BANGABANDHU SHEIKH MUJIBUR RAHMAN SCIENCE AND TECHNOLOGY UNIVERSITY

Name of the Assignment

Matrix(Definition of Matrix, Matrix Operation, Transpose Matrix, Inverse of Matrix, Rank of Matrix.

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Matrix

Definition of matrix:

A matrix is a rectangular array of numbers (real or complex) enclosed by a pair of brackets and the numbers in the array are called the entries or the elements of the matrix. that is a rectangular array numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a matrix. The numbers $a_{12}, a_{12} \cdots a_{mn}$ are called the entries or the elements of the matrix. The above matrix has m rows and n columns and its called an $(m \times n)$ matrix. The matrix of m rows and n columns is said to be of order "m by n" or $m \times n$. The above matrix is also denoted by $[a_{ij}]$ $i = 1, 2, \cdots m$, $j = 1, 2 \cdots n$

A matrix consisting of a single row is called a row matrix and a matrix of a single column is called a column matrix.

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Matrix Operation:

Addition and scalar multiplication of matrix:

Addition of matrices is defined only for the matrices having same number of rows and columns. Let A and B be the two matrices having m rows and n columns.

$$A = |a_{ij}| = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and}$$

$$B = |b_{ij}| = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

Then the sum of A and B is

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots\cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{21}+b_{21} & \cdots\cdots & b_{2n}+b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots\cdots & a_{mn}+b_{mn} \end{bmatrix}$$

The multiplication of matrix by numbers (scalars) is defined as follow: The product of an $(m \times n)$ matrix A by a number K is denoted by KA or AK and is the $(m \times n)$ matrix obtained by multiplying every element of A by K. that is

$$KA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

We also define $-A = (-1)A$ and $A-B = A+(-B)$

If the matrices A, B, C are comfortable for addition and if K is any scalar, then we can state that

(i) $A+B = B+A$ (commulative law)

(ii) $(A+B)+C = A+(B+C)$ (Associative law)

(iii) $A+0 = 0+A = A$

(iv) $K(A+B) = KA+KB = (A+B)K$

where 0 is the zero matrix of the same order.

Matrix Multiplication:

Two matrices A and B are comfortable for multiplication if the number of columns in A is equal to the number of rows in B.

Let $A = [a_1, a_2 ---- a_n]$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ --- \\ b_n \end{bmatrix}$

The $AB = [a_1 b_1 + a_2 b_2 + --- a_n b_n] = \left[\sum_{i=1}^{n} a_i b_i\right]$

Again let the $m \times p$ matrix $A = [a_{ij}]$ and the $p \times n$ matrix $B = [b_{ij}]$ then AB is the $m \times n$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$\begin{cases} i = 1, 2 \cdots m \\ j = 1, 2 \cdots n \end{cases}$$

If the matrices A, B, C are comfortable for the indicated sums and products, we have the following properties:

(i) $(AB)C = A(BC)$ (Associative law)

(ii) $A(B+C) = AB + AC$ } Distributive law
(iii) $(A+B)C = AC + BC$

(iv) $k(AB) = (kA)B = A(kB)$ where k is any scalar

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In the matrix product AB. the matrix A is called the pre-multiplier (pre-factor) and B is called the post-multiplier (post-factor)

let $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$

then

$$AB = \begin{bmatrix} 1\cdot1 + (-3)\cdot(-2) + 5\cdot3 & 1\cdot(-1) + (-3)\cdot4 + 5\cdot0 \\ 2\cdot1 + 0\cdot(-2) + (-1)\cdot3 & 2\cdot(-1) + 0\cdot4 + (-1)\cdot0 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -13 \\ -1 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot1 + (-1)\cdot2 & 1\cdot(-3) + (-1)\cdot0 & 1\cdot5 + (-1)\cdot(-1) \\ (-2)\cdot1 + 4\cdot2 & (-2)\cdot(-3) + (4)\cdot0 & (-2)\cdot5 + (4)\cdot(-1) \\ 3\cdot1 + 0\cdot2 & 3\cdot(-3) + 0\cdot0 & 3\cdot5 + 0\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & 6 \\ 6 & 6 & -14 \\ 3 & -9 & 15 \end{bmatrix}$$

$\therefore AB \neq BA$

Transpose of a Matrix:

If A is an $m \times n$ matrix over the real field $\mathbb{R}$ then the $n \times m$ matrix obtained from the matrix A by writting its rows as columns and its columns as rows is called the transpose of A and is denoted by the symbol A^T. That is if $A = [a_{ij}]$ is an $m \times n$ matrix then $A^T = [a_{ji}]$ is an $n \times m$ matrix.

let $A = \begin{bmatrix} 1 & 0 & 5 & -7 \\ 2 & 3 & -1 & 6 \end{bmatrix}$ then

$$A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 5 & -1 \\ -7 & 6 \end{bmatrix}$$

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Complex Conjugate (or, conjugate) of a matrix :

The conjugate of a complex number $z = x+iy$ is the complex number $\bar{z} = x-iy$. If A is an $m \times n$ matrix over the complex field then we say that the conjugate of a matrix A is the matrix $\bar{A}$ whose element are respectively the conjugates of the elements of A. That is if $A = |a_{ij}|$ then $\bar{A} = |\overline{a_{ij}}]$.

if
$$A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{bmatrix}$$

then
$$\bar{A} = \begin{bmatrix} 1 & -i & 1-i \\ i & 2 & 2-3i \\ 5 & 1-2i & -5 \end{bmatrix}$$

Real Matrix :

A matrix A is called real provided it satisfies the relation $A = \bar{A}$

Imaginary Matrix ;

A matrix A is called imaginary provided it satisfies the relation $A = -\bar{A}$

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Conjugate transpose of a complex matrix:-

The conjugate of the transpose of a given complex matrix A is said to be the conjugate transpose of A and is generally denoted by the symbol A^*. That is

if $A = |a_{ij}|$ is complex matrix, then

$$A^* = |\overline{A}^T| = |a_{ij}|$$

if
$$A = \begin{vmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{vmatrix}$$

then
$$A^* = \begin{vmatrix} 1 & i & 5 \\ -i & 2 & 1-2i \\ 1-i & 2-3i & 5 \end{vmatrix}$$

Square Matrix:

A matrix with the same number of rows and columns is called a square matrix.

Rectangular Matrix:

The number of rows and columns of a matrix need not be equal. when $m \neq n$ the number of rows and columns of the array are not equal. then the matrix is known as the rectangular matrix.

Diagonal Matrix:

A square matrix whose elements $a_{ij}=0$ when $i \neq j$ is called a diagonal Matrix.

for example: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A diagonal matrix whose diagonal elements are all equal is called a scalar matrix.

Identity Matrix (Unit Matrix):

A square matrix whose elements $a_{ij}=0$ if $i \neq j$ and $a_{ij}=1$ if $i=j$ is called the identity matrix.

Example:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Zero or null Matrix:

A matrix in which every element is zero is called a null matrix or a zero matrix.

Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Upper and lower triangular matrices:

A square matrix whose elements $a_{ij} = 0$ for $i > j$ is called an upper triangular matrix and a square matrix whose elements $a_{ij} = 0$ for $i < j$ is called a lower triangular matrix.

Ex. $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -2 & 5 & -1 & 0 \\ 3 & 7 & 1 & 6 \end{bmatrix}$

Symmetric Matrix:

A matrix equal to its transpose. a square matrix such that $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$ is said to be symmetric. In short we can say a squre matrix A will be symmetric if $A^T = A$

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Ex.

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \\ -3 & 7 & 3 \end{bmatrix}$$

Singular and non-singular Matrix:-

Let D be the determinant of the square matrix A. then if $D=0$, the matrix A is called the singular matrix and if $D \neq 0$ the matrix A is called the non-singular matrix.

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ -1 & -1 & 7 \end{bmatrix}$ are singular

Since,

$$D_1 = |A| = 0$$
$$D_2 = |B| = 0.$$
$$D_3 = |C| = 0$$

Again. $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ is non-singular matrix

Since,

$$D_1 = |A| = -6 \neq 0$$

Inverse Matrix :-

A square matrix A is said to be inverertible if there exists a unique matrix B such that $AB = BA = I$ where I is the unit matrix. We call such a matrix B the inverse of A and is generally denoted by A^{-1}. Here we have to note that if B is the inverse of A, then A is the inverse of B.

Ex. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Then

$$AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, A and B are invertible and are inverse of each other. That is $A^{-1} = B$ and $B^{-1} = A$.

Adjoint or adjugate Matrix:-

let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

let D be the determinant of the matrix A.

then $D = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

let A_{ij} $(i = 1, 2, 3 \ldots n$ and $j = 1, 2, 3 \ldots n)$ be the co-factors of the determinant D.

Form the matrix $[A_{ij}]$ Then the transpose of the matrix $[A_{ij}]$ is called the adjoint on the adjugate matrix of the matrix. A and is generally denoted by Adj A.

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$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$$

$$= \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Ex.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix} \text{ then}$$

$$\text{adj } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

Process of finding the inverse of a square Matrix:

Let the matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Let D be the determinant of the matrix A. Evaluate the determinant D. if $D = 0$, the matrix A is singular and it has no inverse. If $D \neq 0$ the matrix A is non-singular and

A^{-1} exists. Find the adjoint matrix adj A of the matrix A. then:-

$$A^{-1} = \frac{1}{D} \text{Adj } A = \frac{\text{Adj } A}{|A|}$$

$$A^{-1} = \begin{bmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \cdots & \frac{A_{n1}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \cdots & \frac{A_{n2}}{|A|} \\ \cdots & \cdots & & \cdots \\ \frac{A_{1n}}{|A|} & \frac{A_{2n}}{|A|} & \cdots & \frac{A_{nn}}{|A|} \end{bmatrix}$$

Example: 4:

IF $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

prove that $A^3 + A^2 - 21A - 54I = 0$ where I is the Identity matrix of order 3×3.

Proof:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4+3 & -4+2+6 & 6-12+0 \\ -4+2+6 & 4+1+12 & -6-6+0 \\ 2-4+0 & -2-2+0 & 3+12+0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -22+8+6 & 22+4+12 & -33-24+0 \\ -8+34+12 & 8+17+24 & -12-102+0 \\ 4-8-15 & -4-4-30 & 6+24+0 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix}$$

$$\therefore A^3 + A^2 - 21A - 45I$$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} - 21 \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$- 45 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}$$

$$- \begin{bmatrix} -42 & 42 & -63 \\ 42 & 21 & -126 \\ -21 & -42 & 0 \end{bmatrix} - \begin{bmatrix} 45 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{bmatrix}$$

$$= \begin{bmatrix} -8+11+42-45 & 38+4-42-0 & -57-6+63-0 \\ 38+4-42+0 & 49+17-21-45 & -114-12+126-0 \\ -19-2+21-0 & -38-4+42-0 & 30+15-0-45 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence $A^3 + A^2 - 21A - 45I = 0$

Ex. 8: If $\cancel{A = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 5 \end{vmatrix}}$ $A = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix}$ and

$B = \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix}$ then prove that $\overline{A+B} = \bar{A}+\bar{B}$

Proof:

$$\bar{A} = \begin{bmatrix} 1 & 1-i \\ 2+3i & -i \end{bmatrix} \text{ and } \bar{B} = \begin{bmatrix} 2+i & -i \\ 1-5i & 3 \end{bmatrix}$$

$$\bar{A} + \bar{B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix} \text{ --- (i)}$$

Again,

$$A+B = \begin{bmatrix} 3-i & 1+2i \\ 3+2i & 3+i \end{bmatrix}$$

$$\overline{A+B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix} \text{ --- (ii)}$$

Thus from (i) and (ii), we get

$$\bar{A}+\bar{B} = \overline{A+B}$$

Ex-12(a): Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Sol^n: Let D be the determinant of the matrix then $D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$ so the matrix A is non-singular and hence A^{-1} exists.

Now the cofactor of D are

$A_{11} = 4$, $A_{12} = -3$

$A_{21} = -2$, $A_{22} = 1$

Then $\operatorname{Adj} A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T$

$= \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{D} \operatorname{Adj} A = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

(b) Find the inverse matrix of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{bmatrix}$$

Solⁿ: Let D be the determinant of the matrix.

then $D = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix} = 2(0+3) + 1(8+3) + 3(12+0)$

$= 6 + 11 + 36 = 53 \neq 0$

So, the matrix A is non-singular and A^{-1} exists. Now the co-factors of D are

$A_{11} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3$

$A_{12} = (-1) \begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix} = -11$

$A_{13} = \begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix} = 12$

$A_{21} = (-1) \begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix}$

$= 11$

$A_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$

$= -5$

$A_{23} = (-1) \begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}$

$= -9$

$A_{31} = \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} = 1$

$A_{32} = (-1)\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = 14$

$A_{33} = \begin{vmatrix} 4 & -1 \\ 4 & 0 \end{vmatrix}$

$= 4$

Therefore,

$$\text{Adj } A = \begin{bmatrix} 3 & -11 & 12 \\ 11 & -5 & -9 \\ 1 & 14 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$$

Ex-13: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ by using row canonical form.

Soln: $|AI_2| = \left[\begin{array}{cc:cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right]$ Interchange first and second rows.

$\sim \left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$ we multiply first row by 2 and then subs-tract from the second row.

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & -1 & 1 & -2 \end{array}\right]$$ we multiply second row by 3 and then add with the first row.

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right]$$ we multiply second row by (-1).

$$= [I_2 \; A^{-1}]$$

Hence A is inventible and $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Ex-15: Find the inverse of the following matrix by using row canonical form:

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Soln. $$[A \; I_3] = \left[\begin{array}{ccc:ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$ Interchange first and second rows

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$

we multiply first row by 3 and 2 and then substract from the second and and third rows respectively.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & -1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$

substract third row from the second row.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$

multiply the second row by (-1)

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array}\right]$$

we multiply second row by 5 and then substract from the third row.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{5}{10} & \frac{7}{10} & \frac{4}{10} \end{array}\right]$$

we multiply third row by $(-\frac{1}{10})$

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$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

we multiply third row by 3 then substract from the 1st row

$$= \left[I_3 \; A^{-1} \right]$$

Hence A is inventible and

$$A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ \frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

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Ex-16: Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -6 & 0 & 1 & -2 \\ 8 & 1 & -2 & 1 \end{bmatrix}$$ by using only row transformations to reduce A to I.

Soln:

$$AI_4 = \left[\begin{array}{cccc:cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ -6 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\ 8 & 1 & -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

⇒ We substract first row from second row. We multiply first row by 6 and then add with the third row. Also we multiply first row by 8 and then substract from the fourth row.

$$\sim \left[\begin{array}{cccc:cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -6 & 1 & -2 & 6 & 0 & 1 & 0 \\ 0 & 9 & -2 & 1 & -8 & 0 & 0 & 1 \end{array}\right]$$

We multiply second row by 2 and add with the third row. We also multiply second row by 3 and then substract from the fourth row.

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -5 & -3 & 0 & 1 \end{array}\right]$$

we multiply second row by $\frac{1}{3}$.

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -5 & -3 & 0 & 1 \end{array}\right]$$

We add 2nd row with the first row.

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -5 & -3 & 0 & 0 \end{array}\right]$$

We multiply 3rd row by 2 and then add with the 4th row.

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 3 & 1 & 2 & 1 \end{array}\right]$$

we multiply 4th row by $\left(-\frac{1}{3}\right)$

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 2 & 1 \end{array}\right]$$

we multiply fourth row by 2 and then add with the third row:

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 4/3 & -1/3 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & -1 & -1/3 & -2/3 & -1/3 \end{array}\right] = \left[I_4 \, A^{-1}\right]$$

Hence A is invertible and

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & 4/3 & -1/3 & -\frac{2}{3} \\ -1 & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Rank of a matrix

Canonical Matrix:

A canonical matrix is one in which all terms not of the principal diagonal are zero, all terms on the principle diagonal are zero or one and all ones precedes all zeros.

for instance $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a canonical matrix.

Elementary transformation (or operation)

An elementary transformation of a matrix is any one of the following operations:

(i) The multiplication of each element of a row or a column by some non-zero constant.

(ii) The interchange of two rows or two columns

(iii) The addition of any multiple of the elements of one row or one column to the correspon ding elements of another row or column respectifivelly.

Equivalent Matrix:

Two matrices A and B are called equivalent and is written as $A \sim B$ if one can be obtained from another by means of a finite number of elementary transformations.

If a matrix A is reduced to B by the use of elementary row transformations alone; then B is said to be row equivalent to A and conversely.

Definition of Rank of a Matrix:

The rank of a matrix can be defined in seven equivalent ways. We use the following definitions:

(i) Let A be an arbitrary $m \times n$ matrix over a field F. The rank of the matrix A is the largest value of r for which the exists an $r \times r$ submatrix of A with non-vanishing determinant.

(ii) A matrix A is said to have rank r if it contains at least one r-rowed square submatrix with a non-zero determinant whereas the determinant of any square submatrix having (r+1) or more rows possibly contained in A is zero.

(iii) A non-zero matrix A is said to be have rank at least one of its r-square minors is different from zero while even (r+1) square minor if any is zero.

(iv) The rank of a matrix A is the maximu number linearly independent rows or columns in the matrix.

(v) Let A be an mxn matrix and let A_R be the row echele form of A. Then the rank r of the matrix A is the number non zero rows of A_R.

The rank of a matrix A is denoted by rank (A) or $\rho(A)$.

The rank of a null matrix is zero and the rank of a matrix of order $m \times n$ can not be greater than m or n.

An n-rowed square matrix A has rank $r<n$ if and only if $|A|=0$. In this case A is called a singular matrix. The matrix A has rank $r=n$ if and only if $|A| \neq 0$ and is then called a non-singular matrix.

Ex-1: Find the rank of each of the following matrices.

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12-12 = 0$

So: the rank of the matrix A is 1 (one). since $|A| = 0$ but not every element of A is zero say $|2| \neq 0$

(ii) let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= 1(21-20) - 2(14-12) + 3(10-9)$$

$$= 1 - 4 + 3 = 0$$

So, the ~~matrix~~ rank of the matrix A is less than 3. Now let us take two-rowed minor of A. say.

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3-4 = -1 \neq 0$$

since $|A| = 0$ but $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \neq 0$

therefore the rank of the given matrix is 2.

Ex-2: Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Soln:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 2\times3 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 6 \times 0 = 0$$

(since three rows are equal.)

So, the rank of the matrix A is less than 3. Now let us consider the two rowed minors of A.

say $\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 36-36=0$, $\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 18-18=0$

$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12-12=0$ $\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} = 18-18=0$

$\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 9-9=0$ $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6-6=0$

$\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0$ $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$ $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

Thus every two rowed minors of A is zero. So the rank of A is less than 2. But $|4| = 4 \neq 0$ Hence the rank of A is 1 (one).

Ex. 3: Find the rank of the matrix

$$A = \begin{bmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{bmatrix}$$

Solⁿ: Since the given matrix A is of order 3×4 the rank of the given matrix A cannot be greater than 3. Now we observe that the matrix A has the following largest square submatrices of order 3×3

$$A_1 = \begin{bmatrix} 6 & 2 & 0 \\ -2 & -1 & 3 \\ -1 & -1 & 6 \end{bmatrix} \quad A_2 = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -1 & 4 \\ -1 & -1 & 10 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 6 & 0 & 4 \\ -2 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix}$$

Now.

$$|A_1| = \begin{vmatrix} 6 & 2 & 0 \\ -2 & -1 & 3 \\ -1 & -1 & 6 \end{vmatrix} = 6(-6+3) - 2(-12+3) + 0$$
$$= -18 + 18 = 0$$

$$|A_2| = \begin{vmatrix} 6 & 2 & 4 \\ -2 & -1 & 4 \\ -1 & -1 & 10 \end{vmatrix} = 6(-10+4) - 2(-20+4) + 4(2-1)$$
$$= -36 + 32 + 4 = 0$$

$$|A_3| = \begin{vmatrix} 6 & 0 & 4 \\ -2 & 3 & 4 \\ -1 & 6 & 10 \end{vmatrix} = 6(30-24) + 0 + 4(-12+3)$$
$$= 36 - 36 = 0$$

$$|A_4| = \begin{vmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ -1 & 6 & 10 \end{vmatrix} = 2(30-24) + 0 + 4(-6+3)$$
$$= 12 - 12 = 0$$

So, the rank of the matrix A can not be 3. Let us consider submatrices of order 2×2. There we can at once show that

$$\begin{vmatrix} 6 & 2 \\ -2 & -1 \end{vmatrix} = -6+4 = -2 \neq 0$$

Therefore the rank of the given matrix is 2.

Ex-5: Reduce the matrix A to the normal (or canonical form and hence obtain its rank where

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

Soln: We will apply both elementary column and row operation to the matrix A for reducing it to the normal form.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

We replace C_2 and C_4 by $C_2 - 2C_1$ and $C_4 + C_1$ respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix}$$

we replace C_2 and C_4 by $C_2 + 2C_3$ and $C_4 - 5C_3$ respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 11 & 2 & -7 \end{bmatrix}$$

we replace C_1 by $C_1 + C_3$ and C_4 by $C_4 + \frac{7}{11} C_2$

$$\sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{vmatrix}$$

we replace R_2 by $R_2 - 4R_1$

$$\sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{vmatrix}$$

we replace R_3 by $R_3 - 2R_2$

$$\sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 0 \end{vmatrix}$$

we interchange c_2 and c_3

$$\sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & 0 \end{vmatrix}$$

we replace c_3 by $\frac{1}{11} c_3$

$$\sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} I_3 & 0 \end{bmatrix}$$ where

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the rank of A is 3.

Ex-7. Determine the rank of the matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Soln. Reduce the given matrix to echelon form by means of elementary row transformations and form successive matrices by the equivalence sign $\sim$. Interchange first row and fourth row

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

we substract first row from the second row and multiply the first row by 3 and then substract from the third row.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

we multiply second row by 2 and then substract from the third row. Also we Add second row with the fourth row.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

second row is multiplied by -1.
The matrix is now equivalent to the given matrix A and is in the row echelon form. Since the echelon matrix has two non-zero rows, the rank of the given matrix A is 2.

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(1)

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Definition of matrix

Matrix:- A matrix is a rectangular array of numbers enclosed by a pair of brackets and the numbers in the array are called the entries or the elements of the matrix that is a rectangular array of (real or complex) numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{mn} \end{bmatrix}$$

is called a matrix. The numbers $a_{11}, a_{12} \ldots a_{mn}$ are called the entries or the elements of the matrix. The above matrix has m rows and n colums and is called an $(m \times n)$ matrix.

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* Addition and scalar multiplication of matrix :-

Addition of matrices is defined only for the matrices having same number of rows and the same number of colums.

for example :- $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 7 \\ -5 & 4 \end{bmatrix}$

$$\Rightarrow A+B = \begin{bmatrix} 1+2 & (-2)+7 \\ 3+(-5) & 5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \\ -2 & 9 \end{bmatrix}.$$

* Matrix Multiplication :-

Two matrices A and B are comformable for multiplication if the number of columns in A is equal to the number of rows in B.

If the matrices A, B, C are conformable to the indicated sums and products we have the following properties :-

(i) $(AB)C = A(BC)$ ⇒ Associative law

(ii) $A(B+C) = AB + AC$ } Distributive laws

(iii) $(A+B)C = AC + BC$

(iv) $k(AB) = (kA)B = A(kB)$ where k is any scalar

remarks: In the matrix product AB the matrix A is called the pre-multiplier or (pre-factor) and B is called the postmultiplier or (post-factor)

for example :-

$$A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$$

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$$\Rightarrow AB = \begin{bmatrix} 1+6+15 & -1-12+0 \\ 2+0-3 & -2+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -13 \\ -1 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & 6 \\ 6 & 6 & -14 \\ 3 & -9 & 15 \end{bmatrix}$$

$$\therefore AB \neq BA$$

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* Transpose a matrix :-

If A is an $m \times n$ matrix over the real field $\mathbb{R}$, then the $n \times m$ matrix obtained from the matrix A by writing its rows as colums and its columns as rows is called the transpose of A and is denoted by the symbol A^T. That is if $A = [a_{ij}]$ is an $m \times n$ matrix then $A^T = [a_{ji}]$ is an $n \times m$ matrix.

$\Rightarrow$ If $A = \begin{bmatrix} 1 & 0 & 5 & 7 \\ 2 & 3 & -1 & 6 \end{bmatrix}$

then $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 5 & -1 \\ 7 & 6 \end{bmatrix}$.

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* Inverse matrix :-

A square matrix A is said to be invertible if there exists a unique matrix B such that AB = BA = I where I is the unit matrix. We call such a matrix B the inverse of A & is generally denoted by A^{-1}. Here we have to note that if B is the inverse of A then A is the inverse of B.

Ex :- $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

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then $AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Therefore, A and B are invertible and are inverses of each other. This is $A^{-1} = B$ and $B^{-1} = A$

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* Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow$ Let, D be the determinant of the matrix, then $D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$, so the matrix A is non-singular and hence A^{-1} exist.

Now the cofactors of D are,

$A_{11} = 4 \qquad A_{12} = -3$

$A_{21} = -2 \qquad A_{22} = 1$

then, $\text{Adj } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{D} \text{Adj } A$

$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$ Ans

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Ex-01

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$$

Find the matrices $2A$, $A+B$, and $A-B$

$\Rightarrow$

$$2A = \begin{bmatrix} 2\cdot 1 & 2(-2) & 2\cdot 3 \\ 2\cdot 5 & 2\cdot 1 & 2(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 6 \\ 10 & 2 & 8 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & -2+3 & 3+5 \\ 5+1 & 1+4 & -4+(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 8 \\ 6 & 5 & -6 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix} + \begin{bmatrix} -2 & -3 & -5 \\ -1 & -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -5 & -2 \\ 4 & -3 & -2 \end{bmatrix}$$ Ans

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Ex-02

Let, $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$. $B = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

Compute the matrix products AB and BA

$\Rightarrow$ $AB = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5+0 & 0+0 \\ 0+10 & 0+5 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 \\ 10 & 5 \end{bmatrix}$

$BA = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 5+0 & 0+0 \\ 2+0 & 0+5 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 \\ 2 & 5 \end{bmatrix}$

Ans

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Ex-03

If $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

prove that $A^3 + A^2 - 2[A - 54] = 0$

where I is the Identity matrix of (3×3).

$\Rightarrow A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$\therefore A^2 = AA = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4+4+3 & -4+2+6 & 6-12+0 \\ -4+2+6 & 4+1+12 & -6-6+0 \\ 2-4+0 & -2-2+0 & 3+12+0 \end{bmatrix}$

$= \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}$

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$$A^3 = A^2 A = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -22+8+6 & 22+4+12 & -33-24+0 \\ -8+34+12 & 8+17+24 & -12-102+0 \\ 4-8-15 & -4-4-30 & 6+24+0 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix}$$

$\therefore A^3 + A^2 - 21A - 45I$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} - 21 \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$- 45 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} - \begin{bmatrix} -42 & 42 & -63 \\ 42 & 21 & -126 \\ -21 & -42 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 45 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{bmatrix}.$$

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$$= \begin{bmatrix} -8+11+42-45 & 38+4-42-0 & -57-6+63-0 \\ 38+4-42-0 & 93+17-21-45 & -114-12+126-0 \\ -19-2+21-0 & -38-4+21-0 & 30+15-0-45 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence $A^3 + A^2 - 21A - 45I = 0$ proved

Ex-04

If $A = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix}$, $B = \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix}$

then prove that $\overline{A+B} = \overline{A} + \overline{B}$

$\Rightarrow \overline{A} = \begin{bmatrix} 1 & 1-i \\ 2+3i & -i \end{bmatrix}$

$\overline{B} = \begin{bmatrix} 2+i & -i \\ 1-5i & 3 \end{bmatrix}$

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$$\therefore \bar{A}+\bar{B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix} \quad \text{--- (i)}$$

$$\therefore A+B = \begin{bmatrix} 3-i & 1+2i \\ 3+2i & 3+i \end{bmatrix}$$

$$\therefore \overline{A+B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix} \quad \text{--- (ii)}$$

From (i) and (ii) we get,

$$\overline{A+B} = \bar{A}+\bar{B} \quad \text{(proved)}$$

Ex-5

If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 & 3 \\ 7 & -2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

then prove that $(AB)^T = B^T A^T$

$$\Rightarrow A^T = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 4 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -1 & 7 & 2 \\ 5 & -2 & 0 \\ 3 & 1 & 3 \end{bmatrix}$$

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$$\therefore B^T A^T = \begin{bmatrix} 1 & 7 & 2 \\ 5 & -2 & 0 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 5 & 3 \\ 3 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+14+6 & 2+35-2 & -2+21+8 \\ 5-4+0 & -10-10+0 & 10-6+0 \\ 3+2-9 & -6+5+3 & 6+3-12 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 35 & 27 \\ 1 & -20 & 4 \\ -4 & 2 & -3 \end{bmatrix} \text{ ---- } (1)$$

$$\therefore AB = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 5 & 3 \\ 7 & -2 & 1 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1+14+6 & 5-4+0 & 3+2-9 \\ 2+35-2 & -10-10+0 & -6+5+3 \\ -2+21+8 & 10-6+0 & 6+3-12 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 1 & -4 \\ 35 & -20 & 2 \\ 27 & 4 & -3 \end{bmatrix} = B^T A^T$$

$\therefore$ we get $(AB)^T = B^T A^T$

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Ex: 6

$$A^{-1} = \frac{adj(A)}{|A|}$$

Find the inverse of the matrix, $A = \begin{bmatrix} -2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{bmatrix}$

transpose করে সহগুণক
ভেবে করতে হবে বা
সহগুণক হয় তার transpose
নিতে হবে।

$\Rightarrow$ Let D be the determinant of the matrix.

then, $D = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix}$

$= 2(0+3) + 1(8+3) + 3(12-0)$

$= 6 + 11 + 36$

$= 53 \neq 0$

So the matrix A is not singular and A^{-1} exists. Now the cofactors of D are,

$A_{11} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3$, $A_{12} = (-1)\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix}$.

$A_{13} = \begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix} = 12$, 17/ 18ICTCSE027

$A_{21} = (-1)\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix}$, $A_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$, $A_{23} = (-1)\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}$

$= 11 \qquad = -5 \qquad = -9$

$A_{32} = (-1)\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$, $A_{31} = \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix}$, $A_{33} = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$

$= 1 \qquad = 4$

Therefore,

$$\text{Adj } A = \begin{bmatrix} 3 & -11 & 12 \\ 11 & -5 & -9 \\ 1 & 14 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{53}\begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$$

Ans

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Ex :- 07

Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

$$\Rightarrow A^2 = A \times A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6+1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$$\therefore A^2 = I \quad \text{Ans}$$

(1)

MATRIX ALGEBRA

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Defination of Matrix

Matrix:

A matrix is a rectangular array of numbers (real or complex) enclosed by a pair of brackets (or double vertical rolls) and the numbers in the array are called the centries or the elements of the matrix. that is a rectangular array of (real or complex) numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

is called a matrix.

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Matrix has m rows and n colums and is called an (mxn) matrix.

Example of Matrix

1.A.

$$A = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -3 & 7 \end{bmatrix}$$ is a matrix of order 2x3 over the real field IR and also over the complex field C.

The rows of A are (1, 0, −5) and (2, −3, 7) and its columns are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 7 \end{pmatrix}$.

2.B

$$B = \begin{bmatrix} 2 & 0 & i \\ -i & 1 & 4 \\ 1+i & -5 & 3 \end{bmatrix}$$ is a matrix of order 3x3 over the complex field C.

The rows of B are (2, 0, i), (−i, 1, 4), (1+i, −5, 3) and its columns are $\begin{pmatrix} 2 \\ -i \\ 1+i \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$, $\begin{pmatrix} i \\ 4 \\ 3 \end{pmatrix}$.

MATRIX OPERATIONS ID: 18ICTCSE033

If the matrices A, B, C are comformable for addition and if k is scalar then we can state that,

1. $A+B = B+A$ (Commutative law)
2. $(A+B)+C = A+(B+C)$ (Associative law)
3. $A+O = O+A = A$
4. $k(A+B) = kA+kB = (A+B)k$

where O is the zero matrix of the same order.

matrix multiplication:

Two matrices A and B are comfortable for multiplicatio if the numbers of columns in A is equal to the numb of rows in B.

If the matrices A, B, C are comformable for the indic sums and products, we have the following properties:

1. $(AB)C = A(BC)$ (Associative law)
2. $A(B+C) = AB+AC$ (Distributive laws)
3. $(A+B)C = AC+BC$
4. $k(AB) = (kA)B = A(kB)$ where k is any sca

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Examples :

if, $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ -5 & 4 \end{bmatrix}$

then, $A+B = \begin{bmatrix} 1+2 & (-2)+7 \\ 3+(-5) & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 9 \end{bmatrix}$

$2A = \begin{bmatrix} 2.1 & 2.(-2) \\ 2.3 & 2.5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 6 & 10 \end{bmatrix}$

and $A-B = \begin{bmatrix} 1-2 & -2-7 \\ 3-(-5) & 5-4 \end{bmatrix} = \begin{bmatrix} -1 & -9 \\ 8 & 1 \end{bmatrix}$

let, $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$

then, $AB = \begin{bmatrix} 1.1+(-3).(-2)+5.3 & 1.(-1)+(-3).4+5.0 \\ 2.1+0.(-2)+(-1).3 & 2.(-1)+0.4+(-1).0 \end{bmatrix}$

$= \begin{bmatrix} 1+6+15 & -1-12+0 \\ 2-2-3 & -2+0+0 \end{bmatrix} = \begin{bmatrix} 22 & -13 \\ -1 & -2 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1.1+(-1).2 & 1.(-3)+(-1).0 & 1.5+(-1).(-1) \\ -2.1+4.2 & (-2).(-3)+4.0 & (-2).5+(-2).(-1) \\ 3.1+0.2 & 3.(-3)+3.0 & 3.5+3.(-1) \end{bmatrix}$

$= \begin{bmatrix} 1-2 & -3+0 & 5+1 \\ -2+8 & 6+0 & -10+2 \\ 3+0 & -9+0 & 15-3 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 6 \\ 6 & 6 & -14 \\ 3 & -9 & 15 \end{bmatrix}$

$\therefore AB \neq BA$

Transpose of a matrix

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If A is an $m \times n$ matrix over the real field $\mathbb{R}$, the the $n \times m$ matrix obtained from the matrix A by writing its rows as columns and its columns are rows is called the transpose of A and is denoted by the symbol A^T.

That is, if $A = [a_{ij}]$ is $m \times n$ matrix then,

$A^T = [a_{ji}]$ is an $n \times m$ matrix.

For, examples,

let $A = \begin{bmatrix} 1 & 0 & 5 & -7 \\ 2 & 3 & -1 & 6 \end{bmatrix}$, then, $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 5 & -1 \\ -7 & 6 \end{bmatrix}$.

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Complex conjugate (or conjugate) of a matrix

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The conjugate of a complex number $z = x+iy$ is the complex number $\bar{z} = x-iy$.

If A is an $m \times n$ matrix over the complex field, then we say that the conjugate of a matrix A is the matrix $\bar{A}$ whose elements are respectively the conjugates of the elements of A.

That is, if $A = [a_{ij}]$, then $\bar{A} = [\bar{a}_{ij}]$.

For examples, if $A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{bmatrix}$

then, $\bar{A} = \begin{bmatrix} 1 & -i & 1-i \\ i & 2 & 2-3i \\ 5 & 1-2i & -5 \end{bmatrix}$

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Definition Real matrix ID: 18ICTCSE033

A matrix A is called real provided it satisfies the relation

$A = \bar{A}$

Definition Imaginary matrix

A matrix A is called imaginary provided it satisfies the relation $A = -\bar{A}$.

Conjugate transpose of a complex matrix

The conjugate of the transpose of a given complex matrix A is said to be the conjugate transpose of A and is generally denoted by the symbol A^{ϕ}. That is

if, $A = [a_{ij}]$ is a complex matrix, then

$$A^{\phi} = (\bar{A}^{T}) = [\bar{a}_{ji}].$$

For examples, if $A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{bmatrix}$

then, $A^{\phi} = \begin{bmatrix} 1 & i & 5 \\ i & 2 & 1-2i \\ 1-i & 2-3i & -5 \end{bmatrix}$

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Square matrix

A matrix with the same number of rows and columns is called a square matrix.

For examples

$$\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & -5 \\ 7 & -2 & 6 \end{bmatrix} \text{ are square matrix.}$$

Rectangular matrix:

The number of rows and columns of a matrix need not be equal,

When $m \neq n$.

the number of rows and columns of the array are not equal, then the matrix is known as the rectangular matrix.

For examples, $\begin{bmatrix} 1 & -1 & 2 & 7 \\ -2 & 3 & 5 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -i & 2i & 3 \\ i & 2 & 0 & 5 \end{bmatrix}$

are rectangular matrix.

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Diagonal matrix:

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A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a diagonal matrix.

For examples, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrix.

Scalar matrix

A diagonal matrix whose diagonal elements are all equal is called a scalar matrix.

Identity matrix (or Unit matrix):

A square matrix whose elements $a_{ij} = 0$, if $i \neq j$ and $a_{ij} = 1$ if $i = j$ is called the identity matrix or unit matrix.

For examples, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices.

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Upper and lower triangle matrices : ID: 18ICTCSE033

A square matrix whose elements $a_{ij} = 0$ for $i > j$ is called an upper triangle matrix.

A square matrix whose elements $a_{ij} = 0$ for $i < j$ is called a lower triangle matrix.

For examples,
$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -i & 2 & 3 \\ 0 & -1 & i & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

are upper triangular matrix.

and,
$$\begin{bmatrix} 5 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 7 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ i & 2 & 0 & 0 \\ -2 & 5 & -1 & 0 \\ 3 & 7 & -i & 6 \end{bmatrix}$$

are lower triangular matrix.

Symmetric matrix : A matrix equal to it's transpose a square matrix such that $a_{ij} = a_{ji}$ for $1 \le i, j \le n$ is said to be symmetric.

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In short, we can say a square matrix A will be symmetric if $A^T = A$.

For examples,

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \\ -3 & 7 & 3 \end{bmatrix}$$

are symmetric matrix.

λA and λB are also symmetric if λ is scalar.

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~~Def~~ Defination of matrix:

A matrix is a rectangular array of number closed by a pair of matrix brackets and the number in the array are called the entries or the elements of matrix, that is rectangular array is called a matrix.

Example of matrix

$$A = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -3 & 7 \end{bmatrix}.$$

Addition and scalar multiplication of matri

Addition

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 7 \\ -5 & 4 \end{bmatrix}.$$

$$A+B = \begin{bmatrix} 1+2 & -2+7 \\ 3-5 & 5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \\ -2 & 9 \end{bmatrix}$$

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Matrix multiplication:

$$A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\cdot 1 + (-3\cdot -2) + (5\cdot 3) & 1\cdot 1 + (-3\cdot 4) + 5\cdot 0 \\ 2\cdot 1 + (0\times -2) + (-1\cdot 3) & (2\cdot -1) + 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 6 + 15 & -1 - 12 \\ 2 - 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -13 \\ -1 & -2 \end{bmatrix} \text{(Ans)}$$

<u>Transpose matrix:</u> A^T If A is an $(m \times n)$ matrix over the real field $\mathbb{R}$, then the $(n \times m)$ matrix obtained from the matrix A by writing its rows as columns and its columns as rows is called the transpose of A and is denoted by the symbol A^T.

Let $A = \begin{bmatrix} 1 & 0 & 5 & -7 \\ 2 & 3 & -1 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 5 & -1 \\ -7 & 6 \end{bmatrix}$$

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Definition of Real matrix:

A matrix A is called real provided it satisfies the relation $A = \bar{A}$

For ex:

$$A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & 5 \end{bmatrix}$$

then

$$\bar{A} = \begin{bmatrix} 1 & -i & 1-i \\ i & 2 & 2-3i \\ 5 & 1-2i & 5 \end{bmatrix}$$

Def imaginary matrix:

A matrix A is called imaginary provided it satisfies. $A = -\bar{A}$

* conjugate transpose of a complex matrix

The conjugate of the transpose (matrix) of a given complex matrix.

A is said to be conjugate transpose matrix of $\bar{A}$ and is generally denoted by A^*

$$A^* = \bar{A}^T$$

Example

$$A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{bmatrix}$$

$$\Rightarrow \bar{A} = \begin{bmatrix} 1 & -i & 1-i \\ i & 2 & 2-3i \\ 5 & 1-2i & -5 \end{bmatrix}$$

$$\Rightarrow \bar{A}^T = \begin{bmatrix} 1 & i & 5 \\ -i & 2 & 1-2i \\ 1-i & 2-3i & -5 \end{bmatrix}$$

Square matrix:

A matrix with the same number of rows and coloumn is called square matrix. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Rectangular matrix: The number of rows and columns of a matrix not be equal.

$$\underline{ex:} \; A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Diagonal mat: A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a diagonal matrix.

$$\underline{ex} \; A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Scalar mat: A diagonal matrix whose diagonal elements are all equal is called a scalar matrix.

$$\underline{ex:} \; A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Zero matrix: When all elements are zero its called zero or /null matrix.

upper triangle $a_{ij} = 0$ for $i > j$ its called upper and $a_{ij} = 0$ for $i < j$ its called lower triangle.

Orthogonal matrix:

$$A \cdot A^T = A^T A = I$$

Idempotent matrix: A square matrix A is called an idempotent matrix if $A^2 = A$

Involutory matrix: If $A^2 = I$

Worked out examples.

① $A = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$

Find 2A, A+B, A-B

$$2A = \begin{bmatrix} 2 & -4 & 3\cdot2 \\ 5\cdot2 & 1\cdot2 & -4\cdot2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 6 \\ 10 & 2 & -8 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+2 & -2+3 & 3+5 \\ 5+1 & 1+4 & -4-2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 8 \\ 6 & 5 & -6 \end{bmatrix}$$

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Skew hermation matrix:

if $A=[a_{ij}]$ is a square matrix over the complex field and $A^{*}=\bar{A}^{T}=-A$ i.e $a_{ij}=-\bar{a}_{ij}$ for $i,j=1,2,\dots,n$ then A is called a skew hermation matrix

for example

$$A=\begin{bmatrix} 2i & 2-3i & 3 \\ -2-3i & 5i & 1+i \\ -3 & -1+i & 0 \end{bmatrix}$$

Orthogonal matrix. A real square matrix A is said to be orthogonal matrix if $AA^{T}=A^{T}A=I$.

Idempotent matrix: A square matrix is called an Idempotent matrix. if $A^{2}=A$.

Nilpotent matrix: A square matrix A is called a nilpotent matrix of order m if $A^m = 0$ and $A^{m-1} \neq 0$ where m is positive integer and 0 is the null matrix.

Involutory matrix: A square matrix A is is called an involutory matrix if $A^2 = I$.

Matrix operation

example:

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$

$A+B = \begin{bmatrix} 3 & 1 & 8 \\ 6 & 5 & -6 \end{bmatrix}$

$A-B = \begin{bmatrix} -1 & -5 & -2 \\ 4 & -3 & -2 \end{bmatrix}$

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Example-3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 2 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\cdot1+2\cdot1+3\cdot2 & 2+6+3 & -1+4-3 & 3+10+9 \\ 2+3+10 & 4+9+5 & -2+6-5 & 6+15+15 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 11 & 0 & 22 \\ 15 & 18 & -1 & 36 \end{bmatrix}.$$

Find the inverse matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

let $D = 4-6 = -2 \neq 0$

So the matrix A is non-singular matrix. and hence A^{-1} exists.

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$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A_{11} = 4$ $A_{21} = 2$

$A_{12} = 3$

inverse matrix.

$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ Let A^{-1}

$D = |A| = -1(5-0) - 2(10-0) - 3(-4-4)$

$= -5 - 20 + 24$

$= -1$

Hence

$A_{11} = \begin{bmatrix} 1 & 0 \\ -2 & 5 \end{bmatrix} = 5$ $A_{21} = (-1) \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}$

$A_{12} = (-1) \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix} = -10$ $= -4$

$A_{13} = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix} = -8$

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$A_{22} = \begin{bmatrix} -1 & -3 \\ 4 & 5 \end{bmatrix} = -5+12 = 7$

$A_{23} = (-1)\begin{bmatrix} -1 & 2 \\ 4 & -2 \end{bmatrix} = -(2-8) = 6$

$A_{31} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} = 3$

$A_{32} = (-1)\begin{bmatrix} -1 & -3 \\ 2 & 0 \end{bmatrix} = -6$

$A_{33} = \begin{bmatrix} -1 & 2 \\ 2 & 9 \end{bmatrix} = -5$

$$\begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$A^{-1} = \frac{1}{D} \text{ ADJ } A$

$= \frac{1}{-1} \text{ ADJ } A$

$= \begin{bmatrix} -5 & 10 & -8 \\ & -7 & -6 \\ -4 & & \end{bmatrix}$

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⊞ Inverse Matrix: A square matrix A is said to be invertible if there exists a unique matrix B such that $AB = BA = I$ where I is the unit matrix. We call such that B the inverse of A & is generally denoted by A^{-1}. Here we have to note that if B is the inverse of A, then A is the inverse of B.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Therefore, A and B are invertible and are ~~...~~ inverse of each other. That is, $A^{-1} = B$ and $B^{-1} = A$.

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□ Process of finding the inverse of a square matrix

Let the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{23} \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$

Let D be the determinant of the matrix A.
Evaluate the determinant D; D=0 the matrix A is singular and it has no inverse, if $D \neq 0$ the matrix A is non-singular and A^{-1} exists.
Find the adjoint matrix Adj A of the matrix A

then,
$$A^{-1} = \frac{1}{D} \text{Adj}(A)$$
$$= \frac{\text{Adj}(A)}{|A|}$$

$$A^{-1} = \begin{bmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \cdots & \frac{A_{n1}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \cdots & \frac{A_{n2}}{|A|} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{A_{1n}}{|A|} & \frac{A_{2n}}{|A|} & & \frac{A_{nn}}{|A|} \end{bmatrix}$$

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Example 12(a): Find the inverse of matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution: Let D be the determinant of matrix A; then.

$$D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= 4 - 6$$

$$= -2$$

Here $D \neq 0$. so the matrix A is non-singular and hence A^{-1} exists.

Now the co-factor of D are

$A_{11} = 4$, $A_{12} = -3$, $A_{21} = -2$, $A_{22} = 1$

Then $\text{Adj } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T$

$$= \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} \text{Adj } A$$

$$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Ans.

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Example 12 (b): Find the inverse of matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{bmatrix}$$

Soln: Let D be the determenant of matrix A,

then $D = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix}$

$$= 2(0+3) + 1(8+3) + 3(12-0)$$

$$= 6 + 11 + 36$$

$$= 53$$

Here, $D \neq 0$ so the matrix A is non-singular and A^{-1} exists. Now the co-factors D are,

$A_{11} = 3$ $\quad A_{12} = -11$ $\quad A_{13} = 12$

$A_{21} = 11$ $\quad A_{22} = -5$ $\quad A_{23} = -9$

$A_{31} = 1$ $\quad A_{32} = 14$ $\quad A_{33} = +4$

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Therefore, $\text{Adj } A = \begin{bmatrix} 3 & -11 & 12 \\ 11 & -5 & -9 \\ 1 & 14 & 4 \end{bmatrix}^T$

$= \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}^T$

$\therefore A^{-1} = \frac{1}{D} \text{Adj } A$

$= \frac{1}{53} \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$

Ans

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Example 13: Find the inverse matrix of $A=\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$ by using row canonical form.

Soln: $[AI_2] = \left[\begin{array}{cc:cc}2 & 5 & 1 & 0\\1 & 3 & 0 & 1\end{array}\right]$ Interchange first and second row.

$\sim \left[\begin{array}{cc:cc}1 & 3 & 0 & 1\\2 & 5 & 1 & 0\end{array}\right]$ We multiply first row by 2 and then subtract from the second rows.

$\sim \left[\begin{array}{cc:cc}1 & 3 & 0 & 1\\0 & -1 & 1 & -2\end{array}\right]$ we multiply second row by 3 then Add with first row

$\sim \left[\begin{array}{cc:cc}1 & 0 & 3 & -5\\0 & -1 & 1 & -2\end{array}\right]$ we multiply second row by (-1)

$\sim \left[\begin{array}{cc:cc}1 & 0 & 3 & -5\\0 & 1 & -1 & 2\end{array}\right]$

$= [I_2 A^{-1}]$

Hence A is invertible and $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$

Ans.

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Find the inverse matrix of $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

Solⁿ: Let D be the determinant of matrix A.

Then $D = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix}$

$= -1(5+0) - 2(10-0) - 3(-4-4)$

$= -5 - \cancel{0}20 + 24$

$= -25 + 24$

$= -1$

Here $D \neq 0$ so the matrix A is non-singular and A^{-1} exsists.

Now the co-factors are,

$A_{11} = 5 \quad A_{12} = -10 \quad A_{13} = -8$

$A_{21} = -4 \quad A_{22} = 7 \quad A_{23} = 6$

$A_{31} = 3 \quad A_{32} = -6 \quad A_{33} = -5$

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$$\text{Adj } A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} \text{ Adj}(A)$$

$$= \frac{1}{-1} \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

Ans.

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Example 15: Find the inverse matrix of

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$ by using canonical form.

Soln: $$[A I_3] = \left[\begin{array}{ccc:ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$ Interchange first and second rows.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$ We multiply first row by 3 and 2 and then subtract from the second and third rows respectively.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ subtract third row from second row.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ Multiply the second row by (-1).

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$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ we multiply second row by 5 and then subtract from the third row

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array}\right]$$ we multiply third row by $(-1/10)$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -5/10 & 7/10 & 4/10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{-1}{2} & 7/10 & 2/5 \end{array}\right]$$ we multiply third row by 3 and subtract from the first row.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3/2 & -11/10 & -6/5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right]$$

$= I_3 A^{-1}$

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$$\therefore A^{-1} = \begin{bmatrix} 3/2 & -\frac{11}{10} & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{bmatrix}$$

Ans.

Example 17: Solve the following linear equations with the help of matrix: $\left.\begin{aligned} 2x+y&=1 \\ x-2y&=3 \end{aligned}\right\} \cdots\cdots (1)$

Sol^n: The system of linear equations can be written in matrix - form as:

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdots\cdots (2)$$

Let, $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ then from

(2) we get $AX = b \cdots (3)$

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Let D be the determinant of matrix A then

$$D = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4-1 = -5 \neq 0.$$

So the matrix A is non-singular and A^{-1} exists.

Now the co-factors of D are

$A_{11} = -2$, $A_{12} = -1$

$A_{21} = -1$, $A_{22} = 2$

Therefore, $\text{Adj } A = -\frac{1}{5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2/5 & \frac{1}{5} \\ 1/5 & -2/5 \end{bmatrix}$

We multiply both sides of equation (3) by A^{-1}

$A^{-1}AX = A^{-1}L$

or, $IX = A^{-1}L$ $\left\{ \begin{array}{l} \text{since, } A^{-1}A = I \\ \text{and } IX = X \end{array} \right.$

or, $X = A^{-1}L$

Thus $X = \begin{bmatrix} 2/5 & \frac{1}{5} \\ 1/5 & -2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/5 & +\frac{3}{5} \\ 1/5 & -6/5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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or. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence $\left.\begin{matrix} x = 1 \\ y = -1 \end{matrix}\right\}$ Ans.

□ The augmented of matrix of the given system of linear equations is.

$$[AL] = \left[\begin{array}{cc:c} 2 & 1 & 1 \\ 1 & -2 & 3 \end{array}\right]$$ Interchange first and second rows.

$$\sim \left[\begin{array}{cc:c} 1 & -2 & 3 \\ 2 & 1 & 1 \end{array}\right]$$ We multiply first row by (2) and then subtract from the second row.

$$\sim \left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & 5 & -5 \end{array}\right]$$ we multiply second row by $1/5$.

$$\sim \left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

Now the system is in row canonical form.
Then forming linear system we have $y = -1$
and $x - 2y = -1$
$\therefore x = 2y + 3 \Rightarrow x = -2 + 3 = 1$ Ans.

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Example 18: Solve the linear equations with the help of matrixs.

$$3x + 5y - 7z = 13$$

$$4x + y - 12x = 6$$

$$2x + 9y - 3z = 20$$

Soln: The given linear equations can be written in matrix-form as.

$$\begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix} \quad \text{--- (1)}$$

and ~~Adj~~ $A^{-1} = \frac{1}{D} \text{Adj } A$

$$= \frac{1}{17}\begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}$$

Now from equation no (3) we get.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}\begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 107 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix} \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 1365 & -288 & -1060 \\ -156 & +30 & +160 \\ 442 & -102 & -340 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 1365 & -1348 \\ -156 & +190 \\ 442 & -442 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 17 \\ 34 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Hence $\left.\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 0 \end{aligned}\right\}$ Ans.

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$\boxed{\text{Exercises}}$

Given that,

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Let D be the determinant of matrix A.

Then, $D = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

$= 1(0-0) - 1(0-0) + 1(0+1)$

$= 1$

Here, $D \neq 0$, so the matrix is non-singualar and A^{-1} exists.

Now the co-factors of D are,

$A_{11} = 0$, $A_{12} = 0$, $A_{13} = 1$

$A_{21} = 0$, $A_{22} = -1$, $A_{23} = \cancel{3} -1$

$A_{31} = 1$, $A_{32} = 2$, $A_{33} = 1$

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Now, $\text{Adj } A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}^T$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$\therefore \; A^{-1} = \frac{1}{D} \text{Adj } A$

$$= \frac{1}{1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

Ans

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$\boxed{22}$ Given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Let D be the determinant of matrix A.

Then, $$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{vmatrix}$$

$$= 1(40-0) - 2(16-3) + 3(0-5)$$

$$= 40 - 26 - 15$$

$$= -1$$

Here, $D \neq 0$, so the matrix A is non-singular and A^{-1} exists. Now the co-factors of D are

$A_{11} = 40$ $\quad A_{12} = -16$ $\quad A_{13} = -5$

$A_{21} = -16$ $\quad A_{22} = 5$ $\quad A_{23} = 2$

$A_{31} = -9$ $\quad A_{32} = -3$ $\quad A_{33} = 1$

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Now Adj A = $\begin{bmatrix} 40 & -16 & -5 \\ -16 & 5 & 2 \\ -9 & -3 & 1 \end{bmatrix}^T$

$$= \begin{bmatrix} 40 & -16 & -9 \\ -16 & 5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{-1} \begin{bmatrix} 40 & -16 & -9 \\ -16 & 5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -40 & 16 & 9 \\ 16 & -5 & 3 \\ 5 & -2 & -1 \end{bmatrix}$$

Ans.

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Given that,

$$5x - 6y + 4z = 15$$
$$7x + 4y - 3z = 19$$
$$2x + y + 6z = 46$$

Let D be the determinant of co-efficient of x, y and z. Then,

$$D = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}$$

$$= 5(24+3) + 6(42+6) + 4(7-8)$$

$$= 135 + 288 - 4$$

$$= 419$$

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$$D_x = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 9 & -3 \\ 46 & 1 & 6 \end{vmatrix}$$

$$= 15(24+3) + 6(114 + 138) + 4(19 - 184)$$

$$= 405 + 1512 - 660$$

$$= 1257$$

$$D_y = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix}$$

$$= 5(114 + 138) - 15(\overset{42}{\cancel{38}} + 6) + 4(322 - 38)$$

$$= 1260 - 0720 + 1136$$

$$\cancel{= 135 - 514}$$

$$= 1676$$

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$$D_z = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix}$$

$$= 5(184 - 19) + 6(322 - 38) + 15(7 - 8)$$

$$= 825 + 1704 - 15$$

$$= 2514$$

$$\therefore x = \frac{D_x}{D} = \frac{1257}{419} = 3$$

$$\therefore y = \frac{D_y}{D} = \frac{1676}{419} = 4$$

$$\therefore z = \frac{D_z}{D} = \frac{2514}{419} = 6$$

Hence, $(x, y, z) = (3, 4, 6)$

Ans.

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MAT205 Assignment-01

Example-11 : Show that the matrix $\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory

Proof : $A^2 = A \times A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$\therefore A^2 = I$

Hence the given matrix A is involutory.

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Example-12(a): Find the inverse of Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution: let D be the determinant of the matrix, then

$$D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$$

So the matrix is non-singular and hence A^{-1} exists.

Now the cofactors of D are

$A_{11} = 4 \quad A_{12} = -3$

$A_{21} = -2 \quad A_{22} = 1$

Then $Adj\, A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^{t} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{D} Adj\, A$$

$$= \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

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Example 12(b): Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{bmatrix}$

Solution: Let D be the determinant of the matrix;

then $D = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix}$

$= 2(0+3) + 1(8+3) + 3(12-0)$

$= 6 + 11 + 36$

$= 53$

$\neq 0$

So the matrix A is non-singular and A^{-1} exists.

Now the cofactors of D are

$A_{11} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3$, $A_{12} = (-1)\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix} = -11$, $A_{13} = \begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix} = 12$

$A_{21} = (-1)\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} = 11$, $A_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$, $A_{23} = (-1)\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix} = -9$

$A_{31} = \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} = 1$, $A_{32} = (-1)\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = 14$, $A_{33} = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = 4$

Therefore, $Adj\,A = \begin{bmatrix} 3 & -11 & 12 \\ 11 & -5 & -9 \\ 1 & 14 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{D} Adj\,A = \frac{1}{53}\begin{bmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{bmatrix}$

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Example-13: Find the ~~matrix~~ inverse of the matrix $A=\begin{bmatrix}2&5\\1&3\end{bmatrix}$

b by using row canonical form.

<u>Solution:</u> $[AI_2] = \left[\begin{array}{cc:cc}2&5&1&0\\1&3&0&1\end{array}\right]$ Interchange first and second rows

$\sim \left[\begin{array}{cc:cc}1&3&0&1\\2&5&1&0\end{array}\right]$ We multiply first row by 2 an ~~second~~ then subtract from the second row.

$\sim \left[\begin{array}{cc:cc}1&3&0&1\\0&-1&1&-2\end{array}\right]$ We multiply second row by and then add with the first row

$\sim \left[\begin{array}{cc:cc}1&0&3&-5\\0&-1&1&-2\end{array}\right]$ We multiply second row by (-

$\sim \left[\begin{array}{cc:cc}1&0&3&-5\\0&1&-1&2\end{array}\right] = [I_2 A^{-1}]$

Hence A is ~~inverted~~ invertible and $A^{-1} = \begin{bmatrix}3&-5\\-1&2\end{bmatrix}$

Example-14: If $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ find $A^{-1}B$

Solution: Given $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

let $D = |A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix}$

$$= -1(5+0) - 2(10-0) - 3(-4-4)$$
$$= -5 - 20 + 24$$
$$= -1$$
$$\neq 0$$

So A is non-singular and hence A^{-1} exists

Cofactors of D are,

$A_{11} = \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 5$, $A_{12} = (-1)\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = -10$, $A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -8$

$A_{21} = (-1)\begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} = -4$, $A_{22} = \begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} = 7$, $A_{23} = (-1)\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} = 6$

$A_{31} = \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 3$, $A_{32} = (-1)\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -6$, $A_{33} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5$

$$\therefore \text{Adj}\,A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D}\text{Adj}\,A = \frac{1}{-1}\begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

$$\text{Thus } A^{-1}B = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -10+0-15 & -5+8-6 & 5+4+9 \\ 20+0+30 & 10-14+12 & -10-7-18 \\ 16+0+25 & 8-12+10 & -8-6-15 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & -3 & 18 \\ 50 & 8 & -35 \\ 41 & 6 & -29 \end{bmatrix}$$

Example-15: Find the inverse of the following matrix by using row canonical form: $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$

Solution:

$$[AI_3] = \left[\begin{array}{ccc:ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$ Interchange first and second rows.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$ We multiply first row by 3 and 2 and then subtract from the second row and third rows respectively.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ Subtract third row from the second row.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & -1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ Multiply the second row by (-1).

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$ We multiply second row by 5 and then subtract from the third row.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array}\right]$$ we multiply third row by $\left(-\frac{1}{10}\right)$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{5}{10} & \frac{7}{10} & \frac{4}{10} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{5}{10} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$ we multiply third row by 3 and then subtract from the first row.

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

$$= \left[I_3 \; A^{-1} \right]$$

Hence A is invertible and $A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$

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Example-16: Find the inverse of the matrix $A=\begin{bmatrix}1 & -1 & 0 & 0\\ 1 & 2 & 0 & 0\\ -6 & 0 & 1 & -2\\ 8 & 1 & -2 & 1\end{bmatrix}$ by using only row transformations to reduce A to I.

Solution:

$$[AI_4]=\left[\begin{array}{cccc:cccc}1 & -1 & 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0\\ -6 & 0 & 1 & -2 & 0 & 0 & 1 & 0\\ 8 & 1 & -2 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

We subtract first row from second row. We multiply first row by 6 and then add with the third row. Also we multiply first row by 8 and then subtract from the fourth row.

$$\sim\left[\begin{array}{cccc:cccc}1 & -1 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0\\ 0 & -6 & 1 & -2 & 6 & 0 & 1 & 0\\ 0 & 9 & -2 & 1 & -8 & 0 & 0 & 1\end{array}\right]$$

We multiply second row by 2 and add with the third row. We also multiply second row by 3 and then subtract from the fourth row.

$$\sim \left[\begin{array}{cccc:cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -5 & -3 & 0 & 1 \end{array}\right]$$

We multiply second row by $\frac{1}{3}$

$$\sim \left[\begin{array}{cccc:cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -5 & -3 & 0 & 1 \end{array}\right]$$

We add second row with the first row.

$$\sim \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 & -5 & -3 & 0 & 1 \end{array}\right]$$

We multiply third row by 2 and then add with the fourth row.

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 3 & 1 & 2 & 1 \end{array}\right]$$

We multiply ~~f~~ fourth row by $\left(-\frac{1}{3}\right)$

$$\sim\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right]$$

We multiply fourth row by 2 and then add with the third row.

$$\sim\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & -1 & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

$$=[I_4\ A^{-1}]$$

Hence A is invertible and

$$A^{-1}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -1 & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Example-17: Solve the following linear equations with the help of matrix:

$$\left.\begin{aligned} 2x+y&=1 \\ x-2y&=3 \end{aligned}\right\}$$

<u>Solution</u>: The augmented matrix of the given system of linear equations is

$$[AL] = \left[\begin{array}{cc:c} 2 & 1 & 1 \\ 1 & -2 & 3 \end{array}\right]$$

Interchange first and second rows.

$$\sim \left[\begin{array}{cc:c} 1 & -2 & 3 \\ 2 & 1 & 1 \end{array}\right]$$

we multiply first row by 2 and then subtract from the second row.

$$\sim \left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & 5 & -5 \end{array}\right]$$

we multiply second row by $\frac{1}{5}$

$$\sim \left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

Now the system is in row canonical form.

Then forming linear system we have $y=-1$ and

$x-2y=3$

বা, $x = 2y+3$

বা, $x = -2+3$

$\therefore x = 1$

Thus the required solution of the system is $x=1$ and $y=-1$.

Example-18: Solve the following linear equations with the the of matrices
$$\left.\begin{aligned}3x+5y-7z&=13\\4x+y-12z&=6\\2x+9-3z&=20\end{aligned}\right\}$$

Solution: The augmented matrix of the given linear equation is

$$[AL]=\left[\begin{array}{ccc:c}3&5&-7&13\\4&1&-12&6\\2&9&-3&20\end{array}\right]$$

We subtract third row from the first row. Also we multiply third row by 2 and then subtract from the third row.

$$\sim\left[\begin{array}{ccc:c}1&-4&-4&-7\\0&-17&-6&-34\\2&9&-3&20\end{array}\right]$$

We multiply first row by 2 and then subtract from the third row.

$$\sim\left[\begin{array}{ccc:c}1&-4&-4&-7\\0&-17&-6&-34\\0&17&5&34\end{array}\right]$$

We add second row with third row

$$\sim\begin{bmatrix} 1 & -4 & -4 & : & -7 \\ 0 & -17 & -6 & : & -34 \\ 0 & 0 & -1 & : & 0 \end{bmatrix}$$

We multiply second row by $\left(-\frac{1}{17}\right)$ and third row by (-1).

$$\begin{bmatrix} 1 & -4 & -4 & : & -7 \\ 0 & 1 & \frac{6}{17} & : & 2 \\ 0 & 0 & 1 & : & 0 \end{bmatrix}$$

Now the system is row canonical form. Then forming the linear system, we have $z=0$

$$y+\frac{6}{17}z=2, \quad x-4y-4z=-7$$

or, $z=0$, $y=2$, $x=8-7=1$

Thus $x=1, y=2, z=0$ is a solution of the given equations.

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Problem: Determine the rank of matrix B.

$$\text{where } B = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

Solution: Reduce the given matrix to echelon form by means of elementary row transformations and Join successive matrices by the equivalence sign~ Inehange first row and fourth row.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

(ii) we multiply second row by 2 and then subtract from with the the third row. Also we add second row from fourth row. (i) Before we subtracted first row from the second row and multiply the first row by 3 and then subtracted from the third row.

i)

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

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ii)

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Second row is multiplied by -1.

this matrix now is equivalent to the given matrix B and is in the row echelon form. Since the echelon matrix has two non-zero rows. so the rank of given matrix is 2.

Problem : Find the rank of the matrix.

$$A = \begin{bmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & 1 & 6 & 10 \end{bmatrix}$$

Solution : Since the given matrix is A of order 3×4 the rank of givenl matrix can't be greatherthan 3 Now we ~~are~~ observe that the matrix A has the following largest square submatrices of order 3×3

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$$A_1 = \begin{bmatrix} 6 & 2 & 0 \\ -2 & -1 & 3 \\ -1 & -1 & 6 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -1 & 4 \\ -1 & -1 & 10 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 6 & 0 & 4 \\ -2 & 3 & 4 \\ 1 & 6 & 10 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ 1 & 6 & 10 \end{bmatrix}$$

Now we get. $|A_1| = 6(-6+3) - 2(-12+3) + 0 = 0$

$|A_2| = 6(-10+4) - 2(-20+4) + 4(2-1) = 0$

$|A_3| = 6(30-24) + 0 + 4(-12+3) = 0$

$|A_4| = 2(30-24) + 0 + 4(-6+3) = 0$

So the rank of matrix can't be 3. Let us consider the square of submatrices of order 2×2.

we get $\begin{bmatrix} 6 & 2 \\ -2 & -1 \end{bmatrix} = -6+4 = -2 \neq 0$

therefore the rank of given matrix is 2.

Matrix operations : Matrix operations mainly involve three algebraic operations which are addition of matrices, subtraction of matrices and multiplication of matrices. Matrix is a rectangular array of number or expressions of matrices can be found in mathmatics.

Addition subtraction and multiplication are the basic operations of matrices, these must be of identical order and for multiplication, the sub number of colums in the first matrix equals the number of rows in the second matrix.

Addition of matrices

If $A[a_{ij}]_{m \times n}$ and $B[b_{ij}]_{m \times n}$ are two matrices of the same order then their sum $A+B$ is a matrix and each element of that matrix is the sum of the corresponding elements. i.e $A+B = [a_{ij}+b_{ij}]_{m}$

Properties of matrix addition : if A, B and c are matrices of same order, then

a) Communicative law $A+B=B+A$

b) Associative Law : $(A+B)+C = A+(B+C)$

c) Identity of the matrix : $A+O = O+A = A$, where O is zero matrix which is additive identity of the matrix.

d) Additive inverse : $A+(-A) = O = (-A)+A$, where $(-A)$ is obtained by changing of the sign of every element of A which is additive inverse of matrix.

e) $\left.\begin{array}{l} A+B = A+C \\ B+A = C+A \end{array}\right\} \Rightarrow B = C$

f) $tr(A \pm B) = tr(A) \pm tr(B)$

g) If $A+B = O = B+A$. then B is called additive inverse of A and also A is called the additive inverse of A.

Subtraction of matrices $A-B = A+(-B)$.

$$A-B = [a_{ij} - b_{ij}]_{m\times n}$$

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Multiplication of matrices :

If $A = [a_{ij}]_{m\times n}$. and $B = [b_{ij}]_{n\times p}$

$\therefore$ $AB = C = [c_{ij}]_{m\times p}$ willbe a matrix of order $m\times p$ where $(AB)_{ij} = c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$

Properties of matrix multiplication

a) $AB \neq BA$

b) $(AB)C = A(BC)$

c) $A\cdot(B+C) = A\cdot B + A\cdot C$ and $(A+B)C = AC + BC$

D) $I_m A = A = A I_n$

f) If O is a null matrix then $A_{m\times n} \cdot O_{n\times p} = O_{m\times p}$

g) If $AB = AC$, $B \neq C$

h) $tr(AB) = tr(BA)$ and $IA = AI = A$

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Problem : Solve the following linear equations with the help of matrices :

$$\left.\begin{aligned} 3x+5y-7z&=13 \\ 4x+y-12x&=6 \\ 2x+9y-3z&=20 \end{aligned}\right\}$$

Solution - the given linear equations can be written in matrix form as.

$$\begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix} \quad \text{——— (i)}$$

Suppose that $A = \begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $L = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$

then the equation given by (1) reduces to $AX = L$ -(ii)

Let D the determinant of the matrix A, then

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$$D = \begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix} = 3(-3+108) - 5(-12+24) - 7(36-2)$$

$$= 315 - 60 - 238 = 17 \neq 0$$

So the matrix A is non-singular and hence A^{-1} exists

we multiply both sides of equation no (ii) by A^{-1} on the left.

we get $A^{-1} A X = A^{-1} L$

or, $I X = A^{-1} L$

$\therefore X = A^{-1} L$ —— (iii)

Now the cofactors of D are,

$$A_{11} = \begin{vmatrix} 1 & -12 \\ 9 & -3 \end{vmatrix} = 105$$

$A_{12} = -12$, $A_{13} = 34$, ~~$A_{14} =$~~

$A_{21} = -48$, $A_{22} = 5$, $A_{23} = -17$.

$A_{31} = -53$ $A_{32} = 8$ $A_{33} = -17$

therefore Adj A $= \begin{bmatrix} 105 & -12 & 34 \\ -48 & 5 & -17 \\ -53 & 8 & -17 \end{bmatrix}^T = \begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & 17 & -17 \end{bmatrix}$

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now, $A^{-1} = \frac{1}{D} \text{ Adj } A = \frac{1}{12} \begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 39 & 12 & -12 \end{bmatrix}$

from equation (iii) we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 105 & -98 & -53 \\ -12 & 5 & 8 \\ 39 & 12 & -12 \end{bmatrix} \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 1365 - 288 - 1060 \\ -156 + 30 + 160 \\ 442 - 102 - 390 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 1365 - 1398 \\ -156 + 190 \\ 442 - 442 \end{bmatrix}$$

and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 12 \\ 39 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

So, we get $x = 1$

$y = 2$

and $z = 0$

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Alternative Process: the agumented matrix of the given linear equation is

$$[AL] = \left[\begin{array}{ccc:c} 3 & 5 & -7 & 13 \\ 4 & 1 & -12 & 6 \\ 2 & 9 & -3 & 20 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & -4 & -4 & -7 \\ 0 & -12 & -6 & -34 \\ 2 & 9 & -3 & 20 \end{array}\right] \quad \begin{array}{l} r_3 - r_1 = r_1' \\ 2r_3 - r_2 = r_2' \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -4 & -4 & -7 \\ 0 & -12 & -6 & -34 \\ 0 & 12 & -5 & 34 \end{array}\right] \quad 2r_1 - r_3 = r_1''$$

$$\sim \left[\begin{array}{ccc:c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 0 & 0 & -1 & 0 \end{array}\right] \quad r_2 + r_3 = r_3' \quad \sim \left[\begin{array}{ccc:c} 1 & -4 & -4 & -7 \\ 0 & 1 & \frac{6}{12} & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$(-1/12)\, r_2$ & $(-1) r_3$

Suppose that, $A = \begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $L = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$

by (i) $AX = L$, Now the system is in row canonical form, then forming the linear system, we have $z = 0$

$y + \frac{6}{12} z = 2$, $x - 4y - 4z = -7$

So, we get $y = 2$, $x = 1$ and $z = 0$

Definition of Matrix

~~Statement:~~ **Statement:**

Matrix refers to an ordered rectangular arrangement of numbers which are either real or complex or functions. We enclose Matrix by [ ] or ( ).

such as,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

The numbers $a_{11}, a_{12}, \dots \dots a_{mn}$ are called the entries or elements of the matrix. The matrix of m rows and n colums is said to be of order "m by n" or $m \times n$.

The above matrix is also denoted by $[a_{ij}]$, $i = 1, 2, \dots m.$ $j = 1, 2, \dots n.$

Many kinds of Matrix:

(i) Real Matrix.
(ii) Imaginary Matrix.
(iii) Rectangular matrix.
(iv) Digonal matrix.
(v) Identity matrix.
(vi) Square matrix.
(vii) Zero matrix/null matrix.
(viii) Symmetric matrix.
(ix) Upper/Lower triangular matrix.
(x) Orthogonal matrix.
(xi) Hermitian matrix.
(xii) Idempotent matrix., etc.
(xiii) Inverse matrix.

Real matrix

definition:

A matrix A is called real provided it satisfies the relation,, $A = \bar{A}$

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Inverse matrix

definition:

A square matrix A is called to be inventiable if there exists a unique matrix B such that $AB = BA = I$. where, I is the unit matrix. → We call such a matrix B the inverse of A & is generally denoted by A^{-1}.

Here we have to note that if B is the inverse of A, then A is the inverse of B.

Example: 1

Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Then, $AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

and

$BA = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -2+2 \\ -6+6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore A^{-1} = B$ and $B^{-1} = A$.

Imaginary matrix

definition: A matrix A is called imaginary provided it satisfies the relation, $A = -\bar{A}$.

Diagonal matrix

definition: A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a diagonal matrix.

For examples: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 2 \\ 4 & 2 & 0 \end{bmatrix}$

→ A ~~diagonal~~ diagonal matrix whose diagonal elements are all equal is called a scalar matrix.

~~Reg~~ Rectangular

definition: The number of rows and columns of matrix need not be equal. when, $m \neq n$ i.e the number of rows and columns of the array are not equal, then the matrix is known as the rectangular matrix.

For examples: $\begin{bmatrix} 1 & 2 & -4 & 0 \\ 2 & 3 & 5 & 10 \end{bmatrix}_{4\times 2}$

and $\begin{bmatrix} 10 & 5 & -7 & -\frac{1}{2} \\ 4 & 3 & 0 & 0 \end{bmatrix}_{4\times 2}$

Zero/null matrix

definition: A matrix in which every elements is zero ~~or~~ is called a nu matrix or a zero matrix.

For examples: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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Symmetric matrix

definition: A matrix equal to its transpose i.e a square matrix such that $a_{ij} = a_{ji}$ for $1 \leq i; j \leq n$ is said to be symmetric

In short we can say a square matrix A will be symmetric if $A^T = A$.

For examples, $A = \begin{bmatrix} a & h & f \\ g & i & b \\ c & d & g \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \end{bmatrix}$

are symmetric matrices.

Idempotent matrix

definition: A square matrix A is called an Idempotent matrix if, $A^2 = A$.

For examples: $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

are idempotent matrices.

Transpose of matrix (7)

definition:

If A is an $m \times n$ matrix over the real field $\mathbb{R}$, then the $n \times m$ matrix obtained from the matrix A by writing its rows as columns and its columns as rows is called the transpose of a matrix.

A is denoted by the symbol, A^T.
That is, if $A = [a_{ij}]$ is an $m \times n$ matrix then $A^T = [a_{ji}]$ is $n \times m$ matrix.

For examples:

let, $A = \begin{bmatrix} 1 & 0 & 5 & -7 \\ 2 & 3 & -1 & 6 \end{bmatrix}$

then, $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 5 & -1 \\ -7 & 6 \end{bmatrix}$

The transpose of matrix A is represented by A' or A^T.

Properties of Transpose of a matrix

To understand the properties of transpose matrix, we will take two matrix A and B which have equal order. Some properties of transpose of a matrix are given below.

(i) **Addition Property of Transpose:**

Transpose of an addition of two matrix A and B obtained will be exactly equal to the sum of transpose of individual matrix A and B.

This means,

$$(A+B)^T = A^T + B^T.$$

ex: if $P = \begin{bmatrix} 2 & -3 & -8 \\ 21 & 6 & -6 \\ 9 & -33 & 19 \end{bmatrix}$

and $Q = \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & [illegible] \end{bmatrix}$

$$P+Q = \begin{bmatrix} 2 & -3 & 8 \\ 21 & 6 & -6 \\ 4 & -33 & 19 \end{bmatrix} + \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & -3+(-29) & 8+(-8) \\ 21+2 & 6+0 & -6+3 \\ 4+17 & -33+15 & 19+4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -32 & 0 \\ 23 & 6 & -3 \\ 21 & -18 & 23 \end{bmatrix}$$

$$(P+Q)^T = \begin{bmatrix} 3 & 23 & 21 \\ -32 & 6 & -18 \\ 0 & -3 & 23 \end{bmatrix}$$

$$P^T + Q^T = \begin{bmatrix} 2 & 21 & 4 \\ -3 & 6 & -33 \\ 8 & -6 & 19 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 17 \\ -29 & 0 & 15 \\ -8 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 23 & 21 \\ -32 & 6 & -18 \\ 0 & -3 & 23 \end{bmatrix}$$

$$= (P+Q)^T$$

So, we can observe that,

$$(P+Q)^T = P^T + Q^T.$$

(ii) Multiplication by constant:

if a matrix is multiplied by a constant and its transpose is taken, then the matrix obtained is equal to transpose of original matrix multiplied by that constant.

That is $(KA)^T = kA^T$, [Here, k is a ~~cons~~ constant].

ex:

if $P = \begin{bmatrix} 2 & 8 & 9 \\ 11 & -15 & -13 \end{bmatrix}_{2\times3}$ and k is a constant, then $(kP)^T = \begin{bmatrix} 2k & 11k \\ 8k & -15k \\ 9k & -13k \end{bmatrix}_{2\times3}$

$\therefore kP^T = k\begin{bmatrix} 2 & 11 \\ 8 & -15 \\ 9 & -13 \end{bmatrix}_{2\times3} = \begin{bmatrix} 2k & 11k \\ 8k & -15k \\ 9k & -13k \end{bmatrix}_{2\times3} = (kP)^T$

We can observe that, $(kP)^T = kP^T$.

(iii) Multiplication Property of Transpose:

Transpose of the product of two matrices is equal to the product of transpose of the two matrices in reverse order.
That is, $(AB)' = B'A'$

ex: $A = \begin{bmatrix} 9 & 8 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$

Let us find $A \times B$.

$\therefore A \times B = \begin{bmatrix} 44 & 18 \\ 5 & 4 \end{bmatrix} \Rightarrow (AB)' = \begin{bmatrix} 44 & 5 \\ 18 & 4 \end{bmatrix}$

$B'A' = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix}$

$= \begin{bmatrix} 44 & 5 \\ 18 & 4 \end{bmatrix} = (AB)'$

$\therefore (AB)' = B'A'$.

But, $A'B' = \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 40 & 9 \\ 26 & 8 \end{bmatrix}$

We can clearly observe from here that, $(AB)' \neq A'B'$.

Rank of a Matrix (13)

☐ Necessary definitions applied in Rank Echelon Matrix.

— An echelon matrix is a matrix which have the property that if in any of it's rows the first element distinct from zero is in the kth position, then in all the ~~foo~~ following rows there are across in the first k positions or equivalently.

— A matrix $A = [a_{ij}]$ is an echelon matrix. A matrix is said to be in echelon form if it satisfies the following two properties.

(i) The first rows k are non-zero; the other rows are zero.

(ii) The first non-zero element in each non-zero row is 1 and it appears in a column two the right of the first non-zero element of any preceding row.

Rank of a Matrix

Necessary definitions applied in Rank Echelon Matrix.

- An echelon matrix is a matrix which have the property that if in any of it's rows the first element distinct from zero is in the kth position, then in all the following rows there are zeros in the first k positions or equivalently.

- A matrix $A = [a_{ij}]$ is an echelon matrix. A matrix is said to be in echelon form if it satisfies the following two properties.

(i) The first rows k are non-zero; the other rows are zero.

(ii) The first non-zero element in each non-zero row is 1 and it appears in a column two the right of the first non-zero element of any preceding row.

- A matrix which is in echelon form and the first non-zero element in each non-zero row is the only non-zero element in it's column is said to be in **reduced echelon form.**

Examples of echelon matrices and matrices of reduced echelon form are given below:

(i)
$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
(Echelon matrix)

(ii)
$$\begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
(Echelon matrix)

(iii)
$$\begin{bmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$
(Reduced echolon form)

(iv)
$$\begin{bmatrix} 0 & 1 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where each * denotes some scalar.

Canonical Matrix

A canonical matrix is one in which all terms not of the principal diagonal are zero, all terms on the principle diagonal are zero or one and all ones precedes all zeros.

For instance $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a canonical matrix.

Equivalent Matrix

Two matrices A and B are called equivalent and is written $A \sim B$ if one can be obtained from the other by means of a finite number of elementary transformations.

If a matrix A is reduced to B by the use of elementary row transformations alone; then B is said to be row equivelently to A and conversly.

Trace of a matrix and Properties of trace

difinition:

Let A be a square matrix of order n

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then the sum of the diagonal elements of A is called the trace of A and is written as $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

$$= \sum_{i=1}^{n} a_{ii}$$

Properties of trace:

(i) If $A = [a_{ij}]_{n \times n}$ and λ is a scalar (real number) then $tr(\lambda A) = \lambda\, tr(A)$.

Proof: $tr(\lambda A) = \sum_{i=1}^{n} \lambda a_{ii} = \lambda \sum_{i=1}^{n} a_{ii} = \cancel{\lambda\, tr(A)}$

$$= \lambda\, tr(A).$$

(2) Let A and B be $n \times n$ matrices.

Then $tr(\alpha A + \beta B = \alpha\, tr(A) + \alpha\, tr(B)$

[where α and β are scalars.

Proof:

$$\alpha\, tr(A) = \alpha \sum_{i=1}^{n} a_{11} = \sum_{i=1}^{n} \alpha a_{11}$$

$$\beta\, tr(B) = \beta \sum_{i=1}^{n} b_{11} = \sum_{i=1}^{n} \beta b_{11}$$

Now, since multiplying a matrix by a scalar number is equivalent to multiplying every element of the matrix by that number, we have,

$$\alpha\, tr(A) + \beta\, tr(B) = \sum_{i=1}^{n} \alpha a_{11} + \sum_{i=1}^{n} \beta b_{11}$$

$$= \sum_{i=1}^{n} (\alpha a_{11} + \beta b_{11})$$

$$= tr(\alpha A + \beta B)$$

For the special case where, $\alpha = \beta = 1$

We have, $tr(A+B) = tr(A) + tr(B)$.

Similarly,

We have, $tr(\alpha A - \beta B) = \alpha\, tr(A) - \beta\, tr(B)$.

(3)

if A and B are matrices such that AB and BA co-exist then $tr(AB) = tr(BA)$.

Proof:

Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times m}$.

such that AB and BA co-exist.

L.H.S. $tr(AB) = tr\left\{ [a_{ij}]_{m\times n} [b_{ij}]_{n\times m} \right\}$

$$= tr\left\{ \sum_{k=1}^{n} [a_{ik} b_{kj}]_{m\times m} \right\}$$

$$= tr\,[c_{ij}]_{m\times m} \quad \Big| \text{ where, } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{i=1}^{m} c_{ii} = \sum_{i=1}^{m} \left[\sum_{k=1}^{n} a_{ik} \cdot b_{ki} \right]$$

$$= \sum_{i=1}^{m} \left[a_{i1} b_{1i} + a_{i2} b_{2i} + \cdots + a_{in} b_{ni} \right]$$

↓

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R.H.S, $= tr(BA) = tr[b_{ij}]_{n\times m} [a_{ij}]_{m\times n}$

$= tr\left\{ \sum_{k=1}^{m} [b_{ik}\, a_{kj}]_{n\times n} \right\}$

$= tr[d_{ij}]_{n\times m}$ $\boxed{\text{where } d_{ij} = \sum_{k=1}^{m} b_{ik} \cdot a_{kj}}$

$= \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} \left[\sum_{k=1}^{m} b_{ik} \cdot a_{kj}\right]$

$= \sum_{i=1}^{n} [b_{i1} a_{1i} + b_{i2} a_{2i} + a \cdots + b_{im} a_{mi}]$

$= b_{11} a_{11} + b_{21} a_{12} + \cdots + b_{n1} a_{1n} +$

$b_{12} a_{21} + b_{22} a_{22} + \cdots + b_{n2} a_{2n} + \cdots$

$\cdots \quad \cdots + b_{1m} a_{1m} + b_{2m} a_{m2} +$

$b_{nm} a_{mn}.$

$= a_{11} b_{11} + b_{n2} a_{2n} + \cdots + b_{m1} a_{m1} + a_{12} b_{21}$

$+ a_{22} b_{22} + \cdots + a_{m2} b_{2m} + \cdots \quad \cdots \quad \cdots$

$+ a_{1n} b_{n1} + a_{2n} b_{n2} + \cdots + a_{mn} b_{nm}.$

$= \sum_{i=1}^{m} [a_{i1} b_{1i} + a_{i2} b_{2i} + \cdots + a_{in} b_{ni}]$

$= \sum_{i=1}^{n} \left[\sum_{k=1}^{n} a_{ik} b_{ki}\right] = (L.H.S)$

(9)

Traces of similar matrices are equal i.e; if A and B are two similar matrices, then $tr(A) = tr(B)$.

Proof:

Let $B = P^{-1}AP$ where P is a non-singular matrix.

$$tr(B) = tr(P^{-1}AP)$$
$$= tr(P^{-1}(AP))$$ since $tr(AB) = tr(BA)$
$$= tr((AP)P^{-1}) = tr(A(PP^{-1}))$$
$$= tr(AI)$$
$$= tr(A)$$ since $PP^{-1} = I$

Hence, $tr(B) = tr(A)$.

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(5)

If C is an orthogonal matrix, then $tr(C^T AC) = tr(A)$.

Proof:

$$tr(C^T AC) = tr(C^T (AC))$$
$$= tr((AC) C^T)$$
$$= tr(A (CC^T))$$
$$= tr(AI) \quad \Big| \text{ since } CC^T = I$$
$$= tr(A).$$

Since, $tr(AB) = tr(BA)$ and $CC^T = I$, where C is orthogonal.

(6)

If A is symmetric matrix and B is a skew-symmetric matrix, then $tr(AB) = 0$

Proof:

$$tr(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j a_{ij} b_{ji}$$

$= \sum_i \sum_j b_{ji} a_{ij} = -\sum_i \sum_j b_{ij} a_{ij}$ | since, $b_{ji} = -b_{ij}$

$= -\sum_i \sum_j b_{ij} a_{ij}$ | since, $a_{ij} = a_{ji}$

$= -\sum_i (BA)_{ii}$

$= -tr(BA)$

$= -tr(AB)$.

$\therefore 2tr(AB) = 0$

or, $tr(AB) = 0$ | since, $2 \neq 0$.

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Defination of rank of a Matrix:

one of the most important attribute of a matrix is its rank. The rank of the matrix can be defined in several ways. Two of them are:

(i) Let A be an arbitrary $m \times n$ matrix over field f. The rank of the matrix A is the longest value of r for which there exists an $r \times r$ sub matrix of A with non-vanishing determinant.

(ii) The rank of the matrix A is the maximum number of linearly independant rows or columns in the matrix.

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(2)

The rank of a matrix A is denoted by $r(A)$ or $\rho(A)$.

The rank of a null matrix is zero and rank of a matrix m×n can't be larger than m or n.

A n-rowed matrix A has rank $r<n$ if and only if $|A|=0$ and A is a square matrix, then it's a singular matrix. The matrix has a rank of $r=n$ if and only if $|A| \neq 0$, the its called as non-singular matrix.

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Determination of a rank of a matrix:

The rank of a matrix can be determined by the following process;

① **First process:** Let A be an $m \times n$ matrix. when $m=n$, evaluate the value of the determinant of the matrix. if $|A| \neq 0$, the rank of the matrix is n. But if $|A|=0$ we have to find n^2 minors of the matrix A. when the value of any of the minor is non-zero, the rank of the matrix is $n-1$.

when $m<n$, the rank of the matrix can't greater than m. we have to find the largest square submatrices of order m. If value of

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the determinant of any of them is non-zero, then rank of the matrix is m. If every determinant is zero, we have to find square submatrices of order (m-1, m-2, m-.....), until we get a non-singular submatrix ~~when~~ and it's order will be the rank of the given matrix.

Second process: Reduce the given matrix A to elechon form using elementary row operations. Since the non-~~row~~ zero rows of a matrix is ecelon form are linearly independant the number of non-zero rows of the echelon matrix is the rank of the given matrix.

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Theorem 4.1: The rank of the transpose matrix is the same of the original matrix.

proof: ~~The~~ Let A be the matrix and $rank(A) = r$. Let R be an r-rowed square submatrix of A with $\det R \neq 0$. clearly R^T is a submatrix of A^T. $\det R^T = \det R$. Thus $rank(A^T) \geq R$. On the other hand, if A contains $(r+1)$ rowed square submatrix ~~of~~ S. and $\det S = 0$. Since S corresponds to S^T in A^T and $\det S^T = 0$. A^T can't contain an $(r+1)$ rowed square submatrix with a non zero determinant. consequently $rant(AT) \leq r$. Altogether. $rank(A^T) = r$.

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Theorem 4.2 : Let A be an n-square matrix. Then A is invertible if and only if rank(A) = n

proof: The rows of the n-square identity matrix I_n are linearly independant. Since, I_n is in echelon form, hence rank $(I_n) = n$. Now, If A is invertible, then A is now equivalent to I_n hence rank(A) = n. But if A is not invertible then A is now equivalent to a matrix with zero row; hence rank(A) < n. That is, A is invirtible if and only if rank(A) = N.

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Row rank: The maximum number of linearly independent rows of a matrix is called as row rank of that matrix.

Column Rank: The maximum number of linearly independent columns of a matrix is called column rank of that matrix.

Theorem 4.3 : The row and column rank of a matrix is ~~eqal~~ equal.

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4.4 Reduction of a matrix to the normal form.

Theorem 4.4 Every non-zero matrix of rank r can be reduced to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary transformation.

Proof: Let A be a matrix of rank r.

Since $A \neq 0$, It has at least one element.

Let $A_{ij} = k \neq 0$.

By interchanging the ith row with the the first row and jth column with the first column, we obtain an equivalent matrix B such that $B_{11} = k \neq 0$. By elementary transformation $R_1(\frac{1}{k})$ on B we obtain an equivalent matrix C such that $C_{11} = 1$.

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subtracting from the element of the jth column of C, the product of the corresponding elements of the first column by c_{ij}, we get matrix D whose All elements are zero except the first element.

Again subtracting from the elements of ith row of D, the product of the corresponding of the first row by d_{11} we obtain an equivalent matrix E such that all the elements of the first row and column are zero except the first element is unity, thus E is of the form

$$\begin{bmatrix} I & 0 \\ 0 & A_I \end{bmatrix}$$

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(10)

Theorem 9.5: Let A be a matrix of order $m \times n$ and $Rank(A) = r$. Then there exists two non-singular square submatrices P of order $m \times m$ and Q of order $n \times n$ such that

$$PAQ = \begin{bmatrix} I_{r \times r} & O_{r \times n-r} \\ O_{m-r \times r} & O_{m-r \times n-r} \end{bmatrix}$$

where $I_{r \times r}$ is Identity matrix of order r.

proof: Since A is a non-singular matrix of rank r, by a chain of Elemenetary transformations it can be reduced to normal form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

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Now every Elementary row/column transformation on a matrix can be affected by pre multiplication with corresponding elementary matrix of suitable order.

Let P_i $(i = 1, 2, 3, \cdots, s)$ denote the elementary matrix corresponding to the elementary row/~~col~~ transformation and P_2 $(i = 1, 2, 3, \cdots, t)$ denote the matrix with corresponding column transformation

Then since $A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, we can write

$$P_s, \cdots\cdots, P_2 P_1, \quad AQ, Q_2, \cdots\cdots Q_t$$

$$= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

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or,

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$ where

$P = P_s P_{s-1} \cdots \cdots P_2 P_1$ and $Q = Q_1 Q_2 \cdots \cdots Q_t$

Since every elementary matrix is non-singular, P and Q as defined avobe being the product of non singular matrices are also non-singular.

Thus we obtain $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Hence the theorem is proved.

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Definitation: ~~Two n~~ Two matrices A and B are said to be equivalent if there exists non singular matrices P, Q such that $PAQ = B$

Theorem: if A is an $m \times n$ matrix of rank r then there exists a non-singular matrix P of order m such that $PA = \begin{bmatrix} B \\ O \end{bmatrix}$ where B is an $r \times n$ matrix of rank r and O is the null matrix of order $(m-r) \times n$.

proof: Since A is an $m \times n$ matrix with rank of r, there exists non singular matrix P and Q of orders m and n respectively such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ — (1)

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Further, a non-singular matrix can be expressed as elementary matrix, let $Q = Q_1, Q_2, Q_3 \cdots Q_t$ are elementary matrices. Moreover, elementary matrices are non-singular. Q is also non-singular and hence has an inverse given by

$$Q^{-1} = \{Q_1 Q_2 Q_3 \cdots Q_t\}^{-1} \text{ ——— (ii)}$$

post multiplying (i) by Q^{-1} and using (ii) we get,

$$PA = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \cdot Q_{t-1}^{-1} \cdots Q_2^{-1} Q_1^{-1} \text{ ——— (iii)}$$

we shall finally obtain on the right hand side of (3) the matrix of the form $\begin{bmatrix} B \\ 0 \end{bmatrix}$, Hence the theorem is proved.

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SYLVESTER'S LAW:

Theorem 4.5: If A and B are square matrix of order n and $\rho(A) = r$, $\rho(B) = s$, then $\rho(AB) \geq r+s-n$ i, e $\rho(AB) \geq \rho(A) + \rho(B) - n$ where ρ denotes the rank of the matrix.

Proof: The equality holds when A and B are non-regular matrices. Since $\rho(A) = r$, there exists two non-singular matries P and Q such that,

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A = P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Let

$$C = P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^{-1}, \text{ then we have}$$

$$A+C = P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^{-1}$$

Now $\rho(A+C) = n$, $\rho(A) = r$ and $\rho(C) = n-r$.
Again since $A+C$ is non-singular, we have
$\rho\{(A+C)B\} = \rho(B)$

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Therefore, $\rho(B) = \rho(AB + CB) \leq \rho(AB) - \rho(CB)$ — (1)

[since for any two matrices $(A)_{m\times n}$ and $(B)_{m\times n}$

$$\rho(A+B) \leq \rho(A) + \rho(B)]$$

Also $\rho(CB) \leq \rho(C)$

[since for any two matrices $(A)_{m\times n}$ $(B)_{n\times r}$

we have $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$]

Thus from (1), we get $\rho(B) \leq \rho(AB) + \rho(C) = \rho(AB) + n - r$

or $\rho(B) \leq \rho(AB) + n - \rho(A)$

or $\rho(AB) \geq \rho(A) + \rho(B) - n$ i,e $\rho(AB) \geq r + s - n$.

4.6 Condition for consistency of a system of linear equations.

Consider the system of linear equations $Ax = b$, where A is an $m\times n$ matrix. Then the $m\times(n+1)$ matrix $(A \mid b)$, obtained by adjoining the column vector b to the matrix A on the right, is called the augmented matrix of the system $Ax = b$

Theorem 1.10: The system of linear equations $Ax = b$ is consistent if and only if the rank of the augmented matrix $(A|b)$ is equal to the rank of the matrix A.

Proof: Let A be an $m \times n$ matrix and let rank $(A) = r$

Then A has r linearly independent columns. Now we can take the first r columns $A^1, A^2, \dots$ A^r. to be linearly independent. Then for each $j = r+1, r+2, \dots n$, A_j is a linear combination of $A^1, A^2, \dots \dots A^r$. Now the augmented matrix $(A|b)$ has just one column, namely b, in addition to the n columns of A. Hence, the maximum numbers of linearly independent columns in $(A|B)$ is either r or $r+1$. Therefore, rank $(A|b)$ is either r or $r+1$.

suppose that, rank of $(A|b) = r$. Then the column vectors $A^1, A^2 \dots A^r$. b are linearly dependent.

Therefore, there exit scalars $\alpha_1, \alpha_2 \cdots \cdots \alpha_{r+1}$ (not all zero) such that $\alpha_1 A_1 + \alpha_2 A^2_r + \alpha_r A^r + \alpha_{r+1} b = 0$

Now, $\alpha_{r+1} \neq 0$, otherwise the column vectors $A^1, A^2 \cdots A^r$ would be linearly dependent, contrary to our hypothasis.

Hence
$$b = -\frac{\alpha_1}{\alpha_{r+1}} A^1 - \frac{\alpha_2}{\alpha_{r+1}} A^2 - \frac{\alpha_r}{\alpha_{r+1}} A^r = A$$

where
$$\beta_j = -\frac{\alpha_j}{\alpha_{r+1}} \quad j = 1, 2, \cdots, r$$

Thus $x = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution of the given system which is therefore consistent.

Conversely, suppose that the system $Ax = b$ is consistent and let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a solution.

Then $b = A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 A^1 + x_2 A^2 + \cdots + x_n A^n$

Now A^j, $j = r+1, r+2, \ldots n$ is a linear combination of $A^1, A^2 \ldots A^r$. Hence b is also a linear combination of $A^1, A^2 \ldots A^r$. Therefore $(r+1)$ column vectors $A^1, A^2 \ldots A^r, b$ are linearly dependent. Consequently, $(A|b)$ has only r linearly independent columns.

Thus rank $(A|b) = r$

Corollary: If A is an $m \times n$ matrix of rank m, the system $Ax = b$ is consistent.

Example-1: Find the rank of each of the following matrices:

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ Then

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$$

So the rank of the matrix A is 1 since $|A| = 0$ but not every element of A is zero. say $|2| \neq 0$

(ii) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \text{ Then}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= 1(21-20) - 2(14-12) + 3(10-9)$$

$$= 1 - 4 + 3 = 0$$

So the rank of the matrix A is less than 3. Now let us take two rowed minor of A, say.

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$$

since $|A| = 0$ but $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \neq 0$

Therefore, the rank of the given matrix is 2.

Example 2: Find the rank of the matrix A

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= 2\times 3 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

$= 6\times 0 = 0$ (Since three rows are equal).

So the rank of a matrix A is less than 3. Now let us consider the two rowed minors of A.

Say, $\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 36-36=0$, $\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 18-18=0$,

$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12-12=0$, $\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} = 18-18=0$, $\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 9-9=0$

$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6-6=0$, $\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 12-12=0$, $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6-6=0$

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4=0$. Thus the two ranked minor of A is zero. So the rank of a matrix A is less than 2. But $|4| = 4 \neq 0$. Hence the matrix is 1.

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Example 4: Find the echelon form and the row reduced echelon form of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

First let us reduce the matrix A to echelon form by the elementary row operations. We multiply 1st row by 2 and 3 and then substrac form 2nd & 3rd rows respectively.
We multiply 3rd row by 3.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 15 & -36 & 6 \end{bmatrix}$$

We multiply 2nd row by 5 and then substract from the 3rd row.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix}$$

This matrix is in row echelon form.

We substract 3rd row from the 2nd row.

Also we multiply 3rd row by $-1/6$.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix}$$

We multiply 2nd row by $1/3$.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix}$$

We add 2nd row by the $1/3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix}$$

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we add 2nd row with the 1st row

$$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix}$$

we multiply 3rd row by 2 and then substract from the 1st row.

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix}$$

This matrix is in row reduced echelon form.

Exercise : 3

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1. a) Given that,

$$A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 5 & 2 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & -1 & -2 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$$

$$\therefore 3A = 3 \begin{bmatrix} 1 & 0 & 3 & 4 \\ 5 & 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 9 & 12 \\ 15 & 6 & 0 & 3 \end{bmatrix}$$

$$\therefore A+B = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 5 & 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & -1 & -2 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -1 & 1 & 9 \\ 6 & 2 & -3 & 5 \end{bmatrix}$$

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$$\therefore A - B = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 5 & 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 6 & -1 & -2 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 1 & 5 & -1 \\ 4 & 2 & 3 & -3 \end{bmatrix}$$

$$3A - 2B = 3\begin{bmatrix} 1 & 0 & 3 & 4 \\ 5 & 2 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 6 & -1 & -2 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 9 & 12 \\ 15 & 6 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 12 & -2 & -4 & 10 \\ 2 & 0 & -6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 2 & 13 & 2 \\ 13 & 6 & 6 & -5 \end{bmatrix}$$

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2. Given that,

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(i)

$$B + C = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$L.H.S. = A(B+C)$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ 1-4 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 1+0 \\ 0-6 & 1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -6 & 3 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & -1+0 \\ 1+2 & -1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$$

$\therefore$ R.H.S. $= AB + AC$

$$= \begin{bmatrix} 0 & 1 \\ -6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}$$

$\therefore$ L.H.S. = R.H.S.

3. (a) Given that,

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$$A = \begin{bmatrix} -1 & 3 & 2 \\ 4 & -2 & 5 \\ 6 & 1 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\cancel{AB}$$

$$\therefore AB = \begin{bmatrix} -1+6+10 & 2+9+4 & -1-3+2 \\ 4-4+25 & -8\,\cancel{6}-6+10 & 4+2+5 \\ 6+2-15 & -12+3-6 & 6-1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 15 & -2 \\ 25 & -4 & 11 \\ -7 & -15 & 2 \end{bmatrix}$$

$$\therefore BA = \begin{bmatrix} -1-8+6 & 3+4+1 & 2-10-3 \\ -2+\cancel{0}12-6 & 6-6-1 & 4+15+3 \\ -5+8+6 & 15-4+1 & 10+10-63 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 8 & -11 \\ 4 & -1 & 22 \\ 9 & 12 & 17 \end{bmatrix}$$

4. Given that,

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1-2+1 & 2-4+2 & 3-6+3 \\ -3+4-1 & -6+8-2 & -9+6-3 \\ -2+2+0 & -8+4+0 & -6+6+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & -4 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1-6-6 & -1+4+3 & 1-2+0 \\ 2-12-12 & -2+8+6 & 2-4+0 \\ 1-6-6 & -1+4+3 & 1-2+0 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$$

$\therefore AB \neq BA$ (Showed)

11. (a)(ii) Given that,

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 5 & 1 \\ -4 & 2 \end{bmatrix}^{T}$$

$$= \begin{bmatrix} 5 & -4 \\ 1 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix}$$

$$= 10 + 4$$

$$= 14$$

$$\therefore A^{-1} = \frac{1}{14} \begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix}$$

(ii) $B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\text{Adj}.B = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1(0-1) - 0 + 1(-1-0)$$

$$= -2$$

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$$\therefore B^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

(iii) Given that,

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$$

$$\text{Adj. } C = \begin{bmatrix} -1+i & 0 & -1 \\ 1 & \cancel{0}\ i-1 & -1 \\ 0 & 0 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1+i & 0 & 0 \\ 0 & i-1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$|C| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1-i \end{vmatrix}$$

$$= -1(i - i^2)$$

$$= -(i+1)$$

$$= -(1+i)$$

$$\therefore C^{-1} = \frac{1}{-(1+i)} \begin{bmatrix} 1+i & 0 & 0 \\ 0 & i-1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & \frac{1+i}{2} \end{bmatrix}$$

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Example (5) Reduce the matrix A to the normal (or Canonical form and hence obtain its rank where

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

Solution: We will apply both elementary column and row operations to the matrix A for reducing it to the normal form.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

We replace C_2 and C_4 by $C_2 - 2C_1$ and $C_4 + 2C_1$ respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix}$$

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we replace C_2 and C_4 by C_2+2C_3 and C_4-5C_3 respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix}$$

we replace C_1 by C_1+C_3 and C_4 by $C_4+\frac{7}{11}C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$$

we replace R_2 by R_2-4R_1

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$$

we replace R_3 by R_3-2R_2

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 0 \end{bmatrix}$$

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We interchange C_2 and C_4.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & 0 \end{bmatrix}$$

we replace C_3 by $\frac{1}{11} C_3$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim [I_3$$

where $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Hence the Rank of A is 3.

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Example: 6 Find the rank of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

by reducing it to the normal (or canonical) form.

Solution: We will apply both elementary row and column operations to the matrix A for reducing it to the normal form and will join successive matrices by the equivalence sign ~.

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

We replace R_3 and R_4 by $R_3 - R_1$ and $R_4 - R_3$ respectively.

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$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

we replace R_2 and R_1 by $R_2 - R_4$ and $R_3 - R_4$ respectively.

$$\sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

we replace C_1 by $\frac{1}{2} C_1$ and R_3 by $\frac{1}{2} R_3$.

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

we replace R_3 by $R_3 - R_2$.

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

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We interchange R_3 and R_4.

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We replace C_2, C_3 and C_4 by C_2+C_1, C_3-3C_1 and C_4-4C_1 respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We interchange C_3 and C_4.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence the rank of A is 3.

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Example (7) Determine the rank of the matrix A.

where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solution: Reduce the given matrix to echelon form by means of elementary row transformations and join successive matrices by the equivalence sign $\sim$. Interchange first row and fourth row

Interchange first row and fourth row.

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

we substract first row from the second row and multiply the first row by 3 and then subtract from the third row.

Example. 8 Determine the rank of the matrix.

$$\begin{bmatrix} 2 & 3 & 5 & -3 & -2 \\ 3 & 4 & 3 & -1 & -3 \\ 5 & 6 & -1 & 3 & -5 \end{bmatrix}$$

Solution:

Reduce the given matrix to row echelon form by means of elementary row transformations and join successive matrics by the equivalence sign.

We substract the first row from the second and interchange these rows.

$$\sim \begin{bmatrix} 2 & 3 & 5 & -3 & -2 \\ 1 & 1 & -2 & 2 & -1 \\ 5 & 6 & -1 & 3 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & 1 \\ 2 & 3 & 5 & -3 & -2 \\ 5 & 6 & -1 & 3 & -5 \end{bmatrix}$$

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we multiply the first row by 2 and by 5 then subtract from the second and third rows respectively.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 0 & 1 & 9 & -7 & 0 \\ 0 & 1 & 9 & -7 & 0 \end{bmatrix}$$

We subtract the second row from the third row.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & 1 \\ 0 & 1 & 9 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

this matrix is now equivalent to the given matri[x] and is in the row echelon form. since the echelon matrix has two non-zero rows. the rank of the matrix is 2.

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Example 9. Find the rank of the matrix A.

where $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

Solution:

Reduce the given matrix to row echelon from by means of the elementary row transformations and join successive matrices by the equivalence sign $\sim$.

We substract the first row from the second row and also multiply the first row by 2 and by 3 and then subtract from third row and the fourth row respectively.

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$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix}$$

we multiply second row by 3 and add with third row. Also add second row with the fourth row.

$$\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is row equivalent to the given matrix and is in the row echolon form.

Scienee the echelon matrix has two non-zero rows.

the rank of the given matrix is 2.

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Example (10) Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 & 0 & 4 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 3 & 0 & 2 \end{bmatrix}$$

Solution: Reduce the given matrix to row echelon from by means of elementary row transformations and join successive matrices by the equivalence sign $\sim$.

Interchange first and third rows and also fourth and fifth rows.

$$\sim \begin{bmatrix} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 2 & 2 & 0 & 4 & 0 & 4 \\ -1 & 1 & -2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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we multiply first row by 2 and then subtract the third row. Also we add first row with fourth row.

$$\sim \begin{bmatrix} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 0 & 0 & -12 & 2 & 0 & 4 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we substract second row from the fo row.

$$\sim \begin{bmatrix} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 0 & 0 & -12 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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We now multiply second and third rows by $\frac{1}{2}$ and by $\left(-\frac{1}{12}\right)$ respectively.

$$\sim \begin{bmatrix} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1/6 & 0 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix is 3.

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Example (11) Prove that the following system of linear equations is inconsistent:

$$\left.\begin{aligned} 3x_1 + 4x_2 - x_3 + 2x_4 &= 1 \\ x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\ 3x_1 + 14x_2 - 11x_3 + x_4 &= 3 \end{aligned}\right\}$$

Proof: From the augmented matrix

$$(A|b) = \begin{bmatrix} 3 & 4 & -1 & 2 & : & 1 \\ 1 & -2 & 3 & 1 & : & 2 \\ 3 & 14 & -11 & 1 & : & 3 \end{bmatrix}$$

Reduce the augmented matrix to row echelon from by the elementary row operations and s successive matrices by the equivalence sign $\sim$.

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We multiply second row by 2 and then substruct from the third row.

$$\sim \left[\begin{array}{cccc:c} 1 & -2 & 3 & 1 & 2 \\ 0 & 10 & -10 & -1 & -5 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right]$$

Thus the augmented matrix is reduced to row-echelon form. Here we observe that rank $(A) = 2$ and rank $(A|b) = 3$.

Therefore, rank $A \neq$ rank $(A|b)$.

Hence the given system of linear equations is inconsistent.

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Example (12) Solve the following system of linear equations:

$$
\left.
\begin{aligned}
x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 1 \\
2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 &= -1 \\
3x_1 + 5x_2 + 6x_3 + 7x_4 + 4x_5 &= 2 \\
4x_1 + 7x_2 + 10x_3 + 13x_4 + 16x_5 &= 1 \\
5x_1 + 8x_2 + 9x_3 + 10x_4 + 3x_5 &= 3
\end{aligned}
\right\}
$$

Solution:

From the augmented matrix

$$
(A|b) = \left[\begin{array}{ccccc|c}
1 & 2 & 3 & 4 & 5 & 1 \\
0 & -1 & -2 & -3 & -4 & -3 \\
0 & -1 & -3 & -5 & -11 & -1 \\
0 & -1 & -2 & -3 & -4 & -3 \\
0 & -2 & -6 & -10 & -22 & -2
\end{array}\right]
$$

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We subtract 2nd row from the 4th row. and also we multiply 3rd row by 2 and then subtract from the 5th row.

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & -1 & -2 & -3 & -4 & -3 \\ 0 & -1 & -3 & -5 & -11 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we subtract 2nd row from the 3rd row.

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & -1 & -2 & -3 & -4 & -3 \\ 0 & 0 & -1 & -2 & -7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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We multiply 3rd row by 2 and then subtract from the second row.

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \vdots & 1 \\ 0 & -1 & 0 & 1 & 10 & \vdots & -7 \\ 0 & 0 & -1 & -2 & -7 & \vdots & 2 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

We multiply both 2nd and 3rd rows by (-1).

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \vdots & 1 \\ 0 & 1 & 0 & -1 & -10 & \vdots & 7 \\ 0 & 0 & 1 & 2 & 7 & \vdots & -2 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

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$\boxed{R_2 = 2 \cdot R_2 - R_1}$ then,

$$\sim \left[\begin{array}{ccccc:c} 1 & 0 & 3 & 6 & 25 & -13 \\ 0 & 1 & 0 & -1 & -10 & 7 \\ 0 & 0 & 1 & 2 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{R_3 = 3 \cdot R_3 - R_1}$ then,

$$\sim \left[\begin{array}{ccccc:c} 1 & 0 & 0 & 0 & 4 & -7 \\ 0 & 1 & 0 & -1 & -10 & 7 \\ 0 & 0 & 1 & 2 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in row-echelon form. Therefore, rank(A) = rank(A|b) = 3.

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Hence the given system is consistent and transforms into linear system.

$$\left.\begin{aligned} x_1 \quad\quad\quad\quad + 4x_5 &= -7 \\ x_2 \quad - x_4 - 10x_5 &= 7 \\ x_3 + 2x_4 + 7x_5 &= -2 \end{aligned}\right\}$$

This system is in echelon form and it has three equations in five unknowns. Therefore, the system has $5-3=2$ free variables which are x_4 and x_5.

Hence the system has infinite number of non-zero solutions.

Let,

$x_4 = a$ and $x_5 = b$.

then,

$$x_3 = -2-2a-7b$$

$$x_2 = 7+a+10b$$

$$x_1 = -7-9b$$

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thus the general Solution is $x_1 = -7-9b$

$$x_2 = 7+a+10b$$

$$x_3 = -2-2a-7b$$

$$x_4 = a$$

$$x_5 = b$$

where a and b are arbitary constants.

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