

Lecture 10: Graph Search

Harvard SEAS - Fall 2022

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1 Announcements

Recommended Reading:

- Roughgarden II Sec 8.1–8.2
- CLRS 22.2
- Reminder: solutions posted on Canvas
- Pset 4 released
- Sender-Receiver exercise on Tuesday.
- Midterm feedback at <https://tinyurl.com/cs120midtermfeedback>

2 Shortest Walks

Motivated by a (simplified version) of the Google Maps problem, we wish to design an algorithm for the following computational problem:

<p>Input : A digraph $G = (V, E)$ and two vertices $s, t \in V$</p> <p>Output : A <i>shortest walk</i> from s to t in G, if any walk from s to t exists</p>
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Computational Problem ShortestWalk

Definition 2.1. Let $G = (V, E)$ be a directed graph, and $s, t \in V$.

- A *walk* w from s to t in G is a sequence v_0, v_1, \dots, v_ℓ of vertices such that $v_0 = s$, $v_\ell = t$, and $(v_{i-1}, v_i) \in E$ for $i = 1, \dots, \ell$.
- The *length* of a walk w is $\text{length}(w) =$ the number of edges in w (the number ℓ above).
- The *distance* from s to t in G is

$$\text{dist}_G(s, t) = \begin{cases} \min\{\text{length}(w) : w \text{ is a walk from } s \text{ to } t\} & \text{if a walk exists} \\ \infty & \text{otherwise} \end{cases}$$

- A *shortest walk* from s to t in G is a walk w from s to t with $\text{length}(w) = \text{dist}_G(s, t)$

Q: An algorithm immediate from the definition?

A: Enumerate over all walks from s in order of length, and terminate after finding the first that ends at t .

But when can we stop this algorithm to conclude that there is no walk? The following lemma allows us to stop at walks of length $n - 1$.

Lemma 2.2. *If w is a shortest walk from s to t , then all of the vertices that occur on w are distinct. That is, every shortest walk is a path — a walk in which all vertices are distinct.*

Proof.

Suppose for contradiction that there is a shortest walk $w = (s = v_0, v_1, \dots, v_\ell = t)$ that does *not* satisfy this property, i.e. $v_i = v_j$ for some $i < j$. But then we can cut out the loop $(v_i, v_{i+1}, \dots, v_j)$ and produce the walk $w' = (s = v_0, \dots, v_{i-1}, v_i = v_j, v_{j+1}, \dots, v_\ell)$. We have the length of w' is strictly less than that of w and has the same start and endpoints. But then w is not a shortest walk, so we have a contradiction. \square

Q: With this lemma, what is the runtime of exhaustive search?

A: $(n-1)! \cdot O(n) = O(n!)$

There is one choice for the first vertex, $n-1$ choices for the second vertex, $n-2$ choices for the third vertex, and so on, for a total of $(n-1)!$ possible paths. For each path, it takes $O(n)$ time to check that it is a correct path.

3 Breadth-First Search

We can get a faster algorithm using *breadth-first search (BFS)*. For simplicity we'll start by presenting the algorithm for the following simpler computational problem:

Input : A directed graph $G = (V, E)$ and two vertices $s, t \in V$
Output : The distance from s to t in G

Computational Problem DistanceInGraph

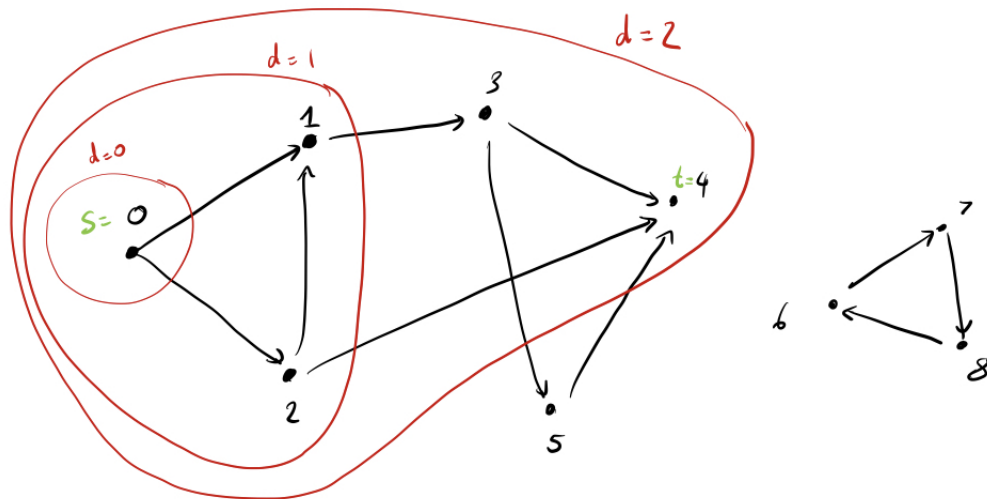
```

1 BFSv0( $G, s, t$ )
   Input : A directed graph  $G = (V, E)$  and two vertices  $s, t \in V$ 
   Output : The distance from  $s$  to  $t$  in  $G$ 
2  $S = \{s\};$ 
3 /* loop invariant:  $S$  contains the vertices at distance  $\leq d$  from  $s$  */
4 foreach  $d = 0, \dots, n-1$  do
5   | if  $t \in S$  then return  $d;$ 
6   |  $S = S \cup \{v \in V : \exists u \in S \text{ s.t. } (u, v) \in E\}$ 
7 return  $\infty$ 
```

Example:

Consider the graph with vertices $V = [9]$ and edges $E = \{(0, 1), (0, 2), (1, 3), (2, 1), (2, 4), (3, 4), (3, 5), (4, 5), (6, 7), (7, 8), (8, 6)\}$.

For that graph, the lists of neighbors of the vertices are **Nbr** = $[[1, 2], [3], [1, 4], [4, 5], [], [4], [7], [8], [6]]$.



Q: What is happening at every iteration of the loop?

We have a set of S which is the set of vertices that have been visited previously. At each iteration, we construct a new S' that is the union of S and the set of vertices that can be visited from all the vertices in S by one additional edge. This allows us to include the new vertices that can be visited now that we update the distance d .

Q: How do we perform the update of Line 6?

We'll iterate over all edges of G . We assume we are given the graph as an *adjacency list*: for each vertex v , we keep a neighbor array $\text{Nbr}[v] = \{u : (v, u) \in E\}$ holding the neighbors of v . We are also given the length of each such array $\text{Nbr}[v]$ —we could compute these lengths ourselves, but they're so often useful that we'll save time by assuming the representation of the graph comes with them.

¹ So we iterate over all edges of G by iterating over all those lists.

Q: How do we prove correctness?

Establish the loop invariant from start of iteration d .

$$S = \{v \in V : \text{dist}_G(s, v) \leq d\}$$

We now can prove that the loop invariant holds by induction on d .

Base Case: At $d = 0$, $S = \{s\}$.

Inductive Step: If $\text{dist}_G(s, v) = d$, then for $d > 0$, $\exists u$ s.t. $\text{dist}_G(s, u) = d - 1$ and $(u, v) \in E$. Thus, we add everything at distance d . Conversely, if $\text{dist}_G(s, u) \leq d - 1$ and $(u, v) \in E$, then $\text{dist}_G(s, v) \leq d$, so we did not add any extra vertices.

Then the loop invariant establishes that if the shortest path from s to t is of length k , we will add t to S at exactly the k th iteration.

Q: What is the runtime of the algorithm, in terms of the number of vertices n and the number of edges m ?

¹Other ways of representing a graph are sometimes useful, and discussed in classes like CS 124. In CS 120, we'll always represent graphs by adjacency lists.

We have n iterations, and in each iteration, we enumerate over all possible edges (u, v) in E . We can iterate over all possible edges by iterating over all n of the neighbor arrays, the sum of whose lengths is m , which takes time $O(m + n)$.

(In order to be able to check whether $u \in S$ and add possibly add v to S in constant time, we can maintain S as a bitvector, i.e. an array of n bits, where the u 'th entry is 1 iff $u \in S$.)

This gives a total runtime $O(n(m + n))$.

4 Improving BFS

Observations:

- S only grows due to edges that cross the *frontier* from S to $V - S$.
- Every edge in E crosses the frontier in at most one loop iteration.

```

1 BFS( $G, s, t$ )
   Input    : A directed graph  $G = (V, E)$  and two vertices  $s, t \in V$ 
   Output   : The distance from  $s$  to  $t$  in  $G$ 
2  $S = \{s\}$ ;
3  $F = \{s\}$  ;                               /* the frontier vertices */
4  $d = 0$ ;
5 /* loop invariant:  $S$  = vertices at distance  $\leq d$  from  $s$ ,  $F$  = vertices at
   distance  $d$  from  $s$  */
6 while  $F \neq \emptyset$  do
7   | if  $t \in F$  then return  $d$ ;
8   |  $F = \{v \in V - S : \exists u \in F \text{ s.t. } (u, v) \in E\}$ ;
9   |  $S = S \cup F$ ;
10  |  $d = d + 1$ ;
11 return  $\infty$ 

```

Theorem 4.1. $\text{BFS}(G)$ correctly solves *DistanceInGraph* and can be implemented in time $O(n+m)$, where n is the number of vertices in G and m is the number of edges.

Proof. 1. Correctness:

Proof is similar to the prior argument.

2. Runtime:

To carry out the update in Line 8, we can enumerate over every *vertex* u in the frontier F , and try every edge (u, v) leaving that vertex and check if v lies in S . If we maintain S as a bitvector and maintain the frontier F as a linked list (e.g. a queue of vertices), then this will take time:

$$O\left(\sum_{u \in F} (1 + d_{\text{out}}(u))\right)$$

Then when we sum over all iterations of the loop, we use that each vertex only appears in at most one frontier, i.e. if we let F_d be the frontier at the d 'th iteration, then the sets F_d are

all disjoint. Thus, our total runtime is

$$\begin{aligned}
O\left(\sum_{d=0}^{\infty} \sum_{u \in F_d} (1 + d_{out}(u))\right) &\leq O\left(\sum_{u \in V} (1 + d_{out}(u))\right) \\
&= O\left(\sum_{u \in V} 1 + \sum_{v \in V: (u,v) \in E} 1\right) \\
&= O\left(n + \sum_{u \in V} \sum_{v \in V: (u,v) \in E(u)} 1\right) \\
&= O(n + m).
\end{aligned}$$

□

Above we used the following definition:

Definition 4.2. For a digraph $G = (V, E)$ and a vertex v , we define the *out-degree* of v to be

$$d_{out}(v) = |\{w : (v, w) \in E\}|$$

and the *in-degree* of v to be

$$d_{in}(v) = |\{u : (u, v) \in E\}|.$$

For an undirected graph, we have $d_{out}(v) = d_{in}(v)$, so we just call this the *degree* of v , denoted $d(v)$.

Q: How would we calculate the out-degree of v from the adjacency-list representation of a graph?

The out-degree $d_{out}(v)$ is just the length of $\text{Nbr}(v)$. We specified that we stored the length of each such list as part of the representation of G , so we can just read that number.

5 More Graph Search

Q: How to actually find a shortest *path*, not just the distance?

Note that, by Lemma 2.2, shortest walks are paths, so we can use the terms “shortest paths” and “shortest walks” interchangeably.

Maintain an auxiliary array A_{pred} of size $|V|$, where $A_{pred}[v]$ holds the vertex u that we “discovered” v from. That is, if we add v to the frontier when exploring the neighbors of u , set $A_{pred}[v] = u$. After the completion of BFS, we can reconstruct the path from s to t using this predecessor array.

Observation: BFS actually solves the following computational problem:

Input : A digraph $G = (V, E)$ and a vertex $s \in V$
Output : For every vertex v , $\text{dist}_G(s, v)$ and, if $\text{dist}_G(s, v) < \infty$, a path p_v from s to v of length $\text{dist}_G(s, v)$ (implicitly represented through a predecessor array as above)

Computational Problem SingleSourceShortestPaths

We have proven:

Theorem 5.1. *There is an algorithm that solves `SingleSourceShortestPaths` in time $O(n + m)$ on digraphs with n vertices and m edges in adjacency list representation.*

The algorithm we have seen (BFS) only works on unweighted graphs; algorithms for weighted graphs are covered in CS124.

6 (Optional) Other Forms of Graph Search

Another very useful form of graph search that you may have seen is *depth-first search* (DFS). We won't cover it in CS120, but DFS and some of its applications are covered in CS124.

We do, however, briefly mention a randomized form of graph search, namely *random walks*, and use it to solve the *decision* problem of STConnectivity on undirected graphs.

Input : A graph $G = (V, E)$ and vertices $s, t \in V$
Output : YES if there is a walk from s to t in G , and NO otherwise

Computational Problem UndirectedSTConnectivity

```

1 RandomWalk( $G, s, \ell$ )
   Input : A digraph  $G = (V, E)$ , a vertices  $s, t \in V$ , and a walk-length  $\ell$ 
   Output : YES or NO
2  $v = s$ ;
3 foreach  $i = 1, \dots, \ell$  do
4   if  $v = t$  then return YES;
5    $j = \text{random}(d_{\text{out}}(v))$ ;
6    $v = j$ 'th out-neighbor of  $v$ ;
7 return  $\infty$ 
```

Q: What is the advantage of this algorithm over BFS?

While BFS needs $\Omega(n)$ words of memory in addition to the space required to store the input, this algorithm uses a *constant* number of words of memory while running.

It can be shown that if G is an *undirected* graph with n vertices and m edges, then for an appropriate choice of $\ell = O(mn)$, with high probability `RandomWalk`(G, s, ℓ) will visit all vertices reachable from s . Thus, we obtain a *Monte Carlo* algorithm for UndirectedSTConnectivity.

Theorem 6.1. *UndirectedSTConnectivity can be solved by a Monte Carlo randomized algorithm with arbitrarily small error probability in time $O(mn)$ using only $O(1)$ words of memory in addition to the input.*