Lecture 1: Construction of finite element spaces

Pr. Ismail Merabet

Univ. of K-M-Ouargla

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Contents

Introduction

In the previous semester we have studied variational problems of the form:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = F(v), \forall v \in V \end{cases} \tag{1}$$

where V is an infinit dimonsional Hilbert space, $a(\cdot,\cdot)$ is a bilinear form and $F(\cdot)$ is a linear form. We have also seen that the Galerkin method replaces the space V by a finite dimensional space V_h.
Finite element methods are Ritz-Galerkin methods where the finite-dimensional trial/test function spaces are constructed by piecing together polynomial functions defined on (small) parts of the domain Ω.
This lecture describes the construction and properties of finite element spaces. We will give examples of conforming and nonconforming finite elements.

The Construction of a Finite Element Space

Let K be a closed bounded subset of $\mathbb{R}^d$ with a nonempty interior and a piecewise smooth boundary.

Definition 1.1 (Ciarlet)

A finite element is a triple ($K, \mathcal{P}, \mathcal{N}$), where:

1. $\mathcal{P}$ is a finite-dimensional vector space of functions defined on K
2. $\mathcal{N}$ is a basis of the dual space $\mathcal{P}'$.

Definition 1.2

- The function space $\mathcal{P}$ is the space of the shape functions.
- The elements of $\mathcal{N}$ are the nodal variables (degrees of freedom).

The Construction of a Finite Element Space

Definition 1.3

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. The basis $\{\phi_1, \phi_2, \ldots, \ \phi_k\}$ of $\mathcal{P}$ dual to $\mathcal{N}$, i.e

$$N_i\left(\phi_j\right) = \delta_{ij}$$

is called the nodal basis of $\mathcal{P}$.

Usually, condition (ii) of Definition 1.1 is the only one that requires much work, and the following simplifies its verification.

Lemma 1.3

Let $\mathcal{P}$ be a d-dimensional vector space and let $\{N_1, N_2, \ldots, \ N_d\}$ be a subset of the dual space $\mathcal{P}'$. Then the following two statements are equivalent.

a. $\{N_1, N_2, \ldots, N_d\}$ is a basis for $\mathcal{P}'$.
b. Given $v \in \mathcal{P}$ with $N_i v = 0$ for $i = 1, 2, \ldots, d$, then $v \equiv 0$.

Proof.

Let $\{\phi_1, \ldots, \phi_d\}$ be some basis for $\mathcal{P}$. $\{N_1, \ldots, N_d\}$ is a basis for $\mathcal{P}'$ iff given any L in $\mathcal{P}'$,

$$L = \alpha_1 N_1 + \ldots + \alpha_d N_d \tag{2}$$

The equation (2) is equivalent to

$$y_i := L(\phi_i) = \alpha_1 N_1(\phi_i) + \ldots + \alpha_d N_d(\phi_i), \quad i = 1, \ldots, d$$

Let $\mathbf{B} = (N_j(\phi_i)), i, j = 1, \ldots, d$. Thus, (a) is equivalent to $\mathbf{B}\alpha = y$ is always solvable, which is the same as $\mathbf{B}$ being invertible.
Given any $v \in \mathcal{P}$, we can write $v = \beta_1\phi_1 + \ldots + \beta_d\phi_d . N_i v = 0$ means that $\beta_1 N_i(\phi_1) + \ldots + \beta_d N_i(\phi_d) = 0$. Therefore, (b) is equivalent to

$$\beta_1 N_i(\phi_1) + \ldots + \beta_d N_i(\phi_d) = 0 \text{ for } i = 1, \ldots, d$$
$$\Longrightarrow \beta_1 = \ldots = \beta_d = 0$$

Let $\mathbf{C} = (N_i(\phi_j)), i, j = 1, \ldots, d$. Then (b) is equivalent to $\mathbf{C}x = 0$ only has trivial solutions, which is the same as $\mathbf{C}$ being invertible. But $\mathbf{C} = \mathbf{B}^T$. Therefore, (a) is equivalent to (b).

1d-Examples

Example ($\mathbb{P}_1$ Lagrange element).

Let $K = [0, 1], \mathcal{P} =$ the set of linear polynomials and $\mathcal{N} = \{N_1, N_2\}$, where

$$N_1(v) = v(0) \quad \text{and} \quad N_2(v) = v(1), \quad \forall v \in \mathcal{P}.$$

Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element and the nodal basis consists of

$$\phi_1(x) = 1 - x \quad \text{and} \quad \phi_2(x) = x.$$

Proof.

Let $v \in \mathbb{P}_1$, means, $v = a + bx$. Then $N_1(v) = N_2(v) = 0$ means

$$\begin{aligned} a &= 0 \\ a + b &= 0. \end{aligned}$$

Hence, $a = b = 0$, i.e., $v \equiv 0$.

$\mathbb{P}_k$ Lagrange element.

Example ($\mathbb{P}_k$ Lagrange element)

In general, we can let $K = [a, b]$ and $\mathcal{P} = \mathbb{P}_k([a, b])$ the set of all polynomials of degree less than or equal to k. Let $\mathcal{N} = \{N_0, N_1, N_2, \ldots, N_k\}$, where

$$N_i(v) = v(a + \frac{(b-a)}{k} i), \quad \forall v \in \mathcal{P} \text{ and } i = 0, 1, \ldots, k.$$

Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element.

Proof.

Example (Hermite finite element).

$K = [0,1], \mathcal{P} = \mathbb{P}_3(K)$, and $\mathcal{N} = \{N_1, N_2, N_3, N_4\}$

$$\begin{aligned} N_1 &: v \mapsto v(0), \\ N_2 &: v \mapsto v'(0), \\ N_3 &: v \mapsto v(1), \\ N_4 &: v \mapsto v'(1). \end{aligned}$$

Proof.

The $\mathbb{P}_1$-Lagrange finite element in two dimensions

The following are examples of two-dimensional finite elements.

Example (Triangular Lagrange Elements)

Let K be a triangle, $\mathcal{P}$ be the space $\mathbb{P}_k$ of polynomials in two variables of degree $\leq k$, and let the set $\mathcal{N}$ consist of evaluations of shape functions at the nodes with barycentric coordinates:

$$z_1 = i/k, \quad z_2 = j/k \quad \text{and} \quad z_3 = \ell/k,$$

where i, j, ℓ are nonnegative integers and $i + j + \ell = k$.
Then $(K, \mathcal{P}, \mathcal{N})$ is the two-dimensional $\mathbb{P}_k$ Lagrange finite element.

The nodal variables for the N_1, N_2, and N_3 Lagrange elements are depicted in Figure 1.

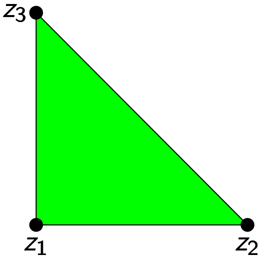


Figure: P1-Lagrange

Here and in the following examples, the symbol • represents pointwise evaluation of shape functions.

$$N_i(v) = v(z_i)$$

Lemma (First factorisation lemma)

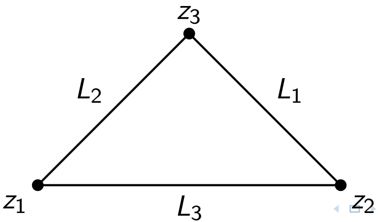
Let P be a polynomial of degree $d \geq 1$ that vanishes on a hyperplane

$$\{x : L(x) = 0\},$$

where $L(x)$ is a non-degenerate linear function.
Then we can write $P = LQ$, where Q is a polynomial of degree $d - 1$.

Example

Suppose P vanishes on each edge of a triangle. Then $P = L_1 L_2 L_3 Q$ for some Q. Here, L_1, L_2 and L_3 are non-trivial linear functions that define the lines on which lie the edges of the triangle.



The $\mathbb{P}_2$-Lagrange finite element in two dimensions

Unisolvence of $\mathbb{P}_2$

Suppose $v \in \mathcal{P}_2(T)$ with all degrees of freedom zero. Restricted to an edge, v is a quadratic polynomial with three roots, hence $v = 0$ on each edge. By the factorization lemma, $v = L_1 L_2 c$ for a constant $c \in \mathbb{R}$. Evaluating both sides on the edge $L_3 = 0$ shows that $c = 0$.

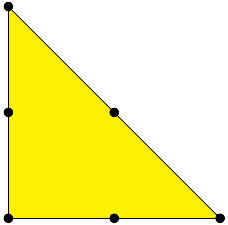


Figure: P2-Lagrange

Lemma (Second factorisation lemma)

Let P be a polynomial of degree $d \geq 2$ such that P and $\nabla P \cdot n$ vanish on a hyperplane $\{x : L(x) = 0\}$, where n is the normal to L. Then we can write $P = L^2 Q$, where Q is a polynomial of degree $d - 2$.

Remark

If P vanishes on $\{x : L(x) = 0\}$, so does $\nabla P \cdot t$ for a tangent vector t.

Proof.

Since P vanishes on $\{x : L(x) = 0\}$, we have $P = L\tilde{Q}$. Calculating,

$$\nabla P \cdot n = \tilde{Q} \nabla L \cdot n + L \nabla \tilde{Q} \cdot n$$

Since L vanishes on the plane, and ∇L is normal to the plane (hence colinear with n), this forces $\tilde{Q} = 0$ on $\{x : L(x) = 0\}$.

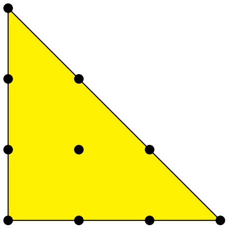


Figure: P3-Lagrange

Let T be a triangle.

Example (Triangular Hermite Elements)

The cubic Hermite element is the triple $(T, \mathbb{P}_3, \mathcal{N})$ where $\mathcal{N}$ consists of evaluations of shape functions and their gradients at the vertices and evaluation of shape functions at the center of T. i.e
$\mathcal{N} = \{N_1, N_2, \ldots, N_{10}\}$,

Here and in the following examples $\odot$ represents pointwise evaluation of gradients of shape functions.

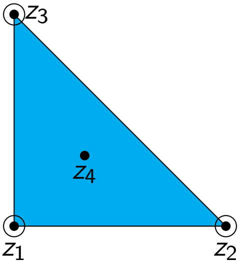


Figure: Cubic Hermite

The triangular Hermite element

Lemma (Unisolvence of the triangular Hermite element).

The Hermite element in two dimensions is unisolvent. i.e $\mathcal{N}$ determine $\mathbb{P}_3$.

Proof.

Let L_1, L_2 and L_3 again be non-trivial linear functions that define the edges of the triangle. Suppose that for a polynomial $P \in \mathcal{P}_3, N_i(P) = 0$ for $i = 1, 2, \ldots, 10$. Restricting P to L_1, we see that z_2 and z_3 are double roots of P since $P(z_2) = 0, P'(z_2) = 0$ and $P(z_3) = 0, P'(z_3) = 0$, where $'$ denotes differentiation along the straight line L_1. But the only third order polynomial in one variable with four roots is the zero polynomial, hence $P \equiv 0$ along L_1. Similarly, $P \equiv 0$ along L_2 and L_3. We can, therefore, write $P = cL_1L_2L_3$. But

$$0 = P(z_4) = cL_1(z_4)\,L_2(z_4)\,L_3(z_4) \quad \Longrightarrow \quad c = 0$$

because $L_i(z_4) \neq 0$ for $i = 1, 2, 3$.

Argyris element

The fifth degree Argyris element is the triple $(K, \mathbb{P}_5, \mathcal{N}_K)$ where $\mathcal{N}_K$ consists of evaluations of the shape functions and their derivatives up to order two at the vertices and evaluations of the normal derivatives at the midpoints of the edges. The nodal variables for the Argyris element are depicted in the third figure in Figure 3, where $\bigcirc$ and $\uparrow$ (here and in the following examples) represent pointwise evaluation of second order derivatives and the normal derivative of the shape functions, respectively.

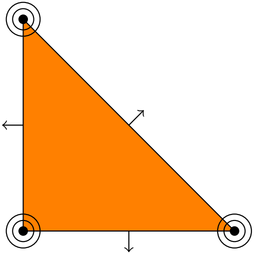


Figure: Quintic Argyris

Proof.

Suppose $u \in \mathbb{P}_5(T)$ with all dofs zero. Along an edge, u is a quintic polynomial with 2 treble roots, so $u = 0$ along each edge. Moreover, $\nabla u \cdot n$ is a quartic polynomial with 2 double roots and a single root, hence zero. Thus, u is divisible by $L_1^2 L_2^2 L_3^2$, which is of degree 6 . Thus $u = 0$.

Remark

The Argyris element with polynomials of degree five and 21 degrees of freedom within a single triangle, is the lowest order C^1 element.

Bell element.

By removing the nodal variables at the midpoints of the edges in the Argyris element and reducing the space of shape functions to

$$\{v \in \mathbb{P}_5(T) : (\partial v/\partial n)|_e \in \mathbb{P}_3(e) \text{ for each edge } e\},$$

we obtain the Bell element.

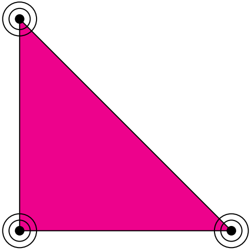


Figure: Bell

the Bell element represents a C^1 element, it has fewer degrees of freedom than the Argyris element.

Morley element

Morley element is $(T, \mathbb{P}_2, \mathcal{N})$, where the set $\mathcal{N}$ consists of evaluations of the shape functions at the vertices of T and the evaluations of the normal derivatives of the shape functions at the midpoints of the edges of T

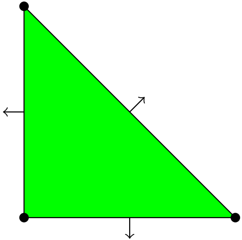


Figure: Morley

The continuity of the Morley element is very weak (non-C^0 continuity element). Morley element is the simplest nonconforming element for fourth order problems. It only consists of piecewise quadratic functions on every triangle

The Zienkiewicz element.

By removing the nodal variable at the center in the cubic Hermite element and reducing the space of shape functions to

$$Z(T) =: \left\{ v \in \mathbb{P}_3(T) : 6v(c) - 2\sum_{i=1}^{3} v\,(p_i) + \sum_{i=1}^{3} (\nabla v)\,(p_i) \cdot (p_i - c) = 0 \right\}$$

where $p_i (i = 1, 2, 3)$ and c are the vertices and center of T respectively, we obtain the Zienkiewicz element.

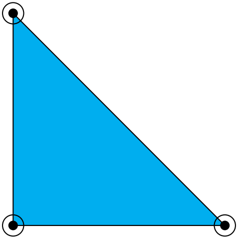


Figure: Zienckiez element