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CSC470 Prof Zhu

Assignment 3

EMPL: 8834

1. (Camera Models- 30 points) Prove that the vector from the viewpoint of a pinhole camera to the vanishing point (which is a point on the image plane) of a set of 3D parallel lines in space is parallel to the direction of that set parallel lines. Please show steps of your proof.

Hint: You can either use geometric reasoning or algebraic calculation.

If you choose to use geometric reasoning, you can use the fact that the projection of a 3D line in space is the intersection of its "interpretation plane" with the image plane. Here the interpretation plane (IP) is a plane passing through the 3D line and the center of projection (viewpoint) of the camera. Also, the interpretation planes of two parallel lines intersect in a line passing through the viewpoint, and the intersection line is parallel to the parallel lines.

If you select to use algebraic calculation, you may use the parametric representation of a 3D line: $P = P0 + tV$, where $P= (X,Y,Z)^T$ is any point on the line (here ^{T} denote for transpose), $P0 = (X0,Y0,Z0)^T$ is a given fixed point on the line, vector $V = (a,b,c)^T$ represents the direction of the line, and t is the scalar parameter that controls the distance (with sign) between P and P0.

Proof (algebraic approach; I used MS Word LaTeX since it was much cleaner than handwriting):

Let's take the point in space $P_0 = (X_0\ Y_0\ Z_0)^T$ and a direction vector V = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Then we get: line $P = P_0 + tV$; where $P_0 = (X_0\ Y_0\ Z_0)$+ t$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

If the pinhole camera viewpoint is not in the same line, then the viewpoint alongside the line defines our interpretation plane.

The intersection of this plane with the image projection plane nets us our image projection of the line.

So, expressing this 3D line parametrically, we get:

$$X = X_0 + ta$$

$$Y = Y_0 + tb$$

$$Z = Z_0 + tc$$

Then applying the perspective equations $x = \frac{fX}{Z}$ and $y = \frac{fY}{Z}$, we get:

$$x = \frac{f(X_0 + ta)}{Z_0 + tc}$$

$$y = \frac{f(Y_0 + tb)}{Z_0 + tc}$$

Given a set of parallel lines, then our vanishing points is a point at infinity in the direction of the lines.

This vanishing point is defined as $t \rightarrow \infty$ where the image points are:

$$x = \frac{f(\frac{X_0}{t} + a)}{\frac{Z_0}{t} + c} \rightarrow \frac{f(\frac{X_0}{\infty} + a)}{\frac{Z_0}{\infty} + c} \rightarrow \frac{fa}{c}$$

$$y = \frac{f(\frac{Y_0}{t} + b)}{\frac{Z_0}{t} + c} \rightarrow \frac{f(\frac{Y_0}{\infty} + b)}{\frac{Z_0}{\infty} + c} \rightarrow \frac{fb}{c}$$

The direction vector defines the vanishing point, which means that a starting point can be varied, and any set of parallel lines will still converge to the same vanishing point.

Since the vector from the pinhole camera viewpoint shares a vanishing point with the set of 3D parallel lines, therefore it parallels the direction of the parallel lines.

QED

2. (Camera Models- 20 points) Show that relation between any image point $(xim, yim)^T$ (in the form of $(x1,x2,x3)^T$ in projective space) of a planar surface in 3D space and its corresponding point $(Xw, Yw, Zw)^T$ on the plane in 3D space can be represented by a 3×3 matrix. You should start from the general form of the camera model $(x1,x2,x3)^T = M_{int}M_{ext}(Xw, Yw, Zw, 1)^T$, where the image center (ox, oy), the focal length f, the scaling factors(sx and sy), the rotation matrix R and the translation vector T are all unknown. Note that in the course slides and the lecture notes, I used a simplified model of the perspective project by assuming ox and oy are known and sx = sy =1, and only discussed the special cases of a plane. So you cannot directly copy those equations I used. Instead you should use the general form of the projective matrix, and the general form of a plane $n_x X_w + n_y Y_w + n_z Z_w = d$.

Proof:

Using the relationship between an image point and its corresponding point in 3D space equation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = M_{int} M_{ext} \begin{pmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{pmatrix} \text{ where } \begin{pmatrix} x_{im} \\ y_{im} \end{pmatrix} = \begin{pmatrix} \frac{x_1}{x_3} \\ \frac{x_2}{x_3} \end{pmatrix} \text{ alongside}$$

$$M_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_x \\ r_{21} & r_{22} & r_{23} & T_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} \text{ and } M_{int} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = M_{int} M_{ext} = \begin{bmatrix} -f_x r_{11} + o_x r_{31} & -f_x r_{12} + o_x r_{23} & -f_x r_{13} + o_x r_{33} & -f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} & -f_y r_{22} + o_y r_{23} & -f_y r_{23} + o_y r_{33} & -f_y T_y + o_y T_z \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$$

Using our given equation of the plane:

$$n_x X_w + n_y Y_w + Z_w = d$$

$$\rightarrow Z_w = d - n_x X_w - n_y Y_w$$

We get:

$$\begin{bmatrix} -f_x r_{11} + o_x r_{31} & -f_x r_{12} + o_x r_{23} & -f_x r_{13} + o_x r_{33} & -f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} & -f_y r_{22} + o_y r_{23} & -f_y r_{23} + o_y r_{33} & -f_y T_y + o_y T_z \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ d - n_x X_w - n_y Y_w \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} (-f_x r_{11} + o_x r_{31}) X_w + (-f_x r_{12} + o_x r_{23}) Y_w + (-f_x r_{13} + o_x r_{33})(d - n_x X_w - n_y Y_w) - f_x T_x + o_x T_z \\ (-f_y r_{21} + o_y r_{31}) X_w + (-f_y r_{22} + o_y r_{23}) Y_w + (-f_y r_{23} + o_y r_{33})(d - n_x X_w - n_y Y_w) - f_y T_y + o_y T_z \\ r_{31} X_w + r_{32} Y_w + r_{33}(d - n_x X_w - n_y Y_w) + T_z \end{bmatrix}$$

$$= \begin{bmatrix} -f_x r_{11} X_w + o_x r_{31} X_w - f_x r_{12} Y_w + o_x r_{23} Y_w - f_x r_{13} d + f_x r_{13} n_x X_w + f_x r_{13} n_y Y_w + o_x r_{33} d - o_x r_{33} n_x X_w - o_x r_{33} n_y Y_w - f_x T_x + o_x T_z \\ -f_y r_{21} X_w + o_y r_{31} X_w - f_y r_{22} Y_w + o_y r_{23} Y_w - f_y r_{23} d + f_y r_{23} n_x X_w + f_y r_{23} n_y Y_w + o_x r_{33} d - o_y r_{33} n_x X_w - o_y r_{33} n_y Y_w - f_y T_y + o_y T_z \\ r_{31} X_w + r_{32} Y_w + r_{33} d - r_{33} n_x X_w - r_{33} n_y Y_w + T_z \end{bmatrix}$$

$$= \begin{bmatrix} X_w(-f_x r_{11} + o_x r_{31} + f_x r_{13} n_x - o_x r_{33} n_x) + Y_w(-f_x r_{12} + o_x r_{23} + f_x r_{13} n_y - o_x r_{33} n_y) - f_x r_{13} d + o_x r_{33} d - f_x T_x + o_x T_z \\ X_w(-f_y r_{21} + o_y r_{31} + f_y r_{23} n_x - o_y r_{33} n_x) + Y_w(-f_y r_{22} + o_y r_{23} + f_y r_{23} n_y - o_y r_{33} n_y) - f_y r_{23} d + o_y r_{33} d - f_y T_y + o_y T_z \\ X_w(r_{31} - r_{33} n_x) + Y_w(r_{32} - r_{33} n_y) + r_{33} d + T_z \end{bmatrix}$$

$$= \begin{bmatrix} -f_x r_{11} + o_x r_{31} + f_x r_{13} n_x - o_x r_{33} n_x & -f_x r_{12} + o_x r_{23} + f_x r_{13} n_y - o_x r_{33} n_y & -f_x r_{13} d + o_x r_{33} d - f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} + f_y r_{23} n_x - o_y r_{33} n_x & -f_y r_{21} + o_y r_{31} + f_y r_{23} n_y - o_y r_{33} n_y & -f_y r_{23} d + o_y r_{33} d - f_y T_y + o_y T_z \\ r_{31} - r_{33} n_x & r_{32} - r_{33} n_y & r_{33} d + T_z \end{bmatrix} \begin{pmatrix} X_w \\ Y_w \\ 1 \end{pmatrix}$$

$$
=
\begin{bmatrix}
-f_x(r_{11} + r_{13}n_x) + o_x(r_{31} - r_{33}n_x) & -f_x(r_{12} - r_{13}n_y) + o_x(r_{23} - r_{33}n_y) & -f_x(-r_{13}d + T_x) + o_x(r_{33}d + T_z) \\
-f_y(r_{21} - r_{32}n_x) + o_y(r_{22} - r_{32}n_y) & -f_y(r_{22} - r_{32}n_y) + o_y(r_{23} - r_{33}n_y) & -f_y(-r_{32}d + T_y) + o_y(r_{33}d + T_z) \\
r_{31} - r_{33}n_x & r_{32} - r_{33}n_y & r_{33}d + T_z
\end{bmatrix}
\begin{pmatrix} X_w \\ Y_w \\ 1 \end{pmatrix}
$$

QED

3. (Calibration- 50 points) Prove the Orthocenter Theorem by geometric arguments: Let T be the triangle on the image plane defined by the three vanishing points of three mutually orthogonal sets of parallel lines in space. Then the image center is the orthocenter of the triangle T (i.e., the common intersection of the three altitudes. Note that you are asked to prove the Orthocenter Theorem rather than that the orthocenter itself as the common interaction of the three altitudes, which you can use as a fact.

(1) Basic proof: use the result of Question 1, assuming the aspect ratio of the camera is 1. (20 points)

Orthocenter Theorem:

Given 3 mutually orthogonal sets of parallel lines in an image, a triangle T is created on the image plane that is defined by the 3 vanishing points of these 3 sets of lines.

The common intersection of the 3 altitudes of T is the image center.

The vanishing points of 3 mutually orthogonal sets of parallel lines are the vertices of triangle T.

- Center of projection of a camera is O
- Three sets of mutually orthogonal parallel lines are L_1, L_2, and L_3
- Vanishing points of triangle T are V_1, V_2, and V_3.

In Q1, we proved that the *"vector from the viewpoint of a pinhole camera to the vanishing point (which is a point on the image plane) of a set of 3D parallel lines in space is parallel to the direction of that set parallel lines".*

Therefore:
OV_1 is perpendicular to V_2V_3,
OV_2 is perpendicular to V_1V_3,
OV_3 is perpendicular to V_1V_2.

Each sides' altitude to the opposite vanishing point are perpendicular to each other.
If our center of image is o, then:
- from the line center of projection of the camera,
- to the center of the image,
- to the image center
– Oo is perpendicular to the image plane.
QED

(2) If you do not know the focal length of the camera, can you still find the image center using the Orthocenter Theorem? Can you further estimate the focal length? For both questions, please show why (and then how) or why not. (20 points)

As already shown in the previous proof, you do not need to know the focal length to find the image center via the Orthocenter Theorem. We don't rely on any of the camera parameters to find the vanishing points, however, you can use these vanishing points to calculate the focal points if needed (which we don't need).

(3) If you do not know the aspect ratio of the camera, can you still find the image center using the Orthocenter Theorem? Show why or why not. (10 points)

The prior proof was done with the assumption of the aspect ratio (shown) since we used orthogonal relations of the altitude.

So, you do need to assume the aspect ratio. If we didn't know the aspect ratio, we would have gotten the incorrect image center (shown in figure 3).

Note: I'm referencing this slide from Calibration chapter (I mainly relied on this chapter for this assignment):

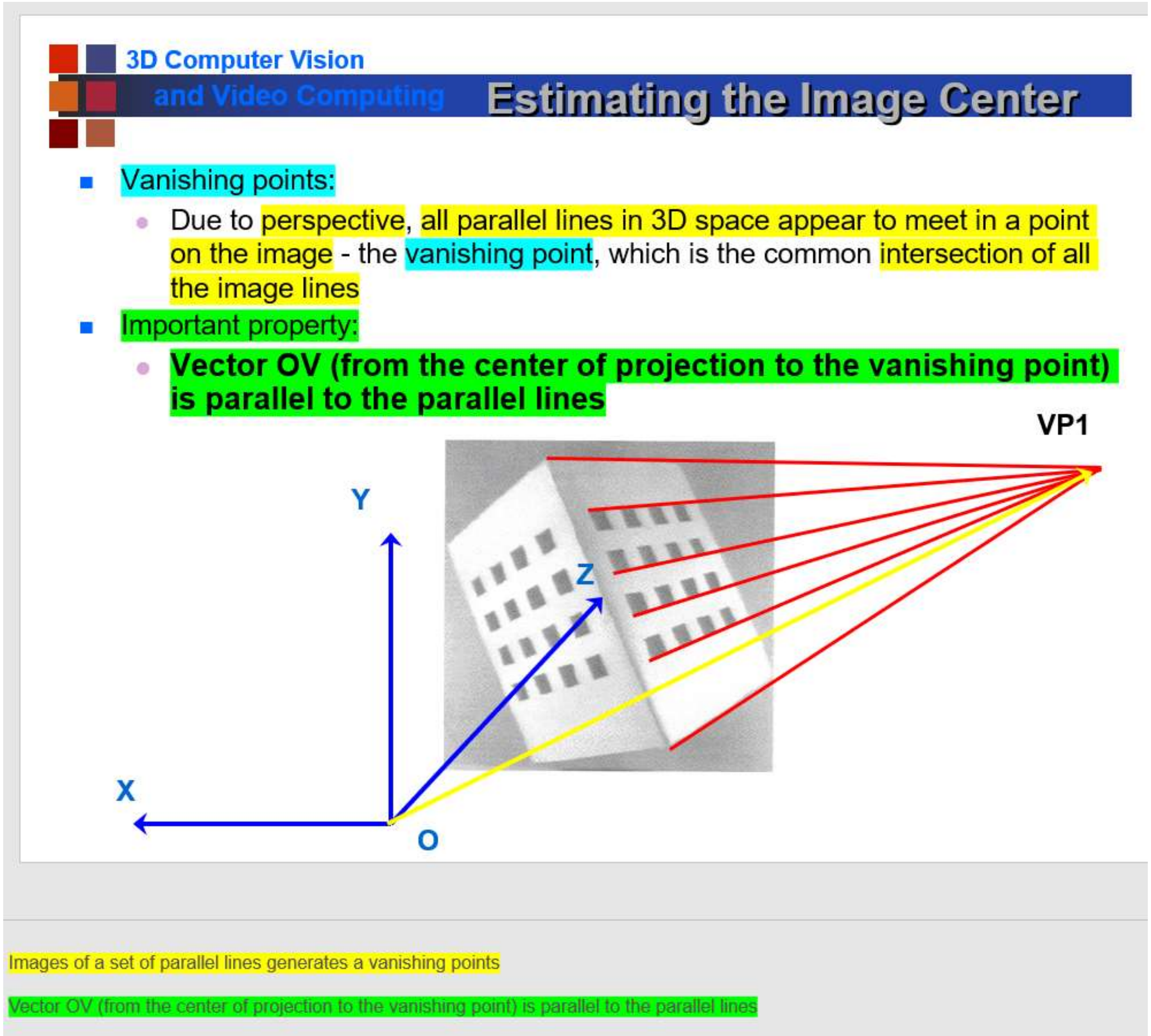


Figure 1. Vanishing points

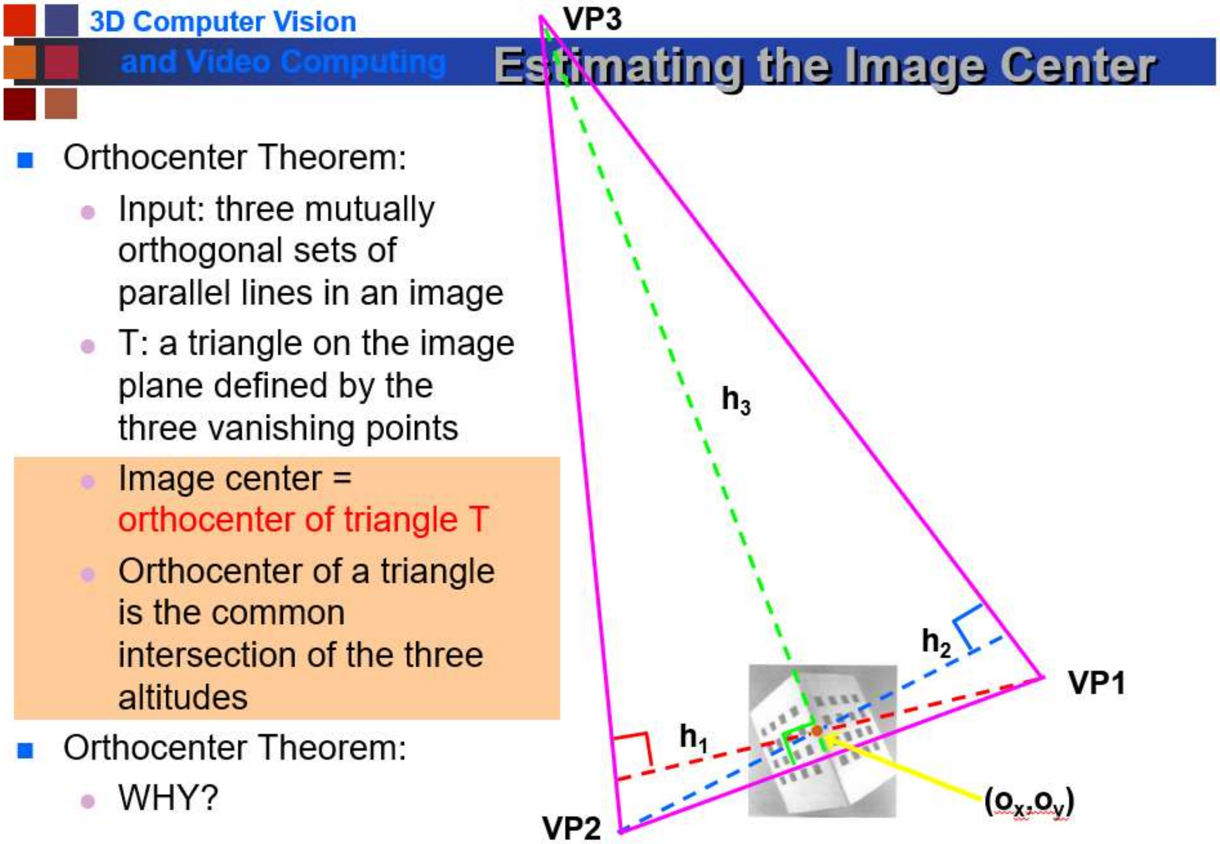
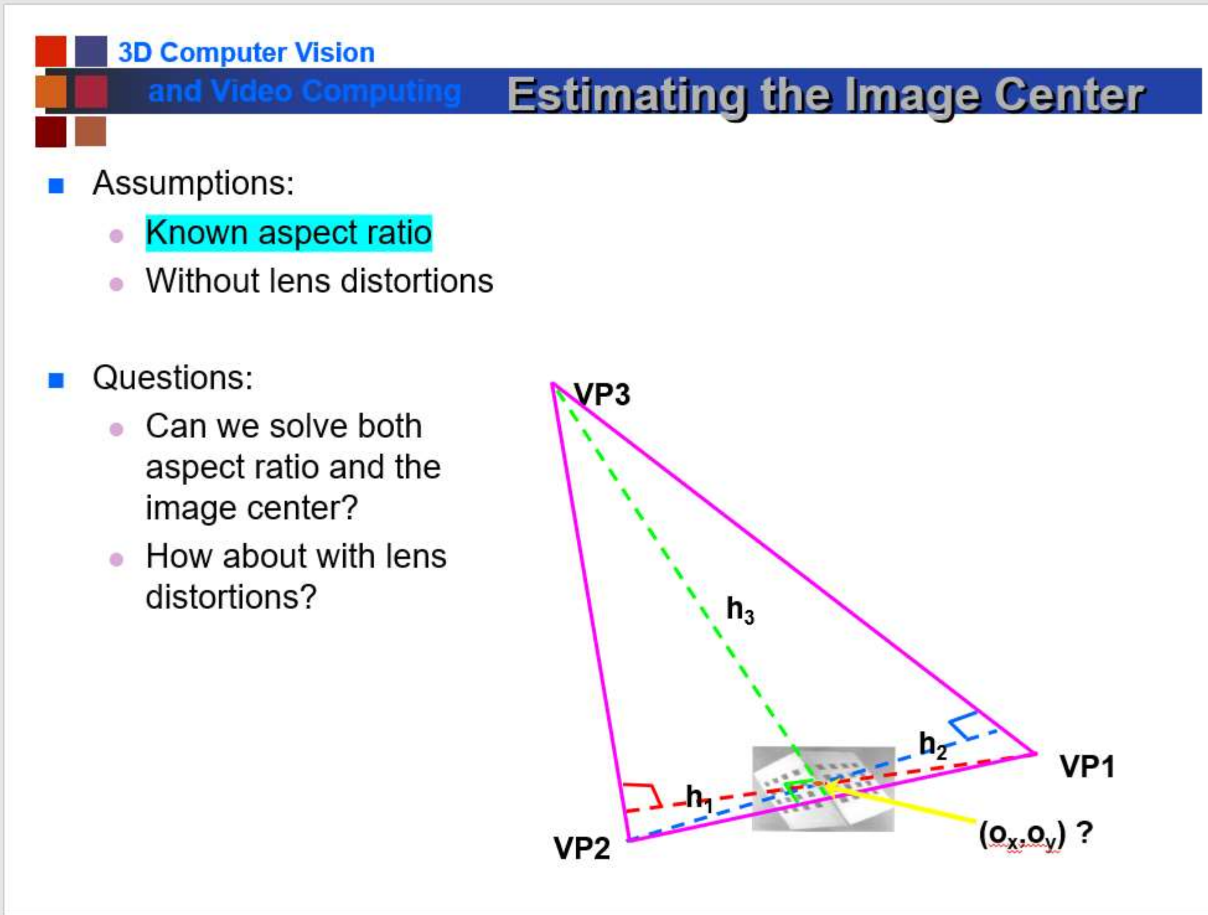


Figure 2. Orthocenter Theorem from lecture slides



Questions:

(1) Do we need to know the aspect ratio in advance?

In the figure, we exaggerate the (wrong) aspect ration. The vanishing points are correct; but the image center will not at (ox,oy) since it is not the altitude any more

Figure 3. Without the correct aspect ratio assumption, we get an incorrect image center.