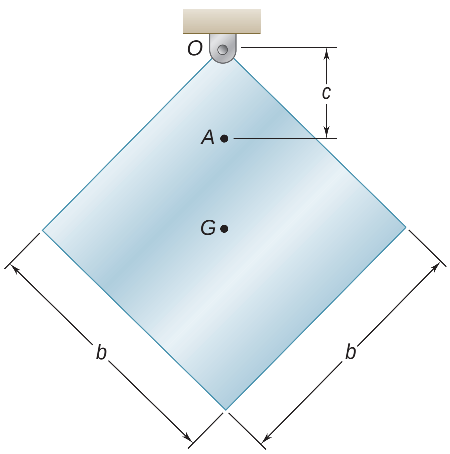


Fig. P19.51

19.52 A *compound pendulum* is defined as a rigid slab which oscillates about a fixed point O, called the center of suspension. Show that the period of oscillation of a compound pendulum is equal to the period of a simple pendulum of length OA, where the distance from A to the mass center G is $GA = \bar{k}^2/\bar{r}$. Point A is defined as the center of oscillation and coincides with the center of percussion defined in Prob. 17.66.

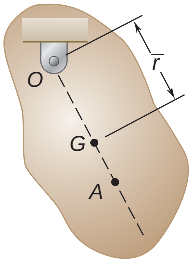


Fig. P19.52 and P19.53

19.53 A rigid slab oscillates about a fixed point O. Show that the smallest period of oscillation occurs when the distance $\bar{r}$ from point O to the mass center G is equal to $\bar{k}$.

19.54 Show that if the compound pendulum of Prob. 19.52 is suspended from A instead of O, the period of oscillation is the same as before and the new center of oscillation is located at O.

19.55 The 8-kg uniform bar AB is hinged at C and is attached at A to a spring of constant $k = 500$ N/m. If end A is given a small displacement and released, determine (*a*) the frequency of small oscillations, (*b*) the smallest value of the spring constant k for which oscillations will occur.

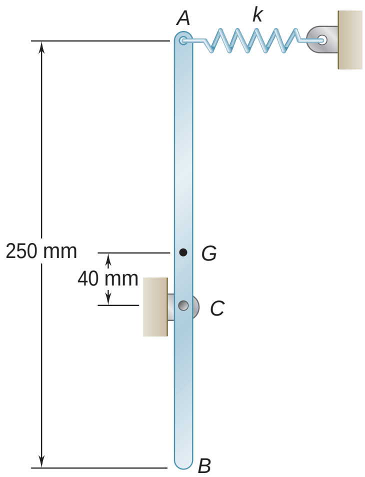


Fig. P19.55

19.56 Two uniform rods, each of mass $m = 12$ kg and length $L = 800$ mm, are welded together to form the assembly shown. Knowing that the constant of each spring is $k = 500$ N/m and that end A is given a small displacement and released, determine the frequency of the resulting motion.

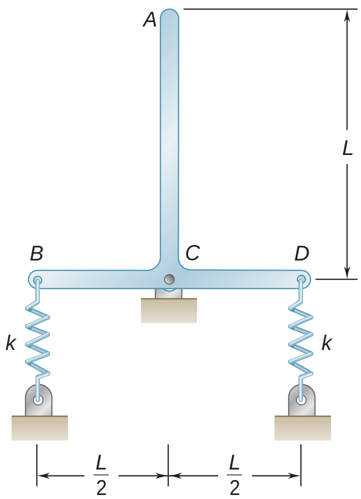


Fig. P19.56

19.57 A 45-lb uniform square plate is suspended from a pin located at the midpoint A of one of its 1.2-ft edges and is attached to springs, each of constant $k = 8$ lb/in. If corner B is given a small displacement and released, determine the frequency of the resulting vibration. Assume that each spring can act in either tension or compression.

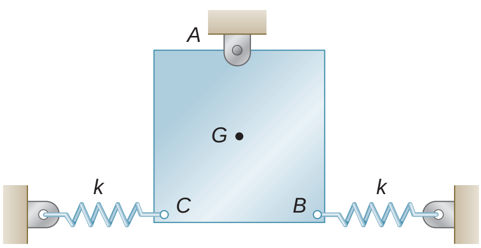


Fig. P19.57

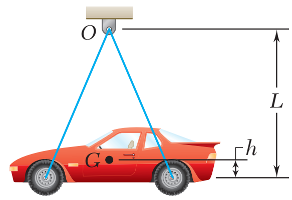


Fig. ***P19.58***

19.58 A 1300-kg sports car has a center of gravity G located a distance h above a line connecting the front and rear axles. The car is suspended from cables that are attached to the front and rear axles as shown. Knowing that the periods of oscillation are 4.04 s when $L = 4$ m and 3.54 s when $L = 3$ m, determine h and the centroidal radius of gyration.

19.59 A 6-lb slender rod is suspended from a steel wire which is known to have a torsional spring constant $K = 1.5$ ft · lb/rad. If the rod is rotated through 180° about the vertical and released, determine (*a*) the period of oscillation, (*b*) the maximum velocity of end A of the rod.

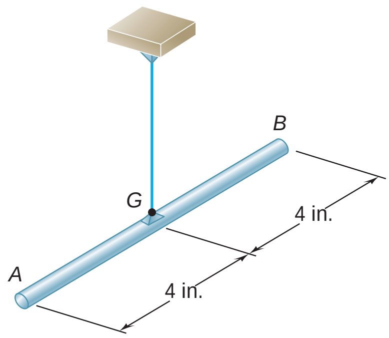


Fig. P19.59

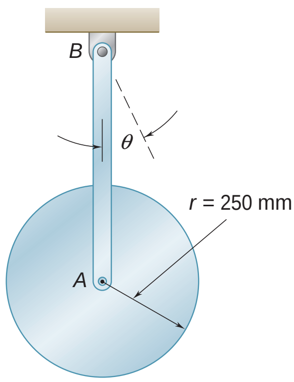


Fig. P19.60

19.60 A uniform disk of radius $r = 250$ mm is attached at A to a 650-mm rod AB of negligible mass which can rotate freely in a vertical plane about B. If the rod is displaced 2° from the position shown and released, determine the magnitude of the maximum velocity of point A, assuming that the disk is (*a*) free to rotate in a bearing at A, (*b*) riveted to the rod at A.

19.61 Two uniform rods, each of mass m and length l, are welded together to form the T-shaped assembly shown. Determine the frequency of small oscillations of the assembly.

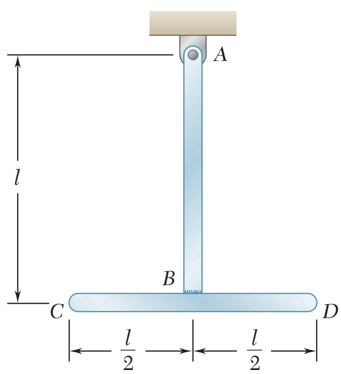


Fig. ***P19.61***

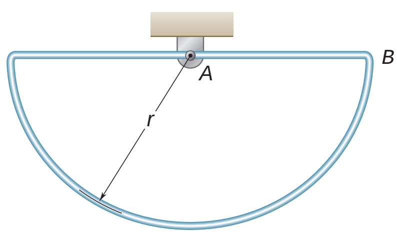


Fig. P19.62

19.62 A homogeneous wire bent to form the figure shown is attached to a pin support at A. Knowing that $r = 220$ mm and that point B is pushed down 20 mm and released, determine the magnitude of the velocity of B, 8 s later.

19.63 A horizontal platform P is held by several rigid bars which are connected to a vertical wire. The period of oscillation of the platform is found to be 2.2 s when the platform is empty and 3.8 s when an object A of unknown moment of inertia is placed on the platform with its mass center directly above the center of the plate. Knowing that the wire has a torsional constant $K = 27 \text{ N} \cdot \text{m/rad}$, determine the centroidal moment of inertia of object A.

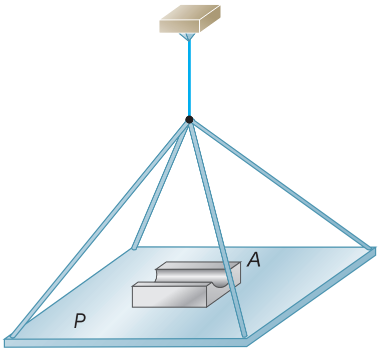


Fig. P19.63

19.64 A uniform disk of radius $r = 120$ mm is welded at its center to two elastic rods of equal length with fixed ends at A and B. Knowing that the disk rotates through an 8° angle when a 500-mN · m couple is applied to the disk and that it oscillates with a period of 1.3 s when the couple is removed, determine (a) the mass of the disk, (b) the period of vibration if one of the rods is removed.

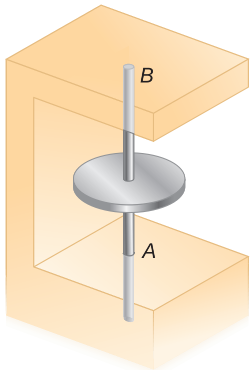


Fig. P19.64

19.65 A 5-kg uniform rod CD of length $l = 0.7$ m is welded at C to two elastic rods, which have fixed ends at A and B and are known to have a combined torsional spring constant $K = 24 \text{ N} \cdot \text{m/rad}$. Determine the period of small oscillations, if the equilibrium position of CD is (a) vertical as shown, (b) horizontal.

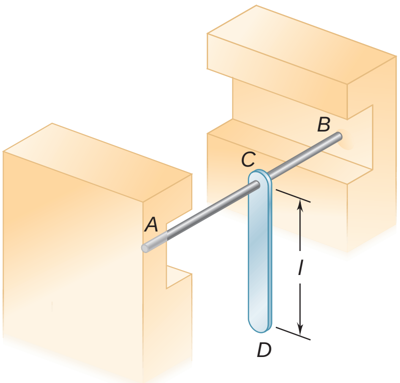


Fig. *P19.65*

19.66 A 1.8-kg uniform plate in the shape of an equilateral triangle is suspended at its center of gravity from a steel wire which is known to have a torsional constant $K = 35 \text{ mN} \cdot \text{m/rad}$. If the plate is rotated 360° about the vertical and then released, determine (a) the period of oscillation, (b) the maximum velocity of one of the vertices of the triangle.

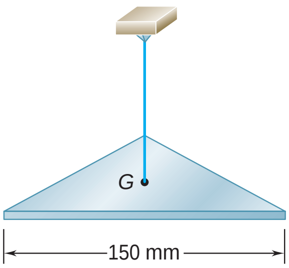


Fig. *P19.66*

19.67 A period of 6.00 s is observed for the angular oscillations of a 4-oz gyroscope rotor suspended from a wire as shown. Knowing that a period of 3.80 s is obtained when a 1.25-in.-diameter steel sphere is suspended in the same fashion, determine the centroidal radius of gyration of the rotor. (Specific weight of steel = 490 lb/ft^3.)

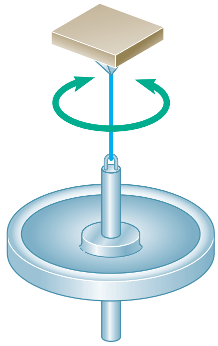


Fig. P19.67

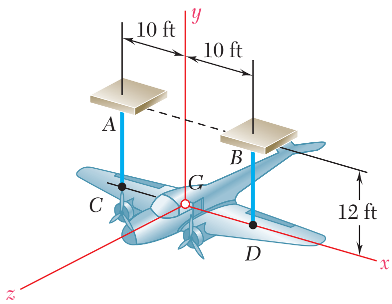


Fig. P19.68

19.68 The centroidal radius of gyration $\bar{k}_y$ of an airplane is determined by suspending the airplane by two 12-ft-long cables as shown. The airplane is rotated through a small angle about the vertical through G and then released. Knowing that the observed period of oscillation is 3.3 s, determine the centroidal radius of gyration $\bar{k}_y$.

19.6 APPLICATION OF THE PRINCIPLE OF CONSERVATION OF ENERGY

We saw in Sec. 19.2 that when a particle of mass m is in simple harmonic motion, the resultant **F** of the forces exerted on the particle has a magnitude proportional to the displacement x measured from the position of equilibrium O and is directed toward O; we write $F = -kx$. Referring to Sec. 13.6, we note that **F** is a *conservative force* and that the corresponding potential energy is $V = \frac{1}{2}kx^2$, where V is assumed equal to zero in the equilibrium position $x = 0$. Since the velocity of the particle is equal to $\dot{x}$, its kinetic energy is $T = \frac{1}{2}m\dot{x}^2$ and we can express that the total energy of the particle is conserved by writing

$$T + V = \text{constant} \qquad \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \text{constant}$$

Dividing through by $m/2$ and recalling from Sec. 19.2 that $k/m = \omega_n^2$, where ω_n is the natural circular frequency of the vibration, we have

$$\dot{x}^2 + \omega_n^2 x^2 = \text{constant} \qquad (19.25)$$

Equation (19.25) is characteristic of a simple harmonic motion, since it can be obtained from Eq. (19.6) by multiplying both terms by $2\dot{x}$ and integrating.

The principle of conservation of energy provides a convenient way for determining the period of vibration of a rigid body or of a system of rigid bodies possessing a single degree of freedom, once it has been established that the motion of the system is a simple harmonic motion or that it can be approximated by a simple harmonic motion. Choosing an appropriate variable, such as a distance x or an angle θ, we consider two particular positions of the system:

1. *The displacement of the system is maximum;* we have $T_1 = 0$, and V_1 can be expressed in terms of the amplitude x_m or θ_m (choosing $V = 0$ in the equilibrium position).
2. *The system passes through its equilibrium position;* we have $V_2 = 0$, and T_2 can be expressed in terms of the maximum velocity $\dot{x}_m$ or the maximum angular velocity $\dot{\theta}_m$.

We then express that the total energy of the system is conserved and write $T_1 + V_1 = T_2 + V_2$. Recalling from (19.15) that for simple harmonic motion the maximum velocity is equal to the product of the amplitude and of the natural circular frequency ω_n, we find that the equation obtained can be solved for ω_n.

As an example, let us consider again the square plate of Sec. 19.5. In the position of maximum displacement (Fig. 19.6*a*), we have

$$T_1 = 0 \qquad V_1 = W(b - b\cos\theta_m) = Wb(1 - \cos\theta_m)$$

or, since $1 - \cos\theta_m = 2\sin^2(\theta_m/2) \approx 2(\theta_m/2)^2 = \theta_m^2/2$ for oscillations of small amplitude,

$$T_1 = 0 \qquad V_1 = \tfrac{1}{2}Wb\theta_m^2 \tag{19.26}$$

As the plate passes through its position of equilibrium (Fig. 19.6*b*), its velocity is maximum and we have

$$T_2 = \tfrac{1}{2}m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 = \tfrac{1}{2}mb^2\dot{\theta}_m^2 + \tfrac{1}{2}\bar{I}\dot{\theta}_m^2 \qquad V_2 = 0$$

or, recalling from Sec. 19.5 that $\bar{I} = \frac{2}{3}mb^2$,

$$T_2 = \tfrac{1}{2}(\tfrac{5}{3}mb^2)\dot{\theta}_m^2 \qquad V_2 = 0 \tag{19.27}$$

Substituting from (19.26) and (19.27) into $T_1 + V_1 = T_2 + V_2$, and noting that the maximum velocity $\dot{\theta}_m$ is equal to the product $\theta_m\omega_n$, we write

$$\tfrac{1}{2}Wb\theta_m^2 = \tfrac{1}{2}(\tfrac{5}{3}mb^2)\theta_m^2\omega_n^2 \tag{19.28}$$

which yields $\omega_n^2 = 3g/5b$ and

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{5b}{3g}} \tag{19.29}$$

as previously obtained.

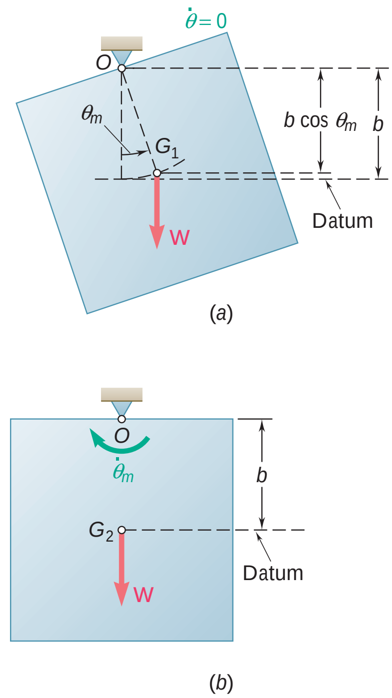
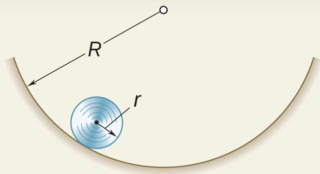
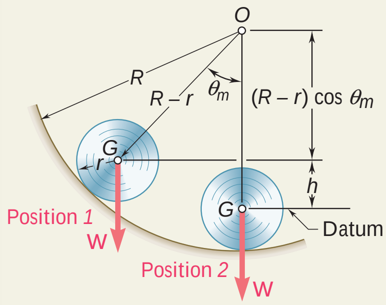


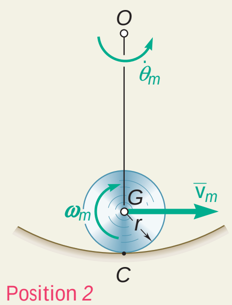
Fig. 19.6



SAMPLE PROBLEM 19.4

Determine the period of small oscillations of a cylinder of radius r which rolls without slipping inside a curved surface of radius R.





SOLUTION

We denote by θ the angle which line OG forms with the vertical. Since the cylinder rolls without slipping, we may apply the principle of conservation of energy between position *1*, where $\theta = \theta_m$, and position *2*, where $\theta = 0$.

Position 1

Kinetic Energy. Since the velocity of the cylinder is zero, $T_1 = 0$.

Potential Energy. Choosing a datum as shown and denoting by W the weight of the cylinder, we have

$$V_1 = Wh = W(R - r)(1 - \cos\theta)$$

Noting that for small oscillations $(1 - \cos\theta) = 2\sin^2(\theta/2) \approx \theta^2/2$, we have

$$V_1 = W(R - r)\frac{\theta_m^2}{2}$$

Position 2. Denoting by $\dot{\theta}_m$ the angular velocity of line OG as the cylinder passes through position 2, and observing that point C is the instantaneous center of rotation of the cylinder, we write

$$\bar{v}_m = (R - r)\dot{\theta}_m \qquad \omega_m = \frac{\bar{v}_m}{r} = \frac{R - r}{r}\dot{\theta}_m$$

Kinetic Energy

$$\begin{aligned} T_2 &= \tfrac{1}{2}m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 \\ &= \tfrac{1}{2}m(R - r)^2\dot{\theta}_m^2 + \tfrac{1}{2}\left(\tfrac{1}{2}mr^2\right)\left(\frac{R - r}{r}\right)^2\dot{\theta}_m^2 \\ &= \tfrac{3}{4}m(R - r)^2\dot{\theta}_m^2 \end{aligned}$$

Potential Energy

$$V_2 = 0$$

Conservation of Energy

$$T_1 + V_1 = T_2 + V_2$$

$$0 + W(R - r)\frac{\theta_m^2}{2} = \tfrac{3}{4}m(R - r)^2\dot{\theta}_m^2 + 0$$

Since $\dot{\theta}_m = \omega_n\theta_m$ and $W = mg$, we write

$$mg(R - r)\frac{\theta_m^2}{2} = \tfrac{3}{4}m(R - r)^2(\omega_n\theta_m)^2 \qquad \omega_n^2 = \frac{2}{3}\frac{g}{R - r}$$

$$\tau_n = \frac{2\pi}{\omega_n} \qquad \tau_n = 2\pi\sqrt{\frac{3}{2}\frac{R - r}{g}} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In the problems which follow you will be asked to use the *principle of conservation of energy* to determine the period or natural frequency of the simple harmonic motion of a particle or rigid body. Assuming that you choose an angle θ to define the position of the system (with $\theta = 0$ in the equilibrium position), as you will in most of the problems in this lesson, you will express that the total energy of the system is conserved, $T_1 + V_1 = T_2 + V_2$, between the position 1 of maximum displacement ($\theta_1 = \theta_m$, $\dot{\theta}_1 = 0$) and the position 2 of maximum velocity ($\dot{\theta}_2 = \dot{\theta}_m$, $\theta_2 = 0$). It follows that T_1 and V_2 will both be zero, and the energy equation will reduce to $V_1 = T_2$, where V_1 and T_2 are homogeneous quadratic expressions in θ_m and $\dot{\theta}_m$, respectively. Recalling that, for a simple harmonic motion, $\dot{\theta}_m = \theta_m \omega_n$ and substituting this product into the energy equation, you will obtain, after reduction, an equation that you can solve for ω_n^2. Once you have determined the natural circular frequency ω_n, you can obtain the period τ_n and the natural frequency f_n of the vibration.

The steps that you should take are as follows:

1. Calculate the potential energy V_1 of the system in its position of maximum displacement. Draw a sketch of the system in its position of maximum displacement and express the potential energy of all the forces involved (internal as well as external) in terms of the maximum displacement x_m or θ_m.

a. The potential energy associated with the weight W of a body is $V_g = Wy$, where y is the elevation of the center of gravity G of the body above its equilibrium position. If the problem you are solving involves the oscillation of a rigid body about a horizontal axis through a point O located at a distance b from G (Fig. 19.6), express y in terms of the angle θ that the line OG forms with the vertical: $y = b(1 - \cos\theta)$. But, for small values of θ, you can replace this expression with $y = \frac{1}{2}b\theta^2$ [Sample Prob. 19.4]. Therefore, when θ reaches its maximum value θ_m, and for oscillations of small amplitude, you can express V_g as

$$V_g = \tfrac{1}{2}Wb\theta_m^2$$

Note that *if G is located above O* in its equilibrium position (instead of below O, as we have assumed), the vertical displacement y will be negative and should be approximated as $y = -\frac{1}{2}b\theta^2$, which will result in a negative value for V_g. In the absence of other forces, the equilibrium position will be unstable, and the system will not oscillate. (See, for instance, Prob. 19.89.)

b. The potential energy associated with the elastic force exerted by a spring is $V_e = \frac{1}{2}kx^2$, where k is the constant of the spring and x its deflection. In problems involving the rotation of a body about an axis, you will generally have $x = a\theta$, where a is the distance from the axis of rotation to the point of the body

(continued)

where the spring is attached, and where θ is the angle of rotation. Therefore, when x reaches its maximum value x_m and θ reaches its maximum value θ_m, you can express V_e as

$$V_e = \tfrac{1}{2}kx_m^2 = \tfrac{1}{2}ka^2\theta_m^2$$

c. The potential energy V_1 of the system in its position of maximum displacement is obtained by adding the various potential energies that you have computed. It will be equal to the product of a constant and θ_m^2.

2. Calculate the kinetic energy T_2 of the system in its position of maximum velocity. Note that this position is also the equilibrium position of the system.

a. If the system consists of a single rigid body, the kinetic energy T_2 of the system will be the sum of the kinetic energy associated with the motion of the mass center G of the body and the kinetic energy associated with the rotation of the body about G. You will write, therefore,

$$T_2 = \tfrac{1}{2}m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2$$

Assuming that the position of the body has been defined by an angle θ, express $\bar{v}_m$ and ω_m in terms of the rate of change $\dot{\theta}_m$ of θ as the body passes through its equilibrium position. The kinetic energy of the body will thus be expressed as the product of a constant and $\dot{\theta}_m^2$. Note that if θ measures the rotation of the body about its mass center, as was the case for the plate of Fig. 19.6, then $\omega_m = \dot{\theta}_m$. In other cases, however, the kinematics of the motion should be used to derive a relation between ω_m and $\dot{\theta}_m$ [Sample Prob. 19.4].

b. If the system consists of several rigid bodies, repeat the above computation for each of the bodies, using the same coordinate θ, and add the results obtained.

3. Equate the potential energy V_1 of the system to its kinetic energy T_2,

$$V_1 = T_2$$

and, recalling the first of Eqs. (19.15), replace $\dot{\theta}_m$ in the right-hand term by the product of the amplitude θ_m and the circular frequency ω_n. Since both terms now contain the factor θ_m^2, this factor can be canceled and the resulting equation can be solved for the circular frequency ω_n.

19.69 A 1.8-kg collar A is attached to a spring of constant 800 N/m and can slide without friction on a horizontal rod. If the collar is moved 70 mm to the left from its equilibrium position and released, determine the maximum velocity and maximum acceleration of the collar during the resulting motion.

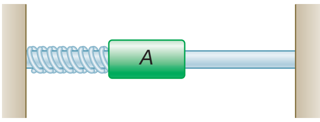


Fig. P19.69

19.70 Two blocks, each of weight 3 lb, are attached to links which are pin-connected to bar BC as shown. The weights of the links and bar are negligible, and the blocks can slide without friction. Block D is attached to a spring of constant $k = 4$ lb/in. Knowing that block A is moved 0.5 in. from its equilibrium position and released, determine the magnitude of the maximum velocity of block D during the resulting motion.

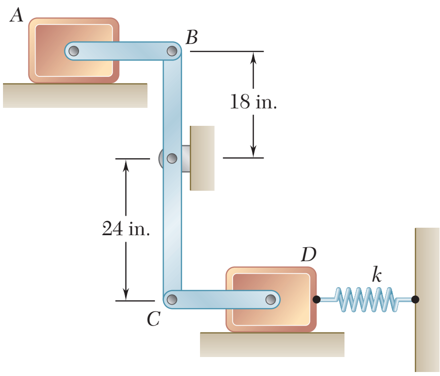


Fig. P19.70

19.71 A 14-oz sphere A and a 10-oz sphere C are attached to the ends of a rod AC of negligible weight which can rotate in a vertical plane about an axis at B. Determine the period of small oscillations of the rod.

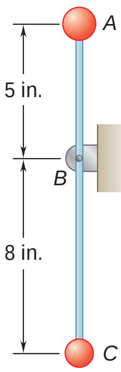


Fig. P19.71

19.72 Determine the period of small oscillations of a small particle which moves without friction inside a cylindrical surface of radius R.

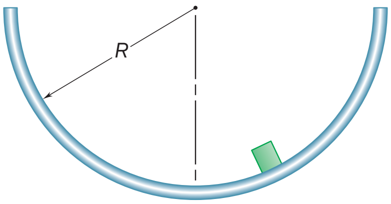


Fig. P19.72

19.73 The inner rim of an 85-lb flywheel is placed on a knife edge, and the period of its small oscillations is found to be 1.26 s. Determine the centroidal moment of inertia of the flywheel.

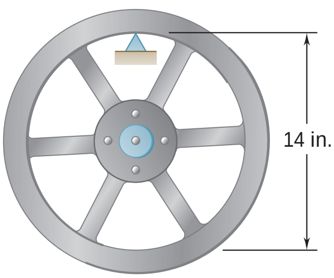


Fig. *P19.73*

19.74 A connecting rod is supported by a knife edge at point *A*; the period of its small oscillations is observed to be 1.03 s. Knowing that the distance r_a is 6 in., determine the centroidal radius of gyration of the connecting rod.

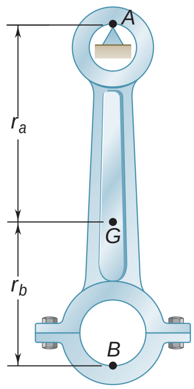


Fig. P19.74

19.75 A uniform rod *AB* can rotate in a vertical plane about a horizontal axis at *C* located at a distance *c* above the mass center *G* of the rod. For small oscillations determine the value of *c* for which the frequency of the motion will be maximum.

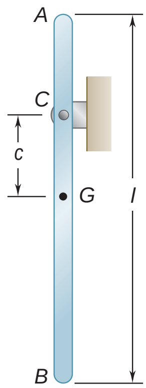


Fig. P19.75

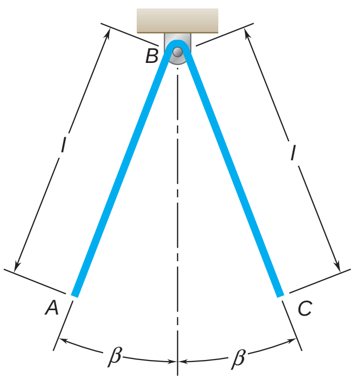


Fig. P19.76

19.76 A homogeneous wire of length $2l$ is bent as shown and allowed to oscillate about a frictionless pin at *B*. Denoting by t_0 the period of small oscillations when $b = 0$, determine the angle b for which the period of small oscillations is $2\ t_0$.

19.77 A uniform disk of radius *r* and mass *m* can roll without slipping on a cylindrical surface and is attached to bar *ABC* of length *L* and negligible mass. The bar is attached to a spring of constant *k* and can rotate freely in the vertical plane about point *B*. Knowing that end *A* is given a small displacement and released, determine the frequency of the resulting oscillations in terms of *m*, *L*, *k*, and *g*.

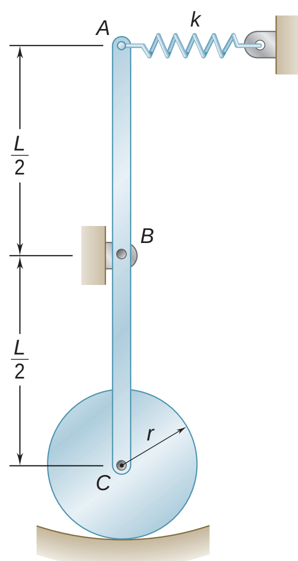


Fig. P19.77

19.78 Two uniform rods, each of weight $W = 1.2$ lb and length $l = 8$ in., are welded together to form the assembly shown. Knowing that the constant of each spring is $k = 0.6$ lb/in. and that end A is given a small displacement and released, determine the frequency of the resulting motion.

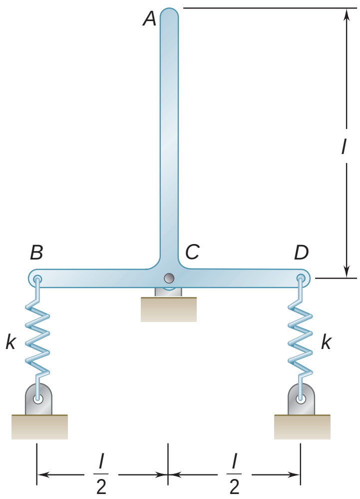


Fig. P19.78

19.79 A 15-lb uniform cylinder can roll without sliding on an incline and is attached to a spring AB as shown. If the center of the cylinder is moved 0.4 in. down the incline and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of the center of the cylinder.

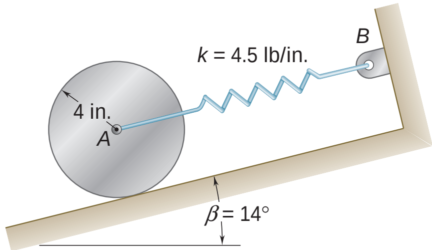


Fig. P19.79

19.80 A 3-kg slender rod AB is bolted to a 5-kg uniform disk. A spring of constant 280 N/m is attached to the disk and is unstretched in the position shown. If end B of the rod is given a small displacement and released, determine the period of vibration of the system.

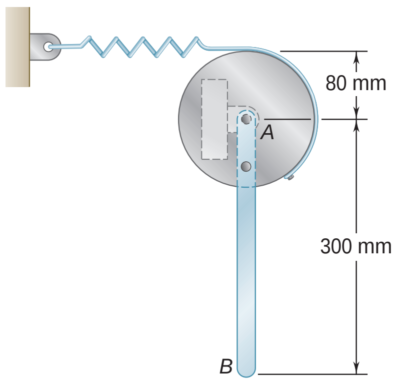


Fig. P19.80

19.81 A slender rod AB of mass m and length l is connected to two collars of negligible mass in a horizontal plane as shown. Collar A is attached to a spring of constant k. Knowing that the collars can slide freely on their respective rods and the system is in equilibrium in the position shown, determine the period of vibration if collar A is given a small displacement and released.

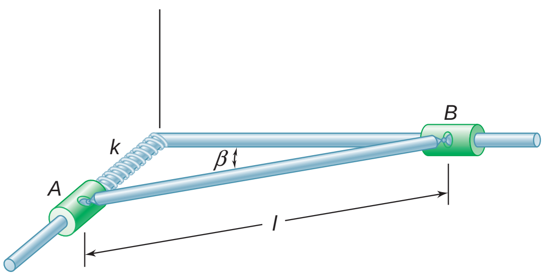


Fig. *P19.81* and *P19.82*

19.82 A slender rod AB of mass m and length l is connected to two collars of mass m_c in a horizontal plane as shown. Collar A is attached to a spring of constant k. Knowing that the collars can slide freely on their respective rods and the system is in equilibrium in the position shown, determine the period of vibration if collar A is given a small displacement and released.

19.83 An 800-g rod AB is bolted to a 1.2-kg disk. A spring of constant $k = 12$ N/m is attached to the center of the disk at A and to the wall at C. Knowing that the disk rolls without sliding, determine the period of small oscillations of the system.

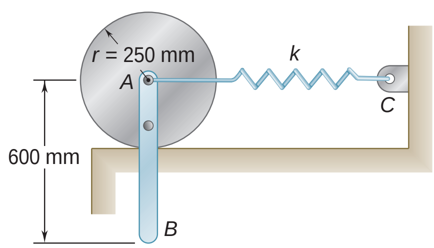


Fig. P19.83

19.84 Three identical rods are connected as shown. If $b = \frac{3}{4}l$, determine the frequency of small oscillations of the system.

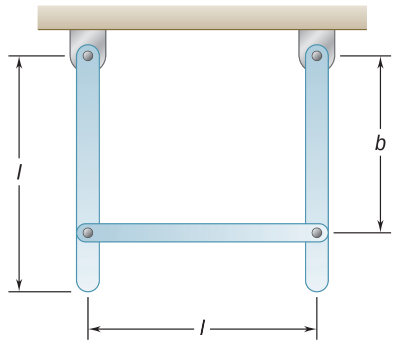


Fig. P19.84

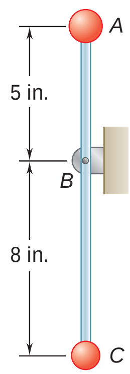


Fig. P19.85

19.85 A 14-oz sphere A and a 10-oz sphere C are attached to the ends of a 20-oz rod AC which can rotate in a vertical plane about an axis at B. Determine the period of small oscillations of the rod.

19.86 A 10-lb uniform rod CD is welded at C to a shaft of negligible mass which is welded to the centers of two 20-lb uniform disks A and B. Knowing that the disks roll without sliding, determine the period of small oscillations of the system.

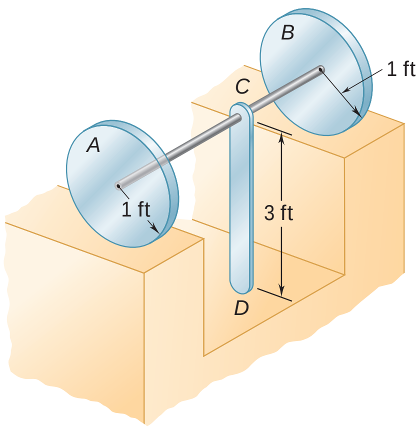


Fig. P19.86

19.87 and 19.88 Two uniform rods AB and CD, each of length l and mass m, are attached to gears as shown. Knowing that the mass of gear C is m and that the mass of gear A is $4m$, determine the period of small oscillations of the system.

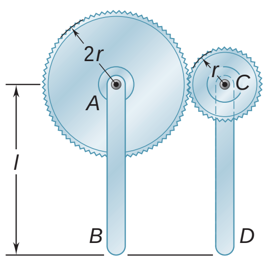


Fig. P19.87

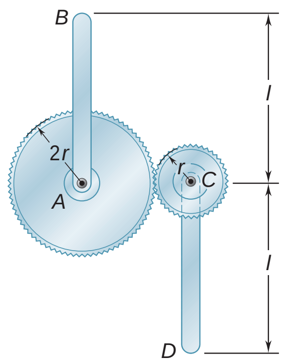


Fig. P19.88

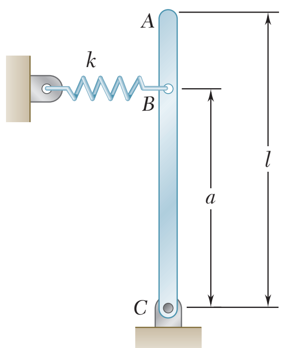


Fig. P19.89

19.89 An inverted pendulum consisting of a rigid bar ABC of length l and mass m is supported by a pin and bracket at C. A spring of constant k is attached to the bar at B and is undeformed when the bar is in the vertical position shown. Determine (a) the frequency of small oscillations, (b) the smallest value of a for which these oscillations will occur.

19.90 Two 12-lb uniform disks are attached to the 20-lb rod AB as shown. Knowing that the constant of the spring is 30 lb/in. and that the disks roll without sliding, determine the frequency of vibration of the system.

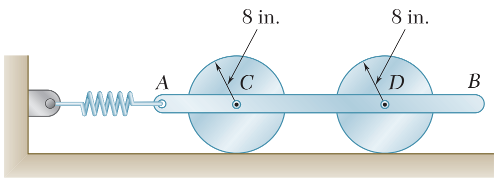


Fig. P19.90

19.91 The 20-lb rod AB is attached to two 8-lb disks as shown. Knowing that the disks roll without sliding, determine the frequency of small oscillations of the system.

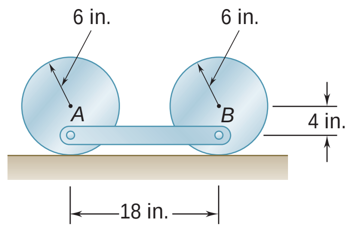


Fig. P19.91

19.92 A half section of a uniform cylinder of radius r and mass m rests on two casters A and B, each of which is a uniform cylinder of radius $r/4$ and mass $m/8$. Knowing that the half cylinder is rotated through a small angle and released and that no slipping occurs, determine the frequency of small oscillations.

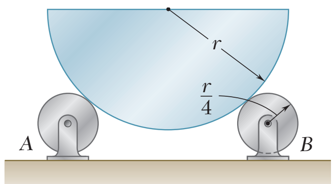


Fig. P19.92

19.93 The motion of the uniform rod AB is guided by the cord BC and by the small roller at A. Determine the frequency of oscillation when the end B of the rod is given a small horizontal displacement and released.

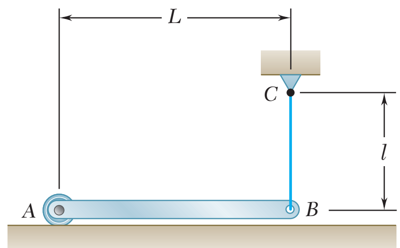


Fig. P19.93

19.94 A uniform rod of length L is supported by a ball-and-socket joint at A and by a vertical wire CD. Derive an expression for the period of oscillation of the rod if end B is given a small horizontal displacement and then released.

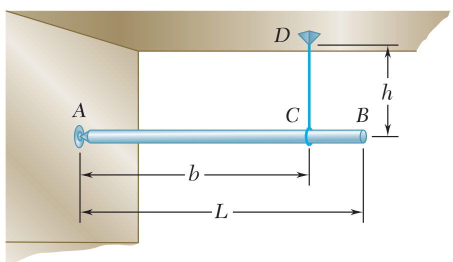


Fig. P19.94

19.95 A section of uniform pipe is suspended from two vertical cables attached at A and B. Determine the frequency of oscillation when the pipe is given a small rotation about the centroidal axis OO' and released.

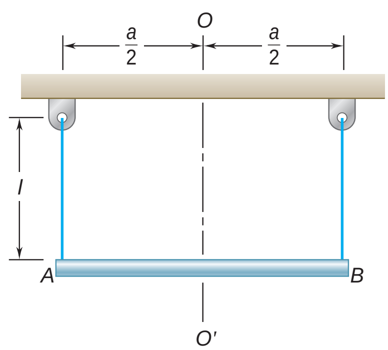


Fig. P19.95

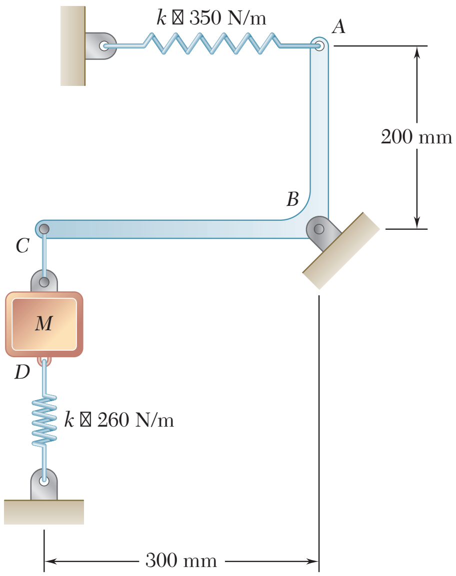


Fig. P19.96

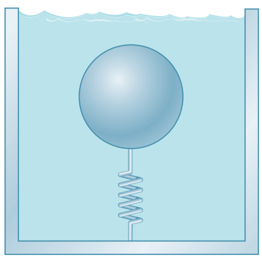


Fig. P19.98

19.96 A 0.6-kg uniform arm *ABC* is supported by a pin at *B* and is attached to a spring at *A*. It is connected at *C* to a 1.4-kg mass *M* which is attached to a spring. Knowing that each spring can act in tension or compression, determine the frequency of small oscillations of the system when the weight is given a small vertical displacement and released.

***19.97** A thin plate of length *l* rests on a half cylinder of radius *r*. Derive an expression for the period of small oscillations of the plate.

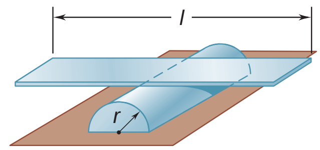


Fig. P19.97

***19.98** As a submerged body moves through a fluid, the particles of the fluid flow around the body and thus acquire kinetic energy. In the case of a sphere moving in an ideal fluid, the total kinetic energy acquired by the fluid is $\frac{1}{4}\mathrm{r}Vv^2$, where r is the mass density of the fluid, *V* is the volume of the sphere, and *v* is the velocity of the sphere. Consider a 500-g hollow spherical shell of radius 80 mm which is held submerged in a tank of water by a spring of constant 500 N/m. (*a*) Neglecting fluid friction, determine the period of vibration of the shell when it is displaced vertically and then released. (*b*) Solve part *a*, assuming that the tank is accelerated upward at the constant rate of 8 m/s^2.

19.7 FORCED VIBRATIONS

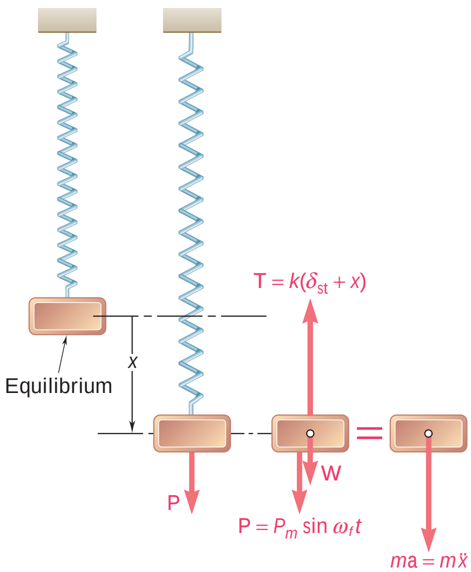


Fig. 19.7

The most important vibrations from the point of view of engineering applications are the *forced vibrations* of a system. These vibrations occur when a system is subjected to a periodic force or when it is elastically connected to a support which has an alternating motion.

Consider first the case of a body of mass *m* suspended from a spring and subjected to a periodic force **P** of magnitude $P = P_m \sin \mathrm{v}_f t$, where v_f is the circular frequency of **P** and is referred to as the *forced circular frequency* of the motion (Fig. 19.7). This force may be an actual external force applied to the body, or it may be a centrifugal force produced by the rotation of some unbalanced part of the body (see Sample Prob. 19.5). Denoting by *x* the displacement of the body measured from its equilibrium position, we write the equation of motion,

$$+\mathrm{w}\Sigma F = ma: \qquad P_m \sin \mathrm{v}_f t + W - k(\mathrm{d}_{st} + x) = m\ddot{x}$$

Recalling that $W = k\mathrm{d}_{st}$, we have

$$m\ddot{x} + kx = P_m \sin \mathrm{v}_f t \tag{19.30}$$

Next we consider the case of a body of mass m suspended from a spring attached to a moving support whose displacement d is equal to d_m sin $v_f t$ (Fig. 19.8). Measuring the displacement x of the body from the position of static equilibrium corresponding to $v_f t = 0$, we find that the total elongation of the spring at time t is $d_{st} + x - d_m$ sin $v_f t$. The equation of motion is thus

$$+w\Sigma F = ma: \qquad W - k(d_{st} + x - d_m \sin v_f t) = m\ddot{x}$$

Recalling that $W = kd_{st}$, we have

$$m\ddot{x} + kx = kd_m \sin v_f t \tag{19.31}$$

We note that Eqs. (19.30) and (19.31) are of the same form and that a solution of the first equation will satisfy the second if we set $P_m = kd_m$.

A differential equation such as (19.30) or (19.31), possessing a right-hand member different from zero, is said to be *nonhomogeneous*. Its general solution is obtained by adding a particular solution of the given equation to the general solution of the corresponding *homogeneous* equation (with right-hand member equal to zero). A *particular solution* of (19.30) or (19.31) can be obtained by trying a solution of the form

$$x_{part} = x_m \sin v_f t \tag{19.32}$$

Substituting x_{part} for x into (19.30), we find

$$-mv_f^2 x_m \sin v_f t + kx_m \sin v_f t = P_m \sin v_f t$$

which can be solved for the amplitude,

$$x_m = \frac{P_m}{k - mv_f^2}$$

Recalling from (19.4) that $k/m = v_n^2$, where v_n is the natural circular frequency of the system, we write

$$x_m = \frac{P_m/k}{1 - (v_f/v_n)^2} \tag{19.33}$$

Substituting from (19.32) into (19.31), we obtain in a similar way

$$x_m = \frac{d_m}{1 - (v_f/v_n)^2} \tag{19.33$'$}$$

The homogeneous equation corresponding to (19.30) or (19.31) is Eq. (19.2), which defines the free vibration of the body. Its general solution, called the *complementary function,* was found in Sec. 19.2:

$$x_{comp} = C_1 \sin v_n t + C_2 \cos v_n t \tag{19.34}$$

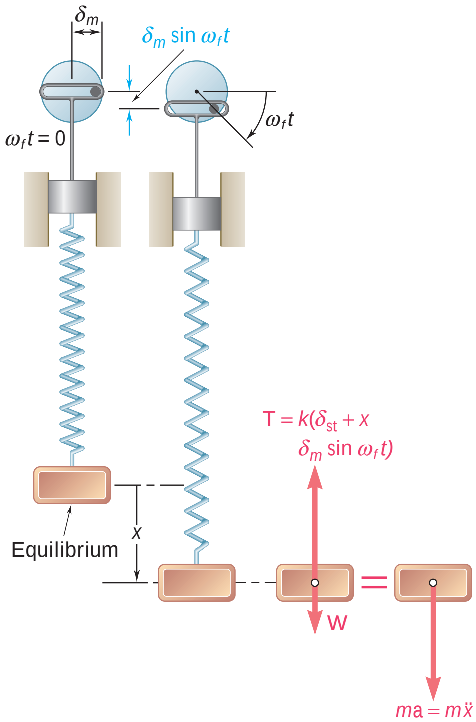


Fig. 19.8

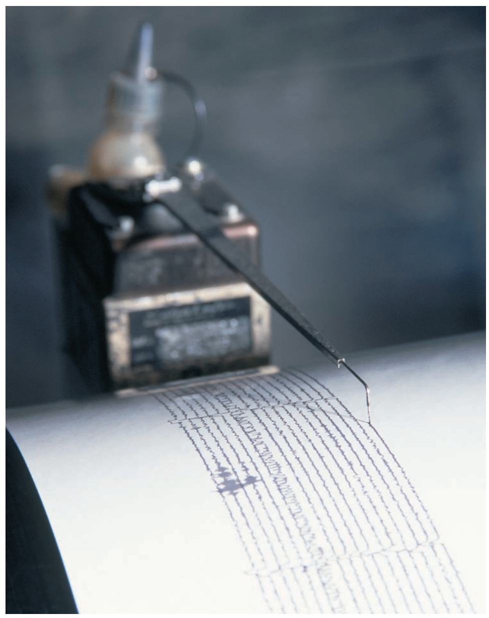


Photo 19.1 A seismometer operates by measuring the amount of electrical energy needed to keep a mass centered in the housing in the presence of strong ground shaking.

Adding the particular solution (19.32) to the complementary function (19.34), we obtain the *general solution* of Eqs. (19.30) and (19.31):

$$x = C_1 \sin \omega_n t + C_2 \cos \omega_n t + x_m \sin \omega_f t \tag{19.35}$$

We note that the vibration obtained consists of two superposed vibrations. The first two terms in Eq. (19.35) represent a free vibration of the system. The frequency of this vibration is the *natural frequency* of the system, which depends only upon the constant k of the spring and the mass m of the body, and the constants C_1 and C_2 can be determined from the initial conditions. This free vibration is also called a *transient* vibration, since in actual practice it will soon be damped out by friction forces (Sec. 19.9).

The last term in (19.35) represents the *steady-state* vibration produced and maintained by the impressed force or impressed support movement. Its frequency is the *forced frequency* imposed by this force or movement, and its amplitude x_m, defined by (19.33) or (19.33′), depends upon the *frequency ratio* ω_f/ω_n. The ratio of the amplitude x_m of the steady-state vibration to the static deflection P_m/k caused by a force P_m, or to the amplitude δ_m of the support movement, is called the *magnification factor.* From (19.33) and (19.33′), we obtain

$$\text{Magnification factor} = \frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{1 - (\omega_f/\omega_n)^2} \tag{19.36}$$

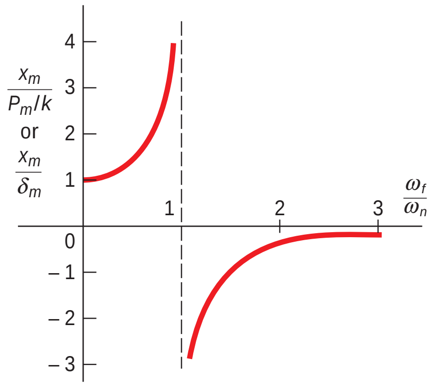
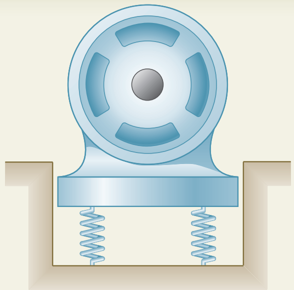


Fig. 19.9

The magnification factor has been plotted in Fig. 19.9 against the frequency ratio ω_f/ω_n. We note that when $\omega_f = \omega_n$, the amplitude of the forced vibration becomes infinite. The impressed force or impressed support movement is said to be in *resonance* with the given system. Actually, the amplitude of the vibration remains finite because of damping forces (Sec. 19.9); nevertheless, such a situation should be avoided, and the forced frequency should not be chosen too close to the natural frequency of the system. We also note that for $\omega_f < \omega_n$ the coefficient of $\sin \omega_f t$ in (19.35) is positive, while for $\omega_f > \omega_n$ this coefficient is negative. In the first case the forced vibration is *in phase* with the impressed force or impressed support movement, while in the second case it is 180° *out of phase.*

Finally, let us observe that the velocity and the acceleration in the steady-state vibration can be obtained by differentiating twice with respect to t the last term of Eq. (19.35). Their maximum values are given by expressions similar to those of Eqs. (19.15) of Sec. 19.2, except that these expressions now involve the amplitude and the circular frequency of the forced vibration:

$$v_m = x_m \omega_f \qquad a_m = x_m \omega_f^2 \tag{19.37}$$



SAMPLE PROBLEM 19.5

A motor weighing 350 lb is supported by four springs, each having a constant of 750 lb/in. The unbalance of the rotor is equivalent to a weight of 1 oz located 6 in. from the axis of rotation. Knowing that the motor is constrained to move vertically, determine (*a*) the speed in rpm at which resonance will occur, (*b*) the amplitude of the vibration of the motor at a speed of 1200 rpm.

SOLUTION

a. Resonance Speed. The resonance speed is equal to the natural circular frequency ω_n (in rpm) of the free vibration of the motor. The mass of the motor and the equivalent constant of the supporting springs are

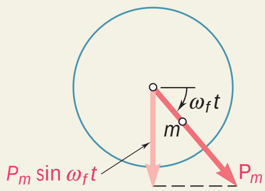
$$m = \frac{350\text{ lb}}{32.2\text{ ft/s}^2} = 10.87\text{ lb}\cdot\text{s}^2\text{/ft}$$

$$k = 4(750\text{ lb/in.}) = 3000\text{ lb/in.} = 36{,}000\text{ lb/ft}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{36{,}000}{10.87}} = 57.5\text{ rad/s} = 549\text{ rpm}$$

Resonance speed = 549 rpm ◀

b. Amplitude of Vibration at 1200 rpm. The angular velocity of the motor and the mass of the equivalent 1-oz weight are



$$\omega = 1200\text{ rpm} = 125.7\text{ rad/s}$$

$$m = (1\text{ oz})\frac{1\text{ lb}}{16\text{ oz}}\,\frac{1}{32.2\text{ ft/s}^2} = 0.001941\text{ lb}\cdot\text{s}^2\text{/ft}$$

The magnitude of the centrifugal force due to the unbalance of the rotor is

$$P_m = ma_n = mr\omega^2 = (0.001941\text{ lb}\cdot\text{s}^2\text{/ft})(\tfrac{6}{12}\text{ ft})(125.7\text{ rad/s})^2 = 15.33\text{ lb}$$

The static deflection that would be caused by a constant load P_m is

$$\frac{P_m}{k} = \frac{15.33\text{ lb}}{3000\text{ lb/in.}} = 0.00511\text{ in.}$$

The forced circular frequency ω_f of the motion is the angular velocity of the motor,

$$\omega_f = \omega = 125.7\text{ rad/s}$$

Substituting the values of P_m/k, ω_f, and ω_n into Eq. (19.33), we obtain

$$x_m = \frac{P_m/k}{1 - (\omega_f/\omega_n)^2} = \frac{0.00511\text{ in.}}{1 - (125.7/57.5)^2} = -0.001352\text{ in.}$$

$x_m = 0.001352$ in. (out of phase) ◀

Note. Since $\omega_f > \omega_n$, the vibration is 180° out of phase with the centrifugal force due to the unbalance of the rotor. For example, when the unbalanced mass is directly below the axis of rotation, the position of the motor is $x_m = 0.001352$ in. above the position of equilibrium.

SOLVING PROBLEMS ON YOUR OWN

This lesson was devoted to the analysis of the *forced vibrations* of a mechanical system. These vibrations occur either when the system is subjected to a periodic force **P** (Fig. 19.7), or when it is elastically connected to a support which has an alternating motion (Fig. 19.8). In the first case, the motion of the system is defined by the differential equation

$$m\ddot{x} + kx = P_m \sin \mathrm{v}_f t \tag{19.30}$$

where the right-hand member represents the magnitude of the force **P** at a given instant. In the second case, the motion is defined by the differential equation

$$m\ddot{x} + kx = k\mathrm{d}_m \sin \mathrm{v}_f t \tag{19.31}$$

where the right-hand member is the product of the spring constant k and the displacement of the support at a given instant. You will be concerned only with the *steady-state* motion of the system, which is defined by a *particular solution* of these equations, of the form

$$x_{\text{part}} = x_m \sin \mathrm{v}_f t \tag{19.32}$$

1. If the forced vibration is caused by a periodic force P, of amplitude P_m and circular frequency v_f, the amplitude of the vibration is

$$x_m = \frac{P_m/k}{1 - (\mathrm{v}_f/\mathrm{v}_n)^2} \tag{19.33}$$

where v_n is the *natural circular frequency* of the system, $\mathrm{v}_n = 1\overline{k/m}$, and k is the spring constant. Note that the circular frequency of the vibration is v_f and that the amplitude x_m does not depend upon the initial conditions. For $\mathrm{v}_f = \mathrm{v}_n$, the denominator in Eq. (19.33) is zero and x_m is infinite (Fig. 19.9); the impressed force **P** is said to be in *resonance* with the system. Also, for $\mathrm{v}_f < \mathrm{v}_n$, x_m is positive and the vibration is *in phase* with **P**, while, for $\mathrm{v}_f > \mathrm{v}_n$, x_m is negative and the vibration is *out of phase.*

a. In the problems which follow, you may be asked to determine one of the parameters in Eq. (19.33) when the others are known. We suggest that you keep Fig. 19.9 in front of you when solving these problems. For example, if you are asked to find the frequency at which the amplitude of a forced vibration has a given value, but you do not know whether the vibration is in or out of phase with respect to the impressed force, you should note from Fig. 19.9 that there can be two frequencies satisfying this requirement, one corresponding to a positive value of x_m and to a vibration in phase with the impressed force, and the other corresponding to a negative value of x_m and to a vibration out of phase with the impressed force.

b. Once you have obtained the amplitude x_m of the motion of a component of the system from Eq. (19.33), you can use Eqs. (19.37) to determine the maximum values of the velocity and acceleration of that component:

$$v_m = x_m\omega_f \qquad a_m = x_m\omega_f^2 \tag{19.37}$$

c. When the impressed force P is due to the unbalance of the rotor of a motor, its maximum value is $P_m = mr\omega_f^2$, where m is the mass of the rotor, r is the distance between its mass center and the axis of rotation, and ω_f is equal to the angular velocity ω of the rotor expressed in rad/s [Sample Prob. 19.5].

2. If the forced vibration is caused by the simple harmonic motion of a support, of amplitude δ_m and circular frequency ω_f, the amplitude of the vibration is

$$x_m = \frac{\delta_m}{1 - (\omega_f/\omega_n)^2} \tag{19.33$'$}$$

where ω_n is the *natural circular frequency* of the system, $\omega_n = \sqrt{k/m}$. Again, note that the circular frequency of the vibration is ω_f and that the amplitude x_m does not depend upon the initial conditions.

a. Be sure to read our comments in paragraphs 1, 1a, and 1b, since they apply equally well to a vibration caused by the motion of a support.

b. If the maximum acceleration a_m of the support is specified, rather than its maximum displacement δ_m, remember that, since the motion of the support is a simple harmonic motion, you can use the relation $a_m = \delta_m\omega_f^2$ to determine δ_m; the value obtained is then substituted into Eq. (19.33′).

PROBLEMS

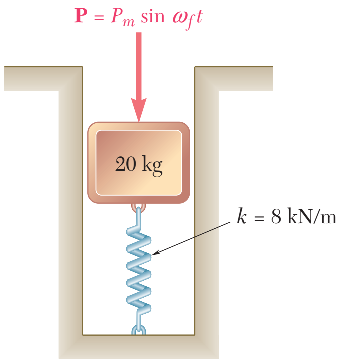


Fig. P19.99 and P19.100

19.99 A 20-kg block is attached to a spring of constant $k = 8$ kN/m and can move without friction in a vertical slot as shown. The block is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 100$ N. Determine the amplitude of the motion of the block if (*a*) $\omega_f = 10$ rad/s, (*b*) $\omega_f = 19$ rad/s, (*c*) $\omega_f = 30$ rad/s.

19.100 A 20-kg block is attached to a spring of constant $k = 8$ kN/m and can move without friction in a vertical slot as shown. The block is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 10$ N. Knowing that the amplitude of the motion is 3 mm, determine the value of ω_f.

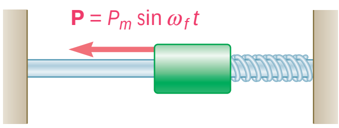


Fig. P19.101 and P19.102

19.101 A 9-lb collar can slide on a frictionless horizontal rod and is attached to a spring of constant k. It is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 2$ lb and $\omega_f = 5$ rad/s. Determine the value of the spring constant k knowing that the motion of the collar has an amplitude of 6 in. and is (*a*) in phase with the applied force, (*b*) out of phase with the applied force.

19.102 A collar of mass m which slides on a frictionless horizontal rod is attached to a spring of constant k and is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$. Determine the range of values of ω_f for which the amplitude of the vibration exceeds two times the static deflection caused by a constant force of magnitude P_m.

19.103 A small 20-kg block A is attached to the rod BC of negligible mass which is supported at B by a pin and bracket and at C by a spring of constant $k = 2$ kN/m. The system can move in a vertical plane and is in equilibrium when the rod is horizontal. The rod is acted upon at C by a periodic force **P** of magnitude $P = P_m \sin \omega_f t$, where $P_m = 6$ N. Knowing that $b = 200$ mm, determine the range of values of ω_f for which the amplitude of vibration of block A exceeds 3.5 mm.

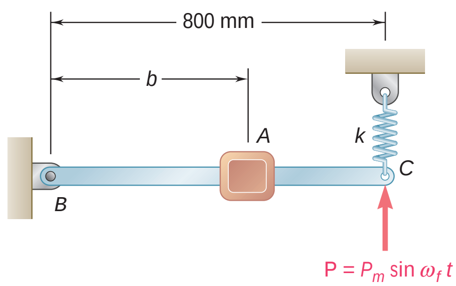


Fig. *P19.103*

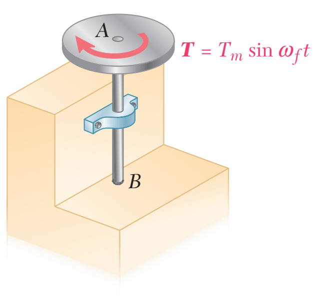


Fig. *P19.104*

19.104 An 8-kg uniform disk of radius 200 mm is welded to a vertical shaft with a fixed end at B. The disk rotates through an angle of 3° when a static couple of magnitude 50 N · m is applied to it. If the disk is acted upon by a periodic torsional couple of magnitude $T = T_m \sin \omega_f t$, where $T_m = 60$ N · m, determine the range of values of ω_f for which the amplitude of the vibration is less than the angle of rotation caused by a static couple of magnitude T_m.

19.105 An 18-lb block A slides in a vertical frictionless slot and is connected to a moving support B by means of a spring AB of constant $k = 10$ lb/in. Knowing that the displacement of the support is $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 6$ in., determine the range of values of ω_f for which the amplitude of the fluctuating force exerted by the spring on the block is less than 30 lb.

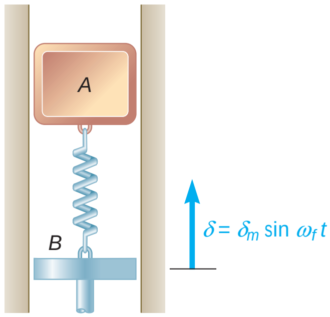


Fig. P19.105

19.106 A cantilever beam AB supports a block which causes a static deflection of 8 mm at B. Assuming that the support at A undergoes a vertical periodic displacement $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 2$ mm, determine the range of values of ω_f for which the amplitude of the motion of the block will be less than 4 mm. Neglect the weight of the beam and assume that the block does not leave the beam.

Fig. P19.106

19.107 Rod AB is rigidly attached to the frame of a motor running at a constant speed. When a collar of mass m is placed on the spring, it is observed to vibrate with an amplitude of 15 mm. When two collars, each of mass m, are placed on the spring, the amplitude is observed to be 18 mm. What amplitude of vibration should be expected when three collars, each of mass m, are placed on the spring? (Obtain two answers.)

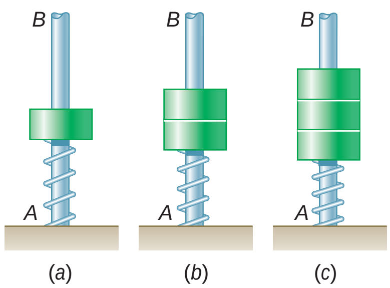


Fig. P19.107

19.108 The crude-oil-pumping rig shown is driven at 20 rpm. The inside diameter of the well pipe is 2 in., and the diameter of the pump rod is 0.75 in. The length of the pump rod and the length of the column of oil lifted during the stroke are essentially the same, and equal to 6000 ft. During the downward stroke, a valve at the lower end of the pump rod opens to let a quantity of oil into the well pipe, and the column of oil is then lifted to obtain a discharge into the connecting pipeline. Thus, the amount of oil pumped in a given time depends upon the stroke of the lower end of the pump rod. Knowing that the upper end of the rod at D is essentially sinusoidal with a stroke of 45 in. and the specific weight of crude oil is 56.2 lb/ft^3, determine (*a*) the output of the well in ft^3/min if the shaft is rigid, (*b*) the output of the well in ft^3/min if the stiffness of the rod is 2210 N/m, the equivalent mass of the oil and shaft is 290 kg, and damping is negligible.

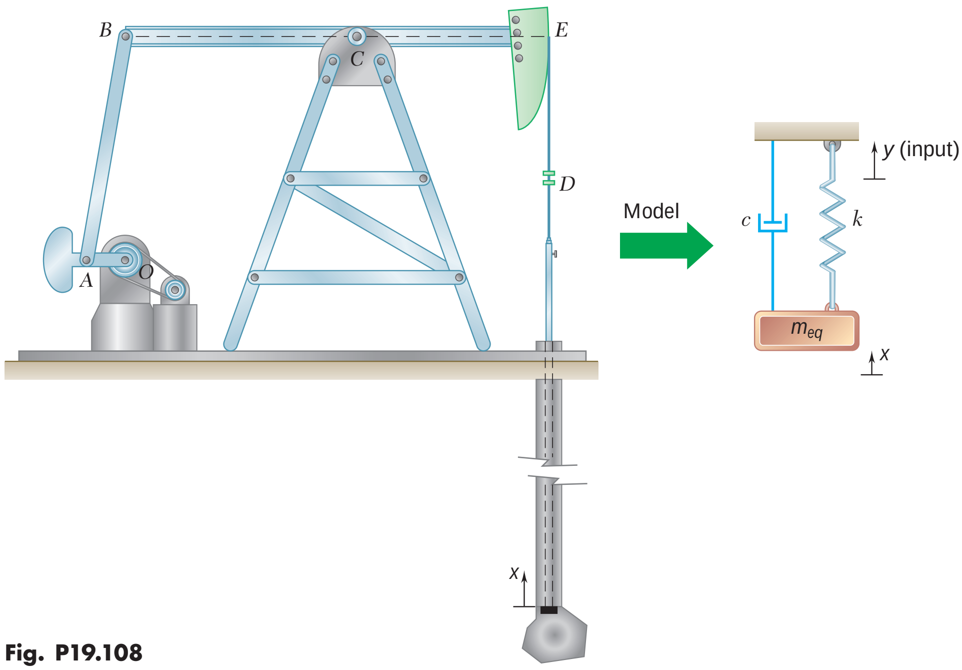


Fig. P19.108

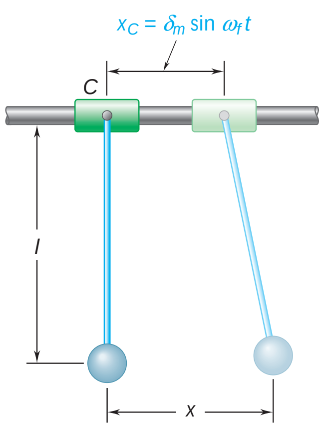


Fig. P19.109 and P19.110

19.109 A simple pendulum of length l is suspended from collar C which is forced to move horizontally according to the relation $x_C = d_m \sin v_f t$. Determine the range of values of v_f for which the amplitude of the motion of the bob is less than d_m. (Assume that d_m is small compared with the length l of the pendulum.)

19.110 The 2.75-lb bob of a simple pendulum of length $l = 24$ in. is suspended from a 3-lb collar C. The collar is forced to move according to the relation $x_C = d_m \sin v_f t$, with an amplitude $d_m = 0.4$ in. and a frequency $f_f = 0.5$ Hz. Determine (*a*) the amplitude of the motion of the bob, (*b*) the force that must be applied to collar C to maintain the motion.

19.111 An 18-lb block A slides in a vertical frictionless slot and is connected to a moving support B by means of a spring AB of constant $k = 8$ lb/ft. Knowing that the acceleration of the support is $a = a_m \sin \omega_f t$, where $a_m = 5$ ft/s^2 and $\omega_f = 6$ rad/s, determine (*a*) the maximum displacement of block A, (*b*) the amplitude of the fluctuating force exerted by the spring on the block.

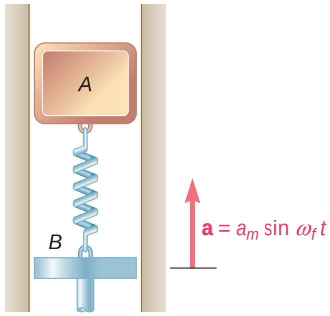


Fig. *P19.111*

19.112 A variable-speed motor is rigidly attached to a beam BC. When the speed of the motor is less than 600 rpm or more than 1200 rpm, a small object placed at A is observed to remain in contact with the beam. For speeds between 600 and 1200 rpm the object is observed to "dance" and actually to lose contact with the beam. Determine the speed at which resonance will occur.

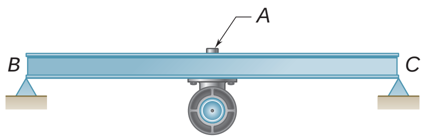


Fig. P19.112

19.113 A motor of mass M is supported by springs with an equivalent spring constant k. The unbalance of its rotor is equivalent to a mass m located at a distance r from the axis of rotation. Show that when the angular velocity of the motor is ω_f, the amplitude x_m of the motion of the motor is

$$x_m = \frac{r(m/M)(\omega_f/\omega_n)^2}{1 - (\omega_f/\omega_n)^2}$$

where $\omega_n = \sqrt{k/M}$.

19.114 As the rotational speed of a spring-supported 100-kg motor is increased, the amplitude of the vibration due to the unbalance of its 15-kg rotor first increases and then decreases. It is observed that as very high speeds are reached, the amplitude of the vibration approaches 3.3 mm. Determine the distance between the mass center of the rotor and its axis of rotation. (*Hint:* Use the formula derived in Prob. 19.113.)

19.115 A motor of weight 40 lb is supported by four springs, each of constant 225 lb/in. The motor is constrained to move vertically, and the amplitude of its motion is observed to be 0.05 in. at a speed of 1200 rpm. Knowing that the weight of the rotor is 9 lb, determine the distance between the mass center of the rotor and the axis of the shaft.

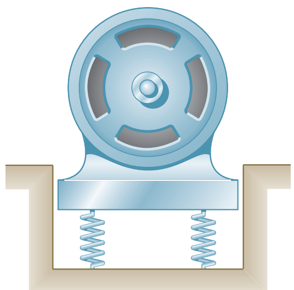


Fig. P19.115

19.116 A motor weighing 400 lb is supported by springs having a total constant of 1200 lb/in. The unbalance of the rotor is equivalent to a 1-oz weight located 8 in. from the axis of rotation. Determine the range of allowable values of the motor speed if the amplitude of the vibration is not to exceed 0.06 in.

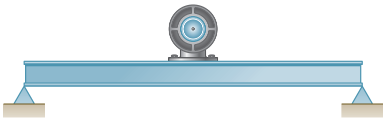


Fig. P19.117

19.117 A 180-kg motor is bolted to a light horizontal beam. The unbalance of its rotor is equivalent to a 28-g mass located 150 mm from the axis of rotation, and the static deflection of the beam due to the weight of the motor is 12 mm. The amplitude of the vibration due to the unbalance can be decreased by adding a plate to the base of the motor. If the amplitude of vibration is to be less than 60 mm for motor speeds above 300 rpm, determine the required mass of the plate.

19.118 The unbalance of the rotor of a 400-lb motor is equivalent to a 3-oz weight located 6 in. from the axis of rotation. In order to limit to 0.2 lb the amplitude of the fluctuating force exerted on the foundation when the motor is run at speeds of 100 rpm and above, a pad is to be placed between the motor and the foundation. Determine (*a*) the maximum allowable spring constant k of the pad, (*b*) the corresponding amplitude of the fluctuating force exerted on the foundation when the motor is run at 200 rpm.

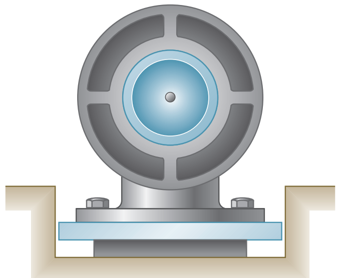


Fig. *P19.118*

19.119 A counter-rotating eccentric mass exciter consisting of two rotating 100-g masses describing circles of radius r at the same speed but in opposite senses is placed on a machine element to induce a steady-state vibration of the element. The total mass of the system is 300 kg, the constant of each spring is $k = 600$ kN/m, and the rotational speed of the exciter is 1200 rpm. Knowing that the amplitude of the total fluctuating force exerted on the foundation is 160 N, determine the radius r.

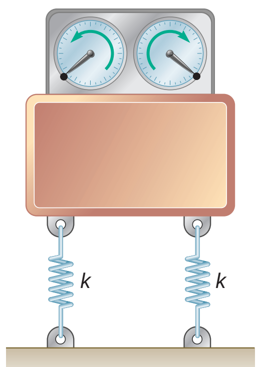


Fig. P19.119

19.120 A 360-lb motor is supported by springs of total constant 12.5 kips/ft. The unbalance of the rotor is equivalent to a 0.9-oz weight located 7.5 in. from the axis of rotation. Determine the range of speeds of the motor for which the amplitude of the fluctuating force exerted on the foundation is less than 5 lb.

19.121 Figures (1) and (2) show how springs can be used to support a block in two different situations. In Fig. (1) they help decrease the amplitude of the fluctuating force transmitted by the block to the foundation. In Fig. (2) they help decrease the amplitude of the fluctuating displacement transmitted by the foundation to the block. The ratio of the transmitted force to the impressed force or the ratio of the transmitted displacement to the impressed displacement is called the *transmissibility.* Derive an equation for the transmissibility for each situation. Give your answer in terms of the ratio ω_f/ω_n of the frequency ω_f of the impressed force or impressed displacement to the natural frequency ω_n of the spring-mass system. Show that in order to cause any reduction in transmissibility, the ratio ω_f/ω_n must be greater than $\sqrt{2}$.

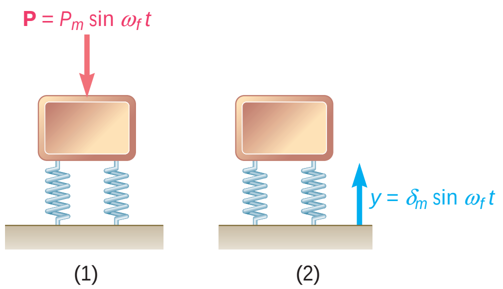


Fig. P19.121

19.122 A vibrometer used to measure the amplitude of vibrations consists essentially of a box containing a mass-spring system with a known natural frequency of 120 Hz. The box is rigidly attached to a surface which is moving according to the equation $y = \delta_m \sin \omega_f t$. If the amplitude z_m of the motion of the mass relative to the box is used as a measure of the amplitude δ_m of the vibration of the surface, determine (*a*) the percent error when the frequency of the vibration is 600 Hz, (*b*) the frequency at which the error is zero.

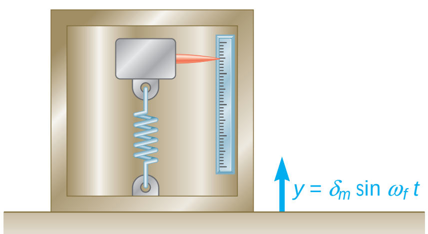


Fig. P19.122 and P19.123

19.123 A certain accelerometer consists essentially of a box containing a mass-spring system with a known natural frequency of 2200 Hz. The box is rigidly attached to a surface which is moving according to the equation $y = \delta_m \sin \omega_f t$. If the amplitude z_m of the motion of the mass relative to the box times a scale factor ω_n^2 is used as a measure of the maximum acceleration $\alpha_m = \delta_m \omega_f^2$ of the vibrating surface, determine the percent error when the frequency of the vibration is 600 Hz.

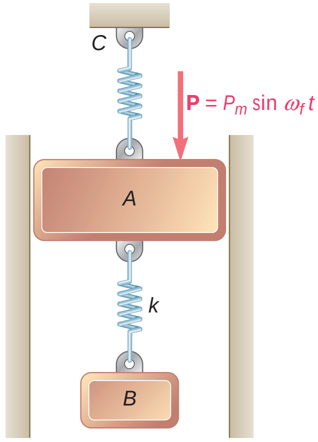


Fig. *P19.124*

19.124 Block A can move without friction in the slot as shown and is acted upon by a vertical periodic force of magnitude $P = P_m \sin \omega_f t$, where $\omega_f = 2$ rad/s and $P_m = 20$ N. A spring of constant k is attached to the bottom of block A and to a 22-kg block B. Determine (*a*) the value of the constant k which will prevent a steady-state vibration of block A, (*b*) the corresponding amplitude of the vibration of block B.

19.125 A 60-lb disk is attached with an eccentricity $e = 0.006$ in. to the midpoint of a vertical shaft AB which revolves at a constant angular velocity ω_f. Knowing that the spring constant k for horizontal movement of the disk is 40,000 lb/ft, determine (*a*) the angular velocity ω_f at which resonance will occur, (*b*) the deflection r of the shaft when $\omega_f = 1200$ rpm.

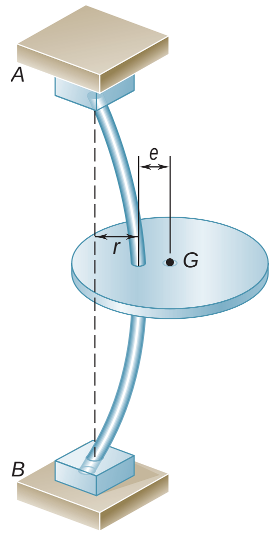


Fig. P19.125

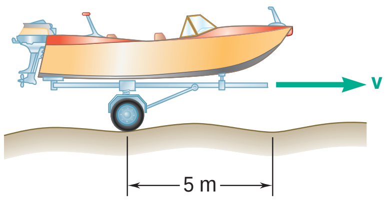


Fig. P19.126

19.126 A small trailer and its load have a total mass of 250 kg. The trailer is supported by two springs, each of constant 10 kN/m, and is pulled over a road, the surface of which can be approximated by a sine curve with an amplitude of 40 mm and a wavelength of 5 m (i.e., the distance between successive crests is 5 m and the vertical distance from crest to trough is 80 mm). Determine (*a*) the speed at which resonance will occur, (*b*) the amplitude of the vibration of the trailer at a speed of 50 km/h.

DAMPED VIBRATIONS

*19.8 DAMPED FREE VIBRATIONS

The vibrating systems considered in the first part of this chapter were assumed free of damping. Actually all vibrations are damped to some degree by friction forces. These forces can be caused by *dry friction,* or *Coulomb friction,* between rigid bodies, by *fluid friction* when a rigid body moves in a fluid, or by *internal friction* between the molecules of a seemingly elastic body.

A type of damping of special interest is the *viscous damping* caused by fluid friction at low and moderate speeds. Viscous damping is characterized by the fact that the friction force is *directly proportional and opposite to the velocity* of the moving body. As an example, let us again consider a body of mass m suspended from a spring of constant k, assuming that the body is attached to the plunger of a dashpot (Fig. 19.10). The magnitude of the friction force exerted on the plunger by the surrounding fluid is equal to $c\dot{x}$, where the constant c, expressed in N · s/m or lb · s/ft and known as the *coefficient of viscous damping*, depends upon the physical properties of the fluid and the construction of the dashpot. The equation of motion is

$$+\downarrow\Sigma F = ma: \qquad W - k(\delta_{st} + x) - c\dot{x} = m\ddot{x}$$

Recalling that $W = k\delta_{st}$, we write

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0} \tag{19.38}$$

Substituting $x = e^{\lambda t}$ into (19.38) and dividing through by $e^{\lambda t}$, we write the *characteristic equation*

$$m\lambda^2 + c\lambda + k = 0 \tag{19.39}$$

and obtain the roots

$$\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{19.40}$$

Defining the *critical damping coefficient* c_c as the value of c which makes the radical in Eq. (19.40) equal to zero, we write

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n \tag{19.41}$$

where ω_n is the natural circular frequency of the system in the absence of damping. We can distinguish three different cases of damping, depending upon the value of the coefficient c.

1. *Heavy damping:* $c > c_c$. The roots λ_1 and λ_2 of the characteristic equation (19.39) are real and distinct, and the general solution of the differential equation (19.38) is

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{19.42}$$

This solution corresponds to a nonvibratory motion. Since λ_1 and λ_2 are both negative, x approaches zero as t increases indefinitely. However, the system actually regains its equilibrium position after a finite time.

2. *Critical damping:* $c = c_c$. The characteristic equation has a double root $\lambda = -c_c/2m = -\omega_n$, and the general solution of (19.38) is

$$x = (C_1 + C_2 t)e^{-\omega_n t} \tag{19.43}$$

The motion obtained is again nonvibratory. Critically damped systems are of special interest in engineering applications since

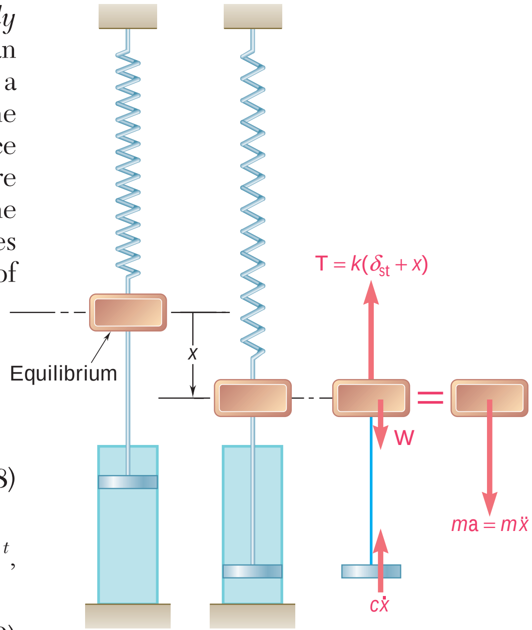


Fig. 19.10

they regain their equilibrium position in the shortest possible time without oscillation.

3. *Light damping:* $c < c_c$. The roots of Eq. (19.39) are complex and conjugate, and the general solution of (19.38) is of the form

$$x = e^{-(c/2m)t}(C_1 \sin \omega_d t + C_2 \cos \omega_d t) \tag{19.44}$$

where ω_d is defined by the relation

$$\omega_d^2 = \frac{k}{m} - \left(\frac{c}{2m}\right)^2$$

Substituting $k/m = \omega_n^2$ and recalling (19.41), we write

$$\omega_d = \omega_n \sqrt{1 - \left(\frac{c}{c_c}\right)^2} \tag{19.45}$$

where the constant c/c_c is known as the *damping factor.* Even though the motion does not actually repeat itself, the constant ω_d is commonly referred to as the *circular frequency* of the damped vibration. A substitution similar to the one used in Sec. 19.2 enables us to write the general solution of Eq. (19.38) in the form

$$x = x_0 e^{-(c/2m)t} \sin (\omega_d t + \phi) \tag{19.46}$$

The motion defined by Eq. (19.46) is vibratory with diminishing amplitude (Fig. 19.11), and the time interval $\tau_d = 2\pi/\omega_d$ separating two successive points where the curve defined by Eq. (19.46) touches one of the limiting curves shown in Fig. 19.11 is commonly referred to as the *period of the damped vibration.* Recalling Eq. (19.45), we observe that $\omega_d < \omega_n$ and, thus, that τ_d is larger than the period of vibration τ_n of the corresponding undamped system.

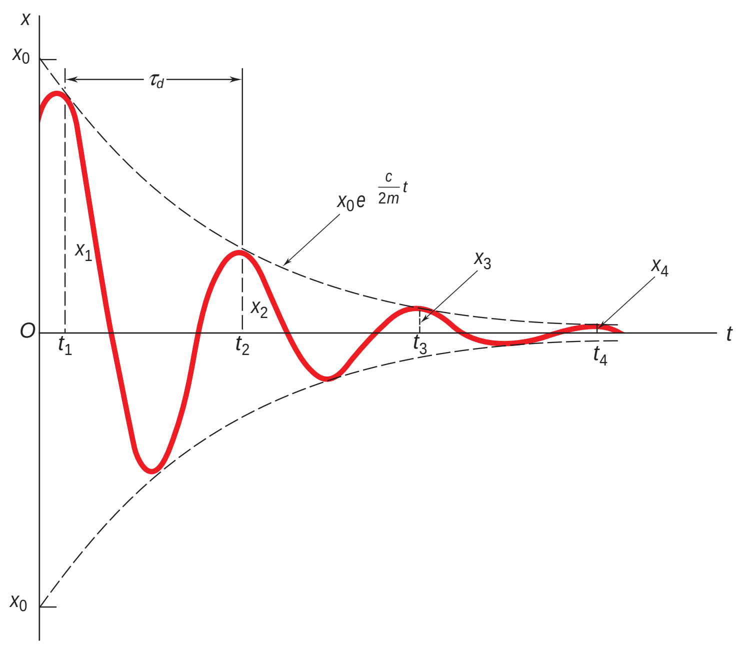


Fig. 19.11

*19.9 DAMPED FORCED VIBRATIONS

If the system considered in the preceding section is subjected to a periodic force **P** of magnitude $P = P_m \sin \omega_f t$, the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t \tag{19.47}$$

The general solution of (19.47) is obtained by adding a particular solution of (19.47) to the complementary function or general solution of the homogeneous equation (19.38). The complementary function is given by (19.42), (19.43), or (19.44), depending upon the type of damping considered. It represents a *transient* motion which is eventually damped out.

Our interest in this section is centered on the steady-state vibration represented by a particular solution of (19.47) of the form

$$x_{\text{part}} = x_m \sin (\omega_f t - \varphi) \tag{19.48}$$

Substituting x_{part} for x into (19.47), we obtain

$$-m\omega_f^2 x_m \sin (\omega_f t - \varphi) + c\omega_f x_m \cos (\omega_f t - \varphi) + kx_m \sin (\omega_f t - \varphi) = P_m \sin \omega_f t$$

Making $\omega_f t - \varphi$ successively equal to 0 and to $\pi/2$, we write

$$c\omega_f x_m = P_m \sin \varphi \tag{19.49}$$

$$(k - m\omega_f^2)\, x_m = P_m \cos \varphi \tag{19.50}$$

Squaring both members of (19.49) and (19.50) and adding, we have

$$[(k - m\omega_f^2)^2 + (c\omega_f)^2]\, x_m^2 = P_m^2 \tag{19.51}$$

Solving (19.51) for x_m and dividing (19.49) and (19.50) member by member, we obtain, respectively,

$$x_m = \frac{P_m}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \qquad \tan \varphi = \frac{c\omega_f}{k - m\omega_f^2} \tag{19.52}$$

Recalling from (19.4) that $k/m = \omega_n^2$, where ω_n is the circular frequency of the undamped free vibration, and from (19.41) that $2m\omega_n = c_c$, where c_c is the critical damping coefficient of the system, we write

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{\sqrt{[1 - (\omega_f/\omega_n)^2]^2 + [2(c/c_c)(\omega_f/\omega_n)]^2}} \tag{19.53}$$

$$\tan \varphi = \frac{2(c/c_c)(\omega_f/\omega_n)}{1 - (\omega_f/\omega_n)^2} \tag{19.54}$$

Photo 19.2 The automobile suspension shown consists essentially of a spring and a shock absorber, which will cause the body of the car to undergo *damped forced vibrations* when the car is driven over an uneven road.

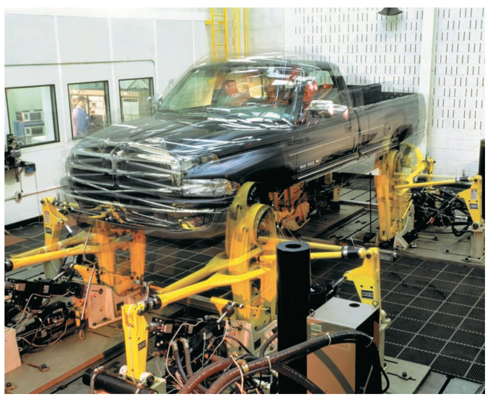


Photo 19.3 This truck is experiencing damped forced vibration in the vehicle dynamics test shown.

Formula (19.53) expresses the magnification factor in terms of the frequency ratio ω_f/ω_n and damping factor c/c_c. It can be used to determine the amplitude of the steady-state vibration produced by an impressed force of magnitude $P = P_m \sin \omega_f t$ or by an impressed support movement $\delta = \delta_m \sin \omega_f t$. Formula (19.54) defines in terms of the same parameters the *phase difference* φ between the impressed force or impressed support movement and the resulting steady-state vibration of the damped system. The magnification factor has been plotted against the frequency ratio in Fig. 19.12 for various values of the damping factor. We observe that the amplitude of a forced vibration can be kept small by choosing a large coefficient of viscous damping c or by keeping the natural and forced frequencies far apart.

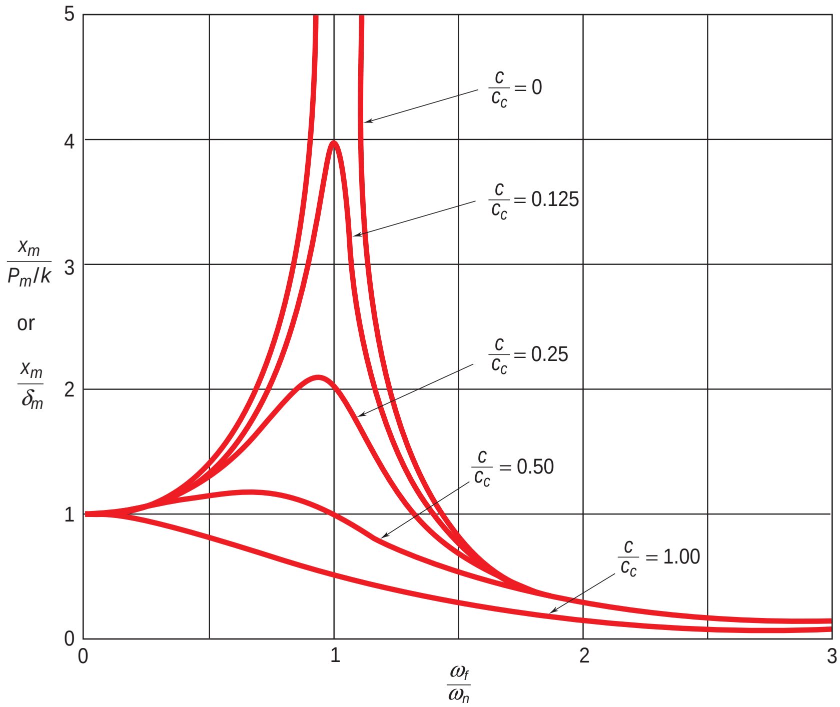
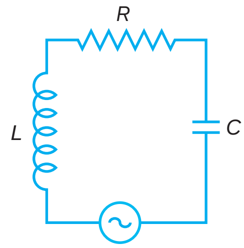


Fig. 19.12

*19.10 ELECTRICAL ANALOGUES

Oscillating electrical circuits are characterized by differential equations of the same type as those obtained in the preceding sections. Their analysis is therefore similar to that of a mechanical system, and the results obtained for a given vibrating system can be readily extended to the equivalent circuit. Conversely, any result obtained for an electrical circuit will also apply to the corresponding mechanical system.

Consider an electrical circuit consisting of an inductor of inductance L, a resistor of resistance R, and a capacitor of capacitance C, connected in series with a source of alternating voltage $E = E_m \sin \omega_f t$ (Fig. 19.13). It is recalled from elementary circuit theory† that if i denotes the current in the circuit and q denotes the electric charge on the capacitor, the drop in potential is $L(di/dt)$ across the inductor, Ri across the resistor, and q/C across the capacitor. Expressing that the algebraic sum of the applied voltage and of the drops in potential around the circuit loop is zero, we write



$E = E_m \sin \omega_f t$

Fig. 19.13

$$E_m \sin \omega_f t - L\frac{di}{dt} - Ri - \frac{q}{C} = 0 \tag{19.55}$$

Rearranging the terms and recalling that at any instant the current i is equal to the rate of change $\dot{q}$ of the charge q, we have

$$\boxed{L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_m \sin \omega_f t} \tag{19.56}$$

We verify that Eq. (19.56), which defines the oscillations of the electrical circuit of Fig. 19.13, is of the same type as Eq. (19.47), which characterizes the damped forced vibrations of the mechanical system of Fig. 19.10. By comparing the two equations, we can construct a table of the analogous mechanical and electrical expressions.

Table 19.2 can be used to extend the results obtained in the preceding sections for various mechanical systems to their electrical analogues. For instance, the amplitude i_m of the current in the circuit of Fig. 19.13 can be obtained by noting that it corresponds to the

TABLE 19.2 Characteristics of a Mechanical System and of Its Electrical Analogue

Mechanical System		Electrical Circuit	
m	Mass	L	Inductance
c	Coefficient of viscous damping	R	Resistance
k	Spring constant	$1/C$	Reciprocal of capacitance
x	Displacement	q	Charge
v	Velocity	i	Current
P	Applied force	E	Applied voltage

†See C. R. Paul, S. A. Nasar, and L. E. Unnewehr, *Introduction to Electrical Engineering*, 2nd ed., McGraw-Hill, New York, 1992.

maximum value v_m of the velocity in the analogous mechanical system. Recalling from the first of Eqs. (19.37) that $v_m = x_m\omega_f$, substituting for x_m from Eq. (19.52), and replacing the constants of the mechanical system by the corresponding electrical expressions, we have

$$i_m = \frac{\omega_f E_m}{\sqrt{\left(\frac{1}{C} - L\omega_f^2\right)^2 + (R\omega_f)^2}}$$

$$i_m = \frac{E_m}{\sqrt{R^2 + \left(L\omega_f - \frac{1}{C\omega_f}\right)^2}} \tag{19.57}$$

The radical in the expression obtained is known as the *impedance* of the electrical circuit.

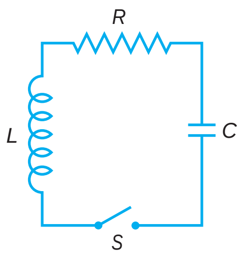


Fig. 19.14

The analogy between mechanical systems and electrical circuits holds for transient as well as steady-state oscillations. The oscillations of the circuit shown in Fig. 19.14, for instance, are analogous to the damped free vibrations of the system of Fig. 19.10. As far as the initial conditions are concerned, we should note that closing the switch S when the charge on the capacitor is $q = q_0$ is equivalent to releasing the mass of the mechanical system with no initial velocity from the position $x = x_0$. We should also observe that if a battery of constant voltage E is introduced in the electrical circuit of Fig. 19.14, closing the switch S will be equivalent to suddenly applying a force of constant magnitude P to the mass of the mechanical system of Fig. 19.10.

The discussion above would be of questionable value if its only result were to make it possible for mechanics students to analyze electrical circuits without learning the elements of circuit theory. It is hoped that this discussion will instead encourage students to apply to the solution of problems in mechanical vibrations the mathematical techniques they may learn in later courses in circuit theory. The chief value of the concept of electrical analogue, however, resides in its application to *experimental methods* for the determination of the characteristics of a given mechanical system. Indeed, an electrical circuit is much more easily constructed than is a mechanical model, and the fact that its characteristics can be modified by varying the inductance, resistance, or capacitance of its various components makes the use of the electrical analogue particularly convenient.

To determine the electrical analogue of a given mechanical system, we focus our attention on each moving mass in the system and observe which springs, dashpots, or external forces are applied directly to it. An equivalent electrical loop can then be constructed to match each of the mechanical units thus defined; the various loops obtained in that way will together form the desired circuit. Consider, for instance, the mechanical system of Fig. 19.15. We observe that the mass m_1 is acted upon by two springs of constants k_1 and k_2 and by two dashpots characterized by the coefficients of viscous damping c_1 and c_2. The electrical circuit should therefore include a loop consisting of an inductor of inductance L_1 proportional to m_1, of two capacitors of capacitance C_1 and C_2 inversely proportional to k_1 and k_2, respectively, and of two resistors of resistance R_1 and R_2, proportional to c_1 and c_2, respectively. Since the mass m_2 is acted upon by the spring k_2 and the dashpot c_2, as well as the force $P = P_m \sin \omega_f t$, the circuit should also include a loop containing the capacitor C_2, the resistor R_2, the new inductor L_2, and the voltage source $E = E_m \sin \omega_f t$ (Fig. 19.16).

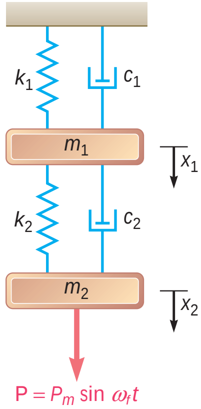


Fig. 19.15

To check that the mechanical system of Fig. 19.15 and the electrical circuit of Fig. 19.16 actually satisfy the same differential equations, the equations of motion for m_1 and m_2 will first be derived. Denoting, respectively, by x_1 and x_2 the displacements of m_1 and m_2 from their equilibrium positions, we observe that the elongation of the spring k_1 (measured from the equilibrium position) is equal to x_1, while the elongation of the spring k_2 is equal to the relative displacement $x_2 - x_1$ of m_2 with respect to m_1. The equations of motion for m_1 and m_2 are therefore

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = 0 \quad (19.58)$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = P_m \sin \omega_f t \quad (19.59)$$

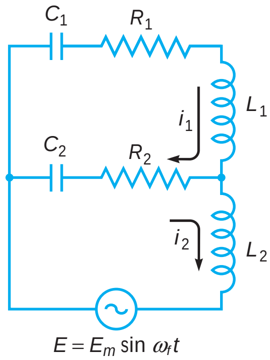


Fig. 19.16

Consider now the electrical circuit of Fig. 19.16; we denote, respectively, by i_1 and i_2 the current in the first and second loops, and by q_1 and q_2 the integrals $\int i_1\, dt$ and $\int i_2\, dt$. Noting that the charge on the capacitor C_1 is q_1, while the charge on C_2 is $q_1 - q_2$, we express that the sum of the potential differences in each loop is zero and obtain the following equations

$$L_1\ddot{q}_1 + R_1\dot{q}_1 + R_2(\dot{q}_1 - \dot{q}_2) + \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_2} = 0 \quad (19.60)$$

$$L_2\ddot{q}_2 + R_2(\dot{q}_2 - \dot{q}_1) + \frac{q_2 - q_1}{C_2} = E_m \sin \omega_f t \quad (19.61)$$

We easily check that Eqs. (19.60) and (19.61) reduce to (19.58) and (19.59), respectively, when the substitutions indicated in Table 19.2 are performed.

SOLVING PROBLEMS ON YOUR OWN

In this lesson a more realistic model of a vibrating system was developed by including the effect of the *viscous damping* caused by fluid friction. Viscous damping was represented in Fig. 19.10 by the force exerted on the moving body by a plunger moving in a dashpot. This force is equal in magnitude to $c\dot{x}$, where the constant c, expressed in N · s/m or lb · s/ft, is known as the *coefficient of viscous damping*. Keep in mind that the same sign convention should be used for x, $\dot{x}$, and $\ddot{x}$.

1. Damped free vibrations. The differential equation defining this motion was found to be

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{19.38}$$

To obtain the solution of this equation, calculate the *critical damping coefficient* c_c, using the formula

$$c_c = 2m\sqrt{k/m} = 2m\omega_n \tag{19.41}$$

where ω_n is the natural circular frequency of the *undamped* system.

a. If $c > c_c$ (heavy damping), the solution of Eq. (19.38) is

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{19.42}$$

where

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{19.40}$$

and where the constants C_1 and C_2 can be determined from the initial conditions $x(0)$ and $\dot{x}(0)$. This solution corresponds to a nonvibratory motion.

b. If $c = c_c$ (critical damping), the solution of Eq. (19.38) is

$$x = (C_1 + C_2 t)e^{-\omega_n t} \tag{19.43}$$

which also corresponds to a nonvibratory motion.

c. If $c < c_c$ (light damping), the solution of Eq. (19.38) is

$$x = x_0 e^{-(c/2m)t} \sin(\omega_d t + \phi) \tag{19.46}$$

where

$$\omega_d = \omega_n \sqrt{1 - \left(\frac{c}{c_c}\right)^2} \tag{19.45}$$

and where x_0 and ϕ can be determined from the initial conditions $x(0)$ *and* $\dot{x}(0)$. This solution corresponds to oscillations of decreasing amplitude and of period $\tau_d = 2\pi/\omega_d$ (Fig. 19.11).

2. Damped forced vibrations. These vibrations occur when a system with viscous damping is subjected to a periodic force **P** of magnitude $P = P_m \sin \omega_f t$ or when it is elastically connected to a support with an alternating motion $\delta = \delta_m \sin \omega_f t$. In the first case the motion is defined by the differential equation

$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t \tag{19.47}$$

and in the second case by a similar equation obtained by replacing P_m with $k\delta_m$. You will be concerned only with the *steady-state* motion of the system, which is defined by a *particular solution* of these equations, of the form

$$x_{\text{part}} = x_m \sin (\omega_f t - \psi) \tag{19.48}$$

where

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{\sqrt{[1 - (\omega_f/\omega_n)^2]^2 + [2(c/c_c)(\omega_f/\omega_n)]^2}} \tag{19.53}$$

and

$$\tan \varphi = \frac{2(c/c_c)(\omega_f/\omega_n)}{1 - (\omega_f/\omega_n)^2} \tag{19.54}$$

The expression given in Eq. (19.53) is referred to as the *magnification factor* and has been plotted against the frequency ratio ω_f/ω_n in Fig. 19.12 for various values of the damping factor c/c_c. In the problems which follow, you may be asked to determine one of the parameters in Eqs. (19.53) and (19.54) when the others are known.

PROBLEMS

19.127 Show that in the case of heavy damping ($c > c_c$), a body never passes through its position of equilibrium O if it is (*a*) released with no initial velocity from an arbitrary position, (*b*) started from O with an arbitrary initial velocity.

19.128 Show that in the case of heavy damping ($c > c_c$), a body released from an arbitrary position with an arbitrary initial velocity cannot pass more than once through its equilibrium position.

19.129 In the case of light damping, the displacements x_1, x_2, x_3, shown in Fig. 19.11 may be assumed equal to the maximum displacements. Show that the ratio of any two successive maximum displacements x_n and x_{n+1} is a constant and that the natural logarithm of this ratio, called the *logarithmic decrement,* is

$$\ln \frac{x_n}{x_{n+1}} = \frac{2\pi(c/c_c)}{\sqrt{1-(c/c_c)^2}}$$

19.130 In practice, it is often difficult to determine the logarithmic decrement of a system with light damping defined in Prob. 19.129 by measuring two successive maximum displacements. Show that the logarithmic decrement can also be expressed as $(1/k)\ \ln(x_n/x_{n+k})$, where k is the number of cycles between readings of the maximum displacement.

19.131 In a system with light damping ($c < c_c$), the period of vibration is commonly defined as the time interval $\tau_d = 2\pi/\omega_d$ corresponding to two successive points where the displacement-time curve touches one of the limiting curves shown in Fig. 19.11. Show that the interval of time (*a*) between a maximum positive displacement and the following maximum negative displacement is $\frac{1}{2}\tau_d$, (*b*) between two successive zero displacements is $\frac{1}{2}\tau_d$, (*c*) between a maximum positive displacement and the following zero displacement is greater than $\frac{1}{4}\tau_d$.

19.132 A loaded railroad car weighing 30,000 lb is rolling at a constant velocity $\mathbf{v}_0$ when it couples with a spring and dashpot bumper system (Fig. 1). The recorded displacement-time curve of the loaded railroad car after coupling is as shown (Fig. 2). Determine (*a*) the damping constant, (*b*) the spring constant. (*Hint:* Use the definition of logarithmic decrement given in 19.129.)

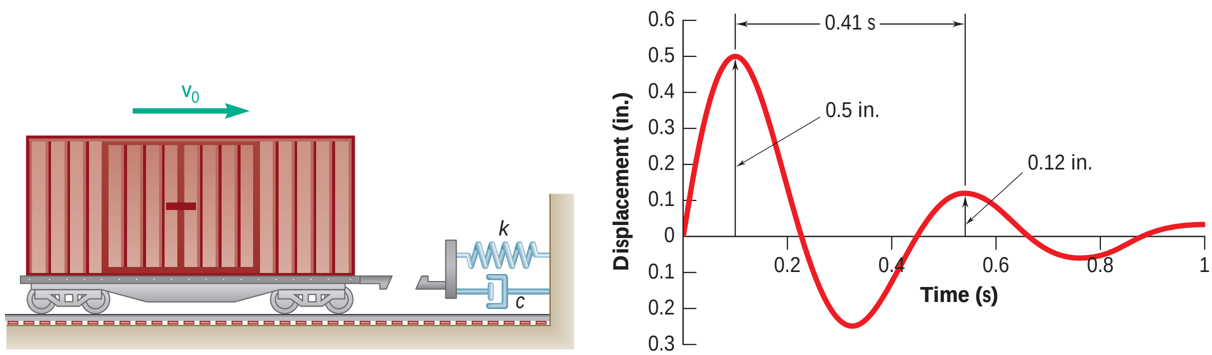


Fig. P19.132 (1) (2)

19.133 A torsional pendulum has a centroidal mass moment of inertia of $0.3\ \text{kg} \cdot \text{m}^2$ and when given an initial twist and released is found to have a frequency of oscillation of 200 rpm. Knowing that when this pendulum is immersed in oil and when given the same initial condition it is found to have a frequency of oscillation of 180 rpm, determine the damping constant for the oil.

19.134 The barrel of a field gun weighs 1500 lb and is returned into firing position after recoil by a recuperator of constant $c = 1100\ \text{lb} \cdot \text{s/ft}$. Determine (*a*) the constant k which should be used for the recuperator to return the barrel into firing position in the shortest possible time without any oscillation, (*b*) the time needed for the barrel to move back two-thirds of the way from its maximum-recoil position to its firing position.

19.135 A platform of weight 200 lb, supported by two springs each of constant $k = 250$ lb/in., is subjected to a periodic force of maximum magnitude equal to 125 lb. Knowing that the coefficient of damping is $12\ \text{lb} \cdot \text{s/in.}$, determine (*a*) the natural frequency in rpm of the platform if there were no damping, (*b*) the frequency in rpm of the periodic force corresponding to the maximum value of the magnification factor, assuming damping, (*c*) the amplitude of the actual motion of the platform for each of the frequencies found in parts *a* and *b*.

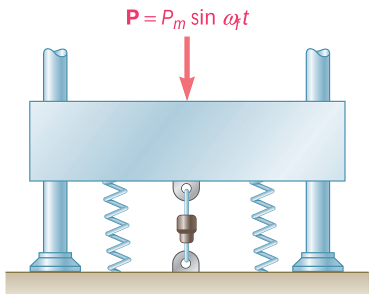


Fig. P19.135

19.136 A 4-kg block A is dropped from a height of 800 mm onto a 9-kg block B which is at rest. Block B is supported by a spring of constant $k = 1500$ N/m and is attached to a dashpot of damping coefficient $c = 230\ \text{N} \cdot \text{s/m}$. Knowing that there is no rebound, determine the maximum distance the blocks will move after the impact.

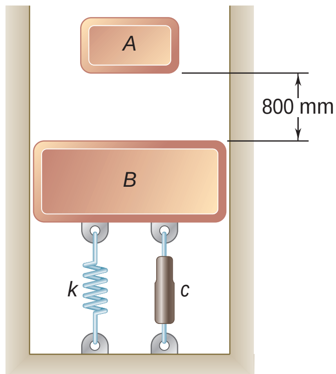


Fig. P19.136

19.137 A 3-kg slender rod AB is bolted to a 5-kg uniform disk. A dashpot of damping coefficient $c = 9\ \text{N} \cdot \text{s/m}$ is attached to the disk as shown. Determine (*a*) the differential equation of motion for small oscillations, (*b*) the damping factor c/c_c.

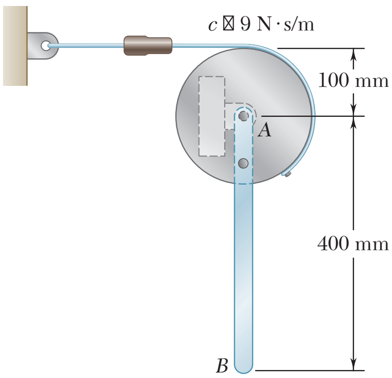


Fig. P19.137

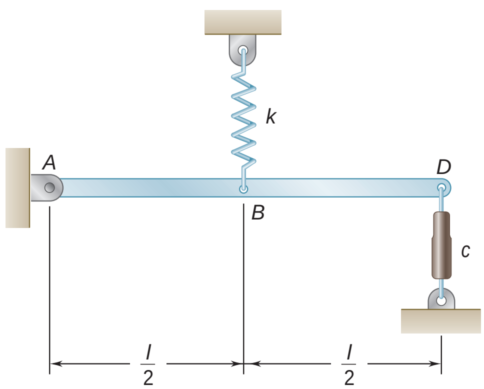


Fig. P19.138

19.138 A uniform rod of mass m is supported by a pin at A and a spring of constant k at B and is connected at D to a dashpot of damping coefficient c. Determine in terms of m, k, and c, for small oscillations, (*a*) the differential equation of motion, (*b*) the critical damping coefficient c_c.

19.139 A machine element weighing 800 lb is supported by two springs, each having a constant of 200 lb/in. A periodic force of maximum value 30 lb is applied to the element with a frequency of 2.5 cycles per second. Knowing that the coefficient of damping is 8 lb · s/in., determine the amplitude of the steady-state vibration of the element.

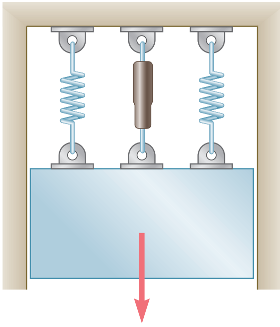


Fig. P19.139

19.140 In Prob. 19.139, determine the required value of the coefficient of damping if the amplitude of the steady-state vibration of the element is to be 0.15 in.

19.141 In the case of the forced vibration of a system, determine the range of values of the damping factor c/c_c for which the magnification factor will always decrease as the frequency ratio ω_f/ω_n increases.

19.142 Show that for a small value of the damping factor c/c_c, the maximum amplitude of a forced vibration occurs when $\omega_f \approx \omega_n$ and that the corresponding value of the magnification factor is $\frac{1}{2}(c/c_c)$.

19.143 A counter-rotating eccentric mass exciter consisting of two rotating 14-oz weights describing circles of 6-in. radius at the same speed but in opposite senses is placed on a machine element to induce a steady-state vibration of the element and to determine some of the dynamic characteristics of the element. At a speed of 1200 rpm a stroboscope shows the eccentric masses to be exactly under their respective axes of rotation and the element to be passing through its position of static equilibrium. Knowing that the amplitude of the motion of the element at that speed is 0.6 in. and that the total mass of the system is 300 lb, determine (*a*) the combined spring constant k, (*b*) the damping factor c/c_c.

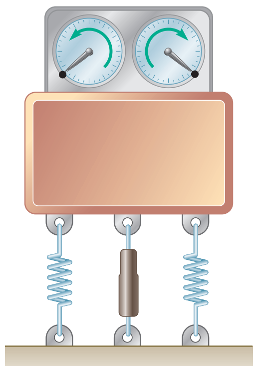


Fig. P19.143

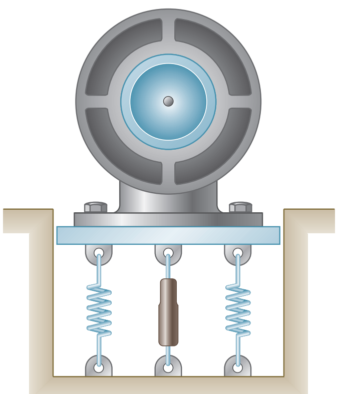


Fig. P19.144 and P19.145

19.144 A 15-kg motor is supported by four springs, each of constant 40 kN/m. The unbalance of the motor is equivalent to a mass of 20 g located 125 mm from the axis of rotation. Knowing that the motor is constrained to move vertically and that the damping factor c/c_c is equal to 0.4, determine the range of frequencies for which the amplitude of the steady-state vibration of the motor is less than 0.2 mm.

19.145 A 220-lb motor is supported by four springs, each of constant 500 lb/in., and is connected to the ground by a dashpot having a coefficient of damping c = 35 lb · s/in. The motor is constrained to move vertically, and the amplitude of its motion is observed to be 0.08 in. at a speed of 1200 rpm. Knowing that the weight of the rotor is 30 lb, determine the distance between the mass center of the rotor and the axis of the shaft.

19.146 A 100-lb motor is directly supported by a light horizontal beam which has a static deflection of 0.2 in. due to the weight of the motor. The unbalance of the rotor is equivalent to a weight of 3.5 oz located 3 in. from the axis of rotation. Knowing that the amplitude of the vibration of the motor is 0.03 in. at a speed of 400 rpm, determine (*a*) the damping factor c/c_c, (*b*) the coefficient of damping c.

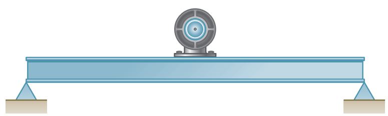


Fig. P19.146

19.147 A machine element is supported by springs and is connected to a dashpot as shown. Show that if a periodic force of magnitude $P = P_m \sin \omega_f t$ is applied to the element, the amplitude of the fluctuating force transmitted to the foundation is

$$F_m = P_m \sqrt{\frac{1 + [2(c/c_c)(\omega_f/\omega_n)]^2}{[1 - (\omega_f/\omega_n)^2]^2 + [2(c/c_c)(\omega_f/\omega_n)]^2}}$$

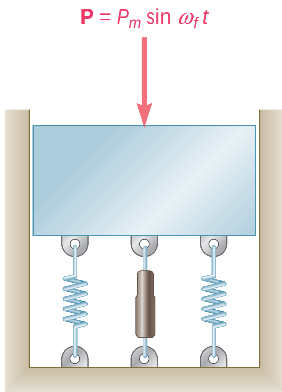


Fig. P19.147 and P19.148

19.148 A 91-kg machine element supported by four springs, each of constant $k = 175$ N/m, is subjected to a periodic force of frequency 0.8 Hz and amplitude 89 N. Determine the amplitude of the fluctuating force transmitted to the foundation if (*a*) a dashpot with a coefficient of damping $c = 365$ N · s/m is connected to the machine element and to the ground, (*b*) the dashpot is removed.

19.149 A simplified model of a washing machine is shown. A bundle of wet clothes forms a weight w_b of 20 lb in the machine and causes a rotating unbalance. The rotating weight is 40 lb (including w_b) and the radius of the washer basket e is 9 in. Knowing the washer has an equivalent spring constant $k = 70$ lb/ft and damping ratio $\zeta = c/c_c = 0.05$ and during the spin cycle the drum rotates at 250 rpm, determine the amplitude of the motion and the magnitude of the force transmitted to the sides of the washing machine.

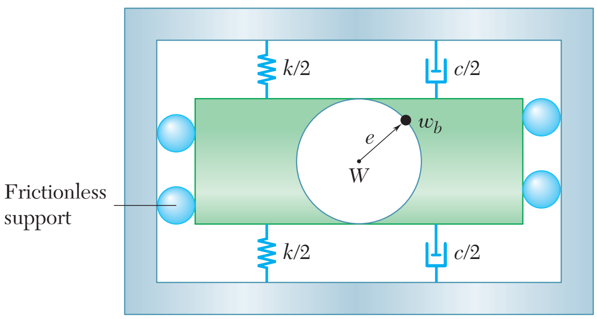


Fig. P19.149

***19.150** For a steady-state vibration with damping under a harmonic force, show that the mechanical energy dissipated per cycle by the dashpot is $E = \pi c x_m^2 \omega_f$, where c is the coefficient of damping, x_m is the amplitude of the motion, and ω_f is the circular frequency of the harmonic force.

***19.151** The suspension of an automobile can be approximated by the simplified spring-and-dashpot system shown. (*a*) Write the differential equation defining the vertical displacement of the mass m when the system moves at a speed v over a road with a sinusoidal cross section of amplitude δ_m and wave length L. (*b*) Derive an expression for the amplitude of the vertical displacement of the mass m.

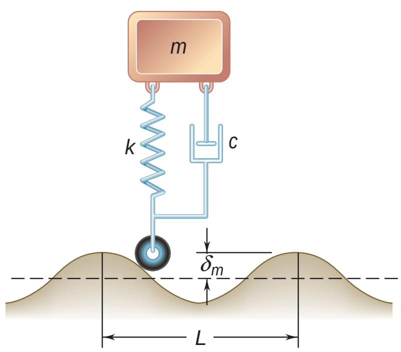


Fig. P19.151

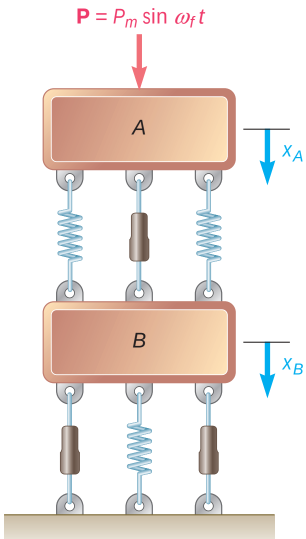


Fig. *P19.152*

***19.152** Two blocks A and B, each of mass m, are supported as shown by three springs of the same constant k. Blocks A and B are connected by a dashpot and block B is connected to the ground by two dashpots, each dashpot having the same coefficient of damping c. Block A is subjected to a force of magnitude $P = P_m \sin \omega_f t$. Write the differential equations defining the displacements x_A and x_B of the two blocks from their equilibrium positions.

19.153 Express in terms of L, C, and E the range of values of the resistance R for which oscillations will take place in the circuit shown when switch S is closed.

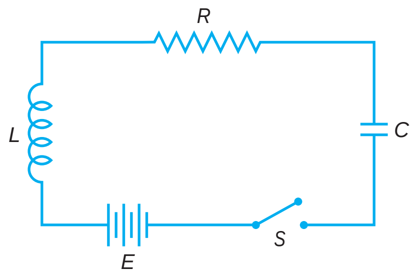


Fig. P19.153

19.154 Consider the circuit of Prob. 19.153 when the capacitor C is removed. If switch S is closed at time $t = 0$, determine (*a*) the final value of the current in the circuit, (*b*) the time t at which the current will have reached $(1 - 1/e)$ times its final value. (The desired value of t is known as the *time constant* of the circuit.)

19.155 and 19.156 Draw the electrical analogue of the mechanical system shown. (*Hint:* Draw the loops corresponding to the free bodies m and A.)

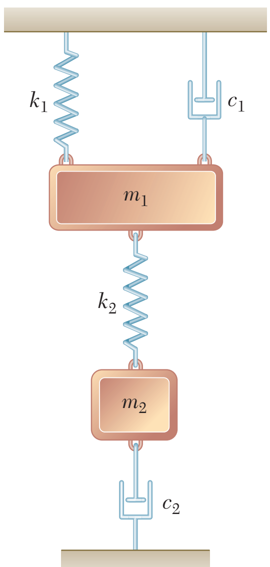


Fig. P19.157 and *P19.158*

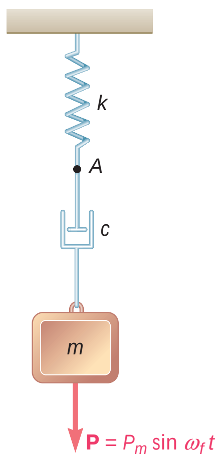


Fig. P19.155 and P19.156

19.157 and *19.158* Write the differential equations defining (*a*) the displacements of the mass m and of the point A, (*b*) the charges on the capacitors of the electrical analogue.

REVIEW AND SUMMARY

This chapter was devoted to the study of *mechanical vibrations,* i.e., to the analysis of the motion of particles and rigid bodies oscillating about a position of equilibrium. In the first part of the chapter [Secs. 19.2 through 19.7], we considered *vibrations without damping*, while the second part was devoted to *damped vibrations* [Secs. 19.8 through 19.10].

Free vibrations of a particle

In Sec. 19.2, we considered the *free vibrations of a particle,* i.e., the motion of a particle P subjected to a restoring force proportional to the displacement of the particle—such as the force exerted by a spring. If the displacement x of the particle P is measured from its equilibrium position O (Fig. 19.17), the resultant $\mathbf{F}$ of the forces acting on P (including its weight) has a magnitude kx and is directed toward O. Applying Newton's second law $F = ma$ and recalling that $a = \ddot{x}$, we wrote the differential equation

$$m\ddot{x} + kx = 0 \qquad (19.2)$$

or, setting $\omega_n^2 = k/m$,

$$\ddot{x} + \omega_n^2 x = 0 \qquad (19.6)$$

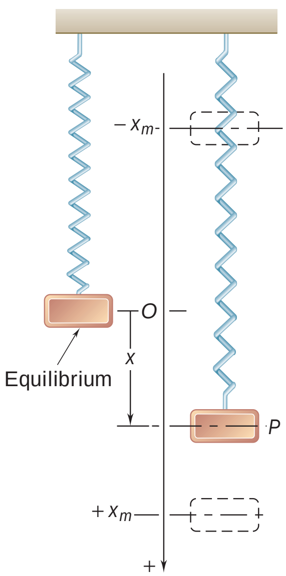


Fig. 19.17

The motion defined by this equation is called a *simple harmonic motion.*

The solution of Eq. (19.6), which represents the displacement of the particle P, was expressed as

$$x = x_m \sin (\omega_n t + \phi) \qquad (19.10)$$

where x_m = amplitude of the vibration
$\omega_n = \sqrt{k/m}$ = natural circular frequency
ϕ = phase angle

The *period of the vibration* (i.e., the time required for a full cycle) and its *natural frequency* (i.e., the number of cycles per second) were expressed as

$$\text{Period} = \tau_n = \frac{2\pi}{\omega_n} \qquad (19.13)$$

$$\text{Natural frequency} = f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} \qquad (19.14)$$

The velocity and acceleration of the particle were obtained by differentiating Eq. (19.10), and their maximum values were found to be

$$v_m = x_m \omega_n \qquad a_m = x_m \omega_n^2 \qquad (19.15)$$

Since all the above parameters depend directly upon the natural circular frequency ω_n and thus upon the ratio k/m, it is essential in any given problem to calculate the value of the constant k; this can be done by determining the relation between the restoring force and the corresponding displacement of the particle [Sample Prob. 19.1].

It was also shown that the oscillatory motion of the particle P can be represented by the projection on the x axis of the motion of a point Q describing an auxiliary circle of radius x_m with the constant angular velocity ω_n (Fig. 19.18). The instantaneous values of the velocity and acceleration of P can then be obtained by projecting on the x axis the vectors $\mathbf{v}_m$ and $\mathbf{a}_m$ representing, respectively, the velocity and acceleration of Q.

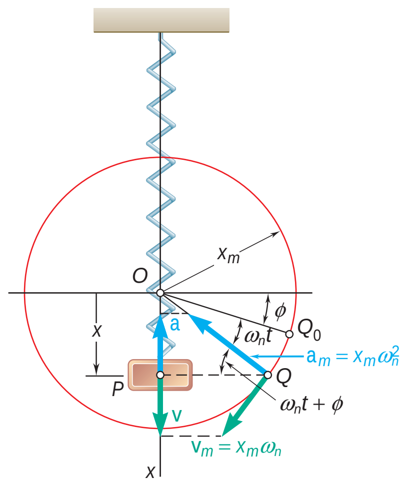


Fig. 19.18

Simple pendulum

While the motion of a *simple pendulum* is not truly a simple harmonic motion, the formulas given above can be used with $\omega_n^2 = g/l$ to calculate the period and natural frequency of the *small oscillations* of a simple pendulum [Sec. 19.3]. Large-amplitude oscillations of a simple pendulum were discussed in Sec. 19.4.

Free vibrations of a rigid body

The *free vibrations of a rigid body* can be analyzed by choosing an appropriate variable, such as a distance x or an angle θ, to define the position of the body, drawing a free-body-diagram equation to express the equivalence of the external and effective forces, and writing an equation relating the selected variable and its second derivative [Sec. 19.5]. If the equation obtained is of the form

$$\ddot{x} + \omega_n^2 x = 0 \qquad \text{or} \qquad \ddot{\theta} + \omega_n^2 \theta = 0 \tag{19.21}$$

the vibration considered is a simple harmonic motion and its period and natural frequency can be obtained *by identifying* ω_n and substituting its value into Eqs. (19.13) and (19.14) [Sample Probs. 19.2 and 19.3].

Using the principle of conservation of energy

The *principle of conservation of energy* can be used as an alternative method for the determination of the period and natural frequency of the simple harmonic motion of a particle or rigid body [Sec. 19.6]. Choosing again an appropriate variable, such as θ, to define the position of the system, we express that the total energy of the system is conserved, $T_1 + V_1 = T_2 + V_2$, between the position of maximum displacement ($\theta_1 = \theta_m$) and the position of maximum velocity ($\dot{\theta}_2 = \dot{\theta}_m$). If the motion considered is simple harmonic, the two members of the equation obtained consist of homogeneous quadratic expressions in θ_m and $\dot{\theta}_m$, respectively.† Substituting $\dot{\theta}_m = \theta_m \omega_n$ in this equation, we can factor out θ_m^2 and solve for the circular frequency ω_n [Sample Prob. 19.4].

Forced vibrations

In Sec. 19.7, we considered the *forced vibrations* of a mechanical system. These vibrations occur when the system is subjected to a periodic force (Fig. 19.19) or when it is elastically connected to a support which has an alternating motion (Fig. 19.20). Denoting by ω_f the forced circular frequency, we found that in the first case, the motion of the system was defined by the differential equation

$$m\ddot{x} + kx = P_m \sin \omega_f t \tag{19.30}$$

and that in the second case it was defined by the differential equation

$$m\ddot{x} + kx = k\delta_m \sin \omega_f t \tag{19.31}$$

The general solution of these equations is obtained by adding a particular solution of the form

$$x_{\text{part}} = x_m \sin \omega_f t \tag{19.32}$$

†If the motion considered can only be *approximated* by a simple harmonic motion, such as for the small oscillations of a body under gravity, the potential energy must be approximated by a quadratic expression in θ_m.

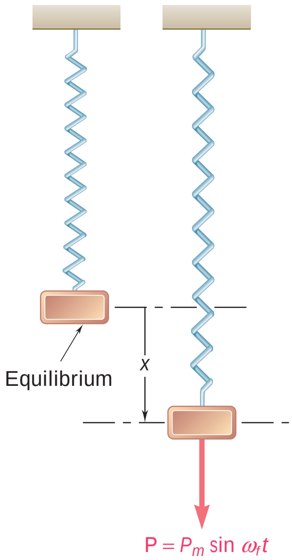


Fig. 19.19

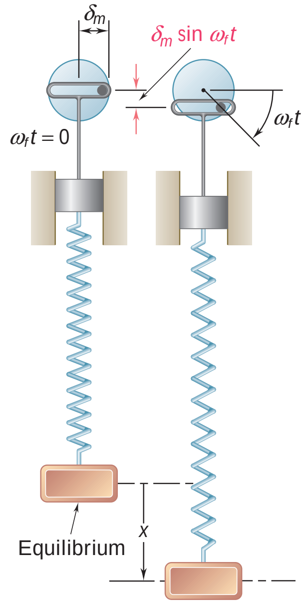


Fig. 19.20

to the general solution of the corresponding homogeneous equation. The particular solution (19.32) represents a *steady-state vibration* of the system, while the solution of the homogeneous equation represents a *transient free vibration* which can generally be neglected.

Dividing the amplitude x_m of the steady-state vibration by P_m/k in the case of a periodic force, or by δ_m in the case of an oscillating support, we defined the *magnification factor* of the vibration and found that

$$\text{Magnification factor} = \frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{1 - (\omega_f/\omega_n)^2} \tag{19.36}$$

According to Eq. (19.36), the amplitude x_m of the forced vibration *becomes infinite when* $\omega_f = \omega_n$, i.e., *when the forced frequency is equal to the natural frequency of the system.* The impressed force or impressed support movement is then said to be in *resonance* with the system [Sample Prob. 19.5]. Actually the amplitude of the vibration remains finite, due to damping forces.

Damped free vibrations

In the last part of the chapter, we considered the *damped vibrations* of a mechanical system. First, we analyzed the *damped free vibrations* of a system with *viscous damping* [Sec. 19.8]. We found that the motion of such a system was defined by the differential equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{19.38}$$

where c is a constant called the *coefficient of viscous damping*. Defining the *critical damping coefficient* c_c as

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n \tag{19.41}$$

where ω_n is the natural circular frequency of the system in the absence of damping, we distinguished three different cases of damping, namely, (1) *heavy damping*, when $c > c_c$; (2) *critical damping*, when $c = c_c$; and (3) *light damping*, when $c < c_c$. In the first two cases, the system when disturbed tends to regain its equilibrium position without any oscillation. In the third case, the motion is vibratory with diminishing amplitude.

Damped forced vibrations

In Sec. 19.9, we considered the *damped forced vibrations* of a mechanical system. These vibrations occur when a system with viscous damping is subjected to a periodic force $\mathbf{P}$ of magnitude $P = P_m \sin \omega_f t$ or when it is elastically connected to a support with an alternating motion $\delta = \delta_m \sin \omega_f t$. In the first case, the motion of the system was defined by the differential equation

$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t \tag{19.47}$$

and in the second case by a similar equation obtained by replacing P_m by $k\delta_m$ in (19.47).

The *steady-state vibration* of the system is represented by a particular solution of Eq. (19.47) of the form

$$x_{\text{part}} = x_m \sin (\omega_f t - \varphi) \tag{19.48}$$

Dividing the amplitude x_m of the steady-state vibration by P_m/k in the case of a periodic force, or by δ_m in the case of an oscillating support, we obtained the following expression for the magnification factor:

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{\sqrt{[1 - (\omega_f/\omega_n)^2]^2 + [2(c/c_c)(\omega_f/\omega_n)]^2}} \tag{19.53}$$

where $\omega_n = \sqrt{k/m}$ = natural circular frequency of undamped system
$c_c = 2m\omega_n$ = critical damping coefficient
c/c_c = damping factor

We also found that the *phase difference* φ between the impressed force or support movement and the resulting steady-state vibration of the damped system was defined by the relation

$$\tan \varphi = \frac{2(c/c_c)(\omega_f/\omega_n)}{1 - (\omega_f/\omega_n)^2} \tag{19.54}$$

Electrical analogues

The chapter ended with a discussion of *electrical analogues* [Sec. 19.10], in which it was shown that the vibrations of mechnical systems and the oscillations of electrical circuits are defined by the same differential equations. Electrical analogues of mechanical systems can therefore be used to study or predict the behavior of these systems.

REVIEW PROBLEMS

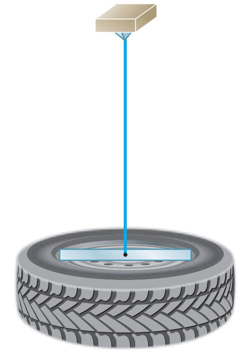


Fig. P19.159

19.159 An automobile wheel-and-tire assembly of total weight 47 lb is attached to a mounting plate of negligible weight which is suspended from a steel wire. The torsional spring constant of the wire is known to be $K = 0.40$ lb · in./rad. The wheel is rotated through 90° about the vertical and then released. Knowing that the period of oscillation is observed to be 30 s, determine the centroidal mass moment of inertia and the centroidal radius of gyration of the wheel-and-tire assembly.

19.160 The period of vibration of the system shown is observed to be 0.6 s. After cylinder B has been removed, the period is observed to be 0.5 s. Determine (*a*) the weight of cylinder A, (*b*) the constant of the spring.

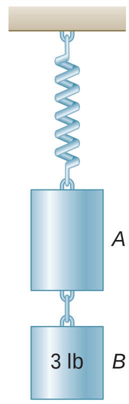


Fig. P19.160

19.161 Disks A and B weigh 30 lb and 12 lb, respectively, and a small 5-lb block C is attached to the rim of disk B. Assuming that no slipping occurs between the disks, determine the period of small oscillations of the system.

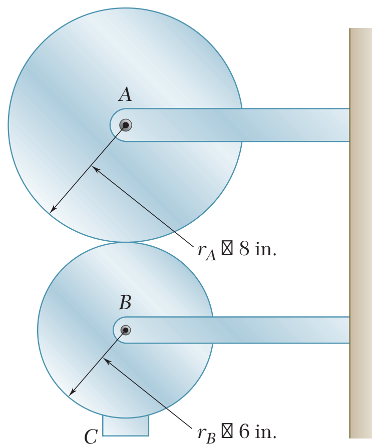


Fig. P19.161

19.162 For the uniform equilateral triangular plate of side $l = 300$ mm, determine the period of small oscillations if the plate is suspended from (*a*) one of its vertices, (*b*) the midpoint of one of its sides.

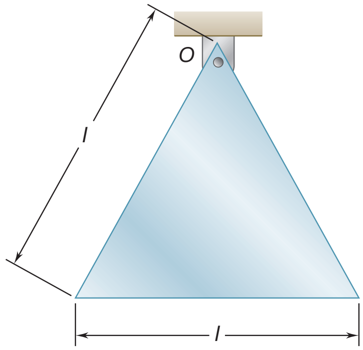


Fig. P19.162

19.163 An 0.8-lb ball is connected to a paddle by means of an elastic cord *AB* of constant $k = 5$ lb/ft. Knowing that the paddle is moved vertically according to the relation $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 8$ in., determine the maximum allowable circular frequency ω_f if the cord is not to become slack.

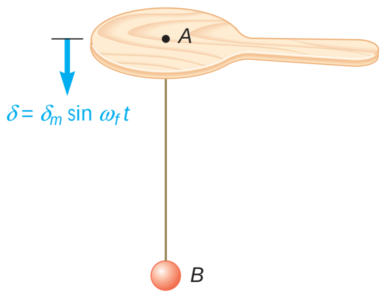


Fig. *P19.163*

19.164 The block shown is depressed 1.2 in. from its equilibrium position and released. Knowing that after 10 cycles the maximum displacement of the block is 0.5 in., determine (*a*) the damping factor c/c_c, (*b*) the value of the coefficient of viscous damping. (*Hint:* See Probs. 19.129 and 19.130.)

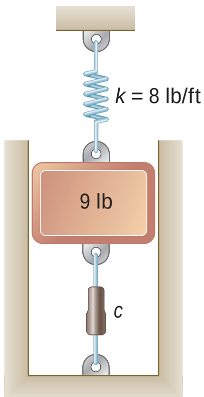


Fig. *P19.164*

19.165 A 4-lb uniform rod is supported by a pin at *O* and a spring at *A* and is connected to a dashpot at *B*. Determine (*a*) the differential equation of motion for small oscillations, (*b*) the angle that the rod will form with the horizontal 5 s after end *B* has been pushed 0.9 in. down and released.

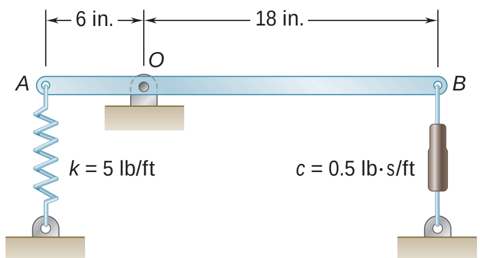


Fig. P19.165

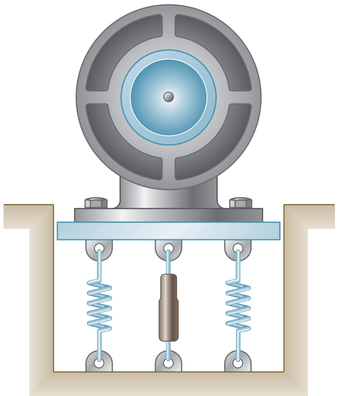


Fig. *P19.166*

19.166 A 400-kg motor supported by four springs, each of constant 150 kN/m, and a dashpot of constant $c = 6500$ N · s/m is constrained to move vertically. Knowing that the unbalance of the rotor is equivalent to a 23-g mass located at a distance of 100 mm from the axis of rotation, determine for a speed of 800 rpm (*a*) the amplitude of the fluctuating force transmitted to the foundation, (*b*) the amplitude of the vertical motion of the motor.

19.167 The compressor shown has a mass of 250 kg and operates at 2000 rpm. At this operating condition, the force transmitted to the ground is excessively high and is found to be $mr\omega_f^2$, where mr is the unbalance and ω_f is the forcing frequency. To fix this problem, it is proposed to isolate the compressor by mounting it on a square concrete block separated from the rest of the floor as shown. The density of concrete is 2400 kg/m^3 and the spring constant for the soil is found to be 80×10^6 N/m. The geometry of the compressor leads to choosing a block that is 1.5 m by 1.5 m. Determine the depth h that will reduce the force transmitted to the ground by 75 percent.

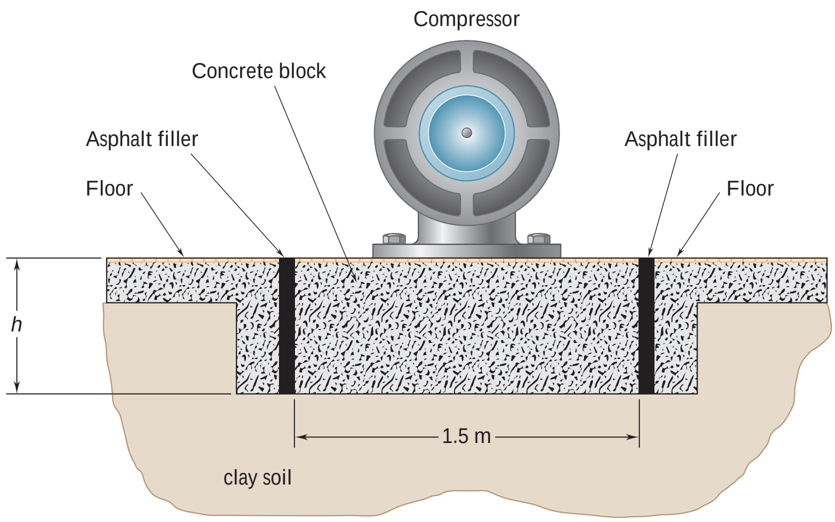


Fig. *P19.167*

19.168 A small ball of mass m attached at the midpoint of a tightly stretched elastic cord of length l can slide on a horizontal plane. The ball is given a small displacement in a direction perpendicular to the cord and released. Assuming the tension T in the cord to remain constant, (*a*) write the differential equation of motion of the ball, (*b*) determine the period of vibration.

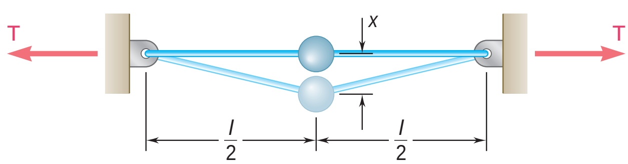


Fig. P19.168

19.169 A certain vibrometer used to measure vibration amplitudes consists essentially of a box containing a slender rod to which a mass m is attached; the natural frequency of the mass-rod system is known to be 5 Hz. When the box is rigidly attached to the casing of a motor rotating at 600 rpm, the mass is observed to vibrate with an amplitude of 0.06 in. relative to the box. Determine the amplitude of the vertical motion of the motor.

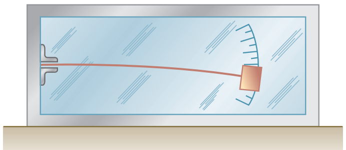


Fig. P19.169

19.170 If either a simple or a compound pendulum is used to determine experimentally the acceleration of gravity g, difficulties are encountered. In the case of the simple pendulum, the string is not truly weightless, while in the case of the compound pendulum, the exact location of the mass center is difficult to establish. In the case of a compound pendulum, the difficulty can be eliminated by using a reversible, or Kater, pendulum. Two knife edges A and B are placed so that they are obviously not at the same distance from the mass center G, and the distance l is measured with great precision. The position of a counterweight D is then adjusted so that the period of oscillation τ is the same when either knife edge is used. Show that the period τ obtained is equal to that of a true simple pendulum of length l and that $g = 4\pi^2 l/\tau^2$.

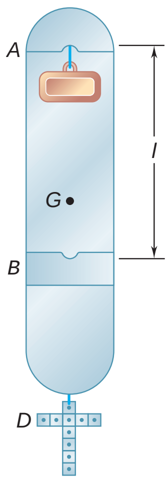


Fig. P19.170

COMPUTER PROBLEMS

19.C1 By expanding the integrand in Eq. (19.19) into a series of even powers of sin ϕ and integrating, it can be shown that the period of a simple pendulum of length l can be approximated by the expression

$$\tau_n = 2\pi \sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 c^2 + \left(\frac{1 \times 3}{2 \times 4}\right)^2 c^4 + \left(\frac{1 \times 3 \times 5}{2 \times 4 \times 6}\right)^2 c^6 + \cdots\right]$$

where $c = \sin \frac{1}{2}\theta_m$ and θ_m is the amplitude of the oscillations. Use computational software to calculate the sum of the series in brackets, using successively 1, 2, 4, 8, and 16 terms, for values of θ_m from 30 to 120° using 30° increments.

19.C2 The force-deflection equation for a class of nonlinear springs fixed at one end is $F = 5x^{1/n}$, where F is the magnitude, expressed in newtons, of the force applied at the other end of the spring and x is the deflection expressed in meters. Knowing that a block of mass m is suspended from the spring and is given a small downward displacement from its equilibrium position, use computational software to calculate and plot the frequency of vibration of the block for values of m equal to 0.2, 0.6, and 1.0 kg and values of n from 1 to 2. Assume that the slope of the force-deflection curve at the point corresponding to $F = mg$ can be used as an equivalent spring constant.

19.C3 A machine element supported by springs and connected to a dashpot is subjected to a periodic force of magnitude $P = P_m \sin \omega_f t$. The *transmissibility* T_m of the system is defined as the ratio F_m/P_m of the maximum value F_m of the fluctuating periodic force transmitted to the foundation to the maximum value P_m of the periodic force applied to the machine element. Use computational software to calculate and plot the value of T_m for frequency ratios ω_f/ω_n equal to 0.8, 1.4, and 2.0 and for damping factors c/c_c equal to 0, 1, and 2. (*Hint:* Use the formula given in Prob. 19.147.)

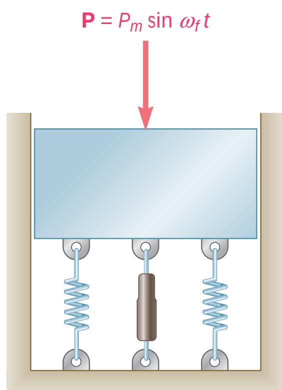


Fig. P19.C3

19.C4 A 15-kg motor is supported by four springs, each of constant 60 kN/m. The unbalance of the motor is equivalent to a mass of 20 g located 125 mm from the axis of rotation. Knowing that the motor is constrained to move vertically, use computational software to calculate and plot the amplitude of the vibration and the maximum acceleration of the motor for motor speeds of 1000 to 2500 rpm.

19.C5 Solve Prob. 19.C4, assuming that a dashpot having a coefficient of damping $c = 2.5$ kN · s/m has been connected to the motor base and to the ground.

19.C6 A small trailer and its load have a total mass of 250 kg. The trailer is supported by two springs, each of constant 10 kN/m, and is pulled over a road, the surface of which can be approximated by a sine curve with an amplitude of 40 mm and a wave length of 5 m (i.e., the distance between successive crests is 5 m and the vertical distance from crest to trough is 80 mm). (*a*) Neglecting the mass of the wheels and assuming that the wheels stay in contact with the ground, use computational software to calculate and plot the amplitude of the vibration and the maximum vertical acceleration of the trailer for speeds of 10 to 80 km/h. (*b*) Determine the range of values of the speed of the trailer for which the wheels will lose contact with the ground.

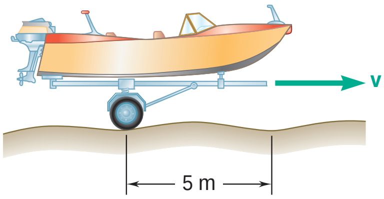


Fig. P19.C6

Appendix

Fundamentals of Engineering Examination

Engineers are required to be licensed when their work directly affects the public health, safety, and welfare. The intent is to ensure that engineers have met minimum qualifications involving competence, ability, experience, and character. The licensing process involves an initial exam, called the *Fundamentals of Engineering Examination;* professional experience; and a second exam, called the *Principles and Practice of Engineering.* Those who successfully complete these requirements are licensed as a *Professional Engineer.* The exams are developed under the auspices of the *National Council of Examiners for Engineering and Surveying.*

The first exam, the *Fundamentals of Engineering Examination,* can be taken just before or after graduation from a four-year accredited engineering program. The exam stresses subject material in a typical undergraduate engineering program, including statics. The topics included in the exam cover much of the material in this book. The following is a list of the main topic areas, with references to the appropriate sections in this book. Also included are problems that can be solved to review this material.

Concurrent Force Systems (2.2–2.9; 2.12–2.14)
Problems: 2.33, 2.35, 2.36, 2.37, 2.75, 2.84, 2.92, 2.93, 2.97

Vector Forces (3.4–3.11)
Problems: 3.17, 3.18, 3.24, 3.33, 3.37, 3.39

Equilibrium in Two Dimensions (2.11; 4.1–4.7)
Problems: 4.1, 4.9, 4.10, 4.17, 4.29, 4.32, 4.67, 4.81

Equilibrium in Three Dimensions (2.15; 4.8–4.9)
Problems: 4.97, 4.99, 4.100, 4.105, 4.113, 4.115, 4.128, 4.129, 4.140

Centroid of an Area (5.2–5.7)
Problems: 5.9, 5.17, 5.28, 5.36, 5.41, 5.56, 5.62, 5.97, 5.100, 5.103, 5.126

Analysis of Trusses (6.2–6.7)
Problems; 6.2, 6.5, 6.33, 6.45, 6.46, 6.50

Equilibrium of Two-Dimensional Frames (6.9–6.11)
Problems: 6.76, 6.82, 6.89, 6.93, 6.94

Shear and Bending Moment (7.3–7.6)
Problems: 7.22, 7.25, 7.31, 7.36, 7.45, 7.49, 7.70, 7.83

Friction (8.2–8.5; 8.10)
Problems: 8.11, 8.15, 8.21, 8.30, 8.50, 8.53, 8.101, 8.104, 8.105

Moments of Inertia (9.2–9.10)
Problems: 9.5, 9.31, 9.32, 9.33, 9.77, 9.78, 9.84, 9.89, 9.101, 9.103

Kinematics (11.1–11.6; 11.9–11.14; 15.2–15.8)
Problems: 11.3, 11.5, 11.35, 11.61, 11.67, 11.97, 15.8, 15.29, 15.39, 15.61, 15.65, 15.87, 15.112, 15.142

Force, Mass, and Acceleration (12.1–12.6; 16.2–16.8)
Problems: 12.6, 12.8, 12.30, 12.32, 12.36, 12.45, 12.52, 12.55, 16.1, 16.3, 16.7, 16.26, 16.27, 16.50, 16.61, 16.63, 16.78, 16.85, 16.137

Work and Energy (13.1–13.6; 13.8; 17.1–17.7)
Problems: 13.5, 13.7, 13.17, 13.22, 13.40, 13.41, 13.47, 13.64, 13.66, 13.68, 17.1, 17.2, 17.16, 17.22

Impulse and Momentum (13.10–13.15; 17.8–17.12)
Problems: 13.119, 13.121, 13.129, 13.134, 13.146, 13.155, 13.160, 13.171, 17.53, 17.58, 17.70, 17.72, 17.96, 17.97, 17.111

Vibration (19.1–19.3; 19.5–19.7)
Problems: 19.1, 19.2, 19.11, 19.18, 19.23, 19.28, 19.49, 19.55, 19.63, 19.74, 19.83, 19.85, 19.101, 19.105, 19.116

Friction (Problems involving friction occur in each of the above subjects)

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Answers to Problems

Answers to problems with a number set in straight type are given on this and the following pages. Answers to problems set in italic are not listed.

CHAPTER 2

2.1 3.30 kN c 66.6°.
2.2 139.1 lb d 67.0°.
2.4 8.03 kips d 3.8°.
2.5 (*a*) 101.4 N.
(*b*) 196.6 N.
2.6 (*a*) 3660 N.
(*b*) 3730 N.
2.7 2600 N c 53.5°.
2.8 (*a*) 853 lb.
(*b*) 567 lb.
2.10 (*a*) 37.1°.
(*b*) 73.2 N.
2.11 (*a*) 392 lb. (*b*) 346 lb.
2.13 (*a*) 368 lb y. (*b*) 213 lb.
2.14 (*a*) 21.1 N w. (*b*) 45.3 N.
2.15 139.1 lb d 67.0°.
2.16 8.03 kips d 3.8°.
2.18 100.3 N d 21.2°.
2.19 104.4 N b 86.7°.
2.21 (80 N) 61.3 N, 51.4 N; (120 N) 41.0 N, 112.8 N; (150 N) −122.9 N, 86.0 N.
2.22 (40 lb) 20.0 lb, −34.6 lb; (50 lb) −38.3 lb, −32.1 lb; (60 lb) 54.4 lb, 25.4 lb.
2.24 (102 lb) −48.0 lb, 90.0 lb; (106 lb) 56.0 lb, 90.0 lb; (200 lb) −160.0 lb, −120.0 lb.
2.25 (*a*) 2190 N. (*b*) 2060 N.
2.26 (*a*) 610 lb. (*b*) 500 lb.
2.28 (*a*) 373 lb. (*b*) 286 lb.
2.30 (*a*) 621 N. (*b*) 160.8 N.
2.31 654 N c 21.5°.
2.32 251 N b 85.3°.
2.33 54.9 lb c 48.9°.
2.35 309 N d 86.6°.
2.36 474 N c 32.5°.
2.37 203 lb a 8.46°.
2.39 (*a*) 21.7°. (*b*) 229 N.
2.40 (*a*) 26.5 N. (*b*) 623 N.
2.42 (*a*) 29.4°. (*b*) 371 lb.
2.43 (*a*) 6.37 kN. (*b*) 12.47 kN.
2.45 (*a*) 1244 lb. (*b*) 115.4 lb.
2.46 (*a*) 172.7 lb. (*b*) 231 lb.
2.48 (*a*) 305 N. (*b*) 514 N.
2.49 T_c = 5.87 kips; T_D = 9.14 kips.
2.51 (*a*) 312 N. (*b*) 144 N.
2.52 $0 < P < 514$ N.
2.53 (*a*) 1213 N. (*b*) 166.3 N
2.54 (*a*) 863 N. (*b*) 1216 N
2.55 F_A = 1303 lb; F_B = 420 lb.
2.57 (*a*) 1081 N. (*b*) 82.5°.
2.58 (*a*) 1294 N. (*b*) 62.5°.
2.59 (*a*) 5.00° a. (*b*) 104.6 lb.
2.61 (*a*) 784 N. (*b*) 71.0°.
2.62 1.250 m.
2.63 (*a*) 10.98 lb. (*b*) 30.0 lb.
2.65 27.4° < a < 222.6°.
2.67 (*a*) 300 lb. (*b*) 300 lb. (*c*) 200 lb. (*d*) 200 lb. (*e*) 150.0 lb.
2.68 (*b*) 200 lb. (*d*) 150.0 lb.
2.69 (*a*) 1293 N. (*b*) 2220 N.
2.71 (*a*) −130.1 N; +816 N; +357 N. (*b*) 98.3°; 25.0°; 66.6°.
2.72 (*a*) +390 N; +614 N; +181.8 N. (*b*) 58.7°; 35.0°; 76.0°.
2.73 (*a*) −175.8 N; −257 N; +251 N. (*b*) 116.1°; 130.0°; 51.1°.
2.74 (*a*) +350 N; −169.0 N; +93.8 N. (*b*) 28.9°; 115.0°; 76.4°.
2.75 (*a*) −1861 lb; +3360 lb; +677 lb. (*b*) 118.5°; 30.5°; 80.0°.
2.77 (*a*) +56.4 lb; −103.9 lb; −20.5 lb. (*b*) 62.0°; 150.0°; 99.8°.
2.79 950 N; 43.4°; 71.6°; 127.6°.
2.81 (*a*) 43.9°. (*b*) F_x = +107.7 lb; F_z = 267 lb; F = 416 lb.
2.82 (*a*) 114.4°. (*b*) F_y = +694 N; F_z = +855 N; F = 1209.1 N.
2.83 (*a*) 194.0 N; 108.0 N. (*b*) 105.1°; 62.0°.
2.85 −1.260 kips; +1.213 kips; +0.970 kips.
2.86 −0.820 kips; +0.978 kips; −0.789 kips.
2.87 −1125 N; 750 N; 450 N.
2.89 240 N; −255 N; 160.0 N.
2.91 940 N; 65.7°; 28.2°; 76.4°.
2.92 940 N; 63.4°; 27.2°; 84.5°.
2.94 913 lb; 50.6°; 117.6°; 51.8°.
2.95 748 N; 120.1°; 52.5°; 128.0°.
2.96 3120 N; 37.4°; 122.0°; 72.6°.
2.97 130.0 lb.
2.99 13.98 kN.
2.101 926 N x.
2.103 2100 lb.
2.104 1868 lb.
2.106 T_{AB} = 571 lb; T_{AC} = 830 lb; T_{AD} = 528 lb.
2.107 960 N.
2.108 $0 \le Q < 300$ N.
2.109 845 N.
2.110 768 N.
2.112 3090 lb.
2.113 T_{AB} = 31.7 lb; T_{AC} = 64.3 lb.
2.115 T_{AB} = 510 N; T_{AC} = 56.2 N; T_{AD} = 536 N.
2.116 T_{AB} = 1340 N; T_{AC} = 1025 N; T_{AD} = 915 N.
2.117 T_{AB} = 1431 N; T_{AC} = 1560 N; T_{AD} = 183.0 N.
2.118 T_{AB} = 1249 N; T_{AC} = 490 N; T_{AD} = 1647 N.
2.119 T_{AB} = 842 lb; T_{AC} = 624 lb; T_{AD} = 1088 lb.
2.121 T_{BAC} = 76.7 lb; T_{AD} = 26.9 lb; T_{AE} = 49.2 lb.
2.122 P = 305 lb; T_{BAC} = 117.0 lb; T_{AD} = 40.9 lb.
2.123 P = 131.2 N; Q = 29.6 N.
2.125 (*a*) 1155 N. (*b*) 1012 N.
2.127 21.8 kN c 73.4°.
2.128 (*a*) 523 lb. (*b*) 428 lb.
2.129 (*a*) 95.1 lb. (*b*) 95.0 lb.

2.131 $F_C = 6.40$ kN; $F_D = 4.80$ kN.
2.133 (*a*) 288 N. (*b*) 67.5°, 30.0°, 108.7°.
2.134 (*a*) 114.4°. (*b*) $F_y = 694$ lb, $F_z = 855$ lb, $F = 1209$ lb.
2.135 515 N; $u_x = 70.2°$; $u_y = 27.6°$; $u_z = 71.5°$.
2.137 (*a*) 125.0 lb. (*b*) 45.0 lb.
2.C2 (1) (*b*) 20°; (*c*) 244 lb. (2) (*b*) −10°; (*c*) 467 lb. (3) (*b*) 10°; (*c*) 163.2 lb.
2.C3 (*a*) 1.001 m. (*b*) 4.01 kN. (*c*) 1.426 kN; 1.194 kN.

CHAPTER 3

3.1 115.7 lb-in. i.
3.2 115.7 lb-in. i.
3.4 (*a*) 196.2 N · m i. (*b*) 199.0 N b 59.5°
3.5 (*a*) 196.2 N · m i. (*b*) 321 N d 35.0°. (*c*) 231 N x at point *D*.
3.6 (*a*) 20.5 N · m l. (*b*) 68.4 mm.
3.7 (*a*) 27.4 N · m l. (*b*) 228 N d 42°.
3.9 1.120 kip · in. l.
3.10 493 lb · in. l.
3.11 (*a*) 760 N · m l. (*b*) 760 N · m l.
3.12 1224 N.
3.17 (*a*) $(-3\mathbf{i} - \mathbf{j} - \mathbf{k})/ 1\overline{11}$. (*b*) $(2\mathbf{j} + 3\mathbf{k})/ 1\overline{13}$.
3.18 2.21 m.
3.20 (*a*) $9\mathbf{i} + 22\mathbf{j} + 21\mathbf{k}$. (*b*) $22\mathbf{i} + 11\mathbf{k}$, (*c*) 0.
3.22 (492 lb · ft)**i** + (144.0 lb · ft)**j** − (372 lb · ft)**k**.
3.23 −(25.4 lb · ft)**i** − (12.60 lb · ft)**j** − (12.60 lb · ft)**k**.
3.24 (1200 N · m)**i** − (1500 N · m)**j** − (900 N · m)**k**.
3.25 (7.50 N · m)**i** − (6.00 N · m)**j** − 10.39 N · m)**k**.
3.27 100.8 mm.
3.28 144.8 mm.
3.29 4.86 ft.
3.30 5.17 ft.
3.32 2.36 m.
3.33 1.491 m.
3.35 **P** · **Q** = +1; **P** · **S** = −11; **Q** · **S** = +10.
3.37 43.6°.
3.38 38.9°.
3.39 77.9°.
3.41 26.8°.
3.42 33.3.
3.43 (*a*) 52.9°. (*b*) 326 N.
3.45 7.
3.46 (*a*) 67. (*b*) 111.
3.47 $M_x = 0$, $M_y = -162.0$ N · m, $M_z = 270$ N · m.
3.48 $M_x = -576$ N · m, $M_y = -243$ N · m, $M_z = 405$ N · m.
3.49 61.5 lb.
3.51 283 lb.
3.53 $P = 125.0$ N; f = 73.7°, u = 53.1°.
3.54 23.0 N · m.
3.55 2.28 N · m.
3.56 −9.50 N · m.
3.57 +207 lb · ft.
3.59 −90 N · m.
3.60 −111.0 N · m.
3.61 $aP/ 1\overline{2}$.
3.64 0.1198 m.
3.66 13.06 in.
3.67 12.69 in.
3.69 0.249 m.
3.70 (*a*) 336 lb · in. l. (*b*) 28.0 in. (*c*) 54.0°.
3.72 1.250 in.
3.73 (*a*) 26.7 N. (*b*) 50.0 N. (*c*) 23.5 N.
3.74 (*a*) 6.19 N · m. i. (*b*) 6.19 N · m i. (*c*) 6.19 N · m i.
3.76 $M = 3.22$ N · m; $u_x = 90.0°$; $u_y = 53.1°$, $u_z = 36.9°$.
3.77 $M = 2.72$ N · m; $u_x = 134.9°$, $u_y = 58.0°$, $u_z = 61.9°$.
3.78 $M = 604$ lb · in.; $u_x = 72.8°$, $u_y = 27.3°$, $u_z = 110.5°$.
3.79 $M = 1170$ lb · in.; $u_x = 81.2°$, $u_y = 13.70°$, $u_z = 100.4°$.
3.80 $M = 4.50$ N · m; $u_x = 90.0°$, $u_y = 177.1°$, $u_z = 87.1°$.
3.81 **F** = 260 lb d 67.4°; $\mathbf{M}_C$ = 200 lb · in. i.
3.82 (*a*) **F** = 30.0 lbw; **M** = 150.0 lb · in. l.
(*b*) **B** = 50.0 lb z ; **C** = 50.0 lb y .
3.83 (*a*) $\mathbf{F}_B$ = 250 N c 60.0°; $\mathbf{M}_B$ = 75.0 N · m i.
(*b*) $\mathbf{F}_A$ = 375 N b 60.0°; $\mathbf{F}_B$ = 625 N c 60.0°.
3.86 $\mathbf{F}_A$ = 389 N c 60.0°; $\mathbf{F}_C$ = 651 N c 60.0°.
3.87 (*a*) **F** = 216 N a 65.0°; **M** = 33.0 N · m i.
(*b*) **F** = 216 N a 65.0° applied to the lever 267 mm to the left of *B*.
3.89 (*a*) **P** = 60.0 lb a 50.0°; 3.24 in. from *A*.
(*b*) **P** = 60.0 lb a 50.0° 3.87 in. below *A*.
3.90 (*a*) 30.0°. (*b*) 65.7°.
3.91 **F** = 900 N w; $x = 50.0$ mm.
3.93 **F** = −(128.0 lb)**i** − (256 lb)**j** + (32.0 lb)**k**;
M = (4.10 kip · ft)**i** + (16.38 kip · ft)**k**.
3.95 **F** = −(28.5 N)**j** + (106.3 N)**k**; **M** = (12.35 N · m)**i** − (19.16 N · m)**j** − (5.13 N · m)**k**.
3.96 **F** = −(1220 N)**i**; **M** = (73.2 N · m)**j** − (122 N · m)**k**.
3.97 $\mathbf{F}_C$ = (5.00 N)**i** + (150.0 N)**j** −(90.0 N)**k**;
$\mathbf{M}_C$ = (77.4 N · m)**i** + (61.5 N · m)**j** + (106.8 N · m)**k**.
3.98 **F** = (36.0 lb)**i** − (28.0 lb)**j** − (6.00 lb)**k**;
M = −(157.0 lb · ft)**i** + (22.5 lb · ft)**j** − (240 lb · ft)**k**.
3.99 (*a*) 135.0 mm. (*b*) $\mathbf{F}_z$ = (42.0 N)**i** + (42.0 N)**j** − (49.0 N)**k**;
$\mathbf{M}_z$ = −(25.9 N · m)**i**. + (21.2 N · m)**j**.
3.101 (*a*) Loading *a* 500 N w; 1000 N · m i.
Loading *b* 500 N x; 500 N · m l.
Loading *c* 500 N w; 500 N · m i.
Loading *d* 500 N w; 1100 N · m i.
Loading *e* 500 N w; 1000 N · m i.
Loading *f* 500 N w; 200 N · m i.
Loading *g* 500 N w; 2300 N · m l.
Loading *h* 500 N w; 650 N · m l.
(*b*) Loadings *a* and *e* are equivalent.
3.102 Equivalent to case *f* of problem 3.101.
3.104 Equivalent force-couple system at *D*.
3.106 (*a*) 39.6 in. to the right of *D*. (*b*) 33.1 in.
3.107 (*a*) 2.00 ft to the right of *C*. (*b*) 2.31 ft to the right of *C*.
3.108 (*a*) 34.0 b 28.0°. (*b*) *AB*: 11.64 in. to the left of *B*; *BC*: 6.20 in. below *B*.
3.109 (*a*) 48.2 lb · in. l. (*b*) 240 lb · in. l. (*c*) 0.
3.111 (*a*) 0.365 m above *G*. (*b*) 0.227 m to the right of *G*.
3.112 (*a*) 0.299 m above *G*. (*b*) 0.259 m to the right of *G*.
3.113 773 lb d 79.0°; 9.54 ft to the right of *A*.
3.114 (*a*) 665 lb a 79.6°; 64.9 in. to the right of *A*.
(*b*) 22.9°.
3.116 (*a*) 1562 N b 50.2°. (*b*) 250 mm to the right of *C* and 300 mm above *C*.
3.117 (*a*) 1308. N a 66.6°. (*b*) 412 mm to the right of *A* and 250 mm to the right of *C*.
3.118 (*a*) **R** = *F* d $\tan^{-1} (a^2/2bx)$;
M = $2Fb^2 (x - x^3/a^2)/ 2\overline{a^4 + 4b^2x^2}$ l. (*b*) 0.369 m.
3.119 **R** = −(21.0 N)**i** −(29.0 N)**j** + (16.00 N)**k**;
M = −(0.870 N · m)**i** + (0.630 N · m)**j** + (0.390 N · m)**k**.
3.120 **R** = (420 N)**j** − (339 N)**k**; **M** = (1.125 N · m)**i** + (163.9 N · m)**j** − (109.9 N · m)**k**.

3.121 $\mathbf{R} = -(420\text{ N})\mathbf{j} - (50.0\text{ N})\mathbf{j} - (250\text{ N})\mathbf{k}$; $\mathbf{M} = (30.8\text{ N}\cdot\text{m})\mathbf{j} - (22.0\text{ N}\cdot\text{m})\mathbf{k}$.
3.122 (*a*) $\mathbf{B} = (2.50\text{ lb})\mathbf{i}$, $\mathbf{C} = (0.1000\text{ lb})\mathbf{i} - (2.47\text{ lb})\mathbf{j} - (0.700\text{ lb})\mathbf{k}$. (*b*) $R_y = -2.47$ lb; $M_x = 1.360$ lb · ft.
3.123 (*a*) $\mathbf{F}_B = -(80.0\text{ N})\mathbf{k}$, $\mathbf{F}_C = -(30.0\text{ N})\mathbf{i} + (40.0\text{ N})\mathbf{k}$. (*b*) $R_y = 0$, $R_z = 40.0$ N. (*c*) When the slot in vertical
3.124 (*a*) 60.0°. (*b*) $(20.0\text{ lb})\mathbf{i} - (34.6\text{ lb})\mathbf{j}$; $(520\text{ lb}\cdot\text{in.})\mathbf{i}$.
3.127 1035 N; 2.57 m from *OG* and 3.05 m from *OE*.
3.128 2.32 m from *OG* and 1.165 m from *OE*.
3.129 405 lb; 12.60 ft to the right of *AB* and 2.94 ft. below *BC*.
3.130 $a = 0.722$ ft; $b = 20.6$ ft.
3.133 (*a*) *P*; $u_x = 90.0°$, $u_y = 90.0°$, $u_z = 0$. (*b*) $5a/2$. (*c*) Axis of the wrench is parallel to the z axis at $x = a$, $y = -a$.
3.134 (*a*) $P\,1\overline{3}$; $u_x = u_y = u_z = 54.7°$. (*b*) $-a$. (*c*) Axis of the wrench is diagonal *OA*.
3.135 (*a*) $-(21.0\text{ lb})\mathbf{j}$. (*b*) 0.571 in. (*c*) Axis of wrench is parallel to the y axis at $x = 0$; $z = 1.667$ in.
3.137 (*a*) $-(84.0\text{ N})\mathbf{j} - (80.0\text{ N})\mathbf{k}$. (*b*) 0.477 m. (*c*) $x = 0.526$ m, $y = 0$, $z = -0.1857$ m.
3.139 (*a*) $3P(2\mathbf{i} - 20\mathbf{j} - \mathbf{k})/25$. (*b*) $-0.0988\,a$. (*c*) $x = 2.00\,a$, $z = -1.990\,a$.
3.142 $\mathbf{R} = (20.0\text{ N})\mathbf{i} + (30.0\text{ N})\mathbf{j} - (10.00\text{ N})\mathbf{k}$; $y = -0.540$ m, $z = -0.420$ m.
3.143 $\mathbf{F}_A = (M/b)\mathbf{i} + R[1+(a/b)]\mathbf{k}$; $\mathbf{F}_B = -(M/b)\mathbf{i} - (aR/b)\mathbf{k}$.
3.147 41.7 N · m l. (b) 147.4 N a 45.0°.
3.148 116.2 lb · ft l.
3.150 27.4°.
3.151 1.252 m.
3.153 (*a*) $\mathbf{F}$ = 560 lb c 20.0°; $\mathbf{M}$ = 7720 lb · ft i. (b) $\mathbf{F}$ = 560 lb c 20.0°; $\mathbf{M}$ = 4290 lb · ft i.
3.154 $(0.227\text{ lb})\mathbf{i} + (0.1057\text{ lb})\mathbf{k}$; 63.6 in. to the right of *B*.
3.156 (*a*) 6.91 m. (*b*) 458 N; 3.16 m to the right of *A*.
3.158 $\mathbf{F}_B$ = 35.0 kips w; $\mathbf{F}_F$ = 25.0 kips w.
3.C3 *4 sides:* b = 10°, a = 44.1°; b = 20°, a = 41.6°; b = 30°, a = 37.8°.
3.C4 u = 0 *rev*: $M = 97.0$ N · m; u = 6 *rev*: $M = 63.3$ N · m; u = 12 *rev*: $M = 9.17$ N · m.
3.C6 $d_{AB} = 36.0$ in.; $d_{CD} = 9.00$ in.; $d_{min} = 58.3$ in.

CHAPTER 4

4.1 (*a*) 6.07 kN x. (*b*) 4.23 kN x.
4.2 (*a*) 4.89 kN x. (*b*) 3.69 kN x.
4.3 (*a*) $\mathbf{A}$ = 20.0 lb w; $\mathbf{B}$ = 150.0 lb x. (*b*) $\mathbf{A}$ = 10.00 lb w; $\mathbf{B}$ = 140.0 lb x.
4.5 (*a*) 37.9 N x. (*b*) 373 N x.
4.6 (*a*) 2.76 N x. (*b*) 391 N x.
4.8 (*a*) 245 lb x. (*b*) 140.0 lb x.
4.9 2.00 in. $\leq a \leq$ 10.00 in.
4.10 150.0 mm $\leq d \leq$ 400 mm.
4.11 0.500 kN $\leq Q \leq$ 11.00 kN.
4.14 6.00 kips $\leq P \leq$ 42.0 kips.
4.15 (*a*) 2.00 kN. (*b*) 2.32 kN a 46.4°.
4.17 (*a*) 150.0 lb. (*b*) 225 lb d 32.3°.
4.18 111.2 lb y.
4.19 (*a*) 600 N. (*b*) 1253 N a 69.8°.
4.21 (*a*) $\mathbf{A}$ = 225 N x; $\mathbf{C}$ = 641 N d 20.6°. (*b*) $\mathbf{A}$ = 365 N a 60.0°; $\mathbf{C}$ = 844 N d 22.0°.
4.23 (*a*) $\mathbf{A}$ = 150.0 N a 30.0°; $\mathbf{B}$ = 150.0 N b 30.0°. (*b*) $\mathbf{A}$ = 433 N c 12.55°; $\mathbf{B}$ = 488 N b 30.0°.
4.25 (*a*) $\mathbf{A}$ = 44.7 b 26.6°; $\mathbf{B}$ = 30.0 lb x. (*b*) $\mathbf{A}$ = 30.2 lb b 41.4°; $\mathbf{B}$ = 34.6 lb b 60.0°.
4.26 (*a*) $\mathbf{A}$ = 20.0 lb x; $\mathbf{B}$ = 50.0 lb b 36.9°. (*b*) $\mathbf{A}$ = 23.1 lb a 60.0°; $\mathbf{B}$ = 59.6 lb b 30.2°.
4.27 (*a*) 190.9 N. (*b*) 142.3 N a 18.43°.
4.28 (*a*) 324 N. (*b*) 270 N y.
4.29 (*a*) 117.0 lb. (*b*) 129.8 lb c 56.3°.
4.30 (*a*) 195.0 lb. (*b*) 255 lb c 45.0°.
4.31 T = 80.0 N; $\mathbf{C}$ = 89.4 N a 26.6°.
4.32 (*a*) 130.0 N. (*b*) 224 N d 2.05°.
4.35 (*a*) 600 N. (*b*) $\mathbf{A}$ = 4.00 kN z; $\mathbf{B}$ = 4.00 kN y.
4.36 (*a*) 105.1 N. (*b*) $\mathbf{A}$ = 147.2 N x; $\mathbf{B}$ = 105. l N z.
4.37 T_{BE} = 50.0 lb; $\mathbf{A}$ = 18.75 lb y; $\mathbf{D}$ = 18.75 lb z.
4.38 $\mathbf{A}$ = 69.3 lb y; $\mathbf{B}$ = 34.6 lb c 60.0°; $\mathbf{C}$ = 173.2 lb b 60.0°.
4.39 T = 80.0 N; $\mathbf{A}$ = 160.0 N c 30.0°; $\mathbf{C}$ = 160.0 N b 30.0°.
4.40 T = 69.3 N; $\mathbf{A}$ = 140.0 N c 30.0°; $\mathbf{C}$ = 180.0 N b 30.0°.
4.43 (*a*) $\mathbf{D}$ = 20.0 lb w; $\mathbf{M}_D$ = 20.0 lb · ft l. (*b*) $\mathbf{D}$ = 10.00 lb w; $\mathbf{M}_D$ = 30.0 lb · ft i.
4.45 (*a*) $\mathbf{A}$ = 78.5 N x; $\mathbf{M}_A$ = 125.6 N · m l. (*b*) $\mathbf{A}$ = 111.0 N a 45.0°; $\mathbf{M}_A$ = 125.6 N · m l. (*c*) $\mathbf{A}$ = 157.0 N x; $\mathbf{M}_A$ = 251 N · m.
4.46 $\mathbf{C}$ = 28.3 N b 45.0°; $\mathbf{M}_C$ = 4.30 N · m i.
4.47 $\mathbf{C}$ = 28.3 N b 45.0°; $\mathbf{M}_C$ = 4.50 N · m i.
4.48 (*a*) $\mathbf{E}$ = 8.80 kips x; $\mathbf{M}_E$ = 36.0 kip · ft i. (*b*) $\mathbf{E}$ = 4.80 kips x; $\mathbf{M}_E$ = 51.0 kip · ft i.
4.50 $\mathbf{A}$ = 1848 N a 82.6°; $\mathbf{M}_A$ = 1431 N · m i.
4.51 (*a*) $u = 2\cos^{-1}\left[\frac{1}{4}\left(\frac{W}{P} \pm \mathrm{B}\,\overline{\frac{W^2}{P^2} + 8}\right)\right]$. (*b*) u = 65.1°.
4.52 (*a*) u = $2\sin^{-1}(W/2P)$. (*b*) u = 29.0°.
4.53 (*a*) $T = \frac{1}{2}W(1 - \tan u)$. (*b*) u = 39.8°.
4.54 (*a*) sin u + cos u = M/pl. (*b*) 17.11° and 72.9°.
4.55 141.1°.
4.56 (*a*) (1 − cos u) tan u = $W/2kl$. (*b*) 49.7°.
4.59 (1) completely constrained; determinate; $\mathbf{A} = \mathbf{C}$ = 196.2 N x. (2) completely constrained; determinate; $\mathbf{B}$ = 0, $\mathbf{C}$ = $\mathbf{D}$ = 196.2 N x. (3) completely constrained; indeterminate; $\mathbf{A}_x$ = 294 N y; $\mathbf{D}_x$ = 294 N z. (4) improperly constrained; indeterminate; no equilibrium. (5) partially constrained; determinate; equilibrium; $\mathbf{C} = \mathbf{D}$ = 196.2 N x. (6) completely constrained; determinate; $\mathbf{B}$ = 294 N y, $\mathbf{D}$ = 491 N b 53.1°. (7) partially constrained; no equilibrium. (8) completely constrained; indeterminate; $\mathbf{B}$ = 196.2 N x, $\mathbf{D}_y$ = 196.2 N x.
4.61 $\mathbf{A}$ = 680 N a 28.1°; $\mathbf{B}$ = 600 N z.
4.62 200 mm.
4.64 T = 289 lb; $\mathbf{A}$ = 577 lb a 60.0°.
4.65 $\mathbf{A}$ = 63.6 lb c 45.0°; $\mathbf{C}$ = 87.5 lb b 59.0°.
4.67 $\mathbf{B}$ = 888 N c 41.3°; $\mathbf{D}$ = 943 N b 45.0°.
4.68 $\mathbf{B}$ = 1001 N b 48.2°; $\mathbf{D}$ = 943 N c 45.0°.
4.69 $\mathbf{A}$ = 778 N w; $\mathbf{C}$ = 1012 N b 77.9°.
4.71 (*a*) 24.9 lb d 30.0°. (*b*) 15.34 lb a 30.0°.
4.72 $\mathbf{A}$ = 37.1 lb a 62.4°; T = 18.57 lb.
4.73 (*a*) 499 N. (*b*) 457 N b 26.6°.
4.75 $\mathbf{A}$ = 163.1 N c 74.1°; $\mathbf{B}$ = 258 N b 65.0°.
4.77 (*a*) 2*P* b 60.0°. (*b*) 1.239*P* c 36.2°.
4.78 (*a*) 1.155*P* b 30.0°. (*b*) 1.086*P* a 22.9°.
4.79 (*a*) $\mathbf{A}$ = 150.0 N a 30.0°; $\mathbf{B}$ = 150.0 N b 30.0°. (*b*) $\mathbf{A}$ = 433 N c 12.55°; $\mathbf{B}$ = 488 N b 30.0°.
4.80 (*a*) 119.3 lb. (*b*) 178.7 lb b 60.5°.

4.81 T = 100.0 lb; **B** = 111.1 lb c 30.3°.
4.83 (*a*) 225 mm. (*b*) 23.1 N. (*c*) 12.21 N y.
4.84 32.5°.
4.87 (*a*) 59.4°. (*b*) **A** = 8.45 lb y; 13.09 lb b 49.8°.
4.88 60.0 mm.
4.89 tan u = 2 tan b.
4.90 (*a*) 49.1°. (*b*) **A** = 45.3 N z; **B** = 90.6 N a 60.0°.
4.91 (*a*) 1200 N. (*b*) **C** = (400 N)**i** + (1200 N)**j**; **D** = −(1600 N)**i** − (480 N)**j**.
4.93 **A** = (24.0 lb)**j** − (2.31 lb)**k**; **B** = (16.00 lb)**j** − (9.24 lb)**k**; **C** = (11.55 lb)**k**.
4.94 **A** = (14.00 lb)**j** + (4.50 lb)**k**; **D** = (6.00 lb)**j** + (10.50 lb)**k**.
4.95 **A** = (120.0 N)**j** + (133.3 N)**k**; **D** = (60.0 N)**j** + (166.7 N)**k.**
4.97 T_A = 23.5 N; T_C = 11.77 N; T_D = 105.9 N.
4.98 (*a*) 0.480 m. (*b*) T_A = 23.5 N; T_C = 0; T_D = 117.7 N.
4.99 T_A = 5.63 lb; T_B = 16.88 lb; T_C = 22.5 lb.
4.101 (*a*) 121.9 N. (*b*) −46.2 N. (*c*) 100.9 N.
4.102 (*a*) 95.6 N. (*b*) −7.36 N. (*c*) 88.3 N.
4.103 T_A = 30.0 lb; T_B = 10.00 lb; T_C = 40.0 lb.
4.105 T_{AD} = 2.60 kN; T_{AE} = 2.80 kN; **C** = (1.800 kN)**j** + (4.80 kN)**k.**
4.106 T_{AD} = 5.20 kN; T_{AE} = 5.60 kN; **C** = (9.60 kN)**k**.
4.107 T_{BD} = T_{BE} = 1100 lb; **A** = (1200 lb)**i** − (560 lb)**j**.
4.108 T_{BE} = T_{BF} = 17.50 kN; **A** = −(7.00 kN)**i** + (22.4 kN)**j**.
4.109 T_{BE} = 6.62 kN; T_{BF} = 25.1 kN; **A** = −(6.34 kN)**i** + (20.3 kN)**j** + (2.96 kN)**k.**
4.112 T_{DE} = T_{DF} = 262 lb; **A** = −(801 lb)**i** + (1544 lb)**j.**
4.113 (*a*) 345 N. (*b*) **A** = (114.5 N)**i** + (377 N)**j** + (144.5 N)**k;** **B** = (113.2 N)**j** + (185.5 N)**k**.
4.115 (*a*) 49.5 lb. (*b*) **A** = −(12.00)**i** +(22.5 lb)**j** − (4.00 lb)**k;** **B** = (15.00 lb)**j** + (34.0 lb)**k**.
4.116 (*a*) 118.8 lb. (*b*) **A** = (93.8 lb)**i** + (22.5 lb)**j** + (70.8 lb)**k;** **B** = (15.00 lb)**j** −(8.33 lb)**k**.
4.117 (*a*) 101.6 N. (*b*) **A** = −(26.3 N)**i**; **B** = (98.1 N)**j**.
4.119 (*a*) 49.5 lb. (*b*) **A** = −(12.00 lb)**i** + (37.5 lb)**j** + (30.0 lb)**k;** $\mathbf{M}_A$ = −(1020 lb · in.)**j** + (450 lb · in.)**k**.
4.120 (*a*) 462 N. (*b*) **C** = (169.1 N)**j** + (400 N)**k**; $\mathbf{M}_C$ = (20.0 N · m)**j** + (151.5 N · m)**k**.
4.121 T_{CF} = 200 N; T_{DE} = 450 N; **A** = (160.0 N)**i** +(270 N)**k;** $\mathbf{M}_A$ = −(16.20 N · m)**i**.
4.122 (*a*) 5.00 lb. (*b*) **C** = −(5.00 lb)**i** + (6.00 lb)**j** − (5.00 lb)**k;** $\mathbf{M}_C$ = (8.00 lb · in.)**j** −(12.00 lb · in.)**k**.
4.125 T_{BD} = 2.18 kN; T_{BE} = 3.96 kN; T_{CD} = 1.500 kN.
4.126 T_{BD} = 0; T_{BE} = 3.96 kN; T_{CO} = 3.00 kN.
4.127 (*a*) T_B = −0.366*P*; T_C = 1.219*P*; T_D = −0.853*P*. (*b*) **F** = −0.345*P***i** + *P***j** − 0.862 *P***k.**
4.129 **A** = (120.0 lb)**j** − (150.0 lb)**k;** **B** = (180.0 lb)**i** + (150.0 lb)**k;** **C** = −(180.0 lb)**i** + (120.0 lb)**j.**
4.130 **A** = (20.0 lb)**j** + (25.0 lb)**k;** **B** = (30.0 lb)**i** − (25.0 lb)**k**; **C** = −(30.0 lb)**i** − (20.0 lb)**j**.
4.131 **B** = (60.0 N)**k;** **C** = (30.0 N)**j** − (16.00 N)**k;** **D** = −(30.0 N)**j** + (4.00 N)**k**.
4.133 373 N.
4.135 (45.0 lb)**j**.
4.136 (*a*) x = 4.00 ft; y = 8.00 ft; (*b*) 10.73 lb.
4.137 (*a*) x = 0 ft; y = 16.00 ft. (*b*) 11.31 lb.
4.138 360 N.
4.140 85.3 lb.
4.141 181.7 lb.
4.142 42.0 N x.
4.143 (*a*) 80.8 lbw. (*b*) 216 lb a 22.0°.
4.145 (*a*) 875 lb. (b) 1584 lb b 45.0°.
4.147 **C** = 1951 N b 88.5°; $\mathbf{M}_C$ = 75.0 N · m i.
4.149 **A** = 170.0 N b 33.9°. **C** = 160.0 N a 28.1°.
4.150 (*a*) T_A = 6.00 lb; T_B = T_C = 9.00 lb. (*b*) 15.00 in.
4.151 T_{BE} = 975 N; T_{CF} = 600 N; T_{DG} = 625 N; **A** = (2100 N)**i** + (175.0 N)**j** − (375 N)**k.**
4.153 (*a*) **A** = 0.745*P* a 63.4°; **C** = 0.471*P* b 45.0°. (*b*) **A** = 0.812*P* a 60.0°; **C** = 0.503*P* d 36.2°. (*c*) **A** = 0.448*P* b 60.0°; **C** = 0.652*P* a 69.9°. (*d*) improperly constrained: no equilibrium.
4.C1 u = *20°*; T = 114.8 lb; u = *70°*; T = 127.7 lb; T_{max} = 132.2 lb at u = 50.4°.
4.C2 x = *600 mm:* P = 31.4 N; x = *150 mm:* P = 37.7 N; P_{max} = 47.2 N at x = 283 mm.
4.C3 u = *30°*: W = 9.66 lb; u = *60°*; W = 36.6 lb; W = 5 lb at u = 22.9° [Also at u = 175.7°].
4.C4 u = *30°*: W = 0.80 lb; u = *60°*: W = 4.57 lb; W = 5 lb at u = 62.6° [Also at u = 159.6°].
4.C5 u = *30°*: m = 7.09 kg; u = *60°*: m = 11.02 kg. When m = 10 kg, u = 51.0°.
4.C6 u = *15°*: T_{BD} = 10.30 kN, T_{BE} = 21.7 kN; u = *30°*: T_{BD} = 5.69 kN, T_{BE} = 24.4 kN; T_{max} = 26.5 kN at u = 36.9°.

CHAPTER 5

5.1 $\bar{X}$ = 1.045 in., $\bar{Y}$ = 3.59 in.
5.2 $\bar{X}$ = 36.0 mm, $\bar{Y}$ = 48.0 mm.
5.3 $\bar{X}$ = 19.27 mm, $\bar{Y}$ = 26.6 mm.
5.4 $\bar{X}$ = 5.67 in., $\bar{Y}$ = 5.17 in.
5.6 $\bar{X}$ = 7.22 in., $\bar{Y}$ = 9.56 in.
5.8 $\bar{X}$ = −10.00 mm, $\bar{Y}$ = 87.5 mm.
5.9 $\bar{X}$ = 92.0 mm, $\bar{Y}$ = 23.3 mm.
5.10 $\bar{X}$ = 0, $\bar{Y}$ = 6.45 in.
5.11 $\bar{X}$ = 0, $\bar{Y}$ = 1.372 in.
5.13 $\bar{X}$ = 50.5 mm, $\bar{Y}$ = 19.34 mm.
5.14 $\bar{X}$ = 3.20 in. $\bar{Y}$ = 2.00 in.
5.17 a/r = 0.508.
5.18 $\bar{Y} = \frac{2}{3}\left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right)\left(\frac{2\cos \mathrm{a}}{\mathrm{p} - 2\mathrm{a}}\right)$.
5.19 $\bar{Y} = \left(\frac{r_1 + r_2}{\mathrm{p} - 2\mathrm{a}}\right)\cos \mathrm{a}$.
5.20 0.235 in^3 for A, −0.235 in^3 for A_2.
5.22 459 N.
5.23 (*a*) $b(c^2 - y^2)/2$. (*b*) y = 0, $Q_x = bc^2/2$.
5.24 $\bar{X}$ = 36.6 mm, $\bar{Y}$ = 47.6 mm.
5.26 $\bar{X}$ = 5.52 in., $\bar{Y}$ = 5.16 in.
5.28 120.0 mm.
5.29 99.5 mm.
5.31 (*a*) 5.09 lb. (*b*) 9.48 lb b 57.5°.
5.32 (*a*) 0.513 *a*. (*b*) 0.691 *a*.
5.34 $\bar{x} = a/3$, $\bar{y} = 2h/3$.
5.36 $\bar{x} = a/2$, $\bar{y} = 2h/5$.
5.37 $\bar{y} = \bar{x} = 9a/20$.
5.38 $\bar{x} = 2a/3(4 - \mathrm{p})$, $\bar{y} = 2b/3(4 - \mathrm{p})$.
5.40 $\bar{x} = 5a/8$, $\bar{y} = b/3$.
5.41 $\bar{x} = 3a/8$, $\bar{y} = b$.
5.43 $\bar{x} = a.$, $\bar{y} = 17b/35$.
5.44 $\bar{x} = 17a/130$, $\bar{y} = 11b/26$.
5.45 −2 1 $\bar{2}r/3\mathrm{p}$.
5.46 $2a/5$.
5.48 $\bar{x} = L/\mathrm{p}$, $\bar{y} = \mathrm{p}a/8$.
5.49 $\bar{x} = -9.27a$, $\bar{y} = 3.09a$.

5.50 $\bar{x} = 1.629$ in., $\bar{y} = 0.1853$ in.

5.51 $a = 1.901$ in. or 3.74 in.

5.52 (*a*) Volume = 248 in^3, Area = 547 in^2.
(*b*) Volume = 72.3 in^3, Area = 169.6 in^2.

5.53 (*a*) Volume = 423×10^3 mm^3, Area = 37.2×10^3 mm^2.
(*b*) Volume = 847×10^3 mm^3, Area = 72.5×10^3 mm^2.

5.54 (*a*) Volume = 2.26×10^6 mm^3, Area = 116.3×10^3 mm^2.
(*b*) Volume = 1.471×10^6 mm^3, Area = 116.3×10^3 mm^2.

5.56 $V = 3470$ mm^3; $A = 2320$ mm^2.

5.58 $V = 3.96$ in^3; $W = 1.211$ lb.

5.60 0.0305 kg.

5.62 0.0900 in^3.

5.63 308 in^2.

5.64 31.9 liters.

5.66 (*a*) **R** = 7.60 kN w, $\bar{x} = 2.57$ m, (*b*) **A** = 4.35 kN x; **B** = 3.25 kN x.

5.67 (*a*) **R** = 1215 lb w, $\bar{x} = 4.33$ ft. (*b*) **A** = 630 lb x; **B** = 585 lb x.

5.68 **A** = 105.0 N x; **B** = 270 N x.

5.69 **A** = 2860 lb x; **B** = 740 lb x.

5.71 **A** = 32.0 kN x; $M_A = 124.0$ kN · m l.

5.72 **A** = 3.00 kN x; $M_A = 12.60$ kN · m l.

5.74 **B** = 3770 lb x; **C** = 429 lb x.

5.75 (*a*) 900 lb/ft. (*b*) 7200 lb x.

5.76 (*a*) 0.536 m. (*b*) **A** = **B** = 761 N x.

5.78 $w_{BC} = 2810$ N/m; $w_{DE} = 3150$ N/m.

5.80 (*a*) **H** = 44.1 kN y ; **V** = 228 kN x. (*b*) 1.159 m to the right of *A*. (*c*) **R** = 59.1 kN d 41.6°.

5.81 (*a*) **H** = 254 kN y ; **V** = 831 kN x. (*b*) 3.25 m to the right of *A* (*c*) **R** = 268 kN d 18.43°.

5.82 12.00 in.

5.83 4.00 in.

5.84 **T** = 67.2 kN z ; **A** = 141.2 kN z .

5.86 3.70 kips x.

5.87 $t = 35.7$ l; gate rotates clockwise.

5.88 0.683 m.

5.89 0.0711 m.

5.91 208 lb.

5.92 **A** = 1197 N b 53.1°; **B** = 1511 N b 53.1°.

5.93 3570 N.

5.94 6.00 ft.

5.96 (*a*) 0.0536 *a* below base of cone.
(*b*) 0.0625 *a* above base of cone.

5.97 (*a*) 0.548 *L*. (*b*) 2 1 $\overline{3}$.

5.98 $-(2h^2 - 3b^2)/2(4h - 3b)$.

5.99 $-a(4h - 2b)/$p$(4h - 3b)$.

5.100 −0.1403 in.

5.101 19.13 mm.

5.103. 3.47 in.

5.104 18.28 mm.

5.106 $\overline{X} = 45.0$ mm; $\overline{Z} = -20.2$ mm.

5.107 $\overline{X} = 0.1402$ m; $\overline{Y} = 0.0944$ m; $\overline{Z} = 0.0959$ m.

5.108 $\overline{X} = 17.00$ in.; $\overline{Y} = 15.68$ in.; $\overline{Z} = 14.16$ in.

5.109 $\overline{X} = 46.5$ mm; $\overline{Y} = 27.2$ mm; $\overline{Z} = 30.0$ mm.

5.110 $\overline{X} = \overline{Z} = 4.21$ in.; $\overline{Y} = 7.03$ in.

5.113 $\overline{X} = 180.2$ mm; $\overline{Y} = 38.0$ mm; $\overline{Z} = 193.5$ mm.

5.114 $\overline{X} = 0$; $\overline{Y} = 10.05$ in.; $\overline{Z} = 5.15$ in.

5.115 $\overline{X} = 0.410$ m; $\overline{Y} = 0.510$ m; $\overline{Z} = 0.1500$ m.

5.116 $\overline{X} = 0.909$ m; $\overline{Y} = 0.1842$ m; $\overline{Z} = 0.884$ m.

5.118 $\overline{X} = \overline{Z} = 0$; $\overline{Y} = 83.3$ mm above the base.

5.120 $\overline{Y} = 0.526$ in. above the base.

5.121 $\overline{X} = 61.1$ mm from the end of the handle.

5.122 $(\bar{x}_1) = 21a/88$; $(\bar{x}_2) = 27a/40$.

5.123 $(\bar{x}_1) = 21h/88$; $(\bar{x}_2) = 27h/40$.

5.124 $(\bar{x}_1) = 2h/9$; $(\bar{x}_2) = 2h/3$.

5.126 $\bar{x} = 2.34$ m; $\bar{y} = \bar{z} = 0$.

5.128 $\bar{x} = 1.297\ a$; $\bar{y} = \bar{z} = 0$.

5.129 $\bar{x} = \bar{z} = 0$; $\bar{y} = 0.374\ b$.

5.132 (*a*) $\bar{x} = \bar{z} = 0$; $\bar{y} = -121.9$ mm. (*b*) $\bar{x} = \bar{z} = 0$; $\bar{y} = -90.2$ mm.

5.134 $\bar{x} = 0$; $\bar{y} = 5h/16$; $\bar{z} = -b/4$.

5.135 $V = 688$ ft^3; $\bar{x} = 15.91$ ft.

5.136 $\bar{x} = a/2$; $y = 8h/25$; $\bar{z} = b/2$.

5.137 $\bar{x} = 1.643$ in.; $\bar{y} = 17.46$ in.

5.138 $\bar{x} = 30.0$ mm; $\bar{y} = 64.8$ mm.

5.140 $\bar{x} = 2a/5$; $\bar{y} = 4h/7$.

5.141 $\bar{x} = 5L/4$; $\bar{y} = 33a/40$.

5.143 **B** = 1360 lb x; **C** = 2360 lb x.

5.144 $w_A = 10.00$ kN/m; $w_B = 50.0$ kN/m.

5.146 (*a*) $b/10$ to the left of base of cone
(*b*) $0.01136b$ to the right of base of cone.

5.147 $\overline{X} = 0.295$ m; $\overline{Y} = 0.423$ m; $\overline{Z} = 1.703$ m.

5.C1 (*b*) **A** = 1220 lb x; **B** = 1830 lb x.
(*c*) **A** = 1265 lb x; **B** = 1601 lb x.

5.C2 (*a*) $\overline{X} = 0$, $\overline{Y} = 0.278$ m. $\overline{Z} = 0.0878$ m.
(*b*) $\overline{X} = 0.0487$ mm, $\overline{Y} = 0.1265$ mm, $\overline{Z} = 0.0997$ mm.
(*c*) $\overline{X} = -0.0372$ m, $\overline{Y} = 0.1659$ m, $\overline{Z} = 0.1043$ m.

5.C3 *d = 1.00 m:* **F** = 5.66 kN c 30°;
d = 3.00 m: **F** = 49.9 kN c 27.7°.

5.C4 (*a*) $\overline{X} = 5.80$ in., $\overline{Y} = 1.492$ in. (*b*) $\overline{X} = 9.11$ in., $\overline{Y} = 2.78$ in.
(*c*) $\overline{X} = 8.49$ in., $\overline{Y} = 0.375$ in.

5.C5 *With n = 40:* (*a*) $\overline{X} = 60.2$ mm, $\overline{Y} = 23.4$ mm.
(*b*) $\overline{X} = 60.2$ mm, $\overline{Y} = 146.2$ mm.
(*c*) $\overline{X} = 68.7$ mm. $\overline{Y} = 20.4$ mm.
(*d*) $\overline{X} = 68.7$ mm, $\overline{Y} = 127.8$ mm.

5.C6 *With n = 40:* (*a*) $\overline{X} = 60.0$ mm, $\overline{Y} = 24.0$ mm.
(*b*) $\overline{X} = 60.0$ mm, $\overline{Y} = 150.0$ mm.
(*c*) $\overline{X} = 68.6$ mm, $\overline{Y} = 21.8$ mm.
(*d*) $\overline{X} = 68.6$ mm, $\overline{Y} = 136.1$ mm.

5.C7 (*a*) $V = 628$ ft^3.
(*b*) $\overline{X} = 8.65$ ft, $\overline{Y} = -4.53$ ft, $\overline{Z} = 9.27$ ft.

CHAPTER 6

6.1 $F_{AB} = 180.0$ kN *T*; $F_{AC} = 156.0$ kN *C*; $F_{BC} = 144.0$ kN *T*.

6.2 $F_{AB} = 720$ lb *T*; $F_{AC} = 1200$ lb *C*; $F_{BC} = 780$ lb *C*.

6.3 $F_{AB} = 52.0$ kN *T*; $F_{AC} = 64.0$ kN *T*; $F_{BC} = 80.0$ kN *C*.

6.5 $F_{AB} = F_{BC} = 0$; $F_{AD} = F_{CF} = 1.000$ kip *C*; $F_{BD} = F_{CF} =$ 6.80 kips *C*; $F_{BE} = 2.40$ kips *T*; $F_{DE} = F_{EF} = 6.00$ kips *T*.

6.7 $F_{AB} = 20.0$ kN *T*; $F_{AD} = 20.6$ kN *C*; $F_{BC} = 30.0$ kN *T*; $F_{BD} = 30.0$ kN *T*; $F_{CD} = 10.00$ kN *T*.

6.8 $F_{AB} = 4.00$ kN *T*; $F_{AD} = 15.00$ kN *T*; $F_{BD} = 9.00$ kN *C*; $F_{BE} = 5.00$ kN *T*; $F_{CD} = 16.00$ kN *C*; $F_{DE} = 4.00$ kN *C*.

6.9 $F_{AB} = F_{FH} = 1500$ lb *C*; $F_{AC} = F_{CE} = F_{EG} = F_{GH} =$ 1200 lb *T*; $F_{BC} = F_{FG} = 0$; $F_{BD} = F_{DF} = 1200$ lb *C*; $F_{BE} = F_{EF} = 60.0$ lb *C*; $F_{DE} = 72.0$ lb *T*.

6.10 $F_{AB} = F_{FH} = 1500$ lb *C*; $F_{AC} = F_{CE} = F_{EG} = F_{GH} =$ 1200 lb *T*; $F_{BC} = F_{FG} = 0$; $F_{BD} = F_{DF} = 1000$ lb *C*; $F_{BE} = F_{EF} = 500$ lb *C*; $F_{DE} = 600$ lb *T*.

6.11 $F_{AB} = 47.2$ kN *C*; $F_{AC} = 44.6$ kN *T*; $F_{BC} = 10.50$ kN *C*; $F_{BD} = 47.2$ kN *C*; $F_{CD} = 17.50$ kN *T*; $F_{CE} = 30.6$ kN *T*; $F_{DE} = 0$.

6.13 F_{AB} = 7.83 kN *C*; F_{AC} = 7.00 kN *T*; F_{BC} = 1.886 kN *C*; F_{BD} = 6.34 kN *C*; F_{CD} = 1.491 kN *T*; F_{CE} = 5.00 kN *T*; F_{DE} = 2.83 kN *C*; F_{DF} = 3.35 kN *C*; F_{EF} = 2.75 kN *T*; F_{EG} = 1.061 kN *C*; F_{EH} = 3.75 kN *T*; F_{FG} = 4.24 kN *C*; F_{GH} = 5.30 kN *C*.
6.15 F_{AB} = 2240 lb *C*; $F_{AC} = F_{CE}$ = 2000 lb *T*; $F_{BC} = F_{EH}$ = 0; F_{BD} = 1789 lb *C*; F_{BE} = 447 lb *C*; F_{DE} = 600 lb *C*; F_{DF} = 2010 lb *C*; F_{DG} = 224 lb *T*; F_{EG} = 1789 lb *T*.
6.17 F_{AB} = 9.90 kN *C*; F_{AC} = 7.83 kN *T*; F_{BC} = 0; F_{BD} = 7.07 kN *C*; F_{BE} = 2.00 kN *C*; F_{CE} = 7.83 kN *T*; F_{DE} = 1.000 kN *T*; F_{DF} = 5.03 kN *C*; F_{DG} = 0.559 kN *C*; F_{EG} = 5.59 kN *T*.
6.18 F_{FG} = 3.50 kN *T*; F_{FH} = 5.03 kN *C*; F_{GH} = 1.677 kN *T*; $F_{GI} = F_{IK} = F_{KL}$ = 3.35 kN *T*; $F_{HI} = F_{IJ} = F_{JK}$ = 0; $F_{HJ} = F_{JL}$ = 4.24 kN *C*.
6.19 $F_{AB} = F_{FG}$ = 7.50 kips *C*; $F_{AC} = F_{EG}$ = 4.50 kips *T*; $F_{BC} = F_{EF}$ = 7.50 kips *T*; $F_{BD} = F_{DF}$ = 9.00 kips *C*; $F_{CD} = F_{DE}$ = 0; F_{CE} = 9.00 kips *T*.
6.21 $F_{AB} = F_{FH}$ = 7.50 kips *C*; $F_{AC} = F_{GH}$ = 4.50 kips *T*; $F_{BC} = F_{FG}$ = 4.00 kips *T*; $F_{BD} = F_{DF}$ = 6.00 kips *C*; $F_{BE} = F_{EF}$ = 2.50 kips *T*; $F_{CE} = F_{EG}$ = 4.50 kips *T*; F_{DE} = 0.
6.22 F_{AB} = 6.25 kips *C*; F_{AC} = 3.75 kips *T*; F_{BC} = 4.00 kips *T*; $F_{BD} = F_{DF}$ = 4.50 kips *C*; F_{BE} = 1.250 kips *T*; F_{CE} = 3.75 kips *T*; $F_{DE} = F_{FG}$ = 0; F_{EF} = 3.75 kips *T*; F_{EG} = F_{GH} = 2.25 kips *T*; F_{FH} = 3.75 kips *C*.
6.23 $F_{AB} = F_{DF}$ = 2.29 kN *T*; $F_{AC} = F_{EF}$ = 2.29 kN *C*; F_{BC} = F_{DE} = 0.600 kN *C*; F_{BD} = 2.21 kN *T*; $F_{BE} = F_{EH}$ = 0; F_{CE} = 2.21 kN *C*; $F_{CH} = F_{EJ}$ = 1.200 kN *C*.
6.26 F_{AB} = 9.39 kN *C*; F_{AC} = 8.40 kN *T*; F_{BC} = 2.26 kN *C*; F_{BD} = 7.60 kN *C*; F_{CD} = 0.128 kN *C*; F_{CE} = 7.07 kN *T*; F_{DE} = 2.14 kN *C*; F_{DF} = 6.10 kN *C*; F_{EF} = 2.23 kN *T*.
6.27 F_{AB} = 31.0 kips *C*; F_{AC} = 28.3 kips *C*; F_{AD} = 15.09 kips *T*; F_{AE} = 9.50 kips *T*; F_{BD} = 21.5 kips *T*; F_{BF} = 28.0 kips *C*; F_{CE} = 41.0 kips *T*; F_{CG} = 42.0 kips *C*; F_{DE} = 22.0 kips *T*; F_{DF} = 33.5 kips *T*; F_{EG} = 0.
6.28 $F_{AB} = F_{BC} = F_{CD}$ = 36.0 kips *T*; F_{AE} = 57.6 kips *T*; F_{AF} = 45.0 kips *C*; $F_{BF} = F_{BG} = F_{CG} = F_{CH}$ = 0; F_{DH} = $F_{FG} = F_{GH}$ = 39.0 kips *C*; F_{EF} = 36.0 kips *C*.
6.29 Truss of prob. 6.33*a* is the only simple truss.
6.30 Trusses of prob 6.31*b* and prob 6.33*b* are simple trusses.
6.31 (*a*) *AI*, *BJ*, *CK*, *DI*, *EI*, *FK*, *GK*. (*b*) *FK*, *IO*.
6.34 (*a*) *GH*, *GJ*, *IJ*. (*b*) *BF*, *BG*, *CG*, *CH*.
6.35 $F_{AB} = F_{AD}$ = 244 lb *C*; F_{AC} = 1040 lb *T*; $F_{BC} = F_{CD}$ = 500 lb *C*; F_{BD} = 280 lb *T*.
6.36 $F_{AB} = F_{AD}$ = 861 N *C*; F_{AC} = 676 N *C*; $F_{BC} = F_{CD}$ = 162.5 N *T*; F_{BD} = 244 N *T*.
6.37 $F_{AB} = F_{AD}$ = 2810 N *T*; F_{AC} = 5510 N *C*; $F_{BC} = F_{CD}$ = 1325 N *T*; F_{BD} = 1908 N *C*.
6.38 $F_{AB} = F_{AC}$ = 1061 lb *C*; F_{AD} = 2500 lb *T*; F_{BC} = 2100 lb *T*; $F_{BD} = F_{CD}$ = 1250 lb *C*; $F_{BE} = F_{CE}$ = 1250 lb *C*; F_{DE} = 1500 lb *T*.
6.39 F_{AB} = 840 N *C*; F_{AC} = 110.6 N *C*; F_{AD} = 394 N *C*; F_{AE} = 0; F_{BC} = 160.0 N *T*; F_{BE} = 200 N *T*; F_{CD} = 225 N *T*; F_{CE} = 233 N *C*; F_{DE} = 120.0 N *T*.
6.40 $F_{AB} = F_{AE} = F_{BC}$ = 0; F_{AC} = 995 N *T*; F_{AD} = 1181 N *C*; F_{BE} = 600 N *T*; F_{CD} = 375 V *T*; F_{CE} = 700 N *C*; F_{DE} = 360 N *T*.
6.43 F_{CD} = 9.00 kN *C*; F_{DF} = 12.00 kN *T*.
6.44 F_{FG} = 5.00 kN *T*; F_{FH} = 20.0 kN *T*.
6.45 F_{CE} = 8000 lb *T*; F_{DE} = 2600 lb *T*; F_{DF} = 9000 lb *C*.
6.46 F_{EG} = 7500 lb *T*; F_{FG} = 3900 lb *C*; F_{FH} = 6000 lb *C*.
6.49 F_{AD} = 13.50 kN *C*; F_{CD} = 0; F_{CE} = 56.1 kN *T*.
6.50 F_{DG} = 75.0 kN *C*; F_{FG} = 56.1 kN *T*; F_{FH} = 69.7 kN *T*.
6.51 F_{AB} = 8.20 kips *T*; F_{AG} = 4.50 kips *T*; F_{FG} 11.60 kips *C*.
6.52 F_{AE} = 17.46 kips *T*; F_{EF} = 11.60 kips *C*; F_{FJ} = 18.45 kips *C*.
6.53 F_{CD} = 20.0 kN *C*; F_{DF} = 52.0 kN *C*.
6.54 F_{CE} = 36.0 kN *T*; F_{EF} = 15.00 kN *C*.
6.55 F_{FG} = 5.23 kN *C*; F_{EG} = 0.1476 kN *C*; F_{EH} = 5.08 kN *T*.
6.56 F_{KM} = 5.02 kN *T*; F_{LM} = 1.963 kN *C*; F_{LN} = 3.95 kN *C*.
6.59 F_{DE} = 25.0 kips *T*; F_{DF} = 13.00 kips *C*.
6.60 F_{EG} = 16.00 kips *T*; F_{EF} = 6.40 kips *C*.
6.63 F_{DG} = 3.75 kN *T*; F_{FI} = 3.75 kN *C*.
6.64 F_{GJ} = 11.25 kN *T*; F_{TK} = 11.25 kN *C*.
6.65 F_{BG} = 5.48 kips *T*; F_{DG} = 1.825 kips *T*.
6.66 F_{CF} = 3.65 kips *T*; F_{CH} = 7.30 kips *T*.
6.67 (*a*) *CJ*. (*b*) 1.026 kN *T*.
6.68 (*a*) *IO*. (*b*) 2.05 kN *T*.
6.69 (*a*) improperly constrained. (*b*) completely constrained, determinate. (*c*) completely constrained, indeterminate.
6.70 (*a*) completely constrained, determine. (*b*) partially constrained. (*c*) improperly constrained.
6.71 (*a*) completely constrained, determinate. (*b*) completely constrained, indeterminate. (*c*) improperly constrained.
6.72 (*a*) partially constrained. (*b*) completely constrained, determinate. (*c*) completely constrained, indeterminate.
6.75 F_{BD} = 780 lb *T*; $\mathbf{C}_x$ = 720 lb z , $\mathbf{C}_y$ = 140.0 lb w.
6.76 F_{BD} = 255 N *C*; $\mathbf{C}_x$ = 120.0 N y , $\mathbf{C}_y$ = 625 N x.
6.77 $\mathbf{A}_x$ = 480 N y , $\mathbf{A}_y$ = 120.0 N x; $\mathbf{B}_x$ = 480 N z , $\mathbf{B}_y$ = 320 N w; $\mathbf{C}$ = 120.0 N w; $\mathbf{D}$ = 320 N x.
6.79 $\mathbf{A}_x$ = 25.0 kips z , $\mathbf{A}_y$ = 20.0 kips x; $\mathbf{B}_x$ = 25.0 kips, $\mathbf{B}_y$ = 10.00 kips w; $\mathbf{C}_x$ = 50.0 kips y , $\mathbf{C}_y$ = 10.00 kips w.
6.81 $\mathbf{A}$ = 375 N; $\mathbf{B}_x$ = 375 N z , $\mathbf{B}_y$ = 150.0 N x; $\mathbf{C}$ = 50.0 N x; $\mathbf{D}$ = 200 N w.
6.82 $\mathbf{A}$ = 150.0 N y ; $\mathbf{B}$ = 0; $\mathbf{C}_x$ = 150.0 N z , $\mathbf{C}_y$ = 100.0 N x; $\mathbf{D}$ = 100.0 N w.
6.83 (*a*) $\mathbf{A}_x$ = 300 N z , $\mathbf{A}_y$ = 660 N x; $\mathbf{E}_x$ = 300 N y , $\mathbf{E}_y$ = 90.0 N x. (*b*) $\mathbf{A}_x$ = 300 N z , $\mathbf{A}_y$ = 150.0 N x; $\mathbf{E}_x$ = 300 N y , $\mathbf{E}_y$ = 600 N x.
6.84 (*a*) $\mathbf{A}_x$ = 450 N z , $\mathbf{A}_y$ = 525 N x; $\mathbf{E}_x$ = 450 N y , $\mathbf{E}_y$ = 225 N x; (*b*) $\mathbf{A}_x$ = 450 N z , $\mathbf{A}_y$ = 150.0 N x; $\mathbf{E}_x$ = 450 N y , $\mathbf{E}_y$ = 600 N x.
6.87 (*a*) $\mathbf{A}$ = 65.0 lb d 22.6°; $\mathbf{C}$ = 120.0 lb y ; $\mathbf{G}$ = 60.0 lb z ; $\mathbf{I}$ = 25.0 lb x. (*b*) $\mathbf{A}$ = 65.0 lb d 22.6°; $\mathbf{C}$ = 60.0 lb y ; $\mathbf{G}$ = 0; *I* = 25.0 lb x.
6.88 (*a*) $\mathbf{A}$ = 65.0 lb d 22.6°; $\mathbf{C}$ = 120.0 lb y ; $\mathbf{G}$ = 60.0 lb z ; $\mathbf{I}$ = 25.0 lb x.
(*b*) $\mathbf{A}$ = 65.0 lb d 22.6°; $\mathbf{C}$ = 100.0 lb y ; $\mathbf{G}$ = 80.0 lb z ; $\mathbf{I}$ = 25.0 lb x.
6.89 (*a*) $\mathbf{A}_x$ = 80.0 lb z , $\mathbf{A}_y$ = 40.0 lb x; $\mathbf{B}_x$ = 80.0 lb y , $\mathbf{B}_y$ = 60.0 lb x. (*b*) $\mathbf{A}_x$ = 0, $\mathbf{A}_y$ = 40.0 lb x; $\mathbf{B}_x$ = 0, $\mathbf{B}_y$ = 60.0 lb x.
6.91 (*a*) $\mathbf{E}_x$ = 2.00 kips z , $\mathbf{E}_y$ = 2.25 kips x.
(*b*) $\mathbf{C}_x$ = 4.00 kips z , $\mathbf{C}_y$ = 5.75 kips x.
6.92 (*a*) $\mathbf{E}_x$ = 3.00 kips z , $\mathbf{E}_y$ = 1.500 kips x.
(*b*) $\mathbf{C}_x$ = 3.00 kips z , $\mathbf{C}_y$ = 6.50 kips x.
6.93 $\mathbf{A}_x$ = 150.0 N z , $\mathbf{A}_y$ = 250 N x; $\mathbf{E}_x$ = 150.0 N y , $\mathbf{E}_y$ = 450 N x.
6.94 $\mathbf{B}_x$ = 700 N z , $\mathbf{B}_y$ = 200 N w; $\mathbf{E}_x$ = 700 N y , $\mathbf{E}_y$ = 500 N x.
6.95 (*a*) $\mathbf{A}$ = 982 lb x; $\mathbf{B}$ = 935 lb x; $\mathbf{C}$ = 733 lb x.
(*b*) ΔB = +291 lb; ΔC = −72.7 lb.
6.96 (*a*) 572 lb. (*b*) $\mathbf{A}$ = 1070 lb x; $\mathbf{B}$ = 709 lb x; $\mathbf{C}$ = 870 lb x.
6.99 $\mathbf{A}_x$ = 13.00 kN z , $\mathbf{A}_y$ = 4.00 kN w; $\mathbf{B}_x$ = 36.0 kN y , $\mathbf{B}_y$ = 6.00 kN x; $\mathbf{E}_x$ = 23.0 kN z , $\mathbf{E}_y$ = 2.00 kN w.

6.100 $\mathbf{A}_x = 2025$ N z , $\mathbf{A}_y = 1800$ N w; $\mathbf{B}_x = 4050$ N y , $\mathbf{B}_y = 1200$ N x; $\mathbf{E}_x = 2025$ N z , $\mathbf{E}_y = 600$ N x.
6.101 $\mathbf{A}_x = 1110$ lb z , $\mathbf{A}_y = 600$ lb x; $\mathbf{B}_x = 1110$ lb z , $\mathbf{B}_y = 800$ lb w; $\mathbf{D}_x = 2220$ lb y , $\mathbf{D}_y = 200$ lb x.
6.102 $\mathbf{A}_x = 660$ lb z , $\mathbf{A}_y = 240$ lb x; $\mathbf{B}_x = 660$ lb z , $\mathbf{B}_y = 320$ lb w; $\mathbf{D}_x = 1320$ lb y , $\mathbf{D}_y = 80.0$ lb x.
6.103 $\mathbf{C}_x = 21.7$ lb y , $\mathbf{C}_y = 37.5$ lb w; $\mathbf{D}_x = 21.7$ lb z , $\mathbf{D}_y = 62.5$ lb x.
6.104 $\mathbf{C}_x = 78.0$ lb y , $\mathbf{C}_j = 28.0$ lb x; $\mathbf{F}_x = 78.0$ lb z , $\mathbf{F}_y = 12.00$ lb x.
6.107 $\mathbf{F}_x = 300$ lb z , $\mathbf{F}_y = 1200$ lb x; $\mathbf{F}_{AE} = 1000$ lb C; $\mathbf{F}_{BD} = 500$ lb T.
6.108 $\mathbf{A} = 327$ lb y ; $\mathbf{B} = 827$ lb z ; $\mathbf{D} = 621$ lb x; $\mathbf{E} = 246$ lb x.
6.109 (*a*) $\mathbf{A}_x = 200$ kN y , $\mathbf{A}_y = 122.0$ kN x. (*b*) $\mathbf{B}_x = 200$ kN z , $\mathbf{B}_y = 10.00$ kN w.
6.110 (*a*) $\mathbf{A}_x = 205$ kN y , $\mathbf{A}_y = 134.5$ kN x. (*b*) $\mathbf{B}_x = 205$ kN z , $\mathbf{B}_y = 5.50$ kN x.
6.112 $\mathbf{F}_{AF} = P/4\ C$; $\mathbf{F}_{BG} = \mathbf{F}_{DG} = P/\ 1\bar{2}\ C$; $\mathbf{F}_{EH} = P/4\ T$.
6.113 $\mathbf{F}_{AG} =\ 1\bar{2}P/6\ C$; $\mathbf{F}_{BF} = 2\ 1\bar{2}P/3\ C$; $\mathbf{F}_{DI} =\ 1\bar{2}P/3\ C$; $\mathbf{F}_{EH} =\ 1\bar{2}P/6\ T$.
6.115 $\mathbf{F}_{AF} = \mathbf{M}_0/4a\ C$; $\mathbf{F}_{BG} = \mathbf{F}_{DG} = \mathbf{M}_0/\ 1\bar{2}a\ T$; $\mathbf{F}_{EH} = 3\mathbf{M}_0/4a\ C$.
6.116 $\mathbf{F}_{AF} = \mathbf{M}_0/6a\ T$; $\mathbf{F}_{BG} =\ 1\bar{2}\mathbf{M}_0/6a\ T$; $\mathbf{F}_{DG} =\ 1\bar{2}\mathbf{M}_0/3a\ T$; $\mathbf{F}_{EH} = \mathbf{M}_0/6a\ C$.
6.117 $\mathbf{E} = P/5$ w; $\mathbf{F} = 8P/5$ x; $\mathbf{G} = 4P/5$ w; $\mathbf{H} = 2P/5$ x.
6.118 $\mathbf{A} = P/15$ x; $\mathbf{D} = 2P/15$ x; $\mathbf{E} = 8P/15$ x; $\mathbf{H} = 4P/15$ x.
6.121 (*a*) $\mathbf{A} = 2.06P$ a 14.04°; $\mathbf{B} = 2.06$ b 14.04°; frame is rigid. (*b*) Frame is not rigid. (*c*) $\mathbf{A} = 1.25P$ b 36.9°; $\mathbf{B} = 1.031P$ a 14.04°; frame is rigid.
6.122 (*a*) 2860 N w. (*b*) 2700 N d 68.5°.
6.123 (*a*) 746 N w. (*b*) 565 N c 61.3°.
6.126 (*a*) $(\mathbf{F}_{BD})_y = 96.0$ lb w. (*b*) $\mathbf{F}_{BD} = 100.0$ lb a 73.7°.
6.127 (*a*) $(\mathbf{F}_{BD})_y = 240$ lb w. (*b*) $\mathbf{F}_{BD} = 250$ lb a 73.7°.
6.128 (*a*) $\mathbf{P} = 109.8$ N y . (*b*) 126.8 N T. (*c*) 139.8 N b 38.3°.
6.129 (*a*) 160.8 N · m l. (*b*) 155.9 N · m l.
6.130 (*a*) 117.8 N · m l. (*b*) 47.9 N · m l.
6.131 (*a*) 21.0 kN z . (*b*) 52.5 kN z .
6.132 (*a*) 1143 N · m i. (*b*) 457 N · m i.
6.133 832 lb · in. l.
6.134 360 lb · in. l.
6.135 18.43 N · m i.
6.136 208 N · m i.
6.139 $\mathbf{F}_{AE} = 800$ N T; $\mathbf{F}_{DG} = 100.0$ N C.
6.140 $\mathbf{P} = 120.0$ N w; $\mathbf{Q} = 110.0$ N z .
6.141 $\mathbf{D} = 30.0$ kN z ; $\mathbf{F} = 37.5$ kN c 36.9°.
6.142 $\mathbf{D} = 150.0$ kN; $\mathbf{F} = 96.4$ kN c 13.50°.
6.144 $\mathbf{F} = 3290$ lb c 15.12°; $\mathbf{D} = 4450$ lb z .
6.145 8.45 kN.
6.147 (*a*) 475 lb. (*b*) 528 lb b 63.3°.
6.148 44.8 kN.
6.149 (*a*) 312 lb. (*b*) 135.0 lb · in. i.
6.150 140.0 N.
6.152 21.3 lb ↘.
6.153 (*a*) 9.29 kN b 44.4°. (*b*) 8.04 kN c 34.4°.
6.155 (*a*) 2.86 kips C. (*b*) 9.43 kips C.
6.156 (*a*) 4.91 kips C. (*b*) 10.69 kips C.
6.159 (*a*) 27.0 mm. (*b*) 40.0 N · m i.
6.160 (*a*) (90.0 N · m)**i**. (*b*) $\mathbf{A} = 0$; $\mathbf{M}_A = -(48.0\text{ N}\cdot\text{m})\mathbf{i}$; $\mathbf{B} = 0$; $\mathbf{M}_B = -(72.0\text{ N}\cdot\text{m})\mathbf{i}$.
6.163 $\mathbf{E}_x = 100.0$ kN y , $\mathbf{E}_y = 154.9$ kN x; $\mathbf{F}_x = 26.5$ kN y , $\mathbf{F}_y = 118.1$ kN w; $\mathbf{H}_x = 126.5$ kN z , $\mathbf{H}_y = 36.8$ kN w.
6.164 $\mathbf{F}_{AB} = \mathbf{F}_{BD} = 0$; $\mathbf{F}_{AC} = 675$ N T; $\mathbf{F}_{AD} = 1125$ N C; $\mathbf{F}_{CD} = 900$ N T; $\mathbf{F}_{CE} = 2025$ N T; $\mathbf{F}_{CF} = 2250$ N C; $\mathbf{F}_{DF} = 675$ N C; $\mathbf{F}_{EF} = 1800$ N T.
6.165 $\mathbf{F}_{AB} = 6.24$ kN C; $\mathbf{F}_{AC} = 2.76$ kN T; $\mathbf{F}_{BC} = 2.50$ kN C; $\mathbf{F}_{BD} = 4.16$ kN C; $\mathbf{F}_{CD} = 1.867$ kN T; $\mathbf{F}_{CE} = 2.88$ kN T; $\mathbf{F}_D = 3.75$ kN C; $\mathbf{F}_{DF} = 0$; $\mathbf{F}_{EF} = 1.200$ kN C.
6.166 $\mathbf{F}_{DF} = 10.48$ kips C; $\mathbf{F}_{DG} = 3.35$ kips C; $\mathbf{F}_{EG} = 13.02$ kips T.
6.168 (*a*) 80.0 lb T. (*b*) 72.1 lb d 16.10°.
6.170 $\mathbf{D}_x = 13.60$ kN y , $\mathbf{D}_y = 7.50$ kN x; $\mathbf{E}_x = 13.60$ kN z , $\mathbf{E}_y = 2.70$ kN w.
6.172 (*a*) 301 lb c 48.4°. (*b*) 375 lb T.
6.173 764 N z .
6.175 25.0 lb w.
6.C1 (*a*) u = *30°*: $W = 472$ lb, $A_{AB} = 1.500$ in^2, $A_{AC} = A_{CE} = 1.299$ in^2, $A_{BC} = A_{BE} = 0.500$ in^2, $A_{BD} = 1.732$ in^2. (*b*) $u_{opt} = 56.8°$: $W = 312$ lb, $A_{AB} = 0.896$ in^2, $A_{AC} = A_{CE} = 0.491$ in^2, $A_{BC} = 0.500$ in^2, $A_{BE} = 0.299$ in^2, $A_{BD} = 0.655$ in^2.
6.C2 (*a*) Fox $x = 9.75$ m, $F_{BH} = 3.19$ kN T. (*b*) For $x = 3.75$ m, $F_{BH} = 1.313$ kN C. (*c*) Fox $x = 6$ m, $F_{GH} = 3.04$ kN T.
6.C3 u = *30°*: $\mathbf{M} = 5860$ lb · ft l; $\mathbf{A} = 670$ lb a 75.5°. (*a*) $M_{max} = 8680$ lb · ft when u = 65.9°. (*b*) $A_{max} = 1436$ lb when u = 68.5°.
6.C4 u = *30°*; $\mathbf{M}_A = 1.669$ N · m l, $F = 11.79$ N, u = *80°*; $\mathbf{M}_A = 3.21$ N · m l, $F = 11.98$ N.
6.C5 $d = 0.40$ *in.*: 634 lb C; $d = 0.55$ *in.*: 286 lb C; $d = 0.473$ *in.*: $F_{AB} = 500$ lb C.
6.C6 u = *20°*: $M = 31.8$ N · m; u = *75°*: $M = 12.75$ N · m; u = 60.0°: $M_{min} = 12.00$ N · m.

CHAPTER 7

7.1 $\mathbf{F} = 720$ lb y ; $V = 140.0$ lb x; $\mathbf{M} = 1120$ lb · in. l (On JC).
7.2 $\mathbf{F} = 1106$ lb d 20.6°; $\mathbf{V} = 386$ lb c 69.4°; $\mathbf{M} = 1650$ lb · ft i (On JD).
7.3 $\mathbf{F} = 125.0$ N a 67.4°; $\mathbf{V} = 300$ N c 22.6°; $\mathbf{M} = 156.0$ N · m i (On BJ).
7.4 $\mathbf{F} = 2330$ N a 67.4°; $\mathbf{V} = 720$ N c 22.6°; $\mathbf{M} = 374$ N · m i (On BJ).
7.7 $\mathbf{F} = 12.50$ lb a 30.0°; $\mathbf{V} = 21.7$ lb b 60.0°; $\mathbf{M} = 75.0$ lb · in. i (On BJ).
7.8 $\mathbf{F} = 108.3$ lb a 60.0°; $\mathbf{V} = 62.5$ b 30.0°; $\mathbf{M} = 100.5$ lb · in. l (On DK).
7.9 $\mathbf{F} = 103.9$ N b 60.0°; $\mathbf{V} = 60.0$ N a 30.0°; $\mathbf{M} = 18.71$ N · m i (On AJ).
7.10 $\mathbf{F} = 60.0$ N d 30.0°; $\mathbf{V} = 103.9$ c 60.0°; $\mathbf{M} = 10.80$ N · m l (On BK).
7.11 $\mathbf{F} = 194.6$ N c 60.0°; $\mathbf{V} = 257$ N a 30.0°; $\mathbf{M} = 24.7$ N · m i (On AJ).
7.12 45.2 N · m for u = 82.9°.
7.15 $\mathbf{F} = 250$ N c 36.9°; $\mathbf{V} = 120.0$ N a 53.1; $\mathbf{M} = 120.0$ N · m l (On BJ).
7.16 $\mathbf{F} = 560$ N z ; $\mathbf{V} = 90.0$ N w; $\mathbf{M} = 72.0$ N · m i (On AK).
7.17 150.0 lb · in. at D.
7.18 105.0 lb · in. at E.
7.19 (*a*) $\mathbf{F} = 500$ N z ; $\mathbf{V} = 500$ N x; $\mathbf{M} = 300$ N · m i (On AJ). (*b*) $\mathbf{F} = 970$ N x; $\mathbf{V} = 171.0$ N z ; $\mathbf{M} = 446$ N · m i (On AK).
7.20 (*a*) $\mathbf{F} = 500$ N z ; $\mathbf{V} = 500$ N x; $\mathbf{M} = 300$ N · m i (On AJ). (*b*) $\mathbf{F} = 933$ N x; $\mathbf{V} = 250$ N z ; $\mathbf{M} = 375$ N · m i (On AK).
7.23 0.0557 Wr l (On AJ).
7.24 0.289 Wr i (On BJ).
7.25 0.1009 Wr for u = 57.3°.

7.26 0.357 Wr for u = 49.3°.
7.29 (*b*) $|V|_{max} = wL$; $|M|_{max} = wL^2/2$.
7.30 (*b*) $|V|_{max} = wL/2$; $|M|_{max} = w_oL^2/6$.
7.31 (*b*) $|V|_{max} = 2P/3$; $|M|_{max} = PL/9$.
7.32 (*b*) $|V|_{max} = P$; $|M|_{max} = PL/2$.
7.35 (*b*) $|V|_{max} = 35.0$ kN; $|M|_{max} = 12.50$ kN · m.
7.36 (*b*) $|V|_{max} = 50.5$ kN; $|M|_{max} = 39.8$ kN · m.
7.39 (*b*) $|V|_{max} = 64.0$ kN; $|M|_{max} = 92.0$ kN · m.
7.40 (*b*) $|V|_{max} = 30.0$ kN; $|M|_{max} = 72.0$ kN · m.
7.41 (*b*) $|V|_{max} = 18.00$ kips; $|M|_{max} = 48.5$ kip · ft.
7.42 (*b*) $|V|_{max} = 15.30$ kips; $|M|_{max} = 46.8$ kip · ft.
7.45 (*b*) $|V|_{max} = 1.800$ kN; $|M|_{max} = 0.225$ kN · m.
7.46 (*b*) $|V|_{max} = 2.00$ kN; $|M|_{max} = 0.500$ kN · m.
7.47 (*a*) $M \geq 0$ everywhere.
(*b*) $|V|_{max} = 4.50$ kips; $|M|_{max} = 13.50$ kip · ft.
7.48 (*a*) $M \leq 0$ everywhere.
(*b*) $|V|_{max} = 4.50$ kips; $|M|_{max} = 13.50$ kip · ft.
7.49 $|V|_{max} = 180.0$ N; $|M|_{max} = 36.0$ N · m.
7.50 $|V|_{max} = 800$ N; $|M|_{max} = 180.0$ N · m.
7.51 $|V|_{max} = 90.0$ lb; $|M|_{max} = 1400$ lb · in.
7.52 $|V|_{max} = 165.0$ lb; $|M|_{max} = 1625$ lb · in.
7.55 (*a*) 54.5°. (*b*) 675 N · m.
7.56 (*a*) 1.236. (*b*) 0.1180 wa^2.
7.57 (*a*) 0.311 m. (*b*) 193.0 N · m.
7.58 (*a*) 0.840 m. (*b*) 1.680 N · m.
7.59 0.207 L.
7.62 (*a*) 0.414 wL; 0.0858 wL^2. (*b*) 0.250 wL; 0.250 wL^2.
7.69 (*b*) $|V|_{max} = 6.40$ kN; $|M|_{max} = 4.00$ kN · m.
7.70 (*b*) $|V|_{max} = 9.00$ kN; $|M|_{max} = 14.00$ kN · m.
7.77 (*b*) 75.0 kN · m, 4.00m from A.
7.78 (*b*) 1.378 kN · m, 1.050m from A.
7.79 (*b*) 40.5 kN · m, 1.800m from A.
7.80 (*b*) 60.5 kN · m, 2.20m from A.
7.81 (*a*) 18.00 kip · ft, 3.00 ft from A.
(*b*) 34.1 kip · ft, 2.25 ft from A.
7.82 (*a*) 12.00 kip · ft at C. (*b*) 6.25 kip · ft, 2.50 ft from A.
7.86 (*a*) $V = (w_0/6L)(L^2 - 3x^2)$; $M = (w_0/6L)(L^2x - x^3)$.
(*b*) 0.0642 $w_0\,L^2$A a = 0.577L.
7.87 (*a*) $V = (w_0/3L)(2x^2 - 3Lx + L^2)$; $M = (w_0/8L)(4x^3 - 9Lx + 6L^2x - L^3)$.
(*b*) $w_0/L^2/72$, at $x = L/2$.
7.89 (*a*) $\mathbf{P} = 4.00$ kN w; $\mathbf{Q} = 6.00$ kN w. (*b*) $M_C = -900$ N · m.
7.90 (*a*) $\mathbf{P} = 2.50$ kN w; $\mathbf{Q} = 7.50$ kN w. (*b*) $M_C = -900$ N · m.
7.91 (*a*) $\mathbf{P} = 1.350$ kips w; $\mathbf{Q} = 0.450$ kips w. (*b*) $V_{max} = 2.70$ kips at A; $M_{max} = 6.345$ kip · ft, 5.40 ft from A.
7.92 (*a*) $\mathbf{P} = 0.540$ kips w; $\mathbf{Q} = 1.860$ kips w.
(*b*) $V_{max} = 3.14$ kips at B; $M_{max} = 7.00$ kip · ft, 6.88 ft from A.
7.93 (*a*) $\mathbf{E}_x = 8.00$ kN y; $\mathbf{E}_y = 5.00$ kN x. (*b*) 9.43 kN.
7.94 2.00 m.
7.95 (*a*) 838 lb b 17.35°. (*b*) 971 lb a 34.5°.
7.96 (*a*) 2670 lb d 2.10°. (*b*) 2810 lb a 18.65.
7.97 (*a*) $d_B = 1.733$ m; $d_D = 4.20$ m. (*b*) 21.5 kN a 3.81°.
7.98 (*a*) 2.80 m. (*b*) $\mathbf{A} = 32.0$ kN b 38.7°; $\mathbf{E} = 25.0$ kN y.
7.101 196.2 N.
7.102 157.0 N.
7.103 (*a*) 240 lb. (*b*) 9.00 ft.
7.104 $a = 7.50$ ft; $b = 17.50$ ft.
7.107 (*a*) 2770 N. (*b*) 75.14 m.
7.109 (*a*) 66,900 kips. (*b*) 4353 ft.
7.110 3.75 ft.
7.111 (*a*) 56,400 kips. (*b*) 4284 ft.
7.112 (*a*) 6.75 m. (*b*) $T_{AB} = 615$ N; $T_{BC} = 600$ N.
7.114 (*a*) $1\overline{3L\Delta/8}$. (*b*) 12.25 ft.
7.115 $h = 27.6$ mm; $u_A = 25.5°$; $u_C = 27.6°$.
7.116 (*a*) 4.05 m. (*b*) 6.41 m. (*c*) $A_x = 5890$ N z, $A_y = 5300$ N x.
7.117 (*a*) 58,900 kips, (*b*) 29.2°.
7.118 (*a*) 16.00 ft to the left of B. (*b*) 2000 lb.
7.125 $Y = h[1 - \cos(px/L)]$; $T_{min} = w_0L^2/hp^2$;
$T_{max} = (w_0L/p)\ 2\,\overline{(L^2/h^2p^2) + 1}$
7.127 (*a*) 9.89 m. (*b*) 60.3 N.
7.128 (*a*) 495 ft. (*b*) 1125 lb.
7.129 (*a*) 35.6 m. (*b*) 49.2 kg.
7.130 199.5 ft.
7.133 (*a*) 5.89 m. (*b*) 10.89 N y.
7.134 10.05 ft
7.135 (*a*) 4.22 ft. (*b*) 80.3°.
7.136 (*a*) 30.2 m. (*b*) 56.6 kg.
7.139 31.8 N.
7.140 29.8 N.
7.143 (*a*) $a = 79.0$ ft; $b = 60.0$ ft. (*b*) 103.9 ft.
7.144 (*a*) $a = 65.8$ ft; $b = 50.0$ ft. (*b*) 86.6 ft.
7.145 119.1 N y.
7.146 177.6 N y.
7.147 3.50 ft.
7.148 5.71 ft.
7.151 0.394 m and 10.97 m
7.152 0.1408.
7.153 (a) 0.338. (b) 56.5°; 0.755 wL.
7.154 $\mathbf{F} = 625$ N w; $\mathbf{V} = 120.0$ N z; $\mathbf{M} = 27.0$ N · m l (on CJ).
7.156 $\mathbf{F} = 23.6$ lb b 76.0°; V = 29.1 lb a 14.04°;
$\mathbf{M} = 540$ lb · in. i (on BJ).
7.157 $\mathbf{F} = 200$ N c 36.9°; $\mathbf{V} = 120.0$ N a 53.1°; $\mathbf{M} = 120$ N · ml (on BJ).
7.158 (*a*) 40.0 kips. (*b*) 40.0 kip · ft.
7.161 (*b*) 12.00 kip · ft, 6.00 ft from A.
7.163 (*a*) 2.28 m. (*b*) $\mathbf{D}_x = 13.67$ kN y; $\mathbf{D}_y = 7.80$ kN x.
(*c*) 15.94 kN.
7.164 (*a*) 138.1 m. (*b*) 602 N.
7.165 (*a*) 56.3 ft. (*b*) 2.36 lb/ft.
7.C1 (*a*) $M_D = +39.8$ kN · m. (*b*) $M_D = +14.00$ kip · ft.
(*c*) $M_D = +1800$ lb · in.
7.C3 $a = 1.923$ m; $M_{max} = 37.0$ kN · m at 4.64 from A.
7.C4 (*b*) $M_{max} = 5.42$ kip · ft when $x = 8.5$ ft and 11.5 ft.
7.C8 $c/L = 0.300$; $h/L = 0.5225$; $s_{AB}/L = 1.532$; $T_0/wL = 0.300$; $T_{max}/wL = 0.823$.

CHAPTER 8

8.1 Equilibrium; $\mathbf{F} = 172.6$ N ↘.
8.2 Block moves; $\mathbf{F} = 279$ N ↖.
8.3 Block moves; $\mathbf{F} = 31.0$ lb. ↘.
8.4 Equilibrium; $\mathbf{F} = 23.5$ lb ↘.
8.5 (*a*) 74.8 lb. (*b*) 59.7 lb. (*c*) 6.76 lb.
8.6 (*a*) 170.5 N. (*b*) 14.04°.
8.8 (*a*) 403 N. (*b*) 229 N.
8.10 143.0 N $\leq P \leq$ 483 N.
8.11 31.0°.
8.12 53.5°.
8.13 (*a*) 353 N z. (*b*) 196.2 N z.
8.14 (*a*) 275 N z. (*b*) 196.2 N z.
8.17 (*a*) 36.0 lb y. (*b*) 30.0 lb y. (*c*) 12.86 lb y.
8.18 (*a*) 36.0 lb y. (*b*) 40.0 in.
8.19 8.34 lb.
8.20 7.50 lb.

8.21 151.5 N · m.
8.22 1.473 kN.
8.23 $6.35 \leq L/a \leq 10.81$.
8.25 0.208.
8.27 664 N w.
8.29 (*a*) Plate in equilibrium. (*b*) Plate moves downward.
8.30 10.00 lb < P < 36.7 lb.
8.32 0.860.
8.34 0.0533.
8.35 (*a*) 1.333. (*b*) 1.192. (*c*) 0.839.
8.36 (*b*) 2.69 lb.
8.37 0.225.
8.39 $168.4 \text{ N} \leq P \leq 308 \text{ N}$.
8.40 $9.38 \text{ N} \cdot \text{m} \leq M \leq 15.01 \text{ N} \cdot \text{m}$
8.41 135.0 lb.
8.43 (*a*) System slides; $P = 62.8$ N. (*b*) System rotates about *B*; $P = 73.2$ N.
8.44 35.8°.
8.45 20.5°.
8.46 1.225 *W*.
8.47 46.4° ≤ u ≤ 52.4° and 67.6° ≤ u ≤ 79.4°.
8.48 (*a*) 620 N z . (*b*) $\mathbf{B}_x = 1390$ N z ; $\mathbf{B}_y = 1050$ N w.
8.49 (*a*) 234 N y . (*b*) $\mathbf{B}_x = 1824$ N z ; $\mathbf{B}_y = 1050$ N w.
8.52 313 lb y .
8.53 297 lb y .
8.54 9.86 kN z .
8.55 9.13 N z .
8.56 (*a*) 28.1°. (*b*) 728 N a 14.04°.
8.57 (*a*) 62.7 lb. (*b*) 62.7 lb.
8.59 67.4 N.
8.60 1.400 lb.
8.62 (*a*) 197.0 lb y . (*b*) Base will not move.
8.63 (*a*) 280 lb z . (*b*) Base moves.
8.64 (*b*) 283 N z .
8.65 0.442.
8.66 0.1103.
8.67 0.1013.
8.71 1068 N · m
8.72 153.1 lb · in.
8.73 41.4 lb · in.
8.75 4.18 N · m.
8.77 (*a*) 0.238. (*b*) 218 N w.
8.78 4.70 kips.
8.79 450 N.
8.80 412 N.
8.81 344 N.
8.82 376 N.
8.84 $T_{AB} = 77.5$ lb; $T_{CD} = 72.5$ lb. $T_{EF} = 67.8$ lb.
8.86 (*a*) 4.80 kN. (*b*) 1.375°.
8.88 22.0 lb z .
8.89 1.948 lb w.
8.90 18.01 lb z .
8.92 0.1670.
8.93 3.75 lb.
8.98 10.87 lb.
8.99 0.0600 in.
8.100 154.4 N.
8.101 300 mm.
8.102 (*a*) 1.288 kN. (*b*) 1.058 kN.
8.103 $73.0 \text{ lb} \leq P \leq 1233 \text{ lb}$.
8.104 (*a*) 0.329. (*b*) 2.67 turns.
8.105 (*a*) 22.8 kg. (*b*) 291 N.
8.106 (*a*) 109.7 kg. (*b*) 828 N.
8.109 44.9 N · m l.
8.110 (*a*) $T_A = 8.40$ lb; $T_B = 19.60$ lb. (*b*) 0.270.
8.111 (*a*) $T_A = 11.13$ lb; $T_B = 20.9$ lb. (*b*) 91.3 lb · in. i.
8.112 35.1 N · m.
8.113 (*a*) 27.0 N · m. (*b*) 675 N.
8.114 (*a*) 39.0 N · m. (*b*) 844 N.
8.117 4.49 in.
8.118 (*a*) 11.66 kg. (*b*) 38.6 kg. (*c*) 34.4 kg.
8.119 (*a*) 9.46 kg. (*b*) 167.2 kg. (*c*) 121.0 kg.
8.120 (*a*) 10.39 lb. (*b*) 58.5 lb.
8.121 (*a*) 28.9 lb. (*b*) 28.9 lb.
8.124 5.97 N.
8.125 9.56 N.
8.126 0.350.
8.128 (*a*) 30.3 lb · in. l. (*b*) 3.78 lb w.
8.129 (*a*) 17.23 lb · in. i. (*b*) 2.15 lb x.
8.133 (*a*) 51.0 N · m. (*b*) 875 N.
8.134 Block moves; $\mathbf{F} = 103.5$ N ↖.
8.136 (*a*) 0.300 *Wr*. (*b*) 0.349 *Wr*.
8.137 (*a*) 136.4°. (*b*) 0.928 *W*.
8.139 0.750.
8.140 $-46.8 \text{ N} \leq P \leq 34.3 \text{ N}$.
8.141 (*a*) $\mathbf{P} = 56.6$ lb z . (*b*) $\mathbf{B}_x = 82.6$ lb z ; $\mathbf{B}_y = 96.0$ lb w.
8.143 169.7 lb · in.
8.144 0.226.
8.C1 $x = 500$ *mm*: 63.3 N; $P_{max} = 67.8$ N at $x = 355$ mm.
8.C2 $W_B = 10$ *lb*: u = 46.4°; $W_B = 70$ *lb*: u = 21.3°.
8.C3 $m_A = 0.25$: $M = 0.0603$ N · m.
8.C4 u = *30°*: 1.336 N · m ≤ M_A ≤ 2.23 N · m.
8.C5 u = *60°*: $\mathbf{P} = 16.40$ lb w; $R = 5.14$ lb.
8.C6 u = *20°*: 10.39 N · m.
8.C7 u = *20°*: 30.3 lb; 13.25 lb.
8.C8 (*a*) $x_0 = 0.600L$; $x_m = 0.604L$; $u_1 = 5.06°$. (*b*) $u_2 = 55.4°$.

CHAPTER 9

9.1 $a^3(h_1 + 3h_2)/12$.
9.2 $3a^3b/10$.
9.3 $ha^3/5$.
9.4 $4a^3b/21$.
9.6 $ab^3/6$.
9.8 $4ab^3/13$.
9.9 $ab^3/28$.
9.10 $(ab^3/3)/(3n + 1)$.
9.11 $0.1056\, ab^3$.
9.12 $a^3b/20$.
9.15 $3ab^3/35$; $b\sqrt{9/35}$.
9.16 $\pi ab^3/8$; $b/2$.
9.17 $3a^3b/35$; $a\sqrt{9/35}$.
9.18 $\pi a^3b/8$; $a/2$.
9.21 $43a^4/48$; $0.773a$.
9.22 $4ab(a^2 + 4b^2)/3$; $\sqrt{(a^2 + 4b^2)/3}$.
9.23 $64\, a^4/15$; $1.265\, a$.
9.25 (*a*) $\pi(R_2^4 - R_1^4)/4$. (*b*) $I_x = I_y = \pi(R_2^4 - R_1^4)/8$.
9.26 (*b*) −10.56%; −2.99%; −0.1248%.
9.28 $bh\,(12h^2 + b^2)/48$; $\sqrt{(12h^2 + b^2)/24}$.
9.31 390×10^3 mm^4; 21.9 mm.
9.32 46.0 in^4; 1.599 in.
9.33 64.3×10^3 mm^4; 8.87 mm.
9.34 46.5 in^4; 1.607 in.
9.37 $\bar{I}_x = 150.0$ in^4; $\bar{I}_y = 300$ in^4.

9.39 $A = 4000\ \text{mm}^2$; $\bar{I} = 500 \times 10^3\ \text{mm}^4$.
9.40 $46.2 \times 10^6\ \text{mm}^4$.
9.41 $\bar{I}_x = 1.874 \times 10^6\ \text{mm}^4$; $\bar{I}_y = 5.82 \times 10^6\ \text{mm}^4$.
9.42 $\bar{I}_x = 479 \times 10^3\ \text{mm}^4$; $\bar{I}_y = 149.7 \times 10^3\ \text{mm}^4$.
9.43 $\bar{I}_x = 191.3\ \text{in}^4$; $\bar{I}_y = 75.2\ \text{in}^4$.
9.44 $\bar{I}_x = 18.13\ \text{in}^4$; $\bar{I}_y = 4.51\ \text{in}^4$.
9.47 (*a*) $11.57 \times 10^6\ \text{mm}^4$. (*b*) $7.81 \times 10^6\ \text{mm}^4$.
9.48 (*a*) $3.13 \times 10^6\ \text{mm}^4$. (*b*) $2.41 \times 10^6\ \text{mm}^4$.
9.49 $\bar{I}_x = 186.7 \times 10^6\ \text{mm}^4$; $\bar{k}_x = 118.6$ mm; $\bar{I}_y = 167.7 \times 10^6\ \text{mm}^4$; $\bar{k}_y = 112.4$ mm.
9.50 $\bar{I}_x = 44.5\ \text{in}^4$; $\bar{k}_x = 2.16$ in.; $\bar{I}_y = 27.7\ \text{in}^4$; $\bar{k}_y = 1.709$ in.
9.51 $\bar{I}_x = 254\ \text{in}^4$; $\bar{k}_x = 4.00$ in.; $\bar{I}_y = 102.1\ \text{in}^4$; $\bar{k}_y = 2.54$ in.
9.52 $\bar{I}_x = 260 \times 10^6\ \text{mm}^4$; $\bar{k}_x = 144.6$ mm; $\bar{I}_y = 17.53\ \text{mm}^4$; $\bar{k}_y = 37.6$ mm.
9.54 $\bar{I}_x = 745 \times 10^6\ \text{mm}^4$; $\bar{I}_y = 91.3 \times 10^6\ \text{mm}^4$.
9.55 $\bar{I}_x = 3.55 \times 10^6\ \text{mm}^4$; $\bar{I}_y = 49.8 \times 10^6\ \text{mm}^4$.
9.57 $h/2$.
9.58 $(a + 3b)\, h/(2a + 4b)$.
9.59 $3\pi b/16$.
9.60 $4h/7$.
9.63 $5a/8$.
9.64 80.0 mm.
9.67 $a^4/2$.
9.68 $b^2h^2/8$.
9.69 $a^2b^2/16$.
9.71 $-1.760 \times 10^6\ \text{mm}^4$.
9.72 $2.40 \times 10^6\ \text{mm}^4$.
9.74 $-0.380\ \text{in}^4$.
9.75 $471 \times 10^3\ \text{mm}^4$.
9.76 $-9010\ \text{in}^4$.
9.78 $1.165 \times 10^6\ \text{mm}^4$.
9.79 (*a*) $\bar{I}_{x'} = 0.482a^4$; $\bar{I}_{y'} = 1.482a^4$; $\bar{I}_{x'y'} = -0.589a^4$. (*b*) $\bar{I}_{x'} = 1.120a^4$; $\bar{I}_{y'} = 0.843a^4$; $\bar{I}_{x'y'} = 0.760a^4$.
9.80 $\bar{I}_{x'} = 2.12 \times 10^6\ \text{mm}^4$; $\bar{I}_{y'} = 8.28 \times 10^6\ \text{mm}^4$; $\bar{I}_{x'y'} = -0.532 \times 10^6\ \text{mm}^4$.
9.81 $\bar{I}_{x'} = 1033\ \text{in}^4$; $\bar{I}_{y'} = 2020\ \text{in}^4$; $\bar{I}_{x'y'} = -873\ \text{in}^4$.
9.83 $\bar{I}_{x'} = 0.236\ \text{in}^4$; $\bar{I}_{y'} = 1.244\ \text{in}^4$; $\bar{I}_{x'y'} = 0.1132\ \text{in}^4$.
9.85 $20.2°$; $1.754a^4$; $0.209a^4$.
9.86 $25.1°$; $\bar{I}_{\max} = 8.32 \times 10^6\ \text{mm}^4$; $\bar{I}_{\min} = 2.08 \times 10^6\ \text{mm}^4$.
9.87 $29.7°$; $2530\ \text{in}^4$; $524\ \text{in}^4$.
9.89 $-23.7°$ and $66.3°$; $1.257\ \text{in}^4$; $0.224\ \text{in}^4$.
9.91 (*a*) $\bar{I}_{x'} = 0.482a^4$; $\bar{I}_{y'} = 1.482a^4$; $\bar{I}_{x'y'} = -0.589a^4$. (*b*) $\bar{I}_{x'} = 1.120a^4$; $\bar{I}_{y'} = 0.843a^4$; $0.760a^4$.
9.92 $\bar{I}_{x'} = 2.12 \times 10^6\ \text{mm}^4$; $\bar{I}_{y'} = 8.28 \times 10^6\ \text{mm}^4$; $\bar{I}_{x'y'} = -0.532 \times 10^6\ \text{mm}^4$.
9.93 $\bar{I}_{x'} = 1033\ \text{in}^4$; $\bar{I}_{y'} = 2020\ \text{in}^4$; $\bar{I}_{x'y'} = -873\ \text{in}^4$.
9.95 $\bar{I}_{x'} = 0.236\ \text{in}^4$; $\bar{I}_{y'} = 1.244\ \text{in}^4$; $\bar{I}_{x'y'} = 0.1132\ \text{in}^4$.
9.97 $20.2°$; $1.754a^4$; $0.209a^4$.
9.98 $25.1°$ counterclockwise at C; $I_{\max} = 8.32 \times 10^6\ \text{mm}^4$; $\bar{I}_{\min} = 2.08 \times 10^6\ \text{mm}^4$.
9.99 $-33.4°$; $22.1 \times 10^3\ \text{in}^4$; $2490\ \text{in}^4$.
9.100 $29.7°$; $2530\ \text{in}^4$; $524\ \text{in}^4$.
9.103 (*a*) $-1.146\ \text{in}^4$. (*b*) $29.1°$ clockwise. (*c*) $3.39\ \text{in}^4$.
9.104 $23.8°$ clockwise; $0.524 \times 10^6\ \text{mm}^4$; $0.0917 \times 10^6\ \text{mm}^4$.
9.105 $19.54°$ counterclockwise; $4.34 \times 10^6\ \text{mm}^4$; $0.647 \times 10^6\ \text{mm}^4$.
9.106 (*a*) $25.3°$. (*b*) $1459\ \text{in}^4$; $40.5\ \text{in}^4$.
9.107 (*a*) $88.0 \times 10^6\ \text{mm}^4$. (*b*) $96.3 \times 10^6\ \text{mm}^4$; $39.7 \times 10^6\ \text{mm}^4$.
9.111 (*a*) $\bar{I}_{AA'} = \bar{I}_{BB'} = ma^2/24$. (*b*) $ma^2/12$.
9.112 (*a*) $5mb^2/4$. (*b*) $5m\,(a^2 + b^2)/4$.
9.113 (*a*) $0.0699\ ma^2$. (*b*) $0.320\ ma^2$.
9.114 (*a*) $25\ mr_2^2/64$. (*b*) $0.1522\ mr_2^2$.
9.117 (*a*) $ma^2/3$. (*b*) $3ma^2/2$.
9.118 (*a*) $7ma^2/6$. (*b*) $ma^2/2$.
9.119 $m(3a^2 + 4L^2)/12$.
9.120 $1.329\ mh^2$.
9.121 (*a*) $0.241\ mh^2$. (*b*) $m(3a^2 + 0.1204\ h^2)$.
9.122 $m(b^2 + 3h^2)/5$.
9.124 $m(a^2 + 3h^2)/6$.
9.126 $I_x = I_y = ma^2/4$; $I_z = ma^2/2$.
9.127 $1.160 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; 0.341 in.
9.128 $837 \times 10^{-9}\ \text{kg} \cdot \text{m}^2$; 6.92 mm.
9.130 $ma^2/2$; $a/\sqrt{2}$.
9.131 (*a*) 2.30 in. (*b*) $20.6 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; 2.27 in.
9.132 (*a*) $\pi p l^2\,[6a^2t(5a^2/3l^2 + 2a/l + 1) + d^2l/4]$. (*b*) 0.1851.
9.133 (*a*) 27.5 mm to the right of A. (*b*) 32.0 mm.
9.135 $I_x = 0.877\ \text{kg} \cdot \text{m}^2$; $I_y = 1.982\ \text{kg} \cdot \text{m}^2$; $I_z = 1.652\ \text{kg} \cdot \text{m}^2$.
9.136 $I_x = 175.5 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$; $I_y = 309 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$; $I_z = 154.4 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.138 $I_x = 745 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_y = 896 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_z = 304\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$.
9.139 $I_x = 344 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_y = 132.1 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_z = 453 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$.
9.141 (*a*) $13.99 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$. (*b*) $20.6 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$. (*c*) $14.30 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.142 $I_x = 28.3 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$; $I_y = 183.8 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$; $k_x = 42.9$ mm; $k_y = 109.3$ mm.
9.143 $0.1785\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$.
9.145 (*a*) $26.4 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$. (*b*) $31.2 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$. (*c*) $8.58 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.147 $I_x = 0.0392\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_y = 0.0363\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_z = 0.0304\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$.
9.148 $I_x = 0.323\ \text{kg} \cdot \text{m}^2$; $I_y = I_z = 0.419\ \text{kg} \cdot \text{m}^2$.
9.149 $I_{xy} = 2.50 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$; $I_{yz} = 4.06 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$; $I_{zx} = 8.81 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.150 $I_{xy} = 286 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$; $I_{yz} = I_{zx} = 0$.
9.151 $I_{xy} = -1.726 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_{yz} = 0.507 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_{zx} = -2.12 \times 10^{-3}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$.
9.152 $I_{xy} = -538 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_{yz} = -171.4 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$; $I_{zx} = 1120 \times 10^{-6}\ \text{lb} \cdot \text{ft} \cdot \text{s}^2$.
9.155 $I_{xy} = -8.04 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$; $I_{yz} = 12.90 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$; $I_{zx} = 94.0 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.156 $I_{xy} = 0$; $I_{yz} = 48.3 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$; $I_{zx} = -4.43 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.157 $I_{xy} = 47.9 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$; $I_{yz} = 102.1 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$; $I_{zx} = 64.1 \times 10^{-6}\ \text{kg} \cdot \text{m}^2$.
9.158 $I_{xy} = -m'\,R_1^3/2$; $I_{yz} = m'\,R_1^3/2$; $I_{zx} = -m'\,R_2^3/2$.
9.159 $I_{xy} = wa^3\,(1 - 5\pi)/g$; $I_{yz} = -11\pi\, wa^3/g$; $I_{zx} = 4wa^3\,(1 + 2\pi)/g$.
9.160 $I_{xy} = -11wa^3/g$; $I_{yz} = wa^3(\pi + 6)/2g$; $I_{zx} = -wa^3/4g$.
9.162 (*a*) $mac/20$. (*b*) $I_{xy} = mab/20$; $I_{yz} = mbc/20$.
9.165 $18.17 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.166 $11.81 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.167 $5Wa^2/18g$.
9.168 $4.41\ \rho ta^4/g$.
9.169 $281 \times 10^{-3}\ \text{kg} \cdot \text{m}^2$.
9.170 $0.354\ \text{kg} \cdot \text{m}^2$.
9.173 (*a*) $1/\sqrt{3}$. (*b*) $\sqrt{7/12}$.
9.174 (*a*) $b/a = 2$; $c/a = 2$. (*b*) $b/a = 1$; $c/a = 0.5$.
9.175 (*a*) 2. (*b*) $\sqrt{2/3}$.
9.179 (*a*) $K_1 = 0.363ma^2$; $K_2 = 1.583ma^2$; $K_3 = 1.720ma^2$. (*b*) $(\theta_x)_1 = (\theta_z)_1 = 49.7°$, $(\theta_y)_1 = 113.7°$; $(\theta_x)_2 = 45.0°$, $(\theta_y)_2 = 90.0°$, $(\theta_z)_2 = 135.0°$; $(\theta_x)_3 = (\theta_z)_3 = 73.5°$, $(\theta_y)_3 = 23.7°$.

9.180 (*a*) $K_1 = 14.30 \times 10^{-3}$ kg · m^2; $K_2 = 13.96 \times 10^{-3}$ kg · m^2; $K_3 = 20.6 \times 10^{-3}$ kg · m^2.
(*b*) $(u_x)_1 = (u_y)_1 = 90.0°$, $(u_z)_1 = 0$; $(u_x)_2 = 3.42°$, $(u_y)_2 = 86.6°$, $(u_z)_2 = 90.0°$; $(u_x)_3 = 93.4°$, $(u_y)_3 = 3.43°$, $(u_z)_3 = 90.0°$.
9.182 (*a*) $K_1 = 0.1639Wa^2/g$; $K_2 = 1.054Wa^2/g$; $K_3 = 1.115Wa^2/g$.
(*b*) $(u_x)_1 = 36.7°$, $(u_y)_1 = 71.6°$, $(u_z)_1 = 59.5°$; $(u_x)_2 = 74.9°$, $(u_y)_2 = 54.5°$, $(u_z)_2 = 140.5°$; $(u_x)_3 = 57.5°$, $(u_y)_3 = 138.8°$, $(u_z)_3 = 112.4°$.
9.183 (*a*) $K_1 = 2.26gta^4/g$; $K_2 = 17.27gta^4/g$; $K_3 = 19.08gta^4/g$.
(*b*) $(u_x)_1 = 85.0°$, $(u_y)_1 = 36.8°$, $(u_z)_1 = 53.7°$; $(u_x)_2 = 81.7°$, $(u_y)_2 = 54.7°$; $(u_z)_2 = 143.4°$; $(u_x)_3 = 9.70°$, $(u_y)_3 = 99.0°$, $(u_z)_3 = 86.3°$.
9.185 $I_x = a^4/8$; $I_y = 3a^4/2$.
9.186 $a^3b/6$; $a/\ 1\overline{3}$.
9.188 $I_x = 48.9 \times 10^3$ mm^4; $I_y = 8.35 \times 10^3$ mm^4.
9.189 (*a*) 80.9×10^6 mm^4. (*b*) 57.4×10^6 mm^4.
9.191 -2.81 in^4.
9.193 (*a*) $5ma^2/18$. (*b*) $3.61ma^2$.
9.195 $I_x = 26.0 \times 10^{-3}$ kg · m^2; $I_y = 38.2 \times 10^{-3}$ kg · m^2; $I_z = 17.55 \times 10^{-3}$ kg · m^2.
9.196 $I_x = 38.1 \times 10^{-3}$ kg · m^2; $k_x = 110.7$ mm.
9.C1 $u = 20°$; $I_{x'} = 14.20$ in^4, $I_{y'} = 3.15$ in^4, $I_{x'y'} = -3.93$ in^4.
9.C3 (*a*) $\bar{I}_{x'} = 371 \times 10^3$ mm^4, $\bar{I}_{y'} = 64.3 \times 10^3$ mm^4; $\bar{k}_{x'} = 21.3$ mm, $\bar{k}_{y'} = 8.87$ mm. (*b*) $\bar{I}_{x'} = 40.4$ in^4, $\bar{I}_{y'} = 46.5$ in^4; $\bar{k}_{x'} = 1.499$ in., $\bar{k}_{y'} = 1.607$ in. (*c*) $\bar{k}_x =$ 2.53 in., $\bar{k}_y = 1.583$ in. (*d*) $\bar{k}_x = 1.904$ in., $\bar{k}_y = 0.950$ in.
9.C5 (*a*) 5.99×10^{-3} kg · m^2. (*b*) 77.4×10^{-3} kg · m^2.
9.C6 (*a*) 74.0×10^{-6} lb · ft · s^2. (*b*) 645×10^{-6} lb · ft · s^2.
(*c*) 208×10^{-6} lb · ft · s^2.

CHAPTER 10

10.1 270 N x.
10.2 60.0 lbw.
10.3 32.4 N · m i.
10.4 600 lb · in. i.
10.5 500 N x.
10.6 750 N x.
10.9 $Q = 3P \tan u$.
10.10 $Q = P\,[(l/a)] \cos^3 u - 1]$.
10.12 $Q = 2P \sin u/\cos(u/2)$.
10.14 $Q = (3P/2) \tan u$.
10.15 $M = Pl/2 \tan u$.
10.16 $M = Pl(\sin u + \cos u)$.
10.17 $M = \frac{1}{2}Wl \tan a \sin u$.
10.18 $M = PR \csc^2 u$.
10.19 85.2 lb · ft i.
10.20 22.8 lb d 70.0°.
10.23 39.2°.
10.26 19.81° and 51.9°.
10.27 36.4°.
10.28 67.1°.
10.29 40.2°.
10.31 15.60 in.
10.32 13.20 in.
10.34 57.2°.
10.35 38.7°.
10.36 60.4°.
10.37 22.6°.
10.38 51.1°.
10.39 59.0°.
10.40 78.7°, 324°, 379°.
10.43 12.03 kN ↘.
10.44 20.4°.
10.45 2370 lb ↖.
10.46 2550 lb ↖.
10.48 300 N · m, 81.8 N · m.
10.49 $h = 1/(1 + m \cot a)$.
10.50 $h = \tan u/\tan(u + f_s)$.
10.52 37.6 N, 31.6 N.
10.53 7.75 kN x.
10.54 $H = 1.361$ kN x; $M_H = 550$ N · m l.
10.57 0.833 in.w.
10.58 0.625 in. y.
10.69 $u = -45.0°$, unstable; $u = 135.0°$, stable.
10.70 $u = -63.4°$, unstable; $u = 116.6°$, stable.
10.71 (*a*) 0, unstable. (*b*) 137.8°, stable.
10.72 $u = 0$ and $u = 180.0°$, unstable; $u = 75.5°$ and $u = 284°$, stable.
10.73 59.0°, stable.
10.74 78.7°, stable; 324°, unstable; 379°, stable.
10.77 357 mm.
10.78 252 mm.
10.80 9.39° and 90.0°, stable; 34.2°, unstable.
10.81 17.11°, stable; 72.9°, unstable.
10.83 49.1°.
10.86 16.88 m.
10.87 54.8°.
10.88 37.4°.
10.89 $P < kl/2$.
10.91 $k > 6.94$ lb/in.
10.92 15.00 in.
10.93 $P < 2kL/9$.
10.94 $P < kL/18$.
10.96 $P < 160.0$ N.
10.98 $P < 764$ N.
10.100 (*a*) $P < 10.00$ lb. (*b*) $P < 20.0$ lb.
10.101 120.0 lb y.
10.102 1200 lb · in. l.
10.103 (*a*) 60.0 N C, 8.00 mmw. (*b*) 300 N C, 40.0 mmw.
10.105 $M = 7Pa \cos u$.
10.107 19.40°.
10.108 7.13 in.
10.110 $P < k\,(l - a)^2/2l$.
10.112 (*a*) 22.0°. (*b*) 30.6°.
10.C1 $u = 60°$: 2.42 in.; $u = 120°$: 1.732 in.; $(M/P)_{max} = 2.25$ in. at $u = 73.7°$.
10.C2 $u = 60°$: 171.1 N *C*. For $32.5° \leq u \leq 134.3°$, $|F| \leq 400$ N.
10.C3 $u = 60°$; 296 N *T*. For $u \leq 125.7°$, $|F| \leq 400$ N.
10.C4 (*b*) $u = 60°$, *datum at C*: $V = -294$ in · lb.
(*c*) 34.2°, stable; 90°, unstable; 145.8°, stable.
10.C5 (*b*) $u = 50°$, *datum at E*: $V = 100.5$ J. $dV/du = 22.9$ J.
(*c*) $u = 0$, unstable; 30.4°.
10.C6 (*b*) $u = 60°$, *datum at B*: 30.0 J.
(*c*) $u = 0$, unstable; 41.4°, stable.
10.C7 (*b*) $u = 60°$, *datum at* $u = 0$: -37.0 J. (*c*) 52.2°, stable.

CHAPTER 11

11.1 11.00 in., -8.00 in/s, -8.00 in/s^2.
11.2 1.000s, 15.00 ft, -6.00 ft/s^2; 2.00 s, 14.00 ft, 6.00 ft/s^2.
11.3 (*a*) 102.9 mm, -35.6 mm/s, -11.40 mm/s^2.
(*b*) -36.1 mm/s, 72.1 mm/s^2.

11.4 (*a*) 0, 960 mm/s y, 9220 mm/s^2 or 9.22 m/s^2 z.
(*b*) 14.16 mm z, 87.9 mm/s y, 3110 mm/s^2 or 3.11 m/s^2 y.
11.5 0.667 s, 0.259m, -8.56 m/s.
11.7 (*a*) 1.000 s and 4.00 s. (*b*) 1.500 m, 24.5 m.
11.9 (*a*) 77.5 ft/s. (*b*) 7.75 s.
11.10 -33.0 in/s, 2.00 in., 87.7 in.
11.11 $x(t) = t^4/108 + 10t + 24$ m.
$v(t) = t^3/27 + 10$ m/s.
11.12 (*a*) 6.00 m/s^4. (*b*) $a = 6t^2$, $v = 2t^3 - 8$, $x = t^4/2 - 8t + 8$.
11.15 800 m/s^2 x.
11.16 (*a*) -2.43×10^6 ft/s^2. (*b*) 1.366×10^{-3} s.
11.17 (*a*) 5.89 ft/s. (*b*) 1.772 ft.
11.18 167.1 mm/s^2 x, 15.19 m/s^2 x.
11.21 (*a*) 1.250 m. (*b*) 0.866 s.
11.22 (*a*) 10.00 ft. (*b*) 1.833 ft/s, 0.440 ft/s^2.
11.23 (*a*) 42.0 ft. (*b*) 12.86 ft/s.
11.24 (*a*) 29.3 m/s. (*b*) 0.947 s.
11.25 (*a*) 3.33 m. (*b*) 2.22 s. (*c*) 1.667 s.
11.26 (*a*) 0.1457 s/m. (*b*) 145.2 m. (*c*) 6.86 m/s.
11.27 (*a*) -0.0525 m/s^2. (*b*) 6.17 s.
11.28 (*a*) 7.15 mi. (*b*) -275×10^{-6} ft/s^2. (*c*) 49.9 min.
11.31 (*a*) 2.36 v_0T, p v_0/T. (*b*) 0.363 v_0.
11.33 (*a*) 9.62 m/s x. (*b*) 29.6. m/s w.
11.34 (*a*) -0.417 m/s^2. (*b*) 18.00 km/h.
11.35 (*a*) 5.28 ft/s^2. (*b*) 8.33 s.
11.36 (*a*) 252 ft/s. (*b*) 1076 ft.
11.39 (*a*) $a_A = -2.10$ m/s^2, $a_B = 2.06$ m/s^2.
(*b*) 2.59 s before *A* reaches the exchange zone.
11.40 (*a*) 1.563 m/s^2. (*b*) 3.13 m/s^2.
11.41 (*a*) 1760 ft. (*b*) 28.6 mi/h.
11.42 (*a*) 15.05 s, 734 ft from the ititial point of *A*.
(*b*) *A*: 42.5 mi/h. *B*: 23.7 mi/h.
11.43 (*a*) $\mathbf{a}_A = 0.767$ ft/s^2 z, $\mathbf{a}_B = 0.834$ ft/s^2 y.
(*b*) 20.7 s. (*c*) 51.8 mi/h.
11.44 (*a*) 1.330 s. (*b*) 4.68 m below the man.
11.47 (*a*) 8.00 m/s x. (*b*) 4.00 m/s x. (*c*) 12.00 m/s x. (*d*) 8.00 m/s x.
11.48 (*a*) $\mathbf{a}_E = 2.40$ ft/s^2 x, $\mathbf{a}_C = 4.80$ ft/s^2 w.
(*b*) 12.00 ft/s x.
11.49 (*a*) 2.00 m/s x. (*b*) 2.00 m/s w. (*c*) 8.00 m/s x.
11.50 (*a*) $\mathbf{a}_A = 24.0$ ft/s^2 y, $\mathbf{a}_B = 8.00$ ft/s^2 w.
(*b*) 16.00 ft/s w, 16.00 ft.
11.51 (*a*) 200 mm/s y. (*b*) 600 mm/s y.
(*c*) 200 mm/s z. (*d*) 400 mm/s y.
11.52 (*a*) $\mathbf{a}_A = 13.33$ mm/s^2 z, $\mathbf{a}_B = 20.0$ mm/s^2 z.
(*b*) 13.33 mm/s^2 y. (*c*) 70.0 mm/s y. 440 mm y.
11.55 (*a*) 1.500 s. (*b*) 3.00 s. (*c*) 10.00 in. x.
11.56 (*a*) 1.000 s. (*b*) 3.00 in. w.
11.57 (*a*) $\mathbf{a}_A = 240$ mm/s^2 w, $\mathbf{a}_B = 345$ mm/s^2 x.
(*b*) $(\mathbf{v}_A)_0 = 43.3$ mm/s x, $(v_C)_0 = 130.0$ mm/s y.
(*c*) 728 mm y.
11.58 (*a*) 10.00 mm/s y. (*b*) $\mathbf{a}_A = 2.00$ mm/s^2 x,
$\mathbf{a}_C = 6.00$ mm/s^2 y. (*c*) 175.0 mm x.
11.63 (*a*) 10 s to 26 s, $a = -5.00$ m/s^2;
41 s to 46 s, $a = 3.00$ m/s^2; otherwise $a = 0$.
(*b*) 1383 m. (*c*) 9.00 s, 49.5 s.
11.64 (*a*) Same as Prob. 11.63. (*b*) 420 m. (*c*) 10.69 s, 40.0 s.
11.65 10.50 s.
11.66 (*a*) 44.8 s. (*b*) 103.3 m/s^2.
11.69 (*a*) 0.600 s. (*b*) 0.200 m/s, 2.84 m.
11.70 (*a*) 60.0 m/s, 1194 m. (*b*) 59.3 m/s.
11.71 (*a*) *A*: 52.2 s, *B*: 52.0 s. (*b*) 1.879 m.
11.72 9.39 s.
11.73 8.54 s, 58.3 mi/h.
11.74 77.5 ft.
11.75 5.67 s.
11.78 (*a*) 18.00 s. (*b*) 178.8 m. (*c*) 34.7 km/h.
11.79 (*a*) 5.01 min. (*b*) 19.18 mi/h.
11.80 (*a*) 2.00 s. (*b*) 1.200 ft/s, 0.600 ft/s.
11.83 (*a*) 2.96 s. (*b*) 224 ft.
11.84 (*a*) 163.0 in/s^2. (*b*) 114.3 in/s^2.
11.89 (*a*) 6.28 m/s c 37.2°. (*b*) 7.49 m.
11.90 (*a*) 67.1 mm/s a 63.4°, 256 mm/s^2 d 69.4°.
(*b*) 8.29 mm/s a 36.2°, 336 mm/s^2 d 86.6°.
11.91 (*a*)(-112.57 in/s)**i**, (-39.5 in/s^2)**j**. (*b*) $y = x^2/8 - 1$.
11.92 (*a*) max: 15.00 ft/s, min: 5.00 ft/s
(*b*) min: $t = 2$pN s, $x = 20$pN ft, $y = 5$ ft, $v_x = 5$ ft/s,
$v_y = 0$, u $= 0$.
max: $t = (2N + 1)$ p s, $x = 20$p$(N + 1)$ ft, $y = 15$ ft,
$v_x = 15$ ft/s, $v_y = 0$, u $= 0$.
11.95 2 $\overline{R^2(1 + w_n^2t^2) + c^2}$, Rw_n 2 $\overline{4 + w_n^2t^2}$.
11.96 (*a*) 3.00 ft/s, 3.61 ft/s^2. (*b*) 3.82 s.
11.97 1140 ft.
11.98 (*a*) 330 m. (*b*) 149.9 m.
11.99 (*a*) 115.3 km/h $\leq v_0 \leq$ 148.0 km/h.
(*b*) $h = 0.788$ m, a $= 6.66°$; $h = 1.068$ m, a $= 4.05°$.
11.100 15.38 ft/s $< v_0 <$ 35.0 ft/s.
11.102 0.244 m $< h <$ 0.386 m.
11.103 (*a*) Ball clears the net. (*b*) 7.01 m from the net.
11.105 22.9 ft/s.
11.106 16.20 m/s $< v_0 <$ 21.0 m/s.
11.107 (*a*) 29.8 ft/s. (*b*) 29.6 ft/s.
11.108 37.7 m/s $< v_0 <$ 44.3 m/s.
11.111 (*a*) 10.38°. (*b*) 9.74°.
11.112 (*a*) 4.17°. (*b*) 285 m. (*c*) 15.89 s.
11.113 (*a*) 14.66°. (*b*) 0.1074 s.
11.114 (*a*) 4.98 m. (*b*) 23.8°.
11.117 17.80 ft/s b 50.9°.
11.118 $\mathbf{v}_A = 125$ mm/s x, $\mathbf{v}_B = 75$ mm/s w, $\mathbf{v}_C = 175$ mm/s w.
11.119 (*a*) 91.0 ft/s d 47.0°. (*b*) 364 ft d 47.0°. (*c*) 293 ft.
11.120 3.20 km/h c 17.8°.
11.123 (*a*) 7.01 in/s d 60.6°. (*b*) 11.69 in/s^2 d 60.6°.
11.124 (*a*) 8.53 in/s b 54.1°. (*b*) 6.40 in/s b 54.1°.
11.125 (*a*) 0.979 m. (*b*) 12.55 m/s c 86.5°.
11.126 (*a*) 0.835 mm/s^2 b 75°. (*b*) 8.35 mm/s b 75°.
11.127 (*a*) 5.18 ft/s b 15°. (*b*) 1.232 ft/s b 15°.
11.128 10.54 ft/s d 81.3°.
11.129 5.96 m/s c 82.8°.
11.131 15.79 km/h c 26.0°.
11.132 1.024 ft/s b 2.07°.
11.133 500 m.
11.134 97.6 km/h.
11.135 444 m/s^2.
11.136 (*a*) 792 ft. (*b*) 51.9 mi/h.
11.137 2.53 ft/s^2.
11.138 (*a*) 10.20 mm/s^2. (*b*) 25.2 s.
11.139 (*a*) 178.9 m. (*b*) 1.118 m/s^2.
11.141 (*a*) 189.5 km/h c 54.0°. (*b*) 21.8 m/s^2 c 5.3°.
11.143 (*a*) 27.2 ft/s a 40°. (*b*) 13.48 ft.
11.144 (*a*) 2.46 m. (*b*) 42.7 mm.
11.145 (*a*) 281 m. (*b*) 209 m.
11.146 (*a*) 27.6 m. (*b*) 34.0 m.
11.147 (*a*) 0.634 m. (*b*) 9.07 m.
11.149 18.17 m/s a 4.04° and 18.17 m/s c 4.04°.
11.151 $(R^2 + C^2)/2w_nR$.

11.152 2.50 ft.
11.153 149.8 Gm.
11.154 1425 Gm.
11.155 16 200 mi/h.
11.156 7740 mi/h.
11.159 1.606 h.
11.161 (*a*) (1.624 in/s)$\mathbf{e}_r$ − (15.56 in/s)$\mathbf{e}_u$
(*b*) (−49.9 in/s^2)$\mathbf{e}_r$ + (−9.74 in/s^2)$\mathbf{e}_u$
(*c*) (−3.25 in/s^2)$\mathbf{e}_r$.
11.162 (*a*) (−2.50 in/s)$\mathbf{e}_r$ + (2.50 in/s)$\mathbf{e}_u$
(*b*) (7.50 in/s^2)$\mathbf{e}_r$ + (−10.00 in/s^2)$\mathbf{e}_u$
(*c*) 7.07 in.
11.163 (*a*) (2p m/s)$\mathbf{e}_u$, − (4p^2 m/s^2)$\mathbf{e}_r$.
(*b*) −(p/2 m/s)$\mathbf{e}_r$ + (p m/s)$\mathbf{e}_u$, −(p^2/2 m/s^2)$\mathbf{e}_r$ − (p^2 m/s^2)$\mathbf{e}_u$.
11.164 (*a*) $v = 2abt$, $a = 2ab$ 2 $\overline{1 + 4b^2t^4}$.
(*b*) $\rho = a$; The path is a circle.
11.165 (*a*) $\mathbf{v} = bk\mathbf{e}_u$, $\mathbf{a} = -(bk^2/2)\mathbf{e}_r$.
(*b*) $\mathbf{v} = 2bk\mathbf{e}_r + 2bk\mathbf{e}_u$, $\mathbf{a} = 2bk^2\mathbf{e}_r + 4bk^2\mathbf{e}_u$.
11.166 (*a*) $a = 4b\dot{u}^2$. (*b*) directed toward point *A*.
11.169 $\dot{r} = 370$ ft/s, $\ddot{r} = 57.9$ ft/s^2, $\dot{u} = -0.0924$ rad/s,
$\ddot{u} = 0.0315$ rad/s^2.
11.170 (*a*) $\dot{r} = -dw/2$, $\dot{u} = w/2$. (*b*) $\ddot{r} = -$ 1 $\overline{3}\,dw^2/4$, $\ddot{u} = 0$.
11.171 185.7 km/h.
11.172 61.8 mi/h, 49.7°.
11.175 bw^2 2 $\overline{4 + u^4}/u^3$.
11.176 $(1 + b^2)w^2e^{bu}$.
11.177 $v = 2p$ 2 $\overline{A^2 + n^2B^2\cos^2 2pnt}$
$a = 4p^2$ 2 $\overline{A^2 + n^4B^2\sin^2 2pnt}$
11.179 (*a*) $v =$ 2 $\overline{A^2 + B^2}$, $a =$ 2 $\overline{(1 + 16p^2)A^2 + B^2}$.
(*b*) $v = 2pA$, $a = 4p^2A$.
11.180 $\tan^{-1}[R(2 + w_n^2t^2)/c$ 2 $\overline{4 + w_n^2t^2}]$.
11.181 (*a*) $u_x = 90°$, $u_y = 123.7°$, $u_z = 33.7°$.
(*b*) $u_x = 103.4°$, $u_y = 134.3°$, $u_z = 47.4°$.
11.182 (*a*) 1.00 s and 4.00 s. (*b*) 1.500 m, 24.5 m.
11.183 $A = -36.8$ m^2, $k = 1.832$ s^{-2}.
11.185 (*a*) 111.4 km/h a 10.50°. (*b*) 2.96 km.
11.187 (*a*) $\mathbf{a}_B = 2.00$ in/s^2 x, $\mathbf{a}_C = 3.00$ in/s^2 w. (*b*) 0.667 s.
(*c*) 0.667 in. x.
11.188 (*a*) 38.1 m/s, 20.4 m. (*b*) 41.1 m/s, 29.6 m.
11.189 (*a*) 3.21 ft/s^2 c 22.4°. (*b*) 6.43 ft/s^2 c 22.4°.
11.190 15.95 ft/s^2.
11.191 (*a*) 23.4 ft/s. (*b*) 103.2 ft.

CHAPTER 12

12.1 (*a*) 844 lb. (*b*) 26.2 slugs.
12.2 (*a*) 0°: 4.987 lb, 45°: 5.000 lb, 90°: 5.013 lb.
(*b*) 5.000 lb. (*c*) 0.1554 lb · s^2/ft.
12.3 2.84 × 10^6 kg · m/s.
12.4 (*a*) 66.8 N. (*b*) Load indicated = 73.6 N, *m* = 6.81 kg.
12.5 0.242 mi.
12.6 (*a*) 20.0 ft/s. (*b*) 0.0621.
12.7 0.414 m/s^2 c 15°.
12.8 (*a*) 110.5 km/h. (*b*) 85.6 km/h. (*c*) 69.9 km/h.
12.9 (*a*) 40.1 m. (*b*) 47.0 m.
12.10 (*a*) 2.22 s. (*b*) 3.32 m.
12.11 51.0 m.
12.12 (*a*) 234 m. (*b*) 3.33 kN (tension).
12.15 (*a*) (1): 10.73 ft/s^2 w, (2): 16.10 ft/s^2 w, (3): 0.749 ft/s^2 w.
(*b*) (1): 14.65 ft/s w, (2): 17.94 ft/s w, (3): 3.87 ft/s w.
(*c*) (1): 1.864 s, (2): 1.242 s, (3): 26.7 s.
12.16 $\mathbf{a}_A = 0.997$ ft/s^2 a 15°, $\mathbf{a}_B = 1.619$ ft/s^2 a 15°.
12.17 (*a*) 765 lb. (*b*) 1016 lb.
12.18 (*a*) 0.986 m/s^2 b 25°. (*b*) 51.7 N.
12.19 (*a*) 1.794 m/s^2 b 25°. (*b*) 58.2 N.
12.20 0.321 m y.
12.23 $\mathbf{a}_1 = 19.53$ m/s^2 a 65°, $\mathbf{a}_2 = 4.24$ m/s^2 d 65°.
12.24 1.598 km.
12.25 $0.347\ m_0v_0^2/F_0$.
12.26 $x = \overline{Pt/K} - \overline{kv/m}$.
12.27 2 $\overline{k/m}$ (2 $\overline{l^2 + x_0^2} - l$)
12.28 (*a*) 10.00 N. (*b*) 103.1 N.
12.29 (*a*) 8.94 ft/s^2 z, 18.06 lb.
(*b*) 12.38 ft/s^2 z, 15.38 lb. (*c*) Same as (*b*).
12.30 (*a*) 33.6 N. (*b*) $\mathbf{a}_A = 4.76$ m/s^2 y,
$\mathbf{a}_B = 3.08$ m/s^2 w, $\mathbf{a}_C = 1.401$ m/s^2 z.
12.31 (*a*) 2.43 lb. (*b*) $\mathbf{a}_A = 3.14$ ft/s^2 y,
$\mathbf{a}_B = 0.881$ m/s^2 y, $\mathbf{a}_C = 5.41$ m/s^2 w.
12.34 (*a*) 2.80 m/s^2 z. (*b*) 8.32 m/s^2 b 25°.
12.35 (*a*) 5.94 m/s^2 c 75.6°. (*b*) 3.74 m/s c 20°.
12.36 (*a*) 49.9°. (*b*) 6.85 N.
12.37 (*a*) 80.4 N. (*b*) 2.30 m/s.
12.38 3.47 m/s.
12.39 3.01 m/s ≤ v ≤ 3.96 m/s.
12.40 3.01 m/s ≤ v ≤ 3.85 m/s.
12.42 9.00 ft/s < v_C < 12.31 ft/s.
12.43 2.42 ft/s < v < 13.85 ft/s.
12.44 (*a*) 122.2 lb. (*b*) 145.6 lb.
12.45 (*a*) 668 ft. (*b*) 120.0 lb x.
12.46 (*a*) 131.7 N. (*b*) 88.4 N.
12.47 (*a*) 4.63 m/s^2. (*b*) 1.962 m/s^2. (*c*) 0.1842 m.s^2.
12.48 24.1° < u < 155.9°.
12.49 (*a*) 2.91 N. (*b*) 13.09°.
12.50 1126 N b 25.6°.
12.51 (*a*) 12.19 m/s. (*b*) 2290 N.
12.53 (*a*) 0.1858 *W*. (*b*) 10.28°.
12.55 7.67 m/s.
12.56 (*a*) 12.00 m/s. (*b*) 2.05 × 10^{-3} N.
12.57 0.236.
12.58 468 mm.
12.61 0.400.
12.62 (*a*) 0.1834. (*b*) left: 10.39°, right 169.6°.
12.63 (*a*) 2.98 ft/s. (*b*) left: 19.29°, right 160.7°.
12.64 $d = eVlL/mdv_0^2$.
12.65 $d/l > 1.054$ 2 $\overline{eV/mv_0^2}$
12.66 (*a*) $F_r = -13.15$ lb, $F_u = 0.520$ lb.; u = 0.
(*b*) $F_r = -2.04$ lb, $F_u = 0.938$ lb; u = 180°.
12.67 (*a*) $F_r = -0.258$ lb, $F_u = -0.0504$ lb.; u = 0.
(*b*) $F_r = -0.00618$ lb, $F_u = 0.278$ lb; u = −54.7°.
12.68 2.00 s.
12.69 (*a*) 72.0 m/s^2 radically outward. (*b*) 1.250 N.
12.71 (*a*) 126.6 N. (*b*) 5.48 m/s^2 y. (*c*) 4.75 m/s^2 w.
12.72 (*a*) 142.7 N. (*b*) 6.18 m/s^2 y. (*c*) 4.10 m/s^2 w.
12.74 $v_r = v_0$ sin 2u/ 1 $\overline{\cos 2u}$, $v_u = v_0$ 1 $\overline{\cos 2u}$.
12.77 (*a*) 0. (*b*) 8m v_0^2/r_0.
12.78 413 × 10^{21} lb · s^2/ft.
12.79 383 × 10^3 km, 238 × 10^3 mi.
12.80 (*a*) 35 800 km, 22 200 mi. (*b*) 3.07 km/s, 10.09 × 10^3 ft/s.
12.81 (*b*) 24.8 m/s^2.
12.82 (*a*) 1.998 × 10^{30} kg. (*b*) 276 m/s^2.
12.85 (*a*) 1684 N. (*b*) 2510 km. (*c*) 1.620 m/s^2.
12.86 (*a*) 1551 m/s. (*b*) −15.8 m/s.
12.87 5000 m/s.
12.88 (*a*) 5280 ft/s. (*b*) 8000 ft/s.

12.89 53 ft/s.
12.90 (*a*) $(a_A)_r = (a_A)_u = 0$. (*b*) 38.4 m/s^2. (*c*) 0.800 m/s.
12.91 (*a*) $(a_B)_r = (a_B)_u = 0$. (*b*) 61.4 ft/s^2. (*c*) 2.98 ft/s.
12.100 (*a*) 10.13 km/s. (*b*) 2.97 km/s.
12.101 1.147.
12.103 $1\overline{2/(2 + a)}$.
12.104 (*a*) 8.00×10^3 m/s. (*b*) 127 m/s.
12.107 (*a*) 52.4×10^3 ft/s. (*b*) *A*: 1318 ft/s, *B′*: 3900 ft/s.
12.108 5.31×10^9 km.
12.109 91.8×10^3 yr.
12.112 4.95 h.
12.113 50 min 55 s.
12.114 $\cos^{-1}[(1 - nb^2)/(1 - b^2)]$.
12.115 (*a*) 4.00 km/s. (*b*) 0.684.
12.124 (*a*) 20.5 ft/s^2 d 30°. (*b*) 17.75 ft/s^2 y.
12.125 (*a*) 1.088 ft/s^2 z. (*b*) 233 lb.
12.126 (*a*) 5.79 m/s^2. (*b*) 2.45 m/s^2. (*c*) 0.230 m/s^2.
12.127 (*a*) $F_r = (5.76\text{ N}) \tan^2 u \sec u$, $F_u = (5.76\text{ N}) \tan u \sec u$.
(*b*) $\mathbf{P} = (5.76\text{ N}) \tan u \sec^2 u$ h u, $\mathbf{Q} = (5.76\text{ N}) \tan^2 u \sec^2 u$ y.
12.128 (*a*) 0.454, down. (*b*) 0.1796 down. (*c*) 0.218, up.
12.129 (*a*) 539 N. (*b*) 47.1 m.
12.132 106.1.
12.133 (*a*) 0.500 m, 0. (*b*) 0.270 m, −84.1 N.

CHAPTER 13

13.1 10.11 GJ.
13.2 (*a*) 140.1 ft · lb, 140.1 ft. (*b*) 140.1 ft · lb, 850 ft.
13.5 10.51 ft/s.
13.6 9.53 ft.
13.7 (*a*) 112.2 km/h. (*b*) 91.6 km/h.
13.8 (*a*) 15.34 m/s. (*b*) 59.9 m/s.
13.9 (*a*) 8.70 m. (*b*) 4.94 m/s d 15°.
13.11 6.71 m.
13.12 (*a*) 2.90 m/s. (*b*) 0.893 m.
13.15 (*a*) 57.8 m. (*b*) 154 N y.
13.16 (*a*) 7.41 kN. (*b*) 5.56 kN (tension)
13.17 (*a*) 124.1 ft. (*b*) *A* to *B*: 19.38 kips (tension);
B to *C*: 8.62 kips (tension).
13.18 (*a*) 279 ft. (*b*) *A* to *B*: 19.38 kips (compression);
B to *C*: 8.62 kips (compression).
13.19 (*a*) 46.0 ft · lb.
(*b*) *A*: 19.76 lb; *B*: 12.10 lb.
13.20 (*a*) 7.43 ft/s. (*b*) 0.800 ft.
13.23 (*a*) 1.218 m/s z. (*b*) 91.0 N.
13.24 1.190 m/s.
13.25 (*a*) 11.35 ft/s d 23.6°.
(*b*) 16.05 ft/s d 23.6°.
13.26 (*a*) 3.29 m/s. (*b*) 1.533 m.
13.27 (*a*) 3.29 m/s. (*b*) 1.472 m.
13.28 (*a*) 13.63 in. (*b*) 8.57 in.
13.29 (*a*) 0.750 in. w. (*b*) 8.51 in/s w.
13.30 (*a*) 0.597 m/s. (*b*) 0.617 m/s.
13.32 $0.759\,1\overline{pAa/m}$.
13.33 (*a*) 13.43 ft. (*b*) 386 ft/s^2.
13.34 *A*: 5.37 in.; *B*: 7.21 in.
13.36 $1/[1 - (v_0^2 - v^2)/2g_m R_m]$.
13.37 (*a*) 0.0314%. (*b*) 25.3%.
13.38 364 m.
13.39 14.00°.
13.40 (*a*) $1\overline{3gl}$. (*b*) $1\overline{2gl}$.
13.41 (*a*) 1.500 *W*. (*b*) 2.50 *W*.
13.44 2.30 m/s.
13.45 (*a*) 27.4°. (*b*) 3.81 ft.
13.46 (*a*) 57.2 kW. (*b*) 269 kW.
13.47 (*a*) 2.75 kW. (*b*) 3.35 kW.
13.48 14.80 kN.
13.51 (*a*) 109.0 kW, 146.2 hp.
(*b*) 530 kW, 711 hp.
13.52 (*a*) 375 kW. (*b*) 5.79 km/h.
13.54 (*a*) 8.00 hp. (*b*) 7.91 hp.
13.55 (*a*) $k_1k_2/(k_1 + k_2)$. (*b*) $k_1 + k_2$.
13.56 (*a*) $v_0\,1\overline{(k_1 + k_2)/k_1k_2}$. (*b*) $v_0\,1\overline{m/(k_1 + k_2)}$.
13.57 (*a*) 4.22 m/s. (*b*) 4.42 m/s.
13.58 (*a*) 11.66 ft/s. (*b*) 15.01 ft/s.
13.59 9.35 ft/s (left and right).
13.62 (*a*) 533 lb/ft. (*b*) 37.0 ft.
13.64 (*a*) 2.48 m/s z. (*b*) 1.732 m/s x.
13.65 (*a*) 3.31 m/s. (*b*) 3.90 m/s.
13.66 (*a*) 43.5°. (*b*) 8.02 ft/s w.
13.67 6.20 ft/s.
13.68 0.269 m.
13.69 0.1744 m.
13.70 731 N.
13.71 (max) 5520 N at *D*; (min) 731 N just above *B*.
13.72 14.34 ft/s, 13.77 lb x.
13.74 Loop 1: (*a*) 25.1 ft/s. (*b*) 1.500 lb z.
Loop 2: (*a*) 24.1 ft/s. (*b*) 1.000 lb
13.76 Loop 1: (*a*) $1\overline{5gr}$. (*b*) 3 W y.
Loop 2: (*a*) $1\overline{4gr}$. (*b*) 2 W y.
13.77 0.488 m.
13.78 (*a*) cot w = 0.243 (12 − *y*)
(*b*) 60.0 lb; $u_x = 85.7°$, $u_y = 71.6°$, $u_z = 161.1°$.
13.80 $V = -\ln xyz$.
13.81 (*a*) $pka^2/4$. (*b*) 0.
13.82 (*a*) $P_x = x/R$, $P_y = y/R$, $P_z = z/R$, where $R = (x^2 + y^2 + z^2)^{1/2}$
(*b*) $U_{OABD} = -\Delta V_{OD} = a\,1\overline{3}$.
13.85 (*a*) 62.5 MJ/kg. (*b*) 11.18 km/s.
13.86 (*a*) 9.56 km/s. (*b*) 2.39 km/s.
13.87 (*a*) 50.1×10^9 ft · lb. (*b*) 115.9×10^9 ft · lb.
13.88 (*a*) 942×10^3 ft · lb/lb. (*b*) 450×10^3 ft · lb/lb.
13.89 25.1 Mm/h.
13.90 6.48 km/s.
13.93 $v_r = \pm 3.87$ m/s, $v_u = 1.000$ m/s.
13.94 (*a*) 0.720 m. (*b*) 0.834 m/s.
13.95 (*a*) $v_r = 3.71$ ft/s, $v_u = 3.00$ ft/s, $v = 4.77$ ft/s.
(*b*) $v_r = 0$, $v_u = 1.129$ ft/s, $v = 1.129$ ft/s.
13.96 (*a*) 14.36 ft/s. (*b*) 1.225 ft.
13.97 (*a*) 4.14 ft/s. (*b*) 16.58 ft/s.
13.100 27.6×10^3 km/h.
13.101 (*a*) 7960 ft/s. (*b*) 4820 ft/s.
13.102 (*a*) 16 800 ft/s. (*b*) 32 700 ft/s.
13.103 14.20 km/s.
13.106 (*a*) 7.35 km/s. (*b*) 45.0°.
13.107 68.9°.
13.108 $r_{max} = r_0(1 + \sin a)$, $r_{min} = (1 - \sin a)r_0$.
13.109 1555 m/s, 79.3°.
13.110 (*a*) 11.32×10^3 ft/s. (*b*) 13.68×10^3 ft/s.
13.111 30.9×10^3 ft/s, 58.9°.
13.115 (*b*) $v_{esc}\,1\overline{a/(1 + a)} < v_0 < v_{esc}\,1\overline{(1 + a)/(2 + a)}$.
13.119 4 min 19 s.
13.120 (*a*) 3.64 s. (*b*) 27.3 s.
13.121 17.86 lb.
13.123 6.77 s.

13.124 (*a*) 2280 lb. (*b*) 3.00 s.
13.125 0.278.
13.126 (*a*) 11.42 s. (*b*) −(125.5 m/s)**j** − (194.5 m/s)**k.**
13.129 (*a*) 5.64 s.
(*b*) *AB*: 19 390 lb (tension); *BC*: 8620 lb (tension).
13.130 (*a*) 12.69 s. (*b*) *AB*: 19 390 lb (compression);
BC: 8620 lb (compression).
13.131 (*a*) 19.60 s. (*b*) 10.20 kN (compression)
13.132 (*a*) 0.549 s. (*b*) 56.8 N.
13.134 (*a*) 3730 lb. (*b*) 7450 lb.
13.136 223 Mpa.
13.138 (*a*) 7.00 s. (*b*) 10.99 ft/s. (*c*) 13.49 s.
13.139 76.9 lb.
13.140 1.449 kips.
13.141 6.21 *W*.
13.142 2.68 kN.
13.145 (*a*) 0.833 km/h z . (*b*) 0.190 s.
13.146 (*a*) car *A*. (*b*) 115. 2 km/h.
13.147 65.0 kN.
13.148 (*a*) 9.32 ft · lb, 0.932 lb · s.
(*b*) 7.99 ft · lb, 0.799 lb · s.
13.149 497 ft/s.
13.150 (*a*) 2.80 ft/s z . (*b*) 0.229 ft/s z .
13.151 (*a*) 1.694 m/s w. (*b*) 0.1619 J.
13.152 1.650 m/s.
13.155 (*a*) $v_A = 0.594$ m/s z , $v_B = 1.156$ m/s y . (*b*) 2.99 J.
13.156 $(1 - e^2)mv^2$.
13.157 $0.728 \leq e \leq 0.762$.
13.158 (*a*) 3.00 lb. (*b*) 2.00 lb $\leq W_B \leq$ 6.00 lb.
13.161 (*a*) $v_A' = v_0(1 - e)/2$, $v_B' = v_0(1 + e)/2$.
(*b*) $v_C' = v_0(1 + e)^2/4$, $v_B'' = v_0(1 - e^2)/4$.
(*c*) $v_n' = v_0(1 + e)^{(n-1)}/2^{(n-1)}$. (*d*) 0.815 v_0.
13.163 0.294 m/s z .
13.164 $\mathbf{v}_A' = 0.711\ v_0$ a 39.3°, $\mathbf{v}_B' = 0.636\ v_0$ c 45°.
13.165 (*a*) 70.0°. (*b*) 0.972 ft/s y .
13.166 $\mathbf{v}_A' = 6.37$ m/s d 77.2°, $\mathbf{v}_B' = 1.802$ m/s a 40°.
13.167 $\mathbf{v}_A' = 1.322$ m/s d 70.9°, $\mathbf{v}_B' = 3.85$ m/s c 27.0°.
13.168 (*a*) $\mathbf{v}_A = 0.878\ v_0$ c 24.2°, $v_B = 0.412\ v_0$ a 61.0°.
13.169 0.837.
13.172 0.156 m.
13.173 (*a*) 22.5°. (*b*) 21.3°.
13.174 (*a*) 20.6 mi/h. (*b*) 0.203.
13.175 (*a*) 0.294 m. (*b*) 54.4 mm.
13.176 (*a*) 0.324. (*b*) 14.30 ft/s.
13.177 (*a*) 2.90 m/s. (*b*) 100.5 J.
13.179 (*a*) 0.0240 m. (*b*) 817 N/m.
13.180 (*a*) 1.000. (*b*) 0.0667 m. (*c*) 0.0762 m.
13.182 (*a*) $\mathbf{v}_A' = 0$, $\mathbf{v}_B' = 0$.
(*b*) $\mathbf{v}_A' = 1.201$ m/s z , $\mathbf{v}_B' = 0.400$ m/s y .
13.183 (*a*) 401 mm. (*b*) 4.10 N · s.
13.184 (*a*) $\mathbf{v}_A' = 1.002$ ft/s x, $\mathbf{v}_B' = 0.695$ ft/s c 10.4°.
(*b*) 0.274 in.
13.185 3.47 in.
13.186 (*a*) 0.923. (*b*) 1.278 m.
13.188 (*a*) $\mathbf{v}_A' = 2.36$ ft/s b 83.8°, $\mathbf{v}_B' = 3.23$ ft/s y . (*b*) 1.97 in.
13.190 102.6 mi/h.
13.191 1.688 ft · lb.
13.194 12 990 ft/s.
13.195 (*a*) 13.31 N y . (*b*) 4.49 N w. (*c*) 13.31 N z .
13.197 (*a*) 217 mm. (*b*) 69.1 mm.
13.198 (*a*) $v_A' = v_B' = v_C' = 1.368$ m/s. (*b*) 0.668 m. (*c*) 1.049 m.
13.200 (*a*) 2.94 m/s. (*b*) 16.14 N.

CHAPTER 14

14.1 (*a*) 4.46 m/s z . (*b*) 0.409 m/s z .
14.2 (*a*) 306 m/s z . (*b*) 0.409 m/s z .
14.3 0.792 ft/s y .
14.4 (*a*) 0.800 oz. (*b*) 900 ft/s y .
14.7 (*a*) 3.79 km/h y , 2.77 km/h y .
(*b*) 5.54 km/h y , 2.77 km/h y .
(*c*) 5.54 km/h y , 3.60 km/h y .
14.8 (*a*) 0.901 m/s y . (*b*) 0.807 m/s y .
14.9 −(4.80 kg · m²/s)**j** + (9.60 kg · m²/s)**k**.
14.10 (*a*) (1.867 m)**i** + (1.533 m)**j** + (0.667 m)**k**.
(*b*) (12.00 kg · m/s)**i** + (28.0 kg · m/s)**j** + (14.00 kg · m/s)**k**.
(*c*) −(2.80 kg · m²/s)**i** + (13.33 kg · m²/s)**j** − (24.3 kg · m²/s)**k**.
14.11 (*a*) $v_x = -0.750$ ft/s, $v_z = 0.438$ ft/s. (*b*) −(3.39 ft · lb · s)**k**.
14.12 (*a*) $v_z = 7.25$ ft/s, $v_x = 8.33$ ft/s. (*b*) −(4.51 ft · lb · s)**k**
(*b*) −(26.0 kg · m/s)**i** + (14.00 kg · m/s)**j** + (14.00 kg · m/s)**k**.
(*c*) −(29.5 kg · m²/s)**i** − (16.75 kg · m²/s)**j** + (3.20 kg · m²/s)**k**.
14.15 (114.4 m)**i** −(76.1 m)**j** + (8.75 m)**k**.
14.16 (1180 m)**i** + (140 m)**j** + (155 m)**k**.
14.19 $x = 45.2$ ft, $y = 54.5$ ft.
14.20 (*a*) 2.00 s. (*b*) 92.8 mi/h.
14.21 (81.5 ft)**i** + (351 ft)**k**.
14.22 (*a*) 8.00 ft/s y . (*b*) 36.6°, $v_C = 10.39$ ft/s, $v_D = 8.72$ ft/s.
14.24 $v_A = 431$ m/s, $v_B = 395$ m/s, $v_C = 528$ m/s.
14.25 $v_A = 646$ m/s, $v_B = 789$ m/s, $v_C = 176$ m/s.
14.26 $v_A = 919$ m/s, $v_B = 717$ m/s, $v_C = 619$ m/s.
14.31 friction: 2.97 J, first impact: 3007 J, second impact: 24.3 J.
14.32 (*a*) 1116 ft · lb. (*b*) 623 ft · lb.
14.33 (woman) 382 ft · lb, (man) 447 ft · lb.
14.34 (*A* hits *B*) 4.86 J, (*B* hits *C*) 3.28 J, (*A* hits *B* again) 0.688 J.
14.35 (*b*) $E_A = 180.0$ kJ, $E_B = 320$ kJ.
14.37 (*a*) $\mathbf{v}_B = \dfrac{m_A v_0}{m_A + m_B}$ y . (*b*) $h = \dfrac{m_A}{m_A + m_B}\dfrac{v_0^2}{2g}$.
14.38 $\mathbf{v}_A = 4.11$ m/s a 46.9°, $\mathbf{v}_B = 17.39$ m/s c 16.7°.
14.39 (*a*) $\mathbf{v}_{B/A} = 11.59$ ft/s d 30°. (*b*) $\mathbf{v}_A = 3.76$ ft/s y .
14.40 $\mathbf{v}_A = 3.11$ ft/s z , $\mathbf{v}_B = 4.66$ ft/s y .
14.41 $v_A = 10.61$ ft/s, $v_B = 5.30$ ft/s, $v_C = 9.19$ ft/s.
14.42 $v_A = 7.50$ ft/s, $v_B = 9.19$ ft/s, $v_C = 9.19$ ft/s.
14.45 (*a*) **L** = (8.75 kg · m/s)**i**, $\mathbf{H}_G$ = −(0.500 kg · m²/s)**k**.
(*b*) $\mathbf{v}_A' = (1.500$ m/s)**i**, $\mathbf{v}_B' = (5.00$ m/s)**i**.
14.46 (200 ft/s)**i** + (172 ft/s)**j** + (1560 ft/s)**k**.
14.47 (*a*) $v_C = 11.00$ ft/s, $v_D = 5.50$ ft/s. (*b*) 0.786.
14.48 $x = 181.7$ mm, $y = 0$, $z = 139.4$ mm.
14.51 (*a*) $\mathbf{v}_B = 2.40$ m/s a 53.1°, $\mathbf{v}_C = 2.56$ m/s y , (*b*) $c = 1.059$ m
14.52 (*a*) $\mathbf{v}_A = 2.40$ m/s w, $\mathbf{v}_B = 3.00$ m/s a 53.1°, (*b*) $a = 1.864$ m
14.55 (*a*) $\mathbf{v}_A = 2.25$ ft x, $\mathbf{v}_B = 2.25$ ft/s w, $\mathbf{v}_c = 3.90$ ft/s y . (*b*) 11.1 in.
14.56 (*a*) 2.00 ft/s y . (*b*) 0.760 ft. (*c*) 5.29 rad/s i.
14.57 rv_1^2 $(A_1 - A_2)$.
14.58 r$A_2\, v_2^2$ − r$A_1\, v_1^2$ cos u.
14.59 64.6 lb.
14.60 18.2 ft/s.
14.61 (*a*) 14.8 kN. (*b*) 27.7 kN.
14.62 90.6 N z .
14.64 $D_x = 329$ N, $D_y = 0$, $C_x = -203$ N, $C_y = 271$ N.
14.66 (*a*) 61.1 m/s. (*b*) $R_x = -38.8$ N, $R_y = 44.7$ N.
14.67 $C_x = 90.0$ N, $C_y = 2360$ N, $D_x = 0$, $D_y = 2900$ N.
14.68 **A** = qv_0 z , **B** = 1$\overline{2gh}$ b 30°.
14.69 100 kg/s.
14.70 7580 lb.
14.71 33.6 kN z .
14.72 7180 lb.

14.74 (*a*) 9690 lb, 3.38 ft. (*b*) 6960 lb, 9.43 ft.
14.76 (*a*) 3.03 m/s^2 a 18°. (*b*) 922 km/h.
14.77 (*a*) 30.6 m/s. (*b*) 96.1 m^3/s. (*c*) 55 100 N · m/s.
14.78 (*a*) 3.23 MW. (*b*) 0.464.
14.79 213 m.
14.80 (*a*) 15 450 hp. (*b*) 28 060 hp. (*c*) 0.551.
14.83 (*b*) friction from gravel slipping on belt.
14.86 (*a*) $m(v^2 + gy)/l$. (*b*) $\mathbf{R} = mg(1 - y/l)$
14.87 (*a*) mgy/l. (*b*) $m[g(l - y) + v^2]/l$ x.
14.88 $1\overline{gh}$ tan h($1\overline{gh}\, t/L$).
14.89 10.10 ft/s.
14.90 4.75 ft/s.
14.91 533 kg/s.
14.94 (*a*) 90.0 m/s^2. (*b*) 35.9 × 10^3 km/h.
14.95 7930 m/s.
14.96 (*a*) 1800 m/s. (*b*) 9240 m/s.
14.99 19.07 mi.
14.100 (*a*) 119.3 mi. (*b*) 14 660 mi/h.
14.101 186.8 km/h.
14.102 (*a*) 31.2 km. (*b*) 197.5 km.
14.106 (*a*) 1.595 m/s. (*b*) 0.370 m.
14.107 (*a*) 5.20 km/h y. (*b*) 4.00 km/h y.
14.108 (*a*) 6.05 ft/s. (*b*) 6.81 ft/s.
14.109 (*a*) 3.11 m/s y. (*b*) 1.356 m.
14.110 $\mathbf{v}_A$ = 15.38 ft/s y, $\mathbf{v}_B$ = 5.13 ft/s z.
14.112 $\mathbf{A}_x$ = 55.5 lb y, $\mathbf{A}_y$ = 20.2 lb w, $\mathbf{m}_A$ = 41.4 lb · ft i.
14.114 (*a*) $m_0 + qt_L = m_0 e^{ql/mv_0}$. (*b*) $v_0 e^{-ql/mv_0}$.
14.115 414 rpm.
14.116 Case 1: (*a*) 0.333 *g* w. (*b*) 0.817 $2\overline{gl}$.
Case 2: (*a*) gy/l w. (*b*) $2\overline{gl}$.

CHAPTER 15

15.1 (*a*) 29.6 rad/s. (*b*) 32.2 rev.
15.2 (*a*) 0, 0, 0. (*b*) 6.00 rad, 4.71 rad/s, −3.70 rad/s^2.
15.3 (*a*) 0, 0.1000 rad/s, −0.250 rad/s^2.
(*b*) 0.211 rad, 0.0472 rad/s, −0.01181 rad/s^2.
(*c*) 0.400 rad, 0, 0.
15.4 (*a*) −3.01 rad/s^2. (*b*) 13 800 rev.
15.5 (*a*) 150 rev. (*b*) 2100 rev.
15.6 (*a*) 0.855 rad/s. (*b*) 3.71°.
15.9 (*a*) 12.73 rev. (*b*) ∞. (*c*) 18.42 s.
15.10 −(0.450 m/s)**i** − (1.200 m/s)**j** + (1.500 m/s)**k**, (12.60 m/s^2)**i** + (7.65 m/s^2)**j** + (9.90 m/s^2)**k**.
15.11 (0.750 m/s)**i** + (1.500 m/s)**k**, (12.75 m/s^2)**i** + (11.25 m/s^2)**j** + (3.00 m/s^2)**k**.
15.12 −(1.333 ft/s)**i** − (4.67 ft/s)**j** − (2.33 ft/s)**k**, (28.0 ft/s^2)**i** + (11.00 ft/s^2)**j** −(38.0 ft/s^2)**k**.
15.13 −(1.333 ft/s)**i** + (2.33 ft/s)**k**.
−(6.67 ft/s^2)**i** − (21.7 ft/s^2)**j** −(10.00 ft/s^2)**k**.
15.16 66 700 mi/h, 19.47 × 10^{-3} ft/s^2.
15.17 (*a*) 1525 ft/s, 0.1112 ft/s^2. (*b*) 1168 ft/s, 0.0852 ft/s^2. (*c*) 0, 0.
15.18 (*a*) 2.50 rad/s l, 1.500 rad/s^2 i. (*b*) 771 mm/s^2 c 76.5°.
15.19 12.00 rad/s^2 l or 12.00 rad/s^2 i.
15.22 left: 3.49 s; middle: 6.98 s; right: 13.96 s.
15.23 (*a*) 0.500 ft/s y, 1.500 ft/s^2 z. (*b*) 4.24 ft/s^2 c 45°.
15.24 (*a*) 300 rpm l, 100 rpm i. (*b*) $\mathbf{a}_B$ = 1974 in/s^2 z, $\mathbf{a}_C$ = 658 in/s^2 y.
15.25 (*a*) *A*: 15.00 rad/s l; *B*: 7.50 rad/s i.
(*b*) *A*: 75.0 ft/s^2 x; *B*: 37.5 ft/s^2 w.
15.26 (*a*) *C*: 120 rpm; *B*: 275 rpm.
(*b*) *A*: 23.7 m/s^2 x; *B*: 19.90 m/s^2 w.
15.27 (*a*) 10.00 rad/s. (*b*) A: 7.50 m/s^2; B: 3.00 m/s^2 w. (*c*) 4.00 m/s^2 w.
15.28 (*a*) 0.400 rad/s^2 i. (*b*) 1.528 rev.
15.29 (*a*) 2.75 rev. (*b*) 1.710 m/s w, 3.11 m w. (*c*) 849 m/s^2 a 32.0°.
15.30 (*a*) 1.152 m/s x, 2.30 m x. (*b*) 1.728 m/s w, 3.46 m w.
15.31 (*a*) 15.28 rev. (*b*) 10.14 s.
15.32 A_A = 5.41 rad/s^2 l, A_B = 1.466 rad/s^2 l.
15.33 (*a*) 10.39 s. (*b*) V_A = 413 rpm i, V_B = 248 rpm l.
15.36 $b\mathsf{v}_0^2/2\mathsf{p}$ y.
15.37 $bv^2/2\mathsf{p}r^3$ i.
15.38 $\mathbf{v}_B$ = 140.8 ft/s y, $\mathbf{v}_C$ = 0, $\mathbf{v}_0$ = 136.0 ft/s a 15°,
$\mathbf{v}_E$ = 99.6 ft/s c 45°.
15.39 (*a*) 0.378 rad/s i. (*b*) 6.42 in/s x.
15.40 (*a*) 1.173 rad/s l. (*b*) 0.998 m/s a 25°.
15.41 (*a*) 3.62 rad/s l. (*b*) 1.963 m/s w.
15.44 (*a*) 2.00 rad/s i. (*b*) (120 mm/s)**i** + (660 mm/s)**j**.
15.45 (*a*) −(240 mm/s)**i** + (300 mm/s)**j**.
(*b*) x = 150 mm, y = −60 mm.
15.47 (*a*) −(4.00 rad/s)**k** or 4.00 rad/s i.
(*b*) −(4.00 in/s)**i**.
15.48 (*a*) $\mathsf{V}_B = \mathsf{V}_C = \mathsf{V}_D = \frac{1}{2}\mathsf{V}_A$ l. (*b*) V_S = 0.25 V_A i.
15.49 (*a*) $\mathsf{V}_B = \mathsf{V}_C = \mathsf{V}_D$ = 150 rpm i. (*b*) V_S = 195 rpm i.
15.50 (*a*) 48.0 rad/s i. (*b*) 3.39 m/s a 45°.
15.51 (*a*) 5.65 m/s x. (*b*) 9000 rpm, (*c*) 1500.
15.53 (*a*) 200 rad/s l. (*b*) 24.0 rad/s i.
15.55 (*a*) (6.00 rad/s)**k** or 6.00 rad/s l.
(*b*) (360 mm/s)**i** −(672 mm/s)**j** or 762 mm/s c 61.8°.
15.56 (*a*) 540 mm/s y. (*b*) 457 mm/s b 61.8°.
15.57 (*a*) $r\mathsf{V}_D$/cos u y. (*b*) V_D tan^2 u l.
15.58 (*a*) 2.90 rad/s i. (*b*) 0.631 rad/s l.
15.59 (*a*) 1.500 rad/s l. (*b*) 18.00 in/s x. (*c*) 11.25 in/s b 53.1°.
15.61 (*a*) $\mathbf{v}_P$ = 0, V_{BD} = 39.3 rad/s l.
(*b*) $\mathbf{v}_P$ = 6.28 m/s w, V_{BD} = 0.
15.62 $\mathbf{v}_P$ = 6.52 m/s w, V_{BD} = 20.8 rad/s l.
15.63 (*a*) 12.00 rad/s l. (*b*) 3.90 m/s d 67.4°.
15.64 V_{DE} = 2.55 rad/s i, V_{BD} = 0.955 rad/s l.
15.65 V_{DE} = 6.40 rad/s i, V_{BD} = 5.20 rad/s i.
15.68 (*a*) 3.33 rad/s l. (*b*) 2.00 m/s c 56.3°.
15.69 (*a*) 1.500 m. (*b*) 5.00 m/s w.
15.70 14.76 in/s y.
15.71 (*a*) 338 mm/s z, 0. (*b*) 710 mm/s z, 2.37 rad/s i.
15.72 $(1 - r_A/r_C)\mathsf{V}_{ABC}$.
15.74 (*a*) 1.00 ft to the right of *A*. (*b*) 4.00 in/s x.
15.75 x = 0, z = 9.34 ft.
15.76 (*a*) 3.00 rad/s l. (*b*) 300 mm/s z. (*c*) 180.0 mm/s (wound).
15.77 (*a*) 3.00 rad/s i. (*b*) 180 mm/s y. (*c*) 300 mm/s (unwound).
15.78 (*a*) 50 mm to the right of the axle.
(*b*) $\mathbf{v}_B$ = 750 mm/s w, $\mathbf{v}_D$ = 1.950 m/s x.
15.79 (*a*) 25 mm to the right of 0. (*b*) 420 mm/s x.
15.80 (*a*) *A*: 300 mm to the left of *A*.
C: 600 mm to the left of *C*.
(*b*) V_A = 4.00 rad/s i, V_C = 2.00 rad/s l.
15.82 (*a*) 0.467 rad/s l. (*b*) 3.49 ft/s a 59.2°.
15.83 (*a*) 3.08 rad/s l. (*b*) 83.3 m/s c 73.9°.
15.86 (*a*) 5.13 rad/s l. (*b*) 0.924 ft/s z. (*c*) 1.870 m/s d 34.7°.
15.87 (*a*) 4.27 rad/s i. (*b*) 1.330 m/s w. (*c*) 1.557 m/s a 34.7°.
15.88 (*a*) (v_A/l) sin b/cos (b − u). (*b*) v_A cos u/cos(b − u).
15.89 (*a*) 3.54 rad/s l. (*b*) 8.68 ft/s a 50°.
15.90 (*a*) 0.900 rad/s i. (*b*) 411 mm/s c 20.5°.
15.94 (*a*) V_{AB} = 1.920 rad/s i, V_{BD} = 1.200 rad/s i.
(*b*) 30.0 in/s d 73.7°.
15.95 (*a*) V_{AB} = 1.200 rad/s i, V_{DE} = 0.450 rad/s i.
(*b*) 5.25 in/s z.

15.96 (*a*) 5.00 rad/s. (*b*) 3.00 m/s w.
15.97 (*a*) 1260 mm/s x. (*b*) 1.250 rad/s l.
15.98 (*a*) V_{AB} = 1.177 rad/s i, V_{DE} = 2.50 rad/s i.
(*b*) 29.4 in/s z .
15.99 Space centrode: quarter circle, r = 15 in, centered at O.
Body centrode: semicircle, r = 7.5 in., centered midway between A and B.
15.100 Space centrode: lower rack.
Body centrode: circumference of gear.
15.102 V_{BD} = 0.955 rad/s i, V_{DE} = 2.55 rad/s l.
15.103 V_{BD} = 5.20 rad/s i, V_{DE} = 6.40 rad/s i.
15.105 (*a*) 0.833 rad/s^2 i, (*b*) 1.083 m/s^2 x.
15.106 $\mathbf{a}_A$ = 2.50 m/s^2 x, $\mathbf{a}_B$ = 0.100 m/s^2 x.
15.107 (*a*) 0.900 m/s^2 y . (*b*) 1.800 m/s^2 z .
15.108 (*a*) 0.600 m from A. (*b*) 0.200 m from A.
15.109 (*a*) 51.3 in/s^2 w. (*b*) 184.9 in/s^2 a 16.1°.
15.110 (*a*) 1.039 rad/s^2 i. (*b*) (2.60 ft/s^2)$\mathbf{i}$ + (4.50 ft/s^2)$\mathbf{j}$ or 5.20 ft/s^2 a 60°.
15.111 (*a*) 1430 m/s^2 w. (*b*) 1430 m/s^2 x, (*c*) 1430 m/s^2 c 60°.
15.112 (*a*) 13.35 in/s^2 d 61.0°. (*b*) 12.62 in/s^2 a 64.0°.
15.113 $\mathbf{a}_A$ = 56.6 in/s^2 b 58.0°, $\mathbf{a}_B$ = 80.0 in/s^2 x,
$\mathbf{a}_C$ = 172.2 in/s^2 b 25.8°.
15.114 $\mathbf{a}_A$ = 48.0 in/s^2 x, $\mathbf{a}_B$ = 85.4 in/s^2 b 69.4°.
$\mathbf{a}_C$ = 82.8 in/s^2 d 65.0°.
15.115 A_A = 96.0 rad/s^2 l, $\mathbf{a}_A$ = 2.40 m/s^2 z .
A_B = 48.0 rad/s^2 l, $\mathbf{a}_B$ = 1.200 m/s^2 z .
15.118 (*a*) 92.5 in/s^2. (*b*) 278 in/s^2.
15.120 148.3 m/s^2 w.
15.121 296 m/s^2 x.
15.122 $\mathbf{a}_D$ = 1558 m/s^2 c 45°. $\mathbf{a}_E$ = 337 m/s^2 a 45°.
15.124 (*a*) 242 in/s^2 z . (*b*) 403 in/s^2 d 72.5°.
15.125 694 in/s^2 z .
15.127 1.745 m/s^2 d 68.2°.
15.128 1.296 m/s^2 z .
15.129 (*a*) 228 rad/s^2 l. (*b*) 92.0 rad/s^2 i.
15.130 (*a*) 138.1 ft/s^2 b 78.6°. (*b*) 203 ft/s^2 a 19.5°.
15.131 (*a*) 10.75 rad/s^2 l. (*b*) 2.30 rad/s^2 l.
15.132 (*a*) 4.18 rad/s^2 i. (*b*) 2.43 rad/s^2 i.
15.133 (*a*) 8.15 rad/s^2 l. (*b*) 0.896 rad/s^2 i.
15.134 (*a*) 3.70 rad/s^2 i. (*b*) 3.70 rad/s^2 i.
15.135 (*a*) 16.53 rad/s^2 l. (*b*) 193.6 in/s^2 d 7.36°.
15.136 $\mathbf{v}_D$ = 1.382 m/s w. $\mathbf{a}_D$ = 0.695 m/s^2 w.
15.138 $\mathbf{v}_B = bV \cos u$, $a_B = bA \cos u - bV^2 \sin u$.
15.139 $\mathbf{v}_B \sin b/l \cos u$.
15.140 $(v_B \sin b/l)^2 (\sin u/\cos^3 u)$.
15.141 $v_x = v[1 - \cos (vt/r)]$. $v_y = v \sin (vt/r)$.
15.142 $V = bv_A(b^2 + x_A^2)$ l, $A = 2bx_A\, v_A^2/(b^2 + x_A^2)^2$ l.
15.143 $v_{B_x} = v_A - \mathrm{l}b^2(b^2 + x_A^2)^{3/2}$ y , $(v_B)_y = \mathrm{l}b\, x_A v_A/(b^2 + x_A{}^2)^{3/2}$ x.
15.144 $V_{BD} = bV(b + l \cos u)/(l^2 + b^2 + 2bl \cos u)$ i,
$\mathbf{v}_E = blV \sin u/(l^2 + b^2 + 2bl \cos u)$ c
$\tan^{-1}[(b \sin u/(l + b \cos u)]$
15.145 $blV^2(l^2 - b^2) \sin u/(l^2 + b^2 + 2bl \cos u)$ l.
15.146 (*a*) $(v_0/b) \sin^2 u$ i. (*b*) $(v_0\, l/b) \sin^2 u \cos u$ y ,
$(v_0 l/b) \sin^3 u$ x. (*c*) $2(v_0/b)^2 \sin^3 u \cos u$ l.
15.147 $V = v_0 \sin^2 u/r \cos u$ l, $A = (v_0/r)^2 (1 + \cos^2 u) \tan^3 u$ l.
15.148 $(v_r)_x = rV\left[\cos\dfrac{rVt}{R - r} - \cos Vt\right]$,
$(v_r)_y = rV\left[\sin\dfrac{rVt}{R - r} + \sin Vt\right]$
15.149 Path is the y axis. $\mathbf{v} = (RV \sin Vt)\mathbf{j}$,
$\mathbf{a} = (RV^2 \cos Vt)\mathbf{j}$.
15.150 2.40 m/s c 73.9°.
15.151 2.87 m/s c 44.8°.
15.152 (*a*) 1.815 rad/s i. (*b*) 16.42 in/s c 20°.
15.153 (*a*) 5.16 rad/s i. (*b*) 1.399 m/s b 60°.
15.154 (*a*) 3.81 rad/s i, 6.53 m/s a 16.26°.
(*b*) 3.00 rad/s i, 4.00 m/s y .
15.155 (*a*) 14.00 rad/s i. (*b*) 28.0 ft/s w.
15.160 15.47 m/s^2 d 77.3°.
15.161 (*a*) 5.88 × 10^{-3} ft/s^2 west. (*b*) 4.47 × 10^{-3} ft/s^2 west.
(*c*) 4.47 × 10^{-3} ft/s^2 west.
15.162 0.0234 m/s^2 west.
15.163 (*a*) 0.809 m/s c 35.5°. (*b*) 1.723 m/s^2 b 67.6°.
15.164 (*a*) 0.520 m/s c 82.6°. (*b*) 50.0 mm/s^2 b 9.8°.
15.165 (*a*) 0.520 m/s c 37.4°. (*b*) 50.0 mm/s^2 d 69.8°.
15.166 (*a*) −(1.32 m/s)$\mathbf{j}$ − (2.70 m/s)$\mathbf{k}$. (*b*) −(1.32 m/s^2)$\mathbf{j}$.
15.167 (*a*) (2.4 m/s)$\mathbf{i}$ − (2.70 m/s)$\mathbf{k}$. (*b*) (2.4 m/s^2)$\mathbf{i}$.
15.168 (1) 303 mm/s^2 y ; (2) 168.5 mm/s^2 d 57.7°.
15.169 (3) 483 mm/s^2 z ; (4) 168.5 mm/s^2 b 57.7°.
15.170 0.750 m/s a 71.3°, 2.13 m/s^2 d 61.9°.
15.171 2.79 rad/s i, 2.13 rad/s^2 i.
15.174 (*a*) 0.621 m/s^2 x. (*b*) 1.733 m/s^2 c 53.6°. (*c*) 2.62 m/s^2 d 67.6°.
15.175 (*a*) 8.09 rad/s^2 i. (*b*) 8.43 m/s^2 d 16.26°.
15.176 7.86 rad/s l, 81.1 rad/s^2 l.
15.177 3.81 rad/s i, 81.4 rad/s^2 i.
15.178 1.526 rad/s i, 57.6 rad/s^2 i.
15.181 (*a*) 3.61 rad/s l. (*b*) 86.6 in/s a 30°. (*c*) 563 in/s^2 d 46.1°.
15.182 (*a*) 3.61 rad/s i. (*b*) 86.6 in/s d 30°. (*c*) 563 in/s^2 d 46.1°.
15.183 51.5 m/s^2 b 44.4°.
15.184 (*a*) (0.480 rad/s)$\mathbf{i}$ − (1.600 rad/s)$\mathbf{j}$ + (0.600 m/s)$\mathbf{k}$.
(*b*) (400 mm/s)$\mathbf{i}$ + (300 mm/s)$\mathbf{j}$ + (480 mm/s)$\mathbf{k}$.
15.185 (*a*) −(0.400 rad/s)$\mathbf{i}$ − (0.360 rad/s)$\mathbf{k}$.
(*b*) (100 mm/s)$\mathbf{i}$ − (90 mm/s)$\mathbf{j}$ + (120 mm/s)$\mathbf{k}$.
15.186 (*a*) (1.5 rad/s)$\mathbf{i}$ − (3.5 rad/s)$\mathbf{j}$ − (3.0 rad/s)$\mathbf{k}$.
(*b*) (640 mm/s)$\mathbf{i}$ − (360 mm/s)$\mathbf{j}$ + (740 mm/s)$\mathbf{k}$.
15.187 (*a*) (30.1 rad/s)$\mathbf{i}$ − (40.2 rad/s)$\mathbf{k}$.
(*b*) (14.4 ft/s)$\mathbf{i}$ + (10.8 ft/s)$\mathbf{k}$.
15.188 (118.4 rad/s^2)$\mathbf{i}$.
15.189 (230 rad/s^2)$\mathbf{i}$ − (2.5 rad/s^2)$\mathbf{k}$.
15.190 (*a*) −(2260 rad/s^2)$\mathbf{k}$. (*b*) (2260 rad/s^2)$\mathbf{j}$.
15.193 (*a*) −(0.600 m/s)$\mathbf{i}$ + (0.750 m/s)$\mathbf{j}$ − (0.600 m/s)$\mathbf{k}$.
(*b*) −(6.15 m/s^2)$\mathbf{i}$ − (3.00 m/s^2)$\mathbf{j}$.
15.195 (*a*) −(20.0 rad/s^2)$\mathbf{j}$. (*b*) −(4.00 ft/s^2)$\mathbf{i}$ + (10.00 ft/s^2)$\mathbf{k}$.
(*c*) −(10.25 ft/s^2)$\mathbf{j}$.
15.196 −(3.46 ft/s^2)$\mathbf{i}$ − (5.13 ft/s^2)$\mathbf{j}$ + (8.66 ft/s^2)$\mathbf{k}$.
15.197 (*a*) (8.00 rad/s^2)$\mathbf{i}$. (*b*) −(19.20 rad/s^2)$\mathbf{k}$.
(*c*) −(1.103 m/s^2)$\mathbf{i}$ − (2.005 m/s^2)$\mathbf{j}$.
15.198 (*a*) (0.0375 rad/s^2)$\mathbf{i}$.
(*b*) −(0.1434 m/s)$\mathbf{i}$ + (0.204 m/s)$\mathbf{j}$ − (0.1228 m/s)$\mathbf{k}$.
(*c*) −(0.696 m/s^2)$\mathbf{i}$ − (0.0358 m/s^2)$\mathbf{j}$ + (0.0430 m/s^2)$\mathbf{k}$.
15.199 (*a*) (28.4 rad/s)$\mathbf{i}$ + (5.24 rad/s)$\mathbf{j}$. (*b*) (25.8 rad/s)$\mathbf{i}$.
15.200 (*a*) (135.1 rad/s^2)$\mathbf{k}$. (*b*) (5.77 m/s^2)$\mathbf{i}$. − (232 m/s^2)$\mathbf{j}$.
15.203 −(33.3 in/s)$\mathbf{j}$.
15.204 (12.00 in/s)$\mathbf{k}$.
15.205 −(34.5 mm/s)$\mathbf{i}$.
15.206 −(30.0 in/s)$\mathbf{j}$.
15.207 (45.7 in/s)$\mathbf{j}$.
15.210 (V_2/cos 25°) (−sin 25°$\mathbf{i}$ + cos 25°)$\mathbf{k}$.
15.211 (V_1 cos 25°) (−sin 25°$\mathbf{i}$ + cos 25°$\mathbf{k}$).
15.212 (*a*) −(0.348 rad/s)$\mathbf{i}$ + (0.279 rad/s)$\mathbf{j}$ + (1.089 rad/s)$\mathbf{k}$.
(*b*) −(30.0 in/s)$\mathbf{j}$.
15.213 (*a*) (1.463 rad/s)$\mathbf{i}$ + (0.1052 rad/s)$\mathbf{j}$ + (0.0841 rad/s)$\mathbf{k}$.
(*b*) −(34.5 mm/s)$\mathbf{j}$.

15.216 $-(45.0 \text{ in/s}^2)\mathbf{j}$.
15.217 $(205 \text{ in/s}^2)\mathbf{j}$.
15.218 $-(9.51 \text{ mm/s}^2)\mathbf{j}$.
15.219 $-(8.76 \text{ mm/s}^2)\mathbf{j}$.
15.220 (*a*) $(24.6 \text{ in/s})\mathbf{i} + (67.7 \text{ in/s})\mathbf{j} - (50.7 \text{ in/s})\mathbf{k}$.
(*b*) $-(423 \text{ in/s}^2)\mathbf{i} + (98.5 \text{ in/s}^2)\mathbf{j} - (147.8 \text{ in/s}^2)\mathbf{k}$.
15.221 (*a*) $(51.6 \text{ in/s})\mathbf{i} + (67.7 \text{ in/s})\mathbf{j} - (50.7 \text{ in/s})\mathbf{k}$.
(*b*) $-(423 \text{ in/s}^2)\mathbf{i} + (98.5 \text{ in/s}^2)\mathbf{j} - (229 \text{ in/s}^2)\mathbf{k}$.
15.222 (*a*) $-(1.215 \text{ m/s})\mathbf{i} + (1.620 \text{ m/s})\mathbf{k}$. (*b*) $-(30.4 \text{ m/s}^2)\mathbf{j}$.
15.223 (*a*) $-(1.215 \text{ m/s})\mathbf{i} - (1.080 \text{ m/s})\mathbf{j} + (1.620 \text{ m/s})\mathbf{k}$.
(*b*) $(19.44 \text{ m/s}^2)\mathbf{i} - (30.4 \text{ m/s}^2)\mathbf{j} - (12.96 \text{ m/s}^2)\mathbf{k}$.
15.224 (*a*) $(1.200 \text{ m/s})\mathbf{i} + (0.500 \text{ m/s})\mathbf{j} - (1.200 \text{ m/s})\mathbf{k}$.
(*b*) $-(7.20 \text{ m/s}^2)\mathbf{i} - (14.40 \text{ m/s}^2)\mathbf{k}$.
15.227 (*a*) $(0.750 \text{ m/s})\mathbf{i} + (1.299 \text{ m/s})\mathbf{j} - (1.732 \text{ m/s})\mathbf{k}$.
(*b*) $(27.1 \text{ m/s}^2)\mathbf{i} + (5.63 \text{ m/s}^2)\mathbf{j} - (15.00 \text{ m/s}^2)\mathbf{k}$.
15.228 (*a*) $(129.9 \text{ mm/s})\mathbf{i} + (75.0 \text{ mm/s})\mathbf{j} + (86.6 \text{ mm/s})\mathbf{k}$.
(*b*) $(45.0 \text{ mm/s}^2)\mathbf{i} - (112.6 \text{ mm/s}^2)\mathbf{j} + (60.0 \text{ mm/s}^2)\mathbf{k}$.
15.230 (*a*) $(1.600 \text{ ft/s})\mathbf{i} + (5.00 \text{ ft/s})\mathbf{j} + (8.80 \text{ ft/s})\mathbf{k}$.
(*b*) $(8.00 \text{ ft/s}^2)\mathbf{i} - (83.5 \text{ ft/s}^2)\mathbf{j} + (20.0 \text{ ft/s}^2)\mathbf{k}$.
15.231 (*a*) $v_1 + (R/r)(v_1 - v_2)\mathbf{k}$. (*b*) $v_1(v_1 - v_2)(R/r)\mathbf{j}$.
15.232 $-(41.6 \text{ in/s}^2)\mathbf{i} - (61.5 \text{ in/s}^2)\mathbf{j} + (103.9 \text{ in/s}^2)\mathbf{k}$.
15.233 (*a*) $(0.0375 \text{ rad/s}^2)\mathbf{i}$.
(*b*) $-(0.143 \text{ m/s})\mathbf{i} + (0.205 \text{ m/s})\mathbf{j} - (0.123 \text{ m/s})\mathbf{k}$.
(*c*) $-(0.0696 \text{ m/s}^2)\mathbf{i} - (0.0358 \text{ m/s}^2)\mathbf{j} + (0.0430 \text{ m/s}^2)\mathbf{k}$.
15.234 (*a*) $(0.600 \text{ m/s})\mathbf{j} - (0.585 \text{ m/s})\mathbf{k}$. (*b*) $-(4.76 \text{ m/s}^2)\mathbf{j}$.
15.235 (*a*) $(0.600 \text{ m/s})\mathbf{i} - (0.225 \text{ m/s})\mathbf{k}$.
(*b*) $-(0.675 \text{ m/s}^2)\mathbf{i} + (3.00 \text{ m/s}^2)\mathbf{j} - (3.60 \text{ m/s}^2)\mathbf{k}$.
15.236. (*a*) $-(1.37 \text{ ft/s})\mathbf{i} + (3.76 \text{ ft/s})\mathbf{j} + (1.88 \text{ ft/s})\mathbf{k}$.
(*b*) $(1.22 \text{ ft/s}^2)\mathbf{i} - (3.42 \text{ ft/s}^2)\mathbf{j} - (0.410 \text{ ft/s}^2)\mathbf{k}$.
15.239 (*a*) $(4.33 \text{ ft/s})\mathbf{i} - (6.18 \text{ ft/s})\mathbf{j} + (5.30 \text{ ft/s})\mathbf{k}$.
(*b*) $(2.65 \text{ ft/s}^2)\mathbf{i} - (2.64 \text{ ft/s}^2)\mathbf{j} - (3.25 \text{ ft/s}^2)\mathbf{k}$.
15.240 (*a*) $(27.2 \text{ in/s}^2)\mathbf{i} - (6.75 \text{ in/s}^2)\mathbf{j}$.
(*b*) $(12.80 \text{ in/s}^2)\mathbf{i} - (7.68 \text{ in/s}^2)\mathbf{k}$.
15.241 (*a*) $-(1.600 \text{ in/s}^2)\mathbf{i} + (6.75 \text{ in/s}^2)\mathbf{j}$.
(*b*) $(12.80 \text{ in/s}^2)\mathbf{i} + (7.68 \text{ in/s}^2)\mathbf{k}$.
15.242 $-(5.04 \text{ m/s})\mathbf{i} - (1.200 \text{ m/s})\mathbf{k}$.
$-(9.60 \text{ m/s}^2)\mathbf{i} - (25.9 \text{ m/s}^2)\mathbf{j} + (57.6 \text{ m/s}^2)\mathbf{k}$.
15.243 $-(0.720 \text{ m/s})\mathbf{i} - (1.200 \text{ m/s})\mathbf{k}$,
$-(9.60 \text{ m/s}^2)\mathbf{i} + (25.9 \text{ m/s}^2)\mathbf{j} - (11.52 \text{ m/s}^2)\mathbf{k}$.
15.244 (*a*) $rv_2^2 \sin 30°\mathbf{j} - (rv_2^2 \cos 30° + 2rv_1v_2)\mathbf{k}$.
(*b*) $-r(v_1^2 + v_2^2 + 2v_1v_2 \cos 30°)\mathbf{i} + rv_1^2 \cos 30°\mathbf{k}$.
(*c*) $-rv_2^2 \sin 30°\mathbf{j} + r(2v_1^2 \cos 30° + v_2^2 \cos 30° + 2v_1v_2)\mathbf{k}$.
15.245 (*a*) $(0.610 \text{ m/s})\mathbf{k}$, $-(0.880 \text{ m/s}^2)\mathbf{i} + (1.170 \text{ m/s}^2)\mathbf{j}$.
(*b*) $(5.20 \text{ m/s})\mathbf{i} - (0.390 \text{ m/s})\mathbf{j} - (1.000 \text{ m/s})\mathbf{k}$,
$-(4.00 \text{ m/s}^2)\mathbf{i} - (3.25 \text{ m/s}^2)\mathbf{k}$.
15.248 (*a*) 6.00 m/s^2. (*b*) 9.98 m/s^2. (*c*) 60.0 m/s^2.
15.249 (*a*) 3.00 rad/s^2 i. (*b*) 4.00 s.
15.252 $A_{BD} = 306 \text{ rad/s}^2$ l, $A_{DE} = 737 \text{ rad/s}^2$ l.
15.253 (*a*) 1080 rad/s^2 i. (*b*) 460 ft/s^2 b 64.9°.
15.255 49.4 m/s^2 c 26.0°.
15.256 (*a*) $(0.450 \text{ m/s})\mathbf{k}$, $(4.05 \text{ m/s}^2)\mathbf{i}$. (*b*) $-(1.350 \text{ m/s})\mathbf{k}$, $-(6.75 \text{ m/s}^2)\mathbf{i}$.
(*b*) AE: 23.5 m/s^2 z, BD: 46.2 m/s^2 d 60°.
15.258 $(40.0 \text{ in/s})\mathbf{k}$.
15.259 $(9.00 \text{ in/s})\mathbf{i} - (7.80 \text{ in/s})\mathbf{j} + (7.20 \text{ in/s})\mathbf{k}$,
$(9.00 \text{ in/s}^2)\mathbf{i} - (22.1 \text{ in/s}^2)\mathbf{j} - (5.76 \text{ in/s}^2)\mathbf{k}$.

CHAPTER 16

16.1 11.72 ft/s^2
16.2 (*a*) 0.897 lb a 20°. (*b*) 4.87 lb b 74.4°.
16.3 (*a*) 25.8 ft/s^2. (*b*) 12.27 ft/s^2. (*c*) 13.32 ft/s^2.
16.4 (*a*) 3.20 m/s^2. (*b*) $\mathbf{A} = 3.82$ N x, $\mathbf{B} = 20.7$ N x.
16.5 (*a*) 4.09 m/s^2. (*b*) 42.5 N.
16.6 (*a*) 5270 N x. (*b*) 4120 N.
16.9 (*a*) 5.00 m/s^2 y. (*b*) $0.311 \text{ m} \le h \le 1.489 \text{ m}$.
16.10 (*a*) 2.55 m/s^2 y. (*b*) $h \le 1.047$ m.
16.11 195.9 kg.
16.14 (*a*) 4.91 m/s^2 d 30°. (*b*) $T_{AD} = 31.0$ N, $T_{BE} = 11.43$ N.
16.15 51.3°.
16.18 $\mathbf{B}_y = 40.3$ lb w, $\mathbf{C}_y = 40.3$ lb w.
16.19 (*a*) 30.6 ft/s^2 c 84.1°. (*b*) $\mathbf{A} = 0.505$ lb a 30°,
$\mathbf{B} = 1.285$ lb a 30°.
16.20 Block: 17.01 ft/s^2 c 58.5°; platform: 31.3 ft/s^2 c 30°.
16.25 5230 rev.
16.26 87.8 lb · ft.
16.27 93.5 rev.
16.28 107.6 rev.
16.29 74.5 s.
16.30 20.4 rad/s^2 i.
16.31 32.7 rad/s^2 l.
16.33 (*a*) 5.66 ft/s^2 w. (*b*) 7.52 ft/s w.
16.34 (1): (*a*) 8.00 rad/s^2 l. (*b*) 14.61 rad/s l.
(2): (*a*) 6.74 rad/s^2 l. (*b*) 13.41 rad/s l.
(3): (*a*) 4.24 rad/s^2 l. (*b*) 10.64 rad/s l.
(4): (*a*) 5.83 rad/s^2 l. (*b*) 8.82 rad/s l.
16.36 (*a*) 6.06 rad/s^2 i. (*b*) 11.28 N ↗.
16.39 (*a*) No slipping on *A*; slipping on *B*.
(*b*) $A_A = 61.8 \text{ rad/s}^2$ l; $A_B = 9.66 \text{ rad/s}^2$ i.
16.40 (*a*) No slipping at either cylinder.
(*b*) $A_A = 15.46 \text{ rad/s}^2$ l; $A_B = 7.73 \text{ rad/s}^2$ i.
16.41 (*a*) $A_A = 12.50 \text{ rad/s}^2$ l, $A_B = 33.3 \text{ rad/s}^2$ l.
(*b*) $V_A = 320$ rpm i, $V_B = 320$ rpm l.
16.42 (*a*) $A_A = 12.50 \text{ rad/s}^2$ l, $A_B = 33.3$ rpm l.
(*b*) $V_A = 90.0$ rpm l, $V_B = 120.0$ rpm i.
16.43 (*a*) $A_A = 9.16 \text{ rad/s}^2$ l, $A_B = 38.2 \text{ rad/s}^2$ l.
(*b*) $\mathbf{C} = 54.9$ N x, $\mathbf{M}_C = 2.64$ N · m l.
16.44 (*b*) $v_0/(1 + m_B/m_A)$ i.
16.48 (*a*) 18.40 ft/s^2 y. (*b*) 9.20 ft/s^2 z. (*c*) $z = 24.0$ in.
16.49 (*a*) 12.0 in. from end *A*. (*b*) 9.20 ft/s^2 y.
16.50 (*a*) 2.50 m/s^2 y. (*b*) 0.
16.52 (*a*) 0, $-1.374 \text{ rad/s}^2\,\mathbf{j}$. (*b*) $-(0.515 \text{ ft/s}^2)\mathbf{i}$, $-1.030 \text{ rad/s}^2\,\mathbf{j}$.
16.55 $rg/\bar{k}^2$.
16.56 (*a*) 0.865 N. (*b*) 72.1 rad/s^2 i.
16.57 $\mathbf{a}_A = 3.33 \text{ ft/s}^2$ w, $\mathbf{a}_B = 14.06 \text{ ft/s}^2$ x.
16.58 (*a*) 0.741 rad/s^2 l. (*b*) 0.857 m/s^2.
16.59 (*a*) 2800 N. (*b*) 15.11 rad/s^2 i.
16.60 $T_A = 359$ lb, $T_B = 312$ lb.
16.63 (*a*) $3\,g/L$ i. (*b*) g x. (*c*) $2\,g$ w.
16.64 (*a*) $2\,g/L$ i. (*b*) $g/3$ x. (*c*) $5\,g/3$ w.
16.65 (*a*) g/L i. (*b*) $0.866\,g$ z. (*c*) $0.5\,g$ w.
16.66 (*a*) $0.25\,g$ x. (*b*) $5\,g/4$ w.
16.67 (*a*) $\frac{1}{2}\,g$ x. (*b*) $\frac{3}{2}\,g$ w.
16.68 (*a*) $g(\mathbf{i} - \mathbf{j})/2 + 1.5\,ga\,(a + b)/(a^2 + b^2)\mathbf{j}$.
(*b*) $g(\mathbf{i} - \mathbf{j})/2 - 1.5\,ga\,(a + b)/(a^2 + b^2)\mathbf{j}$.
16.69 (*a*) $5v_0/2r$ l. (*b*) $v_0/\mu_k g$. (*c*) $v_0^2/2\mu_k g$.
16.70 (*a*) v_0/r l. (*b*) $v_0/\mu_k g$. (*c*) $v_0^2/2\mu_k g$.
16.71 (*a*) 1.597 s. (*b*) 9.86 ft/s. (*c*) 19.85 ft.
16.72 (*a*) 1.863 s. (*b*) 9.00 ft/s. (*c*) 22.4 ft.
16.76 (*a*) 107.1 rad/s^2 i. (*b*) 21.4 N z, 39.2 N x.
16.77 (*a*) 150 mm. (*b*) 125 rad/s^2 i.
16.78 (*a*) 12.08 rad/s^2 i. (*b*) 0.750 lb z, 4.00 lb x.
16.79 (*a*) 8.05 rad/s^2 i. (*b*) 24.0 in.
16.80 (*a*) $vz(l - z/2)v^2$. (*b*) 5.09 N.
16.83 13.64 kN y.

16.84 (*a*) 1.5 gw. (*b*) 0.25 mg x.
16.85 (*a*) 9g/7. (*b*) 4mg/7 x.
16.86 (*a*) 9.02 lb x. (*b*) 74.5 lb z , 8.57 lb x.
16.87 (*a*) 43.6 rad/s^2. (*b*) 21.0 N z , 54.6 N x.
16.88 (*a*) 10.87 rad/s^2. (*b*) 8.49 ft · lb l.
16.94 $r^2 g$ sin b$/(r^2 + \bar{k}^2)$.
16.95 (*a*) 2.27 m (7.46 ft). (*b*) 0.649 m (2.13 ft).
16.98 (*a*) 17.78 rad/s^2 i, 2.13 m/s^2 y . (*b*) 0.122.
16.99 (*a*) 26.7 rad/s^2 i, 3.20 m/s^2 y . (*b*) 0.0136.
16.102 (*a*) no sliding. (*b*) 15.46 rad/s^2 i, 10.30 ft/s^2.
16.103 (*a*) no sliding. (*b*) 23.2 rad/s^2 i, 15.46 ft/s^2.
16.104 (*a*) slides. (*b*) 4.29 rad/s^2 l, 9.66 ft/s^2 y .
16.105 (*a*) slides. (*b*) 12.88 rad/s^2 l, 3.22 ft/s^2 z .
16.107 (*a*) 6.63 ft/s^2 y . (*b*) 3.79 ft/s^2 y .
(*c*) 0.355 ft y .
16.108 (*a*) 72.4 rad/s^2 l. (*b*) 7.24 m/s^{2}w.
16.109 (*a*) $\mathbf{a}_A$ = 1.923 m/s^2 z , $\mathbf{a}_B$ = 1.923 m/s^2 z .
(*b*) 4.33 N z .
16.111 (*a*) 0.298. (*b*) 0.536 g y
16.112 (*a*) 0.322. (*b*) 0.566 g y
16.113 8.26 N z .
16.114 (*a*) 0.125 g/r i. (*b*) 0.125 g y , 0.125 gw.
16.115 $m_B g$ sin u/[2*r* {m_h + m_B (1 + cos u)}]
16.116 3.43 lb a 70.5°, 0.1550 ft · lb i.
16.117 (*a*) 10.94 rad/s^2 i. (*b*) 7.68 lb x. (*c*) 5.75 lb y .
16.118 (*a*) 20.1 lb x. (*b*) 12.42 lb x.
16.119 (*a*) 10.62 rad/s^2 l. (*b*) 4.25 N z .
16.120 *mg* sin u/(1 + 3 sin^2 u).
16.121 (*a*) 10.41 rad/s^2 i. (*b*) 36.8 N. (*c*) 61.3 N.
16.124 6.40 N z .
16.125 7.95 lb a 60°.
16.126 6.02 lb a 60°.
16.127 171.7 N y .
16.128 60.0 N y .
16.129 25.9 N b 60°.
16.131 (*a*) 37.8 ft/s^2 c 26.1°. (*b*) 48.4 lb x.
16.134 0.330 lb z .
16.135 (*a*) 36.3 N · m l. (*b*) 231 N z , 524 N x.
16.136 (*a*) 82.3 N · m l. (*b*) 147.2 N z , 479 N x.
16.137 **B** = 805 N z , **D** = 426 N y .
16.138 **B** = 525 N d 38.1°, **D** = 322 N c 15.7°.
16.139 **A** = 1.565 lb x, **B** = 1.689 lb x.
16.140 **A** = 0.839 lb y + 2.00 lb x,
B = 0.280 lb z + 2.00 lb x.
16.143 (*a*) A: 0.400 *g/r* l, B: 0.400 *g/r* i.
(*b*) 0.200 *mg*. (*c*) 0.800 gw.
16.144 A: 2P/7m y , B: 22P/7m z .
16.145 (*a*) 5.63 m/s^2 c 25°. (*b*) 7.66 rad/s^2 i.
16.146 (*a*) 13.55 m/s^{2}w. (*b*) 2.34 m/s^{2}w.
16.147 (*a*) 6.40 ft/s^2 y . (*b*) 45.4 rad/s^2 l.
***16.148** (*a*) 17.03 ft/s^2 c 20°. (*b*) 42.7 rad/s^2 l.
16.151 M_{max} = 10.39 lb · in. located 20.8 in. below A.
16.153 20.6 ft.
16.154 12.34 ft.
16.156 (*a*) 2mg/(1 + 3m). (*b*) 1.000 g.
16.157 (*a*) 0.513 *g/L* i. (*b*) 0.912 mg x. (*c*) 0.241 mg y .
16.158 (*a*) 1.519 *g/L* i. (*b*) 0.260 gw. (*c*) 0.740 mg x.
16.160 (1): (*a*) 1.200 *g/c* i. (*b*) 0.671 g d 63.4°.
(2): (*a*) 24 *g*/17*c* i, (*b*) 12 *g*/17w.
(3): (*a*) 2.40 *g/c* i, (*b*) 0.500 gw.
16.162 (*a*) 51.2 rad/s^2 i. (*b*) 21.0 N x.
16.163. (*a*) 59.8 rad/s^2 i. (*b*) 20.4 N x.

CHAPTER 17

17.1 188.1 mm.
17.2 58.7 lb · ft.
17.3 4.10 in.
17.4 0.760.
17.5 11.46 rev.
17.6 (*a*) 293 rpm. (*b*) 15.92 rev.
17.7 19.77 rev.
17.10 109.4 lb y .
17.11 (*a*) 6.35 rev. (*b*) 7.14 N.
17.12 (*a*) 2.54 rev. (*b*) 17.86 N.
17.13 80.7 lbw.
17.16 (*a*) 1$\overline{3g/L}$, 2.50 *W* x. (*b*) 5.67 rad/s, 4.50 lb x.
17.17 (*a*) 0.289 *l*. (*b*) 1.861 1$\overline{g/l}$, 2.00 *mg* x.
17.18 11.52 rad/s l.
17.19 4.61 rad/s i.
17.20 (*a*) 3.94 rad/s i, 271 lb b 5.25°.
(*b*) 5.58 rad/s i, 701 lb x.
17.23 7.09 rad/s.
17.24 (*a*) −0.250 rpm. (*b*) 0.249 rpm.
17.25 1$\overline{4gs/3}$.
17.26 1$\overline{gs}$.
17.27 (*a*) 9.77 ft/s. (*b*) 6.67 lb z .
17.29 (*a*) 5.00 rad/s. (*b*) 24.9 N x.
17.30 (*a*) 1.324 1$\overline{g/l}$. (*b*) 2.12 *mg* x.
17.31 (*a*) [10*g* (*R* − *n*) (1 − cos b)/7]$^{1/2}$.
(*b*) *mg*(17 − 10 cos b)/7.
17.32 (*a*) 2.06 ft. (*b*) 4.00 lb.
17.33 (*a*) 7.43 ft/sw. (*b*) 4.00 lb.
17.35 292 mm/s y .
17.36 $\mathbf{v}_A$ = 0.775 1$\overline{gl}$ z , $\mathbf{v}_B$ = 0.775 1$\overline{gl}$ d 60°.
17.37 1.170 rad/s i, 5.07 m/s z .
17.38 [3*g* (cos u_0 − cos u_2)/*L*]$^{1/2}$ i.
17.39 3.71 rad/s l, 7.74 ft/s x.
17.40 4.03 rad/s i, 7.27 ft/sw.
17.42 2.69 m/sw.
17.43 84.7 rpm i.
17.44 110.8 rpm i.
17.45 3.25 m/sw.
17.46 4.43 m/sw.
17.47 0.770 m/s z .
17.48 (*a*) 44.3 hp. (*b*) 118.1 hp.
17.49 (*a*) 39.8 N · m. (*b*) 95.5 N · m. (*c*) 229 N · m.
17.50 1146 rpm.
17.51 0.343 hp.
17.52 179.1 mm.
17.53 0.335 lb · in.
17.54 3.87 rad/s.
17.55 2.84 s.
17.58 3.88 s.
17.59 (1 + m_k^2) *r*v_0/2m_k(1 + m_k)*g*.
17.62 v_0/(1 + m_A/m_B).
17.63 (*a*) V_A = 686 rpm l, V_B = 514 rpm i. (*b*) 4.18 lb · s x.
17.64 (*a*) 5.15 lb. (*b*) 2.01 lb.
17.65 **X** = m**v**, *d* = $\bar{k}^2$v/$\bar{v}$.
17.69 2.79 ft.
17.70 (*a*) $r^2 gt$ sin b$/(r^2 + \bar{k}^2)$ c b.
(*b*) $u_s \geq \bar{k}^2$ tan b$/(r^2 + \bar{k}^2)$.
17.71 (*a*) 2.55 m/s x. (*b*) 10.53 N
17.72 (*a*) 8.05 ft/s y . (*b*) 2.68 ft/s y .
17.74 (*a*) 8.41 m/sw. (*b*) 16.82 N.

17.75 (*a*) 0.557 s. (*b*) 16.82 N.
17.77 (*a*) $2.50\, \bar{v}_0/r$. (*b*) $\bar{v}_0/\mathrm{m}_k g$.
17.78 (*a*) 2.50 s. (*b*) 16.95 ft/s.
17.79 $0.614\ \mathrm{v}_0$.
17.80 84.2 rpm.
17.81 *A* and *B*: 159.1 rpm i; platform 20.9 rpm l.
17.82 18.07 rad/s.
17.83 (*a*) 2.54 rad/s. (*b*) 1.902 J.
17.86 37.2 rpm.
17.87 *A* and *B*: 212 rpm i; platform 27.9 rpm l.
17.88 2.51 m/s.
17.89 18.83 rad/s, $0.0508\ \mathrm{kg \cdot m^2}$.
17.90 (*a*) 31.1 rad/s. (*b*) 18.13 ft/s.
17.91 (*a*) 15.00 rad/s. (*b*) 20.5 ft/s.
17.93 7.45 m/s.
17.94 1.542 m/s.
17.95 2.01 ft/s z .
17.96 $0.400\ r$.
17.97 (*a*) 24.4 rad/s i. (*b*) 1545 lb y .
17.98 (*a*) 10.00 in. (*b*) 22.6 rad/s i.
17.101 (*a*) 2.16 m/s y . (*b*) 4.87 kN a 66.9°.
17.102 (*a*) 79.2 mm. (*b*) 1.992 m/s y .
17.103 (*a*) $0.429\ v_0/L$ i. (*b*) $0.571\ v_0$ y .
17.104 2.40 rad/s i.
17.105 1.667 in.
17.106 $(2 + 5 \cos \mathrm{b})\ \mathrm{v}_1/7$ l, $(2 + 5 \cos \mathrm{b})\bar{v}_1/7$.
17.107 1.414.
17.108 (*a*) mv_0/M y . (*b*) mv_0/MR l.
17.109 (*a*) 1.500 *R*. (*b*) 1.000 *R*.
17.112 $6\ (v_1/L) \sin \mathrm{b}/(1 + 3 \sin^2 \mathrm{b})$.
17.115 2.38 m/s.
17.116 (*a*) 0.366.
17.117 (*a*) $0.437\ 1\overline{g/L}$. (*b*) 5.12°.
17.118 (*a*) $0.250\ \mathrm{v}_0$ i. (*b*) 0.9375. (*c*) 1.50°.
17.119 48.7°.
17.120 1887 ft/s.
17.121 725 mm.
17.122 447 mm.
17.123 (*a*) 3.85 ft/s w. (*b*) 5.13 rad/s i.
17.124 (*a*) 0.256 ft/s x. (*b*) 9.23 rad/s i.
17.127 (*a*) 3.00 rad/s l. (*b*) 0.938 m/s x.
17.128 (*a*) 2.60 rad/s i. (*b*) 1.635 m/s c 53.4°.
17.131 $1.250\ v_0/r$.
17.132 (*a*) $\mathbf{v}_A = 0$, $\mathrm{V}_A = v_1/r$ i, $\mathbf{v}_B = v_1$ y , $\mathrm{V}_B = 0$.
(*b*) $\mathbf{v}'_A = 0.286\ v_1$ y , $v'_B = 0.514\ v_1$ y .
17.133 (*a*) $\mathbf{v}_A = (v_0 \sin \mathrm{u})\mathbf{j}$, $\mathbf{v}_B = (v_0 \cos \mathrm{u})\mathbf{i}$, $\mathrm{V}_A = (v_0/r)\ (-\sin \mathrm{u}\mathbf{i} + \cos \mathrm{u}\mathbf{j})$, $\mathrm{V}_B = 0$.
(*b*) $0.514\ (v_0/r) \cos \mathrm{u}\mathbf{i}$.
17.134 $\mathrm{V}_{AB} = 2.68$ rad/s i, $\mathrm{V}_{BC} = 13.39$ rad/s i.
17.135 (*a*) 106.7 rev. (*b*) 6.98 s.
17.136 70.1 lb w.
17.137 4.81 rad/s.
17.139 (*a*) 53.1°. (*b*) $1.095\ 1\overline{gL}$ c 53.1°.
17.140 $\mathbf{A} = 100.1$ N x, $\mathbf{B} = 43.9$ N y .
17.142 $0.778\ \mathrm{v}_0$.
17.143 (*a*) 418 rpm. (*b*) −20.4 J.
17.145 (*a*) 68.6 rpm. (*b*) 2.82 J.

CHAPTER 18

18.1 $0.250\ \mathrm{m}r^2\ \mathrm{v}_2\mathbf{j} + 0.500\ \mathrm{m}r^2\ \mathrm{v}_1\mathbf{k}$.
18.2 $-(0.408\ \mathrm{slug \cdot ft^2/s})\mathbf{i} + (0.1398\ \mathrm{slug \cdot ft^2/s})\mathbf{j}$.
18.3 $0.247\ \mathrm{slug \cdot ft^2/s}$, $\mathrm{u}_x = 48.6°$, $\mathrm{u}_y = 41.4°$, $\mathrm{u}_z = 90°$.
18.5 $(0.1125\ \mathrm{kg \cdot m^2/s})\mathbf{j} + (0.675\ \mathrm{kg \cdot m^2/s})\mathbf{k}$.
18.7 $0.432\ \mathrm{m}a^2\mathrm{v}$, 20.2°.
18.8 11.88°.
18.9 $(1.843\ \mathrm{lb \cdot ft \cdot s})\mathbf{i} - (0.455\ \mathrm{lb \cdot ft \cdot s})\mathbf{j} + (1.118\ \mathrm{lb \cdot ft \cdot s})\mathbf{k}$.
18.10 $-(1.747\ \mathrm{lb \cdot s \cdot ft})\mathbf{i} + (3.59\ \mathrm{lb \cdot s \cdot ft})\mathbf{j} + (0.0582\ \mathrm{lb \cdot s \cdot ft})\mathbf{k}$.
18.11 $0.500\ \mathrm{m}r^2\mathrm{v}_1\mathbf{i} - m(L^2 + 0.250\ r^2)\ (r\mathrm{v}_1/L)\mathbf{j}$.
18.12 (*a*) 0.485 rad/s. (*b*) 0.01531 rad/s.
18.15 (*a*) $(1.563\ \mathrm{kg \cdot m^2/s})\mathbf{i} - (0.938\ \mathrm{kg \cdot m^2/s})\mathbf{k}$. (*b*) 31.0°.
18.16 (*a*) $(1.563\ \mathrm{kg \cdot m^2/s})\mathbf{i} - (0.938\ \mathrm{kg \cdot m^2/s})\mathbf{k}$.
(*b*) $(1.563\ \mathrm{kg \cdot m^2/s})\mathbf{i} - (0.938\ \mathrm{kg \cdot m^2/s})\mathbf{k}$.
18.17 (*a*) $-(1.041\ \mathrm{lb \cdot ft \cdot s})\mathbf{i} + (1.041\ \mathrm{lb \cdot ft \cdot s})\mathbf{j} + (2.31\ \mathrm{lb \cdot ft \cdot s})\mathbf{k}$.
(*b*) 147.5°.
18.18 (*a*) $-(1.041\ \mathrm{lb \cdot ft \cdot s})\mathbf{i} - (1.041\ \mathrm{lb \cdot ft \cdot s})\mathbf{j} + (2.31\ \mathrm{lb \cdot ft \cdot s})\mathbf{k}$.
(*b*) 32.5°.
18.21 93.6 kg.
18.22 2.57 s.
18.25 (*a*) 0. (*b*) $(3F\Delta t/ma)\ (\mathbf{i} - 4\mathbf{k})$.
18.26 (*a*) $-(F\Delta t/m)\mathbf{i}$. (*b*) $(3\ F\Delta t/8ma)\ (\mathbf{j} + 4\mathbf{k})$.
18.27 (*a*) $-(0.300\ \mathrm{m/s})\mathbf{i}$. (*b*) $-(0.962\ \mathrm{rad/s})\mathbf{i} - (0.577\ \mathrm{rad/s})\mathbf{j}$.
18.28 (*a*) $(0.300\ \mathrm{m/s})\mathbf{j}$.
(*b*) $-(3.46\ \mathrm{rad/s})\mathbf{i} + (1.923\ \mathrm{rad/s})\mathbf{j} - (0.857\ \mathrm{rad/s})\mathbf{k}$.
18.31 (*a*) $0.1250\ \mathrm{v}_0\ (-\mathbf{i} + \mathbf{j})$. (*b*) $0.0884\ a\mathrm{v}_0\mathbf{k}$.
18.32 (*a*) $0.1031\ ma\mathrm{v}_0\mathbf{k}$. (*b*) $-0.01473\ ma\mathrm{v}_0\mathbf{k}$.
18.33 $(0.0248\ \mathrm{rad/s})\mathbf{i} - (0.277\ \mathrm{rad/s})\mathbf{j} - (0.360\ \mathrm{rad/s})\mathbf{k}$.
18.34 (*a*) −0.726 rad/s.
(*b*) $-(2160\ \mathrm{ft/s})\mathbf{i} - (4860\ \mathrm{ft/s})\mathbf{j} + (860\ \mathrm{ft/s})\mathbf{k}$.
18.35 (*a*) *C* and *B*. (*b*) *B*: 4.84 s, *C*: 8.16 s. (*c*) 0.520 s.
18.36 (*a*) *D* and *A*. (*b*) *A*: 1.848 s, *D*: 6.82 s. (*c*) 0.347 s.
18.39 $0.1250\ mr^2\ (\mathrm{v}_2^2 + 2\mathrm{v}_1^2)$.
18.40 0.349 ft · lb.
18.41 0.978 ft · lb.
18.42 12.67 ft · lb.
18.43 15.47 J.
18.44 $0.1250\ ma^2\mathrm{v}^2$.
18.45 $0.203\ ma^2\mathrm{v}^2$.
18.47 9.38 J.
18.49 1.296 J.
18.50 46.2 J.
18.51 $0.1000\ m\bar{v}_0^2$.
18.53 16.75 ft · lb.
18.54 39.9 ft · lb.
18.55 $0.500\ \mathrm{m}r^2\ \mathrm{v}_1\mathrm{v}_2\mathbf{i}$.
18.56 $(0.204\ \mathrm{ft \cdot lb})\mathbf{k}$.
18.57 $(2.22\ \mathrm{ft \cdot lb})\mathbf{k}$.
18.58 $(5.30\ \mathrm{lb \cdot ft})\mathbf{k}$.
18.59 $(3.38\ \mathrm{N \cdot m})\mathbf{i}$.
18.60 $0.0958\ mr^2\mathrm{v}^2\mathbf{k}$.
18.64 $mr^2\ (0.0958\ \mathrm{a}\mathbf{i} + 0.455\ \mathrm{a}\mathbf{j} - 0.0958\ \mathrm{v}^2\mathbf{k})$.
18.65 $\mathbf{C} = 0.1667\ mb\mathrm{v}^2 \sin \mathrm{b} \cos \mathrm{b}\mathbf{i}$.
$\mathbf{D} = -0.1667\ mb\mathrm{v}^2 \sin \mathrm{b} \cos \mathrm{b}\mathbf{i}$.
18.66 $\mathbf{A} = -(4.97\ \mathrm{lb})\mathbf{i}$, $\mathbf{B} = -(1.656\ \mathrm{lb})\mathbf{i}$.
18.67 $\mathbf{A} = -(1.103\ \mathrm{lb})\mathbf{j} - (0.920\ \mathrm{lb})\mathbf{k}$.
$\mathbf{B} = (1.103\ \mathrm{lb})\mathbf{j} + (0.920\ \mathrm{lb})\mathbf{k}$.
18.68 $\mathbf{A} = (14.4\ \mathrm{N})\mathbf{k}$, $\mathbf{B} = -(14.4\ \mathrm{N})\mathbf{k}$.
18.71 (*a*) $3M_0/mb^2 \cos^2 \mathrm{b}$. (*b*) $\mathbf{C} = -\mathbf{D} = (M_0 \tan \mathrm{b}/2b)\mathbf{k}$.
18.72 (*a*) $(14.49\ \mathrm{rad/s^2})\mathbf{j}$. (*b*) $\mathbf{A} = -(1.125\ \mathrm{lb})\mathbf{k}$, $\mathbf{B} = -(0.375\ \mathrm{lb})\mathbf{k}$.
18.73 (*a*) $(0.873\ \mathrm{lb \cdot ft})\mathbf{i}$ (*b*) $\mathbf{A} = -\mathbf{B} = -(0.218\ \mathrm{lb})\mathbf{j} + (0.262\ \mathrm{lb})\mathbf{k}$.
18.74 (*a*) $(2.67\ \mathrm{N \cdot m})\mathbf{i}$. (*b*) $\mathbf{A} = -\mathbf{B} = (2.00\ \mathrm{N})\mathbf{j}$.
18.75 (*a*) $(0.1301\ \mathrm{lb \cdot ft})\mathbf{i}$. (*b*) $\mathbf{A} = -\mathbf{B} = -(0.0331\ \mathrm{lb})\mathbf{i} + (0.0331\ \mathrm{lb})\mathbf{j}$.
18.76 $\mathbf{A} = -\mathbf{B} = -(0.449\ \mathrm{lb})\mathbf{j} - (0.383\ \mathrm{lb})\mathbf{k}$.
18.79 $\mathbf{A} = -\mathbf{B} = (1.527\ \mathrm{N})\mathbf{j}$.

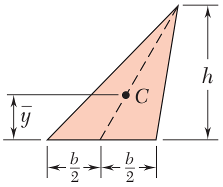
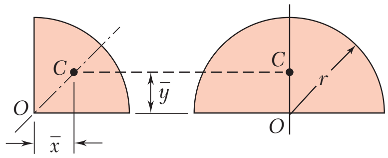
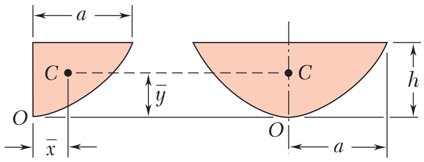
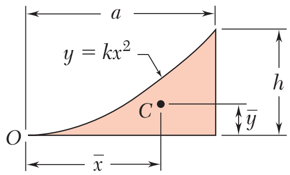
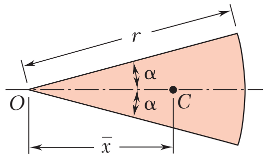
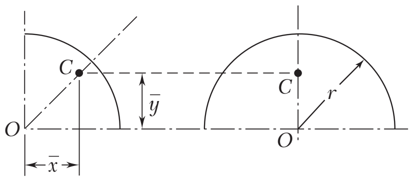
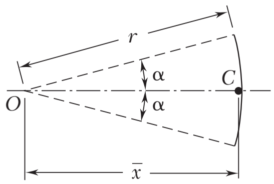
18.81 (*a*) 10.47 N · m. (*b*) 10.47 N · m.
18.82 24.0 N x.
18.84 1.138°; up.
18.85 10.20 rad/s.
18.86 (*a*) 38.1°. (*b*) 11.78 rad/s.
18.87 13.46 rad/s.
18.88 7.00 rad/s.
18.90 5.45 rad/s.
18.91 2.11 N a 18.7°.
18.93 **A** = −**B** = (0.1906 lb)**k**.
18.94 7.87 rad/s.
18.95 (*a*) **C** = −**D** = −(123.4 N)**i**. (*b*) **C** = **D** = 0.
18.96 91.2 rpm.
18.99 −(45.0 N)**i**, (3.38 N · m)**i** + (10.13 N · m)**k**.
18.100 (*a*) **A** = (1.786 kN)**i** + (143.5 kN)**j**,
B = −(1.786 kN)**i** + (150.8 kN)**j**, (*b*) −(35.7 kN · m)**k**.
18.101 **C** = −(7.81 lb)**i** + (7.43 lb)**k**,
D = −(7.81 lb)**i** − (7.43 lb)**k**.
18.102 **C** = −(12.58 lb)**i** + (9.43 lb)**k**,
D = −(12.58 lb)**i** − (9.43 lb)**k**.
18.103 **D** = −(22.0 N)**i** + (26.8 N)**j**, **E** = −(21.2 N)**i** − (5.20 N)**j**.
18.104 (*a*) (0.392 N · m)**k**. (*b*) **D** = −(21.0 N)**i** + (28.0 N)**j**,
E = −(21.0 N)**i** − (4.00 N)**j**.
18.107 1666 rpm.
18.109 45.9 rpm, 533 rpm.
18.111 442 rpm.
18.112 68.1°.
18.113 (*a*) 40.0°. (*b*) 23.5°. (*c*) 85.3°.
18.114 (*a*) 56.1 rad/s. (*b*) 5.30 rad/s.
18.115 23.7°.
18.116 (*a*) 52.7 rad/s. (*b*) 6.44 rad/s.
18.117 (*a*) 4.89 rpm. (*b*) 4.96 rpm, 396 rpm.
18.124 (*a*) 13.19°. (*b*) 1242 rpm (retrograde)
18.125 (*a*) 23.8°. (*b*) precession: 82.6 rpm; spin: 128.8 rpm.
18.126 (*a*) 32.0°. (*b*) precession: 1.126 rpm; spin: 0.344 rpm.
18.127 (*a*) 52.0°. (*b*) precession: 0.1523 rad/s (retrograde);
spin: 0.0338 rad/s.
18.128 (*a*) 109.4 rpm, r_x = 90°, r_y = 100.05°, r_z = 10.05°.
(*b*) u_x = 90°, u_y = 113.9°, u_z = 23.9°.
(*c*) precession: 47.1 rpm; spin: 64.6 rpm.
18.130 (*a*) u_x = 90.0°, u_y = 26.0°, u_z = 64.0°.
(*b*) precession, 0.847 rad/s (retrograde); spin: 0.1593 rad/s.
18.131 (*a*) 40.0° < u < 140.0°. (*b*) 5.31 rad/s. (*c*) 5.58 rad/s.
18.132 (*a*) 2.00 rad/s. (*b*) 8.94 rad/s.
18.135 (*a*) $u_0 \leq u \leq 180° - u_0$. (*b*) $\dot{\varphi}_0 \sin u_0 \cos u_0$.
(*c*) $\dot{\varphi}_0 \sin^2 u_0$.
18.136 (*a*) $(1 + \cos^2 u)\,\dot{\varphi}^2 + \dot{u}^2$ = constant; $\dot{\varphi}(1 + \cos^2 u)$ = constant.
(*b*) $\dot{u} = \dot{\varphi}_0[(1 + \cos^2 u_0)(\cos^2 u - \cos^2 u_0)/(1 + \cos^2 u)]^{1/2}$.
(*c*) $u < u_0$.
18.139 (*a*) 47.0°. (*b*) precession: 15.25 rad/s; spin: 307 rad/s.
18.140 (*a*) 76.3°. (*b*) precession: 9.62 rad/s; spin: 294 rad/s. (*c*) 36.5°.
18.148 (0.234 kg · m²/s)**j** + (1.250 kg · m²/s)**k**.
18.150 (*a*) 0. (*b*) $(F\Delta t/ma)$ (2.50**i** − 1.454**j** + 2.19**k**).
18.151 4.29 kN · m.
18.153 **D** = −(7.12 lb)**j** + (4.47 lb)**k**, **E** = −(1.822 lb)**j** + (4.47 lb)**k**.
18.154 **D** = 0; $\mathbf{M}_D$ = (11.23 N · m) cos² u**i** +
(11.23 N · m) sin u cos u**j** − (2.81 N · m) sin u cos u**k**.
18.155 (*a*) u_x = 52.5°, u_y = 37.5°, u_z = 90°.
(*b*) 53.8 rev/h. (*c*) 6.68 rev/h.
18.156 (*a*) (2.71 lb · ft)**j**. (*b*) **R** = −(5.30 lb)**i** − (1.988 lb)**k**;
$\mathbf{M}_O$ = (2.69 lb · ft)**i** − (4.42 lb · ft)**k**
18.157 (*a*) 4.00 rad/s. (*b*) 5.66 rad/s.

CHAPTER 19

19.1 0.377 m/s, 47.3 m/s².
19.2 4.33 ft/s, 0.551 Hz.
19.3 0.1026 in., 5.16 in/s.
19.4 (*a*) 0.324 s, 3.08 Hz. (*b*) 12.91 mm, 4.84 m/s².
19.5 (*a*) 0.308 s, 3.25 Hz. (*b*) 1.021 m/s, 20.8 m/s².
19.6 (*a*) 267 rpm. (*b*) 5.36 ft/s.
19.7 (*a*) 11.29°. (*b*) 1.933 m/s².
19.8 (*a*) 0.557 Hz. (*b*) 293 mm/s.
19.11 (*a*) 0.0352 s. (*b*) 6.34 ft/s x, 64.4 ft/s² w.
19.12 0.445 ft x, 2.27 ft/s w, 114.7 ft/s² w.
19.13 (*a*) 1.288°. (*b*) 0.874 ft/s, 0.759 ft/s².
19.14 (*a*) 4.91 mm, 5.81 Hz, 0.1791 m/s.
(*b*) 491 N, (c) 0.1592 m/s x.
19.17 2.63 s.
19.18 (*a*) 0.206 s, 4.85 Hz. (*b*) 5.08 ft/s, 154.6 ft/s².
19.19 (*a*) 0.413 s, 2.42 Hz. (*b*) 2.54 ft/s, 38.6 ft/s².
19.20 (*a*) 0.361 s, 2.77 Hz. (*b*) 0.765 m/s, 13.30 m/s².
19.23 (*a*) 35.6 lb/in. (*b*) 5.01 lb.
19.24 (*a*) 6.80 kg. (*b*) 0.583 s.
19.25 192 lb/ft.
19.26 (*a*) 21.7 kg. (*b*) 1011 kg/m³.
19.28 (*a*) 22.3 MN/m. (*b*) 266 Hz.
19.30 (*a*) 858 N/mm. (*b*) 149.5 rpm.
19.31 (*a*) 3.56 kg. (*b*) 43.7 kg.
19.34 16.26°.
19.35 (*a*) 1.737 s. (*b*) 1.864 s. (*c*) 2.05 s.
19.36 28.1 in.
19.37 (*a*) 0.293 s. (*b*) 0.215 m/s.
19.38 (*a*) 1.047 rad/s. (*b*) 16.42 in.
19.39 (*a*) 1.740 s. (*b*) 90.3 mm/s.
19.41 (*a*) 0.491 s. (*b*) 9.60 in/s.
19.42 (*a*) 0.1947 s. (*b*) 171.7 ft/s².
19.44 75.5°.
19.45 0.346 Hz.
19.48 (*a*) 2.28 s. (*b*) 1.294 m.
19.49 (*a*) 1.617 s. (*b*) 1.676 s.
19.50 (*a*) 227 mm. (*b*) 1.352 s.
19.51 (*a*) 1.075 s. (*b*) 3.59 in.
19.55 (*a*) 2.21 Hz. (*b*) 115. 3 N/m.
19.56 0.945 Hz.
19.57 3.03 Hz.
19.59 (*a*) 0.426 s. (*b*) 15.44 ft/s.
19.60 (*a*) 88.1 mm. (*b*) 85.1 mm.
19.62 82.2 mm/s x.
19.63 6.57 kg · m².
19.64 (*a*) 21.3 kg. (*b*) 1.836 s.
19.67 0.672 in.
19.68 8.60 ft.
19.69 1.476 m/s, 31.1 m/s².
19.70 12.11 m/s.
19.71 3.18 s.
19.72 $6.28\ 1\overline{R/g}$.
19.75 $l/\ 1\overline{12}$
19.76 75.5°.
19.77 $0.363\ 1\overline{(2k/3m) + (4g/3L)}$.
19.78 2.10 Hz.
19.79 (*a*) 0.715 s. (*b*) 0.293 ft/s.
19.80 0.821 s.
19.83 1.327 s.
19.85 1.834 s.

19.86 2.39 s.
19.87 $2\pi\sqrt{(12r^2 + 2l^2)/3gl}$.
19.89 (*a*) $\sqrt{(6ka^2 - 3mgl)}/(2\pi)$. (*b*) $\sqrt{mgl/2k}$.
19.90 2.29 Hz.
19.91 0.911 Hz.
19.92 $0.1312\sqrt{g/r}$.
19.93 $0.1125\sqrt{g/l}$.
19.95 $0.276\sqrt{g/l}$.
19.96 2.59 Hz.
19.97 $1.814l/\sqrt{gr}$.
19.98 0.352 s.
19.99 (*a*) 16.67 mm (in-phase). (*b*) 128.2 mm (in-phase).
(*c*) 10.00 mm (out-of-phase).
19.100 15.28 rad/s, 23.8 rad/s.
19.101 (*a*) 10.99 lb/ft. (*b*) 2.99 lb/ft.
19.102 $\sqrt{k/2m} < \omega_f < \sqrt{3k/2m}$.
19.105 $\omega_f < 8.46$ rad/s, (no out-of-phase solution).
19.106 $\omega_f < 24.8$ rad/s (in-phase), $\omega_f > 42.9$ rad/s (out-of-phase).
19.107 22.5 mm, −5.63 mm.
19.108 (*a*) 1.406 ft^3/min. (*b*) 3.31 ft^3/min.
19.109 $\omega_f > \sqrt{2g/l}$.
19.110 (*a*) 1.034 in. (*b*) $-0.1033 \sin \pi t$ (lb).
19.112 651 rpm.
19.114 22.0 mm.
19.115 0.0999 in.
19.116 $\omega_f < 322$ rpm.
19.117 (*a*) −0.00303 in. (*b*) 0.000758 in. (*c*) −0.01990 in.
19.119 149.3 mm.
19.120 (*a*) $\omega_f \leq 286$ rpm. (b) $\omega_f \geq 367$ rpm.
19.121 Force transmissibility: $1/(1 - \omega_f^2/\omega_n^2)$
Displacement transmissiblity: $1/(1 - \omega_f^2/\omega_n^2)$
19.122 (*a*) 4.17%. (*b*) 84.9 Hz.
19.123 8.04%.
19.125 (*a*) 1399 rpm. (*b*) 0.01670 in.
19.126 (*a*) 25.6 km/h. (*b*) −14.25 mm.
19.132 (*a*) 6.49 kip · s/ft. (*b*) 230 kips/ft.
19.133 5.48 N · m · s.
19.134 (*a*) 6490 lb/ft. (*b*) 0.1939 s.
19.135 (*a*) 297 rpm. (*b*) 252 rpm. (*c*) 0.335 in., 0.361 in.
19.136 56.9 mm.
19.137 0.0431.
19.138 (*a*) $\ddot{\theta} + (3c/m)\dot{\theta} + (3k/4m)\theta = 0$
(*b*) $\sqrt{km/3}$
19.139 0.1791 in.
19.140 10.61 lb · s/in.
19.141 ≥ 0.707.
19.143 (*a*) 147 kip/ft. (*b*) 0.0292.
19.144 > 30.8 Hz and < 15.85 Hz.
19.145 0.539 in.
19.146 (*a*) 0.0924. (*b*) 25.2 lb · s/ft.
19.148 (*a*) 71.8 N. (*b*) 39.0 N.
19.149 (*a*) 4.90 in. (*b*) 30.3 lb.
19.151 (*a*) $m\ddot{x} + c\dot{x} + kx = (k \sin \omega_f t + c\omega_f \cos \omega_f t)\delta_m$.
(*b*) $x = x_m \sin (\omega_f t - \varphi + \psi)$, where
$x_m = \delta_m \sqrt{k^2 + (c\omega_f)^2} / \sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}$,
$\tan \varphi = c\omega_f/(k - m\omega_f^2)$, $\tan \psi = c\omega_f/k$.
19.153 $R < 2\sqrt{L/C}$.
19.154 (*a*) E/R. (*b*) L/R.
19.157 (*a*) $C(\dot{x}_A - \dot{x}_m) + kx_A = 0$
$m\ddot{x}_m + c(\dot{x}_m - \dot{x}_A) = P_m \sin \omega_f t$
(*b*) $R(\dot{q}_A - \dot{q}_m) + (1/C)q_A = 0$
$L\ddot{q}_m + R(\dot{q}_m - \dot{q}_A) = E_m \sin \omega_f t$
19.159 0.760 lb · s^2 · ft, 8.66 in.
19.160 (*a*) 6.82 lb. (*b*) 33.4 lb/ft.
19.161 1.785 s.
19.162 (*a*) 2.28 s. (*b*) 1.294 m.
19.165 (*a*) $0.07246\ddot{\theta} + 0.3375\dot{\theta} + 1.25\theta = 0$.
(*b*) -19.05×10^{-6} degrees.
19.168 (*a*) $m\ddot{x} + 2T(2x/l) = 0$. (*b*) $\pi\sqrt{ml/T}$.
19.169 0.045 in.

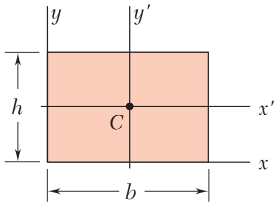
Centroids of Common Shapes of Areas and Lines

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$

Moments of Inertia of Common Geometric Shapes

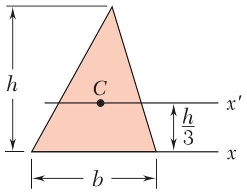
Rectangle

$\bar{I}_{x'} = \frac{1}{12}bh^3$
$\bar{I}_{y'} = \frac{1}{12}b^3h$
$I_x = \frac{1}{3}bh^3$
$I_y = \frac{1}{3}b^3h$
$J_C = \frac{1}{12}bh(b^2 + h^2)$



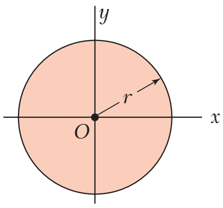
Triangle

$\bar{I}_{x'} = \frac{1}{36}bh^3$
$I_x = \frac{1}{12}bh^3$



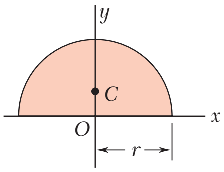
Circle

$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$
$J_O = \frac{1}{2}\pi r^4$



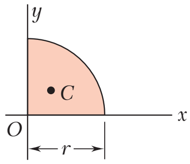
Semicircle

$I_x = I_y = \frac{1}{8}\pi r^4$
$J_O = \frac{1}{4}\pi r^4$



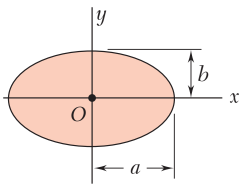
Quarter circle

$I_x = I_y = \frac{1}{16}\pi r^4$
$J_O = \frac{1}{8}\pi r^4$



Ellipse

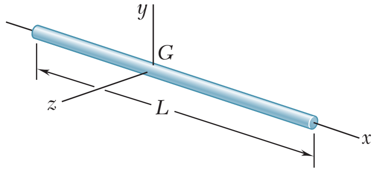
$\bar{I}_x = \frac{1}{4}\pi ab^3$
$\bar{I}_y = \frac{1}{4}\pi a^3 b$
$J_O = \frac{1}{4}\pi ab(a^2 + b^2)$



Mass Moments of Inertia of Common Geometric Shapes

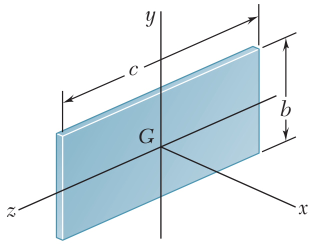
Slender rod

$I_y = I_z = \frac{1}{12}mL^2$



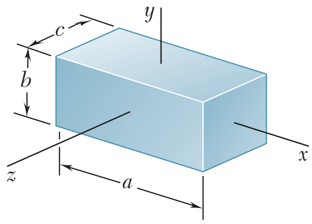
Thin rectangular plate

$I_x = \frac{1}{12}m(b^2 + c^2)$
$I_y = \frac{1}{12}mc^2$
$I_z = \frac{1}{12}mb^2$



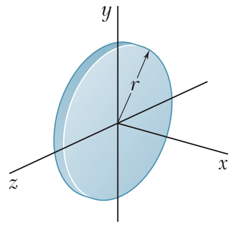
Rectangular prism

$I_x = \frac{1}{12}m(b^2 + c^2)$
$I_y = \frac{1}{12}m(c^2 + a^2)$
$I_z = \frac{1}{12}m(a^2 + b^2)$



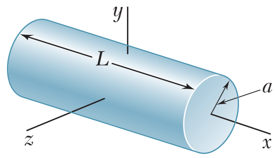
Thin disk

$I_x = \frac{1}{2}mr^2$
$I_y = I_z = \frac{1}{4}mr^2$



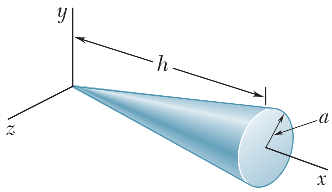
Circular cylinder

$I_x = \frac{1}{2}ma^2$
$I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$



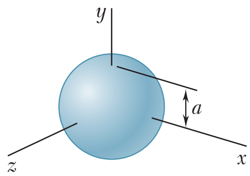
Circular cone

$I_x = \frac{3}{10}ma^2$
$I_y = I_z = \frac{3}{5}m(\frac{1}{4}a^2 + h^2)$



Sphere

$I_x = I_y = I_z = \frac{2}{5}ma^2$



A first course in mechanics should develop a student's ability to analyze and solve problems using well-understood basic principles applied in a simple logical manner. The emphasis of this text focuses on the correct understanding of the principles of mechanics and on their application to the solution of engineering problems. In order to achieve the goal of being able to analyze mechanics problems, the text employs the following pedagogical strategy:

- Practical applications are introduced early
- New concepts are introduced simply
- Fundamental principles are placed in simple contexts

Students are given extensive practice through sample problems, special sections entitled Solving Problems On Your Own, and extensive homework problem sets.

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ISBN 978-0-07-339813-6
MHID 0-07-339813-6

EAN 9 780073 398136 90000

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