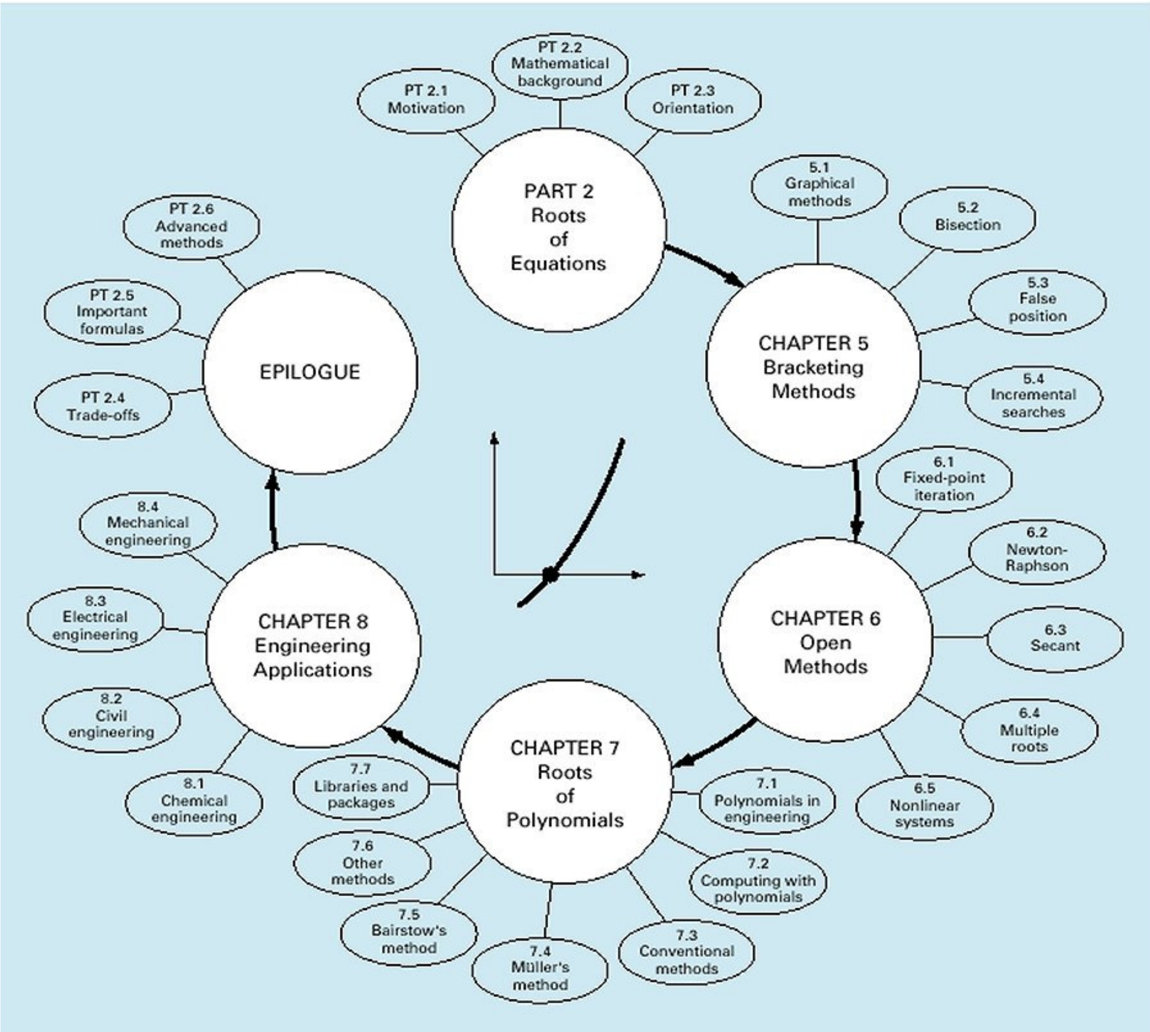
NUMERICAL METHODS FOR ENGINEERS

COEB 223

PART II : ROOTS OF EQUATION

Dr. Hanim Salleh
Universiti Tenaga Nasional
2007/2008

Introduction to Part 2: Roots of equations



Actual beginning of Numerical Methods is from this point.

Polynomial; $f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$

$y = f(x)$ is algebraic if it can be expressed in the form

$$f_n y^n + f_{n-1} y^{n-1} + \ldots + f_1 y + f_0 = 0$$

where f_i is an ith order polynomial

Transcendental equations contain non-algebraic expressions exponential, trigonometric, logarithmic and other functions. For example

$$f(x) = e^{-0.2x} \sin(3x - 0.5)$$

Roots of Equations.:

The value of x which makes $f(x)$=0 are called roots or 'zeros' of the equation.

For quadratic equation roots can be found by a standard formula. Other equations, it is difficult. Two types of problems would be dealt with here.

1. Real roots of algebraic and transcendental equations
2. Complex roots of polynomials.

Methods for finding the roots:

1. Graphical Methods (Ch 5) 2. Bracketing Methods (Ch 5) a) Bisection Method b) False Position Method (Regula Falsi)	3. Open Methods (Ch 6) a) Fixed point iteration b) Newton-Raphson Method c) Secant Method 4. Muller's and Bairstow's methods for polynomial roots (ch 7)

Handout 7

Chapter 5

- 5.1Graphical Methods
- 5.2 Bisection
- 5.3 False Position

5.0 Bracketing Methods

- Method that exploit the fact that a function typically changes sign in the vicinity of a root.
- It is called 'bracketing method' – need 2 initial guesses on either side ('bracketing') of the root

5.1 Graphical method

- of limited practical value since are not precise, however can be employed as starting guesses for numerical method

example 1:

Determine the 'drag coefficient', c, required for a parachutist of mass m = 68.1 kg to have a velocity of 40 m/s after free-falling for a time of t= 10s. Assume g= 9.8 m/s^2.

Solution: The relation between velocity and time and c are given by the relation:

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

It is given that at t= 10s, v= 40 m/s and we should find c.
Notice that c is implicit (cannot rearrange c to one side of the equation). Thus let us define a function $f(c)$ as follows:

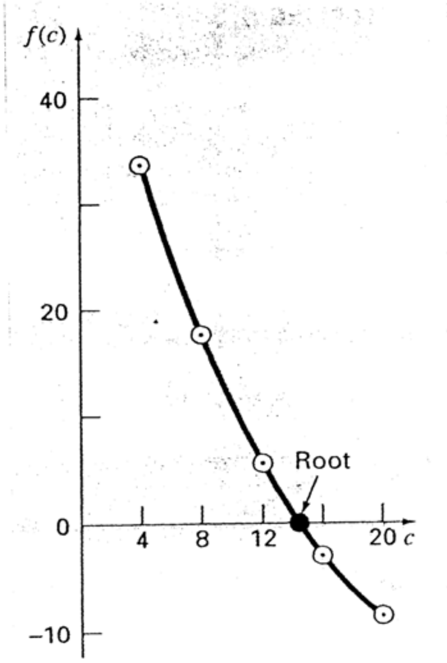
$$f(c) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right) - v$$

the value of 'c' which makes $f(c) = 0$ is the required value i.e. we require the root of $f(c)$. Substitute value,

$$f(c) = \frac{9.8(68.1)}{c}\left(1 - e^{-(c/68.1)10}\right) - 40$$

if c=4, $f(c) = \frac{9.8(68.1)}{4}\left(1 - e^{-(4/68.1)10}\right) - 40 = 34.115$, so do for other values as in the table and plot the graph to see where it intersect. By visual inspection the rough estimate of the root is 14.75. This should be check again by substituting to *f(c)* and *v*, where we want close to *f(c)*=0 and *v*=40.

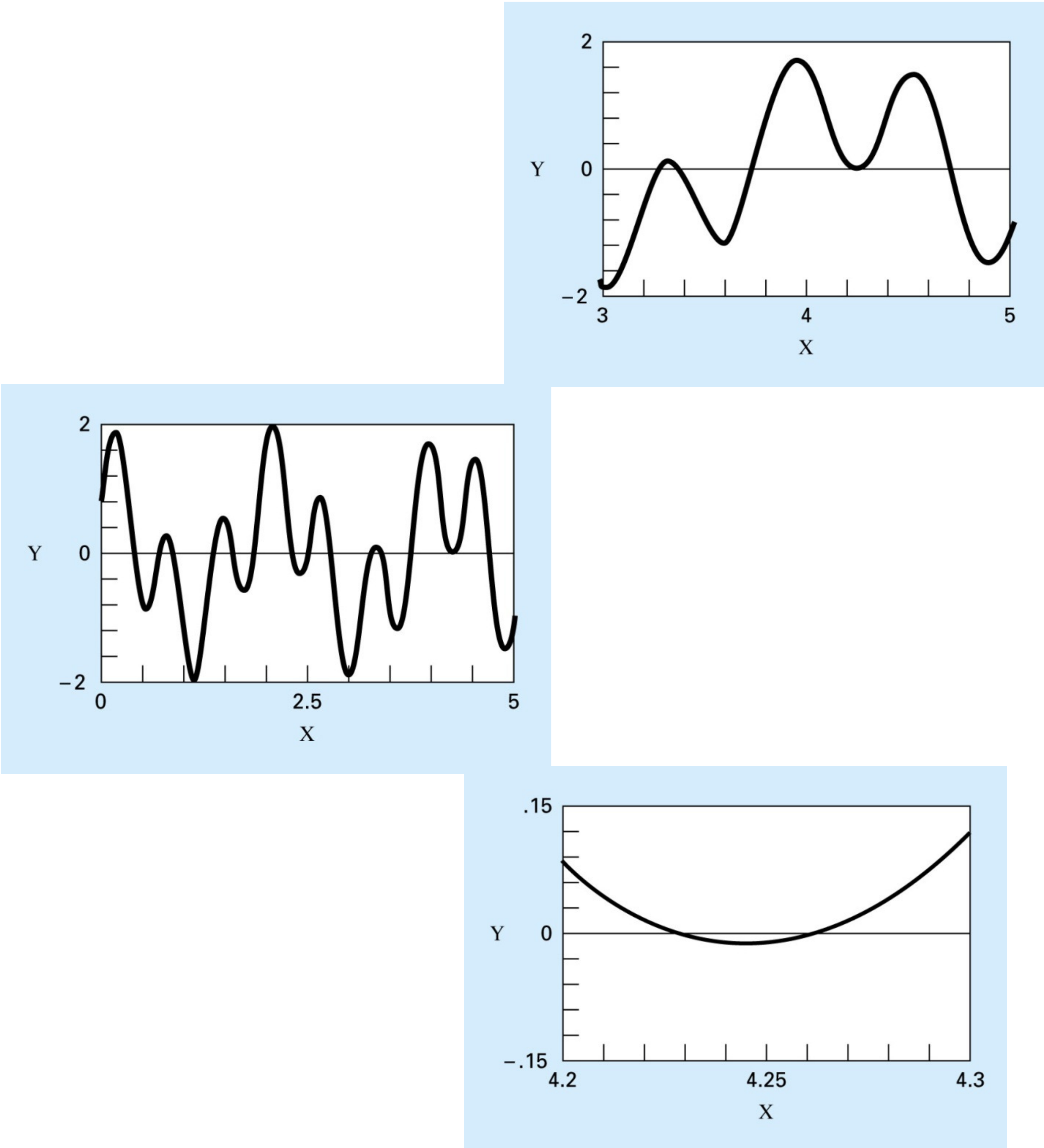
c	f(c)
4	34.115
8	17.653
12	6.067
16	-2.269
20	-8.401



Graphical Methods are not very accurate. However, they are very useful in understanding the nature of the function and also determining which method to be used. Graphical Methods which help you to zoom in can give you interesting results:

Consider the function: f(x) = sin 10x + cos 3x

Note the two roots between x = 4.2 and x = 4.3



2
Y 0
−2
3
4
5
X
2
Y 0
−2
0
2.5
5
X
.15
Y 0
−.15
4.2
4.25
4.3
X

5.2 Bisection Method

(Other names: Binary Chopping, Interval Halving, Bolzano Method.)

It is an incremental Search Method: The interval is refined or narrowed by each increment until the root is found with the necessary accuracy. By graphical method it is observed that the function has a single root between two values x_l and x_u.

1. **Choose starting values of x_l and x_u such that $f(x_l)\,f(x_u) < 0$**
2. **Locate the mid point of interval $x_r = (x_l + x_u)/2$**
3. **Calculate $f(x_r)$**

a) If $f(x_l)\,f(x_r)<0$ then $x_u=x_r$ and return to step 2

b) If $f(x_l)\,f(x_r)>0$ then $x_l=x_r$ and return to step 2

c) If $f(x_l)\,f(x_r)=0$ then x_r is the root.

We can solve the problem of finding 'c' for the falling parachutist.

True value for the root is 14.7802

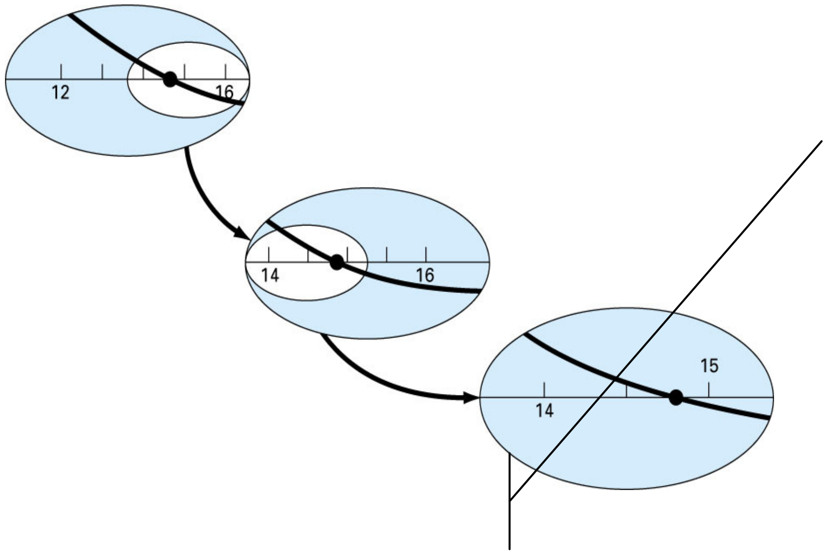
Stopping Criterion $\varepsilon_s = 0.5\%$

Let $x_l = 12$ and $x_u = 16$, $x_r = (12+16)/2 = 14$ for c= 14, $\varepsilon_t = 5.3\%$
f(12) f(14)= 6.067 * 1.569 = 9.517 >0 (no sign change)
Hence root lies between 14 and 16. etc.

note: $f(c) = \dfrac{9.8(68.1)}{c}\left(1 - e^{-(c/68.1)10}\right) - 40$ from example 1.

Iteration	xl	xu	xr	fl	fr	f(xl)*f(xr)	ea(%)	et(%)
1	12	16	14	6.0669	1.5687	9.5172834		5.2787
2	14	16	15	1.5687	-0.425	-0.666438	6.6667	1.4871
3	14	15	14.5	1.5687	0.5523	0.8664426	3.4483	1.8958
4	14.5	15	14.75	0.5523	0.059	0.0325668	1.6949	0.2043
5	14.75	15	14.875	0.059	-0.184	-0.010856	0.8403	0.6414
6	14.75	14.875	14.8125	0.059	-0.063	-0.003707	0.4219	0.2185

Figure for the first three iterations.



5.3 False Position Method

(Other names: Regula Falsi, Linear Interpolation Method)

The Bisection method does not take the magnitude of $f(x_l)$ and $f(x_u)$ in determining the value of x_r. Final value can be reached with less effort if we assume a false position for x_r assuming a straight relation between $f(x_l)$ and $f(x_u)$.

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

This can be solved for x_r which gives:

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Calculate $f(x_r)$ and find its sign. $f(x_r)$ will replace the $f(x_u)$ or $f(x_l)$ whichever has the same sign as $f(x_r)$. Thus x_r will always bracket the root.

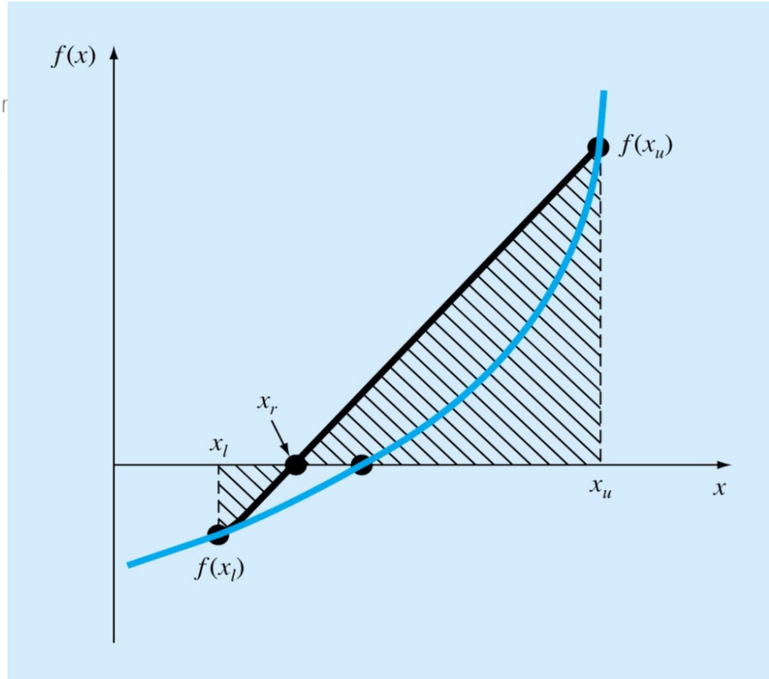
1) Calculate x_r using the above equation

   a) If $f(x_l)\, f(x_r)<0$ then $x_u=x_r$ and return to step 1

   b) If $f(x_l)\, f(x_r)>0$ then $x_l=x_r$ and return to step 1

c) If $f(x_l)\, f(x_r)=0$ then x_r is the root.

FIGURE 5.12
A graphical depiction of the method of false position. Similar triangles used to derive the formula for the method are shaded.



See Sec.5.3.1, Pitfalls of the False-Position Method

Note: Always check by substituting estimated root in the original equation to determine whether $f(x_r) \approx 0$.

See example problem 5.5 and 5.6.

Iteration	xl	xu	fxl	fxu	xr	fr	f(xl)*f(xr)	ea(%)	et(%)
1	12	16	6.06695	-2.26875	14.9113	-0.254277	-1.54269		0.887
2	12	14.9113	6.06695	-0.25426	14.7942	-0.027256	-0.16536	0.7916	0.0947
3	12	14.7942	6.06695	-0.02726	14.7817	-0.002908	-0.01764	0.0845	0.0102
4	12	14.7817	6.06695	-0.00291	14.7804	-0.00031	-0.00188	0.009	0.0011

Example: Problem 5.3

5.3 Determine the real root of

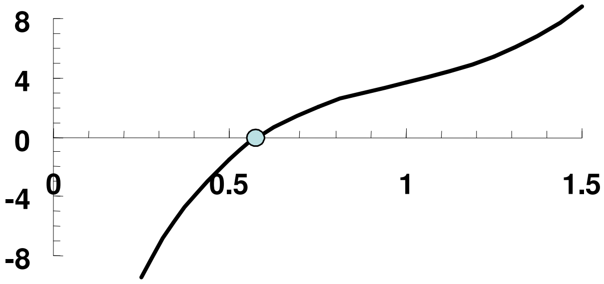
$$f(x) = -25 + 82x - 90x^2 + 44x^3 - 8x^4 + 0.7x^5$$

(a) **Graphically**

(b) **Using bisection,** $x_l = 0.5$, $x_u = 1$, $\varepsilon_s = 10\%$

(c) **Using false-position,** $x_l = 0.5$, $x_u = 1$, $\varepsilon_s = 2\%$

Solution: **(a)** A plot indicates that a single real root occurs at about x = 0.58.



(b) Bisection:

First iteration: $x_r = \dfrac{0.5+1}{2} = 0.75$ $\varepsilon_a = \left|\dfrac{1-0.5}{1+0.5}\right| \times 100\% = 33.33\%$

$$f(0.5)f(0.75) = -1.47813(2.07236) = -3.06321$$

Therefore, the new bracket is $x_l = 0.5$ and $x_u = 0.75$.

iteration	x_l	x_u	x_r	$f(x_l)$	$f(x_r)$	$f(x_l)\times f(x_r)$	ε_a
1	0.50000	1.00000	0.75000				
2							
3							
4							

(c) False position:
First iteration:
$x_l = 0.5$ $f(x_l) = -1.47813$
$x_u = 1$ $f(x_u) = 3.7$

$$x_r = 1 - \frac{3.7(0.5-1)}{-1.47813-3.7} = 0.64273$$

$$f(0.5)f(0.64273) = -1.47813(0.91879) = -1.35808$$

Therefore, the bracket is $x_l = 0.5$ and $x_u = 0.64273$.

Second iteration:

iteration	x_l	x_u	$f(x_l)$	$f(x_u)$	x_r	$f(x_r)$	$f(x_l)\times f(x_r)$	ε_a
1	0.5	1.00000						
2								
3								
4								

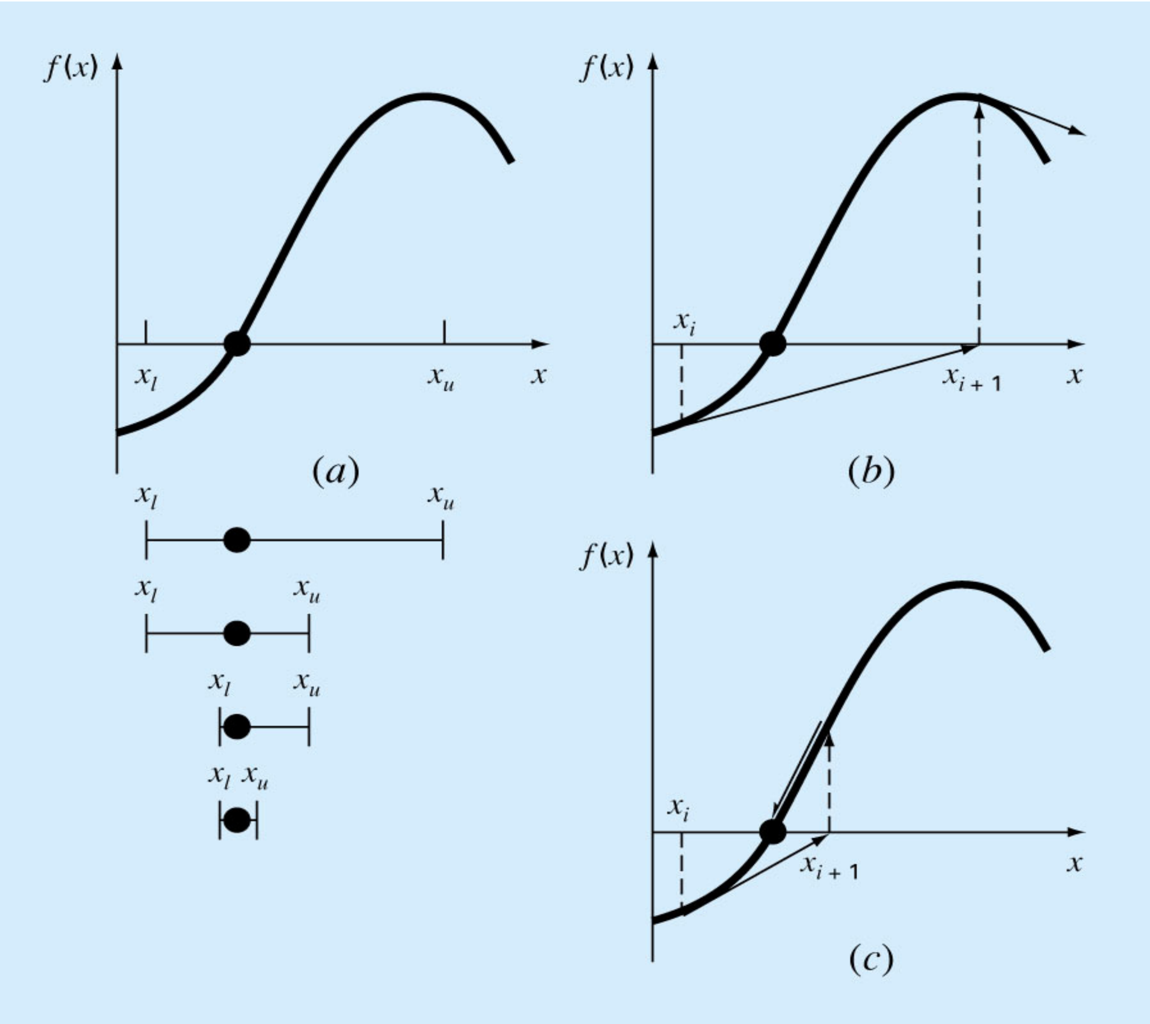
Handout 8

Chapter 6

- 6.1 Simple Fixed-Point Iteration
- 6.2 Newton-Raphson
- 6.3 Secant Methods

6. 0 Open Methods

Figure 6.1: Difference between (a) bracketing and (b),(c) open methods for root location. (b) the method diverge, (c) the method converge, depending on the initial guess.



6.1 Single Fixed-Point Iteration

Rewrite the equation f(x)=0 in the form x=g(x)

Examples: $x^2 - 2x+3=0$ **can be written as** $x = \frac{x^2+3}{2}$

sin x = 0 can be written as x=x+sin x

The advantage is that you get a new value of x and use it to refine the old value.

i.e. $x_{i+1} = g(x_i)$

Approximate relative percent error $\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\,\%$

Example: Find the root of $f(x)=e^{-x} - x$

Rewrite Equation as $x_{i+1} = e^{-x_i}$

True answer is 0.56714329. Results are tabulated below:

i	x_i	e^{-x_i}	ε_a(%)	ε_t(%)
0	0	1	-	100
1	1.0	0.367879	100	76..3
2	0.367879	0.692201	171.8	35.1
3	0.692201	0.500473	46.9	22.1
4	0.500473	0.606244	38.3	11.8
5	0.606244	0.545396	17.4	6.89
6	0.545396	0.579612	11.2	3.83
7	0.579612	0.560115	5.90	2.20
8	0.560115	0.560115	3.48	1.24
9	0.571143	0.564879	1.93	0.705
10	0.564879	0.5684289	1.11	0.399

Linear Convergence – True % relative error decreases by a factor of 0.5 to 0.6

Does the method always converge?

see fig 6.2 and 6.3

6.2 Newton-Raphson method

- Most widely used method.

Based on Taylor series expansion:

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + f''(x_i)\frac{\Delta x^2}{2!} + O\Delta x^3$$

The root is the value of x_{i+1} when $f(x_{i+1}) = 0$

Rearranging,

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

•A convenient method for functions whose derivatives can be evaluated analytically. It may not be convenient for functions whose derivatives cannot be evaluated analytically.

See example 6.3

Figure 6.5

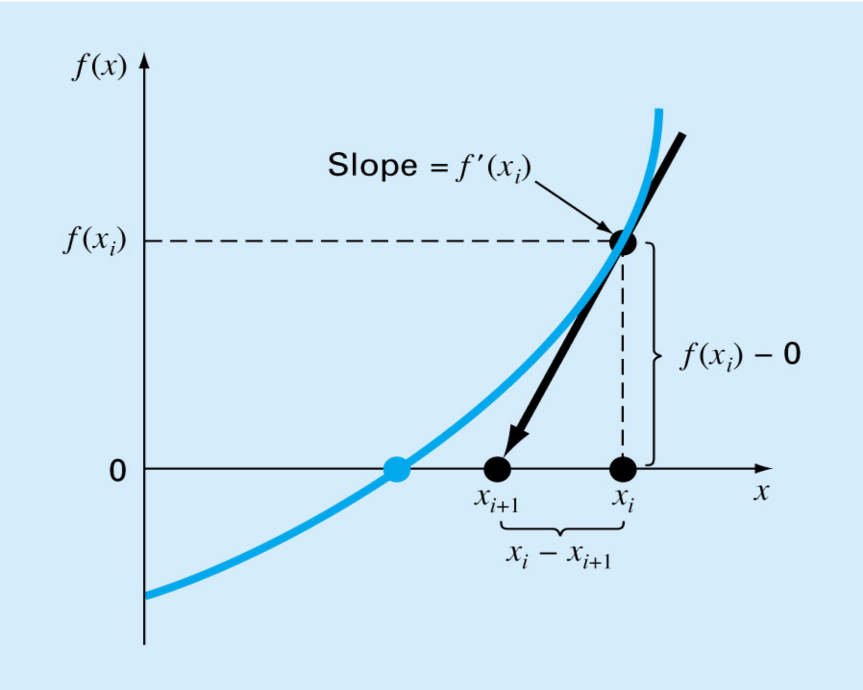
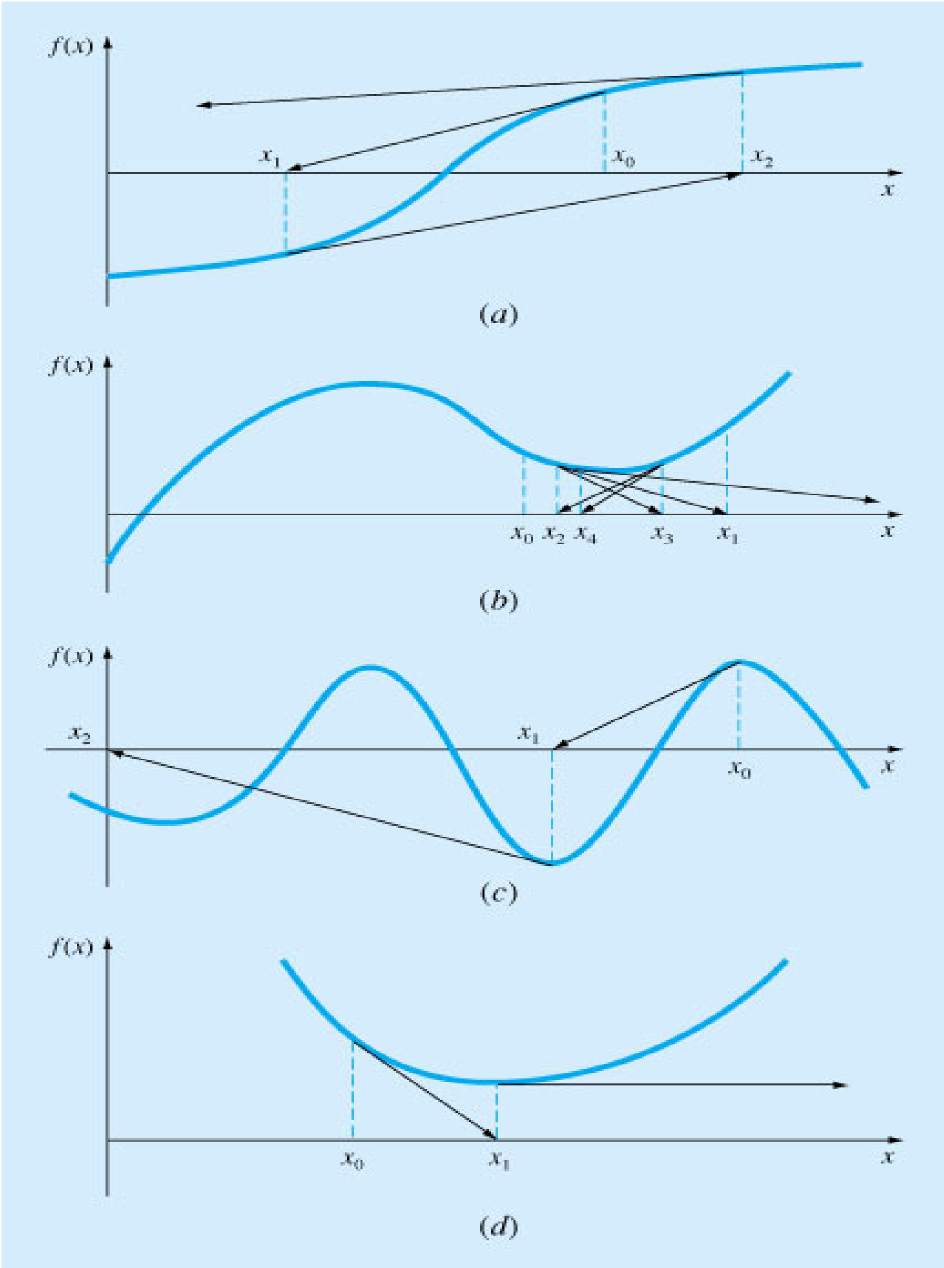


Figure 6.6 Poor convergence of Newton-Raphson



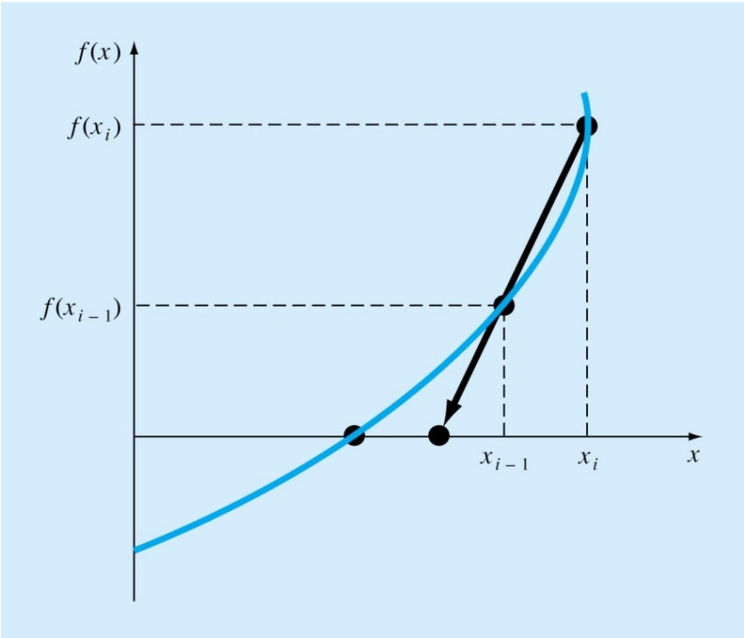
6.3 Secant Method

•A slight variation of Newton's method for functions whose derivatives are difficult to evaluate. For these cases the derivative can be approximated by a backward finite divided difference.

$$f'(x_i) \cong \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \qquad i = 1, 2, 3, \ldots$$

Figure 6.7 : The secant method is similar to Newton-Raphson technique – extrapolate tangent of the function (figure 6.5) , however here a 'difference' is used rather than 'derivative'.

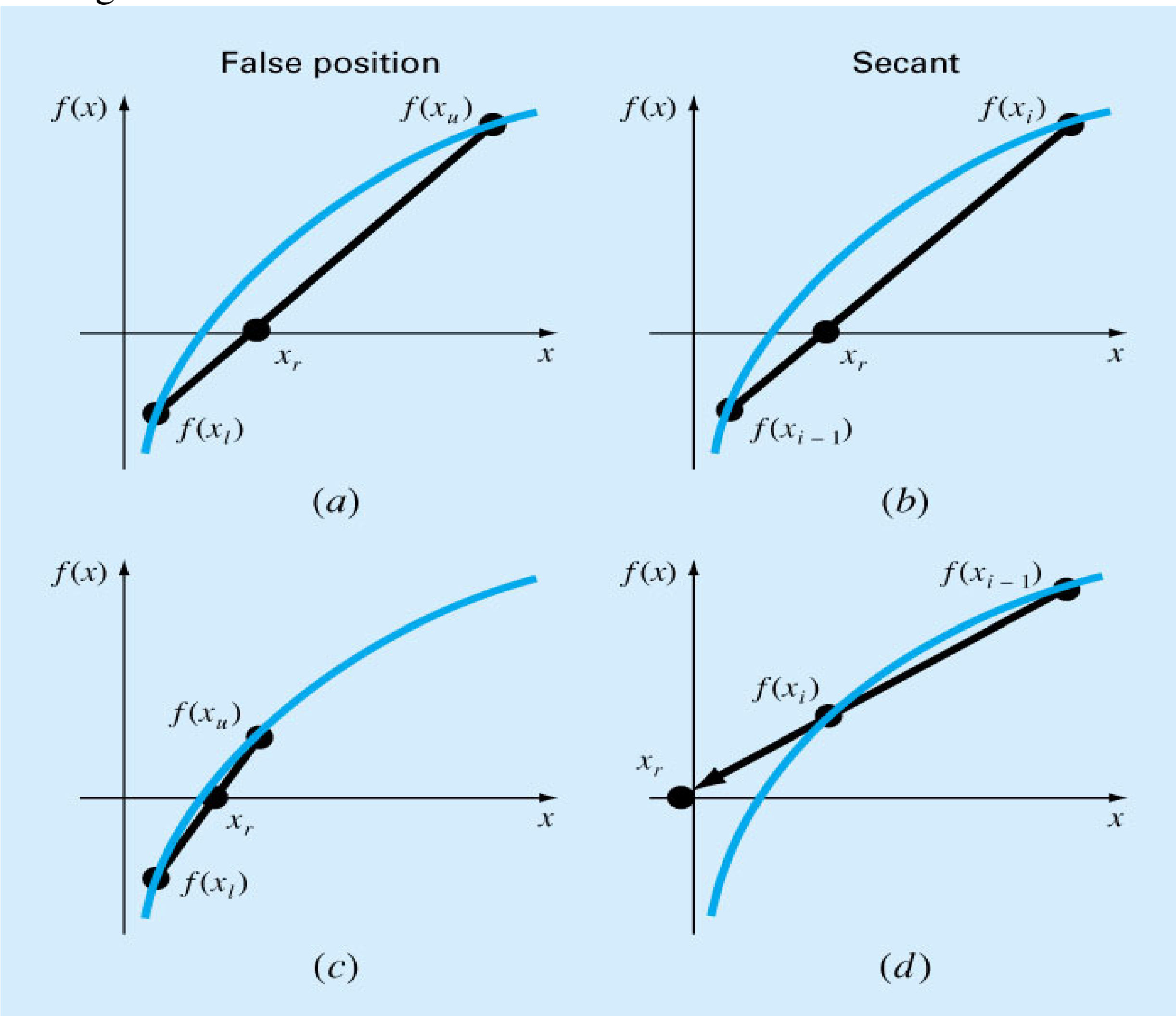


•Requires two initial estimates of x , e.g, x_0, x_1. However, because f(x) is not required to change signs between estimates, it is not classified as a "bracketing" method.

•The secant method has the same properties as Newton's method. Convergence is not guaranteed for all x_o, f(x).

Example 6.6

figure 6.8



Example Problem 6.2

$$f(x) = 2x^3 - 11.7x^2 + 17.7x - 5$$

(b) fixed point iteration - $x_{i+1} = g(x)$

$$x = \frac{5 - 2x^3 + 11.7x^2}{17.7}$$

i	x_i	(%)ε_a
0	3	-
1	3.1808	5.68
2	3.334	4.595
3	3.4425	3.152

(c) Newton-Raphson $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

i	x_i	$f(x_i)$	$f'(x_i)$	ε_a
0	3	-3.2	1.5	-
1	5.1333	48.0882	55.6854	41.5580%
2	4.26975	12.96	27.18	20.14
3	3.7929			12.57

(d) secant $x_{i+1} = x_i - f(x_i)\dfrac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$ $i = 1, 2, 3, \ldots$

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	3		4		-
1	4				20.25
2	3.3265				4.44
3	3.4813				2.93

Handout 9

Chapter 7 : Roots of Polynomial

- 7.4 Muller's Method

7.0 Roots of Polynomials

- ***The roots of polynomials such as***

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Follow these rules:
1.For an *n*th order equation, there are n real or complex roots.
2.If *n* is odd, there is at least one real root.
3.If complex root exist in conjugate pairs (that is, $\lambda+\mu i$ and $\lambda-\mu i$), where *i=sqrt(-1)*.

Conventional Methods

- ***The efficacy of bracketing and open methods depends on whether the problem being solved involves complex roots. If only real roots exist, these methods could be used. However,***
  - **Finding good initial guesses complicates both the open and bracketing methods, also the open methods could be susceptible to divergence.**
- ***Special methods have been developed to find the real and complex roots of polynomials – Müller and Bairstow methods.***

7.4 Müller's method

•Müller's method obtains a root estimate by projecting a parabola to the x axis through three function values.

•The method consists of deriving the coefficients of parabola that goes through the three points.

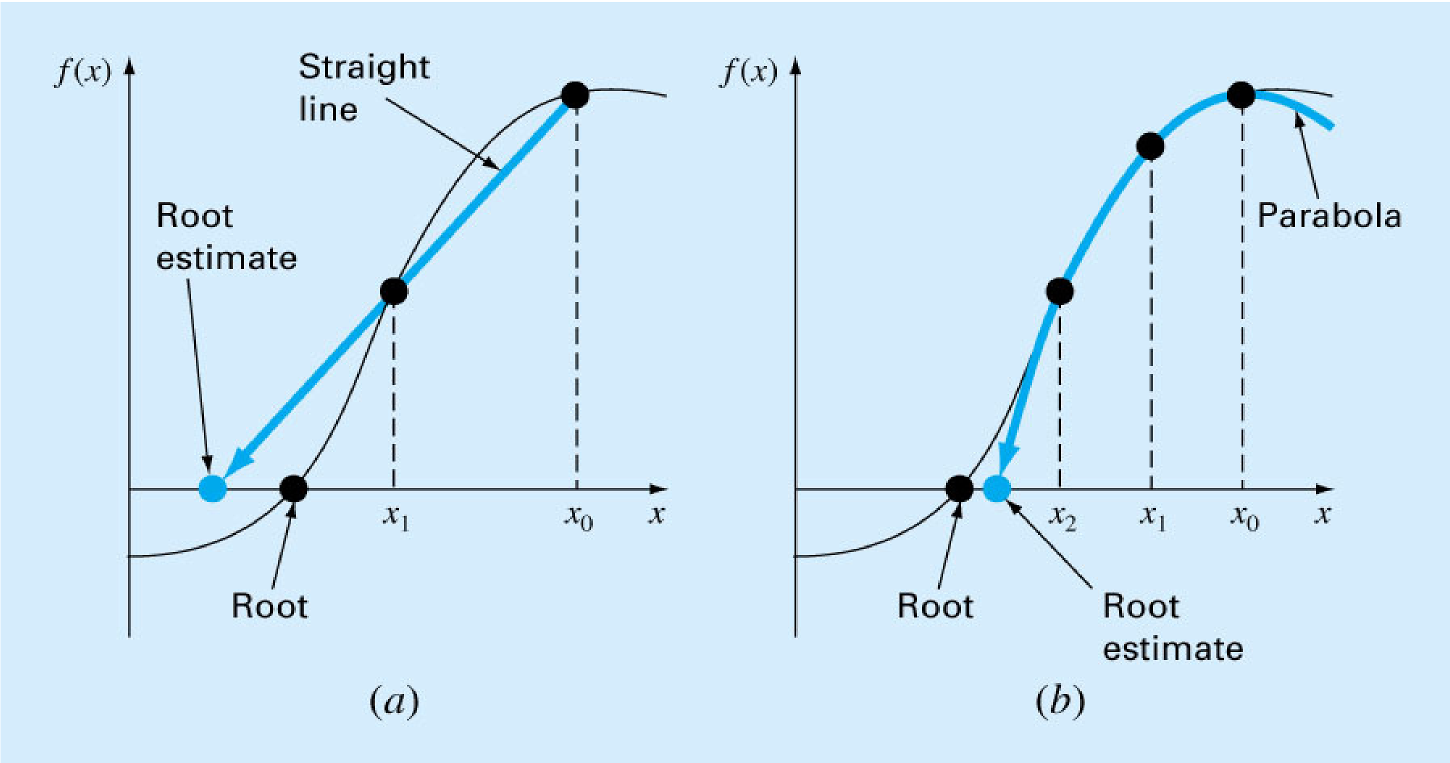


Figure 7.3: comparison between (a) secant method (b) Muller's method

1. Write the equation in a convenient form:

$$f_2(x) = a(x - x_2)^2 + b(x - x_2) + c$$

2.The parabola should intersect the three points *[xo, f(xo)], [x1, f(x1)], [x2, f(x2)]*. The coefficients of the polynomial can be estimated by substituting three points to give

$$f(x_o) = a(x_o - x_2)^2 + b(x_o - x_2) + c$$
$$f(x_1) = a(x_1 - x_2)^2 + b(x_1 - x_2) + c$$
$$f(x_2) = a(x_2 - x_2)^2 + b(x_2 - x_2) + c$$

3.Three equations can be solved for three unknowns, *a, b, c*. Since two of the terms in the 3rd equation are zero, it can be

$$f(x_o) - f(x_2) = a(x_o - x_2)^2 + b(x_o - x_2)$$
$$f(x_1) - f(x_2) = a(x_1 - x_2)^2 + b(x_1 - x_2)$$

If

$$h_o = x_1 - x_o \qquad h_1 = x_2 - x_1$$
$$\delta_o = \frac{f(x_1) - f(x_o)}{x_1 - x_o} \qquad \delta_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$(h_o + h_1)b - (h_o + h_1)^2 a = h_o\delta_o + h_1\delta_1$$
$$h_1 b - h_1^2 a = h_1\delta_1$$

$$a = \frac{\delta_1 - \delta_o}{h_1 + h_o} \qquad b = ah_1 + \delta_1 \qquad c = f(x_2)$$

immediately solved for $c=f(x2)$.

•Roots can be found by applying an alternative form of quadratic formula:

$$x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

•The error can be calculated as

$$\varepsilon_a = \left|\frac{x_3 - x_2}{x_3}\right| 100\%$$

•• ±term yields two roots, the sign is chosen to agree with *b*. This will result in a largest denominator, and will give root estimate that is closest to x2.

•Once *x3* is determined, the process is repeated using the following guidelines:

1.If only real roots are being located, choose the two original points that are nearest the new root estimate, *x3*.

2.If both real and complex roots are estimated, employ a sequential approach just like in secant method, x_1, x_2, and x_3 to replace x_o, x_1, and x_2.

See example 7.2

Using MATLAB to determine all roots:

If $$f(x) = x^3 - x^2 + 3x - 2$$

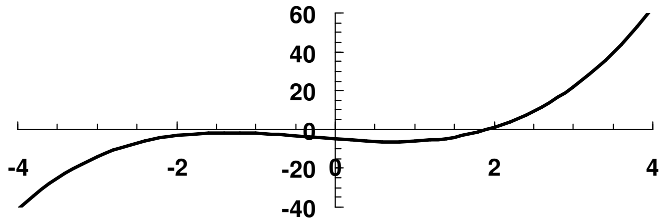
```
>> a=[1 -1 3 -2];
>> roots(a)

ans =
   0.1424 + 1.6661i
   0.1424 - 1.6661i
    0.7152
```

see example 7.6 and 7.7

Example : Problem 7.3 (a) Use Muller method to determine the positive real root:

$$f(x) = x^3 + x^2 - 3x - 5$$

A plot indicates a root at about x = 2.



Try initial guesses of $x_0 = 1$, $x_1 = 1.5$, and $x_2 = 2.5$. Using the same approach as in Example 7.2,

<u>First iteration:</u>

$f(1) = -6$ $f(1.5) = -3.875$ $f(2.5) = 9.375$

$h_0 = 0.5$ $h_1 = 1$

$\delta_0 = 4.25$ $\delta_1 = 13.25$

$$a = \frac{13.25 - 4.25}{1 + 0.5} = 6$$

$$b = 6(1) + 13.25 = 19.25$$

$$c = 9.375$$

$$x_3 = 2.5 + \frac{-2(9.375)}{19.25 + \sqrt{19.25^2 - 4(6)(9.375)}} = 1.901244$$

$$\varepsilon_a = \left| \frac{1.901244 - 2.5}{1.901244} \right| \times 100\% = 31.49\%$$

The iterations can be continued as tabulated below:

i	x_3	ε_a
0	1.901244	31.4929%
1	1.919270	0.9392%
2	1.919639	0.0192%
3	1.919640	0.0000%

7.3 b) for $f(x) = x^3 - 0.5x^2 + 4x - 3$

Try initial guesses of $x_0 = 0.5$, $x_1 = 1$, and $x_2 = 1.5$

?

Handout 10

Chapter 7 : Roots of Polynomial

- 7.5 Bairstow's Method

7.5 Bairstow's Method

- ***Bairstow's method is an iterative approach loosely related to both Müller and Newton Raphson methods.***
- ***It is based on dividing a polynomial by a factor* x-t*:***

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$f_{n-1}(x) = b_1 + b_2 x + b_3 x^2 + \ldots + b_n x^n$$

with a reminder $R = b_o$, the coefficients are calculated by recurrence relationship

$$b_n = a_n$$

$$b_i = a_i + b_{i+1} t \quad i = n-1 \text{ to } 2$$

- ***To permit the evaluation of complex roots, Bairstow's method divides the polynomial by a quadratic factor x^2-rx-s:***

$$f_{n-2}(x) = b_2 + b_3 x + \ldots + b_{n-1} x^{n-3} + b_n x^{n-2}$$

$$R = b_1(x - r) + b_o$$

Using a simple recurrence relationship

$$b_n = a_n$$

$$b_{n\text{-}1} = a_{n\text{-}1} + r b_n$$

$$b_i = a_i + r b_{i+1} + s b_{i+2} \quad i = n\text{-}2 \text{ to } 0$$

- ***For the remainder to be zero,* b_o *and* b_1 *must be zero. However, it is unlikely that our initial guesses at the values of* r *and* s *will lead to this result, a systematic approach can be used to modify our guesses so that* b_o *and* b_1 *approach to zero.***

- *Using a similar approach to Newton Raphson method, both* b_o *and* b_1 *can be expanded as function of both* r *and* s *in Taylor series.*

$$b_1(r+\Delta r, s+\Delta s) = b_1 + \frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s$$

$$b_o(r+\Delta r, s+\Delta s) = b_o + \frac{\partial b_o}{\partial r}\Delta r + \frac{\partial b_o}{\partial s}\Delta s$$

assuming that the initial guesses are adequately close to the values of r and s at roots. The changes in Δs and Δr needed to improve our guesses will be estimated as

$$\frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s = -b_1$$

$$\frac{\partial b_o}{\partial r}\Delta r + \frac{\partial b_o}{\partial s}\Delta s = -b_o$$

- *If partial derivatives of the b's can be determined, then the two equations can be solved simultaneously for the two unknowns* Δr *and* Δs.
- *Partial derivatives can be obtained by a synthetic division of the b's in a similar fashion the b's themselves are derived:*

$$c_n = b_n$$

$$c_{n-1} = b_{n-1} + rc_n$$

$$c_i = b_i + rc_{i+1} + sc_{i+2} \qquad i = n-2 \text{ to } 1$$

where

$$\frac{\partial b_o}{\partial r} = c_1 \qquad \frac{\partial b_o}{\partial s} = \frac{\partial b_1}{\partial r} = c_2 \qquad \frac{\partial b_1}{\partial s} = c_3$$

- *Then*

$$c_2\Delta r + c_3\Delta s = -b_1$$

$$c_1\Delta r + c_2\Delta s = -b_o$$

Solved for Δr and Δs, in turn are employed to improve the initial guesses.

- ***At each step the error can be estimated as***

$$\left|\varepsilon_{a,r}\right| = \left|\frac{\Delta r}{r}\right| 100\%$$

$$\left|\varepsilon_{a,s}\right| = \left|\frac{\Delta s}{s}\right| 100\%$$

When both of these error estimates fall below a prespesified stopping criteria, the roots can be determined

$$x = \frac{r \pm \sqrt{r^2 + 4s}}{2}$$

if the quotient is a first order, since, $f_{n-2}(x) = b_2 + b_3 x = 0$

$$x = \frac{-b_2}{b_3}$$

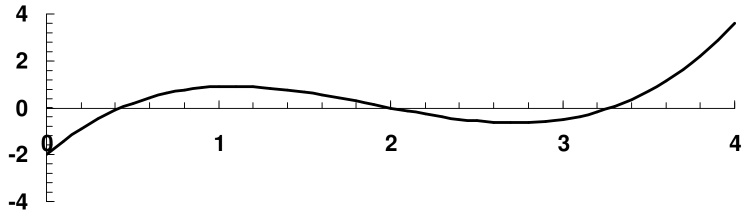
See example 7.3

Refer to Tables pt2.3 and pt2.4

7.5 (a) Use Bairstow's method to determine the roots:

$$f(x) = -2 + 6.2x - 4x^2 + 0.7x^3$$

A plot suggests 3 real roots: 0.44, 2 and 3.3.



$$f(x) = -2 + 6.2x - 4x^2 + 0.7x^3$$

$$a_o = -2 \qquad a_1 = 6.2 \qquad a_2 = -4 \qquad a_3 = 0.7$$

<u>1st iteration:</u>

Try $r = 1$ and $s = -1$

$$b_3 = a_3 =$$

$$b_2 = a_2 + rb_3 =$$

n=3, $b_1 = a_1 + rb_2 + sb_3 =$

$$b_o =$$

solve simultaneous eqn using calculator casio fx-570s:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

MODE MODE MODE
Choose 1 EQN
Unknowns 2
a1? -2.6=
b1?

And so on.. will give x and y which are the Δr *and* Δs

$\Delta r = 1.085$ $\Delta s = 0.887$
$r = 2.085$ $s = -0.1129$

2nd iteration:
$\Delta r = 0.4019$ $\Delta s = -0.5565$
$r = 2.487$ $s = -0.6694$

3rd iteration:
$\Delta r = -0.0605$ $\Delta s = -0.2064$
$r = 2.426$ $s = -0.8758$

4th iteration:
$\Delta r = 0.00927$ $\Delta s = 0.00432$
$r = 2.436$ $s = -0.8714$

$$\text{root}_1 = \frac{2.436 + \sqrt{2.436^2 + 4(-0.8714)}}{2} = 2$$

$$\text{root}_2 = \frac{2.436 - \sqrt{2.436^2 + 4(-0.8714)}}{2} = 0.4357$$

The remaining $\text{root}_3 = 3.279$.