Chapter 1

State of the Art

1.1 Game Theory

Game Theory is the study of rational behavior in situations involving interdependence as it may involve:

- Common interest (coordination);
- Competing interests (rivalry);
- Rational behavior: players can do the best they can, in their own eyes;
- Because of the players' interdependence, a rational decision in a game must be based on a prediction of others' responses;

1.1.1 The parts of a Game

A Game consists of three parts : Players, Actions and Payoffs

Players

Players are the decision makers and they can be : People, Governments or Companies.

Actions

What can the players do ? Decide when to sell a stock, decide how to vote or enter a bid in an auction...

Payoffs

Payoffs can represent the motivation of the players, for example : Do they care about profit ? or Do they care about other players ?

1.1.2 Defining Games

Games can be represented in two methods : Normal forms and Extensive Forms.

1.1.3 Extensive Form

An extensive form game includes timing of moves. Players move sequentially, represented as a tree.

- Chess: white player moves, then black player can see white's move and react...

Keeps track of what each player knows when he or she makes a decision :

- Poker: bet sequentially - what can a given player see when they bet.

1.1.4 Normal Form: Matrix form or strategic form

A normal form represents a list of what players get on function of their actions. Finite, n-person normal form game $\langle N, A, u\rangle$:

- Players: $N = 1, ..., n$ is a finite set of n, indexed by i.
- Actions set for player i A_i

  $a = (a_1, ..., a_n) \in A = A_1 * ... * A_n$ is an action profile.
- Utility function or Payoff function for player $i : u_i : A \to \mathbb{R}$

  $u = (u_1, ..., u_n)$, is a profile of utility functions.

1.1.5 Best Response and Nash Equilibrium

Best Response (Definition): $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$

Nash Equilibrium (Definition): $a =< a_1, ..., a_n >$ is a "**pure strategy**" if $\forall i, a_i \in BR(a_{-i})$

1.1.6 Dominant strategies

let s_i and $s_i^{‘}$ be two strategies for player i, and let S_{-i} be the set of all possible strategy profiles for other players.

- s_i **strictly dominates** $s_i^{‘}$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s_i^{‘}, s_{-i})$
- s_i **very weakly dominates** $s_i^{‘}$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i^{‘}, s_{-i})$
- A strategy is called **dominant** if it dominates all others.
- A strategy profile consisting of dominant strategies for every player must be a Nash Equilibrium.

1.1.7 Pareto Optimality

An outcome o^* is **Pareto-optimal** if there is no other outcome that Pareto-dominates it.

1.2 Mixed Strategies and Nash Equilibrium

A strategy s_i for agent i as any probability distribution over the actions A_i.

- **pure strategy:** only one action is played with positive probability
- **mixed strategy:** more than one action is played with positive probability

  these actions are called the support of the mixed strategy.
- Let the set of all strategies for i be S_i
- let the set of all strategy profiles be $S = S_1 \times ... \times S_n$

1.2.1 Utility in Mixed Strategies

In order to find the payoff if all the players follow mixed strategy profile $s \in S$ we can use the **expected utility** from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) P(a|s) \tag{1.1}$$

$$P(a|s) = \prod_{j \in N} s_j(a_j) \tag{1.2}$$

1.2.2 Best Response and Nash Equilibrium

The definitions of best response and Nash equilibrium are generalized from actions to strategies.

Definition (Best Response):

$$s_i^* \in BR(s_{-i}) \textbf{ if } \forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$$

Definition (Nash Equilibrium)

$$s = < s_1, ..., s_n > \textbf{ is a Nash Equilibrium if } \forall i, s_i \in BR(s_{-i})$$

Theorem (Nash, 1950) Every finite game has a Nash equilibrium.

1.2.3 Computing Nash Equilibrium

Two algorithms for finding NE

- LCP(Linear Complimentary) [Lemke-Howson].
- Support Enumeration Method [Porter et al].

1.2.4 Complexity Analysis

Computing a Nash Equilibrium is a **PPAD-complete**[1] , this theorem has been proven for:

- for games ≥ 4 players;
- for games with 3 players;
- for games with 2 players;

1.2.5 Summary of mixed strategies

- Some games have mixed strategy Nash Equilibria.
- A player must be indifferent between the actions he or she randomizes over.
- Randomization happen in business interactions, society, sports...

1.2.6 Strictly Dominated Strategies

Definition a strategy $a_i \in A_i$ is strictly dominated by $a_i' \in A_i$ if

$$u_i(a_i, a_{-i}) < u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{1.3}$$

1.2.7 Weakly Dominated Strategies

Definition a strategy $a_i \in A_i$ is weakly dominated by $a_i' \in A_i$ if

$$u_i(a_i, a_{-i}) \leq u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{1.4}$$

and

$$u_i(a_i, a_{-i}) < u_i(a_i', a_{-i}), \exists a_{-i} \in A_{-i} \tag{1.5}$$

[1]PPAD : Polynomial Parity Argument on Directed Graphs