No.2 Problem Set

1. 问答:

(1)

$$\ln \frac{p(y=1|x)}{p(y=K|x)} = w_1^T x + b_1 \Rightarrow \frac{p(y=1|x)}{p(y=K|x)} = e^{w_1^T \vec{x} + b_1}$$

$$\vdots$$

$$\ln \frac{p(y=K-1|x)}{p(y=K|x)} = w_{K-1}^T x + b_{K-1} \Rightarrow \frac{p(y=K-1|x)}{p(y=K|x)} = e^{w_{K-1}^T x + b_{K-1}}$$

Set $p(y=K|x)=1$, then use normalization for each probability.

$$\Rightarrow p(y=i|x) = \frac{e^{w_i^T x + b_i}}{1+\sum_{k=1}^{K-1} e^{w_k^T x + b_k}}$$ (当 $i=K$ 时, 分子为 1).

$$\underset{\text{tensor}}{l(\vec{w}, \vec{b})} = \sum_{i=1}^{m}\sum_{j=1}^{K} \ln p(y_i=j|x_i)\cdot \mathbb{I}(y_i=j) \longrightarrow \mathbb{I}(\cdot) = \begin{cases} 1 & \text{if } \cdot \text{ is true} \\ 0 & \text{elsewhere} \end{cases}$$

(i: sample #, j: class #)

$$= \sum_{i=1}^{m}\sum_{j=1}^{K-1} \mathbb{I}(y_i=j)(w_j^T x + b_j) - \sum_{i=1}^{m} \ln(1+\sum_{j=1}^{K-1} e^{w_j^T x + b_j}).$$

(2) Using the same notation from textbook. Set.

$\beta_i = (w_i; b_i)$. $\hat{x} = (x; 1)$.

$$\Rightarrow l(\beta) = \sum_{i=1}^{m}\sum_{j=1}^{K-1} \mathbb{I}(y_i=j)\cdot(\beta_j^T \cdot \hat{x}) - \sum_{i=1}^{m} \ln(1+\sum_{j=1}^{K-1}(e^{\beta_j^T \hat{x}}))$$

$$\Rightarrow \nabla l(\beta) \Rightarrow \frac{\partial}{\partial \beta_i} l(\beta) = \sum_{j=1}^{K-1}\left(\mathbb{I}(y_i=j) - \frac{e^{\beta_j^T \hat{x}}}{1+\sum_{j=1}^{K-1} e^{\beta_j^T \hat{x}}}\right)\hat{x}_i$$

$$= \sum_{j=1}^{K-1}\left[\mathbb{I}(y_i=j) - p(y_i=j|\hat{x}_i)\right]\hat{x}_i$$

2. 解:

由"最优分类"的定义可知，此时其正/反例概率密度相等

$$\frac{1}{(2\pi)^{\frac{k}{2}}(\Sigma_1)^{\frac{1}{2}}} e^{(x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1)} = \frac{1}{(2\pi)^{\frac{k}{2}}(\Sigma_2)^{\frac{1}{2}}} e^{(x-\mu_2)^T \Sigma_2^{-1}(x-\mu_2)}$$

as we have $\Sigma_1 = \Sigma_2$, thus

$$\Rightarrow \quad (x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1) = (x-\mu_2)^T \Sigma_1^{-1}(x-\mu_2).$$

with more calculations

$$x^T \Sigma^{-1}(\mu_2-\mu_1) + (\mu_2-\mu_1)^T \Sigma^{-1} x = \mu_2^T \Sigma^{-1}\mu_2 - \mu_1^T \Sigma^{-1}\mu_1$$

等价

$$2x^T \Sigma^{-1}(\mu_2-\mu_1) = (\mu_2^T+\mu_1^T)\Sigma^{-1}(\mu_2-\mu_1)$$

因为：Σ, Σ^{-1}, $x^T\Sigma^{-1}(\mu_2-\mu_1)$, $\mu_1\Sigma^{-1}\mu_2$ 均为实对称矩阵.

Thus

$$\boxed{(\mu_2-\mu_1)^T \Sigma^{-1}\left(x-\frac{1}{2}(\mu_1+\mu_2)\right) = 0}$$ "最优分类"定义下的分类平面

而对于LDA来说

$$w^T(x-\mu_1) = w^T(\mu_2-x).$$

其中 $w = S_w^{-1}(\mu_1-\mu_2)$, $S_w = \Sigma_1+\Sigma_2 = 2\Sigma$.

$$\Rightarrow S_w^{-1} = \frac{1}{2}\Sigma^{-1}$$

$$\Rightarrow w^T = \frac{1}{2}\Sigma^{-1}(\mu_1-\mu_2)$$

$$\Rightarrow \left(\Sigma^{-1}(\mu_1-\mu_2)\right)^T(x-\mu_1) = \left(\Sigma^{-1}(\mu_1-\mu_2)\right)^T(\mu_2-x).$$

$$\Rightarrow 2(\mu_1-\mu_2)^T\Sigma^{-1}x - (\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1+\mu_2) = 0.$$

$$\Rightarrow \boxed{(\mu_2-\mu_1)^T \Sigma^{-1}\left(x-\frac{1}{2}(\mu_1+\mu_2)\right) = 0}$$ LDA下的分类平面

与之前一致