MATH 415: Assignment 3

Md Ismail Hossain

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Question 1:

1 (a)

Given that,

$$A = \begin{bmatrix} 2 & 3 & 3 & 3 \\ 4 & 0 & 3 & 0 \\ 3 & 4 & 3 & 3 \\ 4 & 4 & 1 & 0 \end{bmatrix}$$

The LU factorization steps of A in MATLAB presented below, with all the codes and outputs:

```
  A = [2,3,3,3;4,0,3,0;3,4,3,3;4,4,1,0]

A =

     2     3     3     3
     4     0     3     0
     3     4     3     3
     4     4     1     0

I4 = eye(4)

I4 =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1

P1 = I4([2 1 3 4],:)

P1 =

     0     1     0     0
     1     0     0     0
     0     0     1     0
     0     0     0     1

P1*A

ans =

     4     0     3     0
     2     3     3     3
     3     4     3     3
     4     4     1     0

L1=I4; L1(2,1)= -0.5; L1(3,1) = -0.75; L1(4,1) = -1

L1 =

    1.0000         0         0         0
   -0.5000    1.0000         0         0
   -0.7500         0    1.0000         0
   -1.0000         0         0    1.0000

L1*P1*A
```

```
ans =

    4.0000         0    3.0000         0
         0    3.0000    1.5000    3.0000
         0    4.0000    0.7500    3.0000
         0    4.0000   -2.0000         0

P2 = I4([1,3,2,4],:)

P2 =

     1     0     0     0
     0     0     1     0
     0     1     0     0
     0     0     0     1

P2*L1*P1*A

ans =

    4.0000         0    3.0000         0
         0    4.0000    0.7500    3.0000
         0    3.0000    1.5000    3.0000
         0    4.0000   -2.0000         0

L2=I4; L2(3,2) = -0.75; L2(4,2) = -1

L2 =

    1.0000         0         0         0
         0    1.0000         0         0
         0   -0.7500    1.0000         0
         0   -1.0000         0    1.0000

L2*P2*L1*P1*A

ans =

    4.0000         0    3.0000         0
         0    4.0000    0.7500    3.0000
         0         0    0.9375    0.7500
         0         0   -2.7500   -3.0000

P3 = I4([1,2,4,3],:)

P3 =

     1     0     0     0
     0     1     0     0
     0     0     0     1
     0     0     1     0

L3 = I4; L3(4,3) = 0.9375/2.75

L3 =

    1.0000         0         0         0
         0    1.0000         0         0
```

```
         0         0    1.0000         0
         0         0    0.3409    1.0000

L3*P3*L2*P2*L1*P1*A

ans =

    4.0000         0    3.0000         0
         0    4.0000    0.7500    3.0000
         0         0   -2.7500   -3.0000
         0    0.0000    0.0000   -0.2727

U = ans

U =

    4.0000         0    3.0000         0
         0    4.0000    0.7500    3.0000
         0         0   -2.7500   -3.0000
         0    0.0000    0.0000   -0.2727

P = P3*P2*P1

P =

     0     1     0     0
     0     0     1     0
     0     0     0     1
     1     0     0     0

L = P*inv(P1)*inv(L1)*inv(P2)*inv(L2)*inv(P3)*inv(L3)

L =

    1.0000         0         0         0
    0.7500    1.0000         0         0
    1.0000    1.0000    1.0000         0
    0.5000    0.7500   -0.3409    1.0000

P*A

ans =

     4     0     3     0
     3     4     3     3
     4     4     1     0
     2     3     3     3

L*U

ans =

     4     0     3     0
     3     4     3     3
     4     4     1     0
     2     3     3     3
```

1(b)

The solution techniques of $Ax = b$ using LU factorization showing below:

```
    %Question 1(b):

b = [24; 12; 28; 16]

b =

    24
    12
    28
    16

% step 1: w = L\P*b
w = L\P*b

w =

   12.0000
   19.0000
  -15.0000
   -1.3636

%step 2: x = U\w
x = U\w

x =

    3.0000
    1.0000
   -0.0000
    5.0000

%Checking: the column should be all zero entry
A*x-b

ans =

   1.0e-14 *

         0
         0
         0
   -0.1776

% Another way of checking the solution:
x = A\b

x =

     3
     1
     0
     5


% The solution that we found using the component of LU factorization
```

```
%is matching with the direct approach
```

1 (c)

The MATLAB steps and outputs for solving $y'A = c'$ using LU factorization presented below:

```
    % Qustion 1(c):

%Step 1: w = (U')\c

c = [26;25;16;9]

c =

    26
    25
    16
     9

w = ((U')\c)

w =

    6.5000
    6.2500
    2.9773
    3.0000

% step 2: v = (L')\w
v = (L')\w

v =

    1.0000
   -0.0000
    4.0000
    3.0000

% step 3: y = P'*v
y = P'*v

y =

    3.0000
    1.0000
   -0.0000
    4.0000

%checking:

y'*A-c'

ans =

     0     0     0     0

% Looks good because all values are 0 !
```

Question 2:

The standard equation form with slack variables:

$$\begin{aligned} \text{minimize} \quad & z = -5x_1 - 6x_2 - 9x_3 - 8x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 5, \\ & x_1 + x_2 + 2x_3 + 3x_4 + x_6 = 3, \\ & \mathbf{x} \geq 0 \end{aligned}$$

All the steps with MATLAB codes and outputs presented below:

```
    % Entering A, b, c
A = [1 2 3 1 1 0;1 1 2 3 0 1];
b=[5; 3];
c = [-5;-6;-9;-8;0;0];
% Initial basis = x5, x6
basis = [5 6];
nonbasis = [1:4];

basisupdate

y =

     0
     0


x =

     0
     0
     0
     0
     5
     3


z =

     0


ans =

    -5    -6    -9    -8
     1     2     3     4

%by Dantzig's rule, x3 will enter the basis
% we want he coefficients of x3 from inv(B)*AN

a = A(:,3);
d = solveBxb(L,U,p,a);
xb

xb =
```

```
     5
     3


% we want to maximize t subject to [5;3] - t*[3;2] >= 0
% x6 will decrease to 0 and leave the basis
% New basis is x3, x5

basis = [3 5];
nonbasis = [1 2 4 6];
basisupdate

y =

   -0.0000
   -4.5000


x =

         0
         0
    1.5000
         0
    0.5000
         0


z =

  -13.5000


ans =

   -0.5000   -1.5000    5.5000    4.5000
    1.0000    2.0000    4.0000    6.0000


% x2  enter by Dantzig's rule
a = A(:,2);
d = solveBxb(L,U,p,a);
xb

xb =

    1.5000
    0.5000



% we want to maximize t subject to [1.5;0.5] - t*[0.5;0.5] >= 0
% x5 will decrease to 0 and leave the basis
% New basis is x2, x3


basis = [2 3];
nonbasis = [1 4 5 6];
```

```
basisupdate

y =

    -3
     0


x =

     0
     1
     1
     0
     0
     0


z =

   -15


ans =

    -2    -5     3     0
     1     4     5     6


% x4  enter by Dantzig's rule
a = A(:,4);
d = solveBxb(L,U,p,a);
xb

xb =

     1
     1


% we want to maximize t subject to [1;1] - t*[-7;5] >= 0
% x3 will decrease to 0 and leave the basis
% New basis is x2, x4

basis = [2 4];
nonbasis = [1 3 5 6];
basisupdate

y =

    -2
    -2


x =

         0
    2.4000
```

```
         0
    0.2000
         0
         0


z =

   -16


ans =

    -1     1     2     2
     1     3     5     6


% x1  enter by Dantzig's rule
a = A(:,1);
d = solveBxb(L,U,p,a);
xb

xb =

    2.4000
    0.2000



% we want to maximize t subject to [2.4;.2] - t*[.4;.2] >= 0
% x4 will decrease to 0 and leave the basis
% New basis is x1,x2

basis = [1 2];
nonbasis = [3 4 5 6];
basisupdate

y =

    -1
    -4


x =

     1
     2
     0
     0
     0
     0


z =

   -17
```

```
ans =

     2     5     1     4
     3     4     5     6

% so found our optimal solution for basis x1 = 1 and x2 = 2 with  z = -17.
```

We got an optimal solution for this LP where objective function $z = -17$ and we have to use $x_1 = 1$, $x_2 = 2$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$ and $x_6 = 0$.

Question 3:

We will use the two-phase Simplex method to solve the given LP problem. The problem is given as follows:

$$\begin{aligned} \text{minimize} \quad & -3x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq -1, \\ & -x_1 - x_2 \leq -3, \\ & 2x_1 - x_2 \leq 2, \\ & \mathbf{x} \geq 0 \end{aligned}$$

The standard equation form with slack variables:

$$\begin{aligned} \text{minimize} \quad & z = -3x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 + x_3 = -1, \\ & -x_1 - x_2 + x_4 = -3, \\ & 2x_1 - x_2 + x_5 = 2, \\ & \mathbf{x} \geq 0 \end{aligned}$$

We cannot use x_3, x_4, x_5 as the initial basis for this problem because the solution would become infeasible in that case (we would get a negative value for x_3 and x_4). As a result, we have to use the two-phase simplex method.

Phase-I

```
% Phase-1


A = [1 -1 1 0 0 -1 0 0;-1 -1 0 1 0 0 -1 0;2 -1 0 0 1 0 0 1]

A =

     1    -1     1     0     0    -1     0     0
    -1    -1     0     1     0     0    -1     0
     2    -1     0     0     1     0     0     1

b = [-1;-3;2]

b =

    -1
    -3
     2

c = [0;0;0;0;0;1;1;1]

c =

     0
     0
     0
     0
     0
     1
     1
     1

%Initial basis x6, x7, x8
basis = [6 7 8]

basis =

     6     7     8

nonbasis = [1:5]

nonbasis =

     1     2     3     4     5

basisupdate

y =

    -1
    -1
     1


x =

     0
```

```
     0
     0
     0
     0
     1
     3
     2


z =

     6


ans =

    -2    -1     1     1    -1
     1     2     3     4     5

%by Dantzig's rule, x1 will enter the basis
% we want he coefficients of x1 from inv(B)*AN

a = A(:,1)

a =

     1
    -1
     2

d = solveBxb(L,U,p,a)

d =

    -1
     1
     2

xb

xb =

     1
     3
     2

% we want to maximize t subject to [1;3;2] - t*[-1;1;2] >= 0
% x8 will decrease to 0 and leave the basis
% New basis is x1, x6, x7

basis = [1 6 7]

basis =

     1     6     7

nonbasis = [2 3 4 5 8]
```

```
nonbasis =

     2     3     4     5     8

basisupdate

y =

    -1
    -1
     0


x =

     1
     0
     0
     0
     0
     2
     2
     0


z =

     4


ans =

    -2     1     1     0     1
     2     3     4     5     8

%by Dantzig's rule, x2 will enter the basis
% we want he coefficients of x2 from inv(B)*AN


a = A(:,2)

a =

    -1
    -1
    -1

d = solveBxb(L,U,p,a)

d =

   -0.5000
    0.5000
    1.5000

xb

xb =
```

```
     1
     2
     2

% we want to maximize t subject to [1;2;2] - t*[-0.5;0.5;1.5] >= 0
% x7 will decrease to 0 and leave the basis
% New basis is x1,x2, x6


basis = [1 2 6]

basis =

     1     2     6

nonbasis = [3 4 5 7 8]

nonbasis =

     3     4     5     7     8

basisupdate

y =

   -1.0000
    0.3333
    0.6667


x =

    1.6667
    1.3333
         0
         0
         0
    1.3333
         0
         0


z =

    1.3333


ans =

    1.0000   -0.3333   -0.6667    1.3333    0.3333
    3.0000    4.0000    5.0000    7.0000    8.0000

%by Dantzig's rule, x2 will enter the basis
% we want he coefficients of x2 from inv(B)*AN

a = A(:,5)
```

```
a =

     0
     0
     1

d = solveBxb(L,U,p,a)

d =

    0.3333
   -0.3333
    0.6667

xb

xb =

    1.6667
    1.3333
    1.3333

% we want to maximize t subject to [1.6667;1.3333;1.3333] - t*[0.3333;-0.3333;0.6667] >= 0
% x6 will decrease to 0 and leave the basis
% New basis is x1,x2, x5

basis = [1 2 5]

basis =

     1     2     5

nonbasis = [3 4 6 7 8]

nonbasis =

     3     4     6     7     8

basisupdate

y =

     0
     0
     0


x =

    1.0000
    2.0000
         0
         0
    2.0000
         0
         0
         0
```

```
z =

     0


ans =

     0     0     1     1     1
     3     4     6     7     8


% we have finished phase 1 because this an optimal solution with z = 0. So,
% for phase 2 we can consider x1, x2, x5 as the starting basis
```

Phase-II

The Phase-I LP is feasible. We will start phase 2 with x_1, x_2, x_5 as the basis:

```
% Phase 2

A = A(:,1:5)

A =

     1    -1     1     0     0
    -1    -1     0     1     0
     2    -1     0     0     1

c = [-3;-1;0;0;0]

c =

    -3
    -1
     0
     0
     0


basis = [1 2 5]

basis =

     1     2     5

nonbasis = [3 4]

nonbasis =

     3     4


basisupdate

y =

    -1
```

```
     2
     0


x =

    1.0000
    2.0000
         0
         0
    2.0000


z =

    -5


ans =

     1    -2
     3     4

%by Dantzig's rule, x4 will enter the basis
% we want he coefficients of x4 from inv(B)*AN

a = A(:,4)

a =

     0
     1
     0

d = solveBxb(L,U,p,a)

d =

   -0.5000
   -0.5000
    0.5000

xb

xb =

    1.0000
    2.0000
    2.0000

% we want to maximize t subject to [1;2;2] - t*[-0.5;-0.5;.5] >= 0
% x5 will decrease to 0 and leave the basis
% New basis is x1,x2, x4

basis = [1 2 4]

basis =
```

```
     1     2     4

nonbasis = [3 5]

nonbasis =

     3     5


basisupdate

y =

    5.0000
   -0.0000
   -4.0000


x =

    3.0000
    4.0000
         0
    4.0000
         0


z =

   -13


ans =

   -5.0000    4.0000
    3.0000    5.0000

%by Dantzig's rule, x3 will enter the basis
% we want he coefficients of x3 from inv(B)*AN

a = A(:,3)

a =

     1
     0
     0

d = solveBxb(L,U,p,a)

d =

    -1
    -2
    -3

xb
```

```
xb =

    3.0000
    4.0000
    4.0000

% we want to maximize t subject to [3;4;4] - t*[-1;-2;-3] >= 0
```

The LP is unbounded because x_3 increase without bound.