The sense in which a "weak measurement" of a spin-$\frac{1}{2}$ particle's spin component yields a value 100

I. M. Duck and P. M. Stevenson
T. W. Bonner Nuclear Laboratory, Physics Department, Rice University, Houston, Texas 77251-1892

E. C. G. Sudarshan
Physics Department, Rice University, Houston, Texas 77251-1892,
Center for Particle Theory, Physics Department, University of Texas, Austin, Texas 78712,
and Institute of Mathematical Sciences, Taramani, Madras 600113, India
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We give a critical discussion of a recent Letter of Aharonov, Albert, and Vaidman. Although their work contains several flaws, their main point is valid: namely, that there is a sense in which a certain "weak measurement" procedure yields values outside the eigenvalue spectrum. Our analysis requires no approximations and helps to clarify the physics behind the effect. We describe an optical analog of the experiment and discuss the conditions necessary to realize the effect experimentally.

I. INTRODUCTION

In a recent, extraordinary paper, Aharonov, Albert, and Vaidman[1] (AAV) claim that a certain procedure, involving post selection as well as preparation of states, can yield a "measurement" of an observable with the "measured value" lying outside the range of the observable's eigenvalues. As their title puts it: ". . . the result of a measurement of a component of the spin of a spin-$\frac{1}{2}$ particle can turn out to be 100."

One's initial reaction is that this is impossible. This prejudice is reinforced when one finds that AAV's paper contains several errors. Nevertheless, after a careful study, we have concluded that AAV's main point does have validity, and our purpose here is to correct the errors and to clarify the nature of the effect.

The phenomenon is not as revolutionary as it sounds—there is nothing here that conflicts with conventional interpretations of quantum mechanics. One could well debate whether the use of the term "measurement" is appropriate. We wish to remain neutral on this semantic and philosophical issue, and we invite the reader to come to his or her own judgment. To avoid circumlocution, however, we shall use AAV's terminology "weak measurement" (in quotation marks) throughout. Our aim is to avoid the metaphysical aspects of measurement theory, and to concentrate on explicating the physics of AAV's effect. We believe the effect is of theoretical and pedagogical interest, and also might possibly have experimental applications for the amplification, and hence detection, of very small signals.[1]

The plan of the paper is as follows. In Sec. II we review AAV's general result, while in Sec. III we analyze their specific example involving spin-$\frac{1}{2}$ particles. Our main concern is to understand how AAV's effect—a "measured value" lying well outside the range of the eigenvalues—can arise from a suitable superposition of the eigenstates. We also show that the effect can be even more dramatic in regions where AAV's approximations break down. An optical analog of AAV's experiment is described in Sec. IV. A summary, and a discussion of the conditions required to realize the effect experimentally, is given in Sec. V.

II. "WEAK" MEASUREMENTS AND THE AAV EFFECT

The starting point of AAV's discussion is the standard von Neumann model of the measurement procedure.[2] The quantum system, whose observable $\hat{A}$ is to be measured, is coupled to a measuring device by a coupling Hamiltonian

$$\hat{H} = -g(t)\hat{q}\hat{A} \ , \tag{1}$$

where $\hat{q}$ is a canonical variable of the measuring device (with conjugate momentum $\hat{p}$) and $g(t)$ is a function with compact support near the time of the measurement (normalized such that its time integral is unity). An *ideal* measuring device has well-defined initial and final values of $\hat{p}$, and the difference $p_f - p_{\rm in}$ is the device's "pointer reading," which registers the value of $\hat{A}$.

More realistically, the measuring device would have some initial spread Δp. Let us suppose that the device has an initial state $|\Phi_{\rm in}\rangle$ whose p-representation wave function $\tilde{\phi}_{\rm in}(p)$ is a Gaussian centered on $p=0$ with width Δp. The q-representation wave function $\phi_{\rm in}(q)$, which is the Fourier transform of $\tilde{\phi}_{\rm in}(p)$, would also be of Gaussian form. Explicitly,[3]

$$|\Phi_{\rm in}\rangle = \begin{cases} \int dq\, \phi_{\rm in}(q)|q\rangle & (q \text{ representation}), \\ \int dp\, \tilde{\phi}_{\rm in}(p)|p\rangle & (p \text{ representation}), \end{cases} \tag{2}$$

where

$$\phi_{\rm in}(q) \equiv \langle q|\Phi_{\rm in}\rangle = \exp\left[-\frac{q^2}{4\Delta^2}\right],$$
$$\tilde{\phi}_{\rm in}(p) \equiv \langle p|\Phi_{\rm in}\rangle = \exp(-\Delta^2 p^2), \tag{3}$$
$$\Delta q \equiv \Delta, \quad \Delta p = 1/(2\Delta),$$

with $\hbar=1$. We shall also assume that the quantum system has been prepared in some definite state $|\Psi_{\rm in}\rangle$. Quite generally, this state may be expanded in terms of the eigenstates of $\hat{A}$:

$$|\Psi_{\rm in}\rangle = \sum_n \alpha_n |A=a_n\rangle . \tag{4}$$

Under the action of the coupling Hamiltonian—which, for the short duration of the measurement, can be assumed to dominate all other terms in the full Hamiltonian—the whole system (quantum system plus measuring device) will evolve to

$$\exp\left[-i\int \hat{H}\,dt\right]|\Psi_{\rm in}\rangle|\Phi_{\rm in}\rangle$$
$$=\sum_n \alpha_n \int dq\, e^{iqa_n}\exp\left[\frac{-q^2}{4\Delta^2}\right]|A=a_n\rangle|q\rangle . \tag{5}$$

Inserting $\hat{1}=\int dp|p\rangle\langle p|$ this can be rewritten in the p representation as equal to

$$\sum_n \alpha_n \int dp\, \exp[-\Delta^2(p-a_n)^2]|A=a_n\rangle|p\rangle . \tag{6}$$

Thus, if $\Delta p = 1/(2\Delta)$ is small compared to the spacing between the a_n's, the measuring device is left in a state consisting of widely separated "spikes," each centered on one of the eigenvalues a_n. Hence, in the limit $\Delta p \to 0$, one has all the properties of an ideal measurement: (i) the measurement always produces one of the eigenvalues a_n; (ii) the probability of the outcome a_n is $|\alpha_n|^2$; (iii) is the measurement yields a_n then the quantum system is left in the eigenstate $|A=a_n\rangle$.

However, let us consider the opposite limit, in which Δp is much larger than the spread of the a_n's. AAV refer to this case as a "weak measurement." The final state of the "measuring" device is then a superposition of strongly overlapping, broad Gaussians. The final value of $\hat{p}$, which indicates the "measured" value of $\hat{A}$, has a probability distribution given by the absolute square of the p-representation wave function of the whole system, i.e., the overlap of (6) with $\langle p|$:

$$P(p)=\sum_n |\alpha_n|^2 \exp[-2\Delta^2(p-a_n)^2] . \tag{7}$$

In the "weak" case (large Δp, small Δ), this will approximate a single, broad Gaussian peaked at the mean value of $\hat{A}$, which is $\langle \hat{A}\rangle = \sum_n |\alpha_n|^2 a_n$. Of course, any single "weak measurement" gives almost no information, since $\Delta p \gg \langle \hat{A}\rangle$. However, by repeating the experiment many times one can map out the whole distribution, and so determine the centroid $\langle \hat{A}\rangle$ to any desired accuracy.[4]

AAV's point is that more interesting effects arise if one makes a *post selection* of the state of the quantum system: that is, immediately after the "weak measurement" of $\hat{A}$, one makes a (strong) measurement of some other observable $\hat{B}$ and one selects a single outcome $B=b$. This puts the quantum system into a definite final state

$$|\Psi_f\rangle = |B=b\rangle = \sum_n \alpha'_n |A=a_n\rangle . \tag{8}$$

[The procedure is exactly analogous to the preparation (or *preselection*) of the initial state $|\Psi_{\rm in}\rangle$]. After the post-selection procedure, the "measuring device" will be left in the state

$$|\Phi_f\rangle = \left\langle \Psi_f \left| \exp\left[-i\int \hat{H}\,dt\right]\right| \Psi_{\rm in}\right\rangle |\Phi_{\rm in}\rangle . \tag{9}$$

This expression can be straightforwardly evaluated using the expansions of $|\Psi_{\rm in}\rangle$ and $|\Psi_f\rangle$. In the q representation,

$$|\Phi_f\rangle = \sum_n \alpha_n \alpha'^*_n \int dq\, e^{iqa_n}\exp\left[-\frac{q^2}{4\Delta^2}\right]|q\rangle \tag{10}$$

and hence

$$|\Phi_f\rangle = \sum_n \alpha_n \alpha'^*_n \int dp\, \exp[-\Delta^2(p-a_n)^2]|p\rangle \tag{11}$$

in the p representation.

AAV evaluate $|\Phi_f\rangle$ in a different manner, invoking some sweeping approximations. They define the quantity

$$A_w \equiv \langle \Psi_f|\hat{A}|\Psi_{\rm in}\rangle / \langle \Psi_f|\Psi_{\rm in}\rangle , \tag{12}$$

which they call the "weak value of $\hat{A}$," and argue as follows:

$$|\Phi_f\rangle = \langle \Psi_f|e^{i\hat{q}\hat{A}}|\Psi_{\rm in}\rangle|\Phi_{\rm in}\rangle \tag{13}$$
$$\simeq (\langle \Psi_f|\Psi_{\rm in}\rangle + i\hat{q}\langle \Psi_f|\hat{A}|\Psi_{\rm in}\rangle + \cdots)|\Phi_{\rm in}\rangle \tag{14}$$
$$= \langle \Psi_f|\Psi_{\rm in}\rangle(1+i\hat{q}A_w+\cdots)|\Phi_{\rm in}\rangle \tag{15}$$
$$\simeq \langle \Psi_f|\Psi_{\rm in}\rangle \int dq\, e^{iqA_w}\exp\left[-\frac{q^2}{4\Delta^2}\right]|q\rangle . \tag{16}$$

In the p representation this becomes

$$|\Phi_f\rangle \simeq \langle \Psi_f|\Psi_{\rm in}\rangle \int dp\, \exp[-\Delta^2(p-A_w)^2]|p\rangle . \tag{17}$$

Obviously, this wave function is a single, broad Gaussian centered on A_w. What makes this disconcerting is the fact that A_w may lie outside—even far outside—the range of the eigenvalues a_n (e.g., consider two nearly orthogonal states $|\Psi_{\rm in}\rangle$ and $|\Psi_f\rangle$ with a sizable matrix element $\langle \Psi_f|\hat{A}|\Psi_{\rm in}\rangle$, so that A_w is very large).

Let us first consider the restrictions that are necessary to justify the manipulations in AAV's calculation above. First, the neglect of the higher terms in (14), with respect to the two terms retained, requires that

$$|q^n\langle \Psi_f|\hat{A}^n|\Psi_{\rm in}\rangle| \ll |\langle \Psi_f|\Psi_{\rm in}\rangle| , \tag{18}$$
$$|q^n\langle \Psi_f|\hat{A}^n|\Psi_{\rm in}\rangle| \ll |q\langle \Psi_f|\hat{A}|\Psi_{\rm in}\rangle| , \tag{19}$$

for all $n \geq 2$. Next, the resummation made in going from (15) to (16) assumes that $|qA_w| \ll 1$. In that case, the condition (19) becomes a much stronger restriction than (18). Finally, since the spread of q values is governed by

Δ, we can effectively replace q by Δ in the above. In summary, the validity of AAV's calculation requires

$$\Delta \ll 1/A_w \tag{20}$$

and

$$\Delta \ll \min_{(n=2,3,\ldots)} \left| \frac{\langle \Psi_f | \hat{A} | \Psi_{\text{in}} \rangle}{\langle \Psi_f | \hat{A}^n | \Psi_{\text{in}} \rangle} \right|^{1/(n-1)} . \tag{21}$$

[Note that the condition quoted by AAV in their Eq. (4) is incorrect in several respects.]

The calculation in Eqs. (13)–(16) above does therefore have a certain region of validity and we must now confront the apparent paradox arising from the two expressions for $|\Phi_f\rangle$: Eqs. (11) and (17). How can $|\Phi_f\rangle$ be a single Gaussian peaked at A_w [Eq. (17)], while simultaneously being a superposition of Gaussians, each peaked at a value a_n [Eq. (11)], when A_w may well be much greater than any of the a_n's? How can a large shift be produced from a superposition of small shifts?

The resolution of the paradox lies in the fact that the superposition of Gaussians in Eq. (11) involves *complex* coefficients. Thus, in contrast with classical probability theory, or with the situation in Eq. (7), where the weights are all positive definite, one may have complicated cancellations between the individual Gaussians. Such cancellations are capable of producing a function whose peak is shifted far to one side, though not by more than of order the width $\Delta p = 1/(2\Delta)$—a fact that is reflected in the restriction $A_w \ll 1/\Delta$ above. The phenomenon is not unique to Gaussians, and would occur for any qualitatively similar distributions. Some explicit, numerical examples will be given at the end of Sec. III and in Figs. 2–4 below.

III. EXAMPLE INVOLVING SPIN-$\frac{1}{2}$ PARTICLES

AAV illustrate their general discussion with the following experiment. A beam of spin-$\frac{1}{2}$ particles moves in the y direction with well-defined velocity. The beam is prepared such that the particles' spins point in the xz plane at an angle α to the x axis. It is assumed that the spatial wave function of the particles has a Gaussian shape of width Δ in the z direction. Consequently, the beam is diverging with a momentum spread $\Delta p_z = 1/(2\Delta)$.

A measurement of the z component of the spin is performed in the usual way, by passing the beam through a Stern-Gerlach magnet. This produces a coupling between the spin operator $\hat{\sigma}_z$ and the z coordinate through the coupling Hamiltonian

$$H = -\lambda g(t) \hat{z} \hat{\sigma}_z , \tag{22}$$

where λ is proportional to the particle's magnetic moment, to $\delta B_z/\delta z$, etc., so that the localized function $g(t)$ (which arises from the passage of the beam particles through the localized region of inhomogeneous magnetic field) is normalized to unity. Using the earlier terminology, the "quantum system's" state $|\Psi\rangle$ corresponds to the particles' spin state, while the state of the "measuring device," $|\Phi\rangle$, corresponds to their spatial wave function: The operator $\hat{A}$ is here $\lambda\hat{\sigma}_z$; the coordinate q is z, and hence p is p_z.

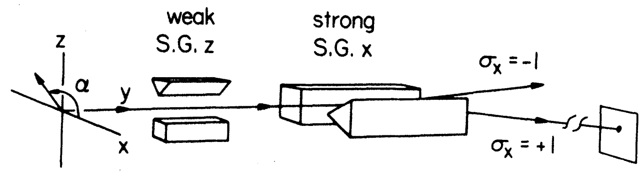

FIG. 1. Stern-Gerlach magnet layout for the AAV experiment.

We shall consider a "weak measurement" of $\lambda\sigma_z$, in which the beam splitting δp_z induced by the Stern-Gerlach magnet is small compared to $\Delta p_z = 1/(2\Delta)$, the overall p_z spread of the beam. Thus, the $\sigma_z = +1$ and -1 components of the beam continue to overlap strongly and are not cleanly separated as they would be in a (strong) measurement. A post selection of the spin state is made by passing the beam through a second magnet with a *strong* Stern-Gerlach field aligned in the x direction. This splits the beam into two well-separated beams and the $\sigma_x = +1$ beam is selected and imaged on a distant screen. (See Fig. 1.)

Thus, the initial spin state is the $+1$ eigenstate of $(\cos\alpha)\sigma_x + (\sin\alpha)\sigma_z$,

$$|\Psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} \end{bmatrix} , \tag{23}$$

and the final state is the $+1$ eigenstate of $\hat{\sigma}_x$:

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{24}$$

[Note that (u,v) is shorthand for $u|\uparrow\rangle + v|\downarrow\rangle$, where $|\uparrow\rangle, |\downarrow\rangle$ are the eigenstates of $\hat{\sigma}_z$.]

Hence,

$$\langle \Psi_f | \Psi_{\text{in}} \rangle = \cos\frac{\alpha}{2}, \quad \langle \Psi_f | \hat{\sigma}_z | \Psi_{\text{in}} \rangle = \sin\frac{\alpha}{2} , \tag{25}$$

so that the "weak" value of the spin component σ_z, in AAV's sense, is

$$A_w = (\lambda\sigma_z)_w = \lambda \tan\frac{\alpha}{2} . \tag{26}$$

The initial spatial wave function is

$$\phi_{\text{in}}(q) \equiv \langle q | \Phi_{\text{in}} \rangle = \exp\left[-\frac{z^2}{4\Delta^2} \right] f(x,y) . \tag{27}$$

The precise x,y dependence is unimportant and we shall ignore it henceforward.

Substituting in Eq. (17), we obtain AAV's prediction for the final (p-representation) wave function:

$$\tilde{\phi}_f(p) \equiv \langle p | \Phi_f \rangle$$
$$\simeq \left[\cos\frac{\alpha}{2}\right] \exp\left[-\Delta^2\left[p_z - \lambda\tan\frac{\alpha}{2}\right]^2\right] . \quad (28)$$

The z distribution of counts observed on the distant screen will, of course, directly reflect the distribution of p_z's in the final beam, which is given by the absolute square of Eq. (28). Thus, AAV's result means that the observed distribution will be Gaussian, centered on the value $\tan\alpha/2$ (on a scale where $\sigma_z = +1$ would correspond to the value $+1$). The surprising feature is that, when α approaches π, the "measured value," $\tan\alpha/2$, can be much greater than unity.

From (16), AAV's result is valid provided that

$$\Delta \ll \lambda^{-1}\min\left[\tan\frac{\alpha}{2}, \cot\frac{\alpha}{2}\right] . \quad (29)$$

This restriction implies that, for a given Δ, one cannot approach too close to $\alpha = \pi$. Nevertheless, with a sufficiently small Δ (i.e., a sufficiently rapidly diverging beam), one could let α be close enough to π to "measure" a value 100, say, for the z-spin component of a spin-$\frac{1}{2}$ particle.

To understand this better let us specialize to the case where $\alpha = \pi - 2\epsilon$ with $\epsilon \ll 1$. AAV's result then reduces to

$$\tilde{\phi}_f(p) \simeq \epsilon \exp[-\Delta^2(p_z - \lambda/\epsilon)^2] , \quad (30)$$

and is valid if

$$\lambda\Delta \ll \epsilon \ll 1 . \quad (31)$$

The corresponding exact result is easily obtained from Eq. (11):

$$\tilde{\phi}_f(p) = \varphi(p_z; \epsilon, \Delta, \lambda) , \quad (32)$$

where φ is a function of p, with parameters ϵ, Δ, and λ, defined by

$$\varphi(p;\epsilon,\Delta,\lambda) \equiv \tfrac{1}{2}\{(1+\epsilon)\exp[-\Delta^2(p-\lambda)^2]$$
$$-(1-\epsilon)\exp[-\Delta^2(p+\lambda)^2]\} . \quad (33)$$

Note that λ could be eliminated by the scaling relation

$$\varphi(p;\epsilon,\Delta,\lambda) = \varphi(p/\lambda;\epsilon,\lambda\Delta,1) . \quad (34)$$

The exact form (33) makes it evident that the wave function is a superposition of two Gaussians, centered on $p = \pm\lambda$, corresponding to the $\sigma_z = \pm 1$ eigenstates. The trick lies in the near cancellation of the two terms, which leaves a small difference function $\varphi(p)$, which turns out to be approximately Gaussian and peaked at $p = \lambda/\epsilon$, in agreement with AAV's form, (30). This fact is illustrated in Fig. 2, which shows $\varphi(p)$ for $\epsilon = 0.2$ and $\lambda\Delta = 0.01$.

The effect becomes more dramatic as ϵ is made smaller (for fixed Δ) (Ref. 5). The rightward shift of the peak increases proportional to $1/\epsilon$: At least, it does so until AAV's approximations break down, which happens when ϵ becomes of order $\lambda\Delta$. The shift of the peak cannot exceed $O(1/\Delta)$ and so the effect saturates. [See Fig. 3(a) for $\epsilon = \lambda = 0.01$.] Note that a "dip" becomes visible on the left-hand side and this becomes more pronounced if ϵ is decreased further. In terms of the probability distribution $|\varphi(p)|^2$ the "dip" appears as a small, secondary peak. [See Fig. 3(b).]

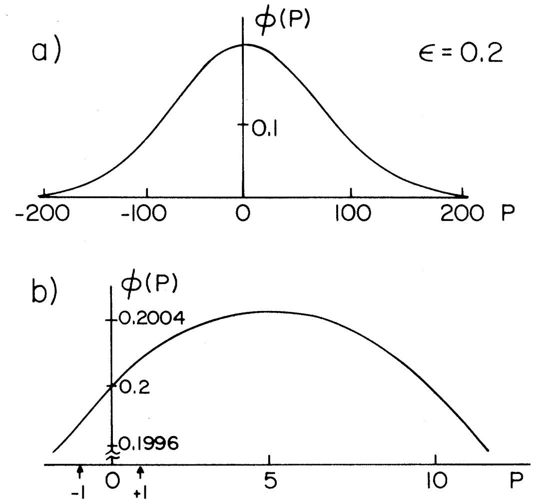

FIG. 2. (a) Graph of the function $\varphi(p;\epsilon,\Delta,\lambda)$ [see Eq. (33)] as a function of $P \equiv p/\lambda$ with $\lambda\Delta = 0.01$ and $\epsilon = 0.2$. Note that the function resembles a single Gaussian whose peak, shown in a closeup in (b), is shifted to $P = 1/\epsilon = 5$.

Ultimately, if ϵ is reduced to zero, one obtains the antisymmetric wave function shown in Fig. 4(a), which produces the "double-humped" probability distribution of Fig. 4(b). AAV's approximations break down completely in this region, of course. However, the effect is still a rather striking one: the two peaks of the distribution are located at $p \simeq \pm 0.7/\Delta$, and not at $\pm\lambda$ (so that $p/\lambda = \pm 70$ rather than ± 1 in the example of the figures). Nevertheless, this distribution arises from a wave function which is a superposition of $\sigma_z = \pm 1$ components only.

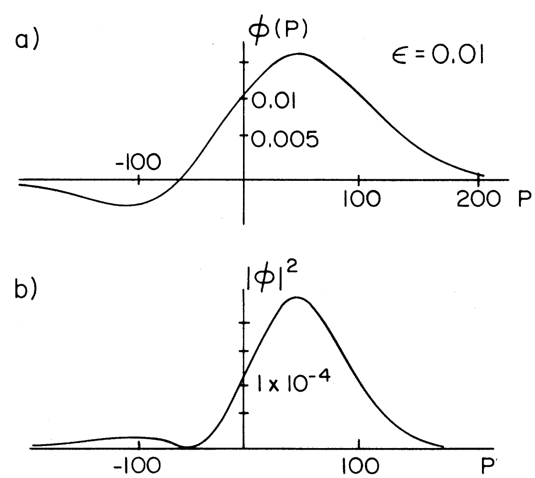

FIG. 3. (a) As Fig. 2, but for $\epsilon = \lambda\Delta = 0.01$. Note that although the component Gaussians peak at $+1$ and -1, the composite function is shifted by about half the width $1/\Delta$. (b) The resulting probability distribution $|\phi(P)|^2$.

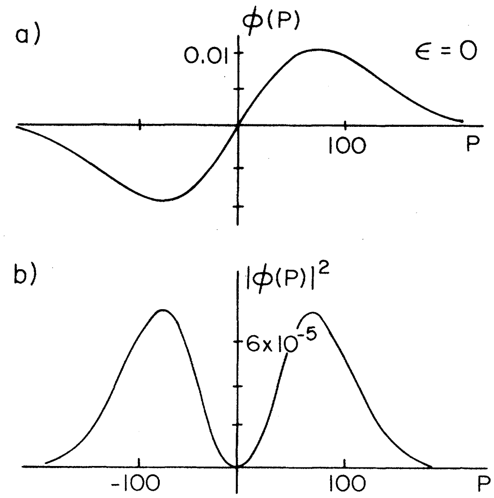

FIG. 4. (a) As in Fig. 2, but for $\epsilon=0$. (b) The resulting probability distribution $|\varphi(P)|^2$.

To conclude this section we note a straightforward generalization of the experiment, in which the initial and final spins are selected to be at angles α and β to the x axis, respectively. (Previously, β was taken to be zero.) The "weak" value of the operator $\lambda\hat{\sigma}_z$ would then be $A_w=\lambda/\epsilon$, where

$$\epsilon=\epsilon_{(1/2)}(\alpha,\beta)\equiv\frac{\cos[\frac{1}{2}(\alpha-\beta)]}{\sin[\frac{1}{2}(\alpha+\beta)]} . \tag{35}$$

AAV's result would be

$$\tilde{\phi}_f(p)=\cos[\tfrac{1}{2}(\alpha-\beta)]\exp[-\Delta^2(p_z-\lambda/\epsilon)^2] , \tag{36}$$

valid if

$$\lambda\Delta\ll\min[\epsilon,1/\epsilon] . \tag{37}$$

The exact result can be expressed as

$$\tilde{\phi}_f(p)=\sin[\tfrac{1}{2}(\alpha+\beta)]\varphi(p_z;\epsilon,\Delta,\lambda) , \tag{38}$$

in terms of the function introduced in Eq. (33). The above formulas are valid for any value of ϵ, but the interesting effects occur when ϵ is small. The ideal region is therefore around $\alpha\simeq\pi$, $\beta\simeq0$ (or vice versa), corresponding to the case considered above.

IV. AN OPTICAL ANALOG

An optical version of the previous experiment can be constructed. Polarized light replaces the spin-$\frac{1}{2}$ particles, and a laser beam, suitably expanded with lenses, can provide the broad, coherent beam needed. The setup is sketched in Fig. 5. A polarizer and analyzer select the initial and final polarizations to be at angles α and β to the x axis, respectively. Between them is placed the "weak measuring device"—a slab of weakly birefringent crystal, which introduces a small lateral displacement between the "ordinary" and "extraordinary" rays[6]—which one

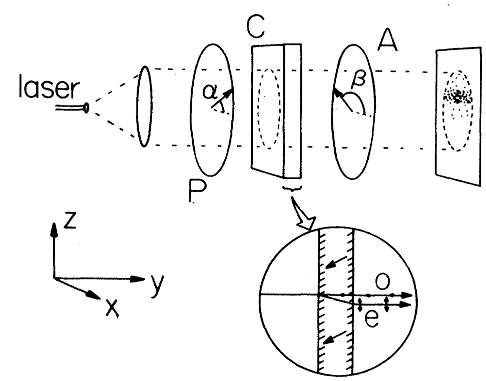

FIG. 5. An optical analog of the AAV experiment. A broad, coherent beam passes through a polarizer (P) and analyzer (A). Between them is a birefringent crystal (C) which produces a small lateral displacement between x (o ray) and z (e ray) polarizations (see inset).

arranges to correspond to the x and z polarizations, respectively. This lateral displacement, easily calculable for a given crystal, needs to be small compared to the width of the beam. Note that, because the displacement produced by the doubly refracting material is *lateral,* rather than an angular deflection, as in the case of a Stern-Gerlach magnet, we shall need to concentrate on the z, rather than the p_z, distribution. This means that Δ and $1/(2\Delta)\equiv\delta$ will exchange roles, relative to our previous discussion.

The initial beam is assumed to have a Gaussian profile, with a large spatial width, $1/(2\delta)$. After passing through the polarizer, its wave function is

$$\langle q|\Phi_{\rm in}\rangle|\Psi_{\rm in}\rangle=\exp(-\delta^2z^2)\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix} , \tag{39}$$

where $|\Psi\rangle$ here corresponds to a two-vector, not a spinor. In passing through the birefringent material the o ray (x polarization) and e ray (z polarization) are displaced by different amounts a_1 and a_2, so the emerging beam will be

$$\begin{bmatrix}\cos\alpha\exp[-\delta^2(z-a_1)^2]\\ \sin\alpha\exp[-\delta^2(z-a_2)^2]\end{bmatrix} . \tag{40}$$

Finally, the analyzer projects out the β component of polarization, producing the spatial wave function

$$\langle q|\Phi_f\rangle=\cos\beta\cos\alpha\exp[-\delta^2(z-a_1)^2] +\sin\beta\sin\alpha\exp[-\delta^2(z-a_2)^2] . \tag{41}$$

This can be rewritten as

$$\phi_f(q)\equiv\langle q|\Phi_f\rangle=\cos(\alpha+\beta)\varphi(z-\bar{a};\epsilon,\delta,\lambda) \tag{42}$$

in terms of the φ function introduced in Eq. (33). The parameters are given by

$$\bar{a}=\tfrac{1}{2}(a_1+a_2),\quad \lambda=\tfrac{1}{2}(a_1-a_2) , \tag{43}$$

and

$$\epsilon = \epsilon_{(1)}(\alpha,\beta) \equiv \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} . \tag{44}$$

Notice that here ϵ involves the whole angles, not the half-angles characteristic of the spinor case [cf. (35)]. Because of this, the interesting region in which ϵ is small now corresponds to $\alpha \simeq 3\pi/4$, $\beta \simeq \pi/4$ (or vice versa), i.e., nearly orthogonal polaroids, each at 45° to the x and z axes. By arranging that $\epsilon \simeq \lambda\delta$, for instance, one can expect to see an intensity distribution resembling Fig. 3(b), in which the beam z value is shifted by $\sim 1/(2\delta)$.

In principle, one can produce a shift of order $1/(2\delta)$ *no matter how small* λ *is.* Thus, one could detect a very small birefringence of the crystal with a photodetector whose position resolution is nowhere near fine enough to detect the lateral separation between the o and e rays. There is a price, of course: One would need to be able to control the angles α and β, and hence ϵ, to great precision. Also, one faces a loss of intensity by a factor of $\lambda^2\delta$, because of the nearly orthogonal polarizers. (Further limitations arise because actual polarizers are not perfectly efficient.) Nevertheless, an experiment of this kind should be perfectly feasible. We hope that it will serve to illustrate the general point that the AAV effect offers a possible means of amplifying and detecting very small signals.

V. SUMMARY

In this paper we have reexamined the problem of "weak measurement" of a quantum dynamical variable $\hat{A}$, in particular the component of a spin, allowing for post selection of the events. We find that the dramatic effect pointed out by Aharonov, Albert, and Vaidman does indeed obtain: the broad distribution of "measured" values can be peaked far outside the range of eigenvalues of $\hat{A}$. This effect obtains under certain exceptional conditions. (1) *Post selection*: the measured events are selected on the basis of a (strong) measurement of some noncommuting observable after the original "weak measurement" is completed. Thus, the result of the "measurement" depends both on the preparation and on the post selection. (2) *Diffuse measurements:* the spread Δp of the pointer readings intrinsic to the "weak measuring device" is large compared with the separation of the expected mean values for the various eigenvalues of A. (3) *Coherent spread.* the spread Δp of the pointer readings must be quantum-mechanically coherent, not merely a probabilistic spread. That is, the state of the "weak measuring device" must be representable by a wave function, rather than by an impure density matrix. This is unfortunately not true of most measuring devices. However, it is not an impossible condition to satisfy. In the case of the experiments described in Secs. III and IV the requirement is that the beam should be coherent across its width.

The surprising effect pointed out by AAV has been shown to be a consequence of constructive and destructive interference between two complex amplitudes. Although surprising, the effect is in no way paradoxical, and involves nothing outside ordinary quantum mechanics. Our analysis, since it dispenses with AAV's approximations, also provides a more complete description of the effect. We have also described an optical analog of AAV's experiment, and emphasized the point that the AAV effect could have applications to the detection of very tiny signals.

Note added. Since this paper was submitted, comments on the AAV Letter by A. J. Leggett, and by A. Peres's, together with a response from Aharonov and Vaidman, have appeared in Phys. Rev. Lett. **62**, 2325 (1989). Leggett disputes the use of the term "measurement"—a thorny issue that we have deliberately avoided. Peres's objection is, we believe, amply answered by our discussion. A new manuscript by Aharonov and Vaidman clarifies the mathematical example originally presented in Ref. 1. We refer the interested reader to this paper, and withdraw our earlier criticism of this example.

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[1]Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).

[2]J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932) [English translation: *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955)]. See also W. E. Lamb, Phys. Today **22**, (4), 23 (1969).

[3]The kets $|p\rangle$ and $|q\rangle$ are eigenstates of the $\hat{p}$ and $\hat{q}$ operators: $\hat{p}|p\rangle = p|p\rangle$ and $\hat{q}|q\rangle = q|q\rangle$. Their overlaps are $\langle p'|p\rangle = \delta(p-p')$, $\langle q'|q\rangle = \delta(q-q')$, and $\langle p|q\rangle = e^{-ipq}$. Note that Δq, Δp refer to the widths of the probability distributions, obtained from the absolute square of the appropriate wave function.

[4]Real measurements have been made in this fashion. One example is the measurement of the τ lepton's lifetime [e.g., G. J. Feldman *et al.*, Phys. Rev. Lett. **48**, 66 (1982); E. Fernandez *et al.*, *ibid.* **54**, 1624 (1985); JADE Collaboration, C. Kleinwort *et al.*, Z. Phys. C **42**, 7 (1989)]. The position resolution of the detectors used is nowhere near good enough to resolve the τ's decay length in any single event. The "apparent decay length" has a very broad distribution, dominated by position-measurement errors. Nevertheless, with a large sample of events, the lifetime can be deduced from the small shift in the mean of the distribution.

[5]The experiment becomes harder as $\epsilon \to 0$ because the intensity of the final beam decreases. In AAV's approximation the intensity vanishes like ϵ^2 as $\epsilon \to 0$. However, the exact result shows that there is a nonzero intensity, of order Δ^2, even when $\epsilon = 0$.

[6]See, for example, R. S. Longhurst, *Geometrical and Physical Optics,* 3rd ed. (Longmans, London, 1984), Chap. 22, or E. Hecht and A. Zajac, *Optics* (Addison-Wesley, Reading, MA, 1974), Chap. 8. The crystal slab could be replaced by a Kerr cell, in which the degree of birefringence is controllable.