Probability Course

Lecture 2

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Session 2

we will discuss:-

- Random variable
- Discrete distributions
- Mean, Variance, Co variance, Correlation.

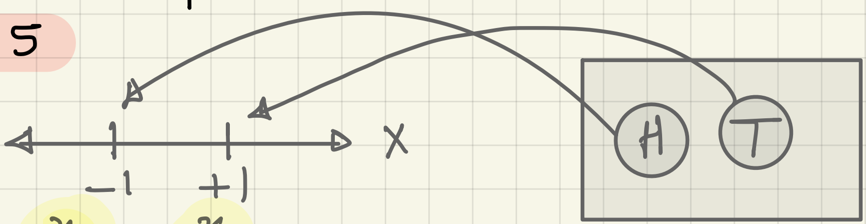
Probability:- is mapping between any event A by a value between 0 and one

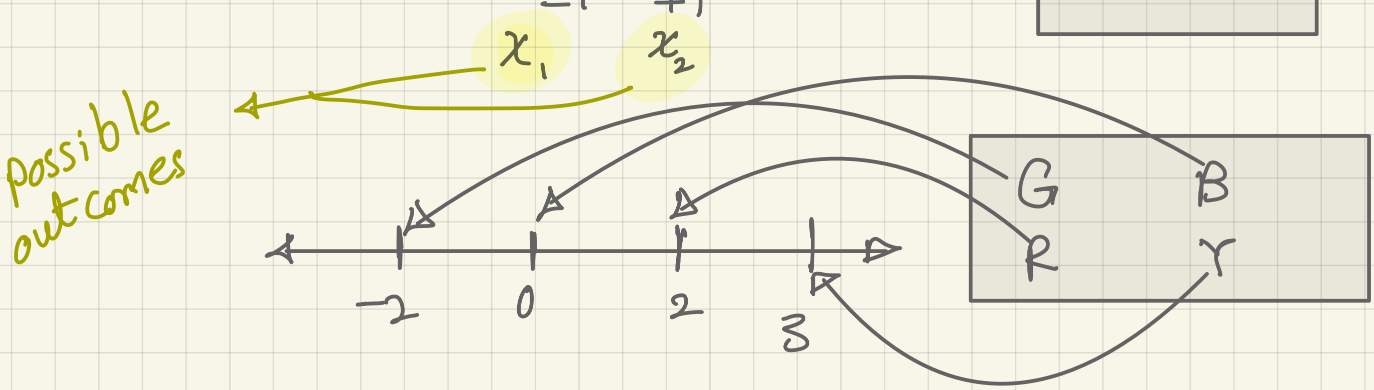
Random variables:-

represent (map) each possible outcome by point in a line

$p(H) = 0.5 \Leftrightarrow p(X = x_1) = 0.5 \Leftrightarrow p(x_1) = ?$

$p(-1) = p(+1) = 0.5$





impulse function has value in a certain point and zero otherwise

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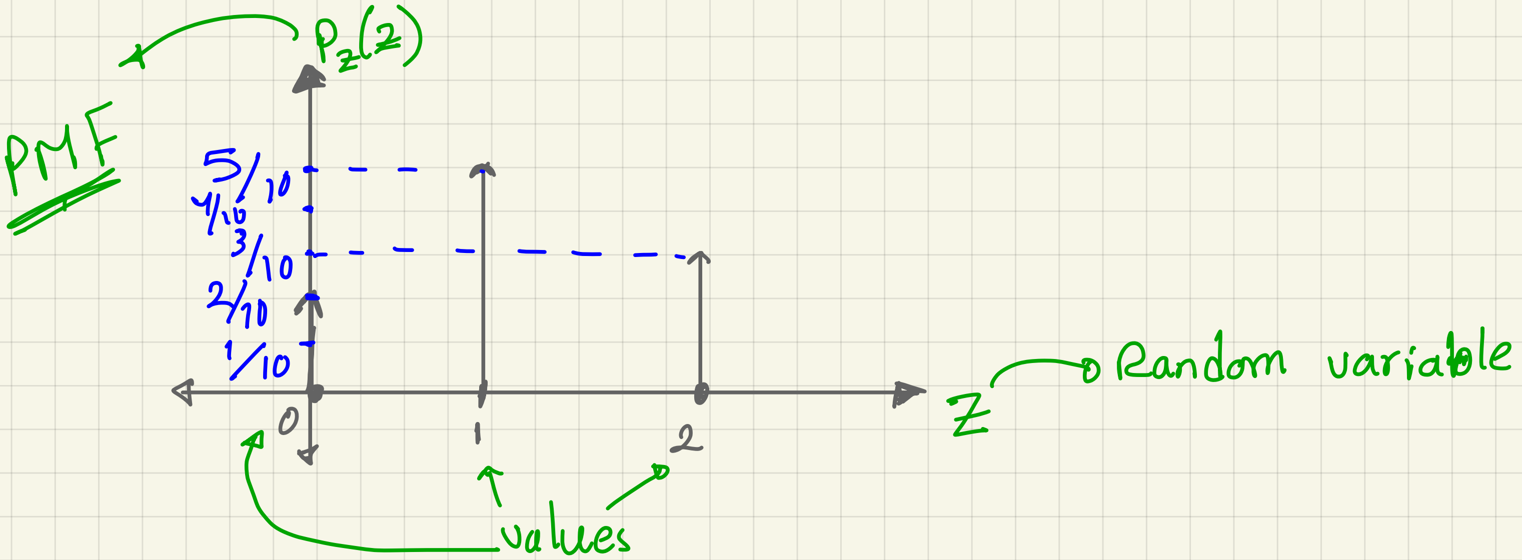
probability mass function (PMF)

box with (2 blue , 5 green , 3 yellow)

$z = 0 \Rightarrow$ outcome is blue

$z = 1 \Rightarrow$ " " Green

$z = 2 \Rightarrow$ " " yellow



$P_Z(z_1) = \frac{2}{10}$, $P_Z(z_2) = \frac{5}{10}$, $P_Z(z_3) = \frac{3}{10}$

z	0	1	2
P(z)	0.2	0.5	0.3

$P_Z(z)$ = function of $z \Rightarrow$ session 3

different ways to represent the probability distributions

$p(0) = 0.2$

$p(0.5) = 0$ (impossible event)

Joint probability:-

= Joint PMF :-

= Joint distribution:-

$P(H_1, H_2) \Longrightarrow$ first is head and second is H too

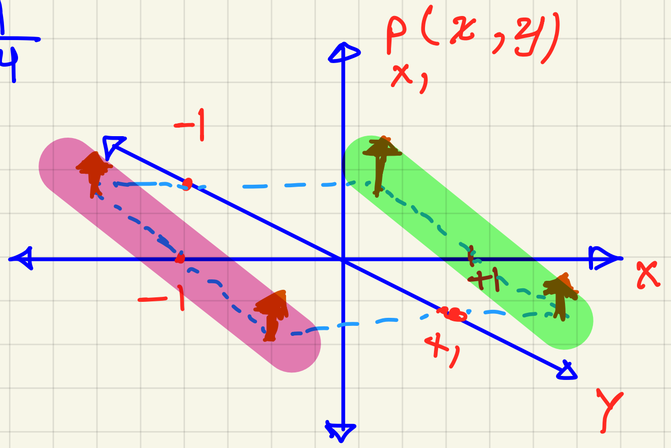
$= \frac{1}{4}$

$p(-1,-1) = \frac{1}{4}$

$p(-1,+1) = \frac{1}{4}$

$p(+1,-1) = \frac{1}{4}$

$p(-1,-1) = \frac{1}{4}$



(this joint probability distribution.)

OR can be represented by a table

Y \ X	−1	+1
−1	1/4	1/4
+1	1/4	1/4

$p(x, y)$

So it can be represented as a

1. table
2. graph
3. List

Joint PMF

* from $P_{X,Y}(x, y)$ we can get $P_X(x)$ & $P_Y(y)$ marginal PMF's

it's the $P_X(H) = P_X(-1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

it's the $P_Y(H) = P_Y(-1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

⇒ given Joint PMF we can find marginal PMF's as follows:-

$$\sum_j P_{x,y}(x_i, y_j) = P_x(x_i)$$

$$\sum_i P_{x,y}(x_i, y_j) = P_y(y_j)$$

$$\boxed{\sum_i \sum_j P_{x,y}(x_i, y_j) = \sum_i P_x(x_i) = 1}$$

Review:- ⇒ Independence of events A, B are independent if $P(A/B) = P(A)$ or $P(B/A) = P(B)$

In dependent $\Longleftrightarrow$ $P(AB) = P(A) \cdot P(B)$

not Independent ⇒ $P(AB) \neq P(A) \cdot P(B)$

→ two R.V.'s x, y are independent if

$$P_{x,y}(x, y) = P_x(x)\, P_y(y)$$

Example 1:-

$y \backslash x$	-1	$+1$
-1	$1/8$	$1/4$
$+1$	$1/8$	$1/2$

$P_X(x_i)$;

$P_X(-1) = \frac{1}{4}$

$P_X(+1) = \frac{3}{4}$

$P_Y(y_j)$;

$P_Y(-1) = \frac{3}{8}$

$P_Y(+1) = \frac{5}{8}$

are they independent or not?

$$P_{X,Y}(x_i, y_j) = P_X(x_i) \cdot P_Y(y_j) \quad ?$$

$$P_{XY}(-1,-1) = \frac{1}{8} \quad , \quad P_X(-1) \cdot P_Y(-1) = \frac{1}{4} \times \frac{3}{8} \neq \frac{1}{8}$$

$\Longrightarrow$ X , Y are not independent.

$\therefore \; P_{X/Y}(x_i / y_j) \neq P_X(x_i)$

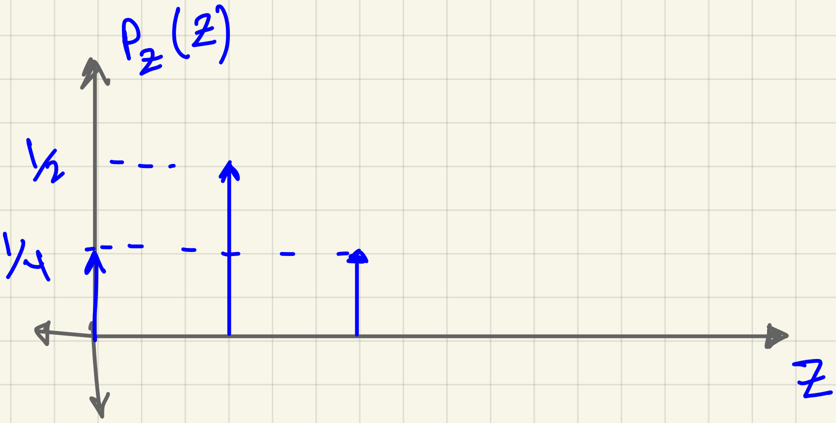
$\therefore \; P_{Y/X}(y_j / x_i) \neq P_Y(y_j)$

So

$$\underbrace{P_{x/y}(x_i, y_j)}_{\text{conditional PMF}} = \underbrace{P_{x/y}(x_i, y_j)}_{\text{Joint PMF}} \div \underbrace{P_y(y_j)}_{\text{marginal PMF}}$$

Example 2:-

tossing 2 coins observing total number of heads.



Bionomial Distribution

$$\begin{cases} P_z(n) = \binom{n}{k} (1/2)^k \left(\frac{1}{2}\right)^{n-k} \\ P_z(0) = \binom{2}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^2 = \frac{1}{4} \\ P_z(1) = \binom{2}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^1 = \frac{1}{2} \\ P_z(2) = \binom{2}{2} (1/2)^2 (1/2)^0 = 1/4 \end{cases}$$

— statistical tools

— Moments

mean, variance, Correlation, covariance
standard diviation, Correlation coefficient
, covariance matrix

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mean, average, expected value, expectation,
expected value;

expected of x

$$m_x = \mu_x = \bar{x} = E[x] = \langle x \rangle$$

$x: 3, 4, 4, 5, 5$

A ctual value in the set

$$\bar{x} = \frac{3+4+4+5+5}{5} = 3 \times \frac{1}{5} + 4 \times \frac{2}{5} + 5 \times \frac{2}{5}$$

$$\bar{x} = \sum_i x_i \frac{n(x_i)}{N}$$

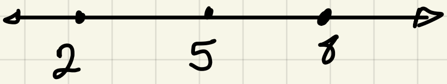
Number of time the number appears

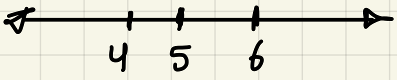
$$\bar{x} = \sum_i x_i P_x(x_i) \Longrightarrow 1^{st} \text{ moment of R.V. "x"}$$

general rule $\Longleftarrow$ $$\overline{\underbrace{g(x)}_{\text{function}}} = \sum_i g(x_i) P_x(x_i)$$

1^{st} moment is the "center" $\Longrightarrow$ "weighted average" one of the problems of the mean is that it losses info.

Ex:-





$\bar{x} = 5$ $\qquad\qquad$ $\bar{x} = 5$

they are not same but there mean is the same

So $\bar{x}$:- measure of Centrality.

we need other measures to describe distribution.

eg; measure of dispertion around $\bar{x}$

2^{nd} Central moment :-

Variance :- σ_x^2 $\qquad$ $\sqrt{\sigma_x^2} = \sigma_x \Rightarrow$ standard diviation

the aim of the square is to remove neagtive signs

$\sigma_x^2 = \overline{(x_i - \bar{x})^2}$ متوسط المتوسط

$$\sigma_x^2 = \frac{(2-5)^2 + (5-5)^2 + (8-5)^2}{n} = \frac{9+0+9}{3} = 6$$

$$\sigma_x^2 = \frac{(4-5)^2 + (5-5)^2 + (6-5)^2}{n} = \frac{1+0+1}{3} = \frac{2}{3} = 0.6667$$

$\sigma_x = \sqrt{6}$ $\qquad\qquad$ $\sigma_y = \sqrt{2/3}$

if data is Normally distributed $\Rightarrow$ I can describe it using mean and standard diviation

$$\sigma_x^2 = \sum_i (x_i - \bar{x})^2 P_x(x_i)$$

$$\sigma_x^2 = \overline{(x_i - \bar{x})^2}$$

$$= \overline{(x_i^2 - 2x_i\bar{x} + \bar{x}^2)}$$

$$= \overline{x_i^2} - \overline{2x_i\bar{x}} + \overline{\bar{x}^2}$$

$$= \overline{x^2} - \cancel{2}\bar{x}^2 + \cancel{\bar{x}^2}$$

mean square value

$\bar{x} = \mu_x$

$\bar{x}^2 = \mu_x^2$

$\overline{\bar{x}^2} = \mu_x^2 = \bar{x}^2$

$2\bar{x}\bar{x} = 2\bar{x}^2$

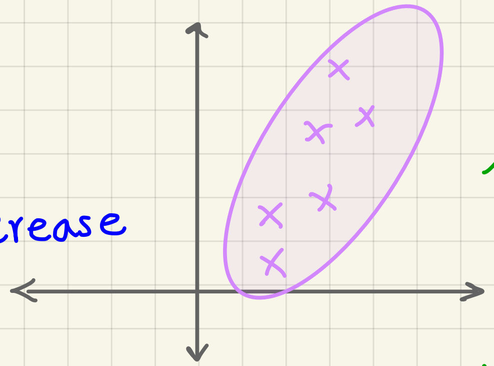
$$\sigma_x^2 = \overline{x^2} - \bar{x}^2$$

Correlation $\neq$ Dependence

Linear correlation

positive correlation

one increase the other increase too

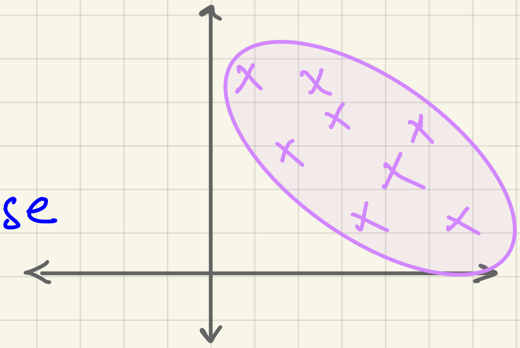


$\rho_{xy} = +ve$

correlated & dependent

negative correlation

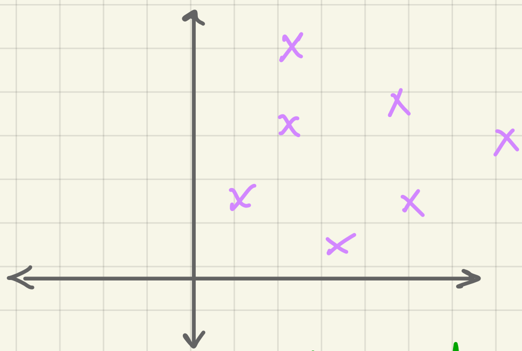
one increase the other decrease



$\rho_{xy} \cong -ve$

correlated & dependent

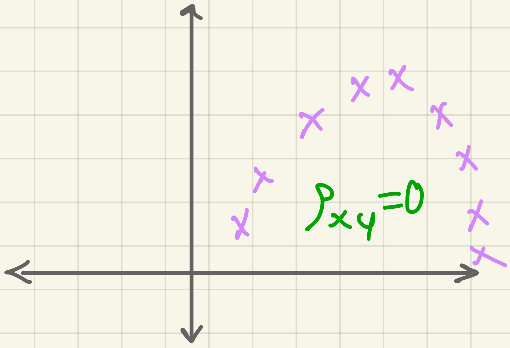
No correlation

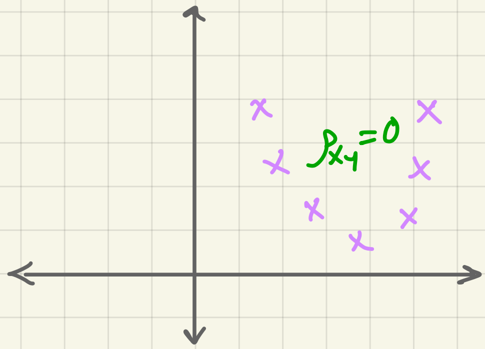


$\rho_{xy} \cong 0$

Independent but uncorrelated

But take care that all cases in are Linear relation





eg $y = (x+b)^2$

→ • Not Linearly correlated

• x, y are not independent
they are dependent

• but not Linearly correlated

Note that usually the word correlated refers to Linearly correlation.

So we can't say that Corelation can tell us if they are dependent or not, as in this example they are not correlated but dependent in the same time

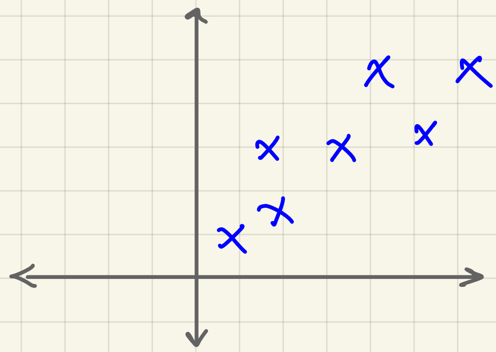
if x, y are independent ⟹ uncorrelated

if x, y are uncorrelated → but not independent
OR → independent

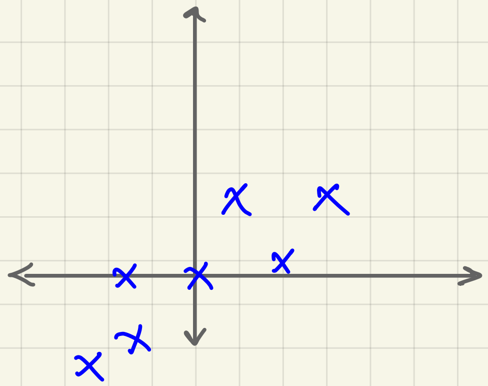
Linearly

So here comes the Covariance

Normal data $\Longrightarrow$



Covariance $\Longrightarrow$



$$\sigma_{xy} = \frac{\Sigma (x_i - \bar{x}) \cdot (y_i - \bar{y})}{n}$$

$$= \overline{(x_i - \bar{x}) \cdot (y_i - \bar{y})}$$

Notes :-

- Variance $= \sigma_x^2 = \sigma_x \, \sigma_x = \sigma_{xx} = \overline{(x - \bar{x}) \cdot (x - \bar{x})} = \overline{(x - \bar{x})^2} = \sigma_x^2$
- Covariance as terminology means the variance between 2 things $\Longrightarrow$ measure the change in one relative to the change in the other one

$\Rightarrow$ we need standarization of covariance

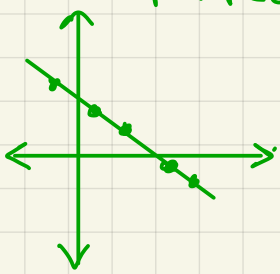
Correlation coefficient :-

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \ \sigma_y}$$

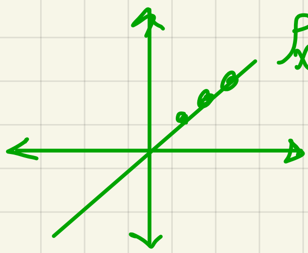
$$-1 \le \rho \le 1$$

-1: perfect Linear Correlation $-$ve

1: perfect Linear correlation $+$ve



$\rho_{xy}=1$



$$\sigma_{xy} = \overline{(x-\bar{x})\cdot(y-\bar{y})}$$

$$= \overline{xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}}$$

$$= \overline{xy} - \bar{x}\,\bar{y} - \cancel{\bar{x}\bar{y}} + \cancel{\bar{x}\bar{y}}$$

$$\sigma_{xy} = \overline{xy} - \bar{x}\,\bar{y}$$

$\Rightarrow$ if x, y are uncorrelated :-

$$\sigma_{xy} = 0 = \overline{xy} - \bar{x}\,\bar{y}$$

$\overline{xy} = \bar{x}\,\bar{y}$ $\Rightarrow$ so to check if it's correlated or not you can multiply each value by the other and get their mean $= \bar{x}\,\bar{y}$?

$\Rightarrow$ if x, y are independent

$$P_{XY}(x,y) = P_X(x) \cdot P_Y(y)$$

if they are independent then there's no Linear correlation

* Resons for outliers $\Rightarrow$ noise & error in measurement

But it Can also be real data

* So you need to study the outliers well to see if I will leave it or keep it