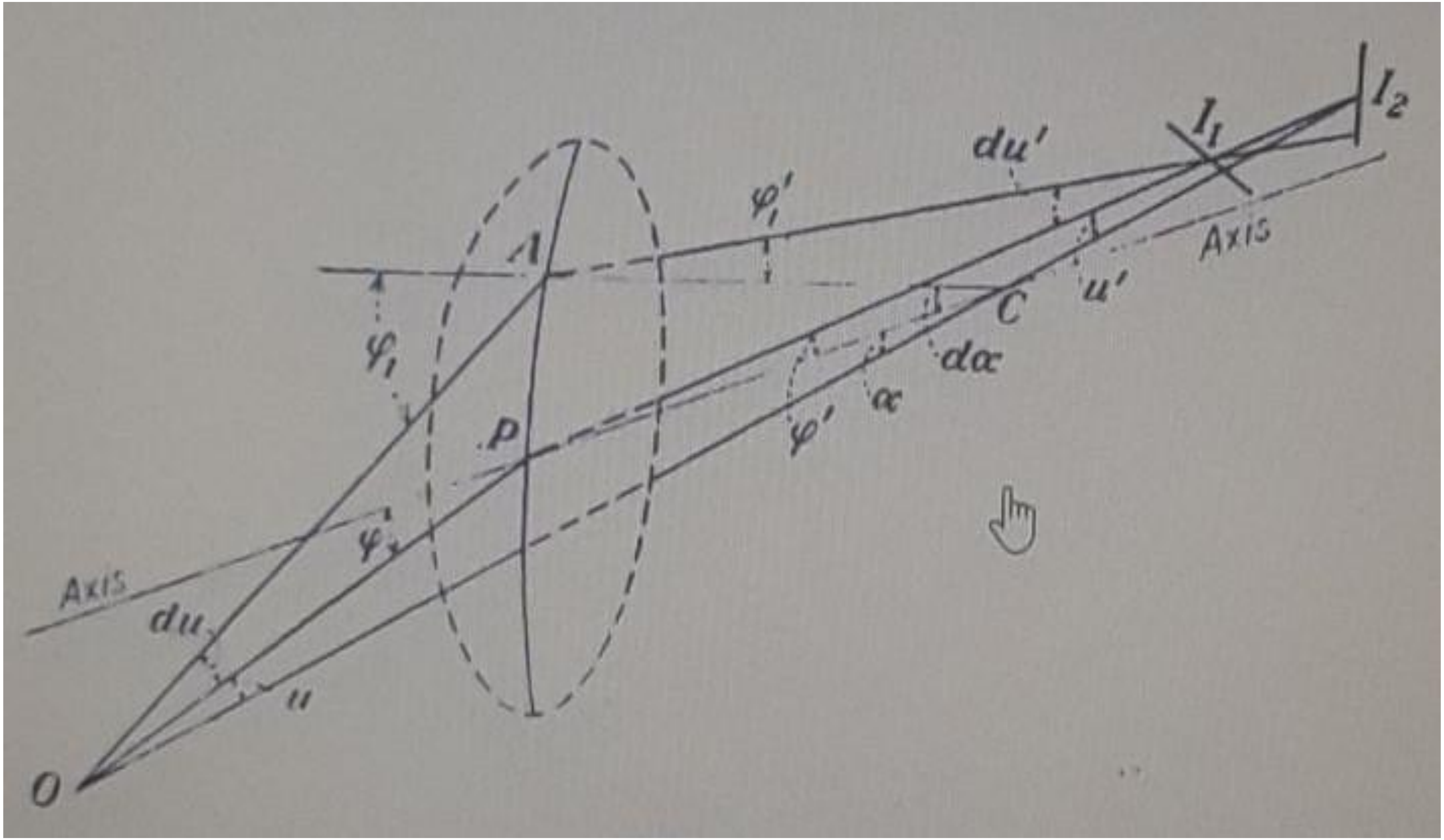
Derivation of equations for astigmatic focal distances at a single refracting surface



In Fig. 1 let O be an object point, not on the axis, in the plane containing the line element AP of a single spherical refracting surface and C, its center of curvature. Then if coma is absent, all the rays which have the same inclination u as OP with OC will intersect the line OC extended in a point such as I_2. Let $OP = s$, and $PI_2 = s_2$. Then from the law of sines, in a triangle OPI_2

$$\frac{s}{sinu'} = \frac{s_2}{sinu} = \frac{OI_2}{sin(\varphi - \varphi')}; \qquad (1)$$

In triangle OPC

$$\frac{s}{sin\alpha} = \frac{r}{sinu} = \frac{OC}{sin\varphi}; \qquad (2)$$

and in triangle PCI_2

$$\frac{r}{sinu'} = \frac{s_2}{sin\alpha} = \frac{CI_2}{sin\varphi'} \qquad (3)$$

From Eq. (2) $$OC = \frac{rsin\varphi}{sinu}; \qquad (4)$$

From Eq. (3) $$CI_2 = \frac{rsin\varphi'}{sinu'}; \qquad (5)$$

Adding eqs. (4) and (5)

$$OC + CI_2 = OI_2 = \frac{r sin\varphi}{sinu} + \frac{r sin\varphi'}{sinu'}. \qquad (6)$$

Substituting this value of OI_2 in eq. (1) and using the first and last terms of eq. (1)

$$\frac{s}{sinu'} = \frac{r}{sin(\varphi - \varphi')}\left[\frac{sin\varphi}{sinu} + \frac{sin\varphi'}{sinu'}\right]. \qquad (7)$$

From the first and second terms of eq. (1),

$$\frac{s}{sinu'} = \frac{s_2}{sinu} \rightarrow sinu = \frac{s_2 sinu'}{s},$$

Whence eq. (7) becomes

$$\frac{s}{sinu'} = \frac{r}{sin(\varphi - \varphi')}\left[\frac{sin\varphi}{\frac{s_2 sinu'}{s}} + \frac{sin\varphi'}{sinu'}\right].$$

$$\frac{s}{sinu'} = \frac{r}{sin(\varphi - \varphi')}\left[\frac{s\ sin\varphi}{s_2 sinu'} + \frac{sin\varphi'}{sinu'}\right]. \tag{8}$$

Expanding $sin(\varphi - \varphi')$ and substituting for $sin\varphi$ its value from Snell's law, i.e.,

$$sin\varphi = \frac{n'}{n} sin\varphi', \tag{9}$$

$$\frac{s}{sinu'} = \frac{r}{(sin\varphi cos\varphi' - cos\varphi sin\varphi')} \frac{1}{sinu'} \left[\frac{s\ sin\varphi}{s_2} + sin\varphi' \right].$$

$$s = \frac{r}{(sin\varphi cos\varphi' - cos\varphi sin\varphi')} \left[\frac{s\ sin\varphi}{s_2} + sin\varphi' \right].$$

$$s = \frac{r}{\left(\frac{n'}{n} sin\varphi' cos\varphi' - cos\varphi sin\varphi' \right)} \left[\frac{s}{s_2} \frac{n'}{n} sin\varphi' + sin\varphi' \right].$$

$$s = \frac{r}{sin\varphi' \left(\frac{n'}{n} cos\varphi' - cos\varphi \right)} \times sin\varphi' \left[\frac{s}{s_2} \frac{n'}{n} + 1 \right].$$

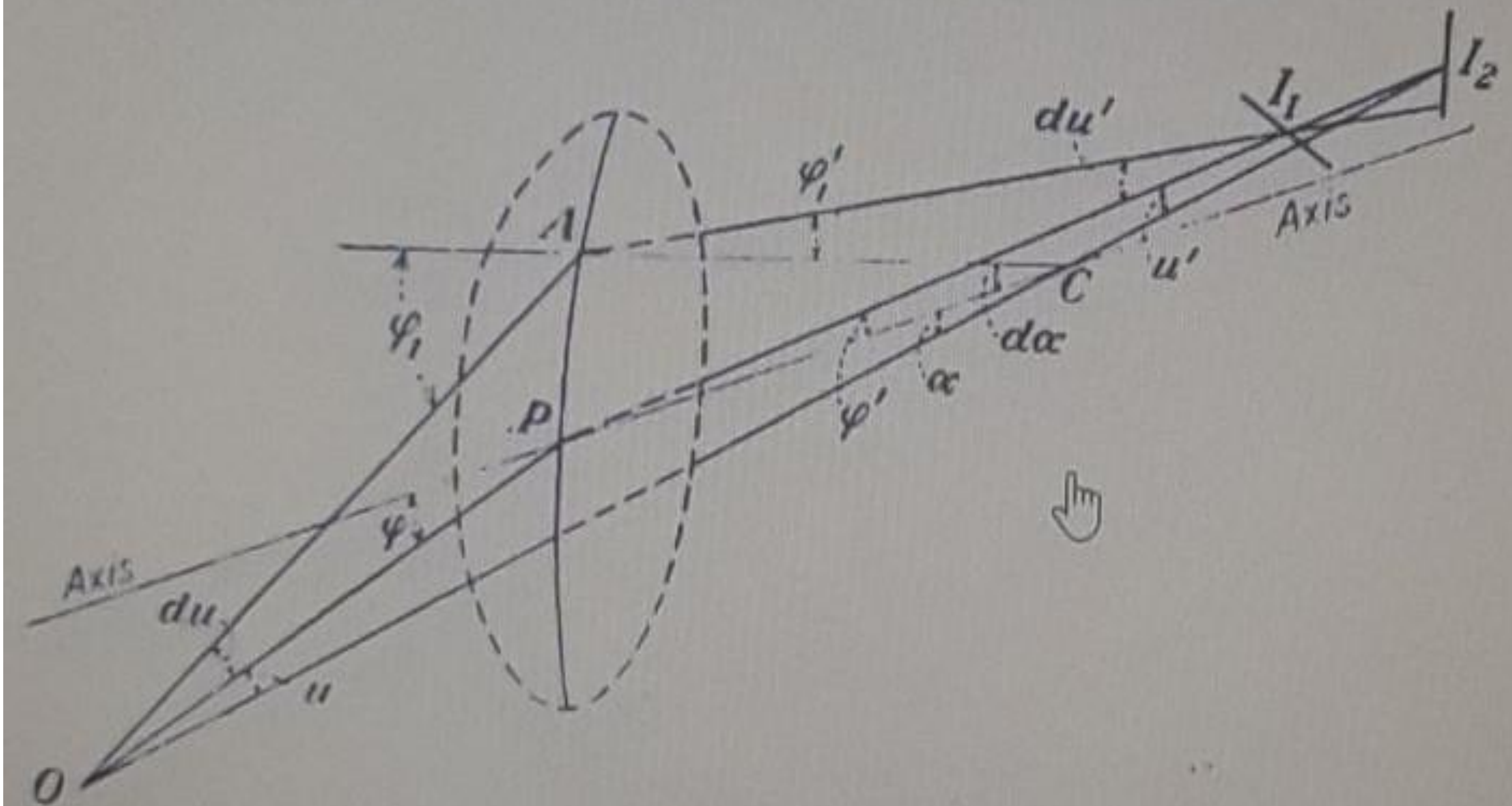
$$s = \frac{r}{\left(\frac{n'}{n} cos\varphi' - cos\varphi\right)} \times \left[\frac{s}{s_2}\frac{n'}{n} + 1\right].$$

$$s\left(\frac{n'}{n} cos\varphi' - cos\varphi\right) = r\left[\frac{s}{s_2}\frac{n'}{n} + 1\right]$$

$$\frac{s}{n}[n' cos\varphi' - ncos\varphi] = r \times \frac{s}{n}\left(\frac{n'}{s_2} + \frac{n}{s}\right)$$

$$[n' cos\varphi' - ncos\varphi] = r\left(\frac{n'}{s_2} + \frac{n}{s}\right)$$

$$\frac{n}{s} + \frac{n'}{s_2} = \frac{n' cos\varphi' - ncos\varphi}{r} \qquad (10)$$



I_2
I_1
du'
φ'_1
Axis
A
u'
C
φ_1
$d\alpha$
α
φ'
P
φ_2
Axis
du
u
O

This gives the distance s_2 measured from the surface, of the sagittal or secondary focus.

Consider next two rays , OP and another adjacent ray OA. Since they are refracted by the surface at different distances from the intersection of OC with the surface they will, after refraction, intersect at a point I_1 not on the OC extended.

Let the angle between OA and OC be $u + du$, that between AI_1 and PI_1 be du', and let $PI_1 = s_1$. Since from the figure

$$\varphi = \alpha + u, \qquad \text{and} \qquad \varphi' = \alpha - u'$$

By differentiation it follows that

$$d\varphi = d\alpha + du, \qquad \text{and} \qquad d\varphi' = d\alpha - du' \tag{11}$$

Considering the angles du, du', and $d\alpha$ to be equal to their sines, it follows that from the law of sines that

$$du = \frac{PAcos\varphi}{s}, \qquad du' = \frac{PAcos\varphi'}{s_1}, \qquad d\alpha = \frac{PA}{r}, \tag{12}$$

whence, in eq. (11),

$$d\varphi = PA\left(\frac{1}{r} + \frac{cos\varphi}{s}\right) \quad \text{and} \quad d\varphi' = PA\left(\frac{1}{r} - \frac{cos\varphi'}{s_1}\right) \tag{13}$$

Differentiation of Snell's law in eq. 9 gives

$$ncos\varphi d\varphi = n'cos\varphi' d\varphi' \qquad (14)$$

and on substituting the values of $d\varphi$ and $d\varphi'$ from eq. (13) this becomes

$$ncos\varphi PA\left(\frac{1}{r}+\frac{cos\varphi}{s}\right) = n'cos\varphi' PA\left(\frac{1}{r}-\frac{cos\varphi'}{s_1}\right)$$

$$ncos\varphi\left(\frac{1}{r}+\frac{cos\varphi}{s}\right) = n'cos\varphi'\left(\frac{1}{r}-\frac{cos\varphi'}{s_1}\right)$$

$$\frac{ncos\varphi}{r}+\frac{ncos^2\varphi}{s} = \frac{n'cos\varphi'}{r}-\frac{n'cos^2\varphi'}{s_1}$$

$$\frac{n\cos^2\varphi}{s} + \frac{n'\cos^2\varphi'}{s_1} = \frac{n'\cos\varphi'}{r} - \frac{n\cos\varphi}{r}$$

$$\frac{n\cos^2\varphi}{s} + \frac{n'\cos^2\varphi'}{s_1} = \frac{n'\cos\varphi' - n\cos\varphi}{r}$$

This gives the distance s_1, measured from the surface, of the tangential or primary focus.