Identifying quantum resources in convex resource theories with Kirkwood-Dirac quasiprobabilities

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We show that the Kirkwood-Dirac quasiprobability distributions can detect the presence of quantum resources in all convex resource theories via strictly negative probability outcomes. This negativity serves not only as a hallmark of resourcefulness in such theories, but also quantifies the geometric distance between a resourceful state and its closest free (i.e., nonresourceful) counterpart. Moreover, we demonstrate that these distributions can be rendered informationally complete, providing both a comprehensive description of the quantum state and a direct indicator of the underlying resource. We further discuss the relationship between convex resource theories, between strong anomalous weak values, and negative quasiprobabilities. Finally, by showing that negativity emerges only under measurement incompatibility, our results identify measurement incompatibility as a fundamental mechanism enabling quantum advantage in resourceful states.

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I. INTRODUCTION

The realization that quantum theory is fundamentally incompatible with classical notions of reality [1,2] has spurred extensive debates, significantly shaping our understanding of fundamental physics [3–9]. This has also catalyzed the development of a diverse range of quantum information processing applications [10–14]. However, despite such advancements, not all quantum states outperform their classical counterparts in these tasks. Many protocols, instead, rely on states possessing specific quantum resources, such as entanglement, which facilitates quantum teleportation or cryptography [13,15]. Identifying whether a given state possesses the requisite resource, though, remains a significant challenge [16].

To address this issue, a variety of tools have been proposed, including entanglement measures based on robustness [17], quantum Fisher information [18], and quasiprobability distributions, such as the renowned Wigner function for continuous-variable systems [19] and the discrete Wigner function [20,21] for finite-dimensional systems. While the Wigner function provides clear signatures of certain resource states [22], it fails to detect the more general manifestations of quantum resourcefulness [23]. This limitation naturally raises the question: Which classes of quantum resources can be identified through a quasiprobability-based approach?

In this work, we investigate the Kirkwood-Dirac (KD) quasiprobability distribution [24,25], a more general framework than the Wigner distribution. Unlike the Wigner function, KD distributions may take on negative and even complex values, and are particularly well suited for discrete quantum systems. Recent studies have highlighted the broad applications of KD distributions, including postselected quantum metrology [26,27], quantum fluctuation theorems [28–30], work extraction [31,32], quantum coherence [33], quantum information scrambling [34–37], and foundational questions involving measurement incompatibility and indefinite causal orders [38–42], among others [23,43–48].

We show that KD-type quasiprobability distribution can identify any quantum resource, provided the underlying resource theory is convex. Specifically, we prove that for every resourceful finite-dimensional quantum state in a convex resource theory, there exists a KD-type distribution that exhibits at least one strictly negative quasiprobability event, whereas all free (nonresourceful) states yield strictly nonnegative values under the same distribution (Sec. III A). Moreover, by suitably tailoring a set of incompatible measurements that witness the underlying resource, we demonstrate that the total negativity of this distribution admits a geometric interpretation, namely, the Frobenius distance between the resourceful state and its nearest free state. This finding suggests that KD-type negativity is at least as powerful as geometric methods for certifying quantum resources (Sec. III B). An explicit construction of the KD distribution is presented in Appendix A and a simple example is provided in Appendix B.

Although negative probabilities lack physical interpretation, quasiprobability distributions can nonetheless be probed experimentally. For instance, recent research demonstrates how sequential measurements at two different times can reveal operational quasiprobabilities in laboratory settings [49]. In Sec. III C, we establish a direct connection between anomalous weak values and quantum resources, which have significant applications in quantum information [50] and are experimentally measurable. Anomalous weak values are closely related to contextuality and macrorealism [51–54]; our

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contribution is to show that any quantum resource indicated by KD negativity necessarily exhibits anomalous weak values, revealing the incompatibility of the underlying measurements as a unifying principle that enables quantum advantage in resourceful states.

II. PRELIMINARIES

A quantum resource theory (QRT) [16] provides a formal framework for characterizing the resourcefulness of quantum states. Here, we briefly describe the basic ingredients that make up a QRT.

A QRT is defined as a tuple $\mathcal{R} = (\mathcal{C}, \mathscr{O})$, where $\mathcal{C}$ a set of quantum states called "free states" and $\mathscr{O}$ is a set of quantum operations called "free operations." The set of free operations $\mathscr{O}$ satisfy the following basic properties: (i) The identity operation should belong to $\mathscr{O}$ and (ii) if $\Phi_1, \Phi_2 \in \mathscr{O}$, then the composite map satisfies $\Phi_1 \circ \Phi_2 \in \mathscr{O}$. An important feature of QRTs, sometimes called the golden rule of QRTs, is the property that if $\rho \in \mathcal{C}$ and $\Phi \in \mathscr{O}$, then $\Phi(\rho) \in \mathcal{C}$. This property justifies the resourcefulness of certain quantum states, because the combination of free operations and free states cannot produce states outside of $\mathcal{C}$. In this sense, states outside of $\mathcal{C}$ are considered resourceful states.

Convex QRTs are a special class of QRTs where both the set of free states $\mathcal{C}$ and $\mathscr{O}$ form closed convex sets. Convex QRTs are frequently encountered because in many physical scenarios, the mixing of quantum states $\Phi(p) = p\rho + (1 - p)\sigma$ where $\sigma \in \mathcal{C}$ is considered a free operation [16]. This necessarily implies $\Phi(p) = p\rho + (1 - p)\sigma \in \mathcal{C}$ when $\rho \in \mathcal{C}$ due to the golden rule of QRTs. The resource theory of entanglement is an example of a convex QRT. In entanglement theory, the free states consist of all separable states (i.e., all possible convex combinations of product states). Any state lying outside this set is entangled and thus resourceful [13,15]. Other examples of convex QRTs include the resource theory of quantum coherence, where free states the set of such incoherent states [55], and general theories of quantum correlations, such as Bell nonlocality [1] and steering [56], where convexity holds at the level of behaviors or assemblages [16,57].

For the remainder of this paper, we will consider the problem of identifying resourceful states in convex QRTs. To this end, consider a finite-dimensional Hilbert space $\mathcal{H}$. A quasiprobability distribution on $\mathcal{H}$ is a function $\mathscr{Q}$ mapping a quantum state $\rho \in \mathscr{L}(\mathcal{H})$ to a set of real numbers $\{q_i\}_{i\in\Omega} \in \mathbb{R}$, labeled by a finite index set Ω, such that $\sum_i q_i = 1$. In general, the values q_i may be negative. When $q_i \geqslant 0\, \forall i$, the distribution is said to be nonnegative, and hence, classical. Moreover, the map $\mathscr{Q}$ is called informationally complete if the index set Ω corresponds to outcomes of a topographically complete measurement, such that $\mathscr{Q}(\rho) = \mathscr{Q}(\sigma)$ implies $\rho = \sigma$.

Given a quantum state ρ and two orthonormal bases $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$, the KD distribution is defined as

$$p(i, j) = \mathrm{Tr}\big(\Pi^a_i \Pi^b_j \rho\big), \tag{1}$$

where $\Pi^a_i = |a_i\rangle\langle a_i|$ and $\Pi^b_j = |b_j\rangle\langle b_j|$ are projective measurement operators, satisfying $\Pi \geqslant 0, \Pi^2 = \Pi$. As long as the projectors form a complete set of measurements, i.e., $\sum_i |a_i\rangle\langle a_i| = \sum_j |b_j\rangle\langle b_j| = \mathbb{1}$, the distribution $p(i, j)$ sums to unity. However, $p(i, j)$ may take negative or complex values and therefore does not represent a proper probability distribution. The real part of this expression yields the Margenau-Hill quasiprobability distribution [58].

In what follows, we consider an extended KD-type quasiprobability distribution defined over a sequence of noncommuting projective measurements [24,27,35,39], rather than the traditional two. Let $\{S_0, \ldots, S_N\}$ be an ordered sequence of sets, where each $S_i = \{\Pi^i_0, \ldots, \Pi^i_{d_i}\}, i = 0, \ldots, N$, is a collection of orthogonal projectors satisfying the completeness, $\sum_{j=0}^{d_i} \Pi^i_j = \mathbb{1}$. The ordering is crucial because, in general, projectors do not commute. For a given input state ρ the quasiprobability distribution is then defined by

$$P(x_0, \ldots, x_N \mid \rho) := \mathrm{Tr}\big[\Pi^0_{x_0} \ldots \Pi^N_{x_N}\, \rho\big], \tag{2}$$

where $x_i \in \{0, \ldots, d_i\}$ and $i = 0, \ldots, N$. It is straightforward to verify that this distribution always sums to 1 using the completeness relation. We refer to each tuple $(x_0, \ldots, x_N)$ as a quasiprobability event, and any union of events forms another event whose quasiprobability is the sum of the individual quasiprobabilities. Consequently, any distribution derived from such unions is considered a valid quasiprobability distribution.

For our purposes, we focus mainly on a specific binary setting where each $S_i = \{\Pi_i, \mathbb{1} - \Pi_i\}$ is determined by a single projector Π_i. To simplify notation, we write $\{\Pi_0, \ldots, \Pi_N\}$ instead of $\{S_0, \ldots, S_N\}$. The corresponding quasiprobability distribution then becomes

$$P(x_0, \ldots, x_N \mid \rho) := \mathrm{Tr}\big[\pi_{x_0}, \ldots, \pi_{x_N} \rho\big], \tag{3}$$

where $x_i \in \{0, 1\}$, $\pi_{x_i=0} := \Pi_i$ and $\pi_{x_i=1} := \mathbb{1} - \Pi_i$. By the appropriate selection of projectors, we ensure that the quasiprobability associated with the all-zero event $P(x_0 = 0, \ldots, x_N = 0 \mid \rho)$ remains real-valued.

III. RESULTS

A. Identifying arbitrary quantum resources in convex resource theories

We now turn to the task of identifying resourceful states within the framework of convex QRTs. As discussed in Sec. II, any such theory is characterized by a closed, convex set of free states $\mathcal{C} \subseteq \mathscr{L}(\mathcal{H})$. The convexity of this set is physically motivated because, in many scenarios, the classical mixtures of free states should be free, so that resourceful states are not generated through classical stochastic processes [16]. A state that lies outside $\mathcal{C}$, i.e., $\rho \notin \mathcal{C}$ is considered resourceful.

Building upon this structure, we show a general connection between the resourceful quantum states and the negativity of a suitably defined quasiprobability distribution in the following theorem.

Theorem 1 (Negative quasiprobabilities). Consider any quantum system with a corresponding closed convex set of free states $\mathcal{C}$ on a finite-dimensional Hilbert space $\mathcal{H}$. Then for any quantum state ρ, there exists a real-valued quasiprobability distribution $P(x \mid \rho)$ that depends on ρ, and satisfies $\sum_x P(x \mid \rho) = 1$, such that $P(x \mid \rho) < 0$ for some x if $\rho \notin \mathcal{C}$, and $p(x \mid \sigma) \geqslant 0$ for all x if $\sigma \in \mathcal{C}$.

Proof. Let $\mathcal{H}$ be the Hilbert space of the system, and consider an auxiliary two-dimensional space $\mathcal{H}_2$. Consider a set of projectors $\{\Pi_0, \ldots, \Pi_N\}$ acting on $\mathcal{H} \otimes \mathcal{H}_2$. Define $P(x_0, \ldots, x_N \mid \rho) := \mathrm{Tr}[\pi_{x_0}, \ldots, \pi_{x_N} \rho \otimes |0\rangle\langle 0|]$, where $x_i \in \{0, 1\}$, $\pi_{x_i=0} := \Pi_i$ and $\pi_{x_i=1} := \mathbb{1} - \Pi_i$. This distribution sums to 1, as a consequence of the completeness relations.

Next, by the hyperplane separation theorem, for any convex set $\mathcal{C}$ and any $\rho \notin \mathcal{C}$, there exists a Hermitian operator W on $\mathcal{H}$ such $\mathrm{Tr}[W\rho] < 0$ for all $\rho \notin \mathcal{C}$ while $\mathrm{Tr}[W\sigma] > 0$ for all $\sigma \in \mathcal{C}$. It is important to note that the hyperplane separation theorem is applicable here because we are considering convex resource theories, where the set of free states $\mathcal{C}$ is convex. Therefore, any resourceful state ρ outside of $\mathcal{C}$ can be separated from $\mathcal{C}$ by a hyperplane described by the Hermitian operator W. All free states in $\mathcal{C}$ lie on one side of the plane, while the resourceful state ρ lies on the other side. Since W is Hermitian, it is measurable, and consequently, so is the KD distribution.

Now, define $W' := W \otimes |0\rangle\langle 0|0$. Observe that W' is a singular matrix because it satisfies $\mathrm{Tr}[W'|\psi\rangle\langle\psi| \otimes |1\rangle\langle 1|] = 0$. Since any matrix in a finite-dimensional Hilbert space, up to a proportionality constant, can be factored into a product of projection operators (see Ref. [59], Theorem 1), we can decompose W' (up to a proportionality constant) as a product of projectors $\{\Pi_0, \ldots, \Pi_N\}$. Consequently, $\mathrm{Tr}[\Pi_0, \ldots, \Pi_N \rho \otimes |0\rangle\langle 0|] < 0$ if $\rho \notin \mathcal{C}$, and $\mathrm{Tr}[\Pi_0, \ldots, \Pi_N \sigma \otimes |0\rangle\langle 0|] > 0$ for any $\sigma \in \mathcal{C}$.

Let $P(x = 0 \mid \rho)$ be the quasiprobability of the event $(x_0, \ldots, x_N) = (0, \ldots, 0)$, and let $P(x \neq 0 \mid \rho)$ be the quasiprobability when the strings are not all zeros. Since $P(x = 0 \mid \rho) + P(x \neq 0 \mid \rho) = 1$, and since $P(x = 0 \mid \rho) = \mathrm{Tr}[\Pi_0, \ldots, \Pi_N \rho \otimes |0\rangle\langle 0|] < 0$, it follows that $P(x \neq 0 \mid \rho) = 1 - P(x = 0 \mid \rho) > 1 > 0$. Hence, we identify at least one negative quasiprobability outcome for any resourceful state $\rho \notin \mathcal{C}$. Conversely, if $\sigma \in \mathcal{C}$, all outcomes remain nonnegative. ■

Theorem 1 establishes KD distributions as a general tool for identifying quantum resources in convex quantum resource theories. Specifically, it demonstrates that for any state $\rho \notin \mathcal{C}$, there exists a KD distribution that identifies ρ as resourceful. However, we emphasize that a single KD distribution does not simultaneously identify every resourceful state outside of $\mathcal{C}$.

B. Geometric interpretation of quasiprobability

Although the KD distribution defined above primarily reveals the presence of a quantum resource through its negativity, it can also be constructed to encode geometric information about ρ in relation to the free states in $\mathcal{C}$. The following corollary formalizes this connection.

Corollary 1 (Geometric interpretation). It is always possible to construct a quasiprobability distribution whose total negativity is directly proportional to

$$\min_{\sigma \in \mathcal{C}} \|\rho - \sigma\|_F,$$

the Frobenius distance to the closest free state.

Proof. Let σ_0 be the free state satisfying $\|\rho - \sigma_0\|_F = \min_{\sigma \in \mathcal{C}} \|\rho - \sigma\|_F$. Define

$$W = \frac{\sigma_0 - \rho - \mathrm{Tr}[\sigma_0(\sigma_0 - \rho)]\mathbb{1}}{\|\rho - \sigma_0\|_F}, \tag{4}$$

and set $W' = W \otimes |0\rangle\langle 0|$. From the proof of Theorem 1, W' can be decomposed as $W' = k\Pi_0, \ldots, \Pi_N$, $k > 0$, giving

$$\mathrm{Tr}[W'\rho \otimes |0\rangle\langle 0|] = -\|\rho - \sigma_0\|_F = -\min_{\sigma \in \mathcal{C}} \|\rho - \sigma\|_F, \tag{5}$$

which implies

$$k\, P(x_0 = 0, \ldots, x_N = 0 \mid \rho) = -\min_{\sigma \in \mathcal{C}} \|\rho - \sigma\|_F.$$

Furthermore, employing the same construction as in the proof of Theorem 1, $P_0 = P(x_0 = 0, \ldots, x_N = 0 \mid \rho)$ is the quasiprobability of the event $(x_0, \ldots, x_N) = (0, \ldots, 0)$, while P_1 is the quasiprobability of the event where the strings are not all zeros. Then $|P_0| = \min_{\sigma \in \mathcal{C}} \|\rho - \sigma\|_F / k$ is the total negativity of the quasiprobability distribution. Therefore, we provide a construction that satisfies the statement in the corollary.

Corollary 1 implies that quasiprobabilities can be as powerful as geometric measures in detecting quantum resources, while also containing additional information about the state (see Theorem 3).

C. Relationship with anomalous weak values

A quantum weak value associated with a Hermitian operator A is defined by $\langle A_w\rangle_{\rho_1}^{\rho_2} := \mathrm{Tr}(\sigma A \rho)/\mathrm{Tr}(\sigma\rho)$ where ρ is the initial state and σ is a postselected state [60]. To simplify the presentation, we will often omit ρ and σ and write $\langle A_w\rangle$. Even when A is Hermitian, $\langle A_w\rangle$ can become complex; its real part may lie outside the operator's eigenvalue range, marking it as anomalous [61–63]. We say that a weak value $\langle A_w\rangle$ is anomalous whenever its real part $\mathrm{Re}(\langle A_w\rangle)$ is smaller than the smallest eigenvalue, or larger than the largest eigenvalue of A. Substituting the spectral decomposition $A = \sum_a a\pi^a$ into the weak value, we get $\mathrm{Re}(\langle A_w\rangle) = \sum_a \mathrm{Re}(\langle \pi_w^a\rangle)$ so anomalous weak values for any observable also implies anomalous weak values for its corresponding projectors. Theorem 2 establishes a direct link between quantum resource theories described by the set of free states $\mathcal{C}$ and measurable weak values. We comment that while Theorem 2 is not a direct statement regarding KD distributions, the arguments are based on similar ideas used in Theorem 1.

Theorem 2 (Anomalous weak values for resourceful states). For any $\rho \notin \mathcal{C}$, there must exist a collection of pairs of projectors $[\pi_1(j), \pi_2(j)]_j$ and a conical combination (sum over positive coefficients) of weak values of the form $\mathcal{W}(\rho) = \sum_j k(j)\, \langle(\pi_2(j))_w\rangle_{\rho\otimes|0\rangle\langle 0|}^{\pi_1(j)}$, $k(j) \geqslant 0$, such that $\mathcal{W}(\rho) < 0$ and $\mathcal{W}(\sigma) \geqslant 0$ for every $\sigma \in \mathcal{C}$.

Moreover, there is at least one j for which the corresponding weak value $\langle(\pi_2(j))_w\rangle_{\rho\otimes|0\rangle\langle 0|}^{\pi_1(j)}$ is anomalous and has a negative real part. The above conical combination of weak values is, therefore, negative only if the anomalous weak values are sufficiently negative and $\rho \notin \mathcal{C}$.

Proof. From Theorem 1, a resource witness W can always be mapped to another operator $W' = W \otimes |0\rangle\langle 0|$ acting on an extended Hilbert space $\mathcal{H} \otimes \mathcal{H}_2$. We write W' as a sum of positive and negative contributions. For simplicity, we first consider $W' = W'_+ + W'_-$, which is comprised of one positive (W'_+) and one negative (W'_-) eigenvalue contribution.

First, we have $W'_+ = a|k_+\rangle\langle k_+| \otimes |0\rangle\langle 0|$ with $a > 0$. This can be rewritten as $\mathrm{Tr}[W'_+ \rho] = a\mathrm{Tr}[|k_+\rangle\langle k_+|\rho]\langle |k_+\rangle\langle k_+| \otimes |0\rangle\langle 0|_W\rangle^{|k_+\rangle\langle k_+|\otimes|0\rangle\langle 0|}_{\rho\otimes|0\rangle\langle 0|}$. This expresses the expectation of W'_+ in terms of a weak value, which is nonnegative in this case. Note that $\pi_1(0) = \pi_2(0) = |k_+\rangle\langle k_+| \otimes |0\rangle\langle 0|$ are projectors.

Next, we have $W'_- = -b|k_-\rangle\langle k_-| \otimes |0\rangle\langle 0|$, where $b > 0$. Define $|\psi_1\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ and $|\psi_2\rangle := (\sqrt{1-p}\,|0\rangle + \sqrt{p}\,|0\rangle)$. One may verify that

$$|0\rangle\,\langle 0|\psi_1\rangle\,\langle\psi_1|\psi_2\rangle\,\langle\psi_2|0\rangle\,\langle 0| = \frac{1-p-\sqrt{p(1-p)}}{2}|0\rangle\langle 0|. \tag{6}$$

From the above Eq. (6), the coefficient $(1-p-\sqrt{p(1-p)})$ is negative when $1/2 < p < 1$. By choosing the value of the parameter p within this range, we obtain

$$-|0\rangle\langle 0| = c\ \ |0\rangle\,\langle 0|\psi_1\rangle\,\langle\psi_1|\psi_2\rangle\,\langle\psi_2|0\rangle\,\langle 0|\,, \tag{7}$$

where $c = 2/(\sqrt{p(1-p)} - 1 + p) > 0$. Substituting this into the expression for W'_-, we get

$$W'_- = bc|k_-\rangle\langle k_-| \otimes |0\rangle\,\langle 0|\psi_1\rangle\,\langle\psi_1|\psi_2\rangle\,\langle\psi_2|0\rangle\,\langle 0|\,, \tag{8}$$

where $b, c > 0$. This gives the expression

$$\begin{aligned}
&\mathrm{Tr}[W'_-\rho \otimes |0\rangle\langle 0|]\\
&\quad = bc\mathrm{Tr}[|k_-\rangle\langle k_-| \otimes |0\rangle\,\langle 0|\psi_1\rangle\,\langle\psi_1|\psi_2\rangle\,\langle\psi_2|0\rangle\\
&\qquad \times \langle 0|\,\rho \otimes |0\rangle\langle 0|]\\
&\quad = bc\mathrm{Tr}[(\mathbb{1} \otimes |0\rangle\langle 0|)|k_-\rangle\langle k_-| \otimes |\psi_1\rangle\,\langle\psi_1|\psi_2\rangle\,\langle\psi_2|\\
&\qquad \times (\mathbb{1} \otimes |0\rangle\langle 0|)\,\rho \otimes |0\rangle\langle 0|]\\
&\quad = bc\mathrm{Tr}[|k_-\rangle\langle k_-| \otimes |\psi_1\rangle\,\langle\psi_1|\psi_2\rangle\,\langle\psi_2|\ \ \rho \otimes |0\rangle\langle 0|]\\
&\quad = d\ \langle(|k_-\rangle\langle k_-| \otimes |\psi_2\rangle\langle\psi_2|)_w\rangle^{|k_-\rangle\langle k_-|\otimes|\psi_1\rangle\langle\psi_1|}_{\rho\otimes|0\rangle\langle 0|},
\end{aligned} \tag{9}$$

where $d := bc\mathrm{Tr}[|k_-\rangle\langle k_-| \otimes |\psi_1\rangle\langle\psi_1|\rho \otimes |0\rangle\langle 0|] > 0$. The expectation value of W'_- is now expressed as a weak value. Furthermore, it is the weak value of the projection operator $|k_-\rangle\langle k_-| \otimes |\psi_2\rangle\langle\psi_2|$ whose spectrum is necessarily nonnegative. If $\rho \notin \mathcal{C}$, then $\mathrm{Tr}[W'_-\rho \otimes |0\rangle\langle 0|] < 0$, which is clearly negative and implies that the corresponding weak value is anomalous. Note that $\pi_1(1) = |k_-\rangle\langle k_-| \otimes |\psi_2\rangle\langle\psi_2|$ and $\pi_1(1) = |k_-\rangle\langle k_-| \otimes |\psi_1\rangle\langle\psi_1|$ are projectors.

The above arguments lead to the expression

$$\mathrm{Tr}[W'\rho \otimes |0\rangle\langle 0|] = k_0\ \langle\pi_1(0)_w\rangle^{\pi_2(0)}_{\rho\otimes|0\rangle\langle 0|} + k_1\ \langle\pi_1(1)_w\rangle^{\pi_2(1)}_{\rho\otimes|0\rangle\langle 0|}, \tag{10}$$

where $k_0, k_1 > 0$ are positive real numbers. This proves the theorem when W' has exactly one positive and one negative eigenvalue contribution. A similar argument can be extended to any number of positive and negative contributions, so the the general case is also true. ■

D. Informational completeness and resource identification

A quasiprobability distribution is said to be informationally complete if it captures the full state description. The Wigner function is a familiar example [19–21]. We now show that one can always construct a quasiprobability from the KD distribution that is both informationally complete and becomes negative if and only if ρ is a resourceful state. It implies that any noninformationally complete KD distribution may be viewed as a postprocessing of a larger KD distribution that fully encodes ρ.

Theorem 3 (Informationally completeness). For any $\rho \notin \mathcal{C}$ there exists a quasiprobability distribution $P(y \mid \rho)$ that is an informationally complete representation of ρ, is nonnegative for every free state such that $P(y \mid \sigma) \geqslant 0$ for every $\sigma \in \mathcal{C}$, and identifies ρ as a resourceful state in the sense that $P(y \mid \rho) < 0$ for some event y.

Proof. The argument is based on the observation that complete information about a state can be measured by a set of projective measurements $\{\Pi_0, \ldots, \Pi_d\}$. For example, symmetric, informationally complete, positive operator-valued measures (SIC-POVMs) are a minimal set of rank 1 projectors that fully describe a quantum state. Recall that an ordered set of projectors can be mapped to a KD distribution. Suppose there is a KD distribution defined by the ordered set of projectors $\{\Pi_{d+1}, \ldots, \Pi_N\}$ that identifies a resourceful state $\rho \notin \mathcal{C}$. We can join the two sets together to form a longer ordered set $\{\Pi_0, \ldots, \Pi_N\}$. This can again be mapped to a KD distribution which performs both tasks simultaneously: (i) it fully describes the quantum state and (ii) it identifies ρ as a resourceful state.

The following is a more formal presentation of the previous argument. Let $\{\Pi_0, \ldots, \Pi_d\}$ be any set of informationally complete, rank-1 projectors (such as SIC-POVMs). Let $\{\Pi_{d+1}, \ldots, \Pi_N\}$ be the set of projectors satisfying $\mathrm{Tr}[\Pi_{d+1}, \ldots, \Pi_N\rho \otimes |0\rangle\langle 0|] < 0\ \forall\rho \notin \mathcal{C}$ and $\mathrm{Tr}[\Pi_{d+1}, \ldots, \Pi_N\sigma \otimes |0\rangle\langle 0|] \geqslant 0\ \ \forall\sigma \in \mathcal{C}$ (see proof of Theorem 1). The concatenation of these two ordered sets, $\{\Pi_0, \ldots, \Pi_N\}$ forms another valid quasiprobability distribution.

Define $P(x_0, \ldots, x_N \mid \rho) := \mathrm{Tr}[\pi_{x_0}, \ldots, \pi_{x_N}\rho \otimes |0\rangle\langle 0|]$, where $x_i \in \{0, 1\}$ and $\pi_{x_i=0} := \Pi_i$ and $\pi_{x_i=1} := \mathbb{1} - \Pi_i$. For any $0 \leqslant k \leqslant d$, we have marginals

$$\begin{aligned}
P(x_k = 0 \mid \rho) &= \sum_{\substack{x_i\\ 0\leqslant i\leqslant N\\ i\neq k}} P(x_0, \ldots, x_k = 0, \ldots x_N \mid \rho)\\
&= \mathrm{Tr}[\Pi_k\ \rho \otimes |0\rangle\langle 0|] \geqslant 0.
\end{aligned} \tag{11}$$

Since $\{\Pi_0, \ldots, \Pi_d\}$ is informationally complete, it is clear that the quasiprobability $P(x_0, \ldots, x_N \mid \rho)$ is also informationally complete.

Furthermore,

$$\begin{aligned}
&P(x_{d+1} = 0, \ldots, x_N = 0 \mid \rho)\\
&\qquad = \sum_{\substack{x_i\\ 0\leqslant i\leqslant d}} P(x_0, \ldots, x_d, x_{d+1} = 0, \ldots, x_N = 0 \mid \rho)\\
&\qquad = \mathrm{Tr}[\Pi_{d+1} \ldots \Pi_N\ \rho \otimes |0\rangle\langle 0|],
\end{aligned} \tag{12}$$

which is negative only if $\rho \notin \mathcal{C}$.

From $\{\Pi_0, \ldots, \Pi_N\}$, the marginals $P(x_k = 0 \mid \rho)$, $0 \leqslant k \leqslant d$ are informationally complete and nonnegative, as well as $P(x_{d+1} = 0, \ldots, x_N = 0 \mid \rho)$, which is negative only when

$\rho \notin \mathcal{C}$. Define the following:

$$P(y = k \mid \rho) := P(x_k = 0 \mid \rho)/(d+1), \quad k = 0, \dots, d,$$
$$P(y = d+1 \mid \rho) := P(x_{d+1} = 0, \dots, x_N = 0 \mid \rho)/(d+1),$$
$$P(y = d+2 \mid \rho) := 1 - \frac{1}{d+1}\sum_{y-0}^{d+1} P(y \mid \rho). \tag{13}$$

Each of the terms $P(y \mid \rho)$ is a sum over quasiprobability events taken from the distribution $P(x_0, \dots, x_N \mid \rho)$, so $P(y \mid \rho)$ is a valid quasiprobability. By construction, $P(y \mid \rho)$ is informationally complete and is also negative given $\rho \notin \mathcal{C}$. Since this construction applies for any $\rho \notin \mathcal{C}$, the theorem is proven. ■

IV. CONCLUDING REMARKS

Quasiprobability distributions are useful indicators of the departure from classical behavior and the emergence of quantum phenomena. In this work, we investigated the extent to which KD quasiprobabilities can detect more general notions of quantum resourcefulness. Our results demonstrate that, in convex resource theories, KD distributions violate Kolmogorov's positivity axiom for resourceful states but remain nonnegative for free states.

Several well-studied quasiprobabilities, such as the Wigner function [19–21], the Glauber-Sudarshan P function [64,65], and Husimi's Q function [66,67], are informationally complete. We demonstrated that, for finite-dimensional Hilbert spaces, KD distributions can likewise yield informationally complete quasiprobabilities without sacrificing their ability to detect resourceful quantum states. Moreover, the total negativity of such distributions can be endowed with a direct geometric interpretation, relating it to the Frobenius distance between a resourceful state and its nearest free state.

Extending the ideas of Theorem 1, we showed that weak-value measurements can be used to identify resourceful states in convex resource theories [68]. This result is consistent with prior findings [23,54] and highlights the mutual incompatibility of negative quasiprobabilities, anomalous weak values, and noncontextual ontological models, indicating a close relationship, if not equivalence, between these quantum signatures. Nevertheless, a precise quantitative mapping between negativity in quasiprobabilities and the onset of anomalous weak values remains an open avenue for future work.

In future work, we hope to generalize our results to settings that involve generalized measurements, also called positive operator-valued measures (POVMs) [69]. In the current framework, the KD quasiprobabilities were constructed using projective measurements, where commutativity underlies nonresourcefulness. In the broader framework of POVMs, joint measurability plays an analogous role [70]. We conjecture that KD-type quasiprobabilities can be built from pairs of jointly measurable POVMs that remain nonnegative for all free states, while resourceful states necessarily yield negativity when measured using incompatible POVMs. This would further strengthen the connection between quasiprobability negativity and measurement incompatibility, and would align with growing evidence that incompatibility is a universal resource [70]. A careful formulation of KD-type representations for POVMs [41,42,47] and their behavior under different resource theories could provide new insights into the operational significance of negativity beyond projective measurements.

Quasiprobabilities have already shown promise in quantum metrology [26,27,35], and it has been established that, for any quantum resource, there exists a parameter estimation task where the resource provides an advantage [18]. Our findings reinforce mounting evidence that incompatibility and contextuality may underpin quantum advantages in metrology [71,72].

A recent result [73] demonstrated that metrological advantage can also be achieved in a postselected scenario using a positive KD distribution. This appears to contradict earlier findings [26] which identified negativity in the KD distribution as a necessary resource for postselected metrological advantage. However, the preparation and postselection scheme in [73] differs from that of [26], indicating that the relationship between KD negativity and metrological gain may not apply to all settings and depends on the KD distribution being considered. We highlight that the physical settings and the measurements used to construct the KD distribution in Theorem 1 are different from those considered in [26,73].

This suggests more work needs to be done to understand the role KD-type quasiprobabilities play in more general metrological settings that involve POVMs, where joint measurability, rather than commutativity, plays the defining role. Clarifying this connection could yield a more comprehensive picture of the operational significance of quasiprobabilities in quantum metrology. We leave this for future investigation.

Finally, our observation that every resourceful quantum state admits (i) an informationally complete quasiprobability representation and (ii) anomalous weak values suggests a deeper connection between seemingly diverse forms of quantum resources. This resonates with the longstanding question [53,74–76] of whether phenomena such as contextuality, nonlocality, indefinite causal order, macrorealism violations (via the Leggett-Garg inequality), and interference can all be traced to a single, more fundamental trait of quantum mechanics.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: EXPLICIT CONSTRUCTION OF KD DISTRIBUTION

A key feature in our approach to KD quasiprobabilities is the ability to map any resource witness W acting on Hilbert space $\mathcal{H}$ onto a new witness operator W' that acts on the extended space $\mathcal{H} \otimes \mathcal{H}_2$. This transformation ensures that we can write $W' = \Pi_0, \dots, \Pi_N$.

We describe an explicit construction for the set of projectors $\{\Pi_0, \dots, \Pi_N\}$ which makes up the KD distribution. Consider the simplest possible witness operator W on $\mathcal{H}$ that has exactly one positive and one negative eigenvalue, such that $W|k_+\rangle = a|k_+\rangle$, $W|k_-\rangle = -b|k_-\rangle$, and $W = a|k_+\rangle\langle k_+| - b|k_-\rangle\langle k_-|$, where $0 < a, b \leqslant \frac{1}{4}$. Perform the map $W \to W' = W \otimes |0\rangle\langle 0|$ so that W' acts on $\mathcal{H} \otimes \mathcal{H}_2$. It is easy to check that $W'|k_\pm\rangle \otimes |1\rangle = 0$, thus ensuring W' has a nontrivial kernel.

Since $|k_+\rangle|0\rangle$, $|k_+\rangle|1\rangle$, and $|k_-\rangle|0\rangle$, $|k_-\rangle|1\rangle$ occupy mutually orthogonal subspaces, consider the positive and negative parts of W'. Define $W'_+ := a|k_+\rangle\langle k_+| \otimes |0\rangle\langle 0|$ and $W'_- := -b|k_-\rangle\langle k_-| \otimes |0\rangle\langle 0|$. Each can be decomposed into products of projectors

$$W'_+ = (|k_+\rangle\langle k_+| \otimes |0\rangle\langle 0|)\,|\phi\rangle\langle\phi|\,(|k_+\rangle\langle k_+| \otimes |0\rangle\langle 0|), \quad \text{(A1)}$$

where $|\phi\rangle = \sqrt{a}\,|k_+\rangle|0\rangle + \sqrt{1-a}\,|k_+\rangle|1\rangle$, and

$$W'_- = (|k_-\rangle\langle k_-| \otimes |0\rangle\langle 0|)\,|\psi_3\rangle\langle\psi_3||\psi_2\rangle\langle\psi_2||\psi_1\rangle\langle\psi_1| \times (|k_-\rangle\langle k_-| \otimes |0\rangle\langle 0|), \quad \text{(A2)}$$

where $|\psi_1\rangle = \sqrt{1-\lambda}\,|k_-\rangle|1\rangle - \sqrt{\lambda}\,|k_-\rangle|0\rangle$, $|\psi_2\rangle = |k_-\rangle|1\rangle$, $|\psi_3\rangle = \sqrt{1-\lambda}\,|k_-\rangle|1\rangle + \sqrt{\lambda}\,|k_-\rangle|0\rangle$, and $\lambda = \frac{1-\sqrt{1-4b}}{2}$. W'_+ and W'_- are both expressed as a product of projectors such that $W'_+ = \Pi_{+,0}\Pi_{+,1}\Pi_{+,0}$ and $W'_- = \Pi_{-,0}\Pi_{-,3}\Pi_{-,2}\Pi_{-,1}\Pi_{-,0}$.

Since W'_+ and W'_- and the projectors that form them act on orthogonal subspaces, we can write $W' = W'_+ + W'_- = \Pi_0\Pi_3\Pi_2\Pi_1\Pi_0$ where $\Pi_0 = |k_-\rangle\langle k_-| \otimes |0\rangle\langle 0| + |k_+\rangle\langle k_+| \otimes |0\rangle\langle 0|$, $\Pi_1 = \Pi_{+,1} + \Pi_{-,1}$, $\Pi_2 = \Pi_{+,1} + \Pi_{-,2}$, and $\Pi_3 = \Pi_{+,1} + \Pi_{-,2}$. It can be directly verified that $\Pi_i^2 = \Pi_i$ $\forall i \in \{0, 1, 2, 3\}$. Thus $\{\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_0\}$ yields the required quasiprobability distribution in KD form.

Although the above construction was illustrated for a witness W with only one positive and one negative eigenvalue, the same reasoning holds for more general W featuring multiple positive and negative eigenvalues. Each eigenvalue block can be decomposed into a product of projectors within its respective orthogonal subspace. Consequently, one can replicate the procedure outlined here for each eigensubspace and build up the full quasiprobability distribution for any such witness W. This construction therefore extends naturally to arbitrary Hermitian witnesses, providing a general method for mapping resource witnesses to KD-type quasiprobability distributions on an appropriately extended Hilbert space.

APPENDIX B: QUBIT EXAMPLE

To illustrate the main concepts discussed above, we consider a single-qubit resource theory in which quantum coherence serves as the resource. In this setting, the set of free states is defined by $\mathcal{C} := \{\sigma = (\mathbb{1} + r\hat{z}\cdot\vec{\sigma})/2 : |r| \leqslant 1\}$ which corresponds to all qubit states aligned with the z axis on the Bloch sphere. Any qubit state not restricted to the z axis possesses quantum coherence and is considered resourceful. This example is particularly instructive because it allows for geometric visualization of resource states in the Bloch sphere, illustrating the key ideas of Corollary 1.

Consider a pure qubit state $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ where θ is the polar angle and ϕ is the azimuthal angle in the Bloch sphere representation. Due to the rotational symmetry about the z axis in $\mathcal{C}$, we may, without loss of generality, set $\phi = 0$. Thus, $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ and $\rho = |\psi\rangle\langle\psi|$. To detect coherence as a resource, we use the resource witness

$$W = \frac{\sigma_0 - \rho - \mathrm{Tr}[\sigma_0(\sigma_0 - \rho)]\mathbb{1}}{\|\rho - \sigma_0\|_F}, \quad \text{(B1)}$$

where σ_0 is the closest state in the set $\mathcal{C}$ to ρ. The state $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ has Cartesian coordinates $(x, y, z) = (\sin\theta, 0, \cos\theta)$ in the Bloch sphere representation, therefore, σ_0 on the z axis has Cartesian coordinates $(0, 0, \cos\theta)$, which gives $\sigma_0 = \frac{1}{2}(\mathbb{1} + \cos\theta\sigma_Z)$. Substituting in the expressions for σ_0 and ρ, we obtain $W = -\sigma_x$.

Following Theorem 1, we embed W in an extended space: $W' = -\frac{1}{4}W \otimes |0\rangle\langle 0|$. Let $W'_+ := \frac{1}{4}|-\rangle\langle -| \otimes |0\rangle\langle 0|$ and $W'_- := -\frac{1}{4}|+\rangle\langle +| \otimes |0\rangle\langle 0|$, so that $W' = W'_+ + W'_-$. We then define the following projectors:

$$\Pi_{+,0} := |-\rangle\langle -| \otimes |0\rangle\langle 0|, \quad \text{(B2)}$$

$$\Pi_{+,1} := \left(\frac{1}{2}|-\rangle|0\rangle + \sqrt{\frac{3}{4}}|-\rangle|1\rangle\right)(\text{H.c.}), \quad \text{(B3)}$$

and

$$\Pi_{-,0} := |+\rangle\langle +| \otimes |0\rangle\langle 0|, \quad \text{(B4)}$$

$$\Pi_{-,1} := \left(\sqrt{\frac{1}{2}}|+\rangle|0\rangle + \sqrt{\frac{1}{2}}|+\rangle|1\rangle\right)(\text{H.c.}), \quad \text{(B5)}$$

$$\Pi_{-,2} := |+\rangle\langle +| \otimes |1\rangle\langle 1|, \quad \text{(B6)}$$

$$\Pi_{-,3} := \left(\sqrt{\frac{1}{2}}|+\rangle|0\rangle - \sqrt{\frac{1}{2}}|+\rangle|1\rangle\right)(\text{H.c.}). \quad \text{(B7)}$$

One may verify that $W' = \Pi_0\Pi_3\Pi_2\Pi_1\Pi_0$ where $\Pi_0 = \mathbb{1} \otimes |0\rangle\langle 0|$, $\Pi_1 = \Pi_{+,1} + \Pi_{-,1}$, $\Pi_2 = \Pi_{+,1} + \Pi_{-,2}$, and $\Pi_3 = \Pi_{+,1} + \Pi_{-,2}$. Moreover, $\mathrm{Tr}[\Pi_0\Pi_3\Pi_2\Pi_1\Pi_0\rho] = -\mathrm{Tr}[\sigma_x\rho] = -\sin\theta < 0$ if $\rho \notin \mathcal{C}$ and $\mathrm{Tr}[\Pi_0\Pi_3\Pi_2\Pi_1\Pi_0\rho]$ for every $\sigma \in \mathcal{C}$, consistent with Theorem 1. Hence, the quasiprobability negativity is $|\sin\theta|$, matching the Frobenius distance $\|\rho - \sigma_0\|_F = \sqrt{\sin^2\theta} = |\sin\theta|$ to the closest incoherent state: a direct geometric interpretation consistent with Corollary 1.

We now compute the expectation value of W'_+ as

$$\mathrm{Tr}[W'_+\rho \otimes |0\rangle\langle 0|] = k_0\,\langle(\pi_2(0))_w\rangle^{\pi_1(0)}_{\rho_\otimes|0\rangle\langle 0|}, \quad \text{(B8)}$$

where $k_0 = \frac{1}{4}\mathrm{Tr}[|-\rangle\langle -|\rho]$ and $\pi_1(0) = \pi_2(0) = |-\rangle\langle -| \otimes |0\rangle\langle 0|$.

Likewise, the expectation value of W'_- becomes

$$\begin{aligned}\mathrm{Tr}[W_-\rho \otimes |0\rangle\langle 0|] &= \frac{c}{4}\mathrm{Tr}[|+\rangle\langle +|\rho \otimes |-\rangle\langle -||\psi\rangle\langle\psi||0\rangle\langle 0|] \\ &= k_1\langle(\pi_2(1))_w\rangle^{\pi_1(1)}_{\rho\otimes|0\rangle\langle 0|}, \end{aligned} \quad \text{(B9)}$$

where $k_1 := c\mathrm{Tr}[|+\rangle\langle +| \otimes |-\rangle\langle -|\rho \otimes |0\rangle\langle 0|]/4$, $\pi_1(1) := |+\rangle\langle +| \otimes |-\rangle\langle -|$, $\pi_2(1) := |+\rangle\langle +| \otimes |\psi\rangle\langle\psi|$. It follows that $\mathrm{Tr}[W'\rho \otimes |0\rangle\langle 0|] = k_0\langle(\pi_2(0))_W\rangle^{\pi_1(0)}_{\rho\otimes|0\rangle\langle 0|} + k_1\langle(\pi_2(1))_W\rangle^{\pi_1(1)}_{\rho\otimes|0\rangle\langle 0|}$, which is a conical combination of weak values.

Choosing $\rho = |+\rangle\langle +|$ (which is resourceful), we find

$$\langle(\pi_2(1))_W\rangle^{\pi_1(1)}_{\rho\otimes|0\rangle\langle 0|} = (1-\sqrt{3})/(2\sqrt{2}) < 0.$$

Since $\pi_2(1)$ has a nonnegative spectrum, this weak value is anomalous. This confirms Theorem 2, demonstrating that every negative KD quasiprobability implies the presence of anomalous weak values in measurable form.

Finally, we will construct the informationally complete quasiprobability discussed in Theorems 1 and 3.

The projectors forming $|\xi_y\rangle\langle\xi_y|\xi_y$ that forms a SIC-POVM of a qubit are specified by

$$\begin{aligned}
|\xi_0\rangle &= |0\rangle\langle 0|0,\\
|\xi_1\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right),\\
|\xi_2\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \frac{e^{i2\pi/3}}{\sqrt{2}}|1\rangle\,,\\
|\xi_3\rangle &= \frac{}{\sqrt{3}}(|0\rangle + \frac{e^{-i2\pi/3}}{\sqrt{2}}|1\rangle\,.
\end{aligned}$$

By concatenating the set of projectors $\{|\xi_y\rangle\langle\xi_y|\xi_y\}_{y=0}^3$ with $\{\Pi_0, \Pi_3, \Pi_2, \Pi_1, \Pi_0\}$, we form a larger ordered set

$$\begin{aligned}
&\{|\xi_0\rangle\langle\xi_0|\xi_0, |\xi_1\rangle\langle\xi_1|\xi_1, |\xi_2\rangle\langle\xi_2|\xi_2, |\xi_3\rangle\langle\xi_3|\xi_3,\\
&\times \Pi_0, \Pi_3, \Pi_2, \Pi_1, \Pi_0\},
\end{aligned}$$

which defines a new KD distribution. To obtain the final quasiprobability, we postprocess the KD distribution by summing up terms (see final paragraph in Proof of Theorem 1 and Eqs. (11)–(13) in the proof of Theorem 3 for details). This yields the final distribution:

$$\begin{aligned}
P(y=0\mid\rho) &= \frac{1}{5}\mathrm{Tr}(|\xi_0\rangle\langle\xi_0|\xi_0\rho) = \frac{1}{5}\cos^2(\theta/2),\\
P(y=1\mid\rho) &= \frac{1}{5}\mathrm{Tr}(|\xi_1\rangle\langle\xi_1|\xi_1\rho)\\
&= \frac{1}{5}\left|\frac{\cos(\theta/2)}{\sqrt{3}} + \frac{\sin(\theta/2)}{\sqrt{2}}\right|^2,\\
P(y=2\mid\rho) &= \frac{1}{5}\mathrm{Tr}(|\xi_2\rangle\langle\xi_2|\xi_2\rho)\\
&= \frac{1}{5}\left|\frac{\cos(\theta/2)}{\sqrt{3}} + \frac{e^{i2\pi/3}\sin(\theta/2)}{\sqrt{2}}\right|^2,\\
P(y=3\mid\rho) &= \frac{1}{5}\mathrm{Tr}(|\xi_3\rangle\langle\xi_3|\xi_3\rho)\\
&= \frac{1}{5}\left|\frac{\cos(\theta/2)}{\sqrt{3}} + \frac{e^{-i2\pi/3}\sin(\theta/2)}{\sqrt{2}}\right|^2,\\
P(y=4\mid\rho) &= \mathrm{Tr}(\Pi_0\Pi_3\Pi_2\Pi_1\Pi_0\rho) = -\frac{\sin\theta}{5},\\
P(y=5\mid\rho) &= 1 - \sum_{y=0}^{4} P(y\mid\rho).
\end{aligned}$$

When $\rho \in \mathcal{C}$, we have $\theta = 0$ which yields the distribution

$$\begin{aligned}
&P(y=0\mid\rho) = \tfrac{1}{5}, \quad P(y=4\mid\rho) = 0, \quad P(y=5\mid\rho) = \tfrac{3}{5},\\
&P(y=1\mid\rho) = P(y=2\mid\rho) = P(y=3\mid\rho) = \tfrac{1}{15},
\end{aligned}$$

which we observe is a nonnegative distribution. However, when $\rho \notin \mathcal{C}$, we have $0 < \theta < \pi$, which implies $P(y=4\mid\rho) = -\sin\theta/5 < 0$. This negative probability correctly identifies that ρ is a resourceful state. Finally, we observe that $P(y\mid\rho) = \frac{1}{5}\mathrm{Tr}(|\xi_y\rangle\langle\xi_y|\xi_y\rho)$ are just expectation values of the SIC-POVMs, which are informationally complete. Therefore, this quasiprobability distribution is informationally complete, and able to identify $\rho \notin \mathcal{C}$, which aligns with the statement in Theorem 3.

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