Phys 512 : PS3

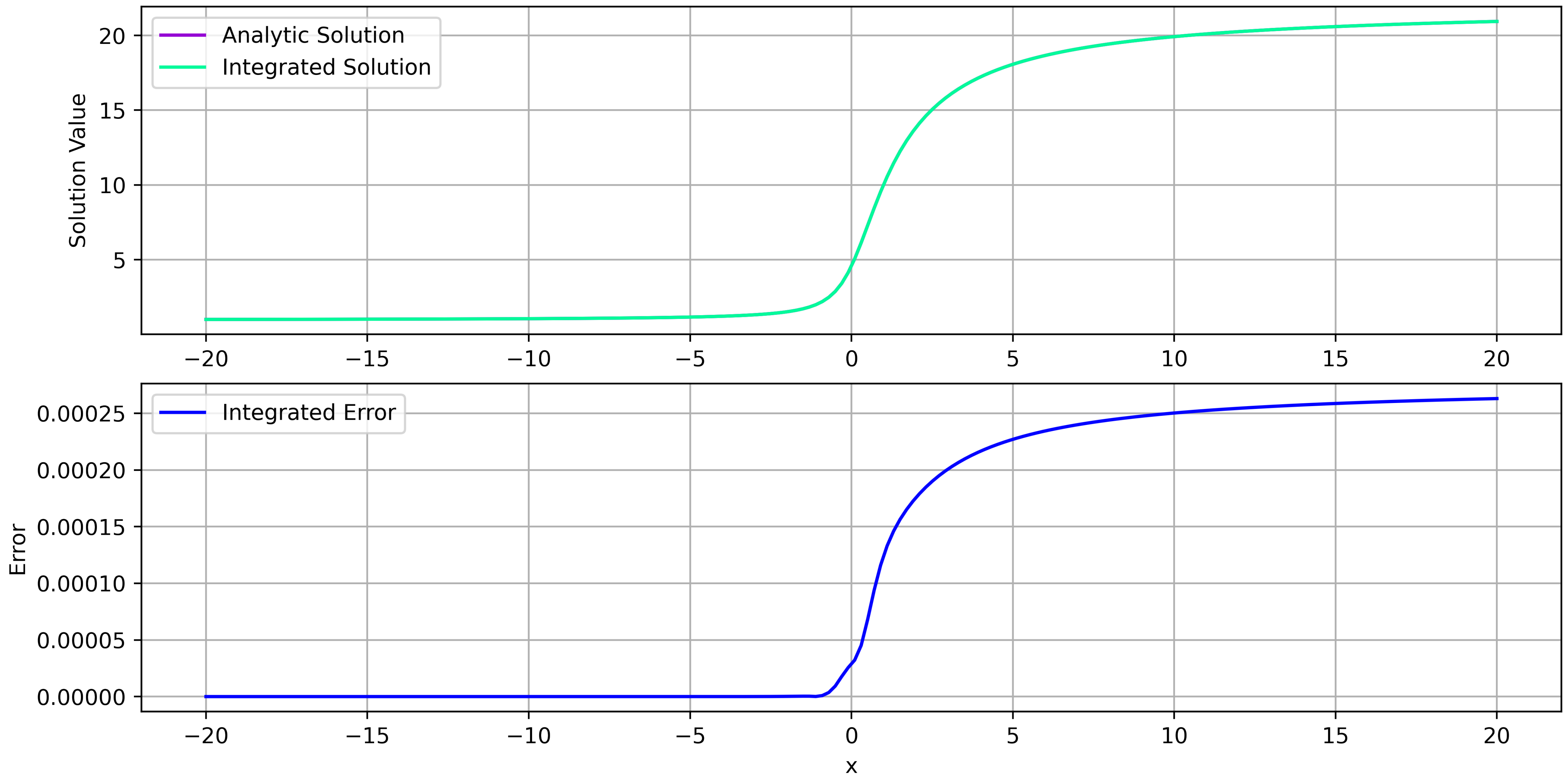
1) $dy/dx = \frac{y}{1+x^2}$, $\int \frac{dy}{y} = \int \frac{dx}{1+x^2}$, $\log y = \arctan x + \tilde{c}$, $\tilde{c} \in \mathbb{C}$

$\Rightarrow y(x) = e^{\tilde{c}} e^{\arctan x} \iff y(x) = C e^{\arctan x}$ as $\tilde{c}$ is arbitrary

$y(-20) = 1 = C e^{\arctan(-20)} \Rightarrow C = 1/e^{\arctan(-20)}$

$\Rightarrow y(x) = \left[1/e^{\arctan(-20)}\right] e^{\arctan x} = e^{\arctan x - \arctan(-20)}$

Using rk4_step, we integrate dy/dx & plot the analytic solution, the integrated solution, and their difference (error):


Analytic Solution
Integrated Solution
Solution Value
20
15
10
5
−20
−15
−10
−5
0
5
10
15
20
Integrated Error
Error
0.00025
0.00020
0.00015
0.00010
0.00005
0.00000
−20
−15
−10
−5
0
5
10
15
20
x

We see that the order of magnitude of difference is $O(10^{-4})$, with the error increasing for large x.

Now we want to compare one step h to two steps h/2 to cancel leading order terms from RK4.

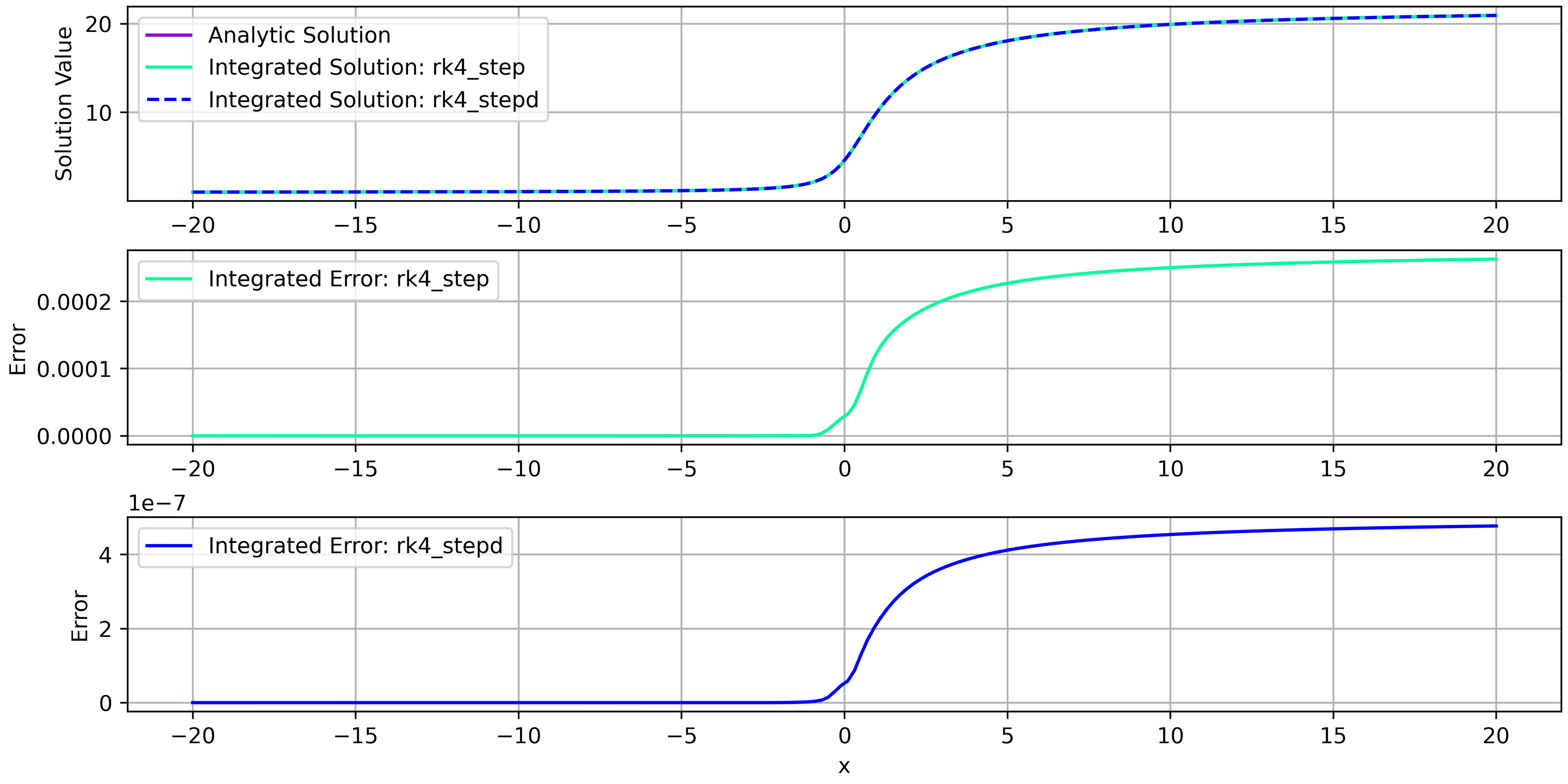
↳ We quote what is derived in Numerical Recipes:

$$y(x+h) = y_2 + \frac{\Delta}{15} + O(h^6)$$

Here y_2 is a solution approximation for two steps of size h/2 ; $\Delta \equiv y_2 - y_1$, & y_1 is a solution approximation for a step of size h.

↳ Each of the three RK steps (k_1, k_2, k_3) requires 4 evaluations of the functions, so 12 calls total. However, the y_1 & y_2 functions share the same starting point so $12 - 1 = 11$ is the amount of function evaluations per step.

Now, using rk4_stepd, we plot the two integrated solutions, the analytic solution, and their respective difference:



Analytic Solution
Integrated Solution: rk4_step
Integrated Solution: rk4_stepd
Solution Value
Integrated Error: rk4_step
Error
1e−7
Integrated Error: rk4_stepd
Error
x

As we can see from above, rk4_stepd has error w/ order of magnitude $O(10^{-7})$ which is 3 order of magnitudes more accurate than rk4_step.

2) a) For this problem we shall make use of the Radau method as it is much faster than Runge-Kutta

↳ The ODE we wish to solve has the following form:

$$\underbrace{d\gamma/dt = \tau}_{\text{matrix w/ decay rates}} \underbrace{\gamma}_{\text{amount of each decay product}}$$

Here $\tau \sim 1/\Gamma$ is calculated via computing the total width of all the decay channels $\Gamma = \sum_i \Gamma_i$ via Feynman diagrams.

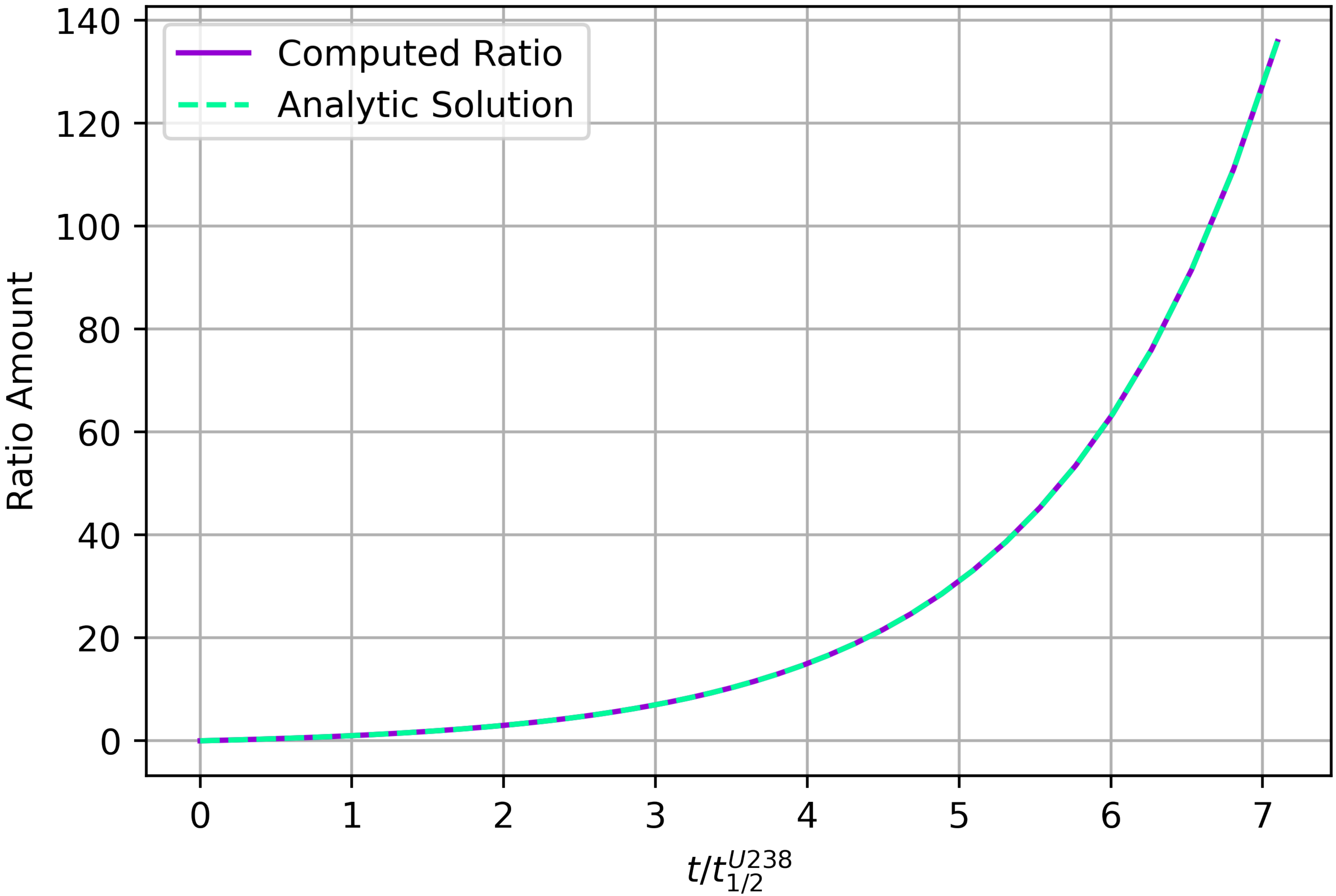
↳ Moreover, we assume we start from a pure sample of U238 so our IC is $\gamma(0) = 1$.

↳ Finally $\tau = \log 2 / t_{1/2}$ where $t_{1/2}$ is the halflife of the intermediate product

b) Looking at Wiki, the analytic form of the ratio of γ_{230} & γ_{238} is:

$$\gamma_{230}/\gamma_{238} = e^{\tau t} - 1$$

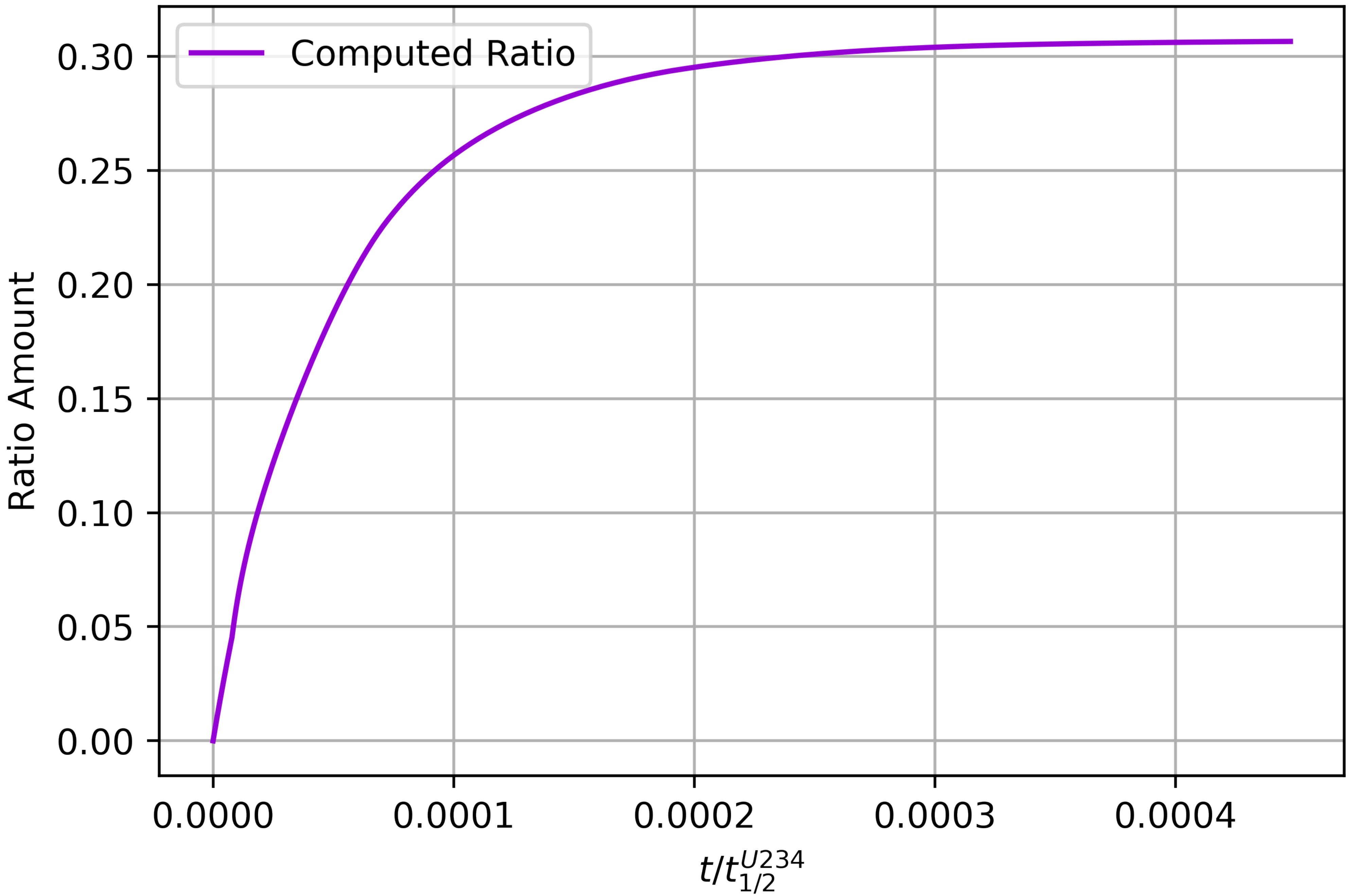
Thus we plot the ratio of Pb206 to U238 via our integrator, plot the analytic result:



140
120
100
80
60
40
20
0
Ratio Amount
Computed Ratio
Analytic Solution
0
1
2
3
4
5
6
7
$t/t_{1/2}^{U238}$

We can see it follows the analytic curve closely & thus it makes analytic sense.

Finally, we plot the ratio of Th230 to U234:


Computed Ratio
Ratio Amount
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.0000
0.0001
0.0002
0.0003
0.0004
$t/t_{1/2}^{U234}$

3)c) As given in the problem, a rotationally symmetric paraboloid has the form:

$$(z - z_0) = a\left((x-x_0)^2 + (y-y_0)^2\right)$$

We wish to linearize this via a new set of parameters.
↳ Let's isolate z & factorize:

$$z = a(x-x_0)^2 + a(y-y_0)^2 + z_0$$
$$= a(x^2 - 2xx_0 + x_0^2 + y^2 - 2yy_0 + y_0^2) + z_0$$
$$= \underbrace{a}_{D}(x^2+y^2) \underbrace{-2ax_0}_{B}x \underbrace{-2ay_0}_{C}y + \underbrace{(ax_0^2 + ay_0^2 + z_0)}_{A}$$

Our new params are thus $D = a$, $B = -2ax_0$, $C = -2ay_0$, $A = z_0 + a(x_0^2 + y_0^2)$ & thus our paraboloid becomes:

$$z = A + Bx + Cy + D(x^2+y^2)$$

b) Now we carry out the fit of the parameters. Our best-fit parameters are:

$x_0 = -1.360$, $y_0 = 58.221$, $z_0 = -1812.877$, $a = 0.000167$

c) Now we wish to compute the noise in the data & the uncertainty in a

↳ We take the focal length to be $f = 1.5\,m$

The computed uncertainty in a is:

$$\sigma_a = 2.66 \times 10^{-8}$$

Moreover, we compute the focal length & its error bar:

$$f = \frac{1}{4a} \doteq 1499.66 \text{ mm} \quad , \quad \sigma_f = \left|\frac{\partial f}{\partial a}\right| \sigma_a = \frac{\sigma_a}{4a^2} = 0.239 \text{ mm}$$