Lecture 5: Noncoercive problems: Babuška theorem

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Introduction

So far we have treated coercive problems. This means that the bilinear form $a(u, v)$ in the linear variational problem we are trying to solve find $u \in V$ such that $a(u, v) = F(v)$ for all $v \in V$ satisfies

$$a(u, u) \geq \alpha \|u\|_V^2$$

for some $\alpha > 0$.
Recall that the best constant α satisfying the definition is given by

$$\alpha := \inf_{\substack{u \in V \\ u \neq 0}} \frac{a(u, u)}{\|u\|_V^2}$$

We now consider noncoercive problems, one for which no such $\alpha > 0$ exists. We will develop more general (necessary and sufficient) criteria for well-posedness of the linear variational problem, the so-called inf-sup or Babuška conditions.

For coercive problems, well-posedness is inherited for $V_h \subset V$. This is not true for noncoercive problems. Well-posedness is not inherited for arbitrary $V_h \subset V$. One must prove the stability of each candidate discretisation individually.

Helmholtz equation

As an example of non coercive, we can show that the Helmholtz equation

$$
\begin{aligned}
-\Delta u - k^2 u = f \text{ in } \Omega \\
u = 0 \text{ on } \partial\Omega
\end{aligned}
$$

is well-posed if k^2 is not an eigenvalue of the Dirichlet Laplacian, but is not coercive for k large enough. For k^2 to be an eigenvalue of the Dirichlet Laplacian, it means that there exists $u \neq 0$ such that

$$
\begin{aligned}
-\Delta u = k^2 u \text{ in } \Omega \\
u = 0 \text{ on } \partial\Omega
\end{aligned}
$$

i.e. $-\Delta - k^2 I$ has a nontrivial kernel.

Mixed Laplacian

Suppose we want to know the flux in the Poisson equation accurately. We can solve the mixed formulation: find $\sigma : \Omega \to \mathbb{R}^n, u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \sigma &= -\nabla u && \text{in } \Omega, \\ \text{div}\sigma &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Solving this formulation will give an accurate approximation of the flux, and allow for the easy implementation of more complicated constitutive laws.

$$\begin{cases} \sigma = -\nabla u & \text{in } \Omega, \\ \text{div}\, \sigma = f & \text{in } \Omega. \end{cases}$$

mixed Laplacian

Let's multiply the first equation by a vector-valued test function v, and the second by a scalar-valued function w :

$$\int_\Omega \sigma \cdot v \,\mathrm{d}x + \int_\Omega \nabla u \cdot v = 0$$
$$\int_\Omega \operatorname{div}(\sigma) w \,\mathrm{d}x = \int_\Omega f w \,\mathrm{d}x$$

Since σ needs to have a divergence, and we want v and σ to come from the same space, let's integrate by parts in the first equation. For symmetry I'll negate the second equation:

$$\int_\Omega \sigma \cdot v \,\mathrm{d}x - \int_\Omega u \operatorname{div}(v) + \int_{\partial\Omega} u v \cdot n \,\mathrm{d}s = 0$$
$$-\int_\Omega \operatorname{div}(\sigma) w \,\mathrm{d}x = -\int_\Omega f w \,\mathrm{d}x$$

What function spaces do we need to make sense of

$$\int_\Omega \sigma \cdot v \, \mathrm{d}x - \int_\Omega u \text{ div } v = 0$$
$$-\int_\Omega \text{ div } \sigma w \, \mathrm{d}x = -\int_\Omega f w \, \mathrm{d}x$$

We don't need any derivatives on u or w, so $u \in L^2(\Omega)$. For σ and v, we need $\sigma \in L^2(\Omega; \mathbb{R}^n)$ and for div $\sigma \in L^2(\Omega)$. This is the space $H(\text{div}, \Omega)$:

$$H(\text{div}, \Omega) = \left\{ \sigma \in L^2(\Omega; \mathbb{R}^n) : \text{ div}(\sigma) \in L^2(\Omega) \right\}$$

Its inner product is

$$(u, v)_{H(\text{div},\Omega)} = \int_\Omega u \cdot v + \text{ div} u \text{ div } v \, \mathrm{d}x$$

Mixed Laplacian

A nice property of variational problems is that we can add the two equations together. The problem is the same as:

$$\begin{cases} \text{Find } (\sigma, u) \in H(\text{ div }, \Omega) \times L^2(\Omega) \text{ such that} \\ \int_\Omega \sigma \cdot v \, \mathrm{d}x - \int_\Omega \text{div}(v) u - \int_\Omega \text{div}(\sigma) w \, \mathrm{d}x = -\int_\Omega f w \, \mathrm{d}x \end{cases}$$

for all $(v, w) \in H(\text{div}, \Omega) \times L^2(\Omega)$.
We define:

$$B(\sigma, u; v, w) := \int_\Omega \sigma \cdot v \, \mathrm{d}x - \int_\Omega \text{div}(v) u - \int_\Omega \text{div}(\sigma) w \, \mathrm{d}x$$

Lax-Milgram certainly won't apply:

$$B(0, u; 0, u) = 0 \text{ for all } u \in L^2(\Omega)$$

Babuška theorem

Theorem

Let V and W be two Hilbert spaces. Let $a : V \times W \to \mathbb{R}$ be a bilinear form for which there exist constants $C < \infty, \gamma > 0, \gamma' > 0$ such that:

1. $$|a(v,w)| \leq C\|v\|_V\|w\|_W \text{ for all } v \in V, w \in W$$

2. $$\inf_{\substack{v \in V \\ v \neq 0}} \sup_{\substack{w \in W \\ w \neq 0}} \frac{a(v,w)}{\|v\|_V\|w\|_W} \geq \gamma$$

3. $$\inf_{\substack{w \in W \\ v \neq 0}} \sup_{\substack{v \in V \\ v \neq 0}} \frac{a(v,w)}{\|v\|_V\|w\|_W} \geq \gamma'$$

Then for all $F \in W'$ there exists exactly one element $u \in V$ such that

$$a(u,w) = F(w) \text{ for all } w \in W.$$

As a first example of how to manipulate inf-sup conditions, let's show that a coercive problem satisfies the inf-sup conditions. Suppose $a(u, v)$ satisfies

$$\alpha \|u\|_V^2 \leq a(u, u) \quad \text{for all } u \in V$$

Dividing both sides of the inequality by $\|u\|_V$ for $u \neq 0$, we have

$$\begin{aligned} \alpha \|u\|_V &\leq \frac{a(u, u)}{\|u\|_V} \\ &\leq \sup_{\substack{v \in V \\ v \neq 0}} \frac{a(u, v)}{\|v\|_V} \end{aligned}$$

Infimising over $u \neq 0$, we have

$$0 < \alpha \leq \inf_{\substack{u \in V \\ u \neq 0}} \sup_{\substack{v \in V \\ v \neq 0}} \frac{a(u, v)}{\|u\|_V \|v\|_V}$$

So the coercivity constant α works for γ (and γ').

Start with find $u \in V$ such that

$$a(u, v) = F(v), \text{ for all } v \in V,$$

and take the Galerkin approximation over closed $V_h \subset V$:
find $u_h \in V_h$ such that

$$a\left(u_h, v_h\right) = F\left(v_h\right) \text{ for all } v_h \in V_h.$$

Note that Galerkin orthogonality still holds. Is the discrete problem well-posed?

Let's check the Babuška conditions.

1. Satisfaction of (1) is inherited.
2. What about (2)?

That is, does there exist $\tilde{\gamma}$ such that

$$\inf_{\substack{u_h \in V_h \\ u_h \neq 0}} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{a\left(u_h, v_h\right)}{\left\|u_h\right\|_V \left\|v_h\right\|_V} \geq \tilde{\gamma} > 0$$

with $\tilde{\gamma}$ independent of the mesh size h ? No! Examples later.

Remark

We don't need to check (3) in this case! The discrete system is square and finite-dimensional, so (2) $\iff$ (3) by rank-nullity.)

For every $v_h \in V_h$, we have

$$\begin{aligned}
\tilde{\gamma} \left\| v_h - u_h \right\|_V &\leq \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{a\left(v_h - u_h, w_h\right)}{\left\| w_h \right\|_V} \qquad \text{(discrete inf-sup)} \\
&= \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{a\left(v_h - u, w_h\right) + a\left(u - u_h, w_h\right)}{\left\| w_h \right\|_V} \\
&= \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{a\left(v_h - u, w_h\right)}{\left\| w_h \right\|_V} \qquad \text{(Galerkin orth.)} \\
&\leq \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{C \left\| v_h - u \right\|_V \left\| w_h \right\|_V}{\left\| w_h \right\|_V}
\end{aligned}$$

Now apply the triangle inequality to $\|u - u_h\|_V$:

$$
\begin{aligned}
\|u - u_h\|_V &\leq \|u - v_h\|_V + \|v_h - u_h\|_V \\
&\leq \|u - v_h\|_V + \frac{C}{\tilde{\gamma}} \|u - v_h\|_V \\
&= \left(1 + \frac{C}{\tilde{\gamma}}\right) \|u - v_h\|_V
\end{aligned}
$$

As before, we can combine this with an approximation result and a regularity result to derive error estimates for finite element discretisations.

Colab account

1. Open a Google Colab account.
2. Paste the following script:

```
try:
    import firedrake
except ImportError:
!wget
"https://fem-on-colab.github.io/releases/firedrake-install-real.sh" -O
"/tmp/firedrake-install.sh" && bash "/tmp/firedrake-install.sh"
import firedrake
```

Mixed Laplacian in 1d

Let's consider the mixed Poisson equation in one dimension. Start with

$$-u'' = f, \quad u(0) = 0 = u(1),$$

and introduce $\sigma = -u'$ to get the system

$$\begin{aligned} \sigma + u' &= 0 \\ \sigma' &= f. \end{aligned}$$

Testing the equations with $(\tau, v) \in V \times Q$, we get

$$\begin{aligned} \int_\Omega \sigma\tau \mathrm{d}x + \int_\Omega u'\tau \mathrm{d}x &= 0 \\ \int_\Omega \sigma' v \, \mathrm{d}x &= \int_\Omega fv \, \mathrm{d}x \end{aligned}$$