Mathematical Modeling and Engineering Problem Solving

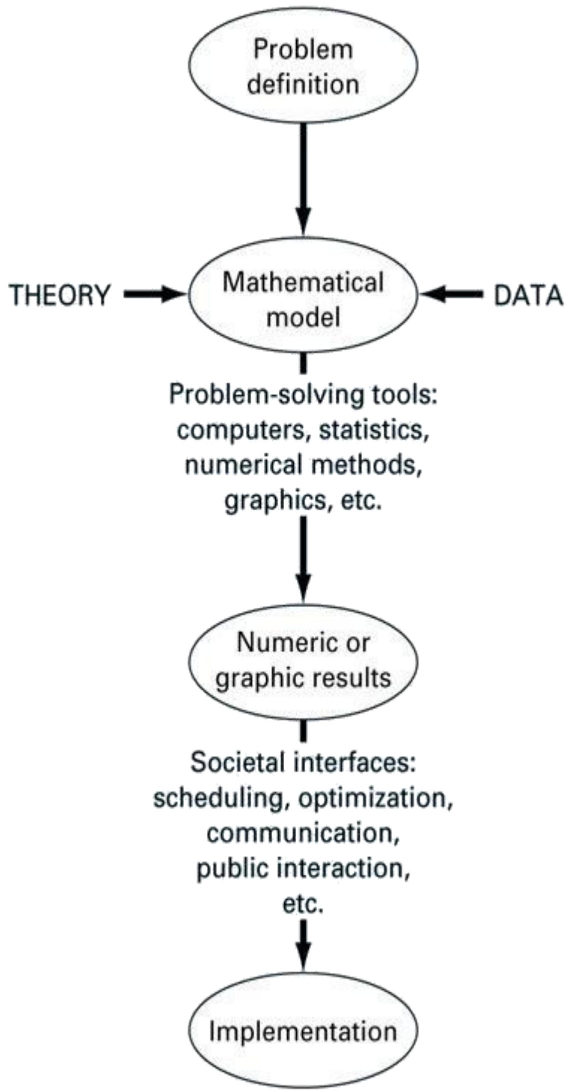
Chapra: Chapter-1

Mathematical Modeling and Engineering Problem Solving

- Requires understanding of engineering systems
    - By observation and experiment
    - Theoretical analysis and generalization

- Computers are great tools, however, without fundamental understanding of engineering problems, they will be useless.

The Engineering Problem Solving Process



The Engineering Problem Solving Process

- A mathematical model is represented as a functional relationship of the form

$$\text{Dependent variable} = f\left(\text{independent variables},\ \text{parameters},\ \text{forcing functions}\right)$$

- ***Dependent variable***: Characteristic that usually reflects the state of the system
- ***Independent variables***: Dimensions such as time and space along which the systems behavior is being determined
- ***Parameters***: reflect the system's properties or composition
- ***Forcing functions***: external influences acting upon the system

Newton's 2nd law of Motion

- States that "*the time rate change of momentum of a body is equal to the resulting force acting on it*."
- The model is formulated as

$$F = m\,a$$

F = net force acting on the body (*N*)

m = mass of the object (kg)

a = its acceleration (m/s^2)

Newton's 2nd law of Motion

- Formulation of Newton's 2nd law has several characteristics that are typical of mathematical models of the physical world:
  - It describes a natural process or system in mathematical terms
  - It represents an idealization and simplification of reality
  - Finally, it yields reproducible results, consequently, can be used for predictive purposes.

Determining Terminal Velocity of Free-falling Body

- Some mathematical models of physical phenomena may be much more complex.

- Complex models may not be solved exactly or require more sophisticated mathematical techniques than simple algebra for their solution.
  - Example, modeling of a falling parachutist:

Determining Terminal Velocity of Free-falling Body

F_U

F_D

$$\frac{dv}{dt} = \frac{F}{m}$$

$$F = F_D + F_U$$

$$F_D = mg$$

$$F_U = -cv$$

$$\frac{dv}{dt} = \frac{mg - cv}{m}$$

c is the proportionality constant called the *drag coefficient(kg/s)*

Exact Solution

- This is a differential equation and is written in terms of the differential rate of change *dv*/*dt* of the variable that we are interested in predicting.
- If the parachutist is initially at rest ($v = 0$ at $t = 0$), using calculus

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

Independent variable

Dependent variable

Forcing function

Parameters

Analytical Solution to the Falling Parachutist Problem

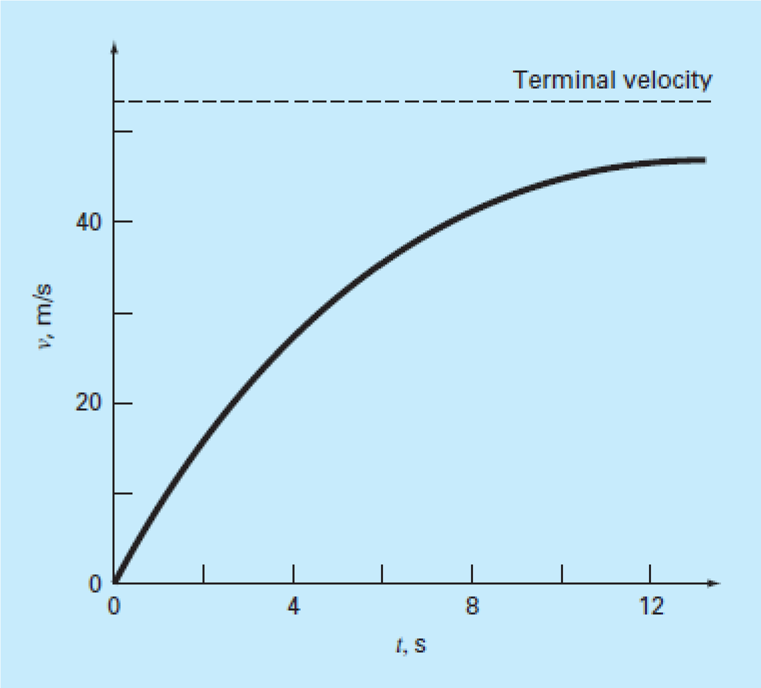
Problem statement: A parachutist of mass 68.1 kg jumps out of a stationary hot air balloon. Compute velocity prior to opening the chute. The drag coefficient is equal to 12.5 kg/s.

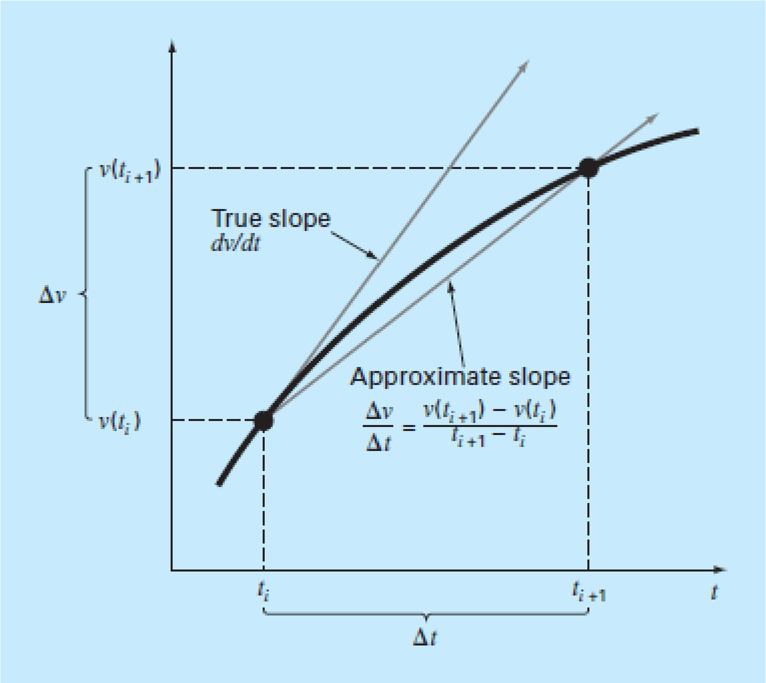
Solution:

$$v(t) = \frac{9.81(68.1)}{12.5}\left(1 - e^{-(12.5/68.1)t}\right) = 53.44\left(1 - e^{-0.18355t}\right)$$

Why Numerical?

- There exists many cases where analytical/exact solution is not possible.
- We can develop a numerical solution that approximates the exact solution.





Numerical Solution

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c}{m} v(t_i)$$

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i) \right] (t_{i+1} - t_i)$$

Find and compare the values of *v(t)* at t={0, 2, 4, 6, 8 …}
Using Exact solution
Using Numerical solution
Compare the results

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

Numerical Solution to the Falling Parachutist Problem

Problem Statement: Perform the same computation as in previous example but use numerical solution to compute the velocity. Employ a step size of 2 s for the calculation.

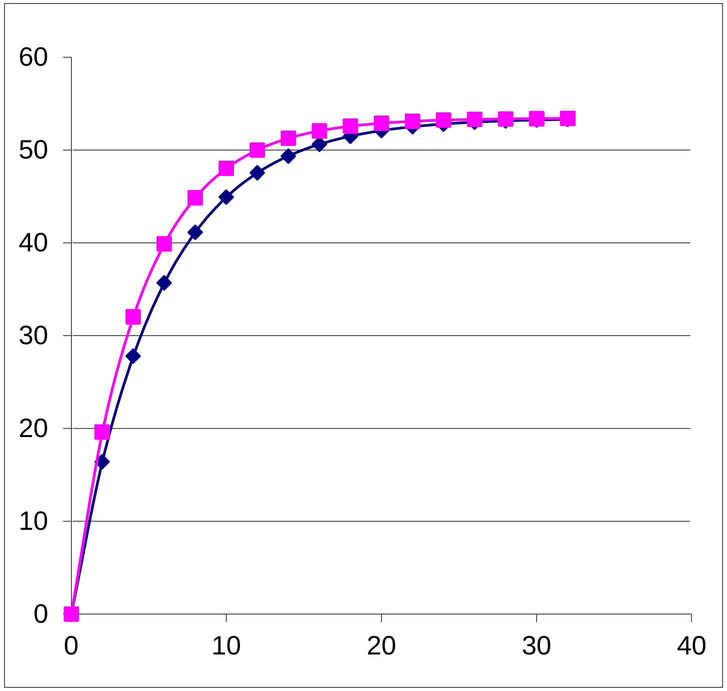
Solution:

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i) \right] (t_{i+1} - t_i)$$

$$v = 0 + \left[9.81 - \frac{12.5}{68.1}(0) \right] 2 = 19.62 \text{ m/s}$$

$$v = 19.62 + \left[9.81 - \frac{12.5}{68.1}(19.62) \right] 2 = 32.04 \text{ m/s}$$

m=	68.1
c=	12.5
g=	9.81
Δt=	2



t	Actual	Estimate
0	0	0
2	16.42172	19.62
4	27.79763	32.03736
6	35.67812	39.89621
8	41.13722	44.87003
10	44.91893	48.01792
12	47.53865	50.01019
14	49.35343	51.27109
16	50.61058	52.06911
18	51.48146	52.57416
20	52.08475	52.89381
22	52.50267	53.09611
24	52.79218	53.22415
26	52.99273	53.30518
28	53.13166	53.35646
30	53.22791	53.38892
32	53.29457	53.40946

Conservation Laws and Engineering

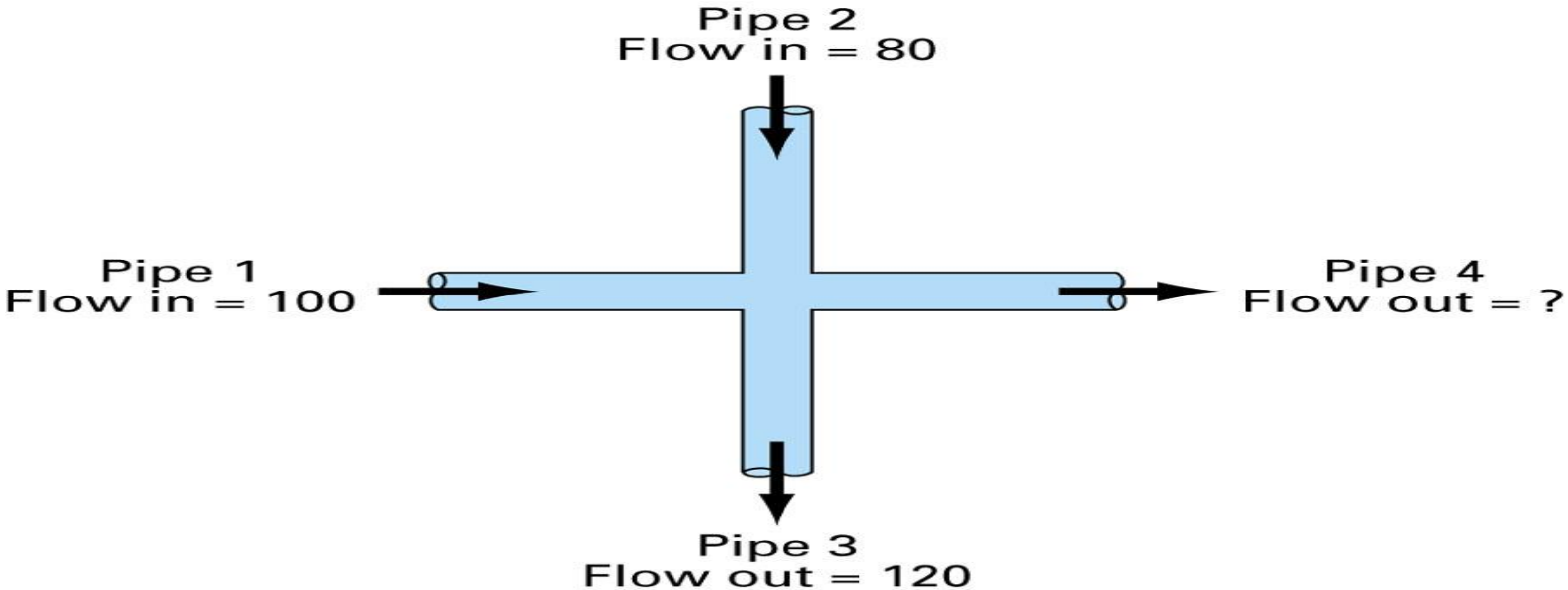
- Conservation laws are the most important and fundamental laws that are used in engineering.

$$\text{Change} = \text{increases} - \text{decreases} \qquad (1.13)$$

- Change implies changes with time (transient). If the change is nonexistent (steady-state), Eq. 1.13 becomes

$$\text{Increases} = \text{Decreases}$$

Conservation Laws and Engineering



- For steady-state incompressible fluid flow in pipes:

$$\text{Flow in} = \text{Flow out}$$

or

$$100 + 80 = 120 + \text{Flow}_4$$

$$\text{Flow}_4 = 60$$

Conservation Laws and Engineering