MATH 415: Assignment 2

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Question 1:

We will use the Simplex method to solve the given LP problem. The problem is given as follows:

$$\begin{aligned}
\text{minimize} \quad & -5x_1 - 6x_2 - 9x_3 - 8x_4 \\
\text{subject to} \quad & x_1 + 2x_2 + 3x_3 + x_4 \leq 5, \\
& x_1 + x_2 + 2x_3 + 3x_4 \leq 3, \\
& \mathbf{x} \geq 0
\end{aligned}$$

The standard equation form with slack variables:

$$\begin{aligned}
\text{minimize} \quad & z = -5x_1 - 6x_2 - 9x_3 - 8x_4 \\
\text{subject to} \quad & x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 5, \\
& x_1 + x_2 + 2x_3 + 3x_4 + x_6 = 3, \\
& \mathbf{x} \geq 0
\end{aligned}$$

Considering the initial basis as x_5 and x_6. The dictionary is:

$$\begin{aligned}
\text{minimize} \quad & z = 0 - 5x_1 - 6x_2 - 9x_3 - 8x_4 \\
\text{subject to} \quad & x_5 = 5 - x_1 - 2x_2 - 3x_3 - x_4, \\
& x_6 = 3 - x_1 - x_2 - 2x_3 - 3x_4, \\
& \mathbf{x} \geq 0
\end{aligned}$$

Solution is:

$$X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \\ 3 \end{bmatrix}$$

and $z = 0$

In the objective function, the highest coefficient is for variable x_3. So, x_3 is the entering variable as a basis.

As $x_3 \to 3/2$, $x_6 \to 0$, x_6 is the leaving basic variable. The new basis will be x_3, x_5. The new dictionary will be:

$$\begin{aligned}
\text{minimize} \quad & z = -13.5 - 0.5x_1 - 1.5x_2 + 5.5x_4 + 4.5x_6 \\
\text{subject to} \quad & x_3 = 1.5 - 0.5x_1 - 0.5x_2 - 1.5x_4 - 0.5x_6, \\
& x_5 = 0.5 + 0.5x_1 - 0.5x_2 + 3.5x_4 + 1.5x_4, \\
& \mathbf{x} \geq 0
\end{aligned}$$

Solution is:

$$X = \begin{bmatrix} 0 \\ 0 \\ 1.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

and $z = -13.5$

In the objective function, between x_1 & x_2, I choose x_2 as the new candidate to enter in the basis because, among these two negative variables, the coefficient for x_2 is higher.

$x_2 \to 1$, $x_5 \to 0$, x_5 is the leaving basic variable. The new basis will be x_2, x_3. The new dictionary will be:

$$\begin{aligned} \text{minimize} \quad & z = -15 - 2x_1 - 5x_4 + 3x_5 \\ \text{subject to} \quad & x_2 = 1 + x_1 + 7x_4 - 2x_5 + 3x_6, \\ & x_3 = 1 - x_1 - 5x_4 + x_5 - 2x_6, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Solution is:

$$X = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $z = -15$

At this stage, x_4 enters the basis, and x_3 leaves because $x_4 \to 1/5$, $x_3 \to 0$.

$$\begin{aligned} \text{minimize} \quad & z = -16 - x_1 + x_3 + 2x_5 + 2x_6 \\ \text{subject to} \quad & x_2 = 2.4 - 0.4x_1 - 1.4x_3 - 0.6x_5 + 0.2x_6, \\ & x_4 = 0.2 - 0.2x_1 - 0.2x_3 + 0.2x_5 - 0.4x_6, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Solution is:

$$X = \begin{bmatrix} 0 \\ 2.4 \\ 0 \\ 0.2 \\ 0 \\ 0 \end{bmatrix}$$

and $z = -16$

As $x_1 \to 1$, $x_4 \to 0$, x_4 is the leaving basic variable. The new basis will be x_1, x_2. The new dictionary will be:

$$\begin{aligned} \text{minimize} \quad & z = -17 + 2x_3 + 5x_4 + x_5 + 4x_6 \\ \text{subject to} \quad & x_1 = 1 - x_3 - 5x_4 + x_5 - 2x_6, \\ & x_2 = 2 - x_3 + 2x_4 - x_5 + x_6, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Solution is:

$$X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $z = -17$

This is the unique optimal solution because none of the variables x is able to reduce the value of an objective function and the value of $z = -17$ and we have to use $x_1 = 1$, $x_2 = 2$ and zero for all other x variables to get the solution.

Question 2:

a

In this LP there are a total of 4 variables and 2 constraints. So, the number of possible bases should be $\binom{4}{2} = 6$. The list of basis are (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), and (x_3, x_4).

b

Now I am going to show the dictionary, basic solution, and objective value for each of the bases shown above:

(1) Basis $= (x_1, x_2)$:

Dictionary:

$$\begin{aligned} \text{minimize} \quad & z = 0 - 2x_1 - 4x_2 \\ \text{subject to} \quad & x_2 = 1 - x_1 + x_3, \\ & x_1 = 2 - 2x_2 - x_4, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Basic solution: Considering $x_3 = 0$ and $x_4 = 0$, the system of equation reduced to the following form and we solved for x_1 and x_2.

$$\left.\begin{aligned} x_2 &= 1 - x_1 \\ x_1 \quad &= 2 - 2x_2 \end{aligned}\right\} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 1 \end{aligned}$$

$$X = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Objective value: $z = -4$

(2) Basis $= (x_1, x_3)$:

Dictionary:

$$\begin{aligned} \text{minimize} \quad & z = 0 - 2x_1 - 4x_2 \\ \text{subject to} \quad & x_3 = -1 + x_1 + x_2, \\ & x_1 = 2 - 2x_2 - x_4, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Basic solution: Considering $x_2 = 0$ and $x_4 = 0$, the system of equation reduced to the following form and we solved for x_1 and x_3.

$$\left.\begin{array}{rl} x_3 &= -1 + x_1 \\ x_1 \qquad &= 2 \end{array}\right\} \quad \begin{array}{l} x_1 = 2 \\ x_3 = 1 \end{array}$$

$$X = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Objective value: $z = -4$

(3) Basis $= (x_1, x_4)$**:**

Dictionary:

$$\begin{aligned} \text{minimize} \quad & z = 0 - 2x_1 - 4x_2 \\ \text{subject to} \quad & x_1 = 1 - x_2 + x_3, \\ & x_4 = 2 - x_1 - 2x_2, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Basic solution: Considering $x_2 = 0$ and $x_3 = 0$ and we solved for x_1 and x_4.

$$\left.\begin{array}{rl} x_1 \qquad &= 1 \\ x_4 &= 2 - x_1 \end{array}\right\} \quad \begin{array}{l} x_1 = 1 \\ x_4 = 1 \end{array}$$

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Objective value: $z = -2$

(4) Basis $= (x_2, x_3)$**:**

Dictionary:

$$\begin{aligned} \text{minimize} \quad & z = 0 - 2x_1 - 4x_2 \\ \text{subject to} \quad & x_3 = -1 + x_1 + x_2, \\ & x_2 = 1 - 0.5x_1 - 0.5x_4, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Basic solution: Considering $x_1 = 0$ and $x_4 = 0$ and we solved for x_2 and x_3.

$$\left.\begin{array}{rl} x_3 &= -1 + x_2 \\ x_2 \qquad &= 1 \end{array}\right\} \quad \begin{array}{l} x_2 = 1 \\ x_3 = 0 \end{array}$$

$$X = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Objective value: $z = -4$

(5) Basis $= (x_2, x_4)$**:**

Dictionary:

$$
\begin{aligned}
\text{minimize} \quad & z = 0 - 2x_1 - 4x_2 \\
\text{subject to} \quad & x_2 = 1 - x_1 + x_3, \\
& x_4 = 2 - x_1 - 2x_2, \\
& \mathbf{x} \geq 0
\end{aligned}
$$

Basic solution: Considering $x_1 = 0$ and $x_3 = 0$ and we solved for x_2 and x_4.

$$
\left.\begin{aligned} x_2 \quad &= 1 \\ x_4 &= 2 - 2x_2 \end{aligned}\right\} \quad \begin{aligned} x_2 &= 1 \\ x_4 &= 0 \end{aligned}
$$

$$
X = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

Objective value: $z = -4$
(6) Basis $= (x_3, x_4)$**:**

Dictionary:

$$
\begin{aligned}
\text{minimize} \quad & z = 0 - 2x_1 - 4x_2 \\
\text{subject to} \quad & x_3 = -1 + x_1 + x_2, \\
& x_4 = 2 - x_1 - 2x_2, \\
& \mathbf{x} \geq 0
\end{aligned}
$$

Basic solution: Considering $x_1 = 0$ and $x_2 = 0$ and we solved for x_3 and x_4.

$$
\left.\begin{aligned} x_3 \quad &= -1 \\ x_4 &= 2 \end{aligned}\right\} \quad \begin{aligned} x_3 &= -1 \\ x_4 &= 2 \end{aligned}
$$

$$
X = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}
$$

Objective value: $z = 0$

c

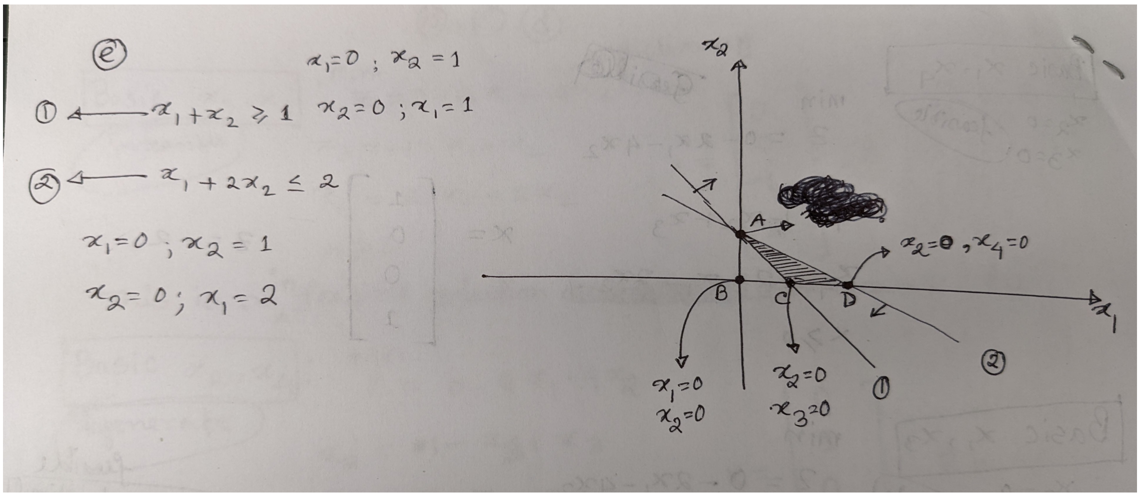
The corresponding basis for feasible, infeasible, and degenerate solutions are given below:

Feasible: (x_1, x_4) and (x_1, x_3) as base.
Infeasible: (x_3, x_4) as the base.
Degenerate: (x_1, x_2), (x_2, x_3), and (x_2, x_4) as the base.

d

Among all of these 6 possible bases, we got the feasible solution for (x_1, x_4) and (x_1, x_3) but the objective value $(z = -4)$ is lowest for (x_1, x_3). So, we got our optimal solution for the basis of (x_1, x_3).

e



In this sketch, the feasible region and the corner points A, B, C and D are marked. At point B the solution is infeasible. At C & D it is feasible and at point A the three degenerate solutions exist.

Question 3:

We will use the two-phase Simplex method to solve the given LP problem. The problem is given as follows:

$$\begin{aligned} \text{minimize} \quad & -3x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq -1, \\ & -x_1 - x_2 \leq -3, \\ & 2x_1 - x_2 \leq 2, \\ & \mathbf{x} \geq 0 \end{aligned}$$

The standard equation form with slack variables:

$$\begin{aligned} \text{minimize} \quad & z = -3x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 + x_3 = -1, \\ & -x_1 - x_2 + x_4 = -3, \\ & 2x_1 - x_2 + x_5 = 2, \\ & \mathbf{x} \geq 0 \end{aligned}$$

We cannot use x_3, x_4, x_5 as the initial basis for this problem because the solution would become infeasible in that case (we would get a negative value for x_3 and x_4). As a result, we have to use the two-phase simplex method.

Phase-I

$$\begin{aligned}
\text{minimize} \quad & W = v_1 + v_2 + v_3 \\
\text{subject to} \quad & x_1 - x_2 + x_3 - v_1 = -1, \\
& -x_1 - x_2 + x_4 - v_2 = -3, \\
& 2x_1 - x_2 + x_5 + v_3 = 2, \\
& \mathbf{x}, \mathbf{v} \geq 0
\end{aligned}$$

Start with the initial basis as v_1, v_2, v_3. The dictionary is:

$$\begin{aligned}
\text{minimize} \quad & W = 6 - 2x_1 - x_2 + x_3 + x_4 - x_5 \\
\text{subject to} \quad & v_1 = 1 + x_1 - x_2 + x_3, \\
& v_2 = 3 - x_1 - x_2 + x_4, \\
& v_3 = 2 - 2x_1 + x_2 - x_5, \\
& \mathbf{x}, \mathbf{v} \geq 0
\end{aligned}$$

Solution is:

$$X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

and $W = 6$

As $x_1 \to 1$, $v_3 \to 0$, so v_3 is the leaving basic variable and x_1 going in as basis. The new basis is (x_1, v_1, v_2) and dictionary will be:

$$\begin{aligned}
\text{minimize} \quad & W = 4 - 2x_2 + x_3 + x_4 + v_3 \\
\text{subject to} \quad & x_1 = 1 + 0.5x_2 - 0.5x_5 - 0.5v_3, \\
& v_1 = 2 - 0.5x_2 + x_3 - 0.5x_5 - 0.5v_3, \\
& v_2 = 2 - 1.5x_2 + x_4 + 0.5x_5 + 0.5v_3, \\
& \mathbf{x}, \mathbf{v} \geq 0
\end{aligned}$$

Solution is:

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

and $W = 4$

As $x_2 \to 4/3$, $v_2 \to 0$, so v_2 out and x_2 in in basis. The new basis is (x_1, x_2, v_1) and dictionary will be:

$$
\begin{aligned}
\text{minimize} \quad & W = 1.33 + x_3 - 0.33x_4 - 0.67x_5 + 1.33v_2 + 0.33v_3 \\
\text{subject to} \quad & x_1 = 1.67 + 0.33x_4 - 0.33x_5 - 0.33v_2 - 0.33v_3, \\
& x_2 = 1.33 + 0.67x_4 + 0.33x_5 - 0.67v_2 + 0.33v_3, \\
& v_1 = 1.33 + x_3 - 0.33x_4 - 0.67x_5 + 0.33v_2 - 0.67v_3, \\
& \mathbf{x}, \mathbf{v} \geq 0
\end{aligned}
$$

Solution is:

$$
X = \begin{bmatrix} 1.67 \\ 1.33 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
V = \begin{bmatrix} 1.33 \\ 0 \\ 0 \end{bmatrix}
$$

and $W = 1.33$

As $x_5 \to 2$, $v_1 \to 0$, so v_1 out and x_5 in in basis. The new basis is (x_1, x_2, x_5) and dictionary will be:

$$
\begin{aligned}
\text{minimize} \quad & W = 0 + v_1 + v_2 + v_3 \\
\text{subject to} \quad & x_1 = 1 - 0.5x_3 + 0.5x_4 + 0.5v_1 - 0.5v_2, \\
& x_2 = 2 + 0.5x_3 + 0.5x_4 - 0.5v_1 - 0.5v_2, \\
& x_5 = 2 + 1.5x_3 - 0.5x_4 - 1.5v_1 + 0.5v_2, \\
& \mathbf{x}, \mathbf{v} \geq 0
\end{aligned}
$$

Solution is:

$$
X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}
$$

$$
V = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

and $W = 0$
So, phase 1 is complete.

Phase-II
The LP is feasible. We will start phase 2 with x_1, x_2, x_5 as the basis:

$$
\begin{aligned}
\text{minimize} \quad & z = -5 + x_3 - 2x_4 \\
\text{subject to} \quad & x_1 = 1 - 0.5x_3 + 0.5x_4, \\
& x_2 = 2 + 0.5x_3 + 0.5x_4, \\
& x_5 = 2 + 1.5x_3 - 0.5x_4, \\
& \mathbf{x} \geq 0
\end{aligned}
$$

Solution is:

$$X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

and $z = -5$

As $x_4 \to 4$, $x_5 \to 0$. So, the new basis is $(x_1, x_2, x_4)$and dictionary will be:

$$\begin{aligned} \text{minimize} \quad & z = -13 - 5x_3 + 4x_5 \\ \text{subject to} \quad & x_1 = 3 + x_3 - x_5, \\ & x_2 = 4 + 2x_3 - x_5, \\ & x_4 = 4 + 3x_3 - 2x_5, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Solution is:

$$X = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

and $z = -13$

Although x_3 has a negative coefficient in the objective function, we could not consider x_3 as one of the basis because none of the existing basis converts to zero even if we consider $x_3 = 0$. So, this is our optimal solution for this LP with $z = -13$ and $x_1 = 3, x_2 = 4, x_3 = 0, x_4 = 4$, and $x_5 = 0$.