Exercises

Ex.1. Consider the following triplet $(K, \mathcal{P}, \mathcal{N})$

i) $K = [0, 1]$.

ii) $\mathcal{P}$ is the linear polynomials on K, i.e $\mathbb{P}_1$.

iii) $\mathcal{N}$ is :

$$N_1(p) = p(0.5), \quad N_2(p) = \int_0^1 p(x)\, dx.$$

Is this finite element unisolvent? Explain your answer

Ex.2. a) Obtain the nodal basis function $\phi_1(x)$ for the finite element $(K = [0, 1], \mathcal{P}_2, \mathcal{N})$, with $\mathcal{N} = (N_1, N_2, N_3)$ given by

$$N_1(f) = f(0),$$
$$N_2(f) = f(1),$$
$$N_3(f) = \int_0^1 f dx.$$

b) What is the global continuity of finite element spaces constructed from the finite element described above ? Explain your answer.

Ex.3. Let $K = [0, 1], \mathcal{P} = \mathcal{P}_3(K)$, and let $\mathcal{N} = \{N_1, N_2, N_3, N_4\}$, where

$$N_1 : v \mapsto v(0),$$
$$N_2 : v \mapsto v(1),$$
$$N_3 : v \mapsto v'(0),$$
$$N_4 : v \mapsto v'(1),$$

This defines the first Hermite finite element in one dimension.

(a) Show that $\mathcal{N}$ determines $\mathcal{P}$.

(b) Construct the nodal basis $\{\phi_0, \phi_1, \phi_0', \phi_1'\}$ for this finite element. Express your answers in the monomial basis.

(c) Consider the problem

$$-u'' + u = f, \quad u'(0) = 0 = u'(1).$$

Discretise $\Omega = [0, 1]$ into N intervals of uniform mesh size $h = 1/N$ and let V_h be the function space constructed by equipping each cell with the Hermite finite element defined above. This induces a linear system

$$Ax = b.$$

(i) State formulae for the components of A and b.

(ii) For a given cell K, what is the local 4×4 matrix A_K of contributions to A ?

(iii) Apply the finite element assembly algorithm cellwise to construct the matrix A for the case $N = 3$.

Ex.4. Let K be the interval $[0, 1]$, and let $\mathcal{P}$ be one-dimensional polynomials of degree 3 or less, with a dual basis $\mathcal{N}$. Let T_h be the corresponding subdivision of the interval $[a, b]$, with elements defined on each subinterval that are affine-equivalent to $(K, \mathcal{P}, \mathcal{N})$.

(a) Determine a dual basis $\mathcal{N}$ on K, such that the corresponding global interpolation operator $\mathcal{I}_{T_h}$ has C^1 continuity. Show that your dual basis determines $\mathcal{P}$.

(b) Determine the corresponding nodal basis for $\mathcal{P}$.

(c) Consider the variational problem for $u \in V$,

$$\int_0^1 u''v''\mathrm{d}x = \int_0^1 fv \; \mathrm{d}x, \quad \forall v \in V$$

where

$$V = \left\{ u : \int_0^1 (u'')^2 \; \mathrm{d}x < \infty, u(0) = u'(0) = u(1) = u'(1) = 0 \right\}.$$

Define a corresponding finite element discretisation based on the nodal basis defined above. Assume that a unique solution u exists the variational problem, and a unique solution u_h to the corresponding finite element discretisation. Prove the Galerkin orthogonality result

$$\int_0^1 (u - u_h)'' \, v''\mathrm{d}x = 0, \quad \forall v \in S$$

for an appropriately defined space S.

Ex.5. What is the choice of the geometric decomposition (allocation of nodal variables to cell and vertex entities) that leads to the maximum possible global continuity of finite element spaces defined on the interval $[0, L]$ constructed from the following one-dimensional elements (K, P, N). Justify your answer.

a) $K = [a, b], P$ is linear polynomials, $N = (N_1, N_2)$ where

$$N_1[u] = u((a + b)/2), \quad N_2[u] = u'((a + b)/2).$$

b) $K=[a,b], P$ is quadratic polynomials, $N=(N_1,N_2,N_3)$ where

$$N_1[u]=u(a), \quad N_2[u]=u(b), \quad N_3[u]=\int_a^b u \, \mathrm{d}x.$$

c) $K=[a,b], P$ is quadratic polynomials, $N=(N_1,N_2,N_3)$ where

$$N_1[u]=u'(a), \quad N_2[u]=u'(b), \quad N_3[u]=u((a+b)/2).$$

Ex.6. Consider the interval $[a,b]$, with points :

$$a=x_0, x_1, x_2, \ldots, x_{n-1}, x_n=b.$$

Let $\mathcal{T}$ be a subdivision (i.e. a 1D mesh) of the interval $[a,b]$ into subintervals

$$I_k=\left[x_k, x_{k+1}\right], k=0, \ldots, N-1.$$

Consider the following three elements.

(a) (K,P,N) where $K=I_k$, P are polynomials of degree $\leqslant 3$, and $N=(N_1,N_2,N_3,N_4)$ with:

$$\begin{aligned} N_1[u] &= u\left(x_k\right), & N_2[u] &= u\left(x_{k+1}\right), \\ N_3[u] &= \int_{x_k}^{x_{k+1}} u dx, & N_4[u] &= u'\left(\left(x_{k+1}+x_k\right)/2\right). \end{aligned}$$

(b) (K,P,N) where $K=I_k$, P are polynomials of degree $\leqslant 3$, and $N=(N_1,N_2,N_3,N_4)$ with:

$$\begin{aligned} N_1[u] &= u\left(x_k\right), & N_2[u] &= u\left(x_{k+1}\right), \\ N_3[u] &= u'\left(x_k\right), & N_4[u] &= u'\left(x_{k+1}\right). \end{aligned}$$

(c) (K,P,N) where $K=I_k$, P are polynomials of degree $\leqslant 3$, and $N=(N_1,N_2,N_3,N_4)$ with:

$$\begin{aligned} & N_1[u]=u\left(\left(x_{k+1}+x_k\right)/2\right), \; N_2[u]=u'\left(\left(x_{k+1}+x_k\right)/2\right), \\ & N_3[u]=u''\left(\left(x_{k+1}+x_k\right)/2\right), N_4[u]=u'''\left(\left(x_{k+1}+x_k\right)/2\right). \end{aligned}$$

i) Which of the three elements above are suitable for the following variational problem? Find $u \in H^1([a,b])$ such that

$$\int_a^b uv+u'v' \mathrm{d}x=\int_a^b fv \, \mathrm{d}x, \quad \forall v \in H^1([a,b]).$$

Justify your answer.

ii) Which of the three elements above are suitable for the following variational problem? Find $u \in H^2([a,b])$ such that

$$\int_a^b uv + u'v' + u''v''\mathrm{d}x = \int_a^b fv \ \mathrm{d}x, \quad \forall v \in H^2([a,b]).$$

Justify your answer.

Ex.7. Consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

- K is a non-degenerate triangle.
- $\mathcal{P}$ is the space of polynomials of degree 2 or less.
- $\mathcal{N} = (N_1, N_2, N_3, N_4, N_5, N_6)$ with

$$\begin{aligned} N_i(v) &= v\left(z_i\right), \qquad i = 1,2,3, \\ N_4(v) &= v\left(\frac{z_1+z_2}{2}\right), \\ N_5(v) &= v\left(\frac{z_1+z_3}{2}\right), \\ N_6(v) &= v\left(\frac{z_2+z_3}{2}\right), \end{aligned}$$

where z_1, z_2 and z_3 are the vertices of K.

Show that $\mathcal{N}$ determines $\mathcal{P}$.

Ex.8. Now consider the finite element $(K, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ where

- K is a non-degenerate triangle (with boundary ∂K).
- $\hat{\mathcal{P}}$ is the space spanned by polynomials of degree 2 or less, plus the cubic "bubble" function $B(x)$ satisfying $B(x) = 0$ for all $x \in \partial K$, and $\int_K B(x)\mathrm{d}x = 1$.
- $\hat{\mathcal{N}} = (N_1, N_2, N_3, N_4, N_5, N_6, N_7)$ with N_i as above for $i = 1, \ldots, 6$, and
- $$N_7(v) = v\left(\frac{z_1+z_2+z_3}{3}\right).$$

Show that $\hat{N}$ determines $\hat{P}$.