Non-Classical States of the Electromagnetic Field

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Non-Classical States of the Electromagnetic Field*

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Abstract

A number of general features of non-classical states of light are reviewed. The phenomena of photon antibunching and sub-Poissonian photon statistics are then treated in more detail, including the experimental observations. Quantum effects in spontaneous parametric down-conversion are discussed, and recent experiments dealing with the time intervals between down-converted photons, with the intensity dependence of the cross-correlation function, and with the detection of a localized one-photon state are described. Finally, some quantum effects in the interference of light are discussed.

1. Introduction

In these lectures we shall discuss certain quantum effects in optics that have either been observed in recent years, or are the subject of current experimental investigations.

That of course immediately brings up the question what one means by a quantum phenomenon. There are actually two quite distinct ways in which the state of an electromagnetic field can be nonclassical:

(a) If the average photon occupation number per mode is less than unity, then we are in the quantum domain and the field cannot be treated classically for some purposes. This is the most familiar condition based on the correspondence principle.

(b) If we make a diagonal coherent state $|\{v\}\rangle$ representation [1–3] of the density operator $\hat{\varrho}$ for the field, by writing

$$\hat{\varrho} = \int \phi(\{v\})|\{v\}\rangle\langle\{v\}|\,\mathrm{d}\{v\}, \tag{1}$$

where $\phi(\{v\})$ is some weight functional or phase space density and the integral is to be taken over all values of the set of complex amplitudes $\{v\}$, and if $\phi(\{v\})$ is not a probability density, then the state is nonclassical. In general $\phi(\{v\})$ has to be regarded as a generalized function, which can be highly singular, even more singular than a tempered distribution.

An optical field behaves as a classical wave field in all respects only when both conditions are violated, i.e., when $\phi(\{v\})$ is a probability density and when the average photon occupation numbers are large.

2. Nonclassical states and the diagonal coherent state representation

Although it might appear from the possible singular form of $\phi(\{v\})$ that other representations of the density operator may be preferable, it is just this property of $\phi(\{v\})$ that makes it so valuable as an indicator of quantum or classical behavior. For the coherent state corresponds as closely as possible to a classical state of definite complex amplitude. When $\phi(\{v\})$ is a probability density, then the state $\hat{\varrho}$ given by eq. (1) corresponds to an ensemble of different complex amplitudes with ensemble density $\phi(\{v\})$, which is just how an optical field is described classically. But when $\phi(\{v\})$ is not a probability density the classical analogy fails completely, and we have a purely quantum mechanical state.

Nor is it always necessary to know $\phi(\{v\})$ explicitly. Sometimes the character of $\phi(\{v\})$ can be inferred from certain expectations with the help of a useful relation that has been called the "optical equivalence theorem" [1, 3]. For simplicity we consider a single-mode field. Let $:f(\hat{a}, \hat{a}^\dagger):$ be some function of annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ in normal order (we label all Hilbert space operators by the caret $\hat{}$)

$$:f(\hat{a}, \hat{a}^\dagger): \equiv \sum_{n,m} c_{nm}\hat{a}^{\dagger n}\hat{a}^m. \tag{2}$$

Then the expectation is given by

$$\begin{aligned}\langle :f(\hat{a}, \hat{a}^\dagger):\rangle &= \mathrm{Tr}(:f(\hat{a}, \hat{a}^\dagger):\,\hat{\varrho}) \\ &= \mathrm{Tr}\int \phi(v)\sum_{n,m} c_{nm}\hat{a}^{\dagger n}\hat{a}^m|v\rangle\langle v|\,\mathrm{d}^2v \\ &= \int \phi(v)\sum_{n,m} c_{nm}v^{*n}v^m \\ &= \langle f(v, v^*)\rangle_\phi \end{aligned} \tag{3}$$

when we make use of the fact that the coherent state is the right eigenstate of $\hat{a}$ and the left eigenstate of $\hat{a}^\dagger$. Hence $\phi(v)$ can be used under an integral just like a probability density to evaluate an average, whether or not it is a true probability density. This is so provided the expectation value exists, which of course imposes some limitations on the test function $f(v, v^*)$, when $\phi(v)$ is regarded as a distribution. It is sometimes possible to determine that $\phi(v)$ cannot be a probability density by inspection of an average like $\langle f(v, v^*)\rangle_\phi$, without explicit determination of $\phi(v)$. As an example, we note that the probability $p(n)$ of having n photons in the field is given by

$$p(n) = \langle n|\hat{\varrho}|n\rangle,$$

and with the help of eq. (1) for a single mode we obtain

$$\begin{aligned} p(n) &= \int \phi(v)\langle n|v\rangle\langle v|n\rangle\,\mathrm{d}^2v \\ &= \int \phi(v)\,\mathrm{e}^{-|v|^2}\frac{|v|^{2n}}{n!}\,\mathrm{d}^2v. \end{aligned} \tag{4}$$

As the integrand without the $\phi(v)$ is positive for all $v \neq 0$, it follows that $p(n)$ cannot be zero for any n when $\phi(v)$ is a probability density, with the exception of the vacuum state. Except for the vacuum, any state with $p(n) = 0$ must be nonclassical.

The question whether $\phi(\{v\})$ always exists was the subject

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of controversy in the 1960's. For example, Fourier integral representations of $\phi(\{v\})$ have been given [4, 5], and for a single-mode field, one such representation takes the form [5]

$$\phi(v) = \mathrm{e}^{|v|^2} \frac{1}{\pi^2} \int \mathrm{e}^{|u|^2} \langle -u|\hat{\varrho}|u\rangle \, \mathrm{e}^{-uv^* + u^* v} \, \mathrm{d}^2 u, \tag{5}$$

but this may not exist as an ordinary function. For example, for the Fock state $|n\rangle$, eq. (5) leads to

$$\begin{aligned} \phi(v) &= \mathrm{e}^{|v|^2} \frac{1}{\pi^2} \int \frac{(-u^* u)^n}{n!} \mathrm{e}^{-uv^* + u^* v} \, \mathrm{d}^2 u \\ &= \frac{\mathrm{e}^{|v|^2}}{n!} \frac{\partial^{2n}}{\partial v^n \partial v^{*n}} \delta^2(v), \end{aligned} \tag{6}$$

which is a tempered distribution. Under an integral, and combined with a sufficiently well-behaved test function, it is a perfectly acceptable form of phase space density.

However, much more singular distributions may be encountered. As an example we consider the pure superposition state with density operator

$$\hat{\varrho} = (c'|v'\rangle + c''|v''\rangle)(c'^*\langle v'| + c''^*\langle v''|), \tag{7}$$

in which the complex numbers c', c'', v', v'' are chosen so that $\hat{\varrho}$ is normalized to unity. Such a state has no classical description, because superposition states do not exist in classical optics. If we substitute for $\hat{\varrho}$ in eq. (5) and formally carry out the integration, we arrive at

$$\begin{aligned} \phi(v) = {} & |c'|^2 \delta^2(v - v') + |c''|^2 \delta^2(v - v'') \\ & + \exp\,[|v|^2 - (1/2)|v'|^2 - (1/2)|v''|^2] \Big\{ c' c''^* \\ & \times \exp\left[\tfrac{1}{2}(v'^* - v''^*) \frac{\partial}{\partial(v^* - \frac{1}{2}v'^* - \frac{1}{2}v''^*)}\right] \\ & \times \exp\left[\tfrac{1}{2}(v'' - v') \frac{\partial}{\partial(v - \frac{1}{2}v' - \frac{1}{2}v'')}\right] \\ & \times \delta^2(v - \tfrac{1}{2}v' - \tfrac{1}{2}v'') + c'^* c'' \\ & \times \exp\left[\tfrac{1}{2}(v''^* - v'^*) \frac{\partial}{\partial(v^* - \frac{1}{2}v'^* - \frac{1}{2}v''^*)}\right] \\ & \times \exp\left[\tfrac{1}{2}(v' - v'') \frac{\partial}{\partial(v - \frac{1}{2}v' - \frac{1}{2}v'')}\right] \\ & \times \delta^2(v - \tfrac{1}{2}v' - \tfrac{1}{2}v'') \Big\}, \end{aligned} \tag{8}$$

which is much more singular than a tempered distribution, and contains infinitely high order derivatives of the δ-function. Evidently $\phi(v)$ is not a probability density and represents a highly nonclassical state. Nevertheless, when it operates on a sufficiently well-behaved test function it yields correct answers. For example, it follows from the theorem in eq. (3) that

$$\langle \hat{n} \rangle = \int v^* v \phi(v) \, \mathrm{d}^2 v. \tag{9}$$

If we substitute for $\phi(v)$ from eq. (8), we obtain after somewhat lengthy integration by parts, with the help of the Campbell–Baker–Hausdorff identity,

$$\begin{aligned} \langle \hat{n} \rangle = {} & |c'|^2 |v'|^2 + |c''|^2 |v''|^2 + c'^* c'' v'^* v'' \\ & \times \exp\,(-\tfrac{1}{2}|v'|^2 - \tfrac{1}{2}|v''|^2 + v'^* v'') + \text{c.c.}, \end{aligned} \tag{10}$$

which is exactly the answer given by eq. (7) directly. This shows that even extremely singular phase space densities are perfectly usable, while they act as useful indicators of nonclassical behavior.

3. States of low photon occupation number

When the average occupation number per mode is much less than unity, then the dispersion of the photon number $\hat{n}$ is also very small necessarily. It then follows from the uncertainty relations connecting the photon number and the phase of the field [6–8], that the phase does not have a value, so that the description of the field in terms of classical complex amplitudes is then not meaningful. For example, for thermal light derived from some source at temperature T by passive linear filtering the phase space density is Gaussian, with the dispersion of mode $\boldsymbol{k}$, s given by

$$\langle \hat{n}_{ks} \rangle \leq \frac{1}{\mathrm{e}^{\hbar\omega/\kappa T} - 1}, \tag{11}$$

where ω is the frequency of mode $\boldsymbol{k}$, s and κ is Boltzmann's constant. It is then tempting to represent the real electric field $E(t)$ as a Gaussian noise, as shown in Fig. 1. But if T is no more than a few thousand degrees Kelvin, then $(\mathrm{e}^{\hbar\omega/\kappa T} - 1)^{-1} \ll 1$, so that the phase of the field does not exist, and no such pictorial representation is valid. This is so despite the fact that $\phi(v)$ is a probability density for this state. The reason why the contradictions implicit in Fig. 1 are usually not very apparent, is that one rarely attempts to make measurements that involve the absolute phase of the field.

We will now go on to consider some examples of fields that exhibit non-classical behavior.

4. Photon antibunching

When light falls on a photodetector, there is a probability $P_1(t)$ that a photoelectron is emitted at time t within a short interval δt, and there is a joint probability $P_2(t, t + \tau)$ that one photoelectron is emitted at time t within δt and another one at time $t + \tau$ within δt. Under conditions of near normal incidence and for a small detector area $\mathscr{A}$, so that the field looks almost like a plane wave over $\mathscr{A}$, one finds [2, 9]

$$P_1(t) \propto \int_t^{t+\delta t} \langle \hat{I}(t') \rangle \mathscr{A} \, \mathrm{d}t' \tag{12}$$

$$P_2(t, t + \tau) \propto \int_t^{t+\delta t} \int_\tau^{\tau+\delta\tau} \langle \mathscr{T} : \hat{I}(t') \hat{I}(t' + \tau') : \rangle \mathscr{A}^2 \, \mathrm{d}t' \, \mathrm{d}\tau'. \tag{13}$$

$\mathscr{T}$ stands for time ordering and : : for normal ordering. The intensity $\hat{I}(t)$ is given by

$$\hat{I}(t) = \hat{\boldsymbol{E}}^{(-)}(t) \cdot \hat{\boldsymbol{E}}^{(+)}(t), \tag{14}$$

Fig. 1. A realization of a classical field

where $\hat{E}^{(+)}$, $\hat{E}^{(-)}$ are positive frequency and negative frequency parts of an electromagnetic field vector $\hat{E}$ (not necessarily the electric field), with mode expansions of the form

$$\hat{E}^{(+)}(r, t) = \frac{1}{L^{3/2}} \sum_{k,s} l(\omega) \hat{a}_{ks} \varepsilon_{ks} \, e^{i(k \cdot r - \omega t)}. \tag{15}$$

$l(\omega)$ is some simple function of frequency [$l(\omega) = i(\hbar\omega/2\varepsilon_0)^{1/2}$ for the electric field] that depends on which field vector is being expanded. In the steady state, in which the correlation functions depend only on τ and not on t, it is often convenient to write

$$\langle \mathscr{T}\!:\hat{I}(t)\hat{I}(t+\tau)\!:\rangle = \langle \hat{I} \rangle^2 [1 + \lambda(\tau)], \tag{16}$$

where $\lambda(\tau)$ is a normalized correlation function.

When the field is describable classically, the operators become c-numbers and the expectations are interpreted as averages over the classical ensemble. Now it follows from the laws of classical probability, with the help of the Schwarz inequality, that

$$\lambda(\tau) \leq \lambda(0), \tag{17}$$

so that from eqs. (13) and (16),

$$P_2(t, t) \geq P_2(t, t + \tau). \tag{18}$$

If we plot the joint probability $P_2(t, t + \tau)$ of detecting two photoelectric pulses separated by time τ against τ, the resulting graph can fall below its initial value, but can never rise above its initial value. If it falls as illustrated in Fig. 2, then the tendency of photoelectric pulses to bunch in time is predicted. This has been observed in numerous experiments since the pioneering work of Hanbury Brown and Twiss [10], and is a phenomenon that is understandable in terms of fluctuating electromagnetic waves, without field quantization.

Antibunching, as the name implies, is the opposite of bunching, and describes a situation in which fewer photons appear close together than further apart. The condition for antibunching is the opposite of eq. (17), and reads

$$\lambda(\tau) > \lambda(0). \tag{19}$$

Because this violates classical probability, antibunching is a quantum phenomenon without classical description [the condition for antibunching has been widely misquoted in the literature], and the corresponding state of the field cannot be given a diagonal coherent state representation (1) with ϕ in the form of a probability density.

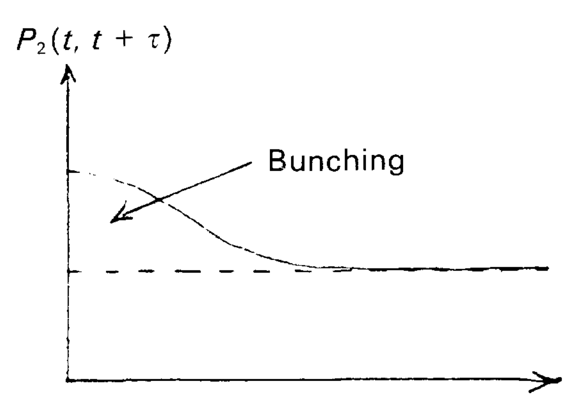

Fig. 2. Behavior of the detection probability $P_2(t, t + \tau)$ for a typical thermal field.

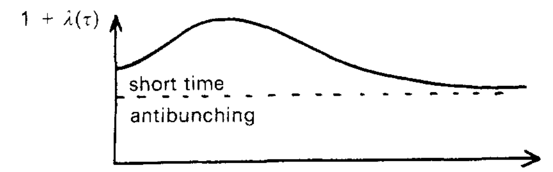

Fig. 3. Illustrating short-time antibunching.

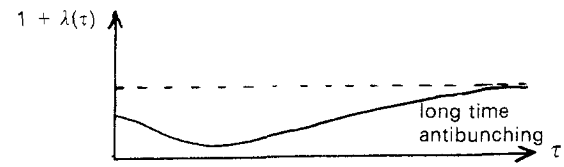

Fig. 4. Illustrating long-time antibunching.

Several different kinds of antibunching are possible. In Fig. 3 condition (19) is satisfied for short times τ, so that the graph describes short time antibunching. In Fig. 4 condition (19) holds for long times τ, so that this graph describes long-time antibunching. Of course, both effects may occur together.

Historically, the oldest example of antibunching occurs in the process of resonance fluorescence, when a single atom is coherently excited close to resonance by a laser beam. It can be shown that the fluorescent light radiated by the driven two-level atom satisfies the relation [11]

$$\langle \mathscr{T}\!:\hat{I}(t)\hat{I}(t+\tau)\!:\rangle = \langle \hat{I}(t) \rangle \langle \hat{I}(\tau) \rangle_G, \tag{20}$$

where the second factor describes the field after time τ when the atom starts in the ground state at time 0. As $\langle \hat{I}(0) \rangle_G = 0$ and $\langle \hat{I}(\tau) \rangle_G$ grows with τ from $\tau = 0$, it is clear that eq. (20) describes antibunching of photons. Both sides of the equation vanish when $\tau = 0$, because the atom makes a quantum jump to the ground state in the process of emitting the first photon. In the special case of exact resonance between the atom and the exciting field, $\langle \hat{I}(\tau) \rangle_G$ takes the form [11, 12]

$$\begin{aligned} \langle \hat{I}(\tau) \rangle_G &= \langle \hat{I} \rangle [1 + \lambda(\tau)] \\ &= \left(\frac{(1/2)\Omega^2/\beta^2}{(1/2)\Omega^2/\beta^2 + 1} \right) \{1 - e^{-3\beta\tau/2} \\ &\quad \times [\cos \Omega'\beta\tau + (3/2\Omega') \sin \Omega'\beta\tau]\}, \end{aligned} \tag{21}$$

$$\Omega' \equiv (\Omega^2/\beta^2 - 1/4)^{1/2}.$$

Here β is half the Einstein A-coefficient and Ω is the atomic Rabi frequency.

Figure 5 shows the results of two experiments in which the time intervals between photons were measured, superimposed on the theoretical curves obtained from eqs. (20) and (21). The vertical axis gives the value of $1 + \lambda(\tau)$. It is quite evident that the photons exhibit antibunching. Moreover, the fact that there are no photon pairs with zero time separation indicates that the atom makes an instantaneous quantum jump to the ground state when it emits the first photon, rather than decaying exponentially.

5. Sub-Poissonian photon statistics

The quantum character of the field can also be exhibited by straight photon counting measurements, rather than by measurements of time intervals between detected photons. To

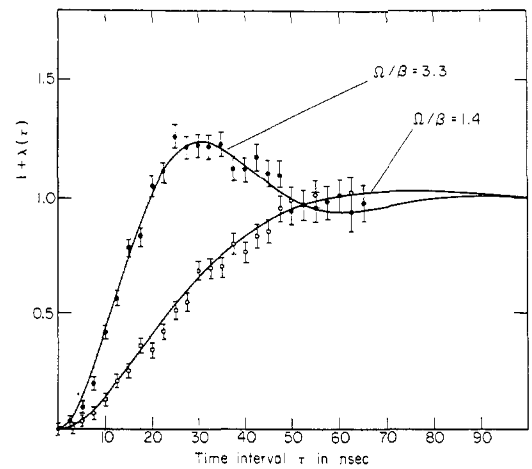

Fig. 5. Results of two-photon correlation measurements. [Reproduced from Dagenais, M. and Mandel, L., Phys. Rev. **A18**, 2217 (1978).]

show this we start from the general expression for the probability $p(n)$ of detecting n photons in a counting time with a detector of small area $\mathscr{A}$, and quantum efficiency α [13, 14], when the field looks like a plane wave over $\mathscr{A}$,

$$p(n) = \left\langle \mathscr{T} : \frac{\hat{W}^n}{n!} \mathrm{e}^{-W} : \right\rangle \tag{22}$$

with

$$\hat{W} = \alpha c \mathscr{A} \int_t^{t+\tau} \hat{I}(t')\,\mathrm{d}t'. \tag{23}$$

Here $\hat{I}(t)$ is the photon density $\hat{\boldsymbol{E}}^{(-)}(t) \cdot \hat{\boldsymbol{E}}(t)$. It is associated with a field vector $\hat{E}^{(+)}(t)$ like that in eq. (15) but with $l(\omega) = 1$. From this it follows immediately that for a stationary field

$$\langle n \rangle = \langle \hat{W} \rangle = \alpha c \mathscr{A} \langle \hat{I} \rangle T \tag{24}$$

$$\langle n(n-1) \rangle = \langle \mathscr{T} : \hat{W}^2 : \rangle, \tag{25}$$

so that

$$\langle (\Delta n)^2 \rangle = \langle n \rangle + \langle \mathscr{T} : (\Delta \hat{W})^2 : \rangle. \tag{26}$$

By inserting the integral expression for $\hat{W}$ and transforming the double integral into a single integral in the difference variable we can re-express the last term and write

$$\langle (\Delta n)^2 \rangle = \langle n \rangle + R^2 \int_{-T}^{T} (T - |\tau|)\lambda(\tau)\,\mathrm{d}\tau. \tag{27}$$

$R = \langle n \rangle / T$ is the average counting rate of the detector. In a sufficiently short time interval T the last term reduces to $(RT)^2 \lambda(0)$. It is therefore apparent that the variance will be smaller than the mean, or the statistics will be sub-Poissonian in a short interval, when

$$\lambda(0) < 0. \tag{28}$$

Moreover, inspection of eq. (26) and application of the optical equivalence theorem (3) shows that a field in such a state has no classical description in terms of a phase space density ϕ. This may be regarded as a reflection of the fact that a classical field of constant intensity yields Poissonian photoelectric counting statistics, and any modulation of the classical intensity can only increase the counting fluctuations.

A convenient parameter Q for characterizing the departure from Poisson statistics is [15]

$$Q = \frac{\langle (\Delta n)^2 \rangle - \langle n \rangle}{\langle n \rangle}, \tag{29}$$

which has some value between 0 and -1 when the variance is sub-Poissonian. From eq. (27)

$$Q = R \int_{-T}^{T} (1 - |\tau|/T)\lambda(\tau)\,\mathrm{d}\tau, \tag{30}$$

and with the help of eq. (21) for $\lambda(\tau)$ in resonance fluorescence it is not difficult to show that [15]

$$Q = \frac{-(1/2)\Omega^2/\beta^2}{((1/2)\Omega^2/\beta^2 + 1)^2} \frac{1}{2\beta T} \left\{ 6\beta T + \frac{1}{(1/2)\Omega^2/\beta^2 + 1} \times \left[-7 + \frac{\Omega^2}{\beta^2} + \mathrm{e}^{-3\beta T/2} \left\{ \left(7 - \frac{\Omega^2}{\beta^2}\right) \cos \Omega'\beta T + \frac{9}{2\Omega'} \left(1 - \frac{\Omega^2}{\beta^2}\right) \sin \Omega'\beta T \right\} \right] \right\}. \tag{31}$$

As $T \to \infty$

$$Q \to -\frac{3\Omega^2/\beta^2}{2((1/2)\Omega^2/\beta^2 + 1)^2}, \tag{32}$$

which has a maximum negative value of $-3/4$ when $\Omega/\beta = \sqrt{2}$. The photons emitted by an atom in the process of resonance fluorescence are therefore sub-Poissonian, when the exciting field is on resonance, and this has been confirmed experimentally [16]. However, much smaller numerical values of Q are observed in practice, because each emitted photon has only a small probability of being detected. It is worth noting also that the statistics can be super-Poissonian off resonance [17–19], despite the fact that the photons still exhibit antibunching. By contrast, a single-mode quantum field in a Fock state would show no antibunching, even though the photon statistics are clearly sub-Poissonian. This helps to emphasize that the two phenomena are distinct and need not necessarily occur together.

6. Correlated two-photon states in parametric down-conversion

We consider the process illustrated in Fig. 6, in which a coherent pump beam of frequency ω_0 falls on a nonlinear crystal, and in the interaction between the light and the medium some incident photons fission into two lower frequency photons known as the signal and idler, of wave vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and frequencies ω_1, ω_2. The process is known as spontaneous parametric down-conversion. If the phase matching condition is satisfied, then the three wave vectors and frequencies are related by

$$\left.\begin{aligned} \boldsymbol{k}_0 &= \boldsymbol{k}_1 + \boldsymbol{k}_2 \\ \omega_0 &= \omega_1 + \omega_2, \end{aligned}\right\} \tag{33}$$

which simply express the conservation of momentum and

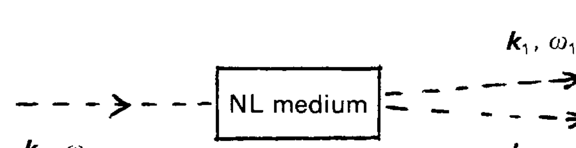

Fig. 6. Spontaneous parametric down-conversion.

energy. Under suitable conditions the down-converted signal and idler beams form cones with their axes centered on the incident pump beam. The phenomenon was first treated in the 1960's [20–23], and has been studied in greater detail many times since then.

If we adopt the simplest form of three-mode Hamiltonian to describe the process [24, 25],

$$\hat{H} = \sum_{j=0}^{2} \hbar\omega_j \hat{n}_j + \mathrm{i}\hbar g[\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_0 - \mathrm{h.c.}], \tag{34}$$

where g is a coupling constant that depends on the second order susceptibility, then it follows immediately that the difference between the signal and idler photon numbers is a constant of the motion,

$$\hat{n}_1(t) - \hat{n}_2(t) = \hat{n}_1(0) - \hat{n}_2(0). \tag{35}$$

Hence

$$\begin{aligned}\langle:\hat{n}_1(t)\hat{n}_2(t):\rangle &= \langle\hat{n}_1(t)\hat{n}_2(t)\rangle \\ &= \langle\hat{n}_1^2(t)\rangle - \langle\hat{n}_1(t)\hat{n}_2(0)\rangle \\ &\quad + \langle\hat{n}_1(t)\hat{n}_2(0)\rangle.\end{aligned} \tag{36}$$

If the initial state of modes 1 and 2 is the vacuum, then the last two terms vanish, and

$$\begin{aligned}\langle:\hat{n}_1(t)\hat{n}_2(t):\rangle &= \langle\hat{n}_1^2(t)\rangle \\ &= \langle:\hat{n}_1^2(t):\rangle + \langle\hat{n}_1(t)\rangle\end{aligned} \tag{37}$$

and from eq. (35),

$$\langle\hat{n}_1(t)\rangle = \langle\hat{n}_2(t)\rangle. \tag{38}$$

Now the left-hand side of eq. (37) is a measure of the joint probability of detecting a signal and an idler photon, whereas $\langle:\hat{n}_1^2:\rangle$ gives the corresponding probability of detecting two signal photons. Evidently for times other than $t = 0$ for which $\langle\hat{n}_1(0)\rangle = 0$, we have inequality

$$\langle:\hat{n}_1(t)\hat{n}_2(t):\rangle > \langle:\hat{n}_1^2(t):\rangle. \tag{39}$$

With the help of the optical equivalence theorem (3), we can express this as a relation between c-number averages

$$\langle|v_1(t)|^2|v_2(t)|^2\rangle_\phi > \langle|v_1(t)|^4\rangle_\phi. \tag{40}$$

Now for two classical random processes $v_1(t)$, $v_2(t)$ with similar statistics it follows from the Schwarz inequality that

$$\langle|v_1(t)|^2|v_2(t)|^2\rangle \leq \sqrt{\langle|v_1(t)|^4\rangle\langle|v_2(t)|^4\rangle} = \langle|v_1(t)|^4\rangle, \tag{41}$$

and comparison with eq. (40) shows clearly that the classical inequality is violated. Indeed, when $\langle\hat{n}_1(t)\rangle \ll 1$ the violation can be very large. It follows that spontaneous parametric down-conversion is a purely quantum mechanical phenomenon.

It is not too difficult to integrate the equations of motion for $\hat{a}_1$ and $\hat{a}_2(t)$, and to obtain an explicit solution in the special case in which the pump mode can be treated classically as a field of constant amplitude v_0 and frequency ω_0. One finds from the Heisenberg equations of motion for $\hat{a}_1(t)$ and $\hat{a}_2(t)$ that [20–22]

$$\left.\begin{aligned}\hat{a}_1(t) &= \hat{a}_1(0)\,\mathrm{e}^{-\mathrm{i}\omega_1 t}\cosh(g|v_0|t) \\ &\quad + \hat{a}_2^\dagger(0)\,\mathrm{e}^{-\mathrm{i}\omega_1 t}\sinh(g|v_0|t) \\ \hat{a}_2(t) &= \hat{a}_2(0)\,\mathrm{e}^{-\mathrm{i}\omega_2 t}\cosh(g|v_0|t) \\ &\quad + \hat{a}_1^\dagger(0)\,\mathrm{e}^{-\mathrm{i}\omega_2 t}\sinh(g|v_0|t)\end{aligned}\right\} \tag{42}$$

from which it follows immediately when we calculate expectations that

$$\langle\hat{n}_1(t)\rangle = \sinh^2(g|v_0|t) = \langle\hat{n}_2(t)\rangle, \tag{43}$$

and

$$\begin{aligned}\langle:\hat{n}_1(t)\hat{n}_2(t):\rangle &= \sinh^2(g|v_0|t)[\cosh^2(g|v_0|t) \\ &\quad + \sinh^2(g|v_0|t)].\end{aligned} \tag{44}$$

When $g|v_0|t \ll 1$, as it usually is in practice, then to a good approxmation

$$\langle\hat{n}_1(t)\rangle \approx \langle:\hat{n}_1(t)\hat{n}_2(t):\rangle \approx \langle\hat{n}_2(t)\rangle. \tag{45}$$

Now $\langle\hat{n}_j(t)\rangle$ is a measure of the probability P_j of detecting a down-converted photon at time t in channel j ($j = 1, 2$) with an ideal detector, and $\langle:\hat{n}_1(t)\hat{n}_2(t):\rangle$ is a measure of the joint probability P_{12} of detecting both a signal and an idler photon at time t. The coincidence of these three probabilities for ideal detectors is a remarkable non-classical feature of the down-conversion process. When the down-converted photon in channel j has only a probability α_j ($\alpha_j \leq 1$) of being detected ($j = 1, 2$), the equality has to be replaced by

$$\alpha_1 P_2 = P_{12} = \alpha_2 P_1, \tag{46}$$

and this has been confirmed experimentally. The confirmation is based on coincidence counting measurements [25, 26], because the signal and idler photons are produced 'simultaneously'. Another way of testing the validity of eq. (46) is to measure the normalized cross-correlation P_{12}/P_1P_2, which should equal α_j/P_j ($j = 1, 2$) according to eq. (46), and should therefore vary inversely with the light intensity. This relationship has also been confirmed [25]. (See Fig. 7.)

Of course an analysis based on three discrete modes cannot yield meaningful answers to questions about the time relationship between the down-converted photons. For that purpose we need to treat the process with a continuum of field modes, and such treatments have been given [27, 28]. They show that the time delays between signal and idler photons are ultimately limited only by the acceptance bandwidth of the light, and this is consistent with experimental results [26, 29]. (See Fig. 8.)

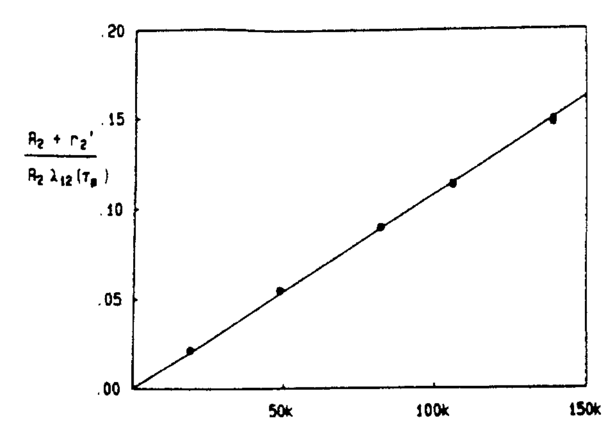

Fig. 7. Results of correlation measurements showing the inverse relationship between the normalized intensity correlation function and the light intensity. [Reproduced from Friberg, S., Hong, C. K. and Mandel, L., Opt. Commun. **54**, 311 (1985).]

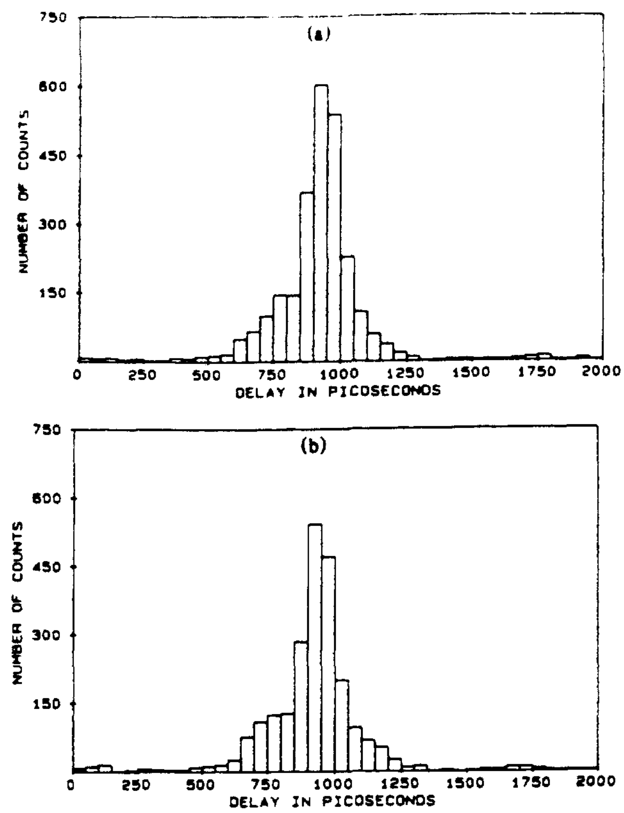

Fig. 8. The measured distributions of time-delays between signal and idler photons (a) for a single-mode pump laser, (b) for a multimode pump laser. [Reproduced from Friberg, S., Hong, C. K. and Mandel, L., Phys. Rev. Lett. **54**, 2011 (1985).]

A paradox surfaces if we express the detection probabilities P_1, P_2, P_{12} as integrals over the short measurement times δt and the small areas $\delta\mathscr{A}$ of the detectors. Then

$$P_j = \alpha_j c \int_{\delta\mathscr{A}} \int_{\delta t} \langle \hat{I}(\mathbf{r}_j, t_j)\rangle \, \mathrm{d}^2 r_j \, \mathrm{d}t_j \quad (j = 1, 2) \tag{47}$$

$$P_{12} = \alpha_1 \alpha_2 c^2 \iint_{\delta\mathscr{A}} \iint_{\delta t} \langle \mathscr{T}:\hat{I}(\mathbf{r}_1, t_1)\hat{I}(\mathbf{r}_2, t_2):\rangle \, \mathrm{d}^2 r_1 \, \mathrm{d}^2 r_2 \, \mathrm{d}t_1 \, \mathrm{d}t_2. \tag{48}$$

It might appear that, for sufficiently short times δt and for sufficiently small detector areas $\delta\mathscr{A}$, P_1 and P_2 would be proportional to $\delta\mathscr{A}\delta t$ and P_{12} would be proportional to $(\delta\mathscr{A})^2(\delta t)^2$. But in that case eq. (46) could not possibly hold, because P_{12} would depend on higher powers of small quantities than P_1 or P_2. The resolution of the paradox depends on the fact that the correlation function $\Gamma \equiv \langle \mathscr{T}:\hat{I}(\mathbf{r}_1, t_1) \times \hat{I}(\mathbf{r}_2, t_2):\rangle$ is highly peaked in space and in time, with a δ-function-like character. For example, when $\mathbf{r}_1$, $\mathbf{r}_2$ are equidistant from the non-linear medium, Γ is almost zero unless $t_1 = t_2$, and there is a similar critical dependence on the positions $\mathbf{r}_1$, $\mathbf{r}_2$. Although the surface and time integrals in eq. (47) can be approximated by multiplying by factors $\delta\mathscr{A}$ and δt, those in eq. (48) cannot. If we make the change of variables $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}$ and $t_2 - t_1 = t$ in eq. (48), then whereas the integrals over $\mathbf{r}_1$ and t_1 can, to a good approximation, be replaced by the factors $\delta\mathscr{A}$ and δt, the integrand varies so rapidly with $\mathbf{r}$ and t that such an approximation is invalid in the remaining integrals. P_{12} has to be expressed as an integral, and it is only in this form that eq. (46) holds.

Finally we point out that in the process of spontaneous down-conversion one comes rather close to realizing a localized one-photon state. Although much more common processes, like the decay of an excited atom, give rise to one-photon states, the photon is then spread over space, and we cannot normally know where or when to detect it. But in the process of down-conversion the detection of a signal photon at some place at some time implies the presence of a conjugate idler photon at another location at the same time. Although the localization of the photon, as always, is limited in accuracy, in position to a region much larger than the wavelength and in time to an interval much longer than the period, we nevertheless come close to the ideal situation. If with the help of a perfect detector we could measure the probability $p_s(n)$ that n idler photons are present within a certain region and within a short time, conditioned on the detection of a signal photon, we should find that $p_s(n) = \delta_{n1}$. If each idler photon has a probability η of being detected, then the measured probability $P_s(n)$ will be related to the photon probability $p_s(n)$ by a Bernouilli convolution

$$P_s(m) = \sum_{n=m}^{\infty} p_s(n) \binom{n}{m} \eta^m (1 - \eta)^{n-m}. \tag{49}$$

In principle this can be inverted to yield $p_s(n)$ from measurements of $P_s(n)$. In practice, when background is present, not all counts can be identified with signal or idler photons, and the relationship becomes more complicated [30]. Nevertheless, the probability $p_s(n)$ can still be derived from photoelectric measurements. Figure 9 shows the results of a recent experiment in which $p_s(n)$ was derived from photoelectric measurements. It is apparent that we come close to realizing a localized one-photon state.

This conclusion has interesting implications for testing certain non-local features of quantum electrodynamics relating to photons. Although a photon has no position in any precise sense [31], one can define a localized photon number operator within a small volume [32–34], provided its linear dimensions are large compared with the wavelength. In this sense the idler photon is in a localized state when the position of the signal photon is known. Neverthless, there is a non-vanishing probability that the photon may actually be detected photoelectrically at another, more distant point, if the measurement involves a coupling through an electromagnetic field vector that is not locally connected to the

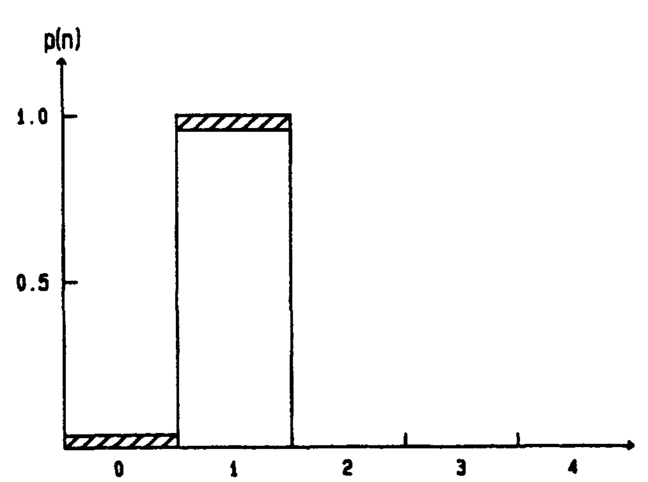

Fig. 9. The measured probability $p(n)$ of n idler photons conditioned on the detection of a signal photon. [Reproduced from Hong, C. K. and Mandel, L., Phys. Rev. Lett. **56**, 58 (1986).]

initial state. This is a paradoxical aspect of QED that has never been tested experimentally.

To illustrate it let us express the localized idler-photon state at the detector as a linear superposition of momentum-polarization states,

$$|\phi\rangle = \frac{1}{L^{3/2}} \sum_{k,s} \phi(\boldsymbol{k}, s)\hat{a}^{\dagger}_{ks}|\text{vac}\rangle, \tag{50}$$

where $\phi(\boldsymbol{k}, s)$ is a suitably normalized weight function that depends on the range of wave vectors and polarizations which the measuring apparatus accepts. Then the probability $\mathscr{P}$ that the photon is localized within a small volume $\mathscr{V}$ at time t can be written [32, 33]

$$\mathscr{P} = \int_{\mathscr{V}} |\boldsymbol{\Phi}(\boldsymbol{r}, t)|^2 \, d^3x, \tag{51}$$

where

$$\begin{aligned}\boldsymbol{\Phi}(\boldsymbol{r}, t) &= \frac{1}{L^3} \sum_{k,s} \phi(\boldsymbol{k}, s)\, \boldsymbol{\varepsilon}_{ks}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \\ &\to \sum_s \frac{1}{(2\pi)^3} \int d^3k\, \phi(\boldsymbol{k}, s)\boldsymbol{\varepsilon}_{ks}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}.\end{aligned} \tag{52}$$

The last expression is obtained in the limit $L \to \infty$. If $|\boldsymbol{\Phi}(\boldsymbol{r}, t)|^2$ is strongly peaked in position and time, then so is $\mathscr{P}$, and we can speak of the idler photon as being approximately localized within $\mathscr{V}$ at time t.

However, if the photon is to be detected with a photoelectric detector that operates through the $e\hat{\boldsymbol{r}} \cdot \hat{E}^{(+)}(\boldsymbol{r}t)$ electric dipole interaction, say, then the probability of detecting it at $\boldsymbol{r}$ at time t is proportional to [2]

$$\left.\begin{aligned}\mathscr{P}' &= \langle\phi|\hat{\boldsymbol{E}}^{(-)}(\boldsymbol{r}, t)\cdot\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}, t)|\phi\rangle \\ &= |\boldsymbol{\Psi}(\boldsymbol{r}, t)|^2,\end{aligned}\right\} \tag{53}$$

where the function $\boldsymbol{\Psi}(\boldsymbol{r}, t)$ is given by

$$\begin{aligned}\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r}, t)|\phi\rangle &= \boldsymbol{\Psi}(\boldsymbol{r}, t)|0\rangle \\ &= i\sqrt{\left(\frac{\hbar}{2\varepsilon_0}\right)} \sum_s \frac{1}{(2\pi)^3} \\ &\quad \times \int d^3k\sqrt{\omega}\, \phi(\boldsymbol{k}, s)\boldsymbol{\varepsilon}_{ks}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}|0\rangle.\end{aligned} \tag{54}$$

Comparison of eqs. (52) and (54) shows that the Fourier transforms of $\boldsymbol{\Phi}(\boldsymbol{r}, t)$ and $\boldsymbol{\Psi}(\boldsymbol{r}, t)$ differ by $\sqrt{\omega}$ apart from constants, so that $\boldsymbol{\Phi}(\boldsymbol{r}, t)$ and $\boldsymbol{\Psi}(\boldsymbol{r}, t)$ are non-locally connected through the convolution [34]

$$\boldsymbol{\Psi}(\boldsymbol{r}, t) = \int d^3r'\, G(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{\Phi}(\boldsymbol{r}, t), \tag{55}$$

in which the Green function $\boldsymbol{\Psi}(\boldsymbol{r}, t) \propto r^{-7/2}$ asymptotically. Because of this non-local connection, the detection probability $\mathscr{P}'$ can be non-negligible at positions where $\mathscr{P}$ is negligibly small. In other words, the photon might be detected where it is not localized. Evidently we are here encountering an aspect of photons that is very unparticle-like.

7. Quantum effects in optical interference

Although the phenomenon of optical interference is commonly treated in completely classical terms, we know that interference also has strong quantum mechanical aspects. Perhaps the simplest way to exhibit interference in optics is to divide a light beam into two with the help of a beam splitter,

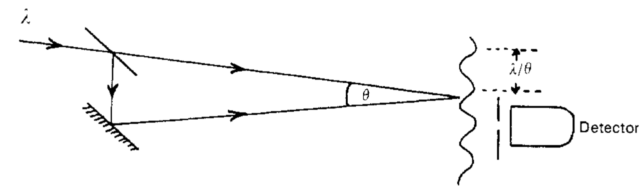

Fig. 10a. Conventional arrangement for observing interference fringes.

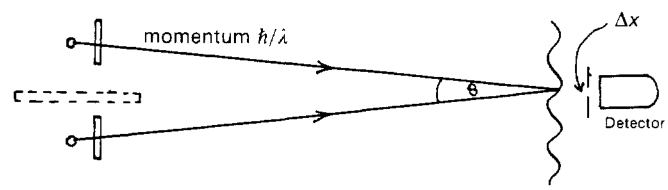

Fig. 10b Arrangement for observing interference between two independent light beams.

and then to recombine the two beams on some distant screen (see Fig. 10a). If λ is the wavelength and θ the small angle between the two beams, and if the path difference is sufficiently small, we expect to see linear interference fringes of periodicity $L = \lambda/\theta$ and 100% modulation. It is well known that the fringes persist no matter how weak the light may be, even if the photons are arriving one at a time with negligible overlap. The explanation given by the conventional quantum mechanical interpretation is that the photons do not interfere with each other. Rather it is the interference of the probability amplitudes associated with the two possible paths of each photon that gives rise to the fringes.

With the development of lasers it became possible to observe interference effects with light beams from two separate sources. Not only are heterodyne experiments commonplace, but it has even proved possible to photograph interference fringes produced by two lasers in a short exposure time [35] (see Fig. 10b). That raises the interesting question whether these interference effects would persist if the photons do not overlap, so that a photon emitted from one or the other source has time to pass through the interferometer and to be absorbed at the detector long before the next photon is emitted, on the average. Because there is no beam splitter in the apparatus to create a superposition state, the answer may not be obvious. However, experiments showed that interference fringes are still observable by photon counting [36].

We can understnd the reason if we apply the uncertainty principle to the photons just prior to their detection. In order to resolve the interference pattern we require a spatial resolution Δx smaller than the fringe spacing $L = \lambda/\theta$. In other words, each photon has to be localized in position to an accuracy

$$\Delta x < \lambda/\theta. \tag{56}$$

By the uncertainty principle this introduces a momentum uncertainty Δp_x in the same transverse direction, such that

$$|\Delta p_x| \gtrsim (\hbar/\lambda)\theta. \tag{57}$$

Now $(\hbar/\lambda)\theta$ is the transverse momentum component of an incoming photon, so that the inequality (57) implies that it is impossible to tell from which of the two sources the photon was emitted. We therefore have to associate a non-vanishing probability amplitude with each of the two photon paths, and these two probability amplitudes of each photon interfere.

There is, however, an important proviso that has to be attached if interference fringes are to be observable in the experiment: the light emitted from the two sources must have well-defined phases. With laser sources the condition is easily satisfied, and it was satisfied in the cited experiments [36]. But with atomic sources of definite excitation, for example, no well-defined phase would exist, and interference effects should not be observable. It is easy to see why. If one source consisted of N_1 excited atoms and the other of N_2 excited atoms, we could, in principle, determine from where each photon was emitted by examining the sources. According to quantum mechanics, no interference fringes should then be observable, because one of the two probability amplitudes would be zero.

It is easy to make this argument a little more quantitative [37]. Let us look on the two light beams in Fig. 10b as two modes 1 and 2 of the field, and let the state of the combined two-mode field be the product state $|\psi_1\rangle|\psi_2\rangle$. If we make a mode decomposition of the photon annihilation operator $\hat{a}$ for the total field by writing

$$\hat{a} = c_1\hat{a}_1 + c_2\hat{a}_2, \tag{58}$$

then the propability of detecting a photon out of the combined two-mode field is proportional to

$$\begin{aligned}\langle\psi_2|\langle\psi_1|\hat{a}^\dagger\hat{a}|\psi_1\rangle|\psi_2\rangle &= |c_1|^2\langle\psi_1|\hat{a}_1^\dagger\hat{a}_1|\psi_1\rangle \\ &+ |c_2|^2\langle\psi_2|\hat{a}_2^\dagger\hat{a}_2|\psi_2\rangle + c_1c_2^*\langle\psi_1|\hat{a}_1|\psi_1\rangle\langle\psi_2|\hat{a}_2^\dagger|\psi_2\rangle + \text{c.c.}\end{aligned} \tag{59}$$

The last two terms describe interference effects, and they are non-zero in a coherent state of the field, such as might be produced by laser sources. However, the terms vanish for a field in a Fock state, such as might be produced by sources consisting of a definite number of fully excited atoms.

It is an immediate consequence of the foregoing that two sources, each of which consists of a single excited atom, will not give rise to interference fringes. Nevertheless, there is a sense in which interference effects of higher order still exist. Let us consider the experimental situation illustrated in Fig. 11, in which each of the two sources is one excited atom and two photoelectric detectors are located at two positions x and x' in the receiving plane. We look for 'simultaneous' detections at x and x' with the help of a coincidence counter as shown, and this is a measurement involving higher order correlation functions.

Let us first see what classical optics has to say about such intensity correlations [38]. Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be the positions of the two sources, and let $\boldsymbol{R}$, $\boldsymbol{R}'$ be the positions of the two detectors corresponding to x, x'. Let $V_1(t)$, $V_2(t)$ be the scalar complex analytic signals describing the optical field at the two sources. Then the field in the receiving plane at position $\boldsymbol{R}$ at time t can be written

$$V(\boldsymbol{R}, t) = V_1(t - \tau_1) + V_2(t - \tau_2), \tag{60}$$

where

$$\tau_1 = |\boldsymbol{r}_1 - \boldsymbol{R}|/c, \quad \tau_2 = |\boldsymbol{r}_2 - \boldsymbol{R}|/c. \tag{61}$$

The instantaneous light intensity at $\boldsymbol{R}$ is

$$\begin{aligned}I(\boldsymbol{R}, t) &= |V(\boldsymbol{R}, t)|^2 \\ &= I_1(t - \tau_1) + I_2(t - \tau_2) \\ &\quad + V_1^*(t - \tau_1)V_2(t - \tau_2) + \text{c.c.},\end{aligned} \tag{62}$$

where $I_j(t) = |V_j(t)|^2$ $(j = 1, 2)$. For quasi-monochromatic fields of midfrequency ω_0, we can make the approximation

$$\begin{aligned}V_j(t - \tau_j) &\approx V_j(t - \tau_0)\, e^{-i\omega_0(\tau - \tau_j)} \\ &\equiv \sqrt{I_j(t - \tau_0)}\, e^{i\phi_j}\, e^{-i\omega_0(\tau_0 - \tau_j)} \quad (j = 1, 2),\end{aligned} \tag{63}$$

where τ_0 is the transit time from $\boldsymbol{r}_1$ or $\boldsymbol{r}_2$ to the symmetry point 0 (see Fig. 11). Then $I(\boldsymbol{R}, t) = I(x, t)$ can be rewritten

$$\begin{aligned}I(x, t) &= I_1(t - \tau_0) + I_2(t - \tau_0) \\ &\quad + 2\sqrt{I_1(t - \tau_0)t_2(t - \tau_0)} \\ &\quad \times \cos[\omega_0(\tau_1 - \tau_2) + \phi_1 - \phi_2].\end{aligned} \tag{64}$$

Finally, on the assumption that the phase difference $\phi_1 - \phi_2$ is completely random, we find that $\langle I(x, t)\rangle$ is independent of position x, whereas the intensity correlation at two points x and x' is given by

$$\begin{aligned}\langle I(x, t)I(x', t)\rangle &= \langle I_1^2\rangle + \langle I_2^2\rangle + 2\langle I_1\rangle\langle I_2\rangle \\ &\quad \times [1 + \cos 2\pi(x - x')/L].\end{aligned} \tag{65}$$

It is to be understood that I_1, I_2 on the right are evaluated at time $t - \tau_0$, and we have written

$$\omega_0(\tau_1 - \tau_2 - \tau_1' + \tau_2') \equiv 2\pi(x - x')/L. \tag{66}$$

L is just the spacing of the interference fringes that would be produced at the receiving plane if there were two coherent sources at $\boldsymbol{r}_1$, $\boldsymbol{r}_2$.

From eq. (65) we see that even two independent sources give rise to a periodic modulation of the two-point intensity correlation function, with a periodicity L that is equal to the fringe spacing produced by two coherent sources. The relative modulation amplitude or 'visibility' is given by

$$\frac{2\langle I_1\rangle\langle I_2\rangle}{\langle I_1^2\rangle + \langle I_2^2\rangle + 2\langle I_1\rangle\langle I_2\rangle} \le \frac{1}{2}, \tag{67}$$

and this has an upper bound of $\frac{1}{2}$ no matter what the statistical properties of the light may be.

Let us now re-examine the problem quantum mechanically [38, 39]. The positive frequency part of the field at $\boldsymbol{R}$, t due to the two atomic sources can be written [11]

$$\begin{aligned}\hat{E}^{(+)}(x, t) &= K[\hat{b}_1(t - \tau_1) \\ &\quad + \hat{b}_2(t - \tau_2) + \hat{E}_{\text{Free}}^{(+)}(x, t),\end{aligned} \tag{68}$$

where $\hat{b}_1(t)$, $\hat{b}_2(t)$ are the atomic lowering operators for the two two-level atoms with resonant frequency ω_0. K is a geometric factor that has been taken to be similar for both sources within the small-angle approximation. As in the

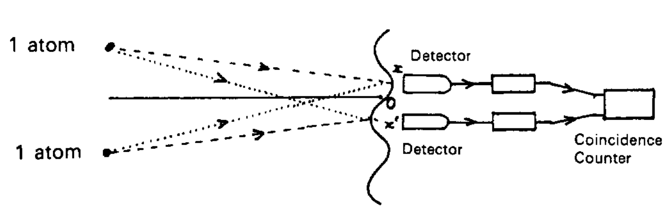

Fig. 11. Arrangement for observing two-photon interference effects by coincidence detection, with light sources each of which consists of a single excited atom.

classical cases we may write approximately,

$$\hat{b}_j(t - \tau_j) \approx \hat{b}_j(t - \tau_0)\, e^{i\omega_0(\tau_j - \tau_0)}, \quad (j = 1, 2). \tag{69}$$

We shall take the initial state $|\rangle$ to be the vacuum state, so that

$$\hat{E}^{(+)}_{\text{Free}}(x, t)|\rangle = 0, \tag{70}$$

and we shall make use of the fact that atomic operators commute with free-field operators at later times, or

$$[\hat{b}_j(t - \tau_0), \hat{E}^{(+)}_{\text{Free}}(x, t)] = 0. \tag{71}$$

Then it follows from eqs. (68)–(71) together with the fact that $\hat{b}_j^2 = 0 = \hat{b}_j^{\dagger 2}$ $(j = 1, 2)$ that the normally ordered two-point intensity correlation function is given by

$$\begin{aligned} \langle :\hat{I}(x, t)\hat{I}(x', t):\rangle &= 2|K|^4\langle \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_2 \hat{b}_1\rangle \\ &\quad \times [1 + \cos\omega_0(\tau_1 - \tau_2 - \tau_1' + \tau_2')] \\ &= 2|K|^4\langle \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_2 \hat{b}_1\rangle[1 + \cos 2\pi(x - x')/L], \end{aligned} \tag{72}$$

where it is understood that the time argument of each $\hat{b}_j$, $\hat{b}_j^\dagger$ is $t - \tau_0$ $(j = 1, 2)$. Even if the atoms are both fully excited and in a product state, so that $\langle \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_2 \hat{b}_1\rangle = 1$ and there are no interference fringes, there is a cosine modulation of the two-point intensity correlation with separation $(x - x')$, with periodicity L. But unlike the classical situation described by eq. (65), the depth of modulation is 100%, in violation of the inequality (67).

It follows from eq. (72) that the joint probability of detecting two photons at two positions x and x' vanishes whenever the two points are separated by an odd number of half interference fringes, or $x - x' = (n + \frac{1}{2})L$ $(n = 0, \pm 1, \pm 2, \ldots)$. This is a phenomenon without classical counterpart. It arises because the measurement has to be described by two two-photon probability amplitudes, corresponding to the detection of the photon from atom 1 at x and that from atom 2 at x', and vice versa. Because the experiment cannot distinguish between the two cases we have to add the corresponding probability amplitudes, which are in anti-phase when $x - x' = (n + \frac{1}{2})L$.

It is worth noting that in looking for simultaneous detections at x and x', we encounter a situation in which the outcome of the measurement at x is strongly influenced by where the other detector is located, even though the two detectors may be sufficiently far apart from the two measurements to be disjoint. This conclusion is in the spirit of the EPR paradox because we encounter certain non-local features of quantum mechanics whenever we create a correlated two-particle state.

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