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Weak measurements, non-classicality and negative probability

Sooryansh Asthana[1] · **V. Ravishankar**[1]

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Abstract
This paper establishes a direct, robust, and intimate connection among (i) non-classicality tests for various quantum features, e.g. non-Boolean logic, quantum coherence, nonlocality, quantum entanglement, quantum discord; (ii) negative probability, and, (iii) anomalous weak values. It has been shown (Adhikary et al. Eur Phys J D 74(68):68, 2020; Asthana et al. Quantum Inform Process 20(1):1–33, 2021) that the nonexistence of a classical joint probability scheme gives rise to sufficiency conditions for nonlocality, a nonclassical feature not restricted to quantum mechanics. The conditions for nonclassical features of quantum mechanics are obtained by employing pseudo-probabilities, which are expectation values of the parent pseudo-projections. The crux of the paper is that the pseudo-probabilities, which can take negative values, can be directly measured as anomalous weak values. We expect that this opens up new avenues for testing nonclassicality via weak measurements and also gives deeper insight into negative pseudo-probabilities, which become measurable. A quantum game, based on violation of a classical probability rule, is also proposed that can be played by employing weak measurements.

Keywords Pseudo-projection · Pseudo-probability · Anomalous weak value · Non-classicality · Non-locality · Entanglement · Non-Boolean logic

1 Introduction

Since its very inception, quantum mechanics has been looked upon in many ways, as may be seen from its different formulations [1–6]. Its unique features have also been explored and expanded in several seminal works [7–10]. Presently, investigations of

✉ Sooryansh Asthana
sooryansh.asthana@physics.iitd.ac.in

V. Ravishankar
vravi@physics.iitd.ac.in

1 Department of Physics, Indian Institute of Technology Delhi, New Delhi 110016, India

fundamental features of quantum mechanics have acquired an importance like never before, for two reasons. Conceptually, there is an improved understanding of the so-called non-classical features of quantum states and, equally importantly, there are new insights into the very concept of the measurement—due to weak measurements in quantum mechanics [11,12]. In terms of practical impact, the importance owes to non-locality, entanglement, quantum discord [9,13,14] of states, which act as non-classical resources for applications to quantum computation [15,16], and quantum information processing [17–21], many of which are being implemented experimentally.

This paper proposes to establish a robust connection between non-classicality, as expressed through notions, such as non-Boolean logic [22], non-locality, entanglement, and quantum discord, and the anomalous weak values, as espoused in the concept of weak measurements. In doing so, the paper aims to elucidate the key notions of quantum probability via weak measurements, through which the so-called negative probability becomes an observable quantity. Thereby, it is hoped that this study illustrates how the traditionally understood non-classical aspects of a quantum state and anomalous weak values are intimately intertwined with each other.

We have two starting points. The first one is the concept of nonclassical probability. It has been shown that various criteria for nonlocality and entanglement emerge as violations of classical probability rules [23–25]. In quantum mechanics, pseudoprobability emerges as the expectation value of a pseudo-projection, introduced recently to explore non-classical features of quantum states directly in the language of quantum probability [24]. A special case, which is also the simplest example, is the Margenau–Hill–Barut distribution [26,27] which serves our purpose in this paper. More generally, in principle—by their very construction—pseudo-projections capture all possible criteria, such as the ones laid out in [14,28–30]. An important feature of this approach is that it does not employ involved algebraic techniques. All the criteria of nonclassicality follow from violations of classical probability rules in a fairly straightforward manner. The task is to identify appropriate sums of pseudoprobabilities. Furthermore, pseudoprojections also capture violation of Boolean logic in the quantum domain.

In fact, numerous entanglement inequalities[1] emerge by imposing nonclassicality conditions on pseuodprobabilities [24,25]. The criteria for nonlocality emerge when a few entries of a joint probability scheme turn negative, without recourse to any model. It is a manifestation of the fact that nonlocality is a nonclassical feature, not restricted to quantum mechanics. In this way, negative probability not only provides us with a unified framework to obtain various criteria for different nonclassical features but also brings out the subtle difference between them.

The second starting point is the concept of weak measurements [11,31], which has expanded the scope of measurements beyond the traditional projective measurements. Weak measurements, which admit experimental implementation, display non-classicality by predicting anomalous weak values, which are otherwise forbidden for projective measurements. In particular, the outcomes of weak measurements may lie beyond the values allowed by the spectrum of the observable. The relationships between anomalous weak values with contextuality, and with counterfactual processes

[1] Entanglement inequalities are violated by all the separable states and obeyed by at least one entangled state.

have been studied in [32] and [33], respectively. Of even greater relevance to us are the weak measurements of noncommuting observables. Schemes for such measurements have been proposed [34], and they have already been implemented experimentally [35–37]. Recently, the weak value of local projection has been shown to appear as the modification in weak couplings in interferometric alignment [38].

Equipped with the weak measurement techniques, we show how they can be employed to observe the violation of Boolean logic in the quantum domain. Furthermore, this paper undertakes the task of expressing non-classicality tests, mainly based on the recent formulation by employing pseudo-projections [24], in terms of weak measurements of appropriate projections, followed by a post-selection. In particular, we focus on non-locality, entanglement inequalities, and condition for discord in two-qubit systems. This opens up new experimental avenues to test quantum features of states. Equally pertinently, we gain a better understanding of the negative pseudo-probability [24] (as also of negative probabilities which were introduced much earlier, in a rather intuitive way, by Dirac [39] and Feynman [40], and which was further made use of in [41]). Negative probabilities are no more confined to the conceptual realm, but become measurable, as negative anomalous weak values.

The paper is organised as follows. The next section (2) summarises the formalisms employed for deriving conditions for non-Boolean logic, quantum coherence, entanglement, quantum discord, and nonlocality. Section 3 presents the basics of weak measurements, to the extent needed for the work and establishes the interrelation between pseudo-probabilities and anomalous weak values. In Sect. 4, we setup notations for an uncluttered discussion. In Sect. 5, we discuss the relation of the present work with earlier works. In Sect. 6.1, anomalous weak value sufficient for a single qubit to be coherent has been identified. In Sect. 6.2, we turn our attention to features of quantum logic that violate classical Boolean logic directly and show that the weak measurements can be used to directly test those violations. In Sect. 6.3, a quantum game, based on violation of classical probability rules to be played employing weak measurements, has been given. The results of the paper involving non-classical correlations are contained in Sects. 7.1, 7.2 and 7.4. In these sections, in the same order, we identify the anomalous weak values that are sufficient for (i) the CHSH non-locality, (ii) families of entanglement inequalities for two-qubit systems, and, (iii) condition for discord (again in two-qubit systems). In Sect. 8, we have discussed how the hierarchical structure among different nonclassical correlations gets reflected in the choices of pseudoprobabilities. The Sect. 9 summarises the results and discusses the scope for further work.

2 Preliminaries

In this section, we discuss our approach for deriving conditions for numerous non-classical features.

2.1 Non-Boolean logic, entanglement, quantum discord, and quantum coherence

Non-Boolean logic, quantum entanglement, quantum discord, and quantum coherence, etc., are nonclassical features of quantum mechanics, which is a new theory of probability. It does not follow Kolmogorov's axioms of classical probability.[2] We discuss here how, by employing standard rules of quantum mechanics, classical indicator functions for joint events map to pseudo-projection operators, and the pseudoprobabilities in quantum mechanics can be defined as expectation values of the pseudoprojections. The logical propositions involving conjunction (AND operation), negation (NOT operation), and disjunction (OR operation) also get represented by these pseudoprojection operators.

In particular, we first recapitulate how Margenau–Hill distribution emerges from a very specific class of Hermitian representatives of indicator functions for joint events. For a detailed discussion, please refer to [24,42].

2.1.1 Joint probabilities and pseudo-projections

2.1.2 For a single observable

Consider the event that an observable M takes a value m. For a classical system, the probability for the event can be determined by, (i) identifying the support for the event in the phase space (and more generally, in the event space) and, (ii) finding the overlap of the indicator function for the support with the given state (probability density). Recall that an indicator function is a dichotomic Boolean observable, taking value 1 in the support and 0 elsewhere. In quantum mechanics, the observable M would be represented by a Hermitian operator[3] M in a Hilbert space, which admits the eigenresolution $M = \sum_i m_i \pi_{m_i}$. The probability that the observable M takes a value m_i, for a system in a state ρ, is given by the overlap, $\mathrm{Tr}(\rho \pi_{m_i})$. The projections, π_{m_i}, are thus the quantum representatives of the parent indicator functions.

2.1.3 For two observables

Classically, the indicator function for joint outcomes of any two observables is simply the product of the respective indicator functions, which is non-vanishing only over the intersection of the two supports. However, in quantum mechanics, observables do not necessarily commute, and there is no projection operator that would represent the classical indicator function for joint outcomes. That is, the indicator function for the intersection does not always map to a projection. Non-classicality can be understood by constructing their quantum representatives.

Thus, consider two observables M, N. Let $\pi_{m_i},\ \pi_{n_j}$ be the projection operators representing the respective indicator functions for the outcomes $M = m_i$ and $N = n_j$. The operator, representing the indicator function for the classical joint outcome, which we term as *pseudo-projection* (PP), is given by the symmetrised product:

[2] For example, in quantum mechanics, probability amplitudes add, and not the probabilities.

[3] For the sake of brevity, we represent both observables and operators by the same symbol throughout.

$$\mathbf{\Pi}_{m_i n_j} = \frac{1}{2}\left\{\pi_{m_i}, \pi_{n_j}\right\}, \tag{1}$$

in accordance with the Weyl prescription [43]. The PP is not idempotent, unless $[\pi_{m_i},\ \pi_{n_j}] = 0$.

In this way, the logical proposition, $\mathcal{L}(M = m_i) \wedge \mathcal{L}(N = n_j)$, gets represented by the pseudoprojection operator $\mathbf{\Pi}_{m_i n_j}$ (the symbol $\wedge$ represents logical conjunction).

2.1.4 For multiple observables

This correspondence suggests that the PP, representing joint outcomes of more than two observables, can also be constructed similarly. It may be done so, but not in a unique way. For, the order in which the noncommuting projections are multiplied matters. Consider the joint outcome, $O_1 = o_1, O_2 = o_2, \ldots, O_N = o_N$, of N observables. If π_{o_i} be the respective projections, their product can be permuted, in general, in $N!$ ways. If all the projections happen to be distinct, it will give rise to $N!/2$ distinct quantum representatives of the form,

$$\mathbf{\Pi}_{o_1 \ldots o_N} = \frac{1}{2}\pi_{o_1}\pi_{o_2}\ldots\pi_{o_N} + \text{ h.c.}, \tag{2}$$

where h.c. represents the Hermitian conjugate. A PP obtained from a given order of N distinct projections has been termed as a *unit pseudoprojection* [24] (That a unit PP has at least one negative eigenvalue is shown in "Appendix A"). Each unit PP is, generally, inequivalent to the others. It is necessary to require that PPs share as many properties as possible with their classical counterparts. Unit PP fails one important test: symmetry in all the observables. This can be restored by summing all distinct unit PP with equal weights. But, not surprisingly, the resultant PP would be more difficult to analyse for nonclassicality. The most general PP is highly non-unique and may be taken to be any point on the manifold of convex sums of all unit PPs.[4] Fortunately, for most of our purposes, the unit PP suffices.

So, the logical proposition, $\mathcal{L}(O_1 = o_1) \wedge \mathcal{L}(O_2 = o_2) \wedge \cdots \wedge \mathcal{L}(O_N = o_N)$, gets represented by different PPs depending on the prescription employed.

2.1.5 Pseudoprobability

We name the expectation of PP as pseudo-probability, i.e. the pseudoprobability for the joint event when the observables M and N take values m_i and n_j, respectively,

[4] For example, the most general PP representing the joint outcome, $O_1 = o_1$, $O_2 = o_2$ and $O_3 = o_3$ is,

$$\mathbf{\Pi} = \nu_1\mathbf{\Pi}_1 + \nu_2\mathbf{\Pi}_2 + \nu_3\mathbf{\Pi}_3;\ 0 \le \nu_i \le 1;\ \sum_{i=1}^{3}\nu_i = 1,$$

where,

$$\mathbf{\Pi}_1 = \frac{1}{2}(\pi_{o_1}\pi_{o_2}\pi_{o_3} + \text{h.c.});\ \mathbf{\Pi}_2 = \frac{1}{2}(\pi_{o_2}\pi_{o_1}\pi_{o_3} + \text{h.c.});\ \mathbf{\Pi}_3 = \frac{1}{2}(\pi_{o_1}\pi_{o_3}\pi_{o_2} + \text{h.c.}).$$

i.e. $M = m_i$ and $N = n_j$, is given by,

$$\mathcal{P}_{m_i n_j} \equiv \mathrm{Tr}(\mathbf{\Pi}_{m_i n_j}\rho). \tag{3}$$

When we employ a unit PP, the resultant pseudo-probability agrees with the Margenau–Hill–Barut distribution [26,27], the object of our study here. Since only the Hermiticity of PP is guaranteed, pseudo-probabilities can take negative values. The PP for a multi-party system is the direct product of PP for individual systems.

The set of pseudo-probabilities, generated for all the possible joint outcomes of N observables, is called a *scheme* [24]. Schemes possess an important property: the marginal, obtained by summing over the outcomes of any one of them, yields the scheme for the remaining $(N-1)$ observables. Importantly, the ultimate marginal, for a single observable, is just the set of quantum probabilities, $p_k = \mathrm{Tr}(\rho\pi_k)$.

2.1.6 Definition of nonclassicality

Definition A state ρ is deemed to be nonclassical with respect to a set of observables $\{O_1, \ldots, O_N\}$, iff at least one pseudoprobability in the scheme assumes a negative value [24].

If all the pseudo-probabilities were to be non-negative, quantum mechanics would be admitting an underlying classical description. That is to say, in an abstract manner, there does exist a classical system giving rise to the same set of nonnegative probabilities. This leads to the attractive thesis that pseudoprobabilities capture all the non-classical features of quantum mechanics. If it be so, one should be able to recover, in the first instance, features of nonclassical logic and, of course, important results such as conditions for entanglement. That it is indeed so, has been shown recently [24,25].

2.1.7 Disjunction

Conjunction is but one logical operation. The PPs, that represent the disjunction (OR), may be obtained by employing standard rules. Thus, for example, the disjunction of two events when the observables M and N, respectively, take values m_i and n_j, i.e. $M = m_i$ OR $N = n_j$, is represented by the PP,

$$\mathbf{\Pi}_{m_i \vee n_j} = \pi_{m_i} + \pi_{n_j} - \mathbf{\Pi}_{m_i n_j}, \tag{4}$$

which is simply the quantum representative of the indicator function for the union: $1_{S_i \cup S_j} = 1_{S_i} + 1_{S_j} - 1_{S_i \cap S_j}$. Here, S_i, S_j and $S_i \cap S_j$ represent the supports for the events, $M = m_i$, $N = n_j$ and ($M = m_i$ AND $N = n_j$), respectively.

2.2 Nonlocality

Since nonlocality is a feature of nonclassicality that is not restricted to quantum mechanics [44], we assume a nonnegative joint probability scheme and show that violation of CHSH inequality is tantamount to the existence of negative entries in the joint probability scheme.

3 Pseudo-probabilities and weak values

The idea of negative probabilities has been around quite a while [39–41]. But what exactly is the physical significance of pseudo-probabilities, so obtained by us, especially when they are negative? We show that, apart from being useful theoretical constructs, they have an operational value, i.e. they can be realised in experiments as anomalous weak values. Conversely, the anomalous outcomes of weak measurements can be given an operator description in the language of PPs. With this, it becomes possible to devise tests, which are distinct from standard correlation measurements, for different manifests of non-classical correlations using weak measurements. This is the main content of the paper. With weak measurements, the experimental scope for testing quantum mechanics gets widened. Further, using weak measurements, the violation of Boolean logic can also be observed in the quantum domain.

The identification of weak measurements that are appropriate to study pseudo-probability is quite straightforward. First, recall that the weak value of an observable A on a pre-selected state $|\psi\rangle$ and a post-selected state $|\phi\rangle$ is given by [11],

$$\langle A_w\rangle_{|\psi\rangle}^{|\phi\rangle} := \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}. \tag{5}$$

If $\langle A_w\rangle_{|\psi\rangle}^{|\phi\rangle}$ lies outside the range of eigenvalues of the observable A, the weak value is termed *anomalous*. Thus, all imaginary values are automatically anomalous. Generalisation to mixed pre-selected state has been done in [45]. The weak value of an observable A in the pre-selected state ρ_1 and post-selected state ρ_2 is given as,

$$\langle A_w\rangle_{\rho_1}^{\rho_2} := \frac{\mathrm{Tr}(\rho_2 A\rho_1)}{\mathrm{tr}(\rho_2\rho_1)}. \tag{6}$$

An alternative interpretation of weak value of an observable as a robust property of a single pre-selected and post-selected quantum system is discussed in [46].

It is pertinent to this work that many schemes have been proposed [34] for joint measurement of non-commuting observables. These techniques facilitate experimental determination of joint probability schemes (for non-commuting observables) [35–37].

Of relevance for us is the relation of pseudo-probabilities with anomalous weak values. Let $\mathbf{\Pi}_{o_1\ldots o_N}$ be a unit PP for a joint event $O_1 = o_1, O_2 = o_2, \ldots, O_N = o_N$. Its associated pseudo-probability may be written, in terms of weak values as,

$$\langle\mathbf{\Pi}_{o_1\ldots o_N}\rangle_\rho = \mathrm{Tr}(\rho\pi_{o_1})\langle\!\langle\pi_{o_2}\ldots\pi_{o_N}\rangle\!\rangle_\rho^{\rho_{o_1}}, \tag{7}$$

where, the notation $\langle\!\langle\ \rangle\!\rangle$—which shall be used everywhere—emphasises that $\langle\!\langle\pi_{o_2}\ldots\pi_{o_N}\rangle\!\rangle_\rho^{\rho_{o_1}}$ represents the real part of the weak value of the product of the remaining $(N-1)$ projections. The pre-selected state is ρ and the post-selected state, ρ_{o_1}. The state ρ_{o_1} is obtained by suitable normalisation of π_{o_1}.

A further generalization to convex sums (including the symmetrised sum) is straightforward.

The weak value in the RHS of Eq. (7) is that of a non-Hermitian operator and merits more description. We employ the resolution $\pi_{o_2} \ldots \pi_{o_N} \equiv H - iJ$ in terms of its Hermitian and anti-Hermitian parts, and state the condition for nonclassicality through the chain of mutual implications,

$$\begin{aligned} \langle \mathbf{\Pi}_{o_1 \ldots o_N} \rangle_\rho < 0 &\iff \langle\!\langle \pi_{o_2} \ldots \pi_{o_N} \rangle\!\rangle_\rho^{\rho_{o_1}} < 0 \\ &\iff \langle\!\langle H \rangle\!\rangle_\rho^{\rho_{o_1}} + \mathrm{Im}\{\langle J \rangle_\rho^{\rho_{o_1}}\} < 0, \end{aligned} \tag{8}$$

which displays the relationship between negative pseudo-probabilities and a combination of the weak values of the two Hermitian observables. The symbol ρ_{o_1} represents the state obtained from π_{o_1} after suitable normalisation, i.e. if π_{o_1} is a d—dimensional projection operator,

$$\rho_{o_1} \equiv \frac{1}{d} \pi_{o_1}. \tag{9}$$

For the special case of pseudoprojections representing joint outcomes of only two observables,

$$\langle \mathbf{\Pi}_{o_1 o_2} \rangle < 0 \iff \langle\!\langle \pi_{o_2} \rangle\!\rangle_\rho^{\rho_{o_1}} < 0, \tag{10}$$

which is expressed entirely in terms of the anomalous weak values of a single Hermitian operator. Indeed, Eq. (10) establishes a strong equivalence between negative pseudo-probabilities and anomalous weak values. This equivalence forms the basis for most of the applications discussed here.

Note that the expectation value $\langle \mathbf{\Pi}_{o_1 o_2} \rangle$ cannot exceed $+1$ and hence, only the negative anomalous weak value corresponds to negative pseudoprobability. It is due to the Cauchy–Schwarz theorem. The expectation value of $\mathbf{\Pi}_{o_1 o_2}$ is upper bounded by $+1$ as shown below,

$$\begin{aligned} |\langle \mathbf{\Pi}_{o_1 o_2} \rangle_\rho| &= \frac{1}{2} \Big| \langle (\pi_{o_1} \pi_{o_2} + \pi_{o_2} \pi_{o_1}) \rangle_\rho \Big| \\ &\leq \frac{1}{2} (|\langle \pi_{o_1} \rangle_\rho| \cdot |\langle \pi_{o_2} \rangle_\rho| + |\langle \pi_{o_2} \rangle_\rho| \cdot |\langle \pi_{o_1} \rangle_\rho|) \leq 1. \end{aligned} \tag{11}$$

The same argument can be extended to the expectation value of a unit pseudoprojection representing joint outcomes of any number of observables.

4 Notation

First, we introduce some notations to make the subsequent expressions less cluttered:

1. We consider only dichotomic observables, with eigenvalues ± 1. For simplicity, if A be such an observable, the respective eigenprojections will be denoted by π_A and $\pi_{\bar{A}}$, corresponding to eigenvalues $+1$ and -1. We also denote the associated states, obtained by suitable normalisation, by ρ_A and $\rho_{\bar{A}}$, as shown in Eq. (9).

2. (a) The events, that the observable A takes value $+1$ and -1, are compactly represented by $\mathcal{E}(A)$ and $\mathcal{E}(\bar{A})$, respectively, i.e.

$$\mathcal{E}(A) \equiv \mathcal{E}(A = +1);\ \ \mathcal{E}(\bar{A}) \equiv \mathcal{E}(A = -1). \tag{12}$$

   (b) The logical propositions, that the observable A takes value $+1$ and -1, are compactly represented by $\mathcal{L}(A)$ and $\mathcal{L}(\bar{A})$, respectively.

3. The symbol $P(A)(P(\bar{A}))$ represents the quantum probability for outcome +1 (-1) of the observable A, i.e.

$$P(A) \equiv P(A = +1);\ \ P(\bar{A}) \equiv P(A = -1). \tag{13}$$

4. The symbol $\mathcal{P}(A_1 A_2 \ldots A_p \bar{A}_{p+1} \bar{A}_{p+2} \ldots \bar{A}_{p+q})$ represents the pseudoprobability for the outcomes of the first p observables to be +1, and of the next q observables to be -1, i.e.

$$\begin{aligned}&\mathcal{P}(A_1 A_2 \ldots A_p \bar{A}_{p+1} \bar{A}_{p+2} \ldots \bar{A}_{p+q})\\ &\equiv \mathcal{P}(A_1 = +1, \ldots, A_p = +1, A_{p+1} = -1, \ldots, A_{p+q} = -1).\end{aligned} \tag{14}$$

5. Observables for subsystems of a bipartite state are denoted by A_i, B_i, i.e. the observables A_i and B_i correspond to the first and second subsystems, respectively.
6. Finally, since all the observables are dichotomic with outcomes ± 1, we employ the following shorthand notation,

$$\begin{aligned}&\mathcal{P}(A_1 = A_2 = B_1 = B_2) \equiv \mathcal{P}(A_1 A_2; B_1 B_2) + \mathcal{P}(\bar{A}_1 \bar{A}_2; \bar{B}_1 \bar{B}_2)\\ =&\mathcal{P}(A_1 = +1, A_2 = +1; B_1 = +1, B_2 = +1)\\ &\quad + \mathcal{P}(A_1 = -1, A_2 = -1; B_1 = -1, B_2 = -1).\end{aligned} \tag{15}$$

7. For a qubit, we employ lowercase letters to represent the observables, i.e. $a_i \equiv \boldsymbol{\sigma}_1 \cdot \hat{a}_i$ and $b_j \equiv \boldsymbol{\sigma}_2 \cdot \hat{b}_j$.

5 Relation with previous works

A nonclassical joint probability distribution was first sought for noncommuting observables by simply taking the product of projection operators [47], which was further studied by Barut [48]. Note that this can lead to complex values. Independently, negative probabilities were introduced by Dirac [39] and Feynmann [40], in a rather ad hoc manner. Subsequently, Margenau and Hill [26] and Barut *et al.* [27] have constructed a nonclassical joint probability distribution by employing a completely symmetrised product of two and three projections. The resulting joint probabilities are real but can take negative values. In contrast to those studies, the encompassing idea of pseudo-projection, as a quantum representative of product of indicator functions, has been introduced in [24]. Their expectation values reduce to Margenau–Hill–Barut distribution for two observables, and also for three observables, for a specific choice of pseudoprojection. In this formalism, conditions for entanglement have been derived, both for two-qubit and multi-qubit systems, by imposing suitable nonclassicality conditions [24,25]. Furthermore, it has also been shown in [24,25] that conditions for nonlocality emerge when suitably chosen sums of joint probabilities turn negative, without recourse to any underlying model.

Coming to weak values, it has been shown by Johansen [49,50] that the Margenau–Hill–Barut distribution, for joint outcomes of two noncommuting observables, can be observed in experiments through weak measurements [11]. In fact, Margenau–Hill–Barut distribution, for joint outcomes of position and momentum (x and p), has also been experimentally observed [35]. In parallel, anomalous weak values have been shown to be signatures of contextuality [32] and violation of Leggett–Garg inequality [51]. What we accomplish in this paper is to identify the anomalous weak values underlying many other forms of nonclassicality, *viz.*, quantum coherence, CHSH nonlocality for a bipartite system, quantum entanglement, and quantum discord in two-qubit systems. Going further, we also show how weak measurements can be used for experimental demonstration of non-Boolean logic, which has not been considered hitherto. Our study, thus, shows that the nonclassical features also show violations of classical probability rules, that can be experimentally demonstrated through weak measurements.

6 Applications-I

Non-classicality is a characteristic of all the quantum states, including the simplest of them all—the single qubit. This is amply reflected in the definitions of both weak measurement and PPs. In fact, a single qubit is quite a rich lab to test quantum mechanics. With this in mind, we propose three applications: a witness for coherence, a direct verification of non-Boolean nature of quantum logic, and a quantum game based on violation of classical probability rule. Weak measurements act as experimental probes in all cases.

6.1 Coherence witness for a single qubit and weak values

In a given basis, a state has coherence if it has nonvanishing off-diagonal elements [52]. Let the state ρ be expressed in the eigenbasis of, say, σ_z. The PP,

$$\mathbf{\Pi}_{a_1a_2} = \frac{1}{4}\Big(1 + \hat{a}_1 \cdot \hat{a}_2 + \boldsymbol{\sigma} \cdot (\hat{a}_1 + \hat{a}_2)\Big), \tag{16}$$

always has a positive overlap with ρ whenever ρ is diagonal, so long as $(\hat{a}_1 + \hat{a}_2)$ lies in the plane orthogonal to $\hat{z}$. On the other hand, if ρ possesses coherence, one may always judiciously choose $\hat{a}_1$ and $\hat{a}_2$, out of the plane such that it has a negative overlap with $\mathbf{\Pi}_{a_1a_2}$. It has been elaborated in “Appendix B”.

Weak measurements can act as detectors of coherence, since,

$$\mathcal{P}(a_1a_2) \equiv \left\langle \mathbf{\Pi}_{a_1a_2} \right\rangle_\rho = \mathrm{Tr}(\pi_{a_1}\rho)\langle\!\langle \pi_{a_2} \rangle\!\rangle_\rho^{\rho_{a_1}}, \tag{17}$$

which implies that,

$$\mathcal{P}(a_1 a_2) < 0 \implies \langle\!\langle \pi_{a_2} \rangle\!\rangle_{\rho}^{\rho_{a_1}} < 0. \tag{18}$$

The preselected and the postselected states are, respectively, ρ and ρ_{a_1}. Note that, an anomalous weak value is necessary and sufficient for the expectation value of the witness to be negative.

6.2 Non-Boolean quantum logic and weak values

Recently, weak measurement techniques have been employed to experimentally observe violation of classical rules such as pigeon-hole paradox [53,54]. In this section, we show how such techniques facilitate experimental demonstration of violation of classical logic in quantum mechanics.

Absurd propositions have null support in event spaces. Thus, their indicator functions are identically zero. We construct two simple examples which illustrate how the quantum representatives of such absurd propositions do not vanish and give rise to nonvanishing weak values. They provide a state-dependent and a state-independent test of violations of Boolean logic, respectively.

6.2.1 State-dependent test of violation of Boolean logic

Consider the logical proposition, $\mathcal{L}(a_1) \wedge \mathcal{L}(a_2) \wedge \mathcal{L}(\overline{a}_1)$. Since this proposition is the conjunction of three events, $\mathcal{E}(a_1 = +1)$, $\mathcal{E}(a_2 = +1)$ and $\mathcal{E}(a_1 = -1)$ (recall $a_1 \to \boldsymbol{\sigma} \cdot \hat{a}_1$), it has a null support. It is because the probability for the conjunction of the three events is identically zero classically, as it involves conjunction of an event $\mathcal{E}(a_1 = +1)$ with its negation $\mathcal{E}(a_1 = -1)$. But, from the rules of quantum mechanics, we find that there is exactly one associated unit PP which is non-vanishing [42]:

$$\mathbf{\Pi}_{a_1 a_2 \bar{a}_1} = \frac{1}{2}\left\{\pi_{a_1}\pi_{a_2}\pi_{\bar{a}_1} + \pi_{\bar{a}_1}\pi_{a_2}\pi_{a_1}\right\} = \frac{1}{8}\boldsymbol{\sigma} \cdot \{\hat{a}_2 - \hat{a}_1(\hat{a}_1 \cdot \hat{a}_2)\}. \tag{19}$$

Its expectation for a given state—the associated pseudo-probability—is nonzero unless the state is completely mixed! On the other hand, classical Boolean logic would demand that the joint probability, corresponding to the absurd event, $\mathcal{E}(a_1 = +1)$ AND $\mathcal{E}(a_2 = +1)$ AND $\mathcal{E}(a_1 = -1)$, is identically equal to zero, i.e.

$$\mathcal{P}_{a_1 a_2 \bar{a}_1} = 0 \ \ \text{(classically)}. \tag{20}$$

Thus, $\mathcal{P}_{a_1 a_2 \bar{a}_1} \neq 0$, is a definite signature of violation of Boolean logic.[5]

[5] This situation can occur elsewhere as well. For example, nonlocality can be detected by both—inequalities and Hardy-type paradoxes [55].

6.2.2 Experimental verification with weak measurements

Note that the form of Eq. (19) suggests weak measurements of all the three projections appearing in it. This can be done by employing three pointers, with ρ as the pre-selected, and the completely mixed state as the post-selected state. The Hamiltonian that couples the system to the pointers may be chosen to be,

$$H_{\text{int}} = g(\pi_{a_1} \otimes P_{1x} + \pi_{a_2} \otimes P_{2y} + \pi_{\bar{a}_1} \otimes P_{3z}), \tag{21}$$

where, P_{1x}, P_{2y}, P_{3z} represent the momenta of the three pointers canonical to their positions x_1, y_2, z_3, respectively, and g is the common coupling constant. If the pointers are initially in the state,

$$\exp(-x_1^2/2\sigma^2) \otimes \exp(-y_2^2/2\sigma^2) \otimes \exp(-z_3^2/2\sigma^2), \tag{22}$$

the correlation among the three pointer readings $\langle x_1 y_2 z_3 \rangle \propto \langle \mathbf{\Pi}_{a_1 a_2 \bar{a}_1} \rangle_\rho$.

6.2.3 State-independent test of violation of Boolean logic

It may be inferred from Eq. (19) that the pseudoprobability of the classically absurd joint event $\mathcal{E}(a_1 a_2 \bar{a}_1)$ would be state-dependent. Furthermore, the completely mixed state eludes violation of Boolean logic as the pseudoprobability $\mathcal{P}(a_1 a_2 \bar{a}_1)$ vanishes. This leads us to the question whether there exists a state-independent test of non-Boolean logic that can be experimentally demonstrated with weak measurement. The answer to this is in the affirmative and can be obtained by constructing the quantum representative of classically absurd proposition, $\mathcal{L}(a_1) \wedge \mathcal{L}(a_2) \wedge \mathcal{L}(\bar{a}_1) \wedge \mathcal{L}(\bar{a}_2)$.

The pseudoprojection, representing the event $\mathcal{E}(a_1 a_2 \bar{a}_1 \bar{a}_2)$, is given by [42]:

$$\mathbf{\Pi}_{a_1 a_2 \bar{a}_1 \bar{a}_2} = \frac{1}{8}\left\{(\hat{a}_1 \cdot \hat{a}_2)^2 - \frac{1}{3}\right\}, \tag{23}$$

whose overlap with any qubit state $\frac{1}{2}(1 + \boldsymbol{\sigma} \cdot \boldsymbol{p})$ is independent of $\boldsymbol{p}$.

6.2.4 Experimental verification with weak measurements

The form of Eq. (23) suggests the weak measurements to be performed. This can be done by employing four pointers, with ρ as the pre-selected state, and the completely mixed state as the post-selected state. The Hamiltonian that couples the system to the pointers may be chosen to be,

$$H_{\text{int}} = g(\pi_{a_1} \otimes P_{1x} + \pi_{a_2} \otimes P_{2y} + \pi_{\bar{a}_1} \otimes P_{3z} + \pi_{\bar{a}_2} \otimes P_{4x}), \tag{24}$$

where, as before, P_{1x}, P_{2y}, P_{3z}, P_{4x} represent the momenta of the four pointers canonical to their positions x_1, y_2, z_3, x_4, respectively, and g is the common coupling constant. If the pointers are initially in the state,

$$\exp(-{x_1}^2/2\sigma^2) \otimes \exp(-{y_2}^2/2\sigma^2) \otimes \exp(-{z_3}^2/2\sigma^2) \otimes \exp\left(-x_4^2/2\sigma^2\right), \tag{25}$$

the correlation among the four pointer readings $\langle x_1 y_2 z_3 x_4 \rangle$ is proportional to the pseudoprobability $\langle \mathbf{\Pi}_{a_1 a_2 \bar{a}_1 \bar{a}_2} \rangle_\rho$.

In these exceptional cases of absurd propositions, the negativity of the pseudoprobability is not a necessary condition for non-classicality. Rather, any nonzero value assumed by them is a signature of non-Boolean nature of quantum mechanics.

6.2.5 Violation of the distributivity law in quantum mechanics

Boolean logic follows the distributivity law, i.e. the following two propositions are equivalent,

$$\mathcal{L}\big(a_1 \wedge (a_2 \vee a_3)\big) \equiv \mathcal{L}(a_1 \wedge a_2) \vee \mathcal{L}(a_1 \wedge a_3). \tag{26}$$

It has been observed in [22] that the distributivity rule is not obeyed in quantum mechanics. In what follows, we show that weak measurements can be employed to observe violation of the distributivity law.

The pseudoprojection representing the event in the LHS of equation (26), i.e. $\mathcal{E}\big(a_1 \wedge (a_2 \vee a_3)\big)$ can be constructed as follows,

$$\begin{aligned}\mathbf{\Pi}_{a_1 \wedge (a_2 \vee a_3)} &\equiv \frac{1}{2}\{\pi_{a_1}, \mathbf{\Pi}_{a_2 \vee a_3}\} = \frac{1}{2}\{\pi_{a_1}, \pi_{a_2} + \pi_{a_3} - \mathbf{\Pi}_{a_2 a_3}\} \\ &= \mathbf{\Pi}_{a_1 a_2} + \mathbf{\Pi}_{a_1 a_3} - \frac{1}{2}\{\pi_{a_1}, \mathbf{\Pi}_{a_2 a_3}\}.\end{aligned} \tag{27}$$

The pseudoprojection representing the event in the RHS of equation (26), i.e. $\mathcal{E}(a_1 \wedge a_2) \vee \mathcal{E}(a_1 \wedge a_3)$ can be constructed as follows,

$$\mathbf{\Pi}_{(a_1 \wedge a_2) \vee (a_1 \wedge a_3)} = \mathbf{\Pi}_{a_1 a_2} + \mathbf{\Pi}_{a_1 a_3} - \mathbf{\Pi}_{a_1 a_2 a_1 a_3}. \tag{28}$$

Subtracting Eq. (28) from Eq. (27), the following equation results:

$$\mathbf{\Pi}_{a_1 \wedge (a_2 \vee a_3)} - \mathbf{\Pi}_{(a_1 \wedge a_2) \vee (a_1 \wedge a_3)} = \mathbf{\Pi}_{a_1 a_2 a_1 a_3} - \frac{1}{2}\{\pi_{a_1}, \mathbf{\Pi}_{a_2 a_3}\}, \tag{29}$$

which is clearly non-vanishing. For example, if we consider a state $\rho_{\bar{a}_1}$, lying in the null space of π_{a_1}, the expectation value of the second term $\frac{1}{2}\{\pi_{a_1}, \mathbf{\Pi}_{a_2 a_3}\}$ vanishes, i.e.

$$\mathrm{Tr}\Big(\rho_{\bar{a}_1} \frac{1}{2}\{\pi_{a_1}, \mathbf{\Pi}_{a_2 a_3}\}\Big) = 0. \tag{30}$$

The first term, $\mathbf{\Pi}_{a_1a_2a_1a_3} = \frac{1}{2}(\pi_{a_2}\pi_{a_1}\pi_{a_3} + \text{h.c.})$, however, has a nonzero expectation value[6] with the state $\rho_{\bar{a}_1}$.

6.2.6 Experimental verification with weak measurements

The pseudoprobabilities, $\langle \mathbf{\Pi}_{a_1a_2a_1a_3}\rangle$ and $\frac{1}{2}\big\langle\{\pi_{a_1}, \mathbf{\Pi}_{a_2a_3}\}\big\rangle$, can be determined with weak measurements, and the difference between the two is a signature of violation of distributivity law in the quantum domain.

6.3 Quantum games via pseudo-projections and weak measurements

In this section, we propose a quantum game, based on violation of rules of classical probability, which may be implemented through weak measurements.

Consider a qubit in the state,

$$\rho = \frac{1}{2}\big(1 + \boldsymbol{\sigma} \cdot \boldsymbol{p}\big), \tag{31}$$

undergoing a unitary transformation induced by the Hamiltonian, $H = -\dfrac{1}{2}\hbar\omega_L\sigma_z$.

The idea is to look at joint pseudo-probabilities for simultaneous outcomes of two incompatible observables, $\boldsymbol{\sigma}\cdot\hat{m}$ and $\boldsymbol{\sigma}\cdot\hat{n}$. We wish to exploit their “anomalous" nature, i.e. they can be negative or the pseudoprobabilities for the complementary event may exceed one. We choose,

$$\hat{m} = \cos\theta\hat{i} + \sin\theta\hat{j}; \quad \hat{n} = -\sin\theta\hat{i} + \cos\theta\hat{j}. \tag{32}$$

Employing the shorthand notations,

$$\mathcal{P}_1 = \mathcal{P}_{++}; \quad \mathcal{P}_2 = \mathcal{P}_{--}; \quad \mathcal{P}_3 = \mathcal{P}_{+-}; \quad \mathcal{P}_4 = \mathcal{P}_{-+}, \tag{33}$$

the expression for the transition matrix, $\mathcal{T}$ for the pseudo-probabilities follows from their time dependence and is given by ($\omega_L = 1$),

[6] The unit pseudoprojection $\mathbf{\Pi}_{a_1a_2a_1a_3}$ is given as,

$$\mathbf{\Pi}_{a_1a_2a_1a_3} = \frac{1}{2}(\pi_{a_2}\pi_{a_1}\pi_{a_1}\pi_{a_3} + \pi_{a_3}\pi_{a_1}\pi_{a_1}\pi_{a_2}) = \frac{1}{2}(\pi_{a_2}\pi_{a_1}\pi_{a_3} + \pi_{a_3}\pi_{a_1}\pi_{a_2}).$$

$$\begin{pmatrix} \mathcal{P}_1(t) \\ \mathcal{P}_2(t) \\ \mathcal{P}_3(t) \\ \mathcal{P}_4(t) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1+2\cos t & 1-2\cos t & 1-2\sin t & 1+2\sin t \\ 1-2\cos t & 1+2\cos t & 1+2\sin t & 1-2\sin t \\ 1+2\sin t & 1-2\sin t & 1+2\cos t & 1-2\cos t \\ 1-2\sin t & 1+2\sin t & 1-2\cos t & 1+2\cos t \end{pmatrix} \begin{pmatrix} \mathcal{P}_1(0) \\ \mathcal{P}_2(0) \\ \mathcal{P}_3(0) \\ \mathcal{P}_4(0) \end{pmatrix}$$
$$\equiv \mathcal{T}(t) \begin{pmatrix} \mathcal{P}_1(0) \\ \mathcal{P}_2(0) \\ \mathcal{P}_3(0) \\ \mathcal{P}_4(0) \end{pmatrix}. \tag{34}$$

$\mathcal{T}(t)$ is an example of the quantum counterparts of the doubly stochastic matrices in classical probability theory. We note that $\mathcal{T}(t)$ satisfies all the properties of doubly stochastic matrix, except that its entries admit negative values. The continuous set, $\{\mathcal{T}(t)\}$, forms a monoid—a semi-group with identity.

If an entry in $\mathcal{P}(t)$ were to become negative, the sum of the other three entries would exceed one. This affords advantages which are not provided classically and will be employed in the game. The rules of the game, that we design, are as follows:

- There are two players. The referee asks them to start and end the game at definite times, t_i and t_f, respectively. They are given the same initial *non-negative* scheme for joint outcomes of a set of two dichotomic observables. The players are to compete to create, by unitary evolution, the scheme such that at some intermediate time T, the sum S, of any of its three entries maximally exceeds one. Weak measurements are to be employed.
- The players are free to choose the initial state and the Hamiltonian.
- In this game, a resource is classical, if for all times, $\mathcal{T}(t)_{ij} \geq 0$.

We provide an explicit illustration of existence of such a pseudo-probability scheme that can be used as a strategy by players. Suppose that $\hat{m} = \hat{x},\ \hat{n} = \hat{y}$, and the initial state (at $t = 0$) is pure with its polarisation along the x axis. The associated initial pseudo-probabilities are,

$$\mathcal{P}_1(0) = 0.5;\ \mathcal{P}_2(0) = 0;\ \mathcal{P}_3(0) = 0.5;\ \mathcal{P}_4(0) = 0.$$

At $T = \dfrac{\pi}{4}$, the pseudo-probabilities satisfy the maximality requirement and are given by,

$$\mathcal{P}_1(T) = 0.25;\ \mathcal{P}_2(T) = 0.25;$$
$$\mathcal{P}_3(T) = 0.25(1+\sqrt{2});\ \mathcal{P}_4(T) = 0.25(1-\sqrt{2}), \tag{35}$$

which lead to the maximal value $S = 0.25(3+\sqrt{2})$. If a player were to employ a different state or a different Hamiltonian, the game would be lost.

7 Applications-II

The definition of nonclassicality, given in Sect. 2.1.6, is broad enough to detect all the quantum states nonclassical.[7] From a resource-theoretic viewpoint, however, specific non-classical features possessed by a state determine the applications, in which it will be useful. This aspect has led to various criteria for detection of those nonclassical features. In order to show that nonclassical probabilities and anomalous weak values underlie them, we start with various non-classical features, e.g. nonlocality, entanglement and discord. We pick up such sums of pseudo-probabilities that the ensuing inequalities get violated by at least one state having that particular non-classical feature. We consider bipartite systems since their nonclassical properties have received the greatest attention.

7.1 CHSH nonlocality and weak values

In this section, we show how the derivation of CHSH inequality, in terms of joint probability given in [24], facilitates identification of anomalous weak values that guarantee a state to be non-local.

Consider any $d_1 \times d_2$ dimensional bipartite system. Let $\{A_1, A_2\}, \{B_1, B_2\}$ be two pairs of dichotomic observables (with outcomes, ± 1), for the respective subsystems. Since CHSH inequality involves terms containing only correlations between the observables of the two subsystems, and not the local terms, we choose the joint probabilities in such a way that all the local terms get cancelled.

The relevant joint probability is the sum,

$$\mathcal{P}_{\text{NL}} = \mathcal{P}(A_1 = B_1 = B_2) + \mathcal{P}(A_2 = B_1 = \overline{B}_2). \tag{36}$$

We rewrite all the joint probabilities in terms of expectation values of observables, by employing dichotomic nature of observables. Classically, it is always true that $\mathcal{P}_{\text{NL}} \geq 0$. Therefore, we demand that $\mathcal{P}_{\text{NL}} < 0$ and obtain the CHSH inequality,

$$\big\langle A_1(B_1 + B_2) + A_2(B_1 - B_2)\big\rangle > -2. \tag{37}$$

The detailed derivation is shown in "Appendix C".

In quantum mechanics, of course, the joint probabilities can be written as expectation values of the parent pseudo-projections. Thus, in quantum mechanics, given the relation between pseudoprobabilities and anomalous weak values, the former can be

[7] Even the completely mixed two-dimensional state has a negative pseudo-probability for the joint event, when $\boldsymbol{\sigma} \cdot \hat{a}_1, \boldsymbol{\sigma} \cdot \hat{a}_2, \boldsymbol{\sigma} \cdot \hat{a}_3$ take value $+1$, where $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are coplanar and at an included angle of $\frac{2\pi}{3}$. The completely symmetrised PP can be constructed as follows:

$$\begin{aligned}\mathbf{\Pi}_{a_1a_2a_3} &= \frac{1}{3!}(\pi_{a_1}\pi_{a_2}\pi_{a_3} + \pi_{a_1}\pi_{a_3}\pi_{a_2} + \pi_{a_2}\pi_{a_1}\pi_{a_3} + \pi_{a_2}\pi_{a_3}\pi_{a_1} + \pi_{a_3}\pi_{a_1}\pi_{a_2} + \pi_{a_3}\pi_{a_2}\pi_{a_1}) \\ &= -\frac{1}{16},\end{aligned}$$

whose overlap with the completely mixed state is negative, $-\frac{1}{16}$.

determined via weak measurements. What would be the corresponding weak measurements? We rewrite the RHS of the Eq. (36) as,

$$\mathcal{P}_{\rm NL} = \langle \pi_{A_1}\pi_{B_1}\rangle \langle\!\langle \pi_{B_2}\rangle\!\rangle_\rho^{\rho_{A_1B_1}} + \langle \pi_{\bar{A}_1}\pi_{\bar{B}_1}\rangle \langle\!\langle \pi_{\bar{B}_2}\rangle\!\rangle_\rho^{\rho_{\bar{A}_1\bar{B}_1}}$$
$$+\langle \pi_{A_2}\pi_{B_1}\rangle \langle\!\langle \pi_{\bar{B}_2}\rangle\!\rangle_\rho^{\rho_{A_2B_1}} + \langle \pi_{\bar{A}_2}\pi_{\bar{B}_1}\rangle \langle\!\langle \pi_{B_2}\rangle\!\rangle_\rho^{\rho_{\bar{A}_2\bar{B}_1}}, \tag{38}$$

where,

$$\pi_A = \frac{1}{2}(1+A);\, \pi_{\bar{A}} = \frac{1}{2}(1-A);\quad \rho_{AB} = \frac{1}{d_{\pi_A}d_{\pi_B}}\pi_A\pi_B, \tag{39}$$

where $d_{\pi_A}(d_{\pi_B})$ represents dimensions of $\pi_A(\pi_B)$.

Note that, if $\mathcal{P}_{\rm NL} < 0$, at least some of the weak values are negative. In fact, for two-qubit nonlocal Werner states, it is necessary that all the four weak values be negative (shown in "Appendix C"). With hindsight, we realise that for pure $2 \otimes 2$ systems, the pseudo-probabilities underlying CHSH inequality have already been experimentally demonstrated in [36].

Our analysis also resolves the apparent paradox posed in problem 17.6 of [56]. The paradox is that apparently, a normalised joint probability distribution exists reproducing all the desired marginals, even when the CHSH inequality gets violated. The resolution is that, though a joint probability distribution does exist for the four observables $\{A_i, B_j\}$, a violation of CHSH inequality requires that the weak values be anomalous. Thus, if Eq. (38) assumes a negative value, it simultaneously signifies that,

1. the underlying state is CHSH nonlocal,
2. the pseudo-probabilities become negative, and,
3. the weak values are anomalous.

In short, we have shown the existence of anomalous weak values and negative probabilities upon violation of CHSH nonlocality. In Sects. 7.2 and 7.4, we show that similar conclusions hold for entanglement and discord in two-qubit systems as well.

7.2 Entanglement inequality for two-qubits and weak values

We next study the interrelation between entanglement inequalities, negative pseudo-probabilities and anomalous weak values. As mentioned in Sect. 1, entanglement inequalities are so formulated that the ensuing inequalities get satisfied by some entangled states and are violated by all the separable states. In order to illustrate the interrelation, we take up two-qubit systems for our study. The same approach can be advanced to multi-qubit as well as to multi-qubit systems, by writing the entanglement inequalities as a sum of PPs with nonnegative weights (by employing the derivation given in [25]). We first derive a set of two entanglement inequalities, which get expressed as linear combinations of pseudo-probabilities. We first list, following [24], the criteria for choosing pseudo-probabilities that lead to entanglement inequalities.

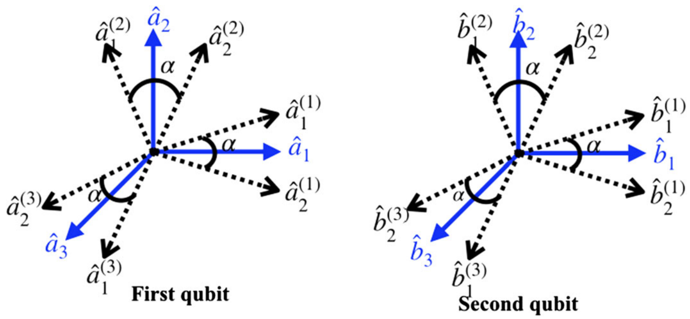

Fig. 1 $\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{b}_1, \hat{b}_2, \hat{b}_3$: Directions for first and second qubit, respectively, that appear in the final entanglement inequality (shown in blue). $\{\hat{a}_i^{(1)}, \hat{a}_i^{(2)}, \hat{a}_i^{(3)}\}$, $\{\hat{b}_i^{(1)}, \hat{b}_i^{(2)}, \hat{b}_i^{(3)}\}$, $i \in \{1, 2\}$: Directions for the first and the second qubit involved in the construction of pseudo-projections (shown in black) (Color figure online)

7.2.1 Rationale underlying choice of pseudoprobabilites yielding entanglement inequalities

1. Entanglement in a two-qubit system refers to correlations between outcomes of one set of locally non-commuting observables on the first qubit with another set of locally non-commuting observables on the second qubit. Guided by this, we choose pseudo-probabilities for joint outcomes of sets of observables, which ensure that the ensuing inequality involves the correlations, such as $\langle \boldsymbol{\sigma}_1 \cdot \hat{a} \boldsymbol{\sigma}_2 \cdot \hat{b} \rangle$ and $\langle \boldsymbol{\sigma}_1 \cdot \hat{a}' \boldsymbol{\sigma}_2 \cdot \hat{b}' \rangle$, with the proviso that the commutators $[\boldsymbol{\sigma}_1 \cdot \hat{a}, \boldsymbol{\sigma}_1 \cdot \hat{a}'] \neq 0$; $[\boldsymbol{\sigma}_2 \cdot \hat{b}, \boldsymbol{\sigma}_2 \cdot \hat{b}'] \neq 0$ (please note that the observables $\boldsymbol{\sigma}_1 \cdot \hat{a}, \boldsymbol{\sigma}_1 \cdot \hat{a}'$ and $\boldsymbol{\sigma}_2 \cdot \hat{b}, \boldsymbol{\sigma}_2 \cdot \hat{b}'$ act over the spaces of the first and the second qubit, respectively.).
2. A further requirement is, of course, that, for separable states, the inequality that follows from non-classicality requirement gets automatically violated.

7.3 Geometry of observables

The observables employed for constructing entanglement inequalities obey a common geometry, which is as follows: (i) For each qubit, we have three sets of doublets, $\{a_1^{(i)}, a_2^{(i)}\}$ and $\{b_1^{(i)}, b_2^{(i)}\}$, $i = 1, 2, 3$. Recall that generically, $a \equiv \boldsymbol{\sigma}_1 \cdot \hat{a}$, $b \equiv \boldsymbol{\sigma}_2 \cdot \hat{b}$. (ii) The angles between the two observables in each of the six sets have the same value, which is represented by α, the only free parameter. (iii) For each qubit, the normalised sums of vectors from within each set forms an orthonormal basis and are denoted by $\{\hat{a}_i\}$ and $\{\hat{b}_i\}$, respectively. Thus, $|\hat{a}_1 \cdot (\hat{a}_2 \times \hat{a}_3)| = |\hat{b}_1 \cdot (\hat{b}_2 \times \hat{b}_3)| = 1$. The geometry is completely depicted in Fig. 1.

7.3.1 Linear entanglement inequalities

The set of entanglement inequalities which we set forth are, essentially, refinements over the construction given in [24]. We, then, recast it in the language of weak measurements.

7.3.2 Inequality I

Of interest to us is the following sum of pseudo-probabilities,

$$\mathcal{P}_{E_1} = \sum_{i=1}^{2} \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}). \tag{40}$$

Classical probability mandates the sum to be non-negative. Hence, for non-classicality, $\mathcal{P}_{E_1} < 0$, which yields the criterion,

$$2\cos\frac{\alpha}{2} + \sum_{i=1}^{2} \left\langle \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i \right\rangle < 0. \tag{41}$$

The necessary condition that, $\mathcal{P}_{E_1} > 0$, for all separable states is obeyed provided that $0 < \alpha \leq \frac{2\pi}{3}$. The detailed derivation is given in "Appendix D.2".

In terms of weak values, Eq. (40) has the form,

$$\mathcal{P}_{E_1} = \sum_{i=1}^{2} \left\{ \langle \pi_{a_i} \pi_{b_2^{(i)}} \rangle \langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i} \rho_{b_2^{(i)}}} + \langle \pi_{\overline{a}_i} \pi_{\overline{b}_2^{(i)}} \rangle \langle\!\langle \pi_{\overline{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\overline{a}_i} \rho_{\overline{b}_2^{(i)}}} \right\}, \tag{42}$$

from which we see that four weak measurements are required. Also, the inequality (41) cannot get satisfied if *none* of the four weak values in (42) is anomalous. For the Werner states, detected to be entangled by inequality (41), *all* the four weak values are anomalous, as shown in "Appendix D.2".

7.3.3 Inequality II

The relevant pseudo-probability is the sum [24],

$$\mathcal{P}_{E_2} = \sum_{i=1}^{3} \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}), \tag{43}$$

which, again, is non-negative classically. Hence, the demand that, $\mathcal{P}_{E_2} < 0$, yields the inequality,

$$3\cos\frac{\alpha}{2} + \sum_{i=1}^{3} \left\langle \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i \right\rangle < 0. \tag{44}$$

The further requirement that, $\mathcal{P}_{E_2} \geq 0$, for all separable states imposes the additional constraint, $0 < \alpha \leq \arccos(-7/9) \simeq \pi$. The detailed derivation is given in "Appendix D.3".

In terms of weak values, $\mathcal{P}_{E_2} < 0$ translates to the condition,

$$\mathcal{P}_{E_2} = \sum_{i=1}^{3} \left\{ \langle \pi_{a_i} \pi_{b_2^{(i)}} \rangle \langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i} \rho_{b_2^{(i)}}} + \langle \pi_{\bar{a}_i} \pi_{\bar{b}_2^{(i)}} \rangle \langle\!\langle \pi_{\bar{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\bar{a}_i} \rho_{\bar{b}_2^{(i)}}} \right\} < 0, \tag{45}$$

which again demonstrates that $\mathcal{P}_{E_2} \geq 0$ if none of the weak values is anomalous. In fact, for entangled Werner states, *all* the six weak values are required to be anomalous, as shown in "Appendix D.3".

7.3.4 Nonlinear entanglement inequalities

We have, so far, considered violation of classical probability rules for sums of chosen pseudo-probabilities. In this section, we explore violations of classical probability rules for bilinears in pseudo-probabilities and show how they can be harnessed to yield nonlinear entanglement inequalities.

7.3.5 Inequality I

For construction of first nonlinear entanglement inequality, the combination of pseudo-probabilities is given by,

$$\mathcal{S}_1 = \sum_{i=1}^{2} \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) \mathcal{P}(\bar{a}_i = b_1^{(i)} = b_2^{(i)}). \tag{46}$$

Imposing the non-classicality condition, $\mathcal{S}_1 < 0$, we arrive at the inequality,

$$2 \cos^2 \frac{\alpha}{2} - \sum_{i=1}^{2} \langle \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i \rangle^2 < 0. \tag{47}$$

The further necessary condition that, $\mathcal{S}_1 \geq 0$, for all separable states yields the range $0 < \alpha \leq \dfrac{\pi}{2}$. The detailed derivation is given in "Appendix E.1".

It remains to express Eq. (46) in terms of weak values, which can be verified to be given by,

$$\sum_{i=1}^{2} \Bigg\{ \langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i} \rho_{b_2^{(i)}}} \langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\bar{a}_i} \rho_{b_2^{(i)}}} \langle \pi_{a_i} \pi_{b_2^{(i)}} \rangle \langle \pi_{\bar{a}_i} \pi_{b_2^{(i)}} \rangle + \langle\!\langle \pi_{\bar{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\bar{a}_i} \rho_{\bar{b}_2^{(i)}}} \langle\!\langle \pi_{\bar{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i} \rho_{\bar{b}_2^{(i)}}} \langle \pi_{\bar{a}_i} \pi_{\bar{b}_2^{(i)}} \rangle \langle \pi_{a_i} \pi_{\bar{b}_2^{(i)}} \rangle \Bigg\}. \tag{48}$$

Equation (46) cannot acquire a negative value if *none* of the eight weak values in (48) is anomalous. If a set of four weak values, $\left(\langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i} \rho_{b_2^{(i)}}} \text{ and } \langle\!\langle \pi_{\bar{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\bar{a}_i} \rho_{\bar{b}_2^{(i)}}} \right)$, or $\left(\langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\overline{a}_i}\rho_{b_2^{(i)}}} \text{ and } \langle\!\langle \pi_{\overline{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i}\rho_{\overline{b}_2^{(i)}}}\right)$, are anomalous, the ensuing inequality (47) is bound to get satisfied.

7.3.6 Inequality II

The bilinear combination of pseudo-probabilities is chosen to be,

$$\mathcal{S}_2 = \sum_{i=1}^{3} \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)})\mathcal{P}(\overline{a}_i = b_1^{(i)} = b_2^{(i)}), \tag{49}$$

which, following the same method, yields the inequality for the non-classicality condition $\mathcal{S}_2 < 0$ to be,

$$3\cos^2\frac{\alpha}{2} - \sum_{i=1}^{3}\langle \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i\rangle^2 < 0. \tag{50}$$

As usual, we demand that, $\mathcal{S}_2 \geq 0$, for all separable states which imposes the constraint $0 < \alpha \leq \arccos(-1/3)$. The detailed derivation is given in "Appendix E.2".

Finally, the inequality (49), in terms of weak values, has the form,

$$\sum_{i=1}^{3}\Big\{ \langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i}\rho_{b_2^{(i)}}} \langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\overline{a}_i}\rho_{b_2^{(i)}}} \langle \pi_{a_i}\pi_{b_2^{(i)}}\rangle\langle \pi_{\overline{a}_i}\pi_{b_2^{(i)}}\rangle \\ + \langle\!\langle \pi_{\overline{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\overline{a}_i}\rho_{\overline{b}_2^{(i)}}} \langle\!\langle \pi_{\overline{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i}\rho_{\overline{b}_2^{(i)}}} \langle \pi_{\overline{a}_i}\pi_{\overline{b}_2^{(i)}}\rangle\langle \pi_{a_i}\pi_{\overline{b}_2^{(i)}}\rangle \Big\}. \tag{51}$$

Equation (49) cannot acquire a negative value if *none* of the twelve weak values in (51) is anomalous. If a set of six weak values, $\left(\langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i}\rho_{b_2^{(i)}}} \text{ and } \langle\!\langle \pi_{\overline{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\overline{a}_i}\rho_{\overline{b}_2^{(i)}}}\right)$, or $\left(\langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\overline{a}_i}\rho_{b_2^{(i)}}} \text{ and } \langle\!\langle \pi_{\overline{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i}\rho_{\overline{b}_2^{(i)}}}\right)$, are anomalous, the ensuing inequality (50) is bound to get satisfied, which again shows the intimate connection between anomalous weak values and entanglement.

7.3.7 Inequality III

The last inequality, which we derive below, differs from the previous ones in that it has contributions from both local observables and correlations.

The combination (which involves terms bilinear in pseudo-probabilities) is chosen to be,

$$\begin{aligned}\mathcal{S}_3 = \sum_{i=1}^{3}\Big[&\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) + \frac{1}{2}\Big\{P(a_i)\mathcal{P}(\overline{a}_1^{(i)}, \overline{a}_2^{(i)})\\ &+P(\overline{a}_i)\mathcal{P}(a_1^{(i)}, a_2^{(i)}) + P(b_i)\mathcal{P}(\overline{b}_1^{(i)}, \overline{b}_2^{(i)})\\ &+P(\overline{b}_i)\mathcal{P}(b_1^{(i)}, b_2^{(i)}) + P(a_i)\mathcal{P}(\overline{b}_1^{(i)}, \overline{b}_2^{(i)})\\ &+P(\overline{a}_i)\mathcal{P}(b_1^{(i)}, b_2^{(i)}) + \mathcal{P}(\overline{a}_1^{(i)}, \overline{a}_2^{(i)})P(b_i)\\ &+\mathcal{P}(a_1^{(i)}, a_2^{(i)})P(\overline{b}_i)\Big\}\Big]. \end{aligned} \tag{52}$$

The ensuing expression has the form ($\lambda = \frac{1}{2}\cos\frac{\alpha}{2}$),

$$\mathcal{S}_3 = \lambda\Big\{9\cos\frac{\alpha}{2} + \sum_{i=1}^{3}\Big(\langle\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\rangle - \frac{1}{2}\langle\boldsymbol{\sigma}_1\cdot\hat{a}_i + \boldsymbol{\sigma}_2\cdot\hat{b}_i\rangle^2\Big)\Big\}. \tag{53}$$

The non-classicality condition, $\mathcal{S}_3 < 0$, is enforced on the states with the proviso that, $\mathcal{S}_3 \geq 0$, for all the separable states. This determines the range of the free parameter to be $0 < \alpha \leq \arccos(-79/81)$. The inequality corresponding to the upper limit, $\alpha = \arccos\left(-\frac{79}{81}\right)$, has been obtained by Gühne by using covariance matrix criteria for local observables [57].

Writing Eq. (52) in terms of weak values, the following expression results,

$$\begin{aligned}\mathcal{S}_3 = \sum_{i=1}^{3}\Big[&\Big\{\langle\pi_{a_i}\pi_{b_2^{(i)}}\rangle\langle\!\langle\pi_{b_1^{(i)}}\rangle\!\rangle_\rho^{\pi_{a_i}\pi_{b_2^{(i)}}} + \langle\pi_{\overline{a}_i}\pi_{\overline{b}_2^{(i)}}\rangle\langle\!\langle\pi_{\overline{b}_1^{(i)}}\rangle\!\rangle_\rho^{\pi_{\overline{a}_i}\pi_{\overline{b}_2^{(i)}}}\Big\}\\ &+\frac{1}{2}\Big\{\langle\pi_{a_i}\rangle\langle\pi_{\overline{a}_1^{(i)}}\rangle\langle\!\langle\pi_{\overline{a}_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{\overline{a}_1^{(i)}}} + \langle\pi_{\overline{a}_i}\rangle\langle\pi_{a_1^{(i)}}\rangle\langle\!\langle\pi_{a_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{a_1^{(i)}}} + \langle\pi_{b_i}\rangle\langle\pi_{\overline{b}_1^{(i)}}\rangle\langle\!\langle\pi_{\overline{b}_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{\overline{b}_1^{(i)}}}\\ &+\langle\pi_{\overline{b}_i}\rangle\langle\pi_{b_1^{(i)}}\rangle\langle\!\langle\pi_{b_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{b_1^{(i)}}} + \langle\pi_{a_i}\rangle\langle\pi_{\overline{b}_1^{(i)}}\rangle\langle\!\langle\pi_{\overline{b}_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{\overline{b}_1^{(i)}}} + \langle\pi_{\overline{a}_i}\rangle\langle\pi_{b_1^{(i)}}\rangle\langle\!\langle\pi_{b_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{b_1^{(i)}}}\\ &+\langle\pi_{b_i}\rangle\langle\pi_{\overline{a}_1^{(i)}}\rangle\langle\!\langle\pi_{\overline{a}_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{\overline{a}_1^{(i)}}} + \langle\pi_{\overline{b}_i}\rangle\langle\pi_{a_1^{(i)}}\rangle\langle\!\langle\pi_{a_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{a_1^{(i)}}}\Big\}\Big]. \end{aligned} \tag{54}$$

The detailed derivation is given in "Appendix E.3".

At this juncture, we note that there does exist partial positive transpose (PPT) criterion [28], which identifies all the entangled states in $2\otimes 2$ systems. In this paper, our aim is to systematically study violations of classical probability rules to obtain non-classicality conditions (not restricted only to entanglement), and their experimental implementation through weak measurements. What we have shown here is that the derived conditions also serve to detect entanglement, contingent on appropriate choices of observables.

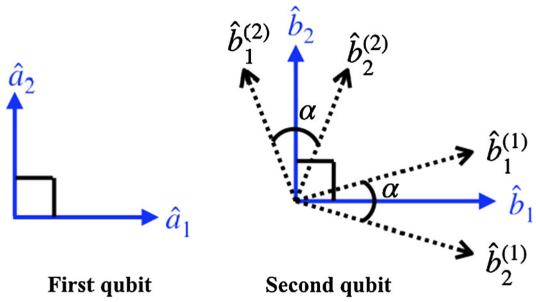

Fig. 2 $\hat{a}_i, \hat{b}_i$: Directions for first and second qubit, respectively, that appear in the condition for discord (shown in blue). $\hat{b}_i^{(1)}, \hat{b}_i^{(2)}, i \in \{1, 2\}$: Directions for second qubit involved in the construction of pseudo-projections (shown in black) (Color figure online)

7.4 Two-qubit quantum discord and weak values

A system displays non-classicality even if any one weak value turns out to be anomalous. This statement is equivalent to recognising non-classicality if even one pseudo-probability becomes negative. We may, therefore, relax the condition that, $\mathcal{P}_E \geq 0$, for separable states, and ask if conditions for quantum discord [14] can be derived. We show below that the answer is in the affirmative. The proof is by explicit construction.

By definition, a state whose discord, $\mathcal{D}^{1\rightarrow 2}$ vanishes has the structure,

$$\rho^{12} = \sum_k p_k |\phi_k^1\rangle\langle\phi_k^1| \otimes \rho_k^2, \tag{55}$$

where, $\sum_k p_k |\phi_k^1\rangle\langle\phi_k^1|$ is the resolution of ρ^1 in its eigenbasis.

7.4.1 Choice of the observables

We choose two orthogonal observables A_1, A_2 for the first subsystem 1 with the stipulation that one of them say, A_1 shares its eigenbasis with ρ^1. The eigen-basis of A_2 is unbiased with respect to that of A_1. Thus, a partial tomography is warranted. For this reason, the condition that we derive does not qualify as a witness. For the subsystem 2, we choose two pairs of observables $\{B_1^{(1)}, B_2^{(1)}\}$, and $\{B_1^{(2)}, B_2^{(2)}\}$. They obey the same conditions that were imposed in entanglement inequality. The geometry is shown in Fig. 2.

Unlike in the earlier cases, we construct two pseudo-probabilities, $\mathcal{P}_D^i,\ i = 1, 2,$

$$\begin{aligned}\mathcal{P}_D^i &= \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) + P(a_i)\mathcal{P}(\overline{b}_1^{(i)}\overline{b}_2^{(i)}) + P(\overline{a}_i)\mathcal{P}(b_1^{(i)}b_2^{(i)}) \\ &\equiv \lambda\Big\{(16\lambda + 2\langle\boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i\rangle - 2\langle\boldsymbol{\sigma}_1 \cdot \hat{a}_i\rangle\langle\boldsymbol{\sigma}_2 \cdot \hat{b}_i\rangle\Big\},\end{aligned} \tag{56}$$

where the parameter $\lambda = \dfrac{1}{4}\cos\dfrac{\alpha}{2}$. If a state is discordant, both the pseudo-probabilities—$\mathcal{P}_D^1$ and $\mathcal{P}_D^2$, become negative for suitable choices of α. The non-discordant states can have a negative value for at most one pseudo-probability. The detailed derivation is given in "Appendix F".

Weak measurements can indeed detect discord. Abstracting the parent PP from the LHS of Eq. (56), we infer the expression, in terms of weak measurements, to be,

$$\mathcal{P}_D^i = \langle \pi_{a_i} \pi_{b_2^{(i)}} \rangle \langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i} \rho_{b_2^{(i)}}} + \langle \pi_{\overline{a}_i} \pi_{\overline{b}_2^{(i)}} \rangle \langle\!\langle \pi_{\overline{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\overline{a}_i} \rho_{\overline{b}_2^{(i)}}}$$
$$+ \langle \pi_{a_i} \rangle \langle \pi_{\overline{b}_2^{(i)}} \rangle \langle\!\langle \pi_{\overline{b}_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{\overline{b}_2^{(i)}}} + \langle \pi_{\overline{a}_i} \rangle \langle \pi_{b_2^{(i)}} \rangle \langle\!\langle \pi_{b_1^{(i)}} \rangle\!\rangle_\rho^{\rho_{b_2^{(i)}}} . \tag{57}$$

It is clear that the existence of quantum discord is concomitant on the existence of negative weak values.

We conclude this section with a final comment. Through these constructions, we have explicitly shown that non-existence of classical joint probability underlies quantum entanglement and discord as well, a conclusion similar to that obtained by Fine [23] for CHSH nonlocality. Weak measurement techniques provide an experimental tool to bring this aspect out through anomalous weak values.

8 Discussion on the choices of pseudoprobabilities for different quantum features

We are now in a position to discuss how the hierarchy among nonlocality, entanglement, discord and coherence gets manifested in the choices of pseudoprobabilities. The hierarchy is manifested in two ways—number of observables used in the pseudoprobability scheme and the way they are combined.

Consider, for example, two-qubit states, in which CHSH nonlocality is the strongest kind of nonclassicality. Thus, only four observables are needed to arrive at CHSH inequality, which is the minimal requirement.

Entanglement is weaker than nonlocality, but stronger than quantum discord. Recall that there are local entangled states and separable discordant states. Consequently, entanglement inequalities emerge from the pseudoprobability schemes involving larger number of observables. For example, the pseudo-probability scheme underlying the entanglement inequality (41) is constructed for joint outcomes of six observables (two observables for the first qubit and four observables for the second qubit). On the other hand, the pseudo-probability scheme, underlying the entanglement inequality (44), requires joint outcomes of nine observables (three observables for the first qubit and six for the second). Derivation of even more stringent nonlinear entanglement inequalities would require more involved sums of pseudo-probabilities.

Finally, detection of quantum discord would also require partial tomography. Thus, the choices of pseudoprobabilities employed by us are not haphazard, but reflect the strength of underlying nonclassical feature. Coherence, of course, is the simplest as it does not involve correlations.

9 Conclusion

In conclusion, we have established the connection between two seemingly different approaches to non-classicality by making use of pseudo-projections [24]—in fact a special case of which leads to the non-classical properties described in [26,27]. Pseudo-probabilities and anomalous weak values share an equivalence which makes the former measurable quantities. Violation of Boolean logic by quantum mechanics can be directly probed by employing weak measurements. We have shown how different sets of anomalous weak values not only serve to detect different nonclassicality manifests in single and two-qubit states but also trace the violations of underlying classical probability rules.

A further extension of our results to multi-party systems constitutes an interesting study. It will integrate conditions for different kinds of nonclassical correlations (e.g. nonlocality, entanglement, etc.), anomalous weak values and negative pseudoprobabilities.

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Author Contributions Both authors contributed equally to this work in all respects.

A Unit pseudo-projections have at least one negative eigenvalue.

Consider a unit PP, $\mathbf{\Pi}$, constructed as the symmetrised product of N mutually non-commuting projections (of any ranks) in a Hilbert space of dimension D, as given by,

$$\mathbf{\Pi} = \frac{1}{2}(\pi_1\pi_2\ldots\pi_N + \text{h.c.}). \tag{58}$$

Let $|a\rangle$ ($|b\rangle$) be a vector in the null space of π_1 (π_N) but not in that of π_N (π_1). Construct the orthogonal basis: $\{|e_1\rangle = |a\rangle, |e_2\rangle = |b\rangle - \langle a|b\rangle|a\rangle\}$ in the two-dimensional subspace spanned by the vectors. The determinant of the minor of $\mathbf{\Pi}$ in this two-dimensional basis is evidently negative. Thus, $\mathbf{\Pi}$ possesses at least one negative eigenvalue. In order to illustrate it, consider a unit PP, $\mathbf{\Pi}$, constructed as symmetrised product of N mutually non-commuting projections $\pi_j = \sum_{i_j} |a_{i_j}\rangle\langle a_{i_j}|;\ j \in \{1,\ldots,N\}$. Thus,

$$\mathbf{\Pi} = \frac{1}{2}\sum_{i_1\ldots i_N}\Big(|a^1_{i_1}\rangle\langle a^1_{i_1}|a^2_{i_2}\rangle\langle a^2_{i_2}|\cdots\langle a^{N-1}_{i_{N-1}}|a^N_{i_N}\rangle\langle a^N_{i_N}| + \text{h.c.}\Big).$$

Let $i_1 \in \{1,\ldots,d\}$, where $d < D$. Since π_1 and π_N are non-commuting, one can always find a state $|a^1_{d+1}\rangle$, lying in the null space of π_1, such that $\pi_N|a^1_{d+1}\rangle \neq 0$. We project $\mathbf{\Pi}$ in the subspace spanned by $\{|a^1_d\rangle, |a^1_{d+1}\rangle\}$,

$$
\begin{aligned}
&\Big(|a_d^1\rangle\langle a_d^1| + |a_{d+1}^1\rangle\langle a_{d+1}^1|\Big)\mathbf{\Pi}\Big(|a_d^1\rangle\langle a_d^1| + |a_{d+1}^1\rangle\langle a_{d+1}^1|\Big)\\
&=\frac{1}{2}\sum_{i_1\dots i_N}\Big\{|a_d^1\rangle\langle a_{i_1}^1|a_{i_2}^2\rangle\langle a_{i_2}^2|\dots\langle a_{i_{N-1}}^{N-1}|a_{i_N}^N\rangle\Big(\langle|a_{i_N}^N|a_d^1\rangle\langle a_d^1| + \langle a_{i_N}^N|a_{d+1}^1\rangle\langle a_{d+1}^1|\Big) + \text{h.c.}\Big\}\\
&=\frac{1}{2}|a_d^1\rangle\Big\{\sum_{i_1\dots i_N}\langle a_{i_1}^1|a_{i_2}^2\rangle\langle a_{i_2}^2|\dots\langle a_{i_{N-1}}^{N-1}|a_{i_N}^N\rangle\langle a_{i_N}^N|a_d^1\rangle + \text{c.c.}\Big\}\langle a_d^1|\\
&=\frac{1}{2}|a_d^1\rangle\Big\{\sum_{i_1\dots i_N}\langle a_{i_1}^1|a_{i_2}^2\rangle\cdots\langle a_{i_{N-1}}^N|a_{i_N}^N\rangle\langle a_{i_N}^N|a_{d+1}^1\rangle\Big\}\langle a_{d+1}^1| + \text{h.c.},
\end{aligned}
\tag{59}
$$

which has a negative determinant, thereby proving that $\mathbf{\Pi}$ has a negative eigenvalue.

B Derivation of coherence witness

In a fixed basis, all the diagonal states are considered to be incoherent and states having nonzero off-diagonal elements are designated as coherent [52]. PPs can act as witnesses for this coherence. To see this, let the state be diagonal in the eigenbasis of σ_z. We are interested in the PP, representing a joint outcome of two incompatible observables $\boldsymbol{\sigma}\cdot\hat{a}_1$ and $\boldsymbol{\sigma}\cdot\hat{a}_2$, which acquires negative values only for states $\rho = \frac{1}{2}(1+\boldsymbol{\sigma}.\boldsymbol{p})$ having nonzero off-diagonal term. Then, the PP, $\mathbf{\Pi}_{a_1a_2} = \frac{1}{4}\Big(1 + \hat{a}_1\cdot\hat{a}_2 + \boldsymbol{\sigma}.(\hat{a}_1 + \hat{a}_2)\Big)$, has positive overlap with all diagonal states whenever $(\hat{a}_1 + \hat{a}_2)$ lies in the $x - y$ plane. Suppose $\hat{a}_1\cdot\hat{a}_2 = \cos\theta$ and we choose $(\hat{a}_1 + \hat{a}_2) \parallel (\cos\lambda\hat{x} + \sin\lambda\hat{y})$. Then,

$$
\mathbf{\Pi}_{a_1a_2} = \frac{1}{2}\cos\frac{\theta}{2}\Big\{\cos\frac{\theta}{2} + (\cos\lambda\sigma_x + \sin\lambda\sigma_y)\Big\}. \tag{60}
$$

The expectation value of this operator with the state ρ is,

$$
\text{Tr}(\mathbf{\Pi}_{a_1a_2}\rho) = \frac{1}{2}\cos\frac{\theta}{2}\Big\{\cos\frac{\theta}{2} + (\cos\lambda p_x + \sin\lambda p_y)\Big\}. \tag{61}
$$

The operator $\mathbf{\Pi}_{a_1a_2}$, for a proper choice of θ and λ, has negative overlap for all states with nonzero p_x and p_y.

C Derivation of CHSH inequality

The sum of pseudoprobabilities for CHSH nonlocality is as follows:

$$
\mathcal{P}(A_1 = B_1 = B_2) + \mathcal{P}(A_2 = B_1 = \bar{B}_2). \tag{62}
$$

Employing dichotomic nature of the observables, we express each pseudoprobability in terms of expectation values of observables. Thus, the following expression results,

$$\begin{aligned}\mathcal{P}_{\mathrm{NL}} \equiv & \mathcal{P}(A_1 = B_1 = B_2) + \mathcal{P}(A_2 = B_1 = \bar{B}_2) \\ = & \mathcal{P}(A_1 = +1; B_1 = +1 B_2 = +1) + \mathcal{P}(A_1 = -1; B_1 = -1 B_2 = -1) \\ + & \mathcal{P}(A_2 = +1; B_1 = +1 B_2 = -1) + \mathcal{P}(A_2 = -1; B_1 = -1 B_2 = +1) \\ = & \frac{1}{8}\Big\langle \big(1 + A_1\big)\big(1 + \{B_1, B_2\} + B_1 + B_2\big) + \big(1 - A_1\big)\big(1 + \{B_1, B_2\} - B_1 - B_2\big) \\ + & \big(1 + A_2\big)\big(1 - \{B_1, B_2\} + B_1 - B_2\big) + \big(1 - A_2\big)\big(1 - \{B_1, B_2\} - B_1 + B_2\big)\Big\rangle \\ = & \frac{1}{4}\Big\langle 2 + A_1(B_1 + B_2) + A_2(B_1 - B_2)\Big\rangle. \end{aligned} \tag{63}$$

Imposing the nonclassicality criteria, $\mathcal{P}_{\mathrm{NL}} < 0$, the CHSH inequality results,

$$\langle A_1(B_1 + B_2) + A_2(B_1 - B_2)\rangle < -2. \tag{64}$$

For the following choice of the observables,

$$\begin{aligned} & A_1 \equiv \boldsymbol{\sigma}_1 \cdot \hat{a}_1 = \sigma_{1x};\ A_2 \equiv \boldsymbol{\sigma}_1 \cdot \hat{a}_2 = \sigma_{1y}; \\ & B_1 \equiv \frac{\sigma_{2x} + \sigma_{2y}}{\sqrt{2}};\ B_2 \equiv \frac{\sigma_{2x} - \sigma_{2y}}{\sqrt{2}}, \end{aligned} \tag{65}$$

the values of the pseudo-probabilities, for two-qubit Werner states, $\rho = \frac{1}{4}(1 - \eta \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$, are given as below,

$$\mathcal{P}(A_1; B_1 B_2) = \mathcal{P}(\bar{A}_1; \bar{B}_1 \bar{B}_2) = \mathcal{P}(A_2; B_1 \bar{B}_2) = \mathcal{P}(\bar{A}_1; \bar{B}_1 B_2) = \frac{1}{8}(1 - \eta\sqrt{2}).$$

Thus, all the four pseudoprobabilities turn negative for all the nonlocal Werner states ($\frac{1}{\sqrt{2}} < \eta \leq 1$), thereby proving that all the weak values also turn negative, i.e. anomalous.

D Derivation of linear entanglement inequalities

D.1 Expression of pseudoprojection operator for joint outcomes of two observables

First, we explicitly calculate the expression of PP representing a joint event, in which $\boldsymbol{\sigma} \cdot \hat{m}_1$ and $\boldsymbol{\sigma} \cdot \hat{m}_2$ both take value $+1$. The PP is given by the symmetrised rule,

$$\begin{aligned}\mathbf{\Pi} &= \frac{1}{2}\Big\{\frac{1}{2}(1+\boldsymbol{\sigma}\cdot\hat{m}_1), \frac{1}{2}(1+\boldsymbol{\sigma}\cdot\hat{m}_2)\Big\} \\ &= \frac{1}{4}\Big(1+\frac{1}{2}\{\boldsymbol{\sigma}\cdot\hat{m}_1, \boldsymbol{\sigma}\cdot\hat{m}_2\}+\boldsymbol{\sigma}\cdot(\hat{m}_1+\hat{m}_2)\Big) \\ &= \frac{1}{4}\Big(1+\hat{m}_1\cdot\hat{m}_2+\boldsymbol{\sigma}\cdot\hat{m}_1+\boldsymbol{\sigma}\cdot\hat{m}_2\Big). \end{aligned} \tag{66}$$

If the included angle between $\hat{m}_1$ and $\hat{m}_2$ is given by α, then,

$$\hat{m}_1\cdot\hat{m}_2 = \cos\alpha \text{ and } \hat{m}+\hat{m}_2 = 2\cos\frac{\alpha}{2}\hat{m}.$$

Here, $\hat{m}$ is a unit vector parallel to $(\hat{m}_1+\hat{m}_2)$. Plugging in these values in equation (66), we obtain,

$$\begin{aligned}\mathbf{\Pi} &= \frac{1}{4}\Big(2\cos^2\frac{\alpha}{2}+2\cos\frac{\alpha}{2}\boldsymbol{\sigma}\cdot\hat{m}\Big) \\ &= \frac{1}{2}\cos\frac{\alpha}{2}\Big(\cos\frac{\alpha}{2}+\boldsymbol{\sigma}\cdot\hat{m}\Big). \end{aligned} \tag{67}$$

We shall thoroughly use this expression to obtain various entanglement inequalities, conditions for discord and coherence witnesses. We start with the derivation of the first linear entanglement inequality.

D.2 Derivation of linear entanglement inequality: I

The sum of pseudoprobabilities for entanglement inequality is as follows:

$$\mathcal{P}_{E_1} = \sum_{i=1}^{2}\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}).$$

Writing each pseudo-probability in terms of expectation of PP, the following expression results,

$$\begin{aligned}\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) &= \mathcal{P}(a_i = +1; b_1^{(i)} = +1 b_2^{(i)} = +1) \\ &\quad + \mathcal{P}(a_i = -1; b_1^{(i)} = -1 b_2^{(i)} = -1) \\ &= \frac{1}{4}\cos\frac{\alpha}{2}\Big\langle(1+\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}+\boldsymbol{\sigma}_2\cdot\hat{b}_i)+(1-\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}-\boldsymbol{\sigma}_2\cdot\hat{b}_i)\Big\rangle \\ &= \frac{1}{2}\cos\frac{\alpha}{2}\Big\langle\cos\frac{\alpha}{2}+\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\Big\rangle. \end{aligned} \tag{68}$$

Substituting the values of pseudoprobabilities in terms of expectation values of observables, we obtain the following expression,

$$\mathcal{P}_{E_1} = \frac{1}{2}\cos\frac{\alpha}{2}\sum_{i=1}^{2}\left\langle 2\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i \right\rangle. \tag{69}$$

Imposing the nonclassicality constraint, $\mathcal{P}_{E_1} < 0$, the following inequality emerges,

$$\sum_{i=1}^{2}\left\langle 2\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i \right\rangle < 0. \tag{70}$$

The range of α is fixed since we demand all the separable states to violate the inequality. The maximum value of $\sum_{i=1}^{2}\langle \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i \rangle$ for a separable state is 1. To show this, consider the two-qubit pure separable state $\rho = \frac{1}{4}(1+\sigma_{1z})(1+\sigma_{2z})$. Then,

$$\begin{aligned}
&\sum_{i=1}^{2}\left\langle 2\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i \right\rangle_\rho \\
=&2\cos\frac{\alpha}{2} + a_{1z}b_{1z} + a_{2z}b_{2z} \geq -1 + 2\cos\frac{\alpha}{2}.
\end{aligned} \tag{71}$$

Note that, $\hat{a}_1 \perp \hat{a}_2$ and $\hat{b}_1 \perp \hat{b}_2$. It implies that $(a_{1z}b_{1z} + a_{2z}b_{2z})$ is upper bounded by 1 in magnitude. Thus, in order that the inequality (70) gets violated by all the separable states, $1 \leq 2\cos\frac{\alpha}{2} < 2$, which, in turn, implies that $0 \leq \alpha < \frac{2\pi}{3}$.

In terms of weak values, the sum of pseudoprobabilities, $\mathcal{P}_{E_1}$, can be written as,

$$\begin{aligned}
\mathcal{P}_{E_1} &= \sum_{i=1}^{2}\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) \\
&= \sum_{i=1}^{2}\left\{\mathcal{P}(a_i = +1; b_1^{(i)} = +1 b_2^{(i)} = +1) + \mathcal{P}(a_i = -1; b_1^{(i)} = -1 b_2^{(i)} = -1)\right\} \\
&= \sum_{i=1}^{2}\langle \pi_{a_i} \otimes \boldsymbol{\Pi}_{b_1^{(i)} b_2^{(i)}} + \pi_{\bar{a}_i} \otimes \boldsymbol{\Pi}_{\bar{b}_1^{(i)} \bar{b}_2^{(i)}} \rangle \\
&= \frac{1}{2}\sum_{i=1}^{2}\langle \pi_{a_i} \otimes \{\pi_{b_1^{(i)}}, \pi_{b_2^{(i)}}\} + \pi_{\bar{a}_i} \otimes \{\pi_{\bar{b}_1^{(i)}}, \pi_{\bar{b}_2^{(i)}}\}\rangle \\
&= \sum_{i=1}^{2}\left(\langle \pi_{a_i}\pi_{b_1^{(i)}}\rangle \langle\!\langle \pi_{b_2^{(i)}} \rangle\!\rangle_\rho^{\rho_{a_i}\rho_{b_1^{(i)}}} + \langle \pi_{\bar{a}_i}\pi_{\bar{b}_1^{(i)}}\rangle \langle\!\langle \pi_{\bar{b}_2^{(i)}} \rangle\!\rangle_\rho^{\rho_{\bar{a}_i}\rho_{\bar{b}_1^{(i)}}}\right).
\end{aligned} \tag{72}$$

For the special choice of observables,

$$\boldsymbol{\sigma}_1 \cdot \hat{a}_1 = \sigma_{1x}; \boldsymbol{\sigma}_1 \cdot \hat{a}_2 = \sigma_{1y}; \boldsymbol{\sigma}_2 \cdot \hat{b}_1 = \sigma_{2x}; \boldsymbol{\sigma}_2 \cdot \hat{b}_2 = \sigma_{2y}, \tag{73}$$

and for the $2 \otimes 2$ Werner states, $\rho = \frac{1}{4}(1 - \eta \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$, the pseudoprobabilities and the inequality assumes the following form,

$$\begin{aligned}
&\mathcal{P}(a_1 = +1; b_1^{(1)} = +1 b_2^{(1)} = +1) = \mathcal{P}(a_1 = -1; b_1^{(1)} = -1 b_2^{(1)} = -1)\\
=&\mathcal{P}(a_2 = +1; b_1^{(2)} = +1 b_2^{(2)} = +1) = \mathcal{P}(a_2 = -1; b_1^{(2)} = -1 b_2^{(2)} = -1)\\
=&\frac{1}{4}\cos\frac{\alpha}{2}\Big(\cos\frac{\alpha}{2} - \eta\Big)\\
&2\cos\frac{\alpha}{2} + \langle \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y}\rangle < 0.
\end{aligned} \tag{74}$$

We consider a particular value of $\alpha = \frac{2\pi}{3}$. For this value, all the four pseudoprobabilities are equal to the following value,

$$\begin{aligned}
&\mathcal{P}(a_1 = +1; b_1^{(1)} = +1 b_2^{(1)} = +1) = \mathcal{P}(a_1 = -1; b_1^{(1)} = -1 b_2^{(1)} = -1)\\
=&\mathcal{P}(a_2 = +1; b_1^{(2)} = +1 b_2^{(2)} = +1) = \mathcal{P}(a_2 = -1; b_1^{(2)} = -1 b_2^{(2)} = -1)\\
=&\frac{1}{8}\Big(\frac{1}{2} - \eta\Big).
\end{aligned} \tag{75}$$

The Werner states are detected to be entangled in the range $\eta \in \left(\frac{1}{2}, 1\right]$ by the inequality (41), which implies that all the four pseudoprobabilities will be negative. This, in turn, implies that all the four weak values are negative, i.e. anomalous.

D.3 Derivation of linear entanglement inequality: II

The sum of pseudoprobabilities for entanglement inequality is as follows:

$$\mathcal{P}_{E_2} = \sum_{i=1}^{3} \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}).$$

The pseudoprobabilities, as before, can be written as expectation values of Pauli operators as follows,

$$\begin{aligned}
\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) &= \frac{1}{4}\cos\frac{\alpha}{2}\Big\langle (1 + \boldsymbol{\sigma}_1 \cdot \hat{a}_i)(\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_2 \cdot \hat{b}_i)\\
&\quad + (1 - \boldsymbol{\sigma}_1 \cdot \hat{a}_i)(\cos\frac{\alpha}{2} - \boldsymbol{\sigma}_2 \cdot \hat{b}_i)\Big\rangle\\
&= \frac{1}{2}\cos\frac{\alpha}{2}\Big\langle \cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i\Big\rangle.
\end{aligned} \tag{76}$$

Substituting the values of pseudoprobabilities in terms of expectation values of observables, we obtain the following expression,

$$\mathcal{P}_{E_2} = \frac{1}{2}\cos\frac{\alpha}{2}\sum_{i=1}^{3}\left\langle 2\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\right\rangle. \tag{77}$$

The nonclassicality condition, $\mathcal{P}_{E_2} < 0$, yields the following inequality,

$$\sum_{i=1}^{3}\left\langle 2\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\right\rangle < 0. \tag{78}$$

The range of α is fixed since we demand all the separable states to violate the inequality (78). The maximum value of $\sum_{i=1}^{3}\langle\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\rangle$ for a separable state is 1. To show this, consider the two-qubit pure separable state $\rho = \frac{1}{4}(1+\sigma_{1z})(1+\sigma_{2z})$. Then,

$$\begin{aligned}&\sum_{i=1}^{3}\left\langle 3\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\right\rangle_\rho\\ =&3\cos\frac{\alpha}{2} + a_{1z}b_{1z} + a_{2z}b_{2z} + a_{3z}b_{3z} \geq -1 + 3\cos\frac{\alpha}{2}.\end{aligned} \tag{79}$$

Thus, in order that the inequality gets violated by all the separable states, $1 \leq 3\cos\frac{\alpha}{2} < 3$, which, in turn, implies that $0 \leq \alpha < \arccos\big(-\frac{7}{9}\big)$.

In terms of weak values, the sum of pseudoprobabilities $\mathcal{P}_{E_2}$ can be written as,

$$\begin{aligned}\mathcal{P}_{E_2} &= \sum_{i=1}^{3}\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)})\\ &= \sum_{i=1}^{3}\left\{\mathcal{P}(a_i = +1; b_1^{(i)} = +1 b_2^{(i)} = +1) + \mathcal{P}(a_i = -1; b_1^{(i)} = -1 b_2^{(i)} = -1)\right\}\\ &= \sum_{i=1}^{3}\langle\pi_{a_i}\otimes\boldsymbol{\Pi}_{b_1^{(i)}b_2^{(i)}} + \pi_{\bar{a}_i}\otimes\boldsymbol{\Pi}_{\bar{b}_1^{(i)}\bar{b}_2^{(i)}}\rangle\\ &= \frac{1}{2}\sum_{i=1}^{3}\langle\pi_{a_i}\otimes\{\pi_{b_1^{(i)}},\pi_{b_2^{(i)}}\} + \pi_{\bar{a}_i}\otimes\{\pi_{\bar{b}_1^{(i)}},\pi_{\bar{b}_2^{(i)}}\}\rangle\\ &= \sum_{i=1}^{3}\left(\langle\pi_{a_i}\pi_{b_1^{(i)}}\rangle\langle\!\langle\pi_{b_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{a_i}\rho_{b_1^{(i)}}} + \langle\pi_{\bar{a}_i}\pi_{\bar{b}_1^{(i)}}\rangle\langle\!\langle\pi_{\bar{b}_2^{(i)}}\rangle\!\rangle_\rho^{\rho_{\bar{a}_i}\rho_{\bar{b}_1^{(i)}}}\right).\end{aligned} \tag{80}$$

For the special choices of observables,

$$\begin{aligned}\boldsymbol{\sigma}_1 \cdot \hat{a}_1 = \sigma_{1x}; \boldsymbol{\sigma}_1 \cdot \hat{a}_2 = \sigma_{1y}; \boldsymbol{\sigma}_1 \cdot \hat{a}_3 = \sigma_{1z}; \boldsymbol{\sigma}_2 \cdot \hat{b}_1 = \sigma_{2x}; \boldsymbol{\sigma}_2 \cdot \hat{b}_2 \\ = \sigma_{2y}, \boldsymbol{\sigma}_2 \cdot \hat{b}_3 = \sigma_{2z},\end{aligned} \tag{81}$$

and for the $2 \otimes 2$ werner states $\rho = \frac{1}{4}(1 - \eta\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$, the pseudoprobabilities and the inequality assume the following form,

$$\begin{aligned}&\mathcal{P}(a_1 = +1; b_1^{(1)} = +1 b_2^{(1)} = +1) = \mathcal{P}(a_1 = -1; b_1^{(1)} = -1 b_2^{(1)} = -1) \\ =&\mathcal{P}(a_2 = +1; b_1^{(2)} = +1 b_2^{(2)} = +1) = \mathcal{P}(a_2 = -1; b_1^{(2)} = -1 b_2^{(2)} = -1) \\ =&\mathcal{P}(a_3 = +1; b_1^{(3)} = +1 b_2^{(3)} = +1) = \mathcal{P}(a_3 = -1; b_1^{(3)} = -1 b_2^{(3)} = -1) \\ =&\frac{1}{4}\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} - \eta\right) \\ &3\cos\frac{\alpha}{2} + \langle \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z} \rangle < 0.\end{aligned} \tag{82}$$

We consider the case $\alpha = \arccos\left(-\frac{7}{9}\right)$, then all the six pseudoprobabilities are equal to the following value,

$$\begin{aligned}&\mathcal{P}(a_1 = +1; b_1^{(1)} = +1 b_2^{(1)} = +1) = \mathcal{P}(a_1 = -1; b_1^{(1)} = -1 b_2^{(1)} = -1) \\ =&\mathcal{P}(a_2 = +1; b_1^{(2)} = +1 b_2^{(2)} = +1) = \mathcal{P}(a_2 = -1; b_1^{(2)} = -1 b_2^{(2)} = -1) \\ =&\mathcal{P}(a_3 = +1; b_1^{(3)} = +1 b_2^{(3)} = +1) = \mathcal{P}(a_3 = -1; b_1^{(3)} = -1 b_2^{(3)} = -1) \\ =&\frac{1}{8}\left(\frac{1}{2} - \eta\right).\end{aligned} \tag{83}$$

The Werner states are detected to be entangled in the range $\eta \in \left(\frac{1}{3}, 1\right]$ by the inequality (44), which implies that all the six pseudoprobabilities will be negative. This, in turn, implies that all the six weak values are negative, i.e. anomalous.

E Derivation of nonlinear entanglement inequalities

E.1 Derivation of nonlinear entanglement inequality: I

The sum of pseudoprobabilities for entanglement inequality is as follows:

$$\mathcal{S}_1 = \sum_{i=1}^{2} \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)})\mathcal{P}(\overline{a}_i = b_1^{(i)} = b_2^{(i)}). \tag{84}$$

As before,

$$\begin{aligned}\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) = &\frac{1}{4}\cos\frac{\alpha}{2}\Big\langle(1+\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}+\boldsymbol{\sigma}_2\cdot\hat{b}_i)\\ &+(1-\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}-\boldsymbol{\sigma}_2\cdot\hat{b}_i)\Big\rangle\\ =&\frac{1}{2}\cos\frac{\alpha}{2}\Big\langle\cos\frac{\alpha}{2}+\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\Big\rangle, \end{aligned} \tag{85}$$

and,

$$\begin{aligned}\mathcal{P}(\bar{a}_i = b_1^{(i)} = b_2^{(i)}) = &\frac{1}{4}\cos\frac{\alpha}{2}\Big\langle(1-\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}+\boldsymbol{\sigma}_2\cdot\hat{b}_i)\Big)\\ &+\frac{1}{4}\Big((1+\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}-\boldsymbol{\sigma}_2\cdot\hat{b}_i)\Big\rangle\\ =&\frac{1}{2}\cos\frac{\alpha}{2}\Big\langle\cos\frac{\alpha}{2}-\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\Big\rangle. \end{aligned} \tag{86}$$

Plugging in the expressions from equations (85) and (86) in Eq. (84),

$$\mathcal{S}_1 = \Big(\frac{1}{2}\cos\frac{\alpha}{2}\Big)^2\sum_{i=1}^{2}\Big(2\cos^2\frac{\alpha}{2}-\langle\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\rangle^2\Big). \tag{87}$$

Imposing the non-classicality condition, $\mathcal{S}_1 < 0$, we arrive at the inequality,

$$2\cos^2\frac{\alpha}{2}-\sum_{i=1}^{2}\langle\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\rangle^2 < 0. \tag{88}$$

The range of α can be fixed by demanding all the separable states to violate this inequality. We can consider the separable state, without any loss of generality, to be $\rho_s = \frac{1}{4}(1+\sigma_{1z})(1+\sigma_{2z})$. For this state, the LHS of the inequality (88) assumes the following form,

$$2\cos^2\frac{\alpha}{2}-\langle a_{1z}b_{1z}\rangle^2-\langle a_{2z}b_{2z}\rangle^2 \geq 2\cos^2\frac{\alpha}{2}-1. \tag{89}$$

The second step follows as, $\hat{a}_1 \perp \hat{a}_2$ and $\hat{b}_1 \perp \hat{b}_2$. In order that all the separable states violate this inequality, $1 \leq 2\cos^2\frac{\alpha}{2} < 2$, which, in turn, implies that, $0 < \alpha \leq \frac{\pi}{2}$. In terms of weak values,

$$\begin{aligned}\mathcal{S}_1 = \sum_{i=1}^{2}\Big\{ &\langle\!\langle\pi_{b_1^{(i)}}\rangle\!\rangle_\rho^{\rho_{a_i}\rho_{b_2^{(i)}}}\langle\!\langle\pi_{b_1^{(i)}}\rangle\!\rangle_\rho^{\rho_{\bar{a}_i}\rho_{b_2^{(i)}}}\langle\pi_{a_i}\pi_{b_2^{(i)}}\rangle\langle\pi_{\bar{a}_i}\pi_{b_2^{(i)}}\rangle+\\ &\langle\!\langle\pi_{\bar{b}_1^{(i)}}\rangle\!\rangle_\rho^{\rho_{\bar{a}_i}\rho_{\bar{b}_2^{(i)}}}\langle\!\langle\pi_{\bar{b}_1^{(i)}}\rangle\!\rangle_\rho^{\rho_{a_i}\rho_{\bar{b}_2^{(i)}}}\langle\pi_{\bar{a}_i}\pi_{\bar{b}_2^{(i)}}\rangle\langle\pi_{a_i}\pi_{\bar{b}_2^{(i)}}\rangle\Big\}. \end{aligned} \tag{90}$$

E.2 Derivation of nonlinear entanglement inequality: II

The sum of pseudoprobabilities for entanglement inequality is as follows:

$$\mathcal{S}_2 = \sum_{i=1}^{3} \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)})\mathcal{P}(\overline{a}_i = b_1^{(i)} = b_2^{(i)}). \tag{91}$$

As before,

$$\begin{aligned}\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) = &\frac{1}{4}\cos\frac{\alpha}{2}\Big\langle(1+\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}+\boldsymbol{\sigma}_2\cdot\hat{b}_i) \\ &+ (1-\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}-\boldsymbol{\sigma}_2\cdot\hat{b}_i)\Big\rangle \\ =&\frac{1}{2}\cos\frac{\alpha}{2}\Big\langle\cos\frac{\alpha}{2}+\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\Big\rangle,\end{aligned} \tag{92}$$

and,

$$\begin{aligned}\mathcal{P}(\bar{a}_i = b_1^{(i)} = b_2^{(i)}) = &\frac{1}{4}\cos\frac{\alpha}{2}\Big\langle(1-\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}+\boldsymbol{\sigma}_2\cdot\hat{b}_i) \\ &+ (1+\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2}-\boldsymbol{\sigma}_2\cdot\hat{b}_i)\Big\rangle \\ =&\frac{1}{2}\cos\frac{\alpha}{2}\Big\langle\cos\frac{\alpha}{2}-\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\Big\rangle.\end{aligned} \tag{93}$$

Imposing the non-classicality condition, $\mathcal{S}_2 < 0$, we arrive at the inequality,

$$3\cos^2\frac{\alpha}{2} - \sum_{i=1}^{3}\langle\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\rangle^2 < 0. \tag{94}$$

The range of α can be fixed by demanding all the separable states to violate this inequality. We can consider the separable state, without any loss of generality, to be $\rho_s = \frac{1}{4}(1+\sigma_{1z})(1+\sigma_{2z})$. For this state, the LHS of the inequality (88) assumes the following form,

$$3\cos^2\frac{\alpha}{2} - \langle a_{1z}b_{1z}\rangle^2 - \langle a_{2z}b_{2z}\rangle^2 - \langle a_{3z}b_{3z}\rangle^2 \geq 3\cos^2\frac{\alpha}{2} - 1. \tag{95}$$

The second step follows as $\hat{a}_1 \perp \hat{a}_2 \perp \hat{a}_3$ and $\hat{b}_1 \perp \hat{b}_2 \perp \hat{b}_3$. In order that all the separable states violate this inequality $1 \leq 3\cos^2\frac{\alpha}{2} < 3$, which, in turn, implies $0 < \alpha \leq \frac{\pi}{2}$.

E.3 Proof of nonlinear entanglement inequality: III

The sum of pseudoprobabilities for entanglement inequality is as follows:

$$\begin{aligned}\mathcal{S}_3 = \sum_{i=1}^{3}\Big[&\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) + \frac{1}{2}\Big\{P(a_i)\mathcal{P}(\overline{a}_1^{(i)}, \overline{a}_2^{(i)})\\
&+ P(\overline{a}_i)\mathcal{P}(a_1^{(i)}, a_2^{(i)}) + P(b_i)\mathcal{P}(\overline{b}_1^{(i)}, \overline{b}_2^{(i)})\\
&+ P(\overline{b}_i)\mathcal{P}(b_1^{(i)}, b_2^{(i)}) + P(a_i)\mathcal{P}(\overline{b}_1^{(i)}, \overline{b}_2^{(i)})\\
&+ P(\overline{a}_i)\mathcal{P}(b_1^{(i)}, b_2^{(i)}) + \mathcal{P}(\overline{a}_1^{(i)}, \overline{a}_2^{(i)})P(b_i)\\
&+ \mathcal{P}(a_1^{(i)}, a_2^{(i)})P(\overline{b}_i)\Big\}\Big]. \end{aligned} \tag{96}$$

We next substitute the values of all the pseudoprobabilities as expectation values of corresponding PP operators as follows:

$$\begin{aligned}\mathcal{S}_3 = &\sum_{i=1}^{3}\frac{1}{2}\cos\frac{\alpha}{2}\Big(\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i\Big) + \frac{1}{2}\Big\{\frac{1}{4}\cos\frac{\alpha}{2}(1+\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2} - \boldsymbol{\sigma}_1\cdot\hat{a}_i)\\
&+ \frac{1}{4}\cos\frac{\alpha}{2}(1-\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1\cdot\hat{a}_i) + \frac{1}{4}\cos\frac{\alpha}{2}(1+\boldsymbol{\sigma}_2\cdot\hat{b}_i)(\cos\frac{\alpha}{2} - \boldsymbol{\sigma}_2\cdot\hat{b}_i)\\
&+ \frac{1}{4}\cos\frac{\alpha}{2}(1-\boldsymbol{\sigma}_2\cdot\hat{b}_i)(\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_2\cdot\hat{b}_i) + \frac{1}{4}\cos\frac{\alpha}{2}(1+\boldsymbol{\sigma}_1\cdot\hat{a}_i)(\cos\frac{\alpha}{2} - \boldsymbol{\sigma}_2\cdot\hat{b}_i)\\
&+ \frac{1}{4}\cos\frac{\alpha}{2}(1-\boldsymbol{\sigma}_2\cdot\hat{a}_i)(\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_2\cdot\hat{b}_i) + \frac{1}{4}\cos\frac{\alpha}{2}(\cos\frac{\alpha}{2} - \boldsymbol{\sigma}_1\cdot\hat{a}_i)(1+\boldsymbol{\sigma}_2\cdot\hat{b}_i)\\
&+ \frac{1}{4}\cos\frac{\alpha}{2}(\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1\cdot\hat{a}_i)(1-\boldsymbol{\sigma}_2\cdot\hat{b}_i)\Big\}\\
&= \frac{1}{2}\cos\frac{\alpha}{2}\Big(9\cos\frac{\alpha}{2} + \sum_{i=1}^{3}\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i - \frac{1}{2}\langle\boldsymbol{\sigma}_1\cdot\hat{a}_i + \boldsymbol{\sigma}_2\cdot\hat{b}_i\rangle^2\Big). \end{aligned} \tag{97}$$

Imposing the nonclassicality condition—$\mathcal{S}_3 < 0$, the following entanglement inequality emerges,

$$9\cos\frac{\alpha}{2} + \sum_{i=1}^{3}\Big(\boldsymbol{\sigma}_1\cdot\hat{a}_i\boldsymbol{\sigma}_2\cdot\hat{b}_i - \frac{1}{2}\langle\boldsymbol{\sigma}_1\cdot\hat{a}_i + \boldsymbol{\sigma}_2\cdot\hat{b}_i\rangle^2\Big) < 0. \tag{98}$$

We demand that all the separable states should violate this inequality. It fixes the range of α to be $0 < \alpha \leq \arccos\Big(-\frac{79}{81}\Big)$.

F Proof of condition for quantum discord

The pseudoprobabilities are as follows:

$$\mathcal{P}_D^i = \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) + P(a_i)\mathcal{P}(\overline{b}_1^{(i)}\overline{b}_2^{(i)}) + P(\overline{a}_i)\mathcal{P}(b_1^{(i)}b_2^{(i)});\, i = 1, 2. \tag{99}$$

Note that the eigenbasis of $a_i \equiv \boldsymbol{\sigma}_1 \cdot \hat{a}_i$ is the same as that of the reduced density matrix of the first subsystem. We rewrite the pseudo-probabilities in terms of expectation values of Pauli observables as below,

$$\begin{aligned}\mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) &= \frac{1}{4}\cos\frac{\alpha}{2}\Big\langle(1 + \boldsymbol{\sigma}_1 \cdot \hat{a}_i)(\cos\frac{\alpha}{2} + \boldsymbol{\sigma}_2 \cdot \hat{b}_i) \\ &\quad + (1 - \boldsymbol{\sigma}_1 \cdot \hat{a}_i)(\cos\frac{\alpha}{2} - \boldsymbol{\sigma}_2 \cdot \hat{b}_i)\Big\rangle \\ &= \frac{1}{2}\cos\frac{\alpha}{2}\Big\langle \cos\frac{\alpha}{2} + \boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i\Big\rangle,\end{aligned} \tag{100}$$

and,

$$\begin{aligned}&P(a_i)\mathcal{P}(\overline{b}_1^{(i)}\overline{b}_2^{(i)}) + P(\overline{a}_i)\mathcal{P}(b_1^{(i)}b_2^{(i)}) \\ &= \frac{1}{4}\cos\frac{\alpha}{2}\Big\{(1 + \langle\boldsymbol{\sigma}_1 \cdot \hat{a}_i\rangle)(\cos\frac{\alpha}{2} - \langle\boldsymbol{\sigma}_2 \cdot \hat{b}_i\rangle) + \cos\frac{\alpha}{2}(1 - \langle\boldsymbol{\sigma}_1 \cdot \hat{a}_i\rangle)(\cos\frac{\alpha}{2} + \langle\boldsymbol{\sigma}_2 \cdot \hat{b}_i\rangle)\Big\} \\ &= \frac{1}{2}\cos\frac{\alpha}{2}\Big(\cos\frac{\alpha}{2} - \langle\boldsymbol{\sigma}_1 \cdot \hat{a}_i\rangle\langle\boldsymbol{\sigma}_2 \cdot \hat{b}_i\rangle\Big).\end{aligned} \tag{101}$$

Thus,

$$\begin{aligned}\mathcal{P}_D^i &= \mathcal{P}(a_i = b_1^{(i)} = b_2^{(i)}) + P(a_i)\mathcal{P}(\overline{b}_1^{(i)}\overline{b}_2^{(i)}) + P(\overline{a}_i)\mathcal{P}(b_1^{(i)}b_2^{(i)}) \\ &\equiv \lambda\Big\{(16\lambda + 2\langle\boldsymbol{\sigma}_1 \cdot \hat{a}_i \boldsymbol{\sigma}_2 \cdot \hat{b}_i\rangle - 2\langle\boldsymbol{\sigma}_1 \cdot \hat{a}_i\rangle\langle\boldsymbol{\sigma}_2 \cdot \hat{b}_i\rangle\Big\},\end{aligned} \tag{102}$$

where $\lambda = \frac{1}{4}\cos\frac{\alpha}{2}$. The two-qubit states having zero discord from $\mathcal{D}^{1\to 2}$ are of the following form:

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i| \otimes \rho_{2i}. \tag{103}$$

Without any loss of generality, we may assume $|\phi_1\rangle = |0\rangle$ and $|\phi_2\rangle = |1\rangle$, where $|0\rangle(|1\rangle)$ are eigenvalues of σ_{1z} with eigenvalues $+1(-1)$. Let $\rho_{2i} = \frac{1}{2}(1 + \boldsymbol{\sigma}_2 \cdot \hat{c}_i)$. Thus, it is quite evident that all the correlation terms in equation (103) have the form $\sigma_{1z}\boldsymbol{\sigma}_2 \cdot \hat{c}_i$ and the local terms have the form σ_{1z} and $\boldsymbol{\sigma}_2 \cdot \hat{c}_i$. Thus, in Eq. (102), if we choose $\hat{a}_1 = \hat{z}$ and $\hat{a}_2 = \hat{x}$ (say), only $\mathcal{P}_D^1$ turns negative and $\mathcal{P}_D^2$ is always non-negative for any value of α and any choice of $\hat{b}_1$ and $\hat{b}_2$ for a state of the form (103).

In order to show that there exist discordant states for which both the pseudoprobabilities, $\mathcal{P}_D^1$ and $\mathcal{P}_D^2$, are negative, consider the two-qubit Werner states $\rho_W = \frac{1}{4}(1 - \alpha\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$. The non-negativity of eigenvalues of ρ_W demands that $\alpha \in \left[-\frac{1}{3}, 1\right]$. For any nonzero value of α, ρ_W has nonzero discord [14]. If we choose, $\hat{a}_1 = \hat{x}, \hat{a}_2 = \hat{y}$ and $\hat{b}_1 = \hat{x}, \hat{b}_2 = \hat{y}$, both the pseudoprobabilities, $\mathcal{P}_D^1$ and $\mathcal{P}_D^2$, acquire negative values for some value of α, which proves our claim.

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