Single neuron models

Reduced models and phase-plane analysis of their dynamics: 3

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Outline

1. Capturing neural dynamics in the abstract

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2. The Fitzhugh-Nagumo model

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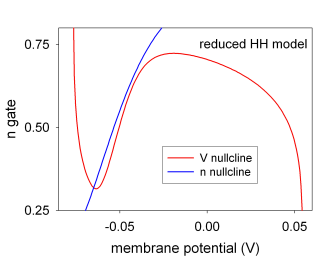
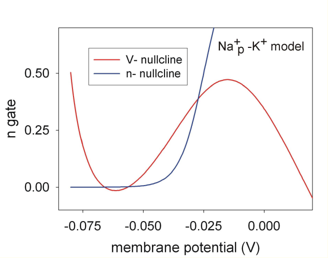
- A new strategy is to establish models, independent of any physiological framework, whose dynamics are equivalent to their biologically grounded counterparts
- This programme requires we describe, as simply as possible, the nullclines of typical neural models

The V-nullcline





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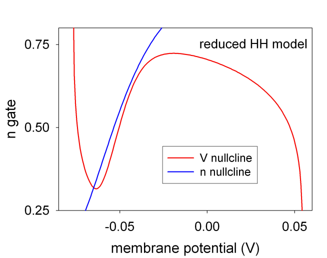
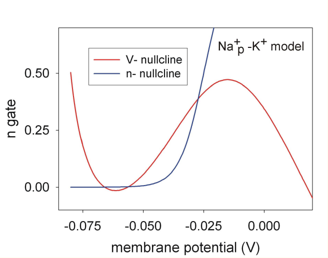




- The V-nullcline of both models we have studied consists of an inverted 'N'-shape
- The simplest function of V which has such a form is

$$y(V) = V - AV^3 + B \quad (1)$$

i.e. a simple 'cubic' with parameters A, B

The V-nullcline

- If the recovery variable in our model is w, then a membrane equation of the form

$$\frac{dV}{dt} = V - AV^3 - w + I \tag{2}$$

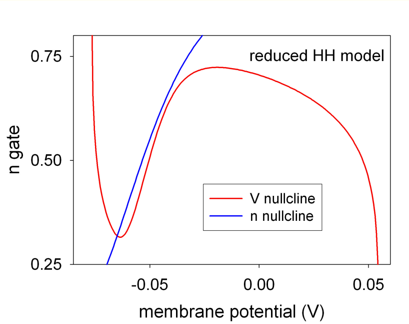
gives rise to a nullcline of the form in (1), for putting $dV/dt = 0$ gives

$$w = V - AV^3 + I \tag{3}$$

where $I \leftrightarrow B$ in (1)

calling the recovery variable w, helps remind us it is an abstract variable and not related to the HH-formalism like n

The w-nullcline



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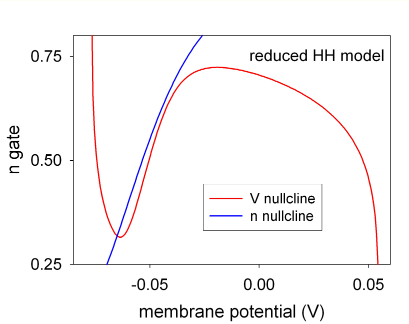
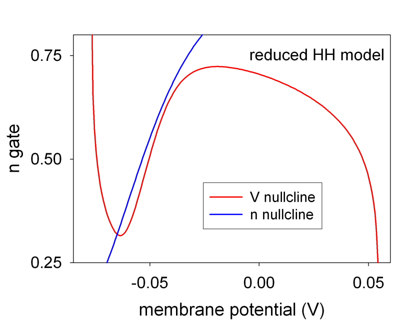


- In the reduced-HH model, the nullcline for the recovery variable is roughly linear over the relevant part of phase space
- if we aim to capture HH model-like dynamics, then we may therefore approximate the nullcline by a linear function

$$w = CV + D \tag{4}$$

The *w*-nullcline



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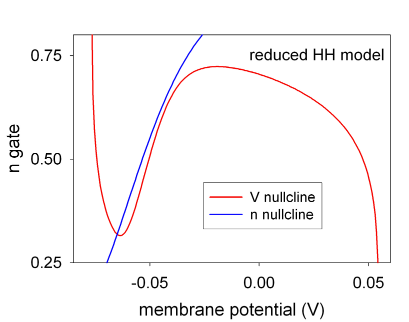

- In fact, a linear *w*-nullcline is sufficient to enable many variations of the dynamics
- The number, and location, of intersections with the V-nullcline is determined by the slope of the *w*-nullcline, and these are key to the dynamics

The w-nullcline

- A recovery variable equation of the form

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- (5) may be written

$$\frac{dw}{dt} = C(V + E - Fw) \tag{6}$$

where the new constants E, F are given in terms of the old ones by $E = D/C, F = 1/C$)

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An HH-like model

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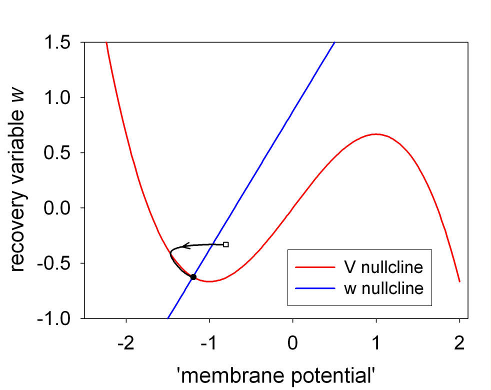
- An implementation of a similar model in electronic hardware by Nagumo in 1962 leads to the joint naming *Fitzhugh-Nagumo* for this model

Phase diagram and resting state



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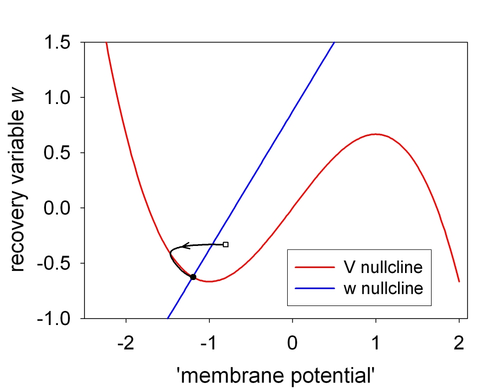

- The nullclines with $I = 0$ are exactly as we expect (they were constructed to be so!)
- A typical trajectory terminating on a a stable focus is shown

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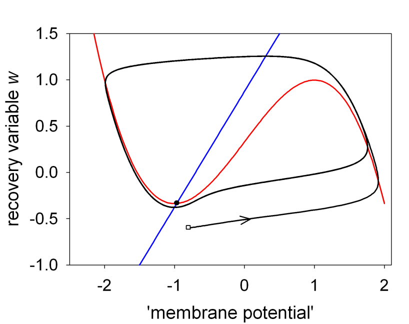
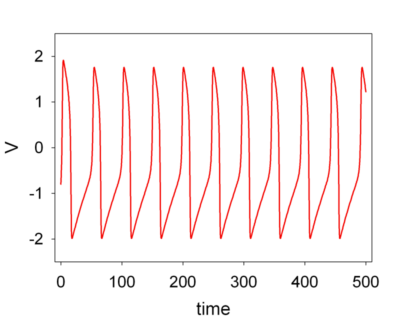
- Physiologically plausible units could be imposed by rescaling all the constants

Limit cycles and spikes





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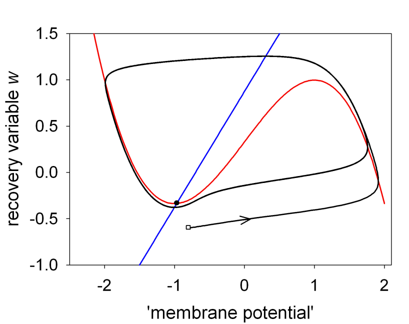

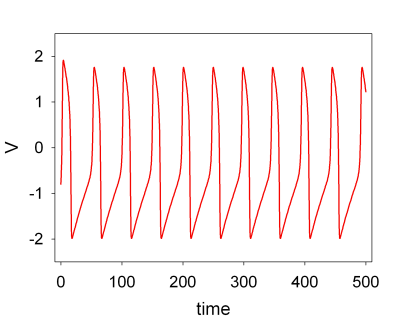

- With a ‘current’ of 0.33, regular spiking is observed associated with a limit cycle
- The model displays the same bifurcations as the reduced HH model and thereby captures the essentials of its dynamics

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- Variable substitutions lead to equivalent forms for the model which involve cubics in the membrane equation and linear (or affine) expressions in the recovery equation
- These would all be classified as Fitzhugh-Nagumo models

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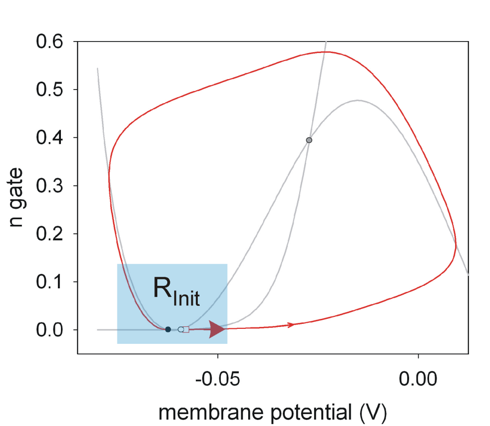
- For neuron models, this corresponds to computing the behaviour during a spike
- Izhikevich has recently proposed a simple 2D model of neural behaviour which circumvents spike computation but which is rich enough to model a wide variety of spiking behaviours

Spike initiation is localised in phase space



spike initiation in the $Na_p^+ - K^+$ model

- In all the reduced 2D models we have seen, the overall spiking behaviour is controlled in a fairly limited part of phase space (R_{init}) around the 'dip' in the V-nullcline

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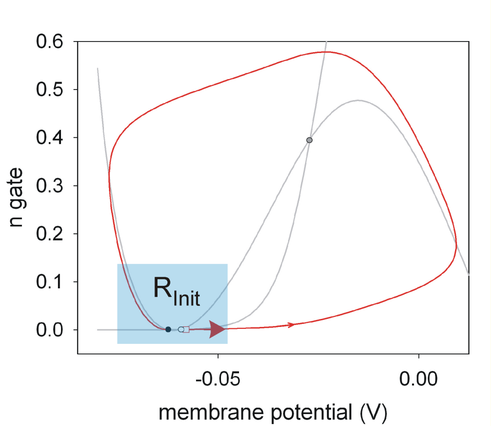

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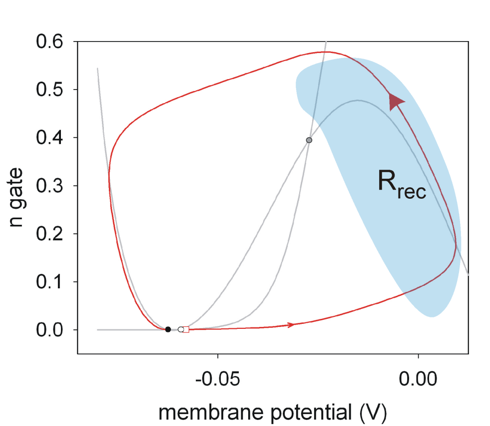
- The vector field here determines if a spike will be initiated, or if the trajectory returns to a stable equilibrium

Spike recovery is localised in phase space



spike recovery in the $Na_p^+ - K^+$ model

- Similarly, the recovery phase (or ‘downstroke’) of the spike is controlled by the rightmost portions (R_{rec}) of the V-nullcline

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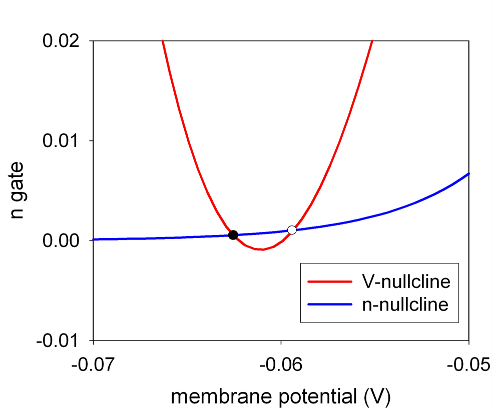
- We now quantify this scheme

Describing nullclines in R_{init}



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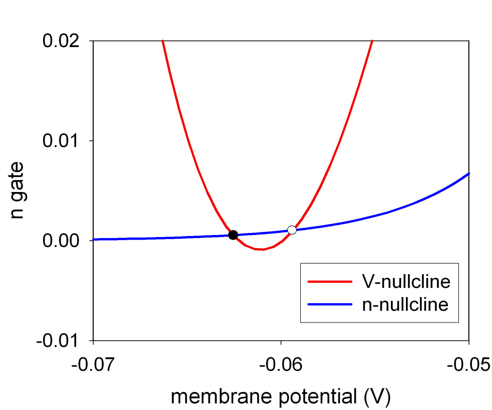

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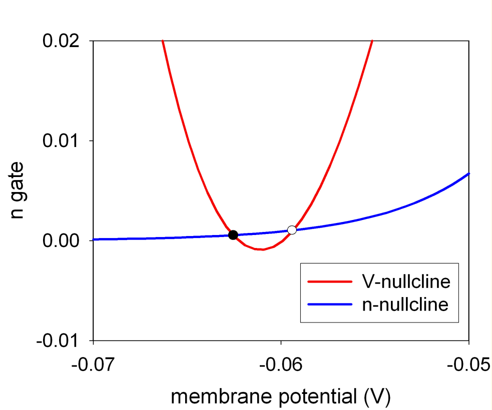

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- The recovery variable nullcline appears to be roughly linear
- We uses these descriptions to define the simple model

The Izhikevich simple model

The simple model: membrane potential V, recovery variable u

$$\frac{dV}{dt} = \frac{1}{C}\left[k(V - V_r)(V - V_t) - u + I\right] \tag{8}$$

Here, C behaves like the membrane capacitance, V_r is the resting potential, I the injection current. Other quantities are illustrated in the next two slides

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$$\text{if } V \geq V_{peak} \text{ then } V \leftarrow c,\ u \leftarrow u + d \tag{10}$$

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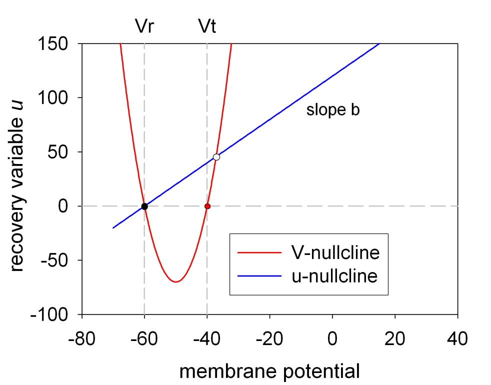
Dynamics with no input



Nullclines and key features of Izhikevich simple model with $I = 0$

Some features with $I = 0$ are:

- In the resting state, $u = 0$

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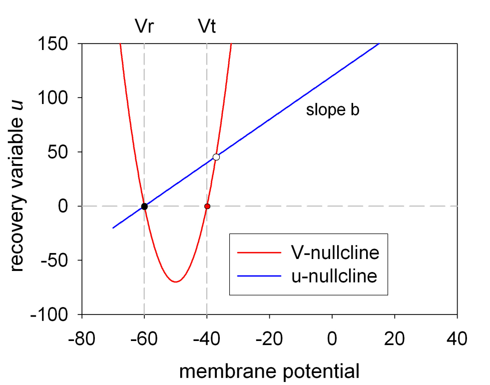

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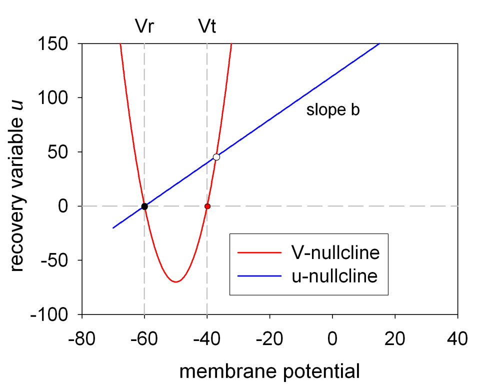

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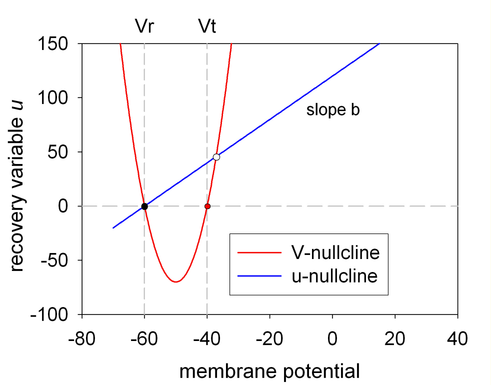

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- V_t, V_r are the values of V when $u = 0$
- The slope of the n-nullcline is the constant b in (8)
- With the parameters chosen here, there is a stable node equilibrium at V_r and a saddle node equilibrium at $V_s > V_t$

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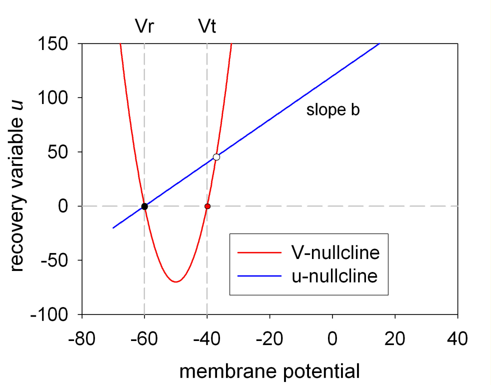

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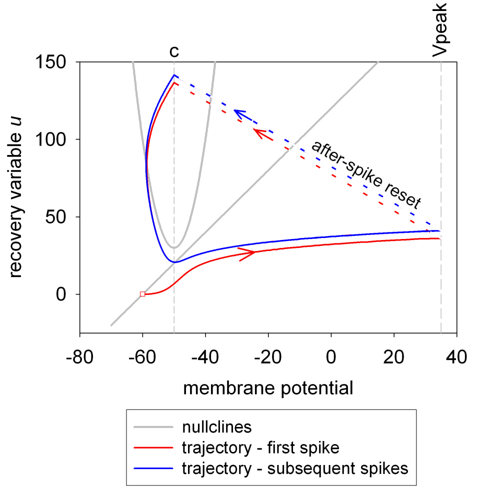
- Note that the parameters have been chosen so that the membrane potential may be plausibly interpreted in mV

Appearance of spikes



Izhikevich simple model with $I = 100$

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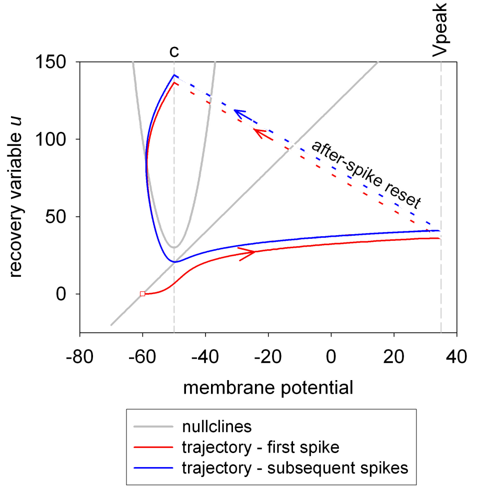

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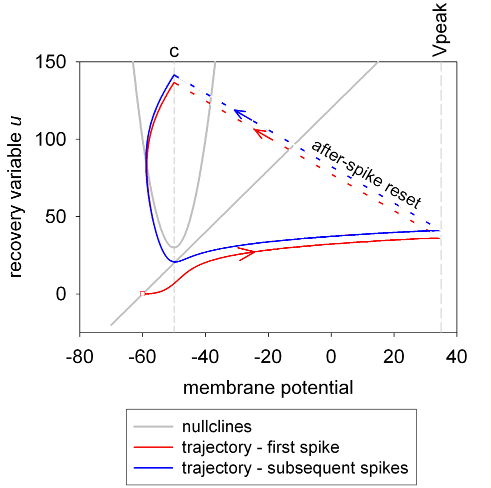

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- This would carry on to indefinitely large values of V unless reset at V_{peak}

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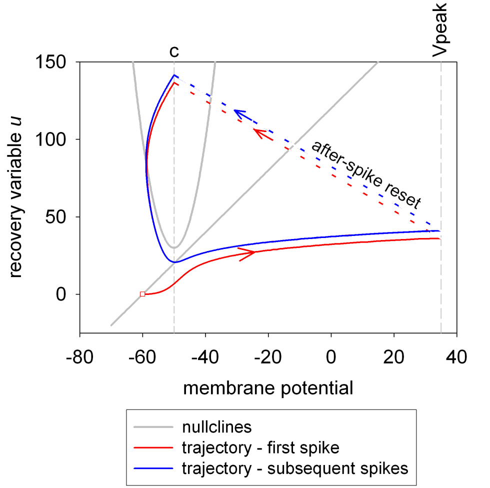
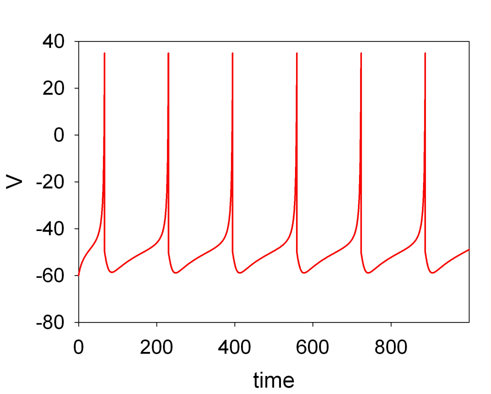


Izhikevich simple model with $I = 100$

- While the reset potential is always c, the reset value of u is $u + d$, which may vary (the initial spike has a smaller value of u at reset)
  - The value of c is at the minimum of the V-nullcline here - this is purely incidental

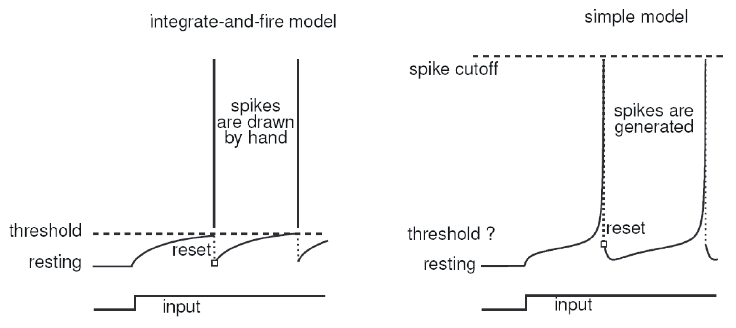
Appearance of spikes



Izhikevich simple model with $I = 100$

- In the time domain, it is clear that there is repetitive spiking

LIF and simple model compared



- In the LIF model, spikes are *initiated* at a threshold

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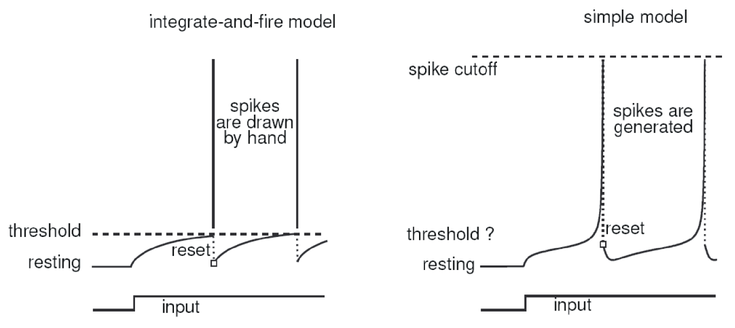
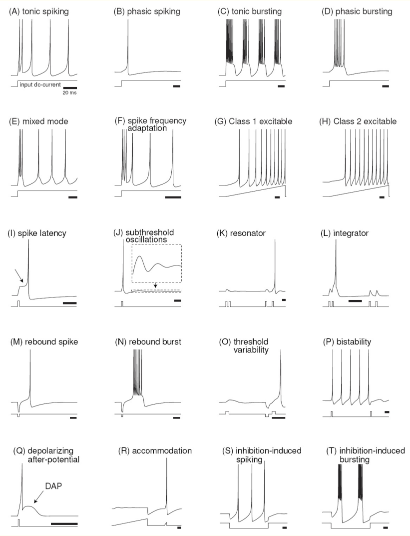


- In the LIF model, spikes are *initiated* at a threshold
- In the simple model they are initiated by the dynamics of the system (as in the more complex reduced models) and *terminated* at a peak value

The simple model is a powerful one



- The simple model is capable of a very wide diversity of firing behaviour using different paramaterisations [figure from ch 8 of (Izhikevich, 2007))

Summary

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- These models have, however, abandoned direct links with any physiological basis
- This strategy gives the Fitzhugh-Nagumo model and the simple models of Izhikevich
- In sum, the analysis of 2D, reduced models is a very powerful approach giving insights into a range of phenomena including, excitability classification, rebound inhibition etc.

Further reading

- Izhikevich and Fitzhugh (2006) have an article on the Fitzhugh-Nagumo model in Scholarpedia which has a nice historical note
- Ch 8 of the book by Izhikevich (2007) deals with the Simple Model
- Matlab code is available on MOLE

References

Izhikevich, E. (2007). *Dynamical systems in neuroscience: The geometry of excitability and bursting.* MIT Press.

Izhikevich, E., & Fitzhugh, R. (2006). Fitzhugh-nagumo model. In *Scholarpedia* (p. 5642). online material. (http://www.scholarpedia.org/wiki/index.php?title=FitzHugh-Nagumo_Model&&oldid=7128)