Quantum theory of successive projective measurements

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We show that a quantum state may be represented as the sum of a joint probability and a complex quantum modification term. The joint probability and the modification term can both be observed in successive projective measurements. The complex modification term is a measure of measurement disturbance. A selective phase rotation is needed to obtain the imaginary part. This leads to a complex quasiprobability: The Kirkwood distribution. We show that the Kirkwood distribution contains full information about the state if the two observables are maximal and complementary. The Kirkwood distribution gives another picture of state reduction. In a nonselective measurement, the modification term vanishes. A selective measurement leads to a quantum state as a non-negative conditional probability. We demonstrate the special significance of the Schwinger basis.

I. INTRODUCTION

Quantum mechanics is a probabilistic theory, in the sense that all experimental predictions are probabilistic. However, it uses concepts unfamiliar from the classical theory of probability. The concept of a quantum state has only partially a direct interpretation in terms of probabilities. A pure state is a unit vector in a complex Hilbert space, and a mixed state is a positive definite Hermitian matrix of trace one. Remarkably, the probability interpretation of quantum mechanics was introduced as a footnote added in proof [1].

On the other hand, a state in classical physics may be represented as a joint probability over a classical logic of proposals. The proposals are basic elements of phase space. This state concept is too narrow to encompass quantum mechanics, as witnessed by the theorems of Bell [2] and Kochen and Specker [3]. The element in quantum mechanics that corresponds to the classical proposal is the projector. The essential difference between quantum and classical physics is that quantum proposals do not commute.

Is there any connection between the state concepts in classical and quantum physics? We demonstrate in this paper that a quantum state may be represented as the sum of a nonnegative joint probability and a quantum modification term. Thus we represent quantum states as a modification of the classical state concept, i.e., as a *quasiprobability*. Nonclassical values of this quasiprobability are due to measurement disturbance. The complex quasiprobability that we obtain was first discovered by Kirkwood as a representation of quantum states over phase space [4]. We shall refer to it as the Kirkwood distribution. It was independently rediscovered and generalized to arbitrary observables by Dirac [5] and Barut [6]. The real part of this distribution was examined in phase space by Terletsky [7] and for arbitrary pairs of observables by Barut [6] and Margenau and Hill [8]. The question of a possible connection between the Kirkwood distribution and measurement disturbance was raised by Prugovečki [9].

The concept of a phase space has been imported from classical physics into quantum mechanics. There are many representations of quantum states in terms of phase space distributions [4,10–14]. Most popular among these is the Wigner distribution [15]. This is a real distribution, but it may take negative values.

It is not possible to establish a joint probability in the classical sense for noncommuting observables [16]. However, one may establish quasiprobabilities for such observables [5]. A fundamental question that remains unanswered is which observables, except for the standard phase space observables, that may provide a complete state description in terms of a quasiprobability.

There exists various generalizations of the Wigner distribution concept beyond the standard continuous phase space (see, e.g., Refs. [17–21]). These distributions are mostly different, since there exists no general agreement on how the Wigner distribution concept is to be generalized. For systems in a finite dimensional Hilbert space, the issue of informational completeness has been dealt with in particular by Wootters [19] and Leonhardt [21]. In infinite dimensional Hilbert space, informationally complete generalizations of the Wigner distribution have been found for photon number and phase [22,23].

We demonstrate in this paper that the Kirkwood distribution over a pair of observables determines the density matrix uniquely provided that the observables are maximal and complementary. A maximal observable has a nondegenerate spectrum. We define two observables as complementary if they have no common eigenvectors, i.e., there is no state where both observables have a well-defined value. This generalizes the concept of informationally complete quasiprobability distributions considerably.

An issue that has been at the center of discussions over the foundations of quantum mechanics for very long time is the question of state reduction or collapse. We shed some light on this issue in this paper by rephrasing the standard formalism of state reduction [16,24] in terms of the Kirkwood distribution. We show that nonselective state reduction, which is represented in the standard representation as a vanishing of offdiagonal elements of the density matrix, is represented here as a vanishing of the complex modification

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term. Furthermore, selective state reduction, which is sometimes called the "collapse," is here represented merely as a conditional probability. The latter result is in correspondence with results based on other approaches to this problem [25].

Projective measurements were the only type of measurements considered in orthodox quantum theory [16,26]. Projective measurements require the preparation of the measurement apparatus in a state with a sharp position of the pointer prior to the measurement interaction [16]. Aharonov *et al.* [27] found a certain regularity in measurements where the measurement apparatus would be prepared in a state with an uncertain pointer position. In particular, they found that successive measurements of incompatible observables using this form of measurement interaction for the first observable gave results that they called "weak values." This type of measurement was soon found to give the first operational significance to the Kirkwood distribution, since the weak values could be expressed as conditional expectation values of the Kirkwood distribution [28]. It was recently shown that weak values also may be reconstructed from projective measurements [29]. In this paper, we explore the operational significance of the Kirkwood distribution in terms of projective measurements.

This paper is organized as follows. In Sec. II we introduce the necessary quantum formalism. We review the standard formalism of state reduction in successive measurements and the derivation of the Wigner formula [15]. In Sec. III we show that the Kirkwood distribution is obtained as a modification of the Wigner formula. We also represent the standard formalism of state reduction in terms of the Kirkwood distribution. In Sec. IV, we determine necessary and sufficient conditions for the informational completeness of the Kirkwood distribution. We also demonstrate the particular significance taken by the Schwinger basis [30].

II. SUCCESSIVE MEASUREMENTS AND THE WIGNER FORMULA

In this section, we establish the notation and review known results concerning successive projective measurements. We define projective measurements in the standard way as defined for observables with nondegenerate spectra by von Neumann [16] and as generalized to observables with degenerate spectra by Lüders [24].

We consider two observables $\hat{A}$ and $\hat{B}$ with spectral resolutions

$$\hat{A} = \sum_m a_m \hat{A}_m, \tag{1a}$$

$$\hat{B} = \sum_n b_n \hat{B}_n, \tag{1b}$$

where a_m ($m=1,2,\ldots N_a$) and b_n ($n=1,2,\ldots N_b$) are eigenvalues and $\hat{A}_m$ and $\hat{B}_n$ are the corresponding eigenprojectors. $N_a \leq N$ and $N_b \leq N$, where N is the dimension of the Hilbert space. These eigenvalues form a rectangular lattice of dimensions $N_a \times N_b$. For the case of nondegenerate eigenvalues, $N_a = N_b = N$.

The projectors are assumed to be orthogonal and idempotent,

$$\hat{A}_m \hat{A}_n = \delta_{mn} \hat{A}_n, \tag{2a}$$

$$\hat{B}_m \hat{B}_n = \delta_{mn} \hat{B}_n. \tag{2b}$$

Eigenvalues may be degenerate, i.e., we have

$$\mathrm{Tr}\, \hat{A}_m \geq 1, \tag{3a}$$

$$\mathrm{Tr}\, \hat{B}_n \geq 1. \tag{3b}$$

The eigenprojectors sum to unity, i.e., we have the projection valued measures

$$\sum_m \hat{A}_m = \hat{1}, \tag{4a}$$

$$\sum_n \hat{B}_n = \hat{1}. \tag{4b}$$

We now consider projective measurements on a system prepared in the state $\hat{\rho}$. According to Born's postulate, the probability of obtaining the eigenvalue a_m in a projective measurement of the observable $\hat{A}$ is

$$P(a_m) = \mathrm{Tr}\, \hat{\rho} \hat{A}_m. \tag{5}$$

This is also the probability of obtaining the value 1 in a projective measurement of the projector $\hat{A}_m$. Note that these are two different ways of measuring this probability.

Since we want to study successive measurements, we also need to take into consideration how the system is affected by the first measurement. This may be done by including into the description a model of the measuring device. Here we will take a simplified approach and represent the state change by the projection postulate [16]. The equivalence of these two methods have been demonstrated for certain models of the measurement apparatus (see, e.g., Refs. [31,32]). We will use the projection postulate in the form proposed by Lüders [24] for observables with possibly degenerate spectra. This type of state reduction has been demonstrated to follow from the requirement of the measurement being repeatable and minimally disturbing [33–35].

We divide projective measurements into two types, selective and nonselective. They can be made with the same measurement interaction. The distinction only applies to the way we treat the ensemble *after* the measurement interaction is over. A selective measurement is one where we only keep the subensemble giving a particular outcome. In a nonselective measurement, the complete ensemble that was prepared initially is considered further as a whole.

The state after the measurement will depend on whether we perform a measurement of the complete observable $\hat{A}$ or whether we perform a measurement of just one projector $\hat{A}_m$. We shall consider here the case of a projector measurement. This is an experiment with only two possible outcomes. For a nonselective projective measurement of the Lüders type of an projector $\hat{A}_m$ of the form (1a), the postmeasurement state is [24],

$$\hat{\rho}' = \hat{A}_m \hat{\rho} \hat{A}_m + (\hat{1} - \hat{A}_m)\hat{\rho}(\hat{1} - \hat{A}_m), \qquad (6)$$

where $\hat{1} - \hat{A}_m$ is the orthogonal complement to $\hat{A}_m$,

$$\hat{A}_m(1 - \hat{A}_m) = (1 - \hat{A}_m)\hat{A}_m = 0. \qquad (7)$$

In case the projector measurement gave as a result the value 1, the selective post-measurement state is [24]

$$\hat{\rho}'_s = \frac{\hat{A}_m \hat{\rho} \hat{A}_m}{\mathrm{Tr}\,\hat{\rho}\hat{A}_m}. \qquad (8)$$

For a rank one projector $\hat{A}_m = |a_m\rangle\langle a_m|$, the selective postmeasurement state is $\hat{\rho}'_s = \hat{A}_m$. This is sometimes referred to as "collapse." In this case, the initial state is completely erased.

We now assume that a projective measurement of the observable $\hat{B}$ is made on the system after a selective measurement of $\hat{A}_m$. The probability of obtaining the eigenvalue b_n on the selective state $\hat{\rho}'_s$ is then [36]

$$P(b_n|a_m) = \mathrm{Tr}\,\hat{\rho}'_s \hat{B}_n = \frac{\mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n\hat{A}_m}{\mathrm{Tr}\,\hat{\rho}\hat{A}_m}. \qquad (9)$$

The joint probability of obtaining successively the eigenvalues a_m and b_n is therefore

$$P(a_m)P(b_n|a_m) = \mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n\hat{A}_m. \qquad (10)$$

This probability for successive measurements was introduced for observables with nondegenerate spectra by Wigner [15]. For this reason, it is sometimes referred to as the Wigner formula. Of course, this is not a joint probability in the classical sense, since reversing the order of operations does not lead to the same probability.

The marginal probabilities obtained from this joint probability are

$$\sum_n \mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n\hat{A}_m = \mathrm{Tr}\,\hat{\rho}\hat{A}_m, \qquad (11a)$$

$$\sum_m \mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n\hat{A}_m = \mathrm{Tr}\,\hat{\rho}'\hat{B}_n. \qquad (11b)$$

We see that the marginal probability for $\hat{A}_m$ is related to the premeasurement state $\hat{\rho}$. However, the marginal probability for $\hat{B}_n$ is given in terms of the nonselective postmeasurement state $\hat{\rho}'$. Thus, the two marginals relate to different states. In this sense, Wigner's formula expresses properties of both the pre- and the postmeasurement state. One may therefore understand that it does not contain complete information about the premeasurement state $\hat{\rho}$.

III. KIRKWOOD DISTRIBUTION

In this section, we investigate the operational significance of the Kirkwood distribution. In the most general form, this distribution is written simply as [4–6]

$$P(a_m, b_n) = \mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n. \qquad (12)$$

Although it is complex, it follows trivially from the completeness relations (4a) and (4b) that it gives correct marginal probabilities,

$$\sum_n \mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n = \mathrm{Tr}\,\hat{\rho}\hat{A}_m, \qquad (13a)$$

$$\sum_m \mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n = \mathrm{Tr}\,\hat{\rho}\hat{B}_n. \qquad (13b)$$

We again consider a projective measurement of the projector $\hat{A}_m$ as described in the previous section. We expand the expression for the nonselective postmeasurement state (6), multiply both sides with $\hat{B}_n$, and compute the trace. After a rearrangement of terms, we arrive at the expression

$$\mathrm{Re}(\mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n) = \mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n\hat{A}_m + \frac{1}{2}\,\mathrm{Tr}(\hat{\rho} - \hat{\rho}')\hat{B}_n. \qquad (14)$$

The left-hand side of this expression is the real part of the Kirkwood distribution. It is often referred to as the Margenau-Hill distribution [8]. It may take negative values, and it gives correct marginal probabilities [8]

$$\sum_m \mathrm{Re}(\mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n) = \mathrm{Tr}\,\hat{\rho}\hat{B}_n, \qquad (15a)$$

$$\sum_n \mathrm{Re}(\mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n) = \mathrm{Tr}\,\hat{\rho}\hat{A}_m. \qquad (15b)$$

We see from Eq. (14) that the Margenau-Hill distribution is expressed in terms of the Wigner formula (10) and a second term. The second term is proportional to the difference in the expectation value of the projector $\hat{B}_n$ measured on the initial state $\hat{\rho}$ and on the nonselective state $\hat{\rho}'$ after a measurement of $\hat{A}_m$. This is therefore proportional to the change imposed on the probability of $\hat{B}_n$ due to an intervening measurement of $\hat{A}_m$.

The second term on the right-hand side of Eq. (14), the quantum modification term, is the cause of any negative values of the Margenau-Hill distribution. It vanishes if the two projectors $\hat{A}_m$ and $\hat{B}_n$ commute. Naturally, commuting projectors are supported by a nonnegative joint probability. Also, it vanishes for states that do not change during a measurement. For example, it vanishes if Eq. (6) is taken as the initial state. We understand that the Margenau-Hill distribution may be reconstructed from the various probabilities that are found on the right-hand side. These probabilities are obtained from measurements of the two projectors $\hat{A}_m$ and $\hat{B}_n$ only.

It may be shown that in general the Margenau-Hill distribution (14) does not determine the density matrix uniquely. It will be shown in the next section that the Kirkwood distribution does determine the density matrix for a wide class of observables. We shall therefore consider also the imaginary part of the Kirkwood distribution.

We start by introducing the operator

$$e^{i\phi\hat{A}_m} = 1 + (e^{i\phi} - 1)\hat{A}_m, \tag{16}$$

where ϕ is a real parameter. By Eq. (2a) we have

$$e^{i\phi\hat{A}_m}\hat{A}_n = [1 + (e^{i\phi} - 1)\delta_{mn}]\hat{A}_n. \tag{17}$$

$e^{i\phi\hat{A}_m}$ may be regarded as a selective phase rotation operator. It affects only a single projector $\hat{A}_m$ in the complete set (4a). $e^{i\phi\hat{A}_m}$ can be implemented at time t_0 by adding to the Hamiltonian a term

$$\Delta\hat{H}_m = -\phi\delta(t - t_0)\hat{A}_m. \tag{18}$$

It may be noted that the nonselective postmeasurement density operator (6) may be written as

$$\hat{\rho}' = \frac{1}{2}[\hat{\rho} + e^{i\pi\hat{A}_m}\hat{\rho}e^{-i\pi\hat{A}_m}]. \tag{19}$$

This shows that the Lüders form of the projection postulate [24] leads to a phase randomization in the measurement of any type of projector, regardless of degeneracy. It may now be verified that

$$\mathrm{Im}(\mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n) = \frac{1}{2}\,\mathrm{Tr}(\hat{\rho} - \hat{\rho}')\hat{B}_n^{\pi/2}, \tag{20}$$

where

$$\hat{B}_n^{\pi/2} = e^{i(\pi/2)\hat{A}_m}\hat{B}_n e^{-i(\pi/2)\hat{A}_m} \tag{21}$$

is a projector obtained by performing a selective phase rotation $e^{i(\pi/2)\hat{A}_m}$ on the projector $\hat{B}_n$.

This shows that the imaginary part of the Kirkwood distribution is obtained by observing the change in the expectation value of the selectively phase rotated projector $\hat{B}_n^{\pi/2}$ due to an intermediate projective measurement of the other projector $\hat{A}_m$.

It follows from the completeness relation (4) that the marginals of the imaginary part vanish,

$$\sum_m \mathrm{Im}(\mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n) = 0, \tag{22a}$$

$$\sum_n \mathrm{Im}(\mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n) = 0. \tag{22b}$$

Of course, the classical equivalent of the imaginary modification term is vanishing.

It may be of interest to examine how the standard formalism of projective measurements, and in particular the von Neumann–Lüders rules of state reduction, are expressed in terms of the Kirkwood distribution. A straightforward calculation shows that the Kirkwood distribution for the nonselective postmeasurement state $\hat{\rho}'$ in Eq. (6) is

$$\mathrm{Tr}\,\hat{\rho}'\hat{A}_m\hat{B}_n = \mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n\hat{A}_m. \tag{23}$$

We recognize this as the non-negative Wigner formula (10), i.e., the joint probability for successive measurements. Thus the effect of a nonselective projective measurement is that the complex modification term vanishes. This is usually associated with a vanishing of the off-diagonal elements of the density matrix [16].

Furthermore, we find that the Kirkwood distribution for the selective postmeasurement state (8) is

$$\mathrm{Tr}\,\hat{\rho}'_s\hat{A}_m\hat{B}_n = \frac{\mathrm{Tr}\,\hat{\rho}\hat{A}_m\hat{B}_n\hat{A}_m}{\mathrm{Tr}\,\hat{\rho}\hat{A}_m}. \tag{24}$$

This is just the conditional probability calculated from the unconditional postmeasurement probability (23). Thus, a selective measurement, often represented as a "collapse" in the standard formulation, is just a classical probability conditionalization in the Kirkwood representation.

IV. INFORMATIONAL COMPLETENESS

In this section, we demonstrate that the Kirkwood distribution determines the density matrix uniquely for a wide class of observables. Although this proof is rather trivial, we have not been able to find it explicitly in the literature, and it is included here for completeness.

We introduce two complete orthonormal bases

$$\sum_m |a_m\rangle\langle a_m| = \hat{1}, \tag{25a}$$

$$\sum_n |b_n\rangle\langle b_n| = \hat{1}. \tag{25b}$$

Therefore, we restrict the attention to observables with nondegenerate spectra. The density matrix in one basis may be expressed in terms of the second basis as

$$\langle b_m|\hat{\rho}|b_n\rangle = \sum_k \langle b_m|\hat{\rho}|a_k\rangle\langle a_k|b_n\rangle. \tag{26}$$

By introducing the Kirkwood distribution

$$\mathrm{Tr}\,\hat{\rho}|a_k\rangle\langle a_k|b_m\rangle\langle b_m| = \langle b_m|\hat{\rho}|a_k\rangle\langle a_k|b_m\rangle, \tag{27}$$

we may write the density matrix in the form

$$\langle b_m|\hat{\rho}|b_n\rangle = \sum_k \frac{\langle a_k|b_n\rangle}{\langle a_k|b_m\rangle}\,\mathrm{Tr}\,\hat{\rho}|a_k\rangle\langle a_k|b_m\rangle\langle b_m|. \tag{28}$$

This equation shows how the density matrix may be obtained by a transformation of the Kirkwood distribution for a pair of nondegenerate observables. The transformation (28) goes through for any bases if $\langle a_k|b_m\rangle \neq 0$ for all (m,n). If $\langle a_k|b_m\rangle = 0$ for at least one pair of indices (m,n), the corresponding term in the sum is indeterminate. Thus the Kirkwood distribution determines the density matrix uniquely for any pair of orthonormal and mutually nonorthogonal bases.

A complete orthonormal basis is maximal, in the sense that there are no more vectors that are orthogonal to this basis. Thus, if a second complete orthonormal basis has one vector which is normal to one vector in the first basis, then this vector must also belong to the first basis. This means that if the two bases have at least one pair of mutually orthogonal vectors, then they must have at least one vector in common. Thus the requirement that the bases should have no mutually

orthogonal vectors is equivalent to the claim that they should have no common vectors. Such bases are sometimes referred to as complementary (see, e.g., Ref. [37]). Such observables cannot both have sharp values. Note that this requirement is stronger than the requirement that the observables should be noncommuting. Noncommuting observables may still have some eigenvectors in common, and if a system is prepared in one of those common eigenvectors, both noncommuting observables have sharp values.

A more strict definition of complementary observables is that sharp knowledge of one observable should imply that all values of the other observable are equally probable. For an N-dimensional Hilbert space, this implies that

$$|\langle a_m|b_n\rangle| = \frac{1}{\sqrt{N}} \tag{29}$$

for all (m,n). Such bases have been called mutually unbiased [38]. Schwinger proposed a particular implementation of mutually unbiased bases as [30]

$$\langle a_m|b_n\rangle = \frac{1}{\sqrt{N}}e^{2\pi i mn/N}. \tag{30}$$

Discrete Wigner functions have been constructed using both the Schwinger bases [20] and more general mutually unbiased bases [19]. For the Schwinger basis, the inversion formula (28) simplifies to

$$\langle b_m|\hat{\rho}|b_n\rangle = \sum_k e^{2\pi i k(n-m)/N} \operatorname{Tr} \hat{\rho}|a_k\rangle\langle a_k|b_m\rangle\langle b_m|. \tag{31}$$

Thus, in this case, the transformation between the Kirkwood distribution and the density matrix is a discrete Fourier transform.

Since we are able to reconstruct the density matrix from complementary bases, we may also transform the Kirkwood distribution between different bases without first transforming to the density matrix. This possibility has been explored in Ref. [39].

V. CONCLUSION

We have shown that the Kirkwood distribution generalizes the classical concept of a state as a joint probability by adding a complex modification term. The joint probability is the Wigner formula, which is obtained in a successive measurement. The complex modification term is obtained from a measurement of the measurement disturbance. We needed the disturbance on the projector itself and on a selectively phase rotated projector.

We demonstrated that the Kirkwood distribution gives a complete description of a quantum state provided that the two observables have nondegenerate spectra and no common eigenvectors. This considerably enlarges the class of informationally complete quasiprobabilities.

We demonstrated that state reduction due to projective measurements gives a different perspective to quantum measurement theory. We found that in a nonselective measurement the complex modification term vanishes, and the quasiprobability reduces to the Wigner formula. In a selective measurement, the quasiprobability reduces to a conditional probability.

It can be mentioned that the Kirkwood distribution may be observed directly as a statistical average in a successive measurement where the measurement interaction for the first observable is weak [40].

The Kirkwood distribution has a well-defined operational meaning both when the interaction with the meter is very strong (i.e., for projective measurements) and when the interaction is very weak. This representation also has a well-defined meaning when the experimenter does not have the possibility of performing maximal measurements. This could be due to a limited resolution of the meter etc. This is a relevant situation, e.g., in a continuous Hilbert space. In this sense, the Kirkwood distribution may find applications as a state representation beyond the standard density matrix representation.

ACKNOWLEDGMENTS

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