Available online at www.sciencedirect.com
ScienceDirect
PHYSICS LETTERS A
Physics Letters A 366 (2007) 374–376
www.elsevier.com/locate/pla

Reconstructing weak values without weak measurements

Lars M. Johansen

Department of Technology, Buskerud University College, N-3601 Kongsberg, Norway

Received 4 February 2007; received in revised form 13 February 2007; accepted 13 February 2007
Available online 16 February 2007
Communicated by P.R. Holland

Abstract

I propose a scheme for reconstructing the weak value of an observable without the need for weak measurements. The post-selection in weak measurements is replaced by an initial projector measurement. The observable can be measured using any form of interaction, including projective measurements. The reconstruction is effected by measuring the change in the expectation value of the observable due to the projector measurement. The weak value may take nonclassical values if the projector measurement disturbs the expectation value of the observable.

PACS: 03.65.Ta

Keywords: Weak values; Projective measurements; Projection postulate; Nonclassicality; Weak measurements

1. Introduction

A weak measurement is achieved either by using a measurement probe with a large uncertainty [1] or by employing a weak measurement interaction [2]. By weakening the interaction, one may obtain an arbitrarily small perturbation of the system state. Aharonov, Albert and Vaidman (AAV) [1] considered an experimental scheme where the weak measurement is followed by a projector measurement, usually called a post-selection. The average of a weak measurement of an observable $\hat{B}$ conditioned on a post-selection represented by a projector $\hat{P}_a^2 = \hat{P}_a$ can be expressed as the real part of a complex "weak value" [1,3]

$$B_w(a) = \frac{\mathrm{Tr}\,\hat{\rho}\hat{P}_a\hat{B}}{\mathrm{Tr}\,\hat{\rho}\hat{P}_a}. \tag{1}$$

AAV came to the surprising conclusion that this average may take values outside the eigenvalue range. The theory predicts effects such as a negative kinetic energy [4], negative photon number [3] and negative probabilities [5–7] for suitably chosen sub-ensembles. The theory of weak measurements has been applied in a number of areas (for a review see Ref. [8]), and it has been confirmed in experiments involving classical intense laser beams [9,10], anomalous pulse propagation [11–13], the quantum box problem [7,14] and single photons [15,16].

Weak values are generally considered to be an artifact of weak measurements with post-selection. Steps towards generalizing the application of weak values to more general types of measurements have been made in Refs. [8,17]. However, it is not known whether weak values can have an operational significance, e.g., in projective measurements. In this Letter, I will demonstrate that weak values may be reconstructed from an initial projector measurement followed by a measurement of the observable. The measurement of the observable may take any form as long as it reproduces expectation values. It could be, e.g., a projective measurement or a weak measurement. The projector measurement disturbs the subsequent measurement of the observable. The reconstruction relies on measuring the change in the expectation value of the observable due to the projector measurement. The imaginary part of the weak value is reconstructed by measuring this change relative to a state that has been subjected to a selective phase rotation of $\pi/2$.

2. Reconstruction of weak values

In weak measurements with post-selection, the post-selection operation is represented by a projection operator $\hat{P}_a^2 = \hat{P}_a$.

E-mail address: lars.m.johansen@hibu.no.

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2007.02.039

In this Letter we propose a measurement scheme where this projector measurement is performed first. A projective measurement of this projector is a binary experiment with only two possible outcomes: yes or no. According to the von Neumann–Lüders projection postulate [18,19], the result of a selective projector measurement is

$$\hat{\rho}_a^s = \frac{\hat{P}_a \hat{\rho} \hat{P}_a}{\operatorname{Tr} \hat{\rho} \hat{P}_a}. \tag{2}$$

This state is represented by a sub-ensemble of the initially prepared ensemble. For a rank-one projector $\hat{P}_a = |a\rangle\langle a|$ the selective state is independent of the initial preparation $\hat{\rho}$ and equals $\hat{\rho}_a^s = |a\rangle\langle a|$.

If the outcome of the projector measurement is disregarded, the state is also changed. This nonselective state is [18,19],

$$\hat{\rho}_a^n = \hat{P}_a \hat{\rho} \hat{P}_a + (1 - \hat{P}_a)\hat{\rho}(1 - \hat{P}_a). \tag{3}$$

This state is represented by the complete initially prepared ensemble.

We now calculate the expectation value of the observable $\hat{B}$ on the nonselective state $\hat{\rho}_a^n$. We expand the right-hand side of Eq. (3), multiply both sides by $\hat{B}$ and take the trace of both sides. After division by $\operatorname{Tr} \hat{\rho} \hat{P}_a$ and rearranging of terms we arrive at the expression

$$\operatorname{Re} B_w(a) = \operatorname{Tr} \hat{\rho}_a^s \hat{B} + \frac{\operatorname{Tr} \hat{\rho} \hat{B} - \operatorname{Tr} \hat{\rho}_a^n \hat{B}}{2 \operatorname{Tr} \hat{\rho} \hat{P}_a}. \tag{4}$$

This expresses the real part of the weak value in terms of the expectation value of $\hat{B}$ on the selective state $\hat{\rho}_a^s$ plus a term which is proportional to the change in the expectation value of the observable $\hat{B}$ due to the initial nonselective projector measurement.

The first term on the right-hand side is bounded by the eigenvalue spectrum. Therefore, the weak value can only exceed the eigenvalue spectrum if the last term on the right-hand side is non-vanishing, which requires the expectation value of the observable $\hat{B}$ to be different on the original state $\hat{\rho}$ than on the nonselective state $\hat{\rho}_a^n$.

All terms on the right-hand side can be measured directly. Measurements must be performed on two identically prepared sub-ensembles. One sub-ensemble is subjected only to a measurement of $\hat{B}$. The other sub-ensemble is subjected to a measurement of $\hat{P}_a$ followed by a measurement of $\hat{B}$. The measurements of $\hat{B}$ may be of any form. It could, e.g., be a projective measurement or a weak measurement.

A case of particular interest is when also the observable $\hat{B}$ is a projector, $\hat{B} = \hat{P}_b$. In this case, the first term on the right-hand side of Eq. (4) is a probability between 0 and 1. The "weak probability" [5–7] on the left-hand side may exceed the classical range only if an intervening measurement of $\hat{P}_a$ disturbs the probability of $\hat{P}_b$. This gives an intuitively pleasing picture of the connection between extended quasi-probabilities in quantum mechanics and measurement disturbance. An early attempt at relating extended probabilities to measurement disturbance was made by Prugovečki [20].

In order to be able to reconstruct also the imaginary part of the weak value, we introduce the unitary operator

$$\hat{R}_a^\phi = 1 + (e^{i\phi} - 1)\hat{P}_a. \tag{5}$$

It is easily checked that $R_a^\phi \hat{P}_a = e^{i\phi}\hat{P}_a$, whereas $R_a^\phi \hat{P}_b = \hat{P}_b$ for any projector $\hat{P}_b$ orthogonal to $\hat{P}_a$. The operator R_a^ϕ imparts a phase change only on the projector $\hat{P}_a$, but does not change any other orthogonal projectors. We shall therefore refer to $\hat{P}_a$ as a selective phase rotation operator. It can be implemented at time t_0, e.g., by adding to the Hamiltonian a term $-\phi\delta(t - t_0)\hat{P}_a$. The state after an arbitrary selective phase rotation is $\hat{\rho}_a^\phi = \hat{R}_a^\phi \hat{\rho} (\hat{R}_a^\phi)^\dagger$.

We may note that the nonselective post-measurement density operator (3) may be written as

$$\hat{\rho}_a^n = \frac{1}{2}\big(\hat{\rho} + \hat{\rho}_a^\pi\big). \tag{6}$$

This is a classical mixture of the initial state and the state where the phase of the vector corresponding to the measured projector has been flipped. This is the well known phase-randomization or decoherence which is associated with nonselective measurements.

It can be shown that the imaginary part of the weak value may be reconstructed by subjecting a third sub-ensemble to a selective phase-rotation $\hat{R}_a^\phi$. In fact, this works for any phase angle ϕ except 0 and π, but the simplest result is obtained for the angle $\pi/2$, for which we get

$$\operatorname{Im} B_w(a) = \frac{\operatorname{Tr} \hat{\rho}_a^{\pi/2} \hat{B} - \operatorname{Tr} \hat{\rho}_a^n \hat{B}}{2 \operatorname{Tr} \hat{\rho} \hat{P}_a}. \tag{7}$$

This expression suggests a way of reconstructing the imaginary part of the weak value. It requires two different state preparations. One where the system is subjected to a nonselective projector measurement, the other where it is subjected to a selective phase rotation of $\pi/2$. If there is a difference in the expectation value of $\hat{B}$ on these two systems, the imaginary part of the weak value is non-vanishing.

In the particular case where the observable $\hat{B}$ is also a projector, $\hat{B} = \hat{P}_b$, the imaginary quasi-probability on the left-hand side is non-vanishing if and only if an intervening measurement of $\hat{P}_a$ disturbs the probability of $\hat{P}_b$, but with respect to the probability of $\hat{P}_b$ on a state that has been subjected to a selective phase rotation of $\pi/2$. This shows that there is a close connection between imaginary quasi-probabilities and measurement disturbance.

3. Conclusion

We have shown that weak values may be reconstructed from a system subjected to an initial projector measurement followed by a measurement of the observable. Whereas the initial projector measurement is a projective measurement in the same manner as the post-selection in weak measurements, the observable itself can be measured using any form of interaction. The weak value is reconstructed by measuring the expectation value of the observable with and without a preceding projector measurement. This opens the possibility of verifying the strange predictions of the theory of weak values without the need for weak measurements. In this way, one may avoid the large experimental inaccuracy and the specific interaction on which weak measurements is grounded. We found that nonclassical weak values, i.e. a real part exceeding the eigenvalue range or a non-vanishing imaginary part, are both directly related to a finite measurement disturbance in this setting.

Acknowledgements

The author is grateful to Pier A. Mello for useful and interesting discussions.

References

[1] Y. Aharonov, D.Z. Albert, L. Vaidman, Phys. Rev. Lett. 60 (14) (1988) 1351.
[2] L.M. Johansen, Phys. Rev. Lett. 93 (12) (2004) 120402.
[3] L.M. Johansen, A. Luis, Phys. Rev. A 70 (2004) 052115.
[4] Y. Aharonov, S. Popescu, D. Rohrlich, L. Vaidman, Phys. Rev. A 48 (6) (1993) 4084.
[5] K. Mølmer, Phys. Lett. A 292 (2001) 151.
[6] Y. Aharonov, A. Botero, S. Popescu, B. Reznik, J. Tollaksen, Phys. Lett. A 301 (2002) 130.
[7] K.J. Resch, J.S. Lundeen, A.M. Steinberg, Phys. Lett. A 324 (2004) 125.
[8] Y. Aharonov, A. Botero, Phys. Rev. A 72 (2005) 052111.
[9] N.W.M. Ritchie, J.G. Story, R.G. Hulet, Phys. Rev. Lett. 66 (9) (1991) 1107.
[10] A.D. Parks, D.W. Cullin, D.C. Stoudt, Proc. R. Soc. Lond. A 454 (1998) 2997.
[11] D. Rohrlich, Y. Aharonov, Phys. Rev. A 66 (2002) 042102.
[12] D.R. Solli, C.F. McCormick, R.Y. Chiao, S. Popescu, J.M. Hickmann, Phys. Rev. Lett. 92 (4) (2004) 043601-1.
[13] N. Brunner, A. Acin, D. Collins, N. Gisin, V. Scarani, Phys. Rev. Lett. 91 (18) (2003) 180402.
[14] Y. Aharonov, L. Vaidman, J. Phys. A: Math. Gen. 24 (1991) 2315.
[15] G.J. Pryde, J.L. O'Brien, A.G. White, T.C. Ralph, H.M. Wiseman, Phys. Rev. Lett. 94 (2005) 220405.
[16] Q. Wang, F.-W. Sun, Y.-S. Zhang, Jian-Li, Y.-F. Huang, G.-C. Guo, Phys. Rev. A 73 (2006) 023814.
[17] J. Tollaksen, Y. Aharonov, quant-ph/0607208.
[18] J. von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton Univ. Press, Princeton, 1955.
[19] G. Lüders, Ann. Phys. 8 (6) (1951) 322.
[20] E. Prugovečki, Can. J. Phys. 45 (1967) 2173.