a b
c d

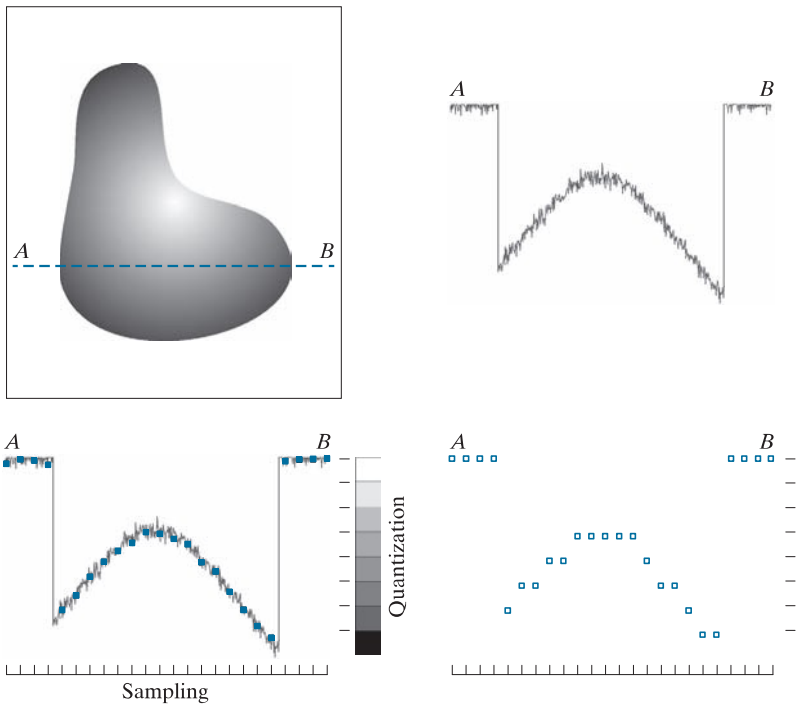
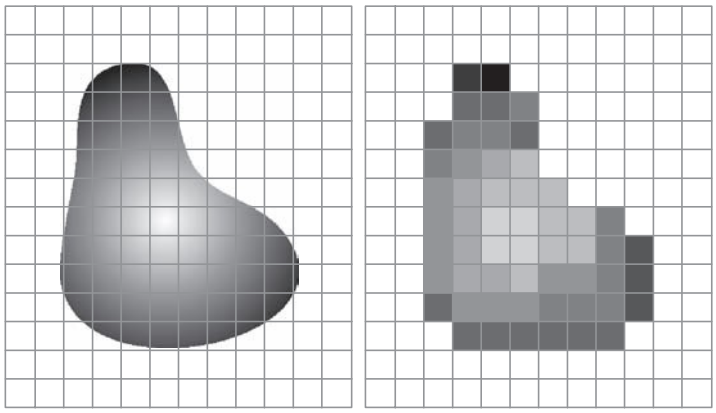


FIGURE 2.16
(a) Continuous image. (b) A scan line showing intensity variations along line AB in the continuous image. (c) Sampling and quantization. (d) Digital scan line. (The black border in (a) is included for clarity. It is not part of the image).

amplitude. To digitize it, we have to sample the function in both coordinates and also in amplitude. Digitizing the coordinate values is called *sampling*. Digitizing the amplitude values is called *quantization*.

The one-dimensional function in Fig. 2.16(b) is a plot of amplitude (intensity level) values of the continuous image along the line segment AB in Fig. 2.16(a). The random variations are due to image noise. To sample this function, we take equally spaced samples along line AB, as shown in Fig. 2.16(c). The samples are shown as small dark squares superimposed on the function, and their (discrete) spatial locations are indicated by corresponding tick marks in the bottom of the figure. The set of dark squares constitute the *sampled* function. However, the *values* of the samples still span (vertically) a continuous range of intensity values. In order to form a digital function, the intensity values also must be converted (*quantized*) into *discrete* quantities. The vertical gray bar in Fig. 2.16(c) depicts the intensity scale divided into eight discrete intervals, ranging from black to white. The vertical tick marks indicate the specific value assigned to each of the eight intensity intervals. The continuous intensity levels are quantized by assigning one of the eight values to each sample, depending on the vertical proximity of a sample to a vertical tick mark. The digital samples resulting from both sampling and quantization are shown as white squares in Fig. 2.16(d). Starting at the top of the continuous image and carrying out this procedure downward, line by line, produces a two-dimensional digital image. It is implied in Fig. 2.16 that, in addition to the number of discrete levels used, the accuracy achieved in quantization is highly dependent on the noise content of the sampled signal.



a b

FIGURE 2.17
(a) Continuous image projected onto a sensor array. (b) Result of image sampling and quantization.

In practice, the method of sampling is determined by the sensor arrangement used to generate the image. When an image is generated by a single sensing element combined with mechanical motion, as in Fig. 2.13, the output of the sensor is quantized in the manner described above. However, spatial sampling is accomplished by selecting the number of individual mechanical increments at which we activate the sensor to collect data. Mechanical motion can be very exact so, in principle, there is almost no limit on how fine we can sample an image using this approach. In practice, limits on sampling accuracy are determined by other factors, such as the quality of the optical components used in the system.

When a sensing strip is used for image acquisition, the number of sensors in the strip establishes the samples in the resulting image in one direction, and mechanical motion establishes the number of samples in the other. Quantization of the sensor outputs completes the process of generating a digital image.

When a sensing array is used for image acquisition, no motion is required. The number of sensors in the array establishes the limits of sampling in both directions. Quantization of the sensor outputs is as explained above. Figure 2.17 illustrates this concept. Figure 2.17(a) shows a continuous image projected onto the plane of a 2-D sensor. Figure 2.17(b) shows the image after sampling and quantization. The quality of a digital image is determined to a large degree by the number of samples and discrete intensity levels used in sampling and quantization. However, as we will show later in this section, image content also plays a role in the choice of these parameters.

REPRESENTING DIGITAL IMAGES

Let $f(s,t)$ represent a *continuous* image function of two continuous variables, s and t. We convert this function into a *digital image* by sampling and quantization, as explained in the previous section. Suppose that we sample the continuous image into a digital image, $f(x,y)$, containing M rows and N columns, where (x,y) are discrete coordinates. For notational clarity and convenience, we use integer values for these discrete coordinates: $x = 0, 1, 2, \ldots, M-1$ and $y = 0, 1, 2, \ldots, N-1$. Thus, for example, the value of the digital image at the origin is $f(0,0)$, and its value at the next coordinates along the first row is $f(0,1)$. Here, the notation $(0, 1)$ is used

to denote the second sample along the first row. It *does not* mean that these are the values of the physical coordinates when the image was sampled. In general, the value of a digital image at any coordinates (x, y) is denoted $f(x, y)$, where x and y are integers. When we need to refer to specific coordinates (i, j), we use the notation $f(i, j)$, where the arguments are integers. The section of the real plane spanned by the coordinates of an image is called the *spatial domain*, with x and y being referred to as *spatial variables* or *spatial coordinates*.

Figure 2.18 shows three ways of representing $f(x, y)$. Figure 2.18(a) is a plot of the function, with two axes determining spatial location and the third axis being the values of f as a function of x and y. This representation is useful when working with grayscale sets whose elements are expressed as triplets of the form (x, y, z), where x and y are spatial coordinates and z is the value of f at coordinates (x, y). We will work with this representation briefly in Section 2.6.

The representation in Fig. 2.18(b) is more common, and it shows $f(x, y)$ as it would appear on a computer display or photograph. Here, the intensity of each point in the display is proportional to the value of f at that point. In this figure, there are only three equally spaced intensity values. If the intensity is normalized to the interval $[0,1]$, then each point in the image has the value $0, 0.5$, or 1. A monitor or printer converts these three values to black, gray, or white, respectively, as in Fig. 2.18(b). This type of representation includes color images, and allows us to view results at a glance.

As Fig. 2.18(c) shows, the third representation is an array (matrix) composed of the numerical values of $f(x, y)$. This is the representation used for computer processing. In equation form, we write the representation of an $M \times N$ numerical array as

$$f(x,y) = \begin{bmatrix} f(0,0) & f(0,1) & \cdots & f(0,N-1) \\ f(1,0) & f(1,1) & \cdots & f(1,N-1) \\ \vdots & \vdots & & \vdots \\ f(M-1,0) & f(M-1,1) & \cdots & f(M-1,N-1) \end{bmatrix} \tag{2-9}$$

The right side of this equation is a digital image represented as an array of real numbers. Each element of this array is called an *image element*, *picture element*, *pixel*, or *pel*. We use the terms *image* and *pixel* throughout the book to denote a digital image and its elements. Figure 2.19 shows a graphical representation of an image array, where the x- and y-axis are used to denote the rows and columns of the array. Specific pixels are values of the array at a fixed pair of coordinates. As mentioned earlier, we generally use $f(i, j)$ when referring to a pixel with coordinates (i, j).

We can also represent a digital image in a traditional matrix form:

$$\mathbf{A} = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & \vdots & & \vdots \\ a_{M-1,0} & a_{M-1,1} & \cdots & a_{M-1,N-1} \end{bmatrix} \tag{2-10}$$

Clearly, $a_{ij} = f(i, j)$, so Eqs. (2-9) and (2-10) denote identical arrays.

Observe that isopreference curves tend to become more vertical as the detail in the image increases. This result suggests that for images with a large amount of detail only a few intensity levels may be needed. For example, the isopreference curve in Fig. 2.26 corresponding to the crowd is nearly vertical. This indicates that, for a fixed value of N, the perceived quality for this type of image is nearly independent of the number of intensity levels used (for the range of intensity levels shown in Fig. 2.26). The perceived quality in the other two image categories remained the same in some intervals in which the number of samples was increased, but the number of intensity levels actually decreased. The most likely reason for this result is that a decrease in k tends to increase the apparent contrast, a visual effect often perceived as improved image quality.

IMAGE INTERPOLATION

Interpolation is used in tasks such as zooming, shrinking, rotating, and geometrically correcting digital images. Our principal objective in this section is to introduce interpolation and apply it to image resizing (shrinking and zooming), which are basically image resampling methods. Uses of interpolation in applications such as rotation and geometric corrections will be discussed in Section 2.6.

Interpolation is the process of using known data to estimate values at unknown locations. We begin the discussion of this topic with a short example. Suppose that an image of size 500×500 pixels has to be enlarged 1.5 times to 750×750 pixels. A simple way to visualize zooming is to create an imaginary 750×750 grid with the same pixel spacing as the original image, then shrink it so that it exactly overlays the original image. Obviously, the pixel spacing in the shrunken 750×750 grid will be less than the pixel spacing in the original image. To assign an intensity value to any point in the overlay, we look for its closest pixel in the underlying original image and assign the intensity of that pixel to the new pixel in the 750×750 grid. When intensities have been assigned to all the points in the overlay grid, we expand it back to the specified size to obtain the resized image.

The method just discussed is called *nearest neighbor interpolation* because it assigns to each new location the intensity of its nearest neighbor in the original image (see Section 2.5 regarding neighborhoods). This approach is simple but, it has the tendency to produce undesirable artifacts, such as severe distortion of straight edges. A more suitable approach is *bilinear interpolation*, in which we use the four nearest neighbors to estimate the intensity at a given location. Let (x, y) denote the coordinates of the location to which we want to assign an intensity value (think of it as a point of the grid described previously), and let $v(x, y)$ denote that intensity value. For bilinear interpolation, the assigned value is obtained using the equation

Contrary to what the name suggests, bilinear interpolation is *not* a linear operation because it involves multiplication of coordinates (which is not a linear operation). See Eq. (2-17).

$$v(x, y) = ax + by + cxy + d \tag{2-17}$$

where the four coefficients are determined from the four equations in four unknowns that can be written using the *four* nearest neighbors of point (x, y). Bilinear interpolation gives much better results than nearest neighbor interpolation, with a modest increase in computational burden.

The next level of complexity is *bicubic interpolation*, which involves the sixteen nearest neighbors of a point. The intensity value assigned to point (x, y) is obtained using the equation

$$v(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j \tag{2-18}$$

The sixteen coefficients are determined from the sixteen equations with sixteen unknowns that can be written using the sixteen nearest neighbors of point (x, y). Observe that Eq. (2-18) reduces in form to Eq. (2-17) if the limits of both summations in the former equation are 0 to 1. Generally, bicubic interpolation does a better job of preserving fine detail than its bilinear counterpart. Bicubic interpolation is the standard used in commercial image editing applications, such as Adobe Photoshop and Corel Photopaint.

Although images are displayed with integer coordinates, it is possible during processing to work with *subpixel accuracy* by increasing the size of the image using interpolation to "fill the gaps" between pixels in the original image.

EXAMPLE 2.4: Comparison of interpolation approaches for image shrinking and zooming.

Figure 2.27(a) is the same as Fig. 2.23(d), which was obtained by reducing the resolution of the 930 dpi image in Fig. 2.23(a) to 72 dpi (the size shrank from 2136×2140 to 165×166 pixels) and then zooming the reduced image back to its original size. To generate Fig. 2.23(d) we used nearest neighbor interpolation both to shrink and zoom the image. As noted earlier, the result in Fig. 2.27(a) is rather poor. Figures 2.27(b) and (c) are the results of repeating the same procedure but using, respectively, bilinear and bicubic interpolation for both shrinking and zooming. The result obtained by using bilinear interpolation is a significant improvement over nearest neighbor interpolation, but the resulting image is blurred slightly. Much sharper results can be obtained using bicubic interpolation, as Fig. 2.27(c) shows.

a b c

FIGURE 2.27 (a) Image reduced to 72 dpi and zoomed back to its original 930 dpi using nearest neighbor interpolation. This figure is the same as Fig. 2.23(d). (b) Image reduced to 72 dpi and zoomed using bilinear interpolation. (c) Same as (b) but using bicubic interpolation.

It is possible to use more neighbors in interpolation, and there are more complex techniques, such as using *splines* or *wavelets*, that in some instances can yield better results than the methods just discussed. While preserving fine detail is an exceptionally important consideration in image generation for 3-D graphics (for example, see Hughes and Andries [2013]), the extra computational burden seldom is justifiable for general-purpose digital image processing, where bilinear or bicubic interpolation typically are the methods of choice.

2.5 SOME BASIC RELATIONSHIPS BETWEEN PIXELS

In this section, we discuss several important relationships between pixels in a digital image. When referring in the following discussion to particular pixels, we use lowercase letters, such as p and q.

NEIGHBORS OF A PIXEL

A pixel p at coordinates (x, y) has two horizontal and two vertical neighbors with coordinates

$$(x+1, y), (x-1, y), (x, y+1), (x, y-1)$$

This set of pixels, called the 4-*neighbors* of p, is denoted $N_4(p)$.

The four *diagonal* neighbors of p have coordinates

$$(x+1, y+1), (x+1, y-1), (x-1, y+1), (x-1, y-1)$$

and are denoted $N_D(p)$. These neighbors, together with the 4-neighbors, are called the 8-*neighbors* of p, denoted by $N_8(p)$. The set of image locations of the neighbors of a point p is called the *neighborhood* of p. The neighborhood is said to be *closed* if it contains p. Otherwise, the neighborhood is said to be *open*.

ADJACENCY, CONNECTIVITY, REGIONS, AND BOUNDARIES

Let V be the set of intensity values used to define adjacency. In a binary image, $V = \{1\}$ if we are referring to adjacency of pixels with value 1. In a grayscale image, the idea is the same, but set V typically contains more elements. For example, if we are dealing with the adjacency of pixels whose values are in the range 0 to 255, set V could be any subset of these 256 values. We consider three types of adjacency:

1. 4-*adjacency*. Two pixels p and q with values from V are 4-adjacent if q is in the set $N_4(p)$.
2. 8-*adjacency*. Two pixels p and q with values from V are 8-adjacent if q is in the set $N_8(p)$.
3. *m-adjacency* (also called *mixed adjacency*). Two pixels p and q with values from V are m-adjacent if

(a) q is in $N_4(p)$, *or*

(b) q is in $N_D(p)$ *and* the set $N_4(p) \cap N_4(q)$ has no pixels whose values are from V.

We use the symbols ∩ and ∪ to denote set intersection and union, respectively. Given sets A and B, recall that their intersection is the set of elements that are members of both A and B. The union of these two sets is the set of elements that are members of A, of B, or of both. We will discuss sets in more detail in Section 2.6.

Mixed adjacency is a modification of 8-adjacency, and is introduced to eliminate the ambiguities that may result from using 8-adjacency. For example, consider the pixel arrangement in Fig. 2.28(a) and let $V = \{1\}$. The three pixels at the top of Fig. 2.28(b) show multiple (ambiguous) 8-adjacency, as indicated by the dashed lines. This ambiguity is removed by using m-adjacency, as in Fig. 2.28(c). In other words, the center and upper-right diagonal pixels are not m-adjacent because they do not satisfy condition (b).

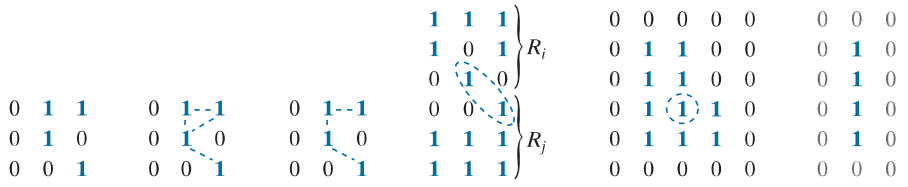
A *digital path* (or *curve*) from pixel p with coordinates (x_0, y_0) to pixel q with coordinates (x_n, y_n) is a sequence of distinct pixels with coordinates

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

where points (x_i, y_i) and (x_{i-1}, y_{i-1}) are adjacent for $1 \le i \le n$. In this case, n is the *length* of the path. If $(x_0, y_0) = (x_n, y_n)$ the path is a *closed* path. We can define 4-, 8-, or m-paths, depending on the type of adjacency specified. For example, the paths in Fig. 2.28(b) between the top right and bottom right points are 8-paths, and the path in Fig. 2.28(c) is an m-path.

Let S represent a subset of pixels in an image. Two pixels p and q are said to be *connected in S* if there exists a path between them consisting entirely of pixels in S. For any pixel p in S, the set of pixels that are connected to it in S is called a *connected component* of S. If it only has one component, and that component is connected, then S is called a *connected set*.

Let R represent a subset of pixels in an image. We call R a *region* of the image if R is a connected set. Two regions, R_i and R_j are said to be *adjacent* if their union forms a connected set. Regions that are not adjacent are said to be *disjoint*. We consider 4- and 8-adjacency when referring to regions. For our definition to make sense, the type of adjacency used must be specified. For example, the two regions of 1's in Fig. 2.28(d) are adjacent only if 8-adjacency is used (according to the definition in the previous



a b c d e f

FIGURE 2.28 (a) An arrangement of pixels. (b) Pixels that are 8-adjacent (adjacency is shown by dashed lines). (c) m-adjacency. (d) Two regions (of 1's) that are 8-adjacent. (e) The circled point is on the boundary of the 1-valued pixels only if 8-adjacency between the region and background is used. (f) The inner boundary of the 1-valued region does not form a closed path, but its outer boundary does.

paragraph, a 4-path between the two regions does not exist, so their union is not a connected set).

Suppose an image contains K disjoint regions, $R_k, k = 1, 2, \ldots, K$, none of which touches the image border.[†] Let R_u denote the union of all the K regions, and let $(R_u)^c$ denote its complement (recall that the *complement* of a set A is the set of points that are not in A). We call all the points in R_u the *foreground*, and all the points in $(R_u)^c$ the *background* of the image.

The *boundary* (also called the *border* or *contour*) of a region R is the set of pixels in R that are adjacent to pixels in the complement of R. Stated another way, the border of a region is the set of pixels in the region that have at least one background neighbor. Here again, we must specify the connectivity being used to define adjacency. For example, the point circled in Fig. 2.28(e) is not a member of the border of the 1-valued region if 4-connectivity is used between the region and its background, because the only possible connection between that point and the background is diagonal. As a rule, adjacency between points in a region and its background is defined using 8-connectivity to handle situations such as this.

The preceding definition sometimes is referred to as the *inner border* of the region to distinguish it from its *outer border*, which is the corresponding border in the background. This distinction is important in the development of border-following algorithms. Such algorithms usually are formulated to follow the outer boundary in order to guarantee that the result will form a closed path. For instance, the inner border of the 1-valued region in Fig. 2.28(f) is the region itself. This border does not satisfy the definition of a closed path. On the other hand, the outer border of the region does form a closed path around the region.

If R happens to be an entire image, then its *boundary* (or *border*) is defined as the set of pixels in the first and last rows and columns of the image. This extra definition is required because an image has no neighbors beyond its border. Normally, when we refer to a region, we are referring to a subset of an image, and any pixels in the boundary of the region that happen to coincide with the border of the image are included implicitly as part of the region boundary.

The concept of an *edge* is found frequently in discussions dealing with regions and boundaries. However, there is a key difference between these two concepts. The boundary of a finite region forms a closed path and is thus a "global" concept. As we will discuss in detail in Chapter 10, edges are formed from pixels with derivative values that exceed a preset threshold. Thus, an edge is a "local" concept that is based on a measure of intensity-level discontinuity at a point. It is possible to link edge points into edge segments, and sometimes these segments are linked in such a way that they correspond to boundaries, but this is not always the case. The one exception in which edges and boundaries correspond is in binary images. Depending on the type of connectivity and edge operators used (we will discuss these in Chapter 10), the edge extracted from a binary region will be the same as the region boundary. This is

[†] We make this assumption to avoid having to deal with special cases. This can be done without loss of generality because if one or more regions touch the border of an image, we can simply pad the image with a 1-pixel-wide border of background values.

intuitive. Conceptually, until we arrive at Chapter 10, it is helpful to think of edges as intensity discontinuities, and of boundaries as closed paths.

DISTANCE MEASURES

For pixels p, q, and s, with coordinates (x, y), (u, v), and (w, z), respectively, D is a *distance function* or *metric* if

(a) $D(p,q) \geq 0 \quad (D(p,q) = 0 \text{ iff } p = q)$,
(b) $D(p,q) = D(q, p)$, and
(c) $D(p,s) \leq D(p, q) + D(q,s)$.

The *Euclidean distance* between p and q is defined as

$$D_e(p, q) = \left[(x - u)^2 + (y - v)^2\right]^{\frac{1}{2}} \tag{2-19}$$

For this distance measure, the pixels having a distance less than or equal to some value r from (x, y) are the points contained in a disk of radius r centered at (x, y).

The D_4 *distance*, (called the *city-block distance*) between p and q is defined as

$$D_4(p, q) = |x - u| + |y - v| \tag{2-20}$$

In this case, pixels having a D_4 distance from (x, y) that is less than or equal to some value d form a diamond centered at (x, y). For example, the pixels with D_4 distance ≤ 2 from (x, y) (the center point) form the following contours of constant distance:

$$\begin{array}{ccccc} & & 2 & & \\ & 2 & 1 & 2 & \\ 2 & 1 & 0 & 1 & 2 \\ & 2 & 1 & 2 & \\ & & 2 & & \end{array}$$

The pixels with $D_4 = 1$ are the 4-neighbors of (x, y).

The D_8 *distance* (called the *chessboard distance*) between p and q is defined as

$$D_8(p, q) = \max(|x - u|, |y - v|) \tag{2-21}$$

In this case, the pixels with D_8 distance from (x, y) less than or equal to some value d form a square centered at (x, y). For example, the pixels with D_8 distance ≤ 2 form the following contours of constant distance:

$$\begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{array}$$

The pixels with $D_8 = 1$ are the 8-neighbors of the pixel at (x, y).

Note that the D_4 and D_8 distances between p and q are independent of any paths that might exist between these points because these distances involve only the coordinates of the points. In the case of m-adjacency, however, the D_m distance between two points is defined as the shortest m-path between the points. In this case, the distance between two pixels will depend on the values of the pixels along the path, as well as the values of their neighbors. For instance, consider the following arrangement of pixels and assume that p, p_2, and p_4 have a value of 1, and that p_1 and p_3 can be 0 or 1:

$$\begin{matrix} & p_3 & p_4 \\ p_1 & p_2 & \\ p & & \end{matrix}$$

Suppose that we consider adjacency of pixels valued 1 (i.e., $V = \{1\}$). If p_1 and p_3 are 0, the length of the shortest m-path (the D_m distance) between p and p_4 is 2. If p_1 is 1, then p_2 and p will no longer be m-adjacent (see the definition of m-adjacency given earlier) and the length of the shortest m-path becomes 3 (the path goes through the points $pp_1p_2p_4$). Similar comments apply if p_3 is 1 (and p_1 is 0); in this case, the length of the shortest m-path also is 3. Finally, if both p_1 and p_3 are 1, the length of the shortest m-path between p and p_4 is 4. In this case, the path goes through the sequence of points $pp_1p_2p_3p_4$.

2.6 INTRODUCTION TO THE BASIC MATHEMATICAL TOOLS USED IN DIGITAL IMAGE PROCESSING

You may find it helpful to download and study the review material dealing with probability, vectors, linear algebra, and linear systems. The review is available in the Tutorials section of the book website.

This section has two principal objectives: (1) to introduce various mathematical tools we use throughout the book; and (2) to help you begin developing a "feel" for how these tools are used by applying them to a variety of basic image-processing tasks, some of which will be used numerous times in subsequent discussions.

ELEMENTWISE VERSUS MATRIX OPERATIONS

An *elementwise operation* involving one or more images is carried out on a *pixel-by-pixel* basis. We mentioned earlier in this chapter that images can be viewed equivalently as matrices. In fact, as you will see later in this section, there are many situations in which operations between images are carried out using matrix theory. It is for this reason that a clear distinction must be made between elementwise and matrix operations. For example, consider the following 2×2 images (matrices):

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

The elementwise product of two matrices is also called the *Hadamard product* of the matrices.

The *elementwise product* (often denoted using the symbol $\odot$ or $\otimes$) of these two images is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \odot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}$$

The symbol $\ominus$ is often used to denote *elementwise division*.

That is, the elementwise product is obtained by multiplying pairs of *corresponding* pixels. On the other hand, the *matrix product* of the images is formed using the rules of matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

We assume elementwise operations throughout the book, unless stated otherwise. For example, when we refer to raising an image to a power, we mean that each individual pixel is raised to that power; when we refer to dividing an image by another, we mean that the division is between corresponding pixel pairs, and so on. The terms *elementwise addition* and *subtraction* of two images are redundant because these are elementwise operations by definition. However, you may see them used sometimes to clarify notational ambiguities.

LINEAR VERSUS NONLINEAR OPERATIONS

One of the most important classifications of an image processing method is whether it is linear or nonlinear. Consider a general operator, $\mathcal{H}$, that produces an output image, $g(x, y)$, from a given input image, $f(x, y)$:

$$\mathcal{H}[f(x, y)] = g(x, y) \tag{2-22}$$

Given two arbitrary constants, a and b, and two arbitrary images $f_1(x, y)$ and $f_2(x, y)$, $\mathcal{H}$ is said to be a *linear operator* if

$$\begin{aligned} \mathcal{H}[af_1(x, y) + bf_2(x, y)] &= a\mathcal{H}[f_1(x, y)] + b\mathcal{H}[f_2(x, y)] \\ &= ag_1(x, y) + bg_2(x, y) \end{aligned} \tag{2-23}$$

This equation indicates that the output of a linear operation applied to the sum of two inputs is the same as performing the operation individually on the inputs and then summing the results. In addition, the output of a linear operation on a constant multiplied by an input is the same as the output of the operation due to the original input multiplied by that constant. The first property is called the property of *additivity,* and the second is called the property of *homogeneity*. By definition, an operator that fails to satisfy Eq. (2-23) is said to be *nonlinear*.

As an example, suppose that $\mathcal{H}$ is the sum operator, Σ. The function performed by this operator is simply to sum its inputs. To test for linearity, we start with the left side of Eq. (2-23) and attempt to prove that it is equal to the right side:

These are image summations, not the sums of all the elements of an image.

$$\begin{aligned} \sum[af_1(x, y) + bf_2(x, y)] &= \sum af_1(x, y) + \sum bf_2(x, y) \\ &= a\sum f_1(x, y) + b\sum f_2(x, y) \\ &= ag_1(x, y) + bg_2(x, y) \end{aligned}$$

where the first step follows from the fact that summation is distributive. So, an expansion of the left side is equal to the right side of Eq. (2-23), and we conclude that the sum operator is linear.

On the other hand, suppose that we are working with the max operation, whose function is to find the maximum value of the pixels in an image. For our purposes here, the simplest way to prove that this operator is nonlinear is to find an example that fails the test in Eq. (2-23). Consider the following two images

$$f_1 = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \qquad \text{and} \qquad f_2 = \begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix}$$

and suppose that we let $a = 1$ and $b = -1$. To test for linearity, we again start with the left side of Eq. (2-23):

$$\max\left\{(1)\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} + (-1)\begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix}\right\} = \max\left\{\begin{bmatrix} -6 & -3 \\ -2 & -4 \end{bmatrix}\right\}$$
$$= -2$$

Working next with the right side, we obtain

$$(1)\max\left\{\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}\right\} + (-1)\max\left\{\begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix}\right\} = 3 + (-1)7 = -4$$

The left and right sides of Eq. (2-23) are not equal in this case, so we have proved that the max operator is nonlinear.

As you will see in the next three chapters, linear operations are exceptionally important because they encompass a large body of theoretical and practical results that are applicable to image processing. The scope of nonlinear operations is considerably more limited. However, you will encounter in the following chapters several nonlinear image processing operations whose performance far exceeds what is achievable by their linear counterparts.

ARITHMETIC OPERATIONS

Arithmetic operations between two images $f(x, y)$ and $g(x, y)$ are denoted as

$$\begin{aligned} s(x, y) &= f(x, y) + g(x, y) \\ d(x, y) &= f(x, y) - g(x, y) \\ p(x, y) &= f(x, y) \times g(x, y) \\ v(x, y) &= f(x, y) \div g(x, y) \end{aligned} \tag{2-24}$$

These are elementwise operations which, as noted earlier in this section, means that they are performed between corresponding pixel pairs in f and g for $x = 0, 1, 2, \ldots, M-1$ and $y = 0, 1, 2, \ldots, N-1$. As usual, M and N are the row and column sizes of the images. Clearly, s, d, p, and v are images of size $M \times N$ also. Note that image arithmetic in the manner just defined involves images of the same size. The following examples illustrate the important role of arithmetic operations in digital image processing.