Chapter 1

State of the Art

1.1 Game Theory

Game Theory is the study of rational behavior in situations involving interdependence as it may involve:

- Common interest (coordination);
- Competing interests (rivalry);
- Rational behavior: players can do the best they can, in their own eyes;
- Because of the players' interdependence, a rational decision in a game must be based on a prediction of others' responses;

1.1.1 The parts of a Game

A Game consists of three parts : Players, Actions and Payoffs

Players

Players are the decision makers and they can be : People, Governments or Companies.

Actions

What can the players do ? Decide when to sell a stock, decide how to vote or enter a bid in an auction...

Payoffs

Payoffs can represent the motivation of the players, for example : Do they care about profit ? or Do they care about other players ?

1.1.2 Defining Games

Games can be represented in two methods : Normal forms and Extensive Forms.

1.1.3 Extensive Form

An extensive form game includes timing of moves. Players move sequentially, represented as a tree.

- Chess: white player moves, then black player can see white's move and react...

Keeps track of what each player knows when he or she makes a decision :

- Poker: bet sequentially - what can a given player see when they bet.

1.1.4 Normal Form: Matrix form or strategic form

A normal form represents a list of what players get on function of their actions. Finite, n-person normal form game $\langle N, A, u \rangle$:

- Players: $N = 1, ..., n$ is a finite set of n, indexed by i.
- Actions set for player i A_i

  $a = (a_1, ..., a_n) \in A = A_1 * ... * A_n$ is an action profile.
- Utility function or Payoff function for player $i : u_i : A \to \mathbb{R}$

  $u = (u_1, ..., u_n)$, is a profile of utility functions.

1.1.5 Best Response and Nash Equilibrium

Best Response (Definition): $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$

Nash Equilibrium (Definition): $a =< a_1, ..., a_n >$ is a "**pure strategy**" if $\forall i, a_i \in BR(a_{-i})$

1.1.6 Dominant strategies

let s_i and $s_i^{`}$ be two strategies for player i, and let S_{-i} be the set of all possible strategy profiles for other players.

- s_i **strictly dominates** $s_i^{`}$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s_i^{`}, s_{-i})$
- s_i **very weakly dominates** $s_i^{`}$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i^{`}, s_{-i})$
- A strategy is called **dominant** if it dominates all others.
- A strategy profile consisting of dominant strategies for every player must be a Nash Equilibrium.

1.1.7 Pareto Optimality

An outcome o^* is **Pareto-optimal** if there is no other outcome that Pareto-dominates it.

1.2 Mixed Strategies and Nash Equilibrium

A strategy s_i for agent i as any probability distribution over the actions A_i.

- **pure strategy:** only one action is played with positive probability
- **mixed strategy:** more than one action is played with positive probability

  these actions are called the support of the mixed strategy.
- Let the set of all strategies for i be S_i
- let the set of all strategy profiles be $S = S_1 \times ... \times S_n$

1.2.1 Utility in Mixed Strategies

In order to find the payoff if all the players follow mixed strategy profile $s \in S$ we can use the **expected utility** from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) P(a|s) \tag{1.1}$$

$$P(a|s) = \prod_{j \in N} s_j(a_j) \tag{1.2}$$

1.2.2 Best Response and Nash Equilibrium

The definitions of best response and Nash equilibrium are generalized from actions to strategies.

Definition (Best Response):

$$s_i^* \in BR(s_{-i}) \textbf{ if } \forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$$

Definition (Nash Equilibrium)

$$s =< s_1, ..., s_n > \textbf{ is a Nash Equilibrium if } \forall i, s_i \in BR(s_{-i})$$

Theorem (Nash, 1950) Every finite game has a Nash equilibrium.

1.2.3 Computing Nash Equilibrium

Two algorithms for finding NE

- LCP(Linear Complimentary) [Lemke-Howson].
- Support Enumeration Method [Porter et al].

1.2.4 Complexity Analysis

Computing a Nash Equilibrium is a **PPAD-complete**[1] , this theorem has been proven for:

- for games ≥ 4 players;
- for games with 3 players;
- for games with 2 players;

1.2.5 Summary of mixed strategies

- Some games have mixed strategy Nash Equilibria.
- A player must be indifferent between the actions he or she randomizes over.
- Randomization happen in business interactions, society, sports...

1.2.6 Strictly Dominated Strategies

Definition a strategy $a_i \in A_i$ is strictly dominated by $a_i' \in A_i$ if

$$u_i(a_i, a_{-i}) < u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{1.3}$$

1.2.7 Weakly Dominated Strategies

Definition a strategy $a_i \in A_i$ is weakly dominated by $a_i' \in A_i$ if

$$u_i(a_i, a_{-i}) \leq u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{1.4}$$

and

$$u_i(a_i, a_{-i}) < u_i(a_i', a_{-i}), \exists a_{-i} \in A_{-i} \tag{1.5}$$

1.3 Extensive Forms

The extensive form is an alternative representation that makes the temporal structure explicit.

- Perfect information extensive form games.
- Imperfect information extensive form games.

A finite perfect information game in extensive form is defined by the tuple $(N, A, H, Z, \chi, \rho, \sigma, u)$ where:

- Players: N is a set of n players.
- Actions: A is set of actions.
- Choice nodes and labels for these nodes:

[1]PPAD : Polynomial Parity Argument on Directed Graphs

  - Choice nodes: H is a set of non-terminal choice nodes.
  - Action function: $\chi : H \to 2^A$ assigns to each choice a set of actions.
  - Player function: $\rho : H \to N$ assigns to each non-terminal node h a player $i \in N$ who chooses an action at h.

- Terminal nodes: Z is a set of terminal nodes, disjoint from H.

- Successor function: $\sigma : H \times A \to H \cup Z$ maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$

- Utility function: $u = (u_1, ..., u_n)$ where $u_i : Z \to R$

1.4 Evolutionary Game Theory

Evolution and Game Theory was introduced by John Maynard Smith in Evolution and The Theory of Games. The Theory was formulated to understand the behavior of animals in game theoretic situations. But it can be applied to modeling human behavior.