MATH 415: Assignment 7

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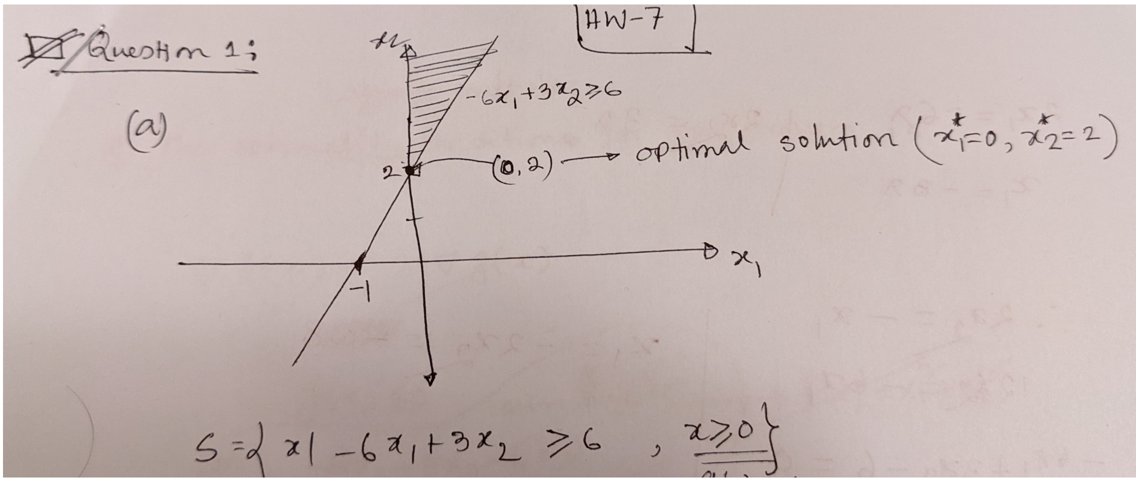
Question 1:

a



b

Constraint optimization problem,

$$\begin{aligned} \text{minimize} \quad & d^2 = x_1^2 + x_2^2 \\ \text{subject to} \quad & -6x_1 + 3x_2 \geq 6, \\ & x \geq 0 \end{aligned}$$

c

We have to,

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1^2 + x_2^2 \\ \text{subject to} \quad & g(x) = -6x_1 + 3x_2 - 6 \geq 0, \\ & x \geq 0 \end{aligned}$$

Lagrange multiplier conditions for the problem are,

$$g(x) = -6x_1 + 3x_2 - 6 \geq 0 \tag{1}$$

$$\nabla f(x) = \lambda \nabla g(x) \tag{2}$$

where, $\nabla f(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$
and
$\nabla g(x) = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$
and λ is called the Lagrange multiplier.

$$\lambda g(x) = 0 \tag{3}$$

Where $\lambda \geq 0$.

d

Since $\nabla f(x)$ is a non-zero constant, there cannot be a minimum point in the interior of the feasible region with $\nabla f(x) = 0$. Need to solve the following equations in Maple:

$$-6x_1 + 3x_2 - 6 = 0 \tag{4}$$

$$2x_1 = -6\lambda \tag{5}$$

$$2x_2 = 3\lambda \tag{6}$$

After using the Maple "solve" command we got the following solution:

```
solve({2*x1=-6*lambda,2*x2=3*lambda,-6*x1+3*x2-6=0})
```

$$\left\{\lambda = \frac{4}{15}, x1 = -\frac{4}{5}, x2 = \frac{2}{5}\right\}$$

$x_1 = -4/5$, $x_2 = 2/5$ and $\lambda = 4/15$. The solution is spurious. As we see x_1 is less than 0 which is violating the condition that $x \geq 0$. So, the solution is infeasible.

e

In Newton's method, we need to use the following values to input into the MATLAB program,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix}$$

$$F(x) = \begin{bmatrix} 2x_1 + 6\lambda \\ 2x_2 - 3\lambda \\ -6x_1 + 3x_2 - 6 \end{bmatrix}$$

$$J(x) = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & -3 \\ -6 & 3 & 0 \end{bmatrix}$$

```
    xstar = newtonnneg(@F1,@J1,[1;2;1],100,1.0e-13)

xstar =

    0.0000
    1.1111
    0.5926
```

After using the function "newtonnneg" in MATLAB we found the numerical solution $x_1 = 0.0000$, $x_2 = 1.1111$, and $\lambda = 0.5926$.

Question 2:

The given quadratic programming problem is,

$$\begin{aligned} &\text{minimize} \quad && f(x) = (1/2)x'Qx + c'x + p \\ &\text{subject to} \quad && Ax = b, \\ & && x \geq 0 \end{aligned}$$

a

The KKT conditions are:

$$\begin{gathered} Ax = b \\ Qx + c = A'y + z \\ x_i z_i = 0, i = 1, 2, ..., n \\ x \geq 0 \\ z \geq 0 \end{gathered}$$

in the form $F(x, y, z) = 0$.

$$F(x,y,z) = \begin{bmatrix} A'y + z - Qx - c \\ Ax - b \\ x_i z_i \end{bmatrix} = 0, \text{ where } i = 1, 2, ..., n$$

$$= \begin{bmatrix} A'y + z - Qx - c \\ Ax - b \\ xz\epsilon \end{bmatrix} = 0 \text{ where } x, z \geq 0$$

$$xz\epsilon = \begin{bmatrix} x_1 z_1 \\ x_2 z_2 \\ ... \\ x_n z_n \end{bmatrix}$$

b

By Newton's method, we need J.

$$J(x,y,z) = \begin{bmatrix} -Q & A' & I \\ A & 0 & 0 \\ z & 0 & x \end{bmatrix}$$

To get Δx, Δy and Δz, we solve:

$$\begin{bmatrix} -Q & A' & I \\ A & 0 & 0 \\ z & 0 & x \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} -A'y - z + Qx + c \\ -Ax + b \\ -xz\epsilon \end{bmatrix}$$

c

Reduce linear system of equations:

$$-Q\Delta x + A'\Delta y + \Delta z = -A'y - z + Qx + c \tag{7}$$

$$z\Delta x + x\Delta z = -xz\epsilon$$

Multiply x^{-1} and solve for Δz:

$$\begin{gathered} \Delta z = x^{-1}(-z\Delta x - xz\epsilon) \\ \Delta z = -x^{-1}z\Delta x - z\epsilon \end{gathered}$$

Now, substitute Δz in equation (7):

$$\begin{gathered}
-Q\Delta x + A'\Delta y - x^{-1}z\Delta x - z\epsilon = -A'y - z + Qx + c \\
A'\Delta y - x^{-1}z\Delta x = -A'y - z + Qx + c + z\epsilon + Q\Delta x \\
A'\Delta y - x^{-1}z\Delta x = -A'y + Q(x + \Delta x) + c \\
-(x^{-1}z + Q)\Delta x + A'\Delta y = -A'y + Qx + c
\end{gathered}$$

We also have another equation:

$$A\Delta x = -Ax + b$$

$$\begin{bmatrix} -(x^{-1}z + Q) & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -A'y + Qx + c \\ -Ax + b \end{bmatrix}$$

This reduced system has $n + m$ equations in the $m + n$ variables Δx and Δy.

Question 3:

$A\Theta A'$ is symmetric

If we can prove, $A\Theta A' = (A\Theta A')'$ the we can say $A\Theta A'$ is symmetric.

$$(A\Theta A')' = (A')'(A\Theta)'$$

We know $(AB)' = B'A'$. So,

$$(A\Theta A')' = A\Theta' A'$$
$$(A\Theta A')' = A\Theta A'$$

Because Θ is a diagonal matrix. So, we can say $A\Theta A'$ is a symmetric matrix.

$A\Theta A'$ is positive definite

To prove $A\Theta A'$ positive definite we need to show $v'(A\Theta A')v > 0$ for all $v > 0$. To do that we can rewrite $A\Theta A'$ in following way:

$$A\Theta A' = A\sqrt{\Theta}.\sqrt{\Theta}A'$$

As we know Θ is a diagonal matrix with positive entries only. Now we can show,

$$v'(A\sqrt{\Theta}.\sqrt{\Theta}A')v$$
$$= (v'A\sqrt{\Theta})(\sqrt{\Theta}A'v)$$
$$= (v'A\sqrt{\Theta})((\sqrt{\Theta}A)'v)'$$

As $\sqrt{\Theta}$ is also a diagonal matrix with positive entries,

$$(v'A\sqrt{\Theta})((\sqrt{\Theta}A)'v)'$$
$$= (v'A\sqrt{\Theta})(v'A\sqrt{\Theta})$$
$$= (v'A\sqrt{\Theta})^2 > 0$$

So, it's proved that $v'(A\Theta A')v > 0$, meaning $A\Theta A'$ is positive definite.