Non-perturbative insights from topological wormholes

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Abstract: Having a *global* definition of a theory is required for non-local effects such as wormholes. The wormhole structures — which usually occur over spacetime manifolds — can emerge in quantized symplectic spaces as *topological wormholes* which connects different independent-particle Hilbert subspaces. These wormholes affect the dynamics of the system non-perturbatively and so are crucial in describing global theories. We review the structure of wormholes in quantum information theory and topological quantum field theory. Furthermore, we initiate a study on the entanglement structure of qubits in potential-well lattices with topological wormholes. In these lattices we demonstrate the relationship between quantum tunneling events and wormholes.

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1 Introduction

A recent topic of interest in theoretical physics has been the proposition of *emergence*; is the macroscopic structure of our universe a consequence of its fine-structure at smaller length scales? In other words, are large scale system behaviours dependent on its constituent parts, or is it independent? These questions have garnered significant attention when describing the nature of our spacetime, and whether or not it possesses some deeper structure as a result of emergence.

The study of spacetime structure is a topic of great interest in the realm of quantum gravity. It can be approached from two distinct perspectives: quantizing classical gravitational theories and investigating theories where quantum effects and spacetime emerge as fundamental properties. The former involves methods like loop quantum gravity, which seeks to capture the discrete nature of spacetime at microscopic scales. On the other hand, the latter explores the intriguing possibility of the universe's geometry emerging through the interplay of fundamental processes, such as phase transitions driven by the renormalization group flow of the theory's coupling constants [1]. Further studies of spacetime emergence have come from the study of entanglement in black holes. Black holes pose interesting contradictions, such as the *black hole information paradox* whereby information input into black hole is lost since what is radiated by the black hole is featureless thermal radiation [2]. This means at some point the system loses information, which violates the conservation of information in an isolated system (our universe). To reconcile this inconsistency, it was proposed that the radiation emitted is entangled with the interior of the black hole [3], meaning that the information is still present in the system globally and there is no violation in the amount of information. The system is described by its *entanglement entropy* which describes the degree of which properties of objects are connected [4]. This can also be viewed as how much information is lost when the system is viewed partially. This picture later came with issues as when considering these black holes in the presence of external matter. As matter falls into the black hole — being that it eventually will be evaporated out from the black hole as radiation — the in-falling matter must be entangled with the radiation. This is problematic as the radiation is already maximally entangled with the inside of the black hole and this violates the monogamy of entanglement; maximal quantum entanglement cannot be shared amongst an arbitrary amount of parties. The reconciliation to this was that monogamy would not be violated if the radiation was *itself* also the interior of the black hole [5]. In this sense the radiation is not truly independent from its source, but is instead tethered to an *island* within the black hole. This presents an interesting example of emergence: the thermal radiation corresponds directly to a piece of space within the black hole. The features of the island emerges by the properties of its entangled radiation elsewhere. What is more is that the implication that the radiation is also the inside of the black hole seems to be contradictory being that they are physically separated in spacetime. However, the spatial separation implies a connection between the two in the form of a non-local extremal geometry: a *wormhole*. We will see further in this introduction that wormholes will be a center focus in probing the global properties of systems in quantum gravity.

The characterization of spacetime as an emergent property can be extended with the use of holography, specifically the AdS/CFT correspondence. This is a conjecture that relates the gravitational partition function in the bulk of the anti de Sitter (AdS) spacetime, to the conformal field theory (CFT) partition function on the conformal boundary of AdS. From this context we can interpret gravitational objects as fields on the boundary. This motivates the idea that the structure of spacetime bulk is an emergent property of a lower dimensional quantum system on the boundary [6], which is the principle of *holography*. A popular example is that pairs entangled particles on the boundary correspond to a wormholes in the bulk [5]. The idea of spacetime emergence from quantum mechanics was extended further in [7] in which pairs of entangled particles corresponds regions of the AdS space. A given amount of entanglement entropy corresponds to certain configuration of the bulk spacetime, which dictates the energy distribution within that space [8]. Furthermore, the entanglement entropy of the quantum particles could also in principle describe the entanglement entropy of matter in the bulk [9], or even endow spacetime with features that match those of general relativity [10].

Emergence of spatial structure is not unique to spacetime however, and also includes configurations of qubits and symplectic manifolds. A configuration of five highly entangled qubits can be used to store the information of a single virtual qubit. This construction is useful in the construction of quantum computers, as if a subset of the physical qubits are damaged, their entanglement makes it so that the information of the virtual qubit is not lost. In this sense we can think of this configuration as a model of emergent space [11]. The virtual qubit emerges as an atom of space from the entanglement structure of the qubits. Additionally, the entanglement structure of qubits turns out to be interesting as they describe different topological wormholes, which are identified by different geometric phases [12]. While our preceding discussions were of spacetime wormholes, *topological wormholes* are extremal geometries arising in the quantized phase spaces of quantum systems [13], which connect different independent system *orbits*. It was shown that topological wormholes occur in a large class of quantum mechanical systems, and not just holographic ones. Much like how there is a correspondence between entangled states and spacetime wormholes, it was also demonstrated that there is a connection between the topological wormhole partition function and the entanglement entropy of quantum states prepared in Euclidean spacetimes [14]. Wormholes are thus integral to describing the global structure of a system, whether it is a spacetime or symplectic manifold. We use the topological wormhole framework to describe entangled systems of qubits in potential-well lattices, and within it relate wormholes to quantum tunneling.

This paper is structured as follows. In section 2 we describe structure of wormholes with topology and look at their correspondences to entanglement with topological quantum field theory. In section 3 we look at wormholes occurring in quantum gravity. In section 4 we look at bipartite qubit systems subject to potential-wells which admit topological wormholes and relate them to qubit quantum tunnelling events. Finally, in section 5 we summarize the discussion.

2 Structure of Wormholes

The geometric construction of spacetime and topological wormholes relies on distinct frameworks. However instead of delving into both, we can gain insight into the nature of wormholes by focusing solely on topology. By employing concepts such as *foliations* and *cobordisms*, we can develop an intuitive understanding of the mechanisms underlying wormholes and their connection to entangled states.

2.1 Equivalence Classes

First, let's review the concept of *equivalence classes*. Equivalences classes are sets where all the elements are equivalent to each other in some way. Given a set X and some element $a \in X$, the equivalence class of a in X is given by:

$$[a] = \{x \in X : x \sim a\}. \tag{2.1}$$

Here $\sim$ is the *equivalence relation* which tells us how two elements are equivalent (the most common equivalence relation is the equal symbol $=$). Thus, the class $[a]$ is the set of elements of X that are equivalent to a, and is known as a *partition* of X. The set of all equivalence classes or partitions of X is known as the *quotient set*, which is defined as:

$$X/\sim = \{[x] : x \in X\}. \tag{2.2}$$

The quotient set is usually defined between two sets, such as X/Y for some other set Y. This specifies the equivalence relation where two elements of a partition of X are equivalent if they differ by an element of Y. What are some examples of equivalence classes, or more precisely, quotient sets? In the context of topology we can look at *homology* and *cohomology* groups over some manifold M to have some intuition, following [15]. We start with what is known as de Rham cohomology. Let C^p be set of closed p-forms ω_p such that $C^p = \{\omega_p : d\omega_p = 0\}$. Moreover, let E^p be the set of exact p-forms ν_p such that $E^p = \{\nu_p : \nu_p = d\alpha_{p-1}\}$, where α_{p-1} is some $(p-1)$-form. We can construct the de Rham cohomology group as the quotient set:

$$H^p(M) = C^p/E^p. \tag{2.3}$$

Here the elements of H^p are equivalence classes of closed p-forms on M, where the forms are considered equivalent if they differ by an exact form:

$$\omega_p \sim \omega_p + d\alpha_{p-1}. \tag{2.4}$$

The cohomology group actually tells us quite a bit about the topology of M, but this might seem too intuitive to think about this in terms of forms. We thus turn our attention to simplicial homology groups. Consider p-dimensional submanifolds of M, $N_i \subset M$, each labelled by an index i. We can consider a *p-chain* a_p as the sum over the submanifolds of M:

$$a_p = \sum_i c_i N_i, \tag{2.5}$$

where $c_i \in \mathbb{C}$ are coefficients. A *p-cycle* is a p-chain that does not have a boundary such that $\partial a_p = 0$. We can now define the sets which we will quotient to form the homology group. Let C_p be the set of cycles such that $C_p = \{a_p : \partial a_p = 0\}$, and B_p be the set of boundaries such that $B_p = \{b_p : b_p = \partial b_{p+1}\}$. The simplicial homology group of M is the quotient set:

$$H_p(M) = C_p/B_p. \tag{2.6}$$

The elements of the group are equivalence classes of p-cycles of M, where two elements of the equivalence classes are equivalent if they differ by a boundary:

$$a_p \sim a_p + \partial c_{p+1}. \tag{2.7}$$

This can be visualized by the following figure [15]:

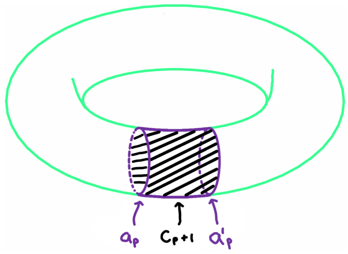

Figure 1. Visualization of the homology of a $(p+1)$-torus. Here a_p and a'_p are p-cycles of the torus, while c_{p+1} is a submanifold of the torus. The cycles a_p and a'_p are equivalent up to the boundary of the submanifold which separates them, given by ∂c_{p+1}. Thus we say a_p is equivalent to $a'_p \equiv a_p + \partial c_{p+1}$ as in equation (2.7).

What does this have to do with topology? Well first off we can construct the topological invariants based on these groups, which are quantities that are preserved under continuous deformations or diffeomorphisms of the space. For example, the dimension of the cohomology groups are the *Betti numbers* given by $b_p = \dim H^p$, which tell us the number of linearly independent harmonic p-forms on M. Additionally, this describes the amount of irreducible p-cycles of M. The connection between the homology and cohomology of M is given by the *Poincaré duality*, which is an isomorphism between the cohomology and homology groups:

$$H^p(M) \cong H_{n-p}(M), \tag{2.8}$$

which holds if M is a compact manifold for $n = \dim M$, and $p \in \mathbb{Z}_+$. Although it might not seem too informative, the cohomology group tells us what forms[†] can exist on M, and the forms correspond to field operators in QFT. These field operators excite the theory's vacuum to give rise to particles, and so we say the topology of the space M tells us exactly what kind of particles can exist on it. An example is the unit 2-sphere S^2 which has the Betti numbers $b_0 = 1, b_1 = 0, b_2 = 1$. Here $b_1 = 0$ tells us that S^2 does not admit a global 1-form or dual vector field, which is a manifestation of the *hairy ball theorem.* This is a direct result of the topology as if we punctured the unit sphere and deformed it to instead be a 2-torus T^2, b_1 would no longer vanish. In essence, the cohomology and homology groups tell us what forms and submanifolds of M are allowed based on its topology.

2.2 Foliations & Cobordisms

Now with some intuition for equivalence classes and topological invariants, we move onto foliations. A *foliation* is an equivalence relation on an n-manifold M through which we decompose M into equivalence classes of its submanifolds. The equivalence classes of submanifolds are known as the *leaves* of the foliation, and we say that M is foliated by

[†]For each p-form we have a corresponding rank p tensor field which is given by the *musical isomorphism* that maps between the cotangent and tangent bundles of M, given by $\sharp : T^*M \to TM$.

the leaves[‡], meaning that we decompose M into its equivalent submanifold constituents. Examples of foliations in physics include foliating a spacetime by decomposing it into constant-time hypersurfaces as seen in figure 2, or the foliation of symplectic spaces when performing geometric quantization.

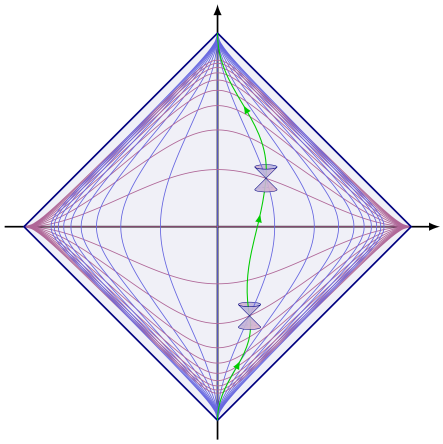

Figure 2. Conformal diagram of the Minkowski spacetime with curvilinear coordinates. The magenta curves represent an equivalence class of constant time slices of the spacetime, while the purple curves represent an equivalence class of constant position slices. The spacetime can be foliated by either equivalence class (the standard convention is foliation with constant time slices). The green curve represents a worldline of an observer with an associated lightcone.

Next, we move onto cobordisms which can be used describe wormholes and (entangled) particle creation in topological quantum field theories. A *cobordism* is an equivalence class of compact manifolds of the same dimension. Two manifolds M, N are *cobordant* if their disjoint union is the boundary of a compact manifold W which is one dimension higher; in other words $M \sqcup N = \partial W$. Furthermore, for these manifolds to be cobordant they must share topological properties such as Pontrjagin and Stifel numbers [16]. From this we can construct *cobordism classes* which consist of all the manifolds that are cobordant to a fixed manifold. More explicitly, two cobordisms in this class are considered equivalent if they can be continuously deformed into each other. *How exactly does this tie into the concepts of wormholes?* Consider the spacetime of a maximally extended Schwarzschild solution (the eternal black hole) foliated by constant-time slices. These time slices are extremal hypersurfaces of minimal area that connect regions of space [17], known as wormholes. In this sense, wormholes are cobordisms of spaces with equal dimension such that their disjoint union (the wormhole throat geometry) has minimal surface area. This describes all the different regions of space we could connect that are related under an equivalence. Wormholes belonging to the same cobordism class represent a set of wormholes that can be continuously deformed from one to another [18]. Each wormhole geometry corresponding

[‡]The word is based on the identical structure of tree leaves occurring in nature, given by equivalence classes of leaves for different tree branches. The inclusion of all the branches (partitions) gives the tree leaf structure (quotient group).

to a constant-time slice of the spacetime is an element within a cobordism class. This interpretation of wormholes holds for both the spacetime and topological kind.

What about the connection between wormholes and (entangled) particle creation? For this we must extend the framework of topology to include quantum fields by looking at 2+1D topological quantum field theories. In such a theory, observables are topological invariants meaning they do not depend on the spacetime geometry and particles are described by topological defects on compact surfaces [19]. Furthermore, the theory contains topological quantum numbers (topological charges) which are a consequence of the space's topological properties. Consider a compact disk over which we define a quantum field with a globally vanishing topological charge. We then puncture the disk such that it now has an internal and external boundary. Even though we have changed the topology of the disk, the global topological charge still vanishes. We can imagine changing the topology again with another puncture to have two internal boundaries within the disk. Despite the fact that a measurement of the topological charge globally would still result in zero, a measurement on each internal boundary separately could result in a non-zero value, meaning the presence of a topological defect [20] (and hence particles). One boundary can have an associated charge λ while the other boundary has the opposite charge $\bar{\lambda}$ which would preserve the condition of the charge vanishing globally but not locally, meaning we have a topological defect. In this sense this is the creation of a particle-antiparticle pair [20]. Furthermore, if these particles are prepared in such a way where their wavefunction cannot be factorized, they are quantum mechanically entangled. If you plot the spacetime history of these disks (the constant time slices of the spacetime history are the separated disks), we get the following [20]:

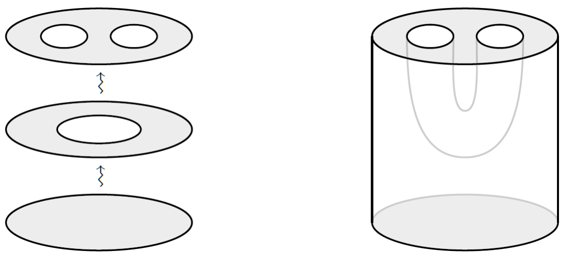

Figure 3. Spacetime history of puncturing a disk. The left plot shows the topological changes of the disk separately, while the right plot describes this change continuously over time as a spacetime history. The resulting configuration for late times corresponds to a pair creation of particles [20].

The spacetime volume can be seen as a cobordism with non-smooth boundary components or corners. The two manifolds which are cobordant in this case are a disk, and a disk with a handle glued to its surface. The cobordism essentially defines an evolution from an initial to final boundary condition of the spacetime. In this sense we could have alternatively taken the disk and created a depression that ends up connecting two pieces of the disk as [20]:

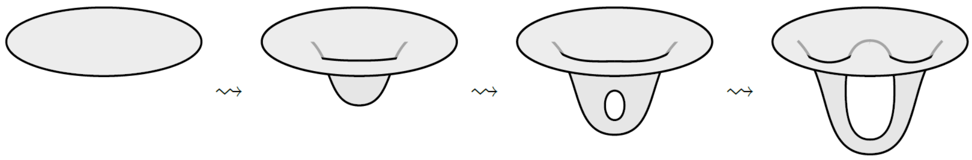

Figure 4. Topological modification of the disk over time. The disk is deformed to have a depression via a diffeomorphism, which is followed by a homeomorphism that allows for a non-trivial genus to form. The resulting geometry connects two regions of the disk via a tunnel [20].

This construction represents a wormhole connecting two regions of the disk. Both the deformations of the disk which gave rise to either particle creation or wormholes result in equivalent spacetime histories. This means that if we plotted figure 4 as a spacetime history, it would result in the identical spacetime volume as in figure 3. This shows us that wormholes correspond to (entangled) particle creation — such as is proposed by the ER=EPR conjecture [5] — via cobordisms.

3 Wormholes in Quantum Gravity

Now we turn our attention to wormhole cobordisms occurring in quantum gravity. First we present a concise overview of spacetime wormholes and their dual entangled states in the context of AdS/CFT. Then we look at when the systems space or bundle admits non-trivial holonomy and how it gives rise to hidden information. Finally, we look topological wormholes which manifest in quantized symplectic spaces.

3.1 Spacetime Wormholes & Holography

Here we look at the spacetime framework of eternal black holes in the context of holography. The eternal black hole spacetime, also known as the *maximally extended Schwarzschild black hole*, has its boundary and singularity identified in multiple universes and exists for all time (even before the big bang in inflationary models). This is due to the spacetime not admitting a global time-like killing vector being that the different definitions of time flow on the boundaries of the black hole generate a topological defect at the event horizon. A constant time slice of the spacetime geometry leads to a hypersurface that connects the spaces of the two universes, a geometry known as a *wormhole* or an *Einstein-Rosen bridge*. The interpretation is that the interior of the internal blackhole is the bulk of the wormhole; to traverse the wormhole you must enter the blackhole. The spaces which the wormhole connects are cobordant if have the same dimension and their disjoint union is the boundary of a compact manifold which is one dimension higher (the 3+1D spacetime slices). In general these wormholes are non-traversable as the wormhole throat shrinks in size as an observer enters the eternal black hole's horizon in their universe and approaches the singularity [5]. What's more is that the wormhole geometry grows with the expansion of the universe, so even if the opening radius is fixed, an observer would be stuck inside the wormhole once the throat ends become causally disconnected. However, it was found that if an interaction is turned on that couples the two boundaries[†], the quantum-matter stress tensor ends up having negative average energy and this prevents the throat opening from closing [21]. After the gravitational field of the wormhole interacts with the background spacetime (gravitational backreaction), the wormhole is rendered traversable. Furthermore, the study also revealed that infinite null geodesics which enter the wormhole must be chronal (sets of points are chronal if any two points can be connected by a timelike curve) and so wormholes *cannot* be used to violate causality or for faster than light travel.

These wormholes are particularly interesting when the spacetimes they connect are anti-de Sitter space (AdS). This is because of the AdS/CFT correspondence, a conjecture which proposes a duality between gravitational theories on the bulk with conformal field theories on the conformal boundary in the form of related partition functions [22]. To illustrate this, consider a massless scalar field ϕ in the bulk of an $(n+1)$-dimensional anti-de Sitter space, AdS_{n+1}. The restriction ϕ to the conformal boundary is denoted as ϕ_0 and is coupled to a conformal field $\mathcal{O}$ (a field that is invariant under the conformal symmetry group) under the coupling $\int_{S^n} \phi_0 \mathcal{O}$, where S^n is the conformal boundary of AdS_{n+1}. We can relate the supergravity partition function Z_S on the bulk associated with ϕ, with the conformal field partition function on the boundary as:

$$Z_S[\phi_0] = \left\langle \exp \int\limits_{S^n} \phi_0 \mathcal{O} \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{CFT}}[\phi_0]. \tag{3.1}$$

This duality allows us to identify gravitational objects in the bulk to conformal fields on the boundary of AdS. Perhaps the most popular correspondence comes from the ER=EPR conjecture which states that wormholes in the spacetime bulk are dual to quantum entangled states living in the CFT on the boundaries. For an external black hole that connects two AdS spaces (where they are denoted as the left and right AdS spaces), we have two copies of CFTs that live on the boundaries of the two connected spacetimes. Such a wormhole connecting the two spacetimes corresponds to an entangled state that live in both CFTs. The entangled state between two identical CFTs is known as the *thermofield double state* (TFD) and is written as:

$$|\Omega\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} \, |n\rangle \otimes |n\rangle^* \,, \tag{3.2}$$

where $Z(\beta) = \mathrm{tr}(e^{\beta H})$ is the thermal partition function of the state for a fixed inverse-temperature β and Hamiltonian H, E_n are the energy eigenvalues of the Hamiltonian, and $|n\rangle^*$ is the CPT conjugate of $|n\rangle$ to account for time flow running in different directions on the conformal boundaries. The energy eigenstates live in the Hilbert vector space $\mathcal{H}_L \otimes \mathcal{H}_R$ over the boundary, where $(\mathcal{H}_L, \mathcal{H}_R)$ are the Hilbert spaces of the left and right CFTs, respectively. The state is prepared via path integrals in the Euclidean AdS spacetime, AdS_E, which is represented as boundary conditions on hypersurfaces of AdS_E [23]. The state is then evolved in time in the usual Lorentzian AdS space. This represents an *entangled state*

[†]The interaction comes from a deformation of the theory's action with an extra term in the form of a product of two single trace operators (a trace of a matrix product of field operators).

as it cannot be factored out into a single product state. The holographic duality gives us a correspondence between wormholes and entanglement.

Finally, spacetime wormholes give extra contributions to the Witten diagrams (roughly speaking they are Feynman diagrams projected onto the Poincaré disks of AdS space) when computing correlations functions in the CFT [24]. This develops in the form of extra terms known as defects, which connect different parts of the conformal diagram via wormholes. This allows for the boundary operators to evolve in different ways, such through the throats of the wormhole. In the following section we look at what happens when the systems space or bundle admits a non-trivial holonomy.

3.2 Hidden Information in Holonomy

There is more to the story when the manifold admits non-trivial holonomy, which in essence captures the non-commutative nature of the space we are working with. If the space is in fact a *fibre bundle* then the holonomy measures how much the endpoints of a closed path in the base space differ when uplifting the path to the total space of fibres [25]. In the case of the eternal black hole, the non-trivial holonomy arises from the topological defect at the event horizon which causes a discontinuity in uplifted time-like Killing paths. This leads to wormhole contributions in the gravitational path integral due to a non-exact symplectic form [26], which will be discussed in the following section. What's more is that in physics then holonomy is often referred to as a *geometric phase* and is attributed to hidden information within a quantum system. This is because manifolds admitting a non-trivial holonomy require multiple coordinate patches and locally a physical observer only perceives one patch of the base manifold. This means that they cannot know that the system admits a geometric phase defined via a path that goes through multiple patches. This phase tells us whether or not the structure is a product space or a fibre bundle and so this information is hidden from a local observer [25]. In the following we look at the consequence of geometric phases and how they distinguish between entangled states with the same entanglement entropy.

Consider the conformal spacetime geometry of an eternal blackhole, consisting of two AdS spacetimes (and their respective conformal boundaries), and the interiors of the black and white holes. If we foliate the spacetime by constant-time hypersurfaces (as in figure 2), the leaf associated with $t = 0$ corresponds to the usual wormhole geometry connecting two spatial regions of the AdS spaces. A geodesic on this geometry is holographically dual to thermofield double state $|\Omega\rangle$ living in the Hilbert space over the conformal boundaries of the leaf. Although the Hilbert space on the boundary factorizes into a tensor product of the Hilbert spaces at each assymptotic boundary, we cannot factorize the bulk Hilbert space over each leaf due to the presence of wormhole structures [25], which is the same reason why we cannot factorize of the wormhole partition function globally. Leaves with $t > 0$ instead correspond to time-shifted wormhole geometries, and the geodesics within those geometries are dual to TFD-like states that have additional phases [12]:

$$|\Omega^\alpha\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{i\alpha_n} e^{-\beta E_n/2} |n\rangle \otimes |n\rangle^* \,, \tag{3.3}$$

where α_n are phases that distinguish the different entangled states and hence distinguish different wormhole leaves. These states are interpreted as the microstates of the eternal black hole [27]. The time-shifted wormholes have the same geometry but have different identifications of boundary times. One musts ask where exactly do these phases come from? The spacetime manifold is not necessarily connected and so must be described by different coordinate charts. Consider the left and right copies of AdS to have the respective coordinate charts (ε_L, L, t_L) and (ε_R, R, t_R). Here ε_L is the asymptotically flat region around the left wormhole throat, L is the boundary of the left AdS space, and t_L is the time coordinate flowing along the left boundary. These charts overlap in the interface of the asymptotic regions around the throats, and at the wormhole horizon (interface of the interior of the wormhole where the left and right throats connect). In other to keep the spacetime manifold well defined, we must make it so that the transition functions between the charts are smooth, resulting in the relations: at the interface of the asymptotic regions $t_L = t_R$, and at the interface at the horizon $t_L = 2\delta - t_R$, where δ is a fundamental degree of freedom of the system [28] (which is actually an element of the system's moduli space; more on this in the qubit section). With this parameter the phases can be computed as $\alpha_n = -2E_n\delta$.

What happens when we transport a particle around a closed geodesic on one of these wormhole leaves? To transport it we require groups that generate symmetries and so evolve the states. Consider a *Lie symmetry group* G which also has the structure of a manifold[†]. We will need a connection if we wish to describe derivatives covariantly and preserve group symmetries. This can be represented as the *Maurer-Cartan* form $\Gamma = g^{-1}dg$ which is the natural connection on a group manifold, where $g \in G$ and $dg \in T_g^*G$. This form carries the basic infinitesimal information about the symmetries and transformations associated with G. Moreover, the tangent space at the identity of the group, $T_{\mathbf{1}}G$, is the vector space over G known as the *Lie Algebra* $\mathfrak{g}$. This consists of generators which encode the dynamics of quantum states in spacetime via the exponentiation of the Lie algebra elements.

With this we move onto parameter spaces, which is the space of external parameters that define the system's Hamiltonian. For example, these parameters could be the components of an external gauge field as in the Aharonov-Bohm effect. When a system evolves adiabatically, the parameters vary slowly such that the system stays in an eigenstate of the Hamiltonian. When the system returns to its initial state after completing a cyclic evolution in parameter space, the quantum state picks up a non-trivial phase known as a *geometric phase* (more commonly known as the *Berry phase*) which is an observable property of the system. This evolution corresponds to parallel transporting a quantum state along a closed loop in parameter space. It is noted that in the case of the Aharonov-Bohm effect the charged particle moves in a loop in physical space which *corresponds* to a closed parallel transport loop in the parameter space which ultimately gives the particle state the phase. Explicitly, the particle moving around the solenoid evolves the system in such a way that a closed loop path is followed in parameter space, and it is **coincidental** that physi-

[†]This can be extended in generality by having a fibre bundle of group manifolds in which the connection endows a covariant derivative on the fibers of the group bundle.

cally the particle also moves in a loop. This is analogous to the case of wormholes where transporting a pair of entangled particles through the wormhole throats in a closed loop happens to change the system's Hamiltonian to go around a loop in parameter space and give us a Berry phase. Being that the geodesic that generates the Berry phase is dual to an entangled state, we must evolve both particles via some *unitary* group elements $U_a \in G$ to preserve probability amplitude normalizations. Moreover, what if one of the entangled particle say interacts with an external field? This gives us distribution of entangled states by deforming the unitary operators of one of the Hilbert subspaces. We define the unitary group representation $U = U_L \otimes U_R$, where $U_{L/R}$ acts on the left/right CFT Hilbert space. We can consider a continuous parameter $\lambda \in [0, 1]$ to deform U_R such that $U_R = \mathbb{1}$ for $\lambda = 0$ and $U_R = U_L$ for $\lambda = 1$ [12]. This gives us a continuous spectrum of entangled states for different values of λ which have the same entanglement entropy (the unitary operators can be moved around in the trace functional and cancel in the definition of the entanglement entropy).

Finally, to describe the Berry phase we must make use of the Berry connection, which is defined in terms of the previously mentioned Maurer-Cartan form $\Gamma = U^{-1}dU$ for a group element $U = U(\lambda)$ that distinguishes different entangled states for different λ. Given the form Γ, we can define the *Berry connection* for a given state which we want to consider (the phase-shifted TFD state $|\Omega^\alpha\rangle$) as:

$$A = i \langle\Omega^\alpha| \Gamma |\Omega^\alpha\rangle . \tag{3.4}$$

Here the Berry phase is the holonomy of the Berry connection. To compute the phase we must define the *Berry curvature* form F which is given by $F = i \langle\Omega^\alpha| dA |\Omega^\alpha\rangle$, where dA is the associated symplectic form of the parameter space which is not globally exact. The *Berry phase* Φ is then given integrating the curvature over a closed loop in parameter space γ:

$$\Phi = \oint_\gamma F(\lambda). \tag{3.5}$$

As long as $\lambda \neq 1$ so that $U_L \neq U_R$, the system admits a non-zero Berry phase. Using the inverse function theorem for a well-behaved function $\Phi(\lambda)$, we can invert this relation to get $\lambda(\Phi)$. Thus, the Berry phases distinguish a class of entangled states with the same entanglement entropy/structure. This corresponds to transporting particles along closed loop geodesics on the different time-shifted wormhole leaf geometries. Furthermore, the class of states with the same entanglement entropy but different Berry phases is a manifestation of the non-factorization of the leaf Hilbert spaces [12], and that the symplectic form cannot be globally exact. Finally, there are different types of Berry phases in holographic CFTs which are classified by the type of bulk diffeomorphisms that are involved [29].

3.3 Topological Wormholes in Symplectic Spaces

We now consider topological wormholes arising in theories of quantum mechanics. We look at how we can arrive at these geometries by quantizing a classical symplectic space via geometric quantization, and looking at orbits occurring in it via symplectic reduction. We first look at quantizing a phase space, and then look at it in the context of wormholes.

3.3.1 Geometric Quantization

Let's start with classical mechanics. Consider a $2n$-dimensional phase space Σ with coordinates $\sigma^a = (q^1, \ldots, q^n, p^1, \ldots, p^n)$, where q^a are the generalized position coordinates, and p^a are the generalized momentum coordinates. Much like how spacetime is endowed with a symmetric bilinear two-form $g_{\mu\nu}$ – the metric tensor – phase spaces are symplectic manifolds which are endowed with a non-vanishing antisymmetric two-form:

$$\Omega = \frac{1}{2}\Omega_{ab} d\sigma^a \wedge d\sigma^b. \tag{3.6}$$

Here Ω_{ab} are the components of the two form, represented in the cotangent bundle basis $d\sigma^a$. Being that Ω vanishes nowhere on Σ — meaning it is non-degenerate — Ω_{ab} has an inverse. The symplectic form encodes the dynamics of the classical system. To see this, consider two functions $f, g \in C^\infty(\Sigma)$, the Poisson bracket of them takes the form [30]:

$$\{f, g\} = \Omega_{ab} \frac{\partial f}{\partial \sigma^a} \frac{\partial g}{\partial \sigma^b}. \tag{3.7}$$

Moreover, we can use this definition of the Poisson bracket to evolve functions in time in the following form:

$$\dot{f}(t, \sigma^a) = \left(\frac{\partial}{\partial t} - \{H, \cdot\} \right) f, \tag{3.8}$$

where H is the Hamiltonian of the system which describes time flow of the system over the symplectic space, and $\{H, \cdot\} f = \{H, f\}$. The Poisson bracket is closely related to the Lie derivative via $\{f, g\} = \mathcal{L}_{X_g} f$ (where X_g is the vector field associated with the function g) and so it gives us information about symmetries and conserved quantities of the system.

Thereafter, we can consider a coordinate system of the phase space which diagonalizes the Hamiltonian, which are called *action angle coordinates*. This allows us to study the normal modes of a system without having to solve the equations of motion. This is a canonical transformation from (q^a, p^a) to (J^m, θ^m) which preserve the structure of the equations of motion. Here J^m are the *action coordinates* given by $J^m = \oint dq^m p^m$ which are conserved quantities relating to the energy of the system and the amplitude of the oscillation modes. θ^m are *angle coordinates* which are the canonical conjugates to J^m, which are related to the phases of the oscillation modes. The intuition for these coordinates can be found when considering orbits. The set of all trajectories of a subsystem in phase space given by the action of a symmetry group are known as *orbits*. These orbits represent the evolution of the system and are usually parametrized by time. Along these orbits J^m is conserved, so the interpretation is that J^m label different orbits in phase space that the system can evolve through and θ^m can parametrize these orbits.

Now, instead of diagonalizing the Hamiltonian via a canonical transformation, we can instead consider an unperturbed diagonal Hamiltonian which is a function of the action angle coordinates $H_0(J)$, in the presence of a perturbing Hamiltonian which is a function of the phase space coordinates $H'(\sigma)$ such that $H' \ll 1$. This gives us a Hamiltonian of the form $H = H_0(J) + H'(\sigma)$ [14] (where H_0 not a function of θ being that its canonical conjugate is conserved) and this allows us to use the framework of perturbation theory. This Hamiltonian results in the symplectic form picking up an extra term:

$$\Omega = \Omega_{ab} d\sigma^a \wedge d\sigma^b + \delta_{mn} dJ^m \wedge d\theta^n, \tag{3.9}$$

where δ_{mn} is the usual Kronecker delta matrix. Moving forward, we want to quantize the phase space so that we can consider a quantum theory. This will be done via geometric quantization, but alternatively deformation quantization works as well. We will present a brief overview of the procedure using natural units, but a rigorous treatment of it can be found in [31]. The procedure is as follows:

We want to lift classical observables from our symplectic manifold Σ to quantum operators in some Hilbert space $\mathcal{H}$ in a way that preserves the algebraic structure of Σ. The first step is known as *pre-quantization* in which we define a pre-quantum Hilbert space $\tilde{\mathcal{H}}$. We begin by defining a line bundle L (a vector bundle where the fibres are one-dimensional vector spaces) L over Σ which is equipped with a $U(1)$-connection such that the curvature form is $i\Omega$. Explicitly, we say the symplectic form is the curvature form of a $U(1)$-principle bundle written as the fibration $L \rtimes \Sigma$. This bundle is called the *pre-quantum line bundle* [31]. This construction requires that Ω obeys the Bohr-Sommerfeld condition which states that $\Omega/2\pi$ forms an integral cohomology class. Essentially this means that integrals of $\Omega/2\pi$ over cycles of Σ must be integers. We define the *pre-quantum Hilbert space* $\tilde{\mathcal{H}}$ as the collection of square-integrable sections of L. We can now begin our construction of pre-quantum operators.

For a classical observable on the phase space given by a smooth function $f \in C^\infty(\Sigma)$, the associated pre-quantum operator is the linear map mapping:

$$Q(f) : \Gamma(L) \to \Gamma(L). \tag{3.10}$$

Here $\Gamma(L)$ is the space of smooth sections of the line bundle, whose elements are the pre-quantum states. If we select $\psi \in \Gamma(L)$, the map Q acts on it in the following way [32]:

$$\psi \mapsto -i\nabla_{v_f}\psi + f \cdot \psi, \tag{3.11}$$

where ∇_{v_f} is the covariant derivative of sections along v_f which is specified by the bundle connection, and v_f is a Hamiltonian vector field† corresponding to the function f. The pre-quantum operators satisfy the following commutator algebra:

$$[Q(f), Q(g)] = iQ(\{f, g\}), \tag{3.12}$$

where $[,]$ are the usual commutator brackets associated with the Lie derivative.

We are ready to move onto the next step of geometric quantization (ignoring the metaplectic correction for non-trivial topologies): *polarization*. In essence, the pre-quantum Hilbert space $\tilde{\mathcal{H}}$ is too big in the sense that phase space is much larger than the physical

†A vector field is *Hamiltonian* if the flow is generates on Σ describes the time evolution of states.

configuration space. So what we do is carefully select a subspace of the pre-quantum Hilbert space $\tilde{\mathcal{H}}$ such that we eliminate redundancies and ensure that the resulting quantum theory captures the relevant degrees of freedom. First, consider the tangent bundle $T\Sigma$ associated with the symplectic phase space Σ. To capture the complex degrees of freedom present in quantum mechanics we complexify the tangent bundle by endowing it with a complex structure given by the two-tensor J^i_j such that $J^2 = -\mathbb{1}$. A *polarization* is a choice of a Lagrangian subbundle of the complexified tangent bundle $T\Sigma^{\mathbb{C}}$, or rather at each point in the complexified tangent space we select a Lagrangian subspace. A subspace is *Lagrangian* if it is isoptropic and is half the dimension of the space it is a subspace of. In this sense these subspaces of the tangent bundle on Σ form an integral distribution of subspaces which foliate Σ. With this we can define the *quantum Hilbert space* $\mathcal{H}$ to be the space of all square-integrable sections of L that are covariantly constant in the direction of the polarization. This might be mysterious at first glance, but the sections are vector fields which don't change orientation when parallel transported about the polarization subspace. This condition ensures that resulting quantum states are compatible with the classical and quantum symmetries of the system. Using the quantum Hilbert space, our pre-quantum operators become quantum operators in the theory $\hat{Q}$ which act on elements of the Hilbert space $\mathcal{H}$. Now that we have a notion of quantizing a classical phase space to produce a quantum theory, we move onto looking at this in the context of topological wormholes.

3.3.2 Symplectic Reduction

Now we can finally look at topological wormholes occurring in quantum theories. For a generic theory of classical mechanics, we can write the partition function of the theory at a fixed inverse temperature β as the following Euclidean path integral representation:

$$Z(\beta) = \int [d\sigma] \exp\left\{ \int\limits_{D} \Omega - \int\limits_{\partial D} dt\ H \right\}, \tag{3.13}$$

where $[d\sigma]$ is the usual product path integral measure associated with phase space coordinates σ^a, Ω is the symplectic two-form, H is the Hamiltonian, and $D \subset \Sigma$ is a two-dimensional submanifold of the symplectic phase space Σ. The argument of the exponential is the action of the system, and the perturbative expansion of the gravitational path integral includes contributions arising from wormhole structures. Now, being that Ω is closed such that $d\Omega = 0$, its integral over D is a topological invariant and so the geometric properties of D don't matter, only its topology [14]. With this we can select the most trivial topology for D, taking the form of a 2-disk which is embedded in Σ. The boundary of this disk is an *orbit*, which is the space of all trajectories through which a state can evolve in phase space that represents the time evolution of a system. Because of this is it natural to parametrize the boundary of D with time t and it represents independent orbit of a particle. One could imagine connecting two disks between two independent particle orbits which otherwise would never intersect, via a minimal surface area hypersurface, a wormhole. In this sense, the *topological wormhole* connects independent particle orbits in phase space which can be extended to connecting n-particle orbits, which forms the *n-fold trumpet geometry.*

This is identical in topology to the *n-fold replica wormhole geometry* which are spacetime wormholes that connect n spacetimes of the same dimension. It is noted that thus far this subsection has been classical.

What about the quantum operators associated with the wormhole on these phase space subspaces? The last step we have to do is known as *symplectic reduction*, a process which involves looking at a subregion of the symplectic space Σ (an orbit) while preserving the structure and symmetries of the system. The Guillemin-Sternberg geometric quantization conjecture (proven in [33]) states that the order of geometric quantization and of symplectic reduction can be interchanged without changing the result, so we chose to reduce and then quantize. This is to simplify calculations which might lead to non-transcendental solutions. The procedure is as follows. As is usual in quantum theories, consider a connected Lie group G which acts on a manifold, which in this case is our symplectic manifold (Σ, Ω) with a symplectic form/structure Ω. The action of a Lie group is said to be *Hamiltonian* when the elements of its associated Lie algebra, denoted as $\mathfrak{g}$, possess associated vector fields that are Hamiltonian. The Hamiltonian action of the lie group on Σ is equivalently described by the *moment map* μ:

$$\mu : \Sigma \longrightarrow \mathfrak{g}^*, \tag{3.14}$$

where $\mathfrak{g}^*$ is the *dual Lie algebra* (much like how the tangent bundle is dual to the cotangent bundle). We consider the identity of $\mathfrak{g}^*$ being $0 \in \mathfrak{g}^*$ and consider the submanifold this corresponds to in Σ given the inverse map $\mu^{-1}(0) = M \subset \Sigma$. The logic behind selecting this region is similar in principle to working near the identity of the lie algebra to generate its elements via an exponential map; there is enough information within M that we don't need to consider the full symplectic space Σ. If G acts freely on M (it has no non-trivial fixed points) then it turns out that if the quotient space M/G is a smooth manifold, then it inherits a non-trivial symplectic structure. Thus we say that M/G is the *symplectic reduction* of Σ that inherits a unique symplectic form ω whose pullback from Σ to M is exactly the restriction of Ω to M. In this case we call M/G the *orbit* of the system which is the set of elements in M that can be moved by elements of G. On these orbits the symplectic form is **not** exact is an indicator of the presence of topological wormholes. This relates direct to non-factorization of the partition function of topological wormhole (we will see in a bit it is also an indicator of entanglement).

Now that we have selected an orbit for the system which will reduce calculation complexity, we quantize it using geometric quantization as we covered before. From the classical observables of the system we can construct the set of quantum operators acting on the Hilbert space as [14]: $\mathcal{O}(\sigma, \theta) = J^m(\sigma), \sigma^a, W(\theta)$. Here σ^a commute with the unperturbed Hamiltonian H_0 as such are *Noether currents*, J^m are *Casimir operators* (such that $J^2 = \sigma^a \sigma_a$), and $W(\theta)$ are *open Wilson lines.* Similar to usual spin eigenkets $|j, m\rangle$ in quantum mechanics, here we have eigenkets $|j, m, s\rangle$ associated with the eigenvalues of the operators $\mathcal{O}(\sigma, \theta)$, where the extra quantum number s comes from a degeneracy of irreducible representations of the σ operator algebra with the same value of the Casimir eigenvalue. The different quantum numbers (j, m, s) correspond to whether or not the orbits that are connected by the topological wormholes are classically correlated, quantum entangled, or classically uncorrelated [13]. It is noted that the different values of the Casimir eigenvalues corresponds to different leaves of the foliation of the complexified tangent bundle when performing a geometric quantization of the orbit.

Finally, much like how there is a correspondence between spacetime wormholes and entangled states when looking at AdS spaces, here we have a correspondence in the form of the partition function of the n-fold topological wormhole is identical to the n-th Rényi entropy (entanglement entropy) of a thermo-mixed double state. A *thermo-mixed double state* is similar to the thermofield double state, with the exception that the two CFTs need not be identical, and that the two systems the wormhole connects are different inverse temperatures β. In this sense spacetime wormholes connect different universes and topological wormholes connect different orbits, both which are dual to entangled thermofield double states. In the following work we look at applying this formalism to qubit systems admitting topological wormholes which are subject to arbitrary well potentials.

4 Wormholes in Qubit Networks

We now look at topological wormholes in the context of qubit systems, and how its entanglement structure may be modified with an arbitrary distribution of potential wells.

4.1 Entanglement Structure of Qubits

There is a lot of mention of how the non-exactness of the symplectic form Ω gives rise to wormhole geometries, but it is not very intuitive. We can have a clearer picture of this manifestation in the context of qubit systems. We will also see how to further characterize different entangled states through different orbits.

In classical computers, the infinitesimal information required for computations are given by *bits* which take a value from the set $\{0, 1\}$. On the other hand, *qubits* are an extension of this where we can have some state $|\psi\rangle$ that it is a superposition of these binary values, given by $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ for some coefficients $\alpha, \beta \in \mathbb{C}$. The advantages of using such a construction for computations in a quantum computer is the non-localization of information through entanglement. For a system admitting n-entangled qubits, if a subset of the system is corrupted, the information is still globally preserved. This is characterized by the entanglement entropy which tells you how much information is lost when looking only at a subset of the system.

Now, to preserve unitarity of the system we require that the state evolves under unitary operators which make the norms of quantum states invariant. Consider a basis in $\mathbb{R}^3$, where on one of the axes we associate the positive direction to be the state $|0\rangle$ and the negative direction to be $|1\rangle$. The state $|\psi\rangle$ is some vector represented in $\mathbb{R}^3$ in the qubit basis where its direction dictated by α, β and has unity norm. We can consider the action of a unitary symmetry group on the state which rotates it, and keeps it in a superposition of $|0\rangle$ and $|1\rangle$. If we apply all possible unitary transformations, this traces out a unit 2-sphere which represents all possible states of $|\psi\rangle$: its Hilbert space $\mathcal{H}$. This is known as the *Bloch sphere* and looks like the following:

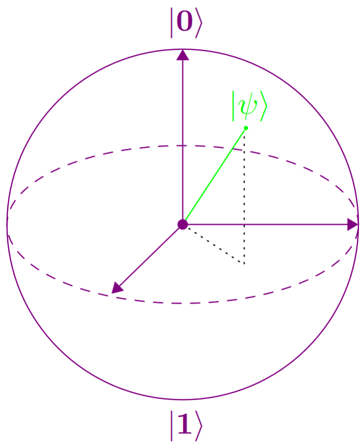

Figure 5. Representation of a spin-1/2 particle with the Bloch sphere in three dimensions.

The Bloch sphere is a Hilbert space for a single particle with spin, and is described by the complex projective space $\mathbb{C}\mathrm{P}^1$ with coordinates z_i. The associated spin operators of the Bloch representation are $S_a = (1/2)z_i^* \sigma_a^{ij} z_j$, where σ_a are the usual spin-1/2 Pauli matrices [12]. For a system of two qubits (which we take to be entangled) with no interaction or external fields, the system is represented as $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. This can be embedded as diagonal blocks in $\mathbb{C}\mathrm{P}^3$ and explain the local properties of the system. However, for *non-local correlations* such as those that arise from wormhole geometries, the embedding is no longer diagonal and the full $\mathbb{C}\mathrm{P}^3$ system must be considered. This is analogous to spacetime metrics no longer being diagonal when some external field is turned on and it breaks some of the spacetime isometries. This gives us a new interpretation of the condition of the symplectic form: globally we cannot make the symplectic form exact nor diagonally embed $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ into $\mathbb{C}\mathrm{P}^3$ due to non-local contributions arising from wormhole geometries. However, locally these contributions are not present and so we can construct the symplectic form to be exact and diagonally embed the two-qubit system. In this case, much like the presence of a non-exact symplectic form, the non-diagonal embedding means the presence of topological wormholes in the phase space of the quantum system.

What about the entanglement structure of these pairs of qubits? Let's consider the case where an entangled pair of qubits lives on the conformal boundaries of the eternal black hole. To look at its entanglement structure we must consider moduli spaces with orbits fibred over them. In quantum gravity, the *moduli space* is the space of all possible solutions or configurations of a physical system. An example of such is the space of all possible metrics endowed on the internal space to be compactified in string theory. Now, for each asymptotic boundary (left and right) we consider a subset of diffeomorphisms that leave the conformal boundary conditions invariant, given by $G_{L/R}$. This gives us a total asymptotic symmetry group: $G_L \times G_R$. Moreover, the set of diffeomorphisms that leave the spacetime bulk information invariant is given by the diagonal subgroup of $G_L \times G_R$ and is given by G_D [25]. Thus, we define the moduli space $\mathcal{G}$ of the system as the quotient of these groups:

$$\mathcal{G} = \frac{G_L \times G_R}{G_D}. \tag{4.1}$$

The moduli space contains parameters or degrees of freedom† which fix the particular bulk spacetime solution. Thereafter, we consider a fibre bundle construction where the base space is the moduli space previously defined, and the total space is the union of all Hilbert space fibres over each point in the moduli space. Following the description of geometric quantization in the previous sections, the fully quantized quantum Hilbert space is the set of all sections of the bundle. *How do we define the Hilbert spaces at each fibre?* For the general case of qubits, instead of considering the projective Hilbert space $\mathbb{C}\mathrm{P}^3$, we instead take submanifolds of $\mathbb{C}\mathrm{P}^{n^2-1}$ for an $(n \times n)$-dimensional bipartite quantum theory (both qubit states have their own n-dimensional Hilbert space) [25]. Like in the case in section 3.3, our space is too large for the configuration space and so we have to select submanifolds of it. The submanifolds of $\mathcal{H}$ are *orbits* which are quotient spaces that are associated to different values of entanglement. These are constructed by quotienting local unitary transformations of the bipartite system — described by $U(n) \times U(n)$ — by symmetries of the entangled state for a given value of entanglement entropy. The quotient space subspaces are known as *entanglement orbits* which describe all possible states of the system. These are closely related to the symplectic orbits in the topological wormholes section which describes all possible evolutions of the system. In either case the contribution to the wormhole partition function is still integrating the symplectic form Ω over an orbit. Much like the case with $\mathbb{C}\mathrm{P}^3$, there is no diagonal embedding of spaces within $\mathbb{C}\mathrm{P}^{n^2-1}$ due to non-local wormhole structures. Instead different orbits of $\mathbb{C}\mathrm{P}^{n^2-1}$ give rise to different entanglement orbits [25] for a given value of entanglement entropy. For example, a product state with vanishing entanglement entropy lives within the orbit $\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}$. We note that the symplectic form Ω of $\mathbb{C}\mathrm{P}^{n^2-1}$ vanishes when restricted to this orbit and so does not give rise to any wormhole contributions to the gravitational path integral. On the other side of the spectrum, states with maximal entanglement entropy live within the orbit $\mathbb{1} \times \mathrm{SU}(n)/\mathbb{Z}_n$. Finally, there is a defined spectra of states that are between vanishing and maximal entanglement entropy which live in the orbit:

$$\frac{U(n)}{U(1)^n} \times \frac{SU(n)}{\mathbb{Z}_n}. \tag{4.2}$$

The intermediate entanglement entropy orbits (and their volumes) are labelled by the external parameters of the theory's Hamiltonian. Both the orbits of maximally entangled and intermediately entangled states will contribute to the gravitational path integral in the form of integrating Ω over the orbits. Being that there are more states with vanishing entanglement entropy, the orbit of the null-entanglement entropy states is much larger (larger symplectic volume) than the orbit for say, the maximally entangled states. Moreover, it is possible define operators such that states can flow between different orbits [25].

†Recall in the previous section that a time parameter δ emerged from the transition functions at the horizon of the black hole. This is in fact a *bulk degree of freedom* and is an element of $\mathcal{G}$.

Does this have any connection to the topological wormholes we saw in the previous section? Yes! The orbits of non-vanishing entanglement entropy are in fact Lagrangian submanifolds of the total projective Hilbert space $\mathbb{C}\mathrm{P}^{n^2-1}$ each endowed with a non-vanishing curvature 2-form. This means these orbits are symplectic spaces (much like the orbits as a result of symplectic reduction) with symplectic forms Ω defined on them. Each of these orbits have a corresponding geometric phase Φ which is obtained via integrating the symplectic form over the orbit volume:

$$\Phi = \int \Omega = \int d\sigma^a \wedge d\sigma^b \, \Omega_{ab}, \tag{4.3}$$

where σ^a are the coordinates on the orbit. This is equivalent to the geometric phase calculated with the Berry curvature form in section 3.3, and contributes wormhole corrections to the gravitational partition function of the system. This is due to the orbits having non-trivial holonomy. Furthermore, being that this is an integral over the symplectic volume, it also characterises the number of states within a specific entanglement orbit [25].

Now that we have looked at the entanglement structure of qubit systems via entanglement orbits, we move on to including potential wells which modify the entanglement structure the system in the following section.

4.2 Entangled Qubits in Potential Wells

(NOTE: this is a draft version and will be updated within the next few days). This section is for considering the entanglement structures of particles within two-well systems, and see the orbits/wormholes/holonomy evolve. We then generalize this to both a discrete set of N-wells, and a continuous distribution of wells.

4.3 Wormhole-Tunnelling Correspondence

(NOTE: this is a draft version and will be updated within the next few days). This section is for relating wormholes to quantum tunneling events in a bipartite systems of entangled qubits in (can ignore spin part, independent) potential-well lattices admiting wormholes.

5 Summary and Discussion

This is for the summary & discussion of the report.

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