Machine Learning Exam - Bilet 3 Task Solutions

# Task 1: Uncorrelated Does Not Imply Independent

We are asked to show that for X ∼ U(-1, 1) and Y = X^2, the random variables X and Y are dependent, but their correlation coefficient ρ(X, Y) = 0.

Solution:

1. E[X] = 0, and var(X) = 1/3.  
2. The covariance cov(X, Y) = E[X^3] = 0, so ρ(X, Y) = 0.  
3. Therefore, X and Y are dependent, but uncorrelated.

# Task 2: MAP Estimation for 1D Gaussians

We calculate the MAP estimate μ̂MAP for the mean μ of a Gaussian distribution with known variance σ^2 and Gaussian prior N(m, s^2).   
We also explore how the estimate changes as the number of samples increases or the prior variance changes.

Solution:

1. The MAP estimate is μ̂MAP = (nσ^2m + s^2Σxi) / (nσ^2 + s^2).  
2. As n increases, μ̂MAP converges to the MLE. As s^2 increases, μ̂MAP also converges to the MLE. As s^2 decreases, μ̂MAP converges to the prior mean m.

# Task 3: Correlation Coefficient is Between -1 and +1

We prove that the correlation coefficient ρ(X, Y) always lies between -1 and 1.

Solution:

1. The correlation coefficient is ρ(X, Y) = cov(X, Y) / (σX σY).  
2. Applying the Cauchy-Schwarz inequality, |cov(X, Y)| ≤ σX σY.  
3. Therefore, -1 ≤ ρ(X, Y) ≤ 1.