Expressive Power and Succinctness of the Positive Calculus of Relations

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Abstract. In this paper, we study the expressive power and succinctness of *the positive calculus of relations*. We show that (1) the calculus has the same expressive power as that of three-variable existential positive (first-order) logic in terms of binary relations, and (2) the calculus is exponentially less succinct than three-variable existential positive logic, namely, there is no polynomial-size translation from three-variable existential positive logic to the calculus, whereas there is a linear-size translation in the converse direction. Additionally, we give a more fine-grained expressive power equivalence between the (full) calculus of relations and three-variable first-order logic in terms of the quantifier alternation hierarchy. It remains open whether the calculus of relations is also exponentially less succinct than three-variable first-order logic.

Keywords: Expressive power · Succinctness · The positive calculus of relations · Existential positive logic

1 Introduction

The calculus of (binary) relations (denoted by CoR, for short), which was revived by Tarski [22], is an algebraic system on binary relations. The calculus of relations and relation algebras have many applications in various areas of computer science, e.g., databases, program development and verification, and program semantics (see [8] for more details and references). Certain properties of binary relations can be simply expressed using (in)equational formulas of CoR; for example, the formula $a \cdot a \leq a$ indicates that the binary relation a is transitive, where the symbol $\cdot$ denotes the composition operator of binary relations and the symbol $\leq$ denotes the inclusion relation on sets. In fact, CoR has a high expressive power, namely, the expressive power of CoR is equivalent to *three-variable first-order logic* (denoted by FO^3) in terms of binary relations [9,15,23]. One of the downsides for this high expressive power is that the equational theory of CoR is undecidable [23], even for terms built only from one variable, union, complement, and composition [18].

In this paper, we focus on *the positive calculus of relations* (denoted by PCoR, for short) [2,19], which is a complement-free fragment of CoR. Namely, PCoR

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terms are built from union, intersection, composition, converse, the identity relation, the empty relation, and the universal relation. PCoR is strictly less expressive than CoR, but its equational theory is decidable [2]. This decidability result also holds when adding a transitive closure operation, thus arriving at Kleene allegory terms [17].

The first contribution of this paper is to show that PCoR has the same expressive power as *three-variable existential positive (first-order) logic* (denoted by EP^3) in terms of binary relations. The standard and linear-size translation from CoR to FO^3 [22, pp. 75–76] naturally specializes into a translation from PCoR to EP^3. Conversely, translations from FO^3 to CoR also exist [23, Sect. 3.9] [9, Sect. 20] [15, Theorem 552], but they generate non-PCoR terms on the EP^3 fragment. Hence, we have to refine them. The translation we propose uses disjunctive/conjunctive normal forms where literals have at most two free variables. We define it from full FO^3 into CoR by relying on a relational sum operation ($\dagger$), which is dual to composition ($\cdot$). By specializing our translation to the various considered fragments, we obtain (1) an exponential-size translation from EP^3 to PCoR, and (2) a perfect match between the quantifier alternation hierarchy in FO^3 and *dot-dagger* ($\cdot$ - $\dagger$) *alternation hierarchy* in CoR. Roughly speaking, it shows that the two operators, $\cdot$ and $\dagger$, from the calculus of relations exactly correspond to the two quantifiers, $\exists$ and $\forall$, of first-order logic.

The second contribution of this paper is to show that PCoR is exponentially less succinct than EP^3, namely, the exponential blowup in translating from EP^3 to PCoR is unavoidable. Hence, the exponential-size translation for EP^3 given in this paper is tight.

Furthermore, we extend the two above results for both transitive closure extensions, namely, we show that PCoR with transitive closure [19] (denoted by PCoR(TC)) has the same expressive power as EP^3 with *variable-confined monadic transitive closure* (denoted by EP^3(v-MTC)) and that PCoR(TC) is exponentially less succinct than EP^3(v-MTC).

Remark 1 (On trade-off between succinctness and tractability). The combined complexity of the (binary-relation) query evaluation problem [24] is the problem to decide for a structure M, a term t, and a pair of nodes in M, whether the pair is in the binary relation denoted by t on the structure M. While PCoR is strictly less succinct than EP^3, PCoR is more tractable than EP^3 in the simple dynamic-programming algorithm for this problem (see, e.g., [14, Proposition 6.6]). While it does not imply a certain computational complexity gap between the two problems, it can be solved in $\mathcal{O}(\|t\| \times \|M\|^2)$-time for PCoR, thanks to this algorithm, if the number of occurrences of $\cdot$ is fixed, while it requires $\mathcal{O}(\|t\| \times \|M\|^3)$-time for EP^3 even if the number of occurrences of $\exists$ is fixed, where $\|t\|$ is the size of t and $\|M\|$ is the cardinality of the domain of M, respectively.

Related Work. *Expressive power* of formal systems is widely studied in mathematical logic and computer science.

An example is that the following systems have the same expressive power in terms of *recognizability over word structures*: regular expressions, deterministic finite automata, non-deterministic finite automata, and monadic second-order logic (MSO) (see, e.g., [6, Sec. 6]). However, these four systems are certainly different in terms of *succinctness*. For example, while there is an (exponential-size) translation from non-deterministic finite automata to deterministic finite automata by the powerset construction, there is no polynomial-size translation [16, Prop. 1]. This can yield significant complexity differences for various problems, e.g., membership, universality, and equivalence testing have different complexity depending on whether we start from a term, an automaton, or a formula. See [12, Thm. 16] and [7, Thm. 11] for the other succinctness gaps among regular expressions, deterministic finite automata, and non-deterministic finite automata. The succinctness gap between MSO and each of the other three systems can be shown by using the computational complexity gap that the equivalence problem is non-elementary for MSO (and even for FO [21, Thm. 5.2]), but is in PSPACE for regular expressions and non-deterministic automata, and almost linear-time for deterministic automata. Additionally, the above like expressive power equivalence is known for FO. The following have the same expressive power: star-free regular expressions, FO^3, and FO (see, e.g., [6, Sec. 6]). In [11], it is shown that FO^3 is exponentially less succinct than FO over unary alphabet words. Another example is that the following classes of formulas have the same expressive power with respect to *boolean queries*: propositional logic formulas, negation normal form formulas, and disjunctive/conjunctive normal form formulas. In [5], the succinctness among a dozen formula classes (including the above ones) is investigated.

In this paper, we compare the succinctness between PCoR and EP^3. To the best of our knowledge, it is the first comparison between the succinctness of the (positive) calculus of relations and those of other systems. Our construction in Sect. 4 is somewhat similar to the construction in [10, Sec. 4.5] in order to show that there is no polynomial-size translation from conjunctive normal form formulas to disjunctive normal form formulas, but is more complicated than the construction. This is because we should consider structures with multiple nodes, whereas it suffices to consider only singleton structures for propositional logic.

Organization. Section 2 provides the definitions of CoR and FO^3 and fragments of them (including PCoR and EP^3), the notions of the expressive power and succinctness, and the standard translation from CoR to FO^3. Section 3 gives a new translation from FO^3 to CoR. Consequently, it is shown that PCoR has the same expressive power as EP^3. Section 4 shows that PCoR is exponentially less succinct than EP^3. Section 5 extends the results in Sects. 3–4 by adding a transitive closure operator. Section 6 concludes this paper.

2 Preliminaries

$\mathbb{N}$ (resp. $\mathbb{N}_+$) denotes the set of all non-negative (resp. positive) integers. For $l, r \in \mathbb{N}$ such that $l \leq r$, $[l, r]$ denotes the set $\{l, \dots, r\}$ and $[r]$ denotes the set $\{1, \dots, r\}$. $\#(A)$ denotes the cardinality of a set A.

Let $\mathscr{A}$ be a countably infinite set of binary relation symbols. A *structure* M *(of binary relations)* is a tuple $\langle |M|, \{a^M\}_{a \in \mathscr{A}} \rangle$, where $|M|$ is a non-empty set, and for each $a \in \mathscr{A}$, $a^M \subseteq |M|^2$ is a binary relation on $|M|$. For two structures, M and M', we say that a function $h \colon |M| \to |M'|$ is a *homomorphism* from M to M' if for every $a \in \mathscr{A}$ and every $v, w \in |M|$, if $\langle v, w \rangle \in a^M$, then $\langle h(v), h(w) \rangle \in a^{M'}$.

The Calculus of Relations and Its Fragments. We introduce the calculus of relations (CoR) [22] and its syntactic fragments: the positive calculus of relations (PCoR) [2,19] and *the primitive positive calculus of relations* (denoted by PPCoR, a.k.a. allegory terms with top [20]). The terms of CoR consist of the following basic operations on binary relations. Let X be a set. For two binary relations R and S on the universe X, the *union* $R \cup S$, *intersection* $R \cap S$, and *complement* R^- are defined as the corresponding set-theoretic operators, respectively. The symbols $\mathbf{0}$ and $\top$ are employed to denote the *empty relation* and the *universal relation*, respectively. *Relational composition* (a.k.a. *relational multiplication*) $R \cdot S$ is defined as $\{\langle v, v' \rangle \in X^2 \mid \exists w. \langle v, w \rangle \in R \wedge \langle w, v' \rangle \in S\}$, and *relational sum* $R \dagger S$ is defined as $\{\langle v, v' \rangle \in X^2 \mid \forall w. \langle v, w \rangle \in R \vee \langle w, v' \rangle \in S\}$. In this paper, the *projection* R^π is defined as $\{\langle v_1, v_2 \rangle \in X^2 \mid \langle v_{\pi(1)}, v_{\pi(2)} \rangle \in R\}$ for each function $\pi \colon [2] \to [2]$. The symbol $\mathbf{1}$ is employed to denote the *identity relation*. We now define the syntax and semantics of CoR. The set of *terms* of CoR/PCoR/PPCoR is given by the following grammar, where $a \in \mathscr{A}$:

$$
\begin{aligned}
t, s \in \mathrm{Term}^{\mathrm{CoR}} &::= t^\pi \mid a \mid \mathbf{1} \mid \top \mid t \cap s \mid t \cdot s \mid \mathbf{0} \mid t \cup s \mid t \dagger s \mid t^- \\
t, s \in \mathrm{Term}^{\mathrm{PCoR}} &::= t^\pi \mid a \mid \mathbf{1} \mid \top \mid t \cap s \mid t \cdot s \mid \mathbf{0} \mid t \cup s \\
t, s \in \mathrm{Term}^{\mathrm{PPCoR}} &::= t^\pi \mid a \mid \mathbf{1} \mid \top \mid t \cap s \mid t \cdot s
\end{aligned}
$$

For $k \in \mathbb{N}$, we use t^k (the *k-th iteration of t*) to denote $t^{k-1} \cdot t$ if $k \geq 1$; and $\mathbf{1}$ if $k = 0$, and use $t^\smile$ (the *converse* of t) to denote $t^{\{1 \mapsto 2, 2 \mapsto 1\}}$. The *semantics* $[\![t]\!]_M$ of a CoR term t on a structure M is a binary relation on $|M|$, which is defined by: $[\![a]\!]_M := a^M$; $[\![\mathbf{1}]\!]_M := \triangle(|M|)$; $[\![\top]\!]_M := |M|^2$; $[\![\mathbf{0}]\!]_M := \emptyset$; $[\![t \cup s]\!]_M := [\![t]\!]_M \cup [\![s]\!]_M$; $[\![t \cap s]\!]_M := [\![t]\!]_M \cap [\![s]\!]_M$; $[\![t^-]\!]_M := |M|^2 \setminus [\![t]\!]_M$; $[\![t^\pi]\!]_M := [\![t]\!]_M^\pi$; $[\![t \cdot s]\!]_M := [\![t]\!]_M \cdot [\![s]\!]_M$; $[\![t \dagger s]\!]_M := [\![t]\!]_M \dagger [\![s]\!]_M$, where $\triangle(X)$ denotes the *diagonal relation* (i.e., $\{\langle v, w \rangle \in X^2 \mid v = w\}$). The *size* $\|t\|$ of a CoR term t is defined by: $\|a\| := \|\mathbf{1}\| := \|\top\| := \|\mathbf{0}\| := 1$, $\|t \cup s\| := \|t \cap s\| := \|t \cdot s\| := \|t \dagger s\| := 1 + \|t\| + \|s\|$, and $\|t^-\| := \|t^\pi\| := 1 + \|t\|$.

Remark 2 (Projection and converse). As usual (e.g., [22]), t^π is defined only when t^π denotes the converse of t. This is because in the other cases, t^π can be expressed by not using the π as follows: $t^{\{1 \mapsto 1, 2 \mapsto 2\}} = t$, $t^{\{1 \mapsto 1, 2 \mapsto 1\}} = (t \cap \mathbf{1}) \cdot \top$, and $t^{\{1 \mapsto 2, 2 \mapsto 2\}} = \top \cdot (t \cap \mathbf{1})$. Nevertheless, we introduce t^π for each function $\pi \colon [2] \to [2]$ for clarifying the relationship between CoR and FO^3 in Sect. 3.1.

Since PCoR has only positive connectives, its terms define monotone operations, and we have:

Proposition 3 (e.g., [2]). *For every* PCoR *term t and every homomorphism h (from M to M'), if $\langle v, w\rangle \in [\![t]\!]_M$, then $\langle h(v), h(w)\rangle \in [\![t]\!]_{M'}$.*

Proposition 3 also implies that PCoR is strictly less expressive than CoR, because this proposition does not hold in general for CoR.

First-Order Logic and Its Fragments. Here, we introduce first-order logic (FO) and its syntactic fragments (see, e.g., [4]): *existential positive logic* (EP) and *primitive positive logic* (PP). Let $\mathscr{V}$ be a countably infinite set of *(first-order) variables*. We use x, y, z, or u to denote these variables. The set of *formulas* of FO/EP/PP is given by the following grammar, where $a \in \mathscr{A}$ and $x, y \in \mathscr{V}$:

$$\begin{aligned}
\varphi, \psi \in \mathrm{Fml}^{\mathrm{FO}} &::= a(x,y) \mid x = y \mid \mathsf{tt} \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \mathsf{ff} \mid \varphi \vee \psi \mid \forall x.\varphi \mid \neg\varphi\\
\varphi, \psi \in \mathrm{Fml}^{\mathrm{EP}} &::= a(x,y) \mid x = y \mid \mathsf{tt} \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \mathsf{ff} \mid \varphi \vee \psi\\
\varphi, \psi \in \mathrm{Fml}^{\mathrm{PP}} &::= a(x,y) \mid x = y \mid \mathsf{tt} \mid \varphi \wedge \psi \mid \exists x.\varphi
\end{aligned}$$

$\mathbf{FV}(\varphi)$ denotes the set of *free variables* occurring in φ. For an FO formula φ and a structure M, we say that a partial function $I\colon \mathscr{V} \rightharpoonup |M|$ is an *interpretation* (of φ on M) if $\mathsf{dom}(I) \supseteq \mathbf{FV}(\varphi)$. Then the *semantics* ($I \models_M \varphi$) of φ on M and an interpretation I is a truth value, which is defined in a standard way as follows: $I \models_M a(x,y) :\Leftrightarrow \langle I(x), I(y)\rangle \in a^M$; $I \models_M x = y :\Leftrightarrow I(x) = I(y)$; $I \models_M \mathsf{tt} :\Leftrightarrow$ true; $I \models_M \mathsf{ff} :\Leftrightarrow$ false; $I \models_M \varphi \vee \psi :\Leftrightarrow (I \models_M \varphi)$ or $(I \models_M \psi)$; $I \models_M \varphi \wedge \psi :\Leftrightarrow (I \models_M \varphi)$ and $(I \models_M \psi)$; $I \models_M \neg\varphi :\Leftrightarrow$ (not $I \models_M \varphi$); $I \models_M \exists x.\varphi :\Leftrightarrow$ for some v, $I[v/x] \models_M \varphi$; and $I \models_M \forall x.\varphi :\Leftrightarrow$ for every v, $I[v/x] \models_M \varphi$, where $I[v/x]$ denotes the I in which the value $I(x)$ has been replaced by v. Here, an FO *(binary-relation-)term* is of the form $[\varphi]_{x,y}$, where x and y are distinct variables; and φ is an FO formula with $\mathbf{FV}(\varphi) \subseteq \{x, y\}$. (In the same manner, for a class $\mathcal{C}$ of formulas, we say that $[\varphi]_{x,y}$ is a $\mathcal{C}$ term if the formula φ is in $\mathcal{C}$.) The *semantics* $[\![[\varphi]_{x,y}]\!]_M$ of an FO term $[\varphi]_{x,y}$ is defined by the binary relation $[\![[\varphi]_{x,y}]\!]_M := \{\langle v, w\rangle \in |M|^2 \mid \{x \mapsto v, y \mapsto w\} \models \varphi\}$. The *size* $\|\varphi\|$ of an FO formula φ is defined by: $\|a(x,y)\| := \|x = y\| := \|\mathsf{tt}\| := \|\mathsf{ff}\| := 1$, $\|\varphi \vee \psi\| := \|\varphi \wedge \psi\| = 1 + \|\varphi\| + \|\psi\|$, and $\|\neg\varphi\| := \|\exists x.\varphi\| := \|\forall x.\varphi\| = 1 + \|\varphi\|$. Also, the *size* $\|[\varphi]_{x,y}\|$ of an FO term $[\varphi]_{x,y}$ is defined as $\|\varphi\|$. For the sake of brevity, we may identify formulas equivalent modulo the commutative and associative laws of $\vee$ and $\wedge$. Also, for a finite set $\Phi = \{\varphi_i \mid i \in I\}$ of formulas, we write $\bigvee \Phi$ (and similarly for $\bigwedge \Phi$) for $\varphi_{i_1} \vee \cdots \vee \varphi_{i_{\#(I)}}$ if $\#(I) > 0$, and for ff otherwise, where $I = \{i_1, \ldots, i_{\#(I)}\}$. Also, let FO^3 be the syntax fragment consisting of FO formulas such that at most three variables appear in the formula. EP^3 and PP^3 are similarly defined.

Remark 4 (Existential positive logics and conjunctive queries). The class of PP (resp. EP) formulas in prenex normal form is also known as the class of *conjunctive queries* (resp. *conjunctive queries with union*), which is a major class in database theory (see e.g., [1, Sec. 4]). However, we do not use prenex normal form because we are interested in the number of variables of formulas.

Expressive Power and Succinctness. We say that two terms t and s are *equivalent*, written $\models t \equiv s$, if for every structure M, $[\![t]\!]_M = [\![s]\!]_M$. We write $\not\models t \equiv s$ if t and s are not equivalent, and write $\models t \leq s$ if for every structure M, $[\![t]\!]_M \subseteq [\![s]\!]_M$. Also, we say that two formulas φ and ψ are *equivalent* if for every $\langle M, I\rangle$ such that $\mathsf{dom}(I) \supseteq \mathbf{FV}(\varphi) \cup \mathbf{FV}(\psi)$, $(I \models_M \varphi)$ iff $(I \models_M \psi)$.

We say that $\mathcal{C}'$ is *at least as expressive as* $\mathcal{C}$, if, for every term t in $\mathcal{C}$, there is a term t' in $\mathcal{C}'$, which is equivalent to t; $\mathcal{C}'$ has *the same expressive power as* $\mathcal{C}$, if $\mathcal{C}'$ is at least as expressive as $\mathcal{C}$ and $\mathcal{C}$ is at least as expressive as $\mathcal{C}'$; and $\mathcal{C}'$ is *strictly more expressive than* $\mathcal{C}$, if C' is at least as expressive as $\mathcal{C}$ and $\mathcal{C}$ is not at least as expressive as $\mathcal{C}'$.

Moreover, for a class F of functions from $\mathbb{N}$ to $\mathbb{N}$, we say that *there is an F-size translation* (preserving the semantics) from $\mathcal{C}$ to $\mathcal{C}'$ (a.k.a. $\mathcal{C}'$ is *F-succinct* than $\mathcal{C}$ [11]) if there is a function $f \in F$ such that for every term t in $\mathcal{C}$, there is a term t' in $\mathcal{C}'$ of size $\|t'\| \leq f(\|t\|)$ that is equivalent to t. In particular, we say that there is a *linear/polynomial/exponential-size* translation from $\mathcal{C}$ to $\mathcal{C}'$ if F is the set of all linear/polynomial/exponential (i.e., $\mathcal{O}(n)/n^{\mathcal{O}(1)}/2^{\mathcal{O}(n)}$) functions. We say that $\mathcal{C}'$ is *exponentially less succinct than* $\mathcal{C}$ if there is *no* $2^{o(n)}$-size translation from $\mathcal{C}$ to $\mathcal{C}'$.

The Standard Translation. We recall that from CoR to FO^3, there is an efficient translation [22]; see Fig. 1. It follows that $[\![[\mathrm{ST}_{x,y}(t)]_{x,y}]\!]_M = [\![t]\!]_M$ by simple induction on the structure of t, hence the following theorem.

$$\begin{array}{llll} \mathrm{ST}_{x,y}(\top) := \mathsf{tt} & \mathrm{ST}_{x,y}(\mathbf{0}) := \mathsf{ff} & \mathrm{ST}_{x,y}(t^-) := \neg \mathrm{ST}_{x,y}(t) & \mathrm{ST}_{x_1,x_2}(t^\pi) := \mathrm{ST}_{x_{\pi(1)},x_{\pi(2)}}(t) \\ \mathrm{ST}_{x,y}(a) := a(x,y) & \mathrm{ST}_{x,y}(t \cup s) := \mathrm{ST}_{x,y}(t) \vee \mathrm{ST}_{x,y}(s) & & \mathrm{ST}_{x,y}(t \cdot s) := \exists z. \mathrm{ST}_{x,z}(t) \wedge \mathrm{ST}_{z,y}(s) \\ \mathrm{ST}_{x,y}(\mathbf{1}) := x = y & \mathrm{ST}_{x,y}(t \cap s) := \mathrm{ST}_{x,y}(t) \wedge \mathrm{ST}_{x,y}(s) & & \mathrm{ST}_{x,y}(t \dagger s) := \forall z. \mathrm{ST}_{x,z}(t) \vee \mathrm{ST}_{z,y}(s) \end{array}$$

Fig. 1. The standard translation, where x, y, and z are all distinct.

Theorem 5 ([22]). *There is a linear-size translation from* CoR *to* FO^3.

The following is also immediate from the standard translation (notice that $\neg$ and $\forall$ do not occur in $\mathrm{ST}_{x,y}(t)$ if $\bullet^-$ and $\dagger$ do not occur in t; furthermore, $\neg$, $\forall$, $\vee$, and ff do not occur in $\mathrm{ST}_{x,y}(t)$ if $\bullet^-$, $\dagger$, $\cup$, and $\mathbf{0}$ do not occur in t).

Corollary 6.

- *There is a linear-size translation from* PCoR *to* EP^3.
- *There is a linear-size translation from* PPCoR *to* PP^3.

3 Expressive Power Equivalence of PCoR and EP^3

In this section, we consider the converse direction of the standard translation, i.e., from FO^3 terms to CoR terms. The aim of this section is to show the following.

Theorem 7.

(1) *There is an exponential-size translation from* FO^3 *to* CoR*.*
(2) *There is an exponential-size translation from* EP^3 *to* PCoR*.*
(3) *There is a linear-size translation from* PP^3 *to* PPCoR*.*

This theorem (combined with Theorem 5 and Corollary 6) implies the following expressive power equivalences.

Corollary 8.

(1) CoR *has the same expressive power as* FO^3 [23]*.*
(2) PCoR *has the same expressive power as* EP^3*.*
(3) PPCoR *has the same expressive power as* PP^3*; furthermore, these two have the same succinctness up to linear factors.*

From here, we prove Theorem 7 by giving a new translation from FO^3 to CoR, which is constructed in the following steps.

(‡1) Translate the given FO^3 term into a term in *negation normal form*.
(‡2) For each sub-formula of the form $\exists z.\psi$ (resp. $\forall z.\psi$), substitute ψ with an equivalent formula, which is a conjunction (resp. disjunction) of formulas having at most two free variables.
(‡3) Push the quantifiers deeper into the formula as much as possible. Then, each sub-formula $\exists z.\varphi$ (resp. $\forall z.\varphi$) is of the form $\exists z.\psi \wedge \rho$ (resp. $\forall z.\psi \vee \rho$) such that $\mathbf{FV}(\psi) \subseteq \{x, z\}$ and $\mathbf{FV}(\rho) \subseteq \{z, y\}$, where x, y, z are three distinct variables.
(‡4) Translate the FO^3 term preprocessed by the above translations to a CoR term by simple structural induction.

In the following, we describe the details of each step. We say that a formula φ is in $\mathrm{FO}^{3(2)}$ if φ is in FO^3 and $\#(\mathbf{FV}(\varphi)) \leq 2$. Note that by the definition of FO^3 term, for every FO^3 term $[\varphi]_{x,y}$, φ is in $\mathrm{FO}^{3(2)}$.

(‡1): We say that an FO formula is in *negation normal form* if it is in the set defined by the following grammar:

$$\varphi, \psi ::= a(x,y) \mid \neg a(x,y) \mid x = y \mid \neg x = y \mid \mathsf{tt} \mid \mathsf{ff} \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \forall x.\varphi.$$

We say that an FO formula φ is an *atomic formula* if φ is of the form $a(x,y)$, $x = y$, or tt; and is a *negated atomic formula* if φ is of the form $\neg a(x,y)$, $\neg x = y$, or ff. Every FO^3 term can be translated to an equivalent FO^3 term in negation normal form by repeatedly applying the De Morgan's law and the double negation elimination law.

Lemma 9. *There is a linear-size translation from* FO^3 *terms to* FO^3 *terms in negation normal form.*

(‡2): We say that a formula is *good* if (a) it is in negation normal form, and (b) for every sub-formula of the form $\exists z.\psi$ (resp. $\forall z.\psi$), ψ is a conjunction (resp. disjunction) of $\mathrm{FO}^{3(2)}$ formulas. According to condition (b), each sub-formula of the form $\exists z.\psi$ (resp. $\forall z.\psi$) can be written as $\exists z.\rho_1 \wedge \rho_2 \wedge \rho_3$ (resp. $\forall z.\rho_1 \vee \rho_2 \vee \rho_3$) by the associativity and commutativity of $\wedge/\vee$, where $\mathbf{FV}(\rho_1) \subseteq \{x, y\}$, $\mathbf{FV}(\rho_2) \subseteq \{y, z\}$, and $\mathbf{FV}(\rho_3) \subseteq \{z, x\}$, and x, y, z are three distinct variables. This property will be fully used in the translation (‡3). In this step, we translate negation normal form FO^3 terms into good FO^3 terms. This is the only step involving an exponential blow-up among (‡1)–(‡4).

Lemma 10. *There is an* exponential-*size translation from* FO^3 *terms in negation normal form to good* FO^3 *terms.*

Proof. We mutually define two functions, $\mathsf{T}_\exists$ and $\mathsf{T}_\forall$, from negation normal form FO^3 formulas to sets of sets of good $\mathrm{FO}^{3(2)}$ formulas and define the function T_- from negation normal form FO^3 formulas to good FO^3 formulas; see Fig. 2. Then the following are shown by simple induction on the structure of φ: φ is equivalent to the formula $\bigvee_{i\in[n]} \bigwedge_{j\in[m_i]} \psi_{i,j}$, where $\mathsf{T}_\exists(\varphi) = \{\{\psi_{i,j} \mid j \in [m_i]\} \mid i \in [n]\}$; and φ is equivalent to the formula $\bigwedge_{i\in[n]} \bigvee_{j\in[m_i]} \psi_{i,j}$, where $\mathsf{T}_\forall(\varphi) = \{\{\psi_{i,j} \mid j \in [m_i]\} \mid i \in [n]\}$. Also the translation T_- (Fig. 2) from negation normal form FO^3 formulas to good FO^3 formulas satisfies that (1) $\mathsf{T}_-(\varphi)$ is equivalent to φ; (2) $\mathbf{FV}(\mathsf{T}_-(\varphi)) \subseteq \mathbf{FV}(\varphi)$; and (3) $\|\mathsf{T}_-(\varphi)\| \leq 2^{2\times\|\varphi\|}$ (hence, T_- is an exponential-size translation). Hence, the desired translation is obtained from T_-. $\square$

- $\mathsf{T}_\bullet(\varphi) := \begin{cases} \varphi & (\bullet = -) \\ \{\{\varphi\}\} & (\bullet = \exists, \forall) \end{cases}$ if φ is an atomic or negated atomic formula.
- $\mathsf{T}_\bullet(\exists z.\varphi) := \begin{cases} \bigvee_{i\in[n]} \exists z. \bigwedge \Phi_i & (\bullet = -) \\ \{\{\bigvee_{i\in[n]} \exists z. \bigwedge \Phi_i\}\} & (\bullet = \exists, \forall) \end{cases}$, where $\mathsf{T}_\exists(\varphi) = \{\Phi_i \mid i \in [n]\}$.
- $\mathsf{T}_\bullet(\forall z.\varphi) := \begin{cases} \bigwedge_{i\in[n]} \forall z. \bigvee \Phi_i & (\bullet = -) \\ \{\{\bigwedge_{i\in[n]} \forall z. \bigvee \Phi_i\}\} & (\bullet = \exists, \forall) \end{cases}$, where $\mathsf{T}_\forall(\varphi) = \{\Phi_i \mid i \in [n]\}$.
- $\mathsf{T}_\bullet(\psi_1 \wedge \psi_2) := \begin{cases} \mathsf{T}_-(\psi_1) \wedge \mathsf{T}_-(\psi_2) & (\bullet = -) \\ \mathsf{T}_\forall(\psi_1) \cup \mathsf{T}_\forall(\psi_2) & (\bullet = \forall) \\ \{\Psi_1 \cup \Psi_2 \mid \Psi_1 \in \mathsf{T}_\exists(\psi_1), \Psi_2 \in \mathsf{T}_\exists(\psi_2)\} & (\bullet = \exists) \end{cases}$
- $\mathsf{T}_\bullet(\psi_1 \vee \psi_2) := \begin{cases} \mathsf{T}_-(\psi_1) \vee \mathsf{T}_-(\psi_2) & (\bullet = -) \\ \mathsf{T}_\exists(\psi_1) \cup \mathsf{T}_\exists(\psi_2) & (\bullet = \exists) \\ \{\Psi_1 \cup \Psi_2 \mid \Psi_1 \in \mathsf{T}_\forall(\psi_1), \Psi_2 \in \mathsf{T}_\forall(\psi_2)\} & (\bullet = \forall) \end{cases}$

Fig. 2. Translation to good $\mathrm{FO}^{3(2)}$ formulas.

Example 11. Let $\varphi = (a(x,z) \vee b(z,x)) \wedge c(x,y)$. Then, the formula $\exists z.\varphi$ is not a good $\mathrm{FO}^{3(2)}$ formula, but the translated formula $\mathsf{T}_{-}(\exists z.\varphi) = (\exists z.a(x,z) \wedge c(x,y)) \vee (\exists z.b(z,x) \wedge c(x,y))$ is a good $\mathrm{FO}^{3(2)}$ formula equivalent to $\exists z.\varphi$. Note that $\mathsf{T}_{-}(\exists z.\varphi)$ is calculated from $\mathsf{T}_{\exists}(\varphi) = \{\{a(x,z), c(x,y)\}, \{b(z,x), c(x,y)\}\}$.

(‡3): In this step, we translate good FO^3 terms into FO^3 terms in the following normal form.

Definition 12. *For two distinct variables x and y, we say that an $\mathrm{FO}^{3(2)}$ formula is ($\{x,y\}$-)*nice *if it is in the set defined by the following grammar, where $w, w' \in \{x,y\}$ and z is the variable distinct from x and y:*

$$\begin{aligned}\varphi^{\{x,y\}}, \psi^{\{x,y\}} ::= a(w,w') \mid \neg a(w,w') \mid w = w' \mid \neg w = w' \mid \mathsf{tt} \mid \mathsf{ff} \mid \varphi^{\{x,y\}} \vee \psi^{\{x,y\}} \\ \mid \varphi^{\{x,y\}} \wedge \psi^{\{x,y\}} \mid \exists z.\varphi^{\{x,z\}} \wedge \psi^{\{z,y\}} \mid \forall z.\varphi^{\{x,z\}} \vee \psi^{\{z,y\}}.\end{aligned}$$

Intuitively, if an FO^3 term is nice, then it is 'almost' a two-variable term (in that, even if a subformula of the term has three free variables, the subformula should be of the form $\varphi^{\{x,z\}} \wedge \psi^{\{z,y\}}$ or $\varphi^{\{x,z\}} \vee \psi^{\{z,y\}}$; hence its immediate subformulas have at most two free variables).

Lemma 13. *There is a linear-size translation from good* FO^3 *terms to nice* FO^3 *terms.*

Proof. Let T be the translation defined as follows: $\mathsf{T}(\varphi) := \varphi$ if φ is an atomic or negated atomic formula; $\mathsf{T}(\psi \vee \rho) := \mathsf{T}(\psi) \vee \mathsf{T}(\rho)$; $\mathsf{T}(\psi \wedge \rho) := \mathsf{T}(\psi) \wedge \mathsf{T}(\rho)$; $\mathsf{T}(\exists z.\psi_1 \wedge \psi_2 \wedge \psi_3) := \mathsf{T}(\psi_1) \wedge \exists z.\mathsf{T}(\psi_2) \wedge \mathsf{T}(\psi_3)$; and $\mathsf{T}(\forall z.\psi_1 \vee \psi_2 \vee \psi_3) := \mathsf{T}(\psi_1) \vee \forall z.\mathsf{T}(\psi_2) \vee \mathsf{T}(\psi_3)$, where $\mathbf{FV}(\psi_1) \subseteq \{x,y\}$, $\mathbf{FV}(\psi_2) \subseteq \{x,z\}$ and $\mathbf{FV}(\psi_3) \subseteq \{y,z\}$. By trivial induction on the size of φ, (1) $\mathsf{T}(\varphi)$ is equivalent to φ; and (2) $\|\mathsf{T}(\varphi)\| \leq \|\varphi\|$ (hence T is a linear-size translation). Also, for every good $\mathrm{FO}^{3(2)}$ formula φ, the formula $\mathsf{T}(\varphi)$ is exactly a nice $\mathrm{FO}^{3(2)}$ formula. Thus the desired translation is obtained from T. $\square$

(‡4): Finally, we give a linear-time translation from nice FO^3 terms to CoR terms by simple structural induction as follows (Fig. 3).

$$\begin{array}{ll}\mathsf{T}([a(x_i,x_j)]_{x_1,x_2}) := a^{\pi} \quad \mathsf{T}([\neg a(x_i,x_j)]_{x_1,x_2}) := (a^{-})^{\pi} & \\ \mathsf{T}([x_i = x_j]_{x_1,x_2}) := \mathbf{1}^{\pi} \quad \mathsf{T}([\neg x_i = x_j]_{x_1,x_2}) := (\mathbf{1}^{-})^{\pi} & \text{where } \pi = \{1 \mapsto i, 2 \mapsto j\}\end{array}$$

$$\begin{array}{ll}\mathsf{T}([\mathsf{tt}]_{x,y}) := \top & \mathsf{T}([\varphi^{\{x,y\}} \wedge \psi^{\{x,y\}}]_{x,y}) := \mathsf{T}([\varphi^{\{x,y\}}]_{x,y}) \cap \mathsf{T}([\psi^{\{x,y\}}]_{x,y}) \\ \mathsf{T}([\mathsf{ff}]_{x,y}) := \mathbf{0} & \mathsf{T}([\varphi^{\{x,y\}} \vee \psi^{\{x,y\}}]_{x,y}) := \mathsf{T}([\varphi^{\{x,y\}}]_{x,y}) \cup \mathsf{T}([\psi^{\{x,y\}}]_{x,y})\end{array}$$

$$\begin{aligned}\mathsf{T}([\exists z.\varphi^{\{x,z\}} \wedge \psi^{\{z,y\}}]_{x,y}) &:= \mathsf{T}([\varphi^{\{x,z\}}]_{x,z}) \cdot \mathsf{T}([\psi^{\{z,y\}}]_{z,y}) \\ \mathsf{T}([\forall z.\varphi^{\{x,z\}} \vee \psi^{\{z,y\}}]_{x,y}) &:= \mathsf{T}([\varphi^{\{x,z\}}]_{x,z}) \dagger \mathsf{T}([\psi^{\{z,y\}}]_{z,y})\end{aligned}$$

Fig. 3. Translation to CoR terms.

Lemma 14. *There is a linear-size translation from nice* FO^3 *terms to* CoR *terms.*

Proof. By induction on the structure of φ, we can show that (1) for every $\{x, y\}$-nice FO^3 term $[\varphi]_{x,y}$, $\mathsf{T}([\varphi]_{x,y})$ is equivalent to $[\varphi]_{x,y}$; and (2) $\|\mathsf{T}([\varphi]_{x,y})\| \leq 2 \times \|[\varphi]_{x,y}\|$ (thus, T is a linear-size translation). Hence, the T is the desired translation. □

Proof (of Theorem 7). Theorem 7(1) has been proved by combining (‡1)–(‡4). Theorem 7(2) holds because, if a term is in EP^3 (i.e. it does not contain $\neg$ nor $\forall$), then the term translated by (‡1)–(‡3) (more precisely, (‡2)–(‡3) are sufficient) is also in EP^3, and thus the CoR term translated by (‡1)–(‡4) does not contain $\bullet^-$ nor $\dagger$, hence the translated term is a PCoR term. Also Theorem 7(3) holds because, if a term is in PP^3, then the term translated by (‡1)–(‡3) (more precisely, (‡3) is sufficient) is also in PP^3, and thus the CoR term translated by (‡1)–(‡4) is a PPCoR term.

3.1 Quantifier Alternation and Dot-Dagger Alternation Hierarchies

In this subsection, we give a more fine-grained expressive power equivalence between CoR and FO^3 in terms of *the quantifier alternation hierarchy*.

Definition 15 (quantifier alternation hierarchy, cf. [3, p. 105]). *The sets* $\{\Sigma_n, \Pi_n\}_{n \in \mathbb{N}}$ *are the minimal sets of* FO *formulas satisfying the following.*

- *If an* FO *formula* φ *contains neither* $\exists$ *nor* $\forall$, *then* $\varphi \in \Sigma_0$ *and* $\varphi \in \Pi_0$.
- *For* $n \geq 0$, $\Sigma_n \subseteq \Sigma_{n+1}$ *and* $\Pi_n \subseteq \Pi_{n+1}$.
- *For* $n \geq 1$, *if* $\varphi, \psi \in \Sigma_n$, *then* $\varphi \vee \psi, \varphi \wedge \psi, \exists x.\varphi \in \Sigma_n$ *and* $\forall x.\varphi \in \Pi_{n+1}$.
- *For* $n \geq 1$, *if* $\varphi, \psi \in \Pi_n$, *then* $\varphi \vee \psi, \varphi \wedge \psi, \forall x.\varphi \in \Pi_n$ *and* $\exists x.\varphi \in \Sigma_{n+1}$.

We also define the sets $\{\Sigma_n^3, \Pi_n^3\}_{n \in \mathbb{N}}$ *as the subclasses of* FO^3 *formulas defined by the same rules.*

We now define *dot-dagger alternation hierarchy* in CoR in the same manner as the quantifier alternation hierarchy in FO, as follows.

Definition 16 (dot-dagger alternation hierarchy). *The subclasses of* CoR *terms,* $\{\Sigma_n^{\mathrm{CoR}}, \Pi_n^{\mathrm{CoR}}\}_{n \in \mathbb{N}}$, *are the minimal sets satisfying the following.*

- *If a* CoR *term* t *contains neither* $\cdot$ *nor* $\dagger$, *then* $t \in \Sigma_0^{\mathrm{CoR}}$ *and* $t \in \Pi_0^{\mathrm{CoR}}$.
- *For* $n \geq 0$, $\Sigma_n^{\mathrm{CoR}} \subseteq \Sigma_{n+1}^{\mathrm{CoR}}$ *and* $\Pi_n^{\mathrm{CoR}} \subseteq \Pi_{n+1}^{\mathrm{CoR}}$.
- *For* $n \geq 1$, *if* $t, u \in \Sigma_n^{\mathrm{CoR}}$, *then* t^π, $t \cup u, t \cap u, t \cdot u \in \Sigma_n^{\mathrm{CoR}}$ *and* $t \dagger u \in \Pi_{n+1}^{\mathrm{CoR}}$.
- *For* $n \geq 1$, *if* $t, u \in \Pi_n^{\mathrm{CoR}}$, *then* t^π, $t \cup u, t \cap u, t \dagger u \in \Pi_n^{\mathrm{CoR}}$ *and* $t \cdot u \in \Sigma_{n+1}^{\mathrm{CoR}}$.

The following shows that the dot-dagger alternation hierarchy in CoR is expressive power equivalent to the quantifier alternation hierarchy in FO^3, uniformly.

Corollary 17. *For each* $n \geq 0$, *the class of terms in* Σ_n^{CoR} *(resp.* Π_n^{CoR}*) and the class of terms in* Σ_n^3 *(resp.* Π_n^3*) have the same expressive power.*

Proof (Sketch). Let us recall the standard translation in Sect. 2 and the translations (‡1)–(‡4) in Sect. 3. In the standard translation, if a given CoR term is in Σ_n^{CoR} (resp. Π_n^{CoR}), then the translated FO^3 term is in Σ_n^3 (resp. Π_n^3). Conversely, for each of (‡1)–(‡3), if a given $\mathrm{FO}^{3(2)}$ formula is in Σ_n^3 (resp. Π_n^3), then the translated nice $\mathrm{FO}^{3(2)}$ formula is also in Σ_n^3 (resp. Π_n^3). Also for (‡4), if a given nice FO^3 term is in Σ_n^3 (resp. Π_n^3), then the translated CoR term is in Σ_n^{CoR} (resp. Π_n^{CoR}). All the above are shown by simple induction on the size of given term/formula. □

Thus, the dot-dagger alternation hierarchy in CoR is also strict as the quantifier alternation hierarchy in FO^3.

Corollary 18 ([3, Lem. 3.9]). $\Sigma_{n+1}^{\mathrm{CoR}}$ *is strictly more expressive than* Σ_n^{CoR}.

Proof. By Corollary 17, it suffices to show that the class of Σ_{n+1}^3 formulas is strictly more expressive than the class of Σ_n^3 formulas. Let us recall the following Σ_{n+1} formula in [3, Lemma 3.9], which is not equivalent to any Σ_n formula:

$$\begin{aligned}&\exists x_0.\exists x_1.\forall x_2.\exists x_3 \cdots .Q x_{n+1}.\\&(\mathrm{Start}(x_0, x_0) \wedge \mathrm{Move}(x_0, x_1) \wedge \mathrm{Move}(x_1, x_2) \wedge \cdots \wedge \mathrm{Move}(x_n, x_{n+1}))\\&\rightarrow \mathrm{Win}(x_{n+1}, x_{n+1}).\end{aligned}$$

Where x_i and x_j are distinct if $i \neq j$; $Q = \exists$ if n is odd and $Q = \forall$ otherwise; the notation $\varphi \rightarrow \psi$ abbreviates $\neg\varphi \vee \psi$; and the unary relation symbols "Start" and "Win" in [3, Lemma 3.9] have been replaced with binary relation symbols, respectively. This formula is equivalent to the following formula in Σ_{n+1}^2:

$$\begin{aligned}&\exists x_0.\mathrm{Start}(x_0, x_0) \rightarrow \exists x_1.\mathrm{Move}(x_0, x_1) \rightarrow \forall x_2.\mathrm{Move}(x_1, x_2) \rightarrow \cdots \rightarrow Q x_{n+1}.\\&\mathrm{Move}(x_n, x_{n+1}) \rightarrow \mathrm{Win}(x_{n+1}, x_{n+1}).\end{aligned}$$

Where x_i and x_j denote the same variable if $i \equiv j \pmod 2$. Therefore Σ_{n+1}^3 is strictly more expressive than Σ_n^3, because there is no Σ_n formula (hence no Σ_n^3 formula) equivalent to this Σ_{n+1}^3 formula (by [3, Lem. 3.9]). □

4 PCoR Is Exponentially Less Succinct Than EP^3

In this section, we show that the exponential blow-up of the translation from EP^3 to PCoR given in Sect. 3 is unavoidable.

Theorem 19. *There is no* $2^{o(n)}$*-size translation from* EP^3 *terms to equivalent* PCoR *terms. (Hence,* PCoR *is exponentially less succinct than* EP^3*.)*

For each $n \in \mathbb{N}_+$, let $\mathfrak{t}_n$ be the following EP^3 term:

$$\left[(x = y) \vee \left(\bigvee_{i \in [n]} \mathsf{a}_i(x, y) \vee \mathsf{b}_i(x, y)\right) \vee \exists z.\left(\bigwedge_{i \in [n]} \mathsf{a}_i(x, z) \vee \mathsf{b}_i(z, y)\right)\right]_{x,y}.$$

Here, x, y, z are three distinct variables and $\mathsf{a}_1, \mathsf{b}_1, \mathsf{a}_2, \mathsf{b}_2, \dots$ are pointwise distinct binary relation symbols in $\mathscr{A}$. To prove Theorem 19, we will actually show that there is no $2^{o(n)}$-size translation from the set $\{\mathsf{t}_n \mid n \in \mathbb{N}_+\}$ to PCoR. Note that each t_n is equivalent to the following PCoR term:

$$\mathbf{1} \cup \left(\bigcup_{i \in [n]} \mathsf{a}_i \cup \mathsf{b}_i \right) \cup \left(\bigcup_{\langle I, J \rangle \in \mathrm{Part}([n])} (\bigcap_{i \in I} \mathsf{a}_i) \cdot (\bigcap_{j \in J} \mathsf{b}_j) \right).$$

Here, $\mathrm{Part}(X)$ denotes the set of all ordered partitions of size 2 of X (i.e., the set of all pairs $\langle I, J \rangle$ s.t. $I \cup J = X$, $I \cap J = \emptyset$, $I \neq \emptyset$, and $J \neq \emptyset$). However, unfortunately, this is not a $2^{o(n)}$-size translation, because $\#(\mathrm{Part}([n])) = 2^n - 2$.

Let us consider the parameter $\mathrm{w}_n(t)$:

$$\mathrm{w}_n(t) := \#(\{\langle I, J \rangle \in \mathrm{Part}([n]) \mid \langle 1, 3 \rangle \in [\![t]\!]_{M_{\langle I,J \rangle}}\}).$$

Here, $M_{\langle I,J \rangle}$ is the structure $\langle [3], \{a^{M_{\langle I,J \rangle}}\}_{a \in \mathscr{A}} \rangle$, where $a^{M_{\langle I,J \rangle}} = \{\langle 1, 2 \rangle\}$ if $a \in \{\mathsf{a}_i \mid i \in I\}$, $a^{M_{\langle I,J \rangle}} = \{\langle 2, 3 \rangle\}$ if $a \in \{\mathsf{b}_j \mid j \in J\}$, and $a^{M_{\langle I,J \rangle}} = \emptyset$ otherwise. Note that $\mathrm{w}_n(\mathsf{t}_n) = \#(\mathrm{Part}([n])) = 2^n - 2$ by the construction of t_n.

The following is the key lemma, which will be shown in the next subsection.

Lemma 20. *For every* PCoR *term s, if $\models s \leq \mathsf{t}_n$, then $\|s\| \geq \mathrm{w}_n(s)/8$.*

Theorem 19 can be proved by Lemma 20.

Proof (of Theorem 19 by using Lemma 20). As a consequence of Lemma 20, for every PCoR term s equivalent to t_n, $\|s\| \geq (2^n - 2)/8 \geq 2^{n-4}$, where $n \geq 2$. Note that $\mathrm{w}_n(s) = \mathrm{w}_n(\mathsf{t}_n)$ since $\models s \equiv \mathsf{t}_n$. We assume, towards contradiction, that there exists a $2^{o(n)}$-size translation f from EP^3 terms to PCoR terms. From this, there exists a monotone function $g \colon \mathbb{N} \to \mathbb{N}$ in $2^{o(n)}$ such that $\|f(\mathsf{t}_n)\| \leq g(\|\mathsf{t}_n\|)$. Also by the construction of the EP^3 term t_n, $\|\mathsf{t}_n\| \leq l(n)$ holds for some linear function $l \colon \mathbb{N} \to \mathbb{N}$. Combining the above, $2^{n-4} \leq \|f(\mathsf{t}_n)\| \leq g(\|\mathsf{t}_n\|) \leq g(l(n)) = (g \circ l)(n)$ and $g \circ l$ is a function in $2^{o(n)}$, but thus reaching a contradiction. $\square$

We prove Lemma 20 in the rest of this section.

4.1 Proof of Lemma 20

We say that a PCoR term is in *projection normal form* if it is in the set defined by the following grammar: $t, s ::= a \mid a^{\smile} \mid \mathbf{1} \mid \top \mid \mathbf{0} \mid t \cup s \mid t \cap s \mid t \cdot s$.

Proposition 21. *There is a linear-size translation l from* PCoR *terms to* PCoR *terms in projection normal form such that $\|l(t)\| \leq 8 \times \|t\|$.*

Proof. First we replace each sub-term t^{π} with $(t \cap \mathbf{1}) \cdot \top$ if $\pi = \{1 \mapsto 1, 2 \mapsto 1\}$; t if $\pi = \{1 \mapsto 1, 2 \mapsto 2\}$; and $\top \cdot (t \cap \mathbf{1})$ if $\pi = \{1 \mapsto 2, 2 \mapsto 2\}$ (then the π of each sub-term t^{π} is converse). Secondly, we push converse operators deeper into the term by the following rewriting rules: $\mathbf{1}^{\smile} \leadsto \mathbf{1}$; $\top^{\smile} \leadsto \top$; $\mathbf{0}^{\smile} \leadsto \mathbf{0}$; $(t^{\smile})^{\smile} \leadsto t$; $(t \cup s)^{\smile} \leadsto t^{\smile} \cup s^{\smile}$; $(t \cap s)^{\smile} \leadsto t^{\smile} \cap s^{\smile}$; and $(t \cdot s)^{\smile} \leadsto s^{\smile} \cdot t^{\smile}$. Note that the first step induces a factor of 4 and that the second step induces a factor of 2. $\square$

From this, to prove Lemma 20, it suffices to prove the following lemma.

Lemma 22. *For every s in projection normal form, if* $\models s \leq \mathsf{t}_n$*,* $\|s\| \geq \mathrm{w}_n(s)$*.*

To prove Lemma 22, we introduce a few notions; and then give a few properties (Lemmas 23 and 24) with respect to t_n.

The *disjoint union* of structures M_1 and M_2, written $M_1 \uplus M_2$, is the structure $\langle |M_1 \uplus M_2|, \{a^{M_1 \uplus M_2}\}_{a \in \mathscr{A}} \rangle$, where $|M_1 \uplus M_2| := \{\langle 1, v \rangle \mid v \in |M_1|\} \cup \{\langle 2, v \rangle \mid v \in |M_2|\}$ and $a^{M_1 \uplus M_2} := \{\langle \langle l, v \rangle, \langle l, v' \rangle \rangle \mid l \in [2], \langle v, v' \rangle \in a^{M_l}\}$. The *quotient* of a structure M w.r.t. an equivalence relation $\sim$, written $M/{\sim}$, is the structure $\langle |M/{\sim}|, \{a^{M/\sim}\}_{a \in \mathscr{A}} \rangle$, where $|M/{\sim}| := \{[v]_\sim \mid v \in |M|\}$ ($[v]_\sim$ denotes the equivalence class of v w.r.t. $\sim$) and $a^{M/\sim} := \{\langle [v]_\sim, [v']_\sim \rangle \mid \langle v, v' \rangle \in a^M\}$.

Lemma 23. *Let* $\mathrm{d}_M(v, v') := \min(\{k \in \mathbb{N} \mid \langle v, v' \rangle \in [\![(\bigcup_{i \in [n]} \mathsf{a}_i \cup \mathsf{b}_i)^k]\!]_M\} \cup \{\omega\})$*.*

- *If* $\mathrm{d}_M(v, v') < 2$*, then* $\langle v, v' \rangle \in [\![\mathsf{t}_n]\!]_M$*.*
- *If* $\mathrm{d}_M(v, v') > 2$*, then* $\langle v, v' \rangle \notin [\![\mathsf{t}_n]\!]_M$*.*

Proof. Immediate from the definition of t_n. □

Lemma 24. *For every two* PCoR *terms s_1 and s_2, the following hold.*

(1) *If* $\models s_1 \cup s_2 \leq \mathsf{t}_n$*, then* $\models s_1 \leq \mathsf{t}_n$ *and* $\models s_2 \leq \mathsf{t}_n$*.*
(2) *If* $\models s_1 \cap s_2 \leq \mathsf{t}_n$*, then* $\models s_1 \leq \mathsf{t}_n$ *or* $\models s_2 \leq \mathsf{t}_n$*.*
(3) *If* $\models s_1 \cdot s_2 \leq \mathsf{t}_n$ *and* $\not\models s_1 \cdot s_2 = \mathbf{0}$*, then* $\models s_1 \leq \mathsf{t}_n$ *and* $\models s_2 \leq \mathsf{t}_n$*.*

Proof.

(1) By $\models s_l \leq s_1 \cup s_2$ for $l \in [2]$.
(2) We show the contraposition. Let $\langle M_l, v_l, v_l' \rangle$ be such that $\langle v_l, v_l' \rangle \in [\![s_l]\!]_{M_l} \setminus [\![\mathsf{t}_n]\!]_{M_l}$ for each $l \in [2]$. Let M be the structure $(M_1 \uplus M_2)/{\sim}$, where $\sim$ is the minimal equivalence relation satisfying $\langle 1, v_1 \rangle \sim \langle 2, v_2 \rangle$ and $\langle 1, v_1' \rangle \sim \langle 2, v_2' \rangle$ (also we let $v = [\langle 1, v_1 \rangle]_\sim$ and $v' = [\langle 1, v_1' \rangle]_\sim$). Then (2-1) $\langle v, v' \rangle \in [\![s_1 \cap s_2]\!]_M$; (2-2) $\mathrm{d}_M(v, v') \geq 2$; and (2-3) $\langle v, v' \rangle \notin [\![\mathsf{t}_n]\!]_M$ hold. For (2-1), it is because $\langle v, v' \rangle \in [\![s_1]\!]_M$ and $\langle v, v' \rangle \in [\![s_2]\!]_M$ by the construction of M and Proposition 3. For (2-2), it is because $\mathrm{d}_M(v, v') = \min(\mathrm{d}_{M_1}(v_1, v_1'), \mathrm{d}_{M_2}(v_2, v_2'))$ by the construction of M; and for $l \in [2]$, $\mathrm{d}_{M_l}(v_l, v_l') \geq 2$ by $\langle v_l, v_l' \rangle \notin [\![\mathsf{t}_n]\!]_{M_l}$ (Lemma 23). For (2-3), by (2-2), it suffices to show that $\langle v, v' \rangle \notin [\![(\bigcap_{i \in I} \mathsf{a}_i) \cdot (\bigcap_{j \in J} \mathsf{b}_j)]\!]_M$ for every $\langle I, J \rangle$ of a partition of $[n]$. We assume, toward contradiction, that $\langle v, v' \rangle \in [\![(\bigcap_{i \in I} \mathsf{a}_i) \cdot (\bigcap_{j \in J} \mathsf{b}_j)]\!]_M$. Let w be such that $\langle v, w \rangle \in [\![\bigcap_{i \in I} \mathsf{a}_i]\!]_M$ and $\langle w, v' \rangle \in [\![\bigcap_{j \in J} \mathsf{b}_j]\!]_M$. Then w is distinct from v and v' by $\mathrm{d}_M(v, v') \geq 2$, so $w = \{\langle l, w_l \rangle\}$ for some l and some w_l. Then $\langle v_l, w_l \rangle \in [\![\bigcap_{i \in I} \mathsf{a}_i]\!]_{M_l}$ and $\langle w_l, v_l' \rangle \in [\![\bigcap_{j \in J} \mathsf{b}_j]\!]_{M_l}$ should hold, so $\langle v_l, v_l' \rangle \in [\![(\bigcap_{i \in I} \mathsf{a}_i) \cdot (\bigcap_{j \in J} \mathsf{b}_j)]\!]_{M_l}$. This contradicts to $\langle v_l, v_l' \rangle \notin [\![\mathsf{t}_n]\!]_{M_l}$. Hence $\langle v, v' \rangle \in [\![s_1 \cap s_2]\!]_M \setminus [\![\mathsf{t}_n]\!]_M$.
(3) We show the contraposition. We only write the case of $\not\models s_1 \leq \mathsf{t}_n$ (the case of $\not\models s_2 \leq \mathsf{t}_n$ is shown by same arguments). Let $\langle M_1, v_1, v_1' \rangle$ be such that $\langle v_1, v_1' \rangle \in [\![s_1]\!]_{M_1} \setminus [\![\mathsf{t}_n]\!]_{M_1}$ and let $\langle M_2, v_2, v_2' \rangle$ be such that $\langle v_2, v_2' \rangle \in [\![s_2]\!]_{M_2}$ (note that $\not\models s_2 = \mathbf{0}$ since $\not\models s_1 \cdot s_2 = \mathbf{0}$). Let M be

the structure $(M_1 \uplus M_2)/{\sim}$, where $\sim$ is the minimal equivalence relation satisfying $\langle 1, v_1' \rangle \sim \langle 2, v_2 \rangle$ (also we let $v = [\langle 1, v_1 \rangle]_\sim$ and $v' = [\langle 2, v_2' \rangle]_\sim$). Then (3-1) $\langle v, v' \rangle \in [\![s_1 \cdot s_2]\!]_M$; (3-2) $\mathrm{d}_M(v, v') \geq 2$; and (3-3) $\langle v, v' \rangle \notin [\![\mathsf{t}_n]\!]_M$ hold. (3-1) is shown by the construction of M and Proposition 3. (3-2) is shown by $\mathrm{d}_M(v, v') = \mathrm{d}_{M_1}(v_1, v_1') + \mathrm{d}_{M_2}(v_2, v_2')$ (by the construction of M) and $\mathrm{d}_{M_1}(v_1, v_1') \geq 2$ (by $\langle v_1, v_1' \rangle \notin [\![\mathsf{t}_n]\!]_{M_l}$ and Lemma 23). For (3-3), by (3-2), it suffices to show that $\langle v, v' \rangle \notin [\![(\cap_{i \in I} \mathsf{a}_i) \cdot (\cap_{j \in J} \mathsf{b}_j)]\!]_M$ for every $\langle I, J \rangle$ of a partition of $[n]$. We assume, toward contradiction, that $\langle v, v' \rangle \in [\![(\cap_{i \in I} \mathsf{a}_i) \cdot (\cap_{j \in J} \mathsf{b}_j)]\!]_M$. Let w be such that $\langle v, w \rangle \in [\![\cap_{i \in I} \mathsf{a}_i]\!]_M$ and $\langle w, v' \rangle \in [\![\cap_{j \in J} \mathsf{b}_j]\!]_M$. Then by $w \neq [\langle 1, v_1' \rangle]_\sim$ (notice $\mathrm{d}_{M_1}(v_1, v_1') \geq 2$), $w = \{\langle 1, w_1 \rangle\}$ for some $w_1 \in |M_1|$. From this, $\langle v_1, w_1 \rangle \in [\![\cap_{i \in I} \mathsf{a}_i]\!]_{M_1}$ and $\langle w_1, v_1' \rangle \in [\![\cap_{j \in J} \mathsf{b}_j]\!]_{M_1}$ should hold, so $\langle v_1, v_1' \rangle \in [\![(\cap_{i \in I} \mathsf{a}_i) \cdot (\cap_{j \in J} \mathsf{b}_j)]\!]_{M_1}$. This contradicts to $\langle v_1, v_1' \rangle \notin [\![\mathsf{t}_n]\!]_{M_1}$. Hence $\langle v, v' \rangle \in [\![s_1 \cdot s_2]\!]_M \setminus [\![\mathsf{t}_n]\!]_M$.

□

We are now ready to prove Lemma 22.

Proof (of Lemma 22). By induction on the structure of s.

Case $s = \mathbf{1}$, $s = \mathbf{0}$, $s = a$, or $s = a^\smile$: By $\mathrm{w}_n(s) = 0$.

Case $s = \top$: By $\not\models \top \leq \mathsf{t}_n$.

Case $s = s_1 \cup s_2$: $\mathrm{w}_n(s) \leq \mathrm{w}_n(s_1) + \mathrm{w}_n(s_2)$ holds by that, for every M, if $\langle v, w \rangle \in [\![s_1 \cup s_2]\!]_M$, then $\langle v, w \rangle \in [\![s_1]\!]_M$ or $\langle v, w \rangle \in [\![s_2]\!]_M$. Therefore by Lemma 24(1) and I.H., $\mathrm{w}_n(s) \leq \mathrm{w}_n(s_1) + \mathrm{w}_n(s_2) \leq \|s_1\| + \|s_2\| \leq \|s\|$.

Case $s = s_1 \cap s_2$: By Lemma 24(2), let l be such that $\models s_l \leq \mathsf{t}_n$. By $\models s \leq s_l$, $\mathrm{w}_n(s) \leq \mathrm{w}_n(s_l)$. Therefore by I.H., $\mathrm{w}_n(s) \leq \mathrm{w}_n(s_l) \leq \|s_l\| \leq \|s\|$.

Case $s = s_1 \cdot s_2$: If $\mathrm{w}_n(s) \leq 1$, then $\mathrm{w}_n(s) \leq \|s\|$ is trivial. Otherwise ($\mathrm{w}_n(s) \geq 2$), let $\Xi(s_1, s_2) := \{\langle \langle I, J \rangle, w \rangle \in \mathrm{Part}([n]) \times [3] \mid \langle 1, w \rangle \in [\![s_1]\!]_{M_{\langle I, J \rangle}} \wedge \langle w, 3 \rangle \in [\![s_2]\!]_{M_{\langle I, J \rangle}}\}$. Note that $\not\models s = \mathbf{0}$ and $\#(\Xi(s_1, s_2)) \geq 2$. Assume, toward contradiction, that, there are $\langle \langle I, J \rangle, v \rangle$ and $\langle \langle I', J' \rangle, v' \rangle$ such that $v \neq v'$. Without loss of generality, we can assume that $v > v'$. Let M be the structure $(M_{\langle I, J \rangle} \uplus M_{\langle I', J' \rangle})/{\sim}$, where $\sim$ is the minimal equivalence relation satisfying $\langle 1, v \rangle \sim \langle 2, v' \rangle$. By the construction of M and Proposition 3, $\langle [\langle 1, 1 \rangle]_\sim, [\langle 1, v \rangle]_\sim \rangle \in [\![s_1]\!]_M$ and $\langle [\langle 2, v' \rangle]_\sim, [\langle 2, 3 \rangle]_\sim \rangle \in [\![s_2]\!]_M$ hold, hence $\langle [\langle 1, 1 \rangle]_\sim, [\langle 2, 3 \rangle]_\sim \rangle \in [\![s]\!]_M$. On the other hand, by $\mathrm{d}_M([\langle 1, 1 \rangle]_\sim, [\langle 2, 3 \rangle]_\sim) = \mathrm{d}_{M_1}(1, v) + \mathrm{d}_{M_2}(v', 3) > 2$ and Lemma 23, $\langle [\langle 1, 1 \rangle]_\sim, [\langle 2, 3 \rangle]_\sim \rangle \notin [\![s]\!]_M$, thus reaching a contradiction.

Let $k \in [3]$ be the unique one such that, if $\langle \langle I, J \rangle, v \rangle \in \Xi(s_1, s_2)$, then $v = k$. We do case analysis on k.

Sub-Case $k = 1$: Then, for every $\langle I, J \rangle$, if $\langle 1, 3 \rangle \in [\![s_1 \cdot s_2]\!]_{M_{\langle I, J \rangle}}$, then $\langle 1, 3 \rangle \in [\![s_2]\!]_{M_{\langle I, J \rangle}}$. Thus $\mathrm{w}_n(s) \leq \mathrm{w}_n(s_2)$. Therefore by I.H. (notice $\models s_2 \leq \mathsf{t}_n$ by Lemma 24(3)), $\mathrm{w}_n(s) \leq \mathrm{w}_n(s_2) \leq \|s_2\| \leq \|s\|$.

Sub-Case $k = 2$: Let $\langle \langle I, J \rangle, 2 \rangle$ and $\langle \langle I', J' \rangle, 2 \rangle$ be *distinct* ones in $\Xi(s_1, s_2)$. Let M be the structure $(M_{\langle I, J \rangle} \uplus M_{\langle I', J' \rangle})/{\sim}$, where $\sim$ is the minimal equivalence relation satisfying $\langle 1, 2 \rangle \sim \langle 2, 2 \rangle$ (see Fig. 4). By the construction of M and Proposition 3, both $\langle [\langle 1, 1 \rangle]_\sim, [\langle 2, 3 \rangle]_\sim \rangle \in [\![s]\!]_M$ and $\langle [\langle 2, 1 \rangle]_\sim, [\langle 1, 3 \rangle]_\sim \rangle \in [\![s]\!]_M$ hold. On the other hand, $I \cup J' \not\subseteq [n]$ or $I' \cup J \not\subseteq [n]$ holds, because $\langle I, J \rangle$ and

$\langle I', J' \rangle$ are *distinct* partitions of $[n]$, and thus $\langle [\langle 1,1 \rangle]_\sim, [\langle 2,3 \rangle]_\sim \rangle \notin [\![\mathsf{t}_n]\!]_M$ or $\langle [\langle 2,1 \rangle]_\sim, [\langle 1,3 \rangle]_\sim \rangle \notin [\![\mathsf{t}_n]\!]_M$ should hold. This contradicts to $\models s \leq \mathsf{t}_n$.

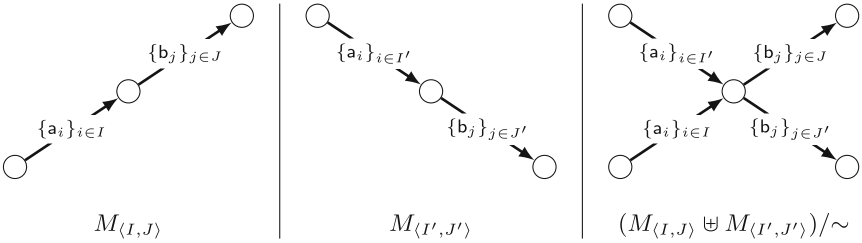

Fig. 4. Construction of $(M_{\langle I,J \rangle} \uplus M_{\langle I',J' \rangle})/\sim$.

Sub-Case $k = 3$: In the same way as Sub-Case $k = 1$. □

As a consequence of Lemma 22 (and Proposition 21), Lemma 20 has been proved.

5 On the Transitive Closure Extension

In this section, we remark that the results in Sects. 3 and 4 can be extended to the positive calculus of relations with transitive closure [19] (denoted by PCoR(TC), for short). We will show that the calculus has the same expressive power as three-variable existential positive (first-order) logic with (variable-confined) monadic transitive closure (denoted by EP^3(v-MTC)) (see, e.g., [6, Sec. 9] for transitive closure logic). The *syntax* of PCoR(TC) is given by: $t, s ::= a \mid \mathbf{1} \mid \top \mid \mathbf{0} \mid t \cup s \mid t \cap s \mid t^\pi \mid t \cdot s \mid t^+$. The *semantics* $[\![t]\!]_M$ and the *size* $\|t\|$ are defined in the same way as for CoR, respectively, where $[\![t^+]\!]_M := \bigcup_{k \in \mathbb{N}_+} [\![t]\!]_M^k$ and $\|t^+\| := 1 + \|t\|$. Also, the *syntax* of EP(v-MTC) is given by the following grammar, where $x, y, z, u \in \mathscr{V}$; z and u are distinct; and each $\mathrm{TC}_{z,u}(\varphi)$ is *variable-confined* (i.e., $\mathbf{FV}(\varphi) \subseteq \{z, u\}$[1]): $\varphi, \psi ::= a(x,y) \mid x = y \mid \mathsf{tt} \mid \mathsf{ff} \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x.\varphi \mid [\mathrm{TC}_{z,u}(\varphi)](x,y)$. The *semantics* $(I \models_M \varphi)$ and the *size* $\|\varphi\|$ are defined in the same way as for EP, where $I \models_M [\mathrm{TC}_{z,u}(\varphi)](x,y) :\Leftrightarrow \langle I(x), I(y) \rangle \in \bigcup_{k \in \mathbb{N}_+} [\![[\varphi]_{z,u}]\!]_M^k$ and $\|[\mathrm{TC}_{z,u}(\varphi)](x,y)\| := 1 + \|\varphi\|$. As in Sects. 3–4, the following are shown. The proofs are proceeded using the same strategy as that in the previous sections, extending the proofs in an appropriate way. In particular, Theorem 26 is proved by that we can extend Lemma 22 for PCoR(TC), because for every PCoR(TC) term t, if $\models t^+ \geq \mathsf{t}_n$, then $\mathrm{w}_n(t^+) = 0$.

[1] Here, z and u in $\mathrm{TC}_{z,u}(\psi)$ are viewed as bound variables (i.e., $\mathbf{FV}([\mathrm{TC}_{z,u}(\psi)](x,y))$ is defined by $\mathbf{FV}([\mathrm{TC}_{z,u}(\psi)](x,y)) := (\mathbf{FV}(\psi) \setminus \{z,u\}) \cup \{x,y\}$). See also [6, Sec. 9].

Theorem 25.

(1) *There is a linear-size translation from* PCoR(TC) *to* EP^3(v-MTC)*.*
(2) *There is an exponential-size translation from* EP^3(v-MTC) *to* PCoR(TC)*.*
(3) *Hence,* PCoR(TC) *has the same expressive power as* EP^3(v-MTC)*.*

Theorem 26. *There is no* $2^{o(n)}$*-size translation from* EP^3(v-MTC) *terms to equivalent* PCoR(TC) *terms.*

6 Conclusion

We have shown that (1) the positive calculus of relations has the same expressive power as three-variable existential positive logic, and (2) the positive calculus of relations is exponentially less succinct than three-variable existential positive logic. To the best of our knowledge, it is open whether the calculus of relations is exponentially less succinct than three-variable first-order logic. It would also be interesting to construct a calculus like the (positive) calculus of relations (or cylindric algebra [13]) such that it has the same expressive power as k-variable (existential positive) first-order logic and there is a succinctness-gap between them.

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