Nineteenth International Mathematical Olympiad, 1977

1977/1 Geometry

1. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AKBK$, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon.

1977/2 Number Theory

2. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

197/3 Number Theory

3. Let n be a given integer >2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, \ldots A$ number $m \epsilon V_n$ is called indecomposable in V_n, if there do not exist numbers p ,$q \epsilon V_n$ such that $pq = m$. Prove that there exists a number $r \epsilon V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (products which differ only in the order of their factors will be considered the same).

1977/4 Trigonometry

4. Four real constants a, b, A, B are given, and

$$f(\theta) = 1 - acos\theta - bsin\theta - Acos2\theta - Bsin2\theta \tag{1}$$

. Prove that if

$$f(\theta) >= 0 \tag{2}$$

,for all real θ, then

$$a^2 + b^2 \leq 2 and A^2 + B^2 \geq= 1 \tag{3}$$

1977/5 Number Theory

5. Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

1977/6 Number Theory

6. Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n+1) > f(f(n)) \tag{4}$$

for each positive integer n, then

$$f(n) = n \tag{5}$$

for each n

Twentieth International Olympiad, 1978

1978/1 Number Theory

7. m and n are natural numbers with $1 \leq m<n$ In their decimal representations, the last three digits of 1978 are equal, respectively, to the last three digits of 1978". Find m and n such that $m+n$ has its least value.

1978/2 Geometry

8. P is a given point inside a given sphere. Three mutually perpendicular rays from Pintersect the sphere at points U, V, and W; Q denotes the vertex diagonally opposite to P in the parallelepiped determined by PU, PV, and PW. Find the locus of Q for all such triads of rays from P

1978/3 Number Theory

9. The set of all positive integers is the union of two disjoint subsets

$$f(1), f(2), \ldots, f(n), \ldots, g(1), g(2), \ldots, g(n), \ldots \tag{6}$$

,where

$$f(1)<f(2)<\ldots<f(n)<\ldots, \tag{7}$$

$$g(1)<g(2)<\ldots<g(n)<\ldots \tag{8}$$

$$, and, \quad g(n)=f(f(n))+1 \tag{9}$$

for all $n \geq 1$. and Determine (240).

1978/4 Geometry

10. In triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at $P.Q$, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC.

19778/5 Number Theory

11. Let a_k $(k = 1, 2, 3. \ldots, n, \ldots)$ be a sequece of distinct positive integers. Prove that for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k} \tag{10}$$

1978/6 Combinatorics

12. An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots,$ 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

Twenty-first International Olympiad, 1979

1979/1 Number Theory

13. Let p and q be natural numbers such that

$$\frac{p}{q} = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319} \tag{11}$$

.Prove that p is divisible by 1979.

1979/2 Geometry

14. A prism with pentagons $A1A2A3A4A5$ and $B1B2B3B4B5$, as top and bottom faces is given. Each side of the two pentagons and each of the line- segments A, B for all $i, j = 1, \ldots, 5$, is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color.

1979/3 Geometry

15. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.

1979/4 Geometry

16. Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $(QP + PA)/QR$ is a maximum.

1979/5 Algebra

17. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5} kx_k = a, \sum k = 15k^3 x_k = a^2, \sum k = 15k^5 x_k = a^3 \qquad (12)$$

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1979/6 Combinatorics

18. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there.. Let a be the number of distinct paths of exactly n jumps ending at E. Prove that

$$a_2 n - 1 = 0, a_{2n} = \frac{1}{\sqrt{2}} \left(x^{n-1} - y^{n-1} \right) \qquad (13)$$

, $n = 1, 2, 3, \ldots,$
where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. Note. A path of a jumps is a sequence of vertices $(P_0 \ldots P_n)$ such that

(a) $PA, P = E$

(b) for every $i, 0 \leq i \leq n - 1, P$ is distinct from E;

(c) for every $i, 0 \leq i \leq n - 1P.$ and P_{i+1} are adjacent.