Game theory Summary

Ismail Fadelli

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1 Game Theory

Game Theroy is the study of rational behaviour in situations involving interdependece as it may involve:

- Common interest (coordination);
- Competing interests (rivalry);
- Rational behaviour: players can do the best they can, in their own eyes;
- Because of the players' interdependence, a rational decision in a game must be based on a prediction of others' responses;

1.1 The parts of a Game

A Game consists of three parts : Players, Actions and Payoffs

1.1.1 Players

Players are the decision makers and they can be : People, Governments or Companies.

1.1.2 Actions

What can the players do ? Decide when to sell a stock, decide how to vote or enter a bid in an auction...

1.1.3 Payoffs

Payoffs can represent the motivation of the players, for example : Do they care about profit ? or Do they care about other players ?

1.2 Defining Games

Games can be represented in two methods : Normal forms and Extensive Forms.

1.2.1 Normal Form : matrix form or strategic form

A normal form represents a list of what players get on function of their actions.

1.2.2 Extensive Form

An extensive forms includes timing of moves. Players move sequentially, represented as a tree.

- Chess: white player moves, then black player can see white's move and react...

Keeps track of what each player knows when he or she makes a decision :

- Poker: bet seqentially - what can a given player see when they bet.

1.3 Normal Form

Finit, n-person normal form game $\langle N, A, u \rangle$:

- Players: $N = 1, ..., n$ is a finite set of n, indexed by i.
- Actions set for player iA_i

  $a = (a_1, ..., a_n) \in A = A_1 * ... * A_n$ is an action profile.
- Utility function or Payoff function for player $i : u_i : A \rightarrow, \mathbb{R}$

  $u = (u_1, ..., u_n)$, is a profile of utility functions.

1.4 Best Response and Nash Equilibrium

1.4.1 Best Response

Definition: $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$

1.4.2 Nash Equilibrium

Definition : $a =< a_1, ..., a_n >$ is a "**pure strategy**" if $\forall i, a_i \in BR(a_{-i})$

1.5 Dominant strategies

let s_i and $s_i^{`}$ be two strategies for player i, and let S_{-i} be the set of all possible strategy profiles for other players.

Definitions:

- s_i **strictly dominates** $s_i^{`}$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \ > u_i(s_i^{`}, s_{-i})$
- s_i **very weakly dominates** $s_i^{`}$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i^{`}, s_{-i})$
- A strategy is called **dominant** if it dominates all others.
- A strategy profile consisting of dominant strategies for every player must be a Nash Equilibrium.

1.6 Pareto Optimality

Definition: An outcome o^* is **Pareto-optimal** if there is no other outcome that Pareto-dominates it.

2 Mixed Strategies and Nash Equilibrium

Definition: A strategy s_i for agent i as any probabiltiy distribution over the actions A_i.

- **pure strategy:** only one action is played with positive probability
- **mixed strategy:** more than one action is played with positive probability

  these actions are called the support of the mixed strategy.
- Let the set of all strategies for i be S_i
- let the set of all strategy profiles be $S = S_1 \times ... \times S_n$

2.1 Utility in Mixed Strategies

In order to find the payoff if all the players follow mixed strategy profile $s \in S$ we can use the **expected utility** from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) P(a|s) \tag{1}$$

$$P(a|s) = \prod_{j \in N} s_j(a_j) \tag{2}$$

2.2 Best Response and Nash Equilibrium

The definitions of best response and Nash equilibrium are generalized from actions to strategies.

Definition (Best Response):

$$s_i^* \in BR(s_{-i}) \textbf{ if } \forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$$

Definition (Nash Equilibrium):

$$s =< s_1, ..., s_n > \textbf{ is a Nash Equilibrium if } \forall i, s_i \in BR(s_{-i})$$

Theorem (Nash, 1950): Every finite game has a Nash equilibrium.

2.3 Computing Nash Equilibrium

Two algorithms for finding NE

- LCP(Linear Complementarity) [Lemke-Howson].
- Support Enumeration Method [Porter et al].

2.4 Complexity Analysis

Theorem: Computing a Nash Equilibrium is a **PPAD-complete**, this theorem has been proven for:

- for games ≥ 4 players;
- for games with 3 players;
- for games with 2 players;

2.5 Summary of mixed strategies

- Some games have mixed strategy Nash Equilibria.
- A player must be indifferent between the actions he or she randomizes over.
- Randomization happen in business interactions, society, sports...

3 Game Theory in Python

Here is some Python code:

Listing 1: Creating a game

```
pip install nashpy
import nashpy as nash
import numpy as np
A =[[2,0], [1,3]]
B =[[3,0], [1,2]]
g = nash.Game(A, B)
g
Bi matrix game with payoff matrices:

Row player:
[[2 0]
 [1 3]]

Column player:
[[3 0]
 [1 2]]
```

4 Strictly Dominated Strategies and Iterated Removal

Apples	Green
Straws	Red
Oranges	Orange

Strictly Dominated Strategies

5 Maxmin Strategies

A **maxmin strategy** for a player i is a strategy that maximizes i's worst case payoff, where all other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to i.

The **maxmin value** of the game for player i is that minimum payoff guaranteed by a maxmin strategy.

Definition (Maxmin) The **maxmin strategy** for player i is $arg\ max_{s_i}$ $min_{s_{-i}}\ u_i(s_1, s_2)$, and **maxmin value** for player i is $max_{s_i} min_{s_{-i}} u_i(s_1, s_2)$.

Minmax Theorem (Von Neuman, 1928) In any finite, two player, zero sum game, in any Nash Equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

5.1 Computing Minmax

Minmax is solvable with LP (for two players)

$$\textbf{subject to} \sum_{K \in A_2} U_1(a_1^j, a_2^k) \times s_2^k \leq U_1 \tag{3}$$

6 Extensive Forms

The extensive form is an alternative representation that makes the temporal structure explicit.

- Perfect information extensive form games.
- Imperfect information extensive form games.

Definition A finite perfect information game in extensive form is defined by the tuple $(N, A, H, Z, \chi, \rho, \sigma, u)$ where:

- Players: N is a set of n players.
- Actions: A is set of actions.
- Choice nodes and labels for these nodes:
  - Choice nodes: H is a set of non-terminal choice nodes.
  - Action function: $\chi : H \to 2^A$ assigns to each choice a set of actions.
  - Player function: $\rho : H \to N$ assigns to each non-terminal node h a player $i \in N$ who chooses an action at h.
- Terminal nodes: Z is a set of terminal nodes, disjoint from H.
- Successor function: $\sigma : H \times A \to H \cup Z$ maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$

7 Transportation processes problem