Lecture 1: A posterirori error analysis by duality

Pr. Ismail Merabet

Univ. of K-M-Ouargla

October 3, 2024

Contents

1. Introduction
2. One-dimensional model problem
3. Numerical Test
4. Conclusion

Introduction

- In the previous year, MEF II, for some elliptic problems we have seen how to show that

$$\|u - u_h\|_{H^1} \leq Ch^k |u|_{H^{k+1}}, \quad \|u - u_h\|_{L^2} \leq Ch^{k+1} |u|_{H^{k+1}}$$

- This means that the error can be arbitrarily reduced by reducing h.

But !

This might be very inefficient for problems that involve:

- corner singularities
- boundary or internal layers

i.e, the error is mostly concentrated in one area of the grid.

Introduction

The aim of this lecture is,

- to derive a computable bound on the error, and
- to demonstrate how such a bound may be implemented into an adaptive mesh-refinement algorithm, capable of reducing the error $u - u_h$ below a certain prescribed tolerance in an automated manner.
- The approach is based on seeking a bound on $u - u_h$ in terms of the computed solution u_h rather than in terms of norms of the unknown analytical solution u.
- A bound on the error in terms of u_h is referred to as an a posteriori error bound, due to the fact that it becomes computable only after the numerical solution u_h has been obtained.

Model problem in 1d

We consider the two-point boundary value problem

$$\begin{cases} -u'' + b(x)u' + c(x)u = f(x), & 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{1}$$

where

- $b \in W^{1,\infty}(0,1)$,
- $c \in L^{\infty}(0,1)$,
- $f \in L^2(0,1)$.

Letting

$$a(w,v) = \int_0^1 \left[w'(x)v'(x) + b(x)w'(x)v(x) + c(x)w(x)v(x) \right] \mathrm{d}x$$

$$\ell(v) = \int_0^1 f(x)v(x)\mathrm{d}x$$

The weak formulation of this problem can be stated as follows:

$$\begin{cases} \text{find } u \in H_0^1(0,1) \text{ such that} \\ a(u,v) = \ell(v), \qquad \text{for all } v \in H_0^1(0,1). \end{cases} \tag{2}$$

Lemma 1

Assuming that

$$c(x) - \frac{1}{2}b'(x) \geq 0, \qquad \text{for } x \in (0,1). \tag{3}$$

Then, there exists a unique weak solution, $u \in H_0^1(0,1)$.

The finite element approximation

We consider a subdivision of the interval $[0, 1]$ by the points

$$0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$$

We let $h_i = x_i - x_{i-1}, i = 1, \ldots, N$, and put $h = \max_i h_i$ and defining the finite element space $V_h \subset H_0^1(0, 1)$ consisting of continuous piecewise linear function.

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = \ell(v_h), \qquad \text{for all } v_h \in V_h. \end{cases} \tag{4}$$

We wish to derive an *a posteriori error* bound; that is, we aim to quantify the size of the global error $u - u_h$ in terms of the mesh parameter h and the computed solution u_h.

Main theorem

Theorem 1

Let u and u_h the solutions of the continuous problem (13) and the discrete problem (4). Then we have the computable a posteriori error bound,

$$\|u - u_h\|_{L_2(0,1)} \leq K_0 \left(\sum_{i=1}^{N} h_i^4 \left\| R\left(u_h\right) \right\|_{L_2(x_{i-1},x_i)}^2 \right)^{1/2}, \tag{5}$$

where:

- K_0 is a positive constant depending only on the coefficients b and c.
- for $i = 1, \ldots, N$,

$$R\left(u_h\right)(x) = f(x) + u_h''(x) - b(x)u_h'(x) - c(x)u_h(x), \quad x \in \left(x_{i-1}, x_i\right).$$

The dual problem

To do so, we consider the following auxiliary boundary value problem

$$\begin{cases} -z'' - (b(x)z)' + c(x)z = (u - u_h)(x), \quad 0 < x < 1, \\ z(0) = 0, \quad z(1) = 0, \end{cases} \tag{6}$$

called the dual or adjoint problem.

Lemma 2

Suppose that z is the solution of the dual problem (6). Then, there exists a positive constant K, dependent only on b, and c, such that

$$\left\| z'' \right\|_{L_2(0,1)} \leq K \left\| u - u_h \right\|_{L_2(0,1)} . \tag{7}$$

Proof of Lemma 2

We have

$$z'' = u_h - u - (bz)' + cz = u_h - u - bz' + \left(c - b'\right) z,$$

and therefore,

$$\left\|z''\right\| \leq \|u - u_h\| + \|b\|_{L_\infty(0,1)} \left\|z'\right\| + \left\|c - b'\right\|_{L_\infty(0,1)} \|z\|. \qquad (8)$$

We shall show that both $\|z'\|_{L_2(0,1)}$ and $\|z\|$ can be bounded in terms of $\|u- u_h\|$ and then, by virtue of (8), we shall deduce that the same is true of $\|z''\|$.
Observe that

$$\left(-z'' - (bz)' + cz, z\right) = (u - u_h, z)\,.$$

Proof of Lemma 2

Integrating by parts and noting that $z(0) = 0$ and $z(1) = 0$ yields

$$\begin{aligned}(-z'' - (bz)' + cz, z) &= (z', z') + (bz, z') + (cz, z) \\ &= \|z'\|^2 + \frac{1}{2}\int_0^1 b(x)\left[z^2(x)\right]' \mathrm{d}x + \int_0^1 c(x)[z(x)]^2 \, \mathrm{d}x\end{aligned}$$

Integrating by parts, again, in the second term on the right gives

$$(-z'' - (bz)' + cz, z) = \|z'\|^2 - \frac{1}{2}\int_0^1 b'(x)\left[z^2(x)\right] \mathrm{d}x + \int_0^1 c(x)[z(x)]^2 \, \mathrm{d}x.$$

Hence,

$$\|z'\|^2 + \int_0^1 \left(c(x) - \frac{1}{2}b'(x)\right)[z(x)]^2 \, \mathrm{d}x = (u - u_h, z),$$

and thereby, noting (3),

$$\|z'\|^2_{L^2(0,1)} \leq (u - u_h, z) \leq \|u - u_h\|_{L_2(0,1)} \|z\|_{L_2(0,1)}. \tag{9}$$

By the Poincaré-Friedrichs inequality,

$$\|z\|_{L_2(0,1)}^2 \le \frac{1}{2}\left\|z'\right\|_{L_2(0,1)}^2 .$$

Thus, (9) gives

$$\|z\|_{L_2(0,1)} \le \frac{1}{2}\|u-u_h\|_{L_2(0,1)} \tag{10}$$

Inserting this into the right-hand side of (9) yields

$$\left\|z'\right\|_{L_2(0,1)} \le \frac{1}{\sqrt{2}}\|u-u_h\|_{L_2(0,1)} \tag{11}$$

Now we substitute (10) and (11) into (8) to deduce that

$$\left\|z''\right\|_{L_2(0,1)} \le K\|u-u_h\|_{L_2(0,1)} . \tag{12}$$

Where

$$K = 1 + \frac{1}{\sqrt{2}}\|b\|_{L_\infty(0,1)} + \frac{1}{2}\left\|c-b'\right\|_{L_\infty(0,1)} .$$

Proof of Theorem 1

The L2-norm of the error is related to the solution of the dual problem via the relation:

$$\|u - u_h\|^2_{L_2(0,1)} = a(u - u_h, z)$$

Indeed,

$$\begin{aligned} \|u - u_h\|^2_{L_2(0,1)} &= (u - u_h, u - u_h) = \left(u - u_h, -z'' - (bz)' + cz\right) \\ &= a(u - u_h, z). \end{aligned}$$

By virtue of the Galerkin orthogonality property,

$$a(u - u_h, z_h) = 0 \quad \forall z_h \in V_h.$$

In particular, choosing $z_h = \mathcal{I}_h z \in V_h$, the continuous piecewise linear interpolant of the function z, we have that

$$a(u - u_h, \mathcal{I}_h z) = 0$$

Thus,

$$\begin{aligned}\|u-u_h\|^2_{L_2(0,1)} &= a\left(u-u_h, z-\mathcal{I}_h z\right) = a\left(u, z-\mathcal{I}_h z\right) - a\left(u_h, z-\mathcal{I}_h z\right) \\ &= \left(f, z-\mathcal{I}_h z\right) - a\left(u_h, z-\mathcal{I}_h z\right). \end{aligned} \tag{13}$$

Now,

$$\begin{aligned} a\left(u_h, z-\mathcal{I}_h z\right) = &\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} u_h'(x)\left(z-\mathcal{I}_h z\right)'(x)\mathrm{d}x \\ &+ \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} b(x)u_h'(x)\left(z-\mathcal{I}_h z\right)(x)\mathrm{d}x \\ &+ \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} c(x)u_h(x)\left(z-\mathcal{I}_h z\right)(x)\mathrm{d}x. \end{aligned}$$

Integrating by parts in each of the $(N-1)$ integrals in the first sum on the right-hand side, noting that

$$(z-\mathcal{I}_h z)(x_i)=0, i=0,\ldots,N,$$

we deduce that

$$a(u_h, z-\mathcal{I}_h z)=\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\left[-u_h''(x)+b(x)u_h'(x)+c(x)u_h(x)\right](z-\mathcal{I}_h z)(x)\mathrm{d}x.$$

Further

$$(f, z-\mathcal{I}_h z)=\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} f(x)(z-\mathcal{I}_h z)(x)\mathrm{d}x.$$

Substituting these two identities into (13), we deduce that

$$\|u-u_h\|_{L_2(0,1)}^2=\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} R(u_h)(x)(z-\mathcal{I}_h z)(x)\mathrm{d}x \tag{14}$$

where, for $i=1,\ldots,N$,

$$R(u_h)(x)=f(x)+u_h''(x)-b(x)u_h'(x)-c(x)u_h(x), \quad x\in(x_{i-1},x_i).$$

The function $R(u_h)$ is called the finite element residual;
it measures the extent to which u_h fails to satisfy the differential equation

$$-u'' + b(x)u' + c(x)u = f(x) \text{ on the interval } (0,1).$$

Now, applying the Cauchy-Schwarz inequality on the right-hand side of (14) yields

$$\|u - u_h\|^2_{L_2(0,1)} \le \sum_{i=1}^{N} \|R(u_h)\|_{L_2(x_{i-1},x_i)} \|z - \mathcal{I}_h z\|_{L_2(x_{i-1},x_i)}.$$

Recalling from the proof of interpolation theorem that

$$\|z - \mathcal{I}_h z\|_{L_2(x_{i-1},x_i)} \le \left(\frac{h_i}{\pi}\right)^2 \|z''\|_{L_2(x_{i-1},x_i)}, \quad i = 1, \dots, N,$$

and consequently,

$$\|u - u_h\|^2_{L_2(0,1)} \le \frac{1}{\pi^2} \left(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|^2_{L_2(x_{i-1},x_i)}\right)^{1/2} \|z''\|_{L_2(0,1)}. \quad (15)$$

Inserting (12) into (15), we arrive at our final result, the computable a posteriori error bound,

$$\|u - u_h\|_{L_2(0,1)} \leq K_0 \left(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|_{L_2(x_{i-1},x_i)}^2 \right)^{1/2}, \qquad (16)$$

where $K_0 = K/\pi^2$.

The name a posteriori stems from the fact that (16) can only be employed to quantify the size of the approximation error that has been committed in the course of the computation after u_h has been computed. In the next section we shall describe the construction of an adaptive mesh refinement algorithm based on the bound (16).

Adaptive method

Suppose that TOL is a prescribed tolerance and that our aim is to compute a finite element approximation u_h to the unknown solution u (with the same definition of u and u_h as in the previous section) so that

$$\|u - u_h\|_{L_2(0,1)} \leq TOL.$$

We shall use the a posteriori error bound (16) to achieve this goal by successively refining the subdivision, and computing a succession of numerical solutions u_h on these subdivisions, until the stopping criterion

$$K_0 \left(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|^2_{L_2(x_{i-1},x_i)} \right)^{1/2} \leq TOL$$

is satisfied.

- Choose an initial subdivision

$$\mathcal{T}_0 : \quad 0 = x_0^{(0)} < x_1^{(0)} < \ldots < x_{N_0-1}^{(0)} < x_{N_0}^{(0)} = 1$$

of the interval $[0,1]$, with $h_i^{(0)} = x_i^{(0)} - x_{i-1}^{(0)}$ for $i = 1, \ldots, N_0$,
- Compute the corresponding solution $u_{h^{(0)}} \in V_{h^{(0)}}$.
- Given a computed solution $u_{h^{(m)}} \in V_{h^{(m)}}$ for some $m \geq 0$, defined on a subdivision $\mathcal{T}_m$, stop if

$$K_0 \left(\sum_{i=1}^{N_m} \left(h_i^{(m)} \right)^4 \| R(u_{h^{(m)}}) \|^2_{L_2\left(x_{i-1}^{(m)}, x_i^{(m)}\right)} \right)^{1/2} \leq TOL.$$

- If not, then determine a new subdivision

$$\mathcal{T}_{m+1} : \quad 0 = x_0^{(m+1)} < x_1^{(m+1)} < \ldots < x_{N_{m+1}-1}^{(m+1)} < x_{N_{m+1}}^{(m+1)} = 1$$

and an associated finite element space $V_{h^{(m+1)}}$, such that

$$K_0 \left(\sum_{i=1}^{N_{m+1}} \left(h_i^{(m+1)} \right)^4 \| R(u_{h^{(m)}}) \|^2_{L_2\left(x_{i-1}^{(m+1)}, x_i^{(m+1)}\right)} \right)^{1/2} = TOL,$$

and continue.

Numerical Test

Let us consider the second-order ordinary differential equation

$$
\begin{aligned}
-\left(a(x)u'\right)' + b(x)u' + c(x)u = f(x), \quad x \in (0,1) \\
u(0) = 0, \quad u(1) = 0
\end{aligned}
$$

Suppose, for example, that

$$a(x) \equiv 1, \quad b(x) \equiv 20, \quad c(x) \equiv 10 \quad \text{and} \quad f(x) \equiv 1$$

In this case, the analytical solution, u, can be expressed in closed form:

$$u(x) = C_1 \mathrm{e}^{\lambda_1 x} + C_2 \mathrm{e}^{\lambda_2 x} + \frac{1}{10}$$

Numerical Test

where λ_1 and λ_2 are the two roots of the characteristic polynomial of the differential equation,

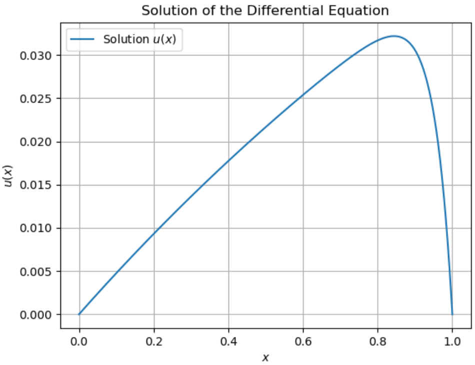
$$-\lambda^2 + 20\lambda + 10 = 0,$$

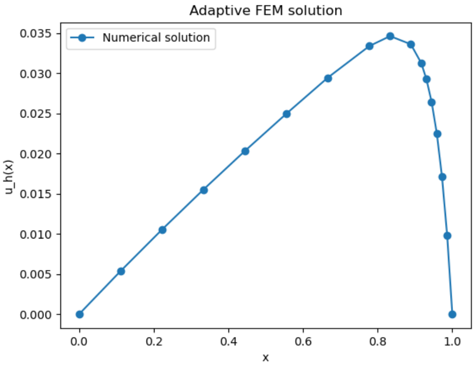
i.e.,

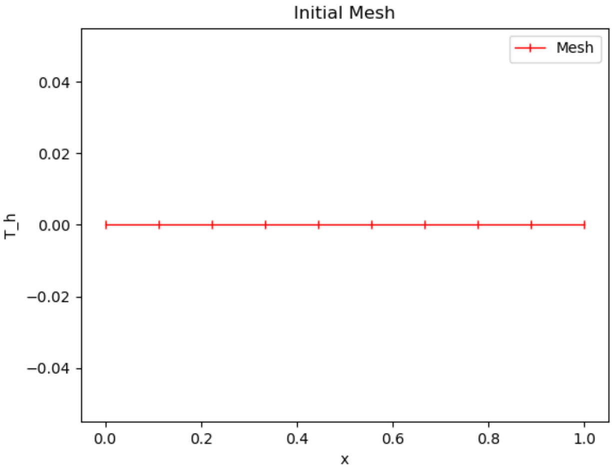
$$\lambda_1 = 10 + \sqrt{110}, \quad \lambda_2 = 10 - \sqrt{110}$$

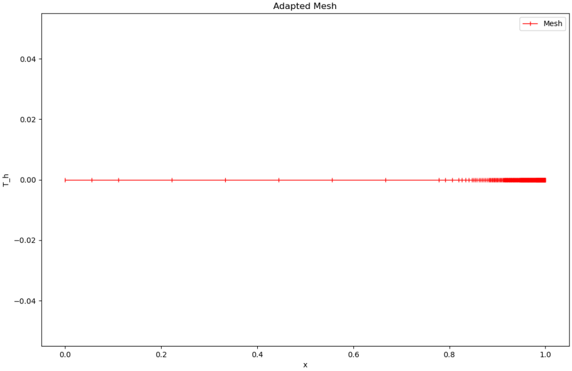
and C_1 and C_2 are constants chosen so as to ensure that $u(0) = 0$ and $u(1) = 0$; hence

$$C_1 = \frac{\mathrm{e}^{\lambda_2} - 1}{10\left(\mathrm{e}^{\lambda_1} - \mathrm{e}^{\lambda_2}\right)}, \quad C_2 = \frac{1 - \mathrm{e}^{\lambda_1}}{10\left(\mathrm{e}^{\lambda_1} - \mathrm{e}^{\lambda_2}\right)}$$


Solution of the Differential Equation
Solution $u(x)$
0.030
0.025
0.020
0.015
0.010
0.005
0.000
$u(x)$
0.0
0.2
0.4
0.6
0.8
1.0
x


Adaptive FEM solution
Numerical solution
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
u_h(x)
0.0
0.2
0.4
0.6
0.8
1.0
x


Initial Mesh
Mesh
0.04
0.02
0.00
−0.02
−0.04
T_h
0.0
0.2
0.4
0.6
0.8
1.0
x


Adapted Mesh
Mesh
0.04
0.02
0.00
−0.02
−0.04
T_h
0.0
0.2
0.4
0.6
0.8
1.0
x

Conclusion

- .
- .
- .