Bangabandhu Sheikh Mujibur Rahman Science And Technology University

Sheikh Hasina Institute of ICT

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18 ICT CSE 001 (Md. Rakibul Haque)

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Fourier analysis is the study of the way general functions may be represented or approximated by sums of smaller trigonometric functions. It is named after Joseph Fourier who showed that representing a function as a sum of trigonometric functions greatly simplifies the study of heat transfer.

In mathematics, a fourier series is a periodic function composed of ~~fully~~ harmonically related sinusoids, combined by a weighted summation. In engineering applications, the fourier series is greatly presumed to converge almost everywhere. This technique could be applied in electrical engineering, vibration analysis, acoustics, optics etc.

Periodic functions

A periodic function is a function that repeats its values of at regular intervals.

A function f is said to be periodic if, for some nonzero constant P, it is the case that

$f(x+P) = f(x)$ for all values of x in the domain.

A nonzero constant for which this is the case is called a period of the function. A function with period P will repeat on intervals of length P, and these intervals are sometimes also referred to as period of the function.

For example, $f(x) = \sin x$ and $f(x) = \cos x$ are periodic functions having period 2π. Also $f(x) = \tan x$ is a periodic function having period π.

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Fourier series

The trigonometric series

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx +$$

$$\dots + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is a fourier series if its coefficients are a_0, a_n and b_n are given by the following formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

where $f(x)$ is any single-valued function defined on the interval $(-\pi, \pi)$

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The fourier series can also be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad (n = 0, 1, 2, 3)$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (n = 0, 1, 2, 3)$

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Problem: Find the complex form of the fourier series of the periodic function $f(x) = \cosh x$ over the interval $-1 < x < 1$

Solution:

The fourier series in complex form is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{in\pi x/l}$$

where l is the half length of the interval size

Here, $b - a = 2 = 2l \quad \therefore \quad l = 1$

$$f(x) = \cosh x = \sum_{n=-\infty}^{\infty} C_n \cdot e^{in\pi x} \quad \cdots\cdots (i)$$

Now,

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x) \cdot e^{-in\pi x/l}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \cosh x \cdot e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{e^{x} + e^{-x}}{2} \cdot e^{-in\pi x}\, dx$$

$$= \frac{1}{4}\int_{-1}^{1}\left\{\left(e^{x} \cdot e^{-in\pi x}\right) + \left(e^{-x} \cdot e^{-in\pi x}\right)\right\} dx$$

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$$= \frac{1}{4}\int_{-1}^{1} e^{x - in\pi x}\,dx + \frac{1}{4}\int_{-1}^{1} e^{-x - in\pi x}\,dx$$

$$= \frac{1}{4}\left| \frac{e^{x(1-in\pi)}}{1-in\pi} \right|_{-1}^{1} + \frac{1}{4}\left| \frac{e^{x(-1-in\pi)}}{-1-in\pi} \right|_{-1}^{1}$$

$$= \frac{1}{4(1-in\pi)}\left[e^{1-in\pi} - e^{-1+in\pi} \right]$$

$$- \frac{1}{4(1+in\pi)}\left[e^{-1-in\pi} - e^{1+in\pi} \right]$$

$$= \frac{e^{1}\cdot e^{-in\pi} - e^{-1}\cdot e^{in\pi}}{4(1-in\pi)} - \frac{e^{-1}\cdot e^{-in\pi} - e^{1}\cdot e^{in\pi}}{4(1+in\pi)}$$

$$= \frac{e^{1}(\cos n\pi - i\sin n\pi) - e^{-1}(\cos n\pi + i\sin n\pi)}{4(1-in\pi)}$$

$$- \frac{e^{-1}(\cos n\pi - i\sin n\pi) - e^{1}(\cos n\pi + i\sin n\pi)}{4(1+in\pi)}$$

$$= \frac{e^{1}\cdot(-1)^n - e^{-1}(-1)^n}{4(1-in\pi)} - \frac{e^{-1}(-1)^n - e^{1}\cdot(-1)^n}{4(1+in\pi)}$$

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$$= \frac{\sinh a\,(-1)^n}{2(1-in\pi)} - \frac{\cosh a\,(-1)^n}{2(1+in\pi)}$$

$$= \frac{\sinh a\,(-1)^n(1+in\pi) - \cosh a\,(-1)^n(1-in\pi)}{2(1-i^2n^2\pi^2)}$$

$$\therefore C_n = \frac{\sinh(-1)^n(1+in\pi) - \cosh(-1)^n(1-in\pi)}{2+2n^2\pi^2}$$

From ①

$$f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh(1+in\pi) - (-1)^n \cosh(1-in\pi)}{2+2n^2\pi^2} \cdot e^{in\pi x}$$

which is the required form of the fourier series

Ans

Change Of Interval

In most of the engineering applications, we require an expansion of a given function over an interval $2l$ other than $2p$.

Let $f(x)$ is a function defined in the interval $c < x < c+2l$. The fourier expansion for $f(x)$ in the interval $c<x<c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where,

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos(n\pi x/l)\,dx \quad \&$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin(n\pi x/l)\,dx$$

Even and Odd Function:

If $f(x)$ is an even function and is defined in the interval $(c, c+2l)$, then

If $f(x)$ is an even function and is defined in the interval $(c, c+2l)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos(n\pi x/l)\,dx$$

If $f(x)$ is an odd function and is defined in the interval $(c, c+2)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin(n\pi x/l)\,dx$$

Half Range Series:

- Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where, $b_n = \frac{2}{l} \int_0^l f(x) \sin(n\pi x/l)\, dx$

- Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where,

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos(n\pi x/l)\, dx$$

Example: 01

Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$, in $0 < x < 3$.

Solve:

Here $2l = 3$.

$l = 3/2$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_0 \cos \frac{2n\pi x}{3} + b_n \sin \frac{2n\pi x}{3} \right)$

where, $a_0 = (2/3) \int_0^3 (2x - x^2)\, dx$

$= (2/3) \int_0^3 2(x^2/2) - (x^3/3)\, dx$

$= 0$

$a_n = (2/3) \int_0^3 (2x - x^2) \cos \frac{2n\pi x}{3}\, dx$

$= (2/3) \int_0^3 (2x - x^2)\, d\left(\frac{\sin(2n\pi x/3)}{2n\pi/3} \right)$

$= \frac{2}{3} \left((2x - x^2) \left(\frac{\sin(2n\pi x/3)}{2n\pi/3} \right) - (2 - 2x) \left(\frac{\cos(2n\pi x/3)}{4n^2\pi^2/9} \right) + (-2) \left(- \frac{\sin(2n\pi x/3)}{8n^3\pi^3/27} \right) \right)_0^3$

$$b_n = (2/3)\left\{-(9/n^2\pi^2) - (9/2n^2\pi^2)\right\} = -9/n^2\pi^2$$

$$b_n = (2/3)\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= (2/3)\int_0^3 (2x - x^2)\, d\left(-\frac{\cos(2n\pi x/3)}{2n\pi/3}\right)$$

$$= (2/3)\Bigg((2-x^2) - \left(\frac{\cos(2n\pi x/3)}{2n\pi/3}\right) - (2-2x) - \left(\frac{\sin(2n\pi x/3)}{4n^2\pi^2/9}\right) + (-2)\left(\frac{\cos(2n\pi x/3)}{8n^3\pi^3/27}\right)\Bigg)_0^3$$

$$= (2/3)\left\{(9/2n\pi x) - (27/4n^3x^3) + (27/4n^3\pi^3)\right\}$$

$$= 3/n\pi$$

Therefor,

$$f(x) = \sum_{n=1}^{\infty}\left(-(9/n^2\pi^2)\cos\frac{2n\pi x}{3} + (3/nx)\sin\frac{2n\pi x}{3}\right)$$

Complex Form of Fourier Series

Let the function $f(x)$ be defined on the interval $[-\pi, \pi]$.

Using the well known Euler's formulas

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i},$$

we can write the Fourier series of the function in the complex form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{inx} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-inx}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Here we have used the following notations:

$$c_0 = \frac{a_0}{2}, \; c_n = \frac{a_n - ib_n}{2}, \; c_{-n} = \frac{a_n + ib_n}{2}.$$

The coefficients c_n are called complex Fourier coef-ficients. They are defined by the formulas

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

In necessary to expand a function $f(x)$ of period $2L$, we can use the following expressions:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}},$$

Where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

The complex from of Fourier series is algebraically simpler and more symmetric. Therefore, it is often used in physic and other science.

Example 1

Using complex form, find the fourier series of the function

$$f(x) = \operatorname{sign} x = \begin{cases} -1, & -\pi \leq x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}.$$

Example 2

using complex form, find the Fourier Series of the function $f(x) = x^2$, defined on the interval $[1,1]$

Example 3

Using the complex form, find the Fourier Series of the function

$$f(x) = \frac{a \sin x}{1 - 2a\cos x + a^2}, \quad |a| < 1.$$

Example 1 solve.

We calculate the coeffceients c_0 and c_n for $n \neq 0$:

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} dx\right]$$

$$= \frac{1}{2\pi}\left[(-x)^{0}\Big|_{-\pi}^{0} + x\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left(\cancel{-\pi}+\cancel{\pi}\right)$$

$$= 0,$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,e^{-inx}\,dx + \int_{0}^{\pi} e^{-inx}\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(e^{-inx})\big|_{-\pi}^{0}}{-in} + \frac{(e^{-inx})\big|_{0}^{\pi}}{-in}\right]$$

$$= \frac{i}{2\pi n}\left[-(1-e^{in\pi}) + e^{in\pi} - 1\right]$$

$$= \frac{i}{2\pi n}\left[e^{in\pi} + e^{-in\pi} - 2\right] = \frac{i}{\pi n}\left[\frac{e^{in\pi}+e^{-in\pi}}{2} - 1\right]$$

$$= \frac{i}{\pi n}\left[\cos n\pi - 1\right]$$

$$= \frac{i}{\pi n}\left[(-1)^n - 1\right].$$

If $n=2K$, then $c_{2K}=0$. If $n=2K-1$, then c_{2K-1}

$$= -\frac{2i}{(2K-1)\pi}$$

Hence, the Fourier series of the function in the complex form is

$$f(x) = \operatorname{sign} x = -\frac{2i}{\pi}\sum_{K=-\infty}^{\infty}\frac{1}{2K-1}e^{i(2K-1)x}.$$

We can transform the series and write it in the real form. Rename: $n=2K-1$, $n=\pm1, \pm2, \pm3, \dots$ Then

$$f(x) = \operatorname{sign} x = -\frac{2i}{\pi}\sum_{K=-\infty}^{\infty}\frac{1}{2K-1}e^{i(2K-1)x}$$

$$= -\frac{2i}{\pi}\sum_{n=-\infty}^{\infty}\frac{e^{inx}}{n} = -\frac{2i}{\pi}\sum_{n=1}^{\infty}\left(\frac{e^{ine}}{-n}+\frac{e^{inx}}{n}\right)$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{inx}-e^{-inx}}{2in} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$= \frac{4}{\pi}\sum_{K=1}^{\infty}\frac{\sin(2K-1)x}{2K-1}.$$

Graph of the function and its Fourier approximation for n = 5 and n = 50 are shown in

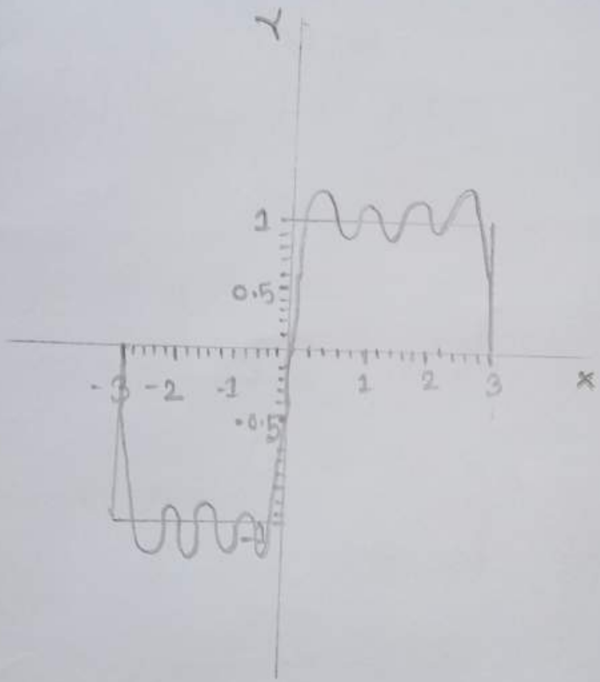

Figure 1, n=5, n=50

Chapter (4A)

Exercise - 03

Find the fourier integral of function,

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Solve:

put $x = -x$

$\Rightarrow f(-x) = f(x)$

The function $f(x)$ is an even function

So, the fourier cosine integral of $f(x)$ is given by $f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t) \cos\lambda t \, dt \, d\lambda$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \left[\int_0^1 1.\cos\lambda t \, dt\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x \left|\frac{\sin\lambda t}{\lambda}\right|_0^1 d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x \left(\frac{\sin\lambda - 0}{\lambda}\right) d\lambda$$

$$= \frac{2}{\pi}\int_0$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x \cdot \sin\lambda}{\lambda}\, d\lambda$$

$$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}\, d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Ans.

Exercise - 04:

Express the function $f(x) = \begin{cases} \sin x & , 0 \le x \le 2\pi \\ 0 & , x > \pi \end{cases}$

as a fourier sine integral and show that

$$\int_0^\infty \frac{\sin \lambda x - \sin \lambda\pi}{1-\lambda^2} d\lambda = \frac{\pi}{2} \sin x , 0 \le x \le \pi$$

Solution:

The fourier sine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty \sin x \sin \lambda t \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \cdot \frac{1}{2} \int_0^\infty 2 \sin \lambda t \sin t \, dt \, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \sin \lambda x \int_0^\pi [\cos (\lambda-1)t - \cos(\lambda+1)t] \, dt \, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \sin \lambda x \left| \frac{\sin(\lambda-1)t}{\lambda-1} - \frac{\sin(\lambda+1)t}{\lambda+1} \right|_0^\pi d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \sin \lambda x \left[\frac{\sin(\lambda\pi-\pi)}{\lambda-1} - \frac{\sin(\lambda\pi+\pi)}{\lambda+1} \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \sin \lambda x \left[-\frac{\sin(\pi - \lambda\pi)}{\lambda - 1} - \frac{\sin(\pi + \lambda\pi)}{\lambda + 1} \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \sin \lambda x \left[-\frac{\sin \lambda\pi}{\lambda - 1} + \frac{\sin \lambda\pi}{\lambda + 1} \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \sin \lambda x \; \sin \lambda\pi \left(\frac{1}{\lambda+1} - \frac{1}{\lambda-1} \right) d\lambda$$

$$\therefore f(x) = \frac{1}{\pi} \int_0^\infty \sin \lambda x \; \sin \lambda\pi \left(\frac{1}{\lambda+1} - \frac{1}{\lambda-1} \right) d\lambda$$

$$\Rightarrow \int_0^\infty \frac{\sin \lambda x . \sin \lambda\pi}{1 - \lambda^2} \, d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

(proved)

*

yeasin 03

Physical Interpretation

physically, $U(x,t)$ represents the temperature at any point x at any time t in solid bounded by the planes $x=0$ and $x=4$ (or a bar on the x-axis with the ends $x=0$ and $x=4$, whose surface is insulated — laterally. The condition $U(0,t)=0$ and $U(4,t)=0$ - implies that the ends are kept at zero temperature while $U(x,0)=2x$ implies that the initial temperature is a function of x.

Example – 8 Solve the boundary value problem

$$\frac{\delta u}{\delta t}=\frac{\delta^2 U}{dx^2},\ U(0,t)=1,\ U(\pi,t)=3$$

$U(x,0)=1$ where $0<x<\pi$, $t>0$

Soln: Given, $$\frac{\delta U}{\delta t}=\frac{\delta^2 U}{\delta x^2} \quad \dots \quad \dots (i)$$

Taking the finit Fourier sine transform of (i)

$$\int_0^\pi \frac{\delta U}{\delta t} \sin nx\, dx=\int_0^\pi \frac{\delta^2 U}{\delta x^2} \sin nx\, dx \quad \dots \quad \dots (i)$$

let, $u = u(n,t) = \int_0^\pi U(x,t)\sin nx\,dx$

then $\frac{du}{dt} = \int_0^\pi \frac{\delta U}{\delta t}\sin nx\,dx$

$= \int_0^\pi \frac{\delta^2 U}{\delta x^2}\sin nx\,dx$ [by (ii)]

$= \left[\sin nx\frac{\delta U}{\delta x}\right]_0^\pi - n\int_0^\pi \cos nx\frac{\delta U}{\delta x}dx$

$= 0 - n\left[\cos nx\,U(x,t)\right]_0^\pi - n^2\int_0^\pi \sin nx\,U(x,t)\,dx$

$= -n\left(\cos n\pi\,U(\pi,t) - U(0,t)\right) - n^2\int_0^\pi U(x,t)\sin nx\,dx$

$= -n(3\cos n\pi - 1) - n^2 u$

$= n(1 - 3\cos n\pi) - n^2 u$

$\Rightarrow \frac{du}{dt} = n(1 - 3\cos n\pi) - n^2 u$

$\Rightarrow \frac{du}{dt} + n^2 u = n(1 - 3\cos n\pi)$ ...... (iii)

which is a linear differential eqn of 1st order.

$I.F = e^{\int n^2 dt} = e^{n^2 t}$

Therefore Sol^n of (iii) is

$$ue^{n^2t} = n(1-3\cos n\pi)\int e^{n^2t}\,dt$$

$$= \frac{n(1-3\cos n\pi)}{n^2} e^{n^2t} + A$$

$$\Rightarrow u = u(n,t) = \frac{1}{n}(1-3\cos n\pi) + A\,e^{-n^2t} \quad \text{— (iv)}$$

when $t=0$, $u(n,0) = \int_0^{\pi} U(x,0)\sin nx\,dx$

$$= \int_0^{\pi} 1.\sin nx\,dx$$

$$= -\frac{1}{n}\left[\cos nx\right]_0^{\pi}$$

$$= -\frac{1}{n}\left[\cos n\pi - 1\right]$$

$$= \frac{1}{n}(1-\cos n\pi)$$

again, when $t=0$ from (iv) we get,

$$u(n,0) = \frac{1}{n}(1-3\cos n\pi) + A$$

$$\therefore \frac{1}{n}(1-\cos n\pi) = \frac{1}{n}(1-3\cos n\pi) + A$$

$$\Rightarrow A = \frac{1}{n}(1-\cos n\pi - 1 + 3\cos n\pi) = \frac{2\cos n\pi}{n}$$

putting the value of A in (4)

$$u = u(n,t) = \frac{1}{n}(1-3\cos n\pi) + \frac{2\cos n\pi}{n} e^{-n^2t}$$

taking inverse finite Fourier sine transform,

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1-3\cos n\pi)}{n} \sin nx + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{2\cos n\pi}{n} e^{-n^2 t} \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1-3\cos n x)}{n} \sin nx + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} e^{-n^2 t} \sin nx \longrightarrow (v)$$

the finite Fourier sine transform of $F(x)$, $0<x<l$ is defined as $f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx$

Here, $l=\pi$

$$\therefore f_s\{F(x)\} = \int_0^{\pi} F(x) \sin nx \, dx$$

$F(x)=1$ $\quad f_s(1) = \int_0^{\pi} 1 . \sin nx \, dx$

$$= -\frac{1}{n}[\cos nx]_0^{\pi}$$

$$= \frac{1}{\pi}(1-\cos n\pi)$$

$\therefore$ Taking inverse finite Fourier sine transform, we get

$$1 = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n} \sin nx$$

$\boxed{F(x)=x}$ $\quad f_s(x) = \int_0^{\pi} x \sin nx \, dx$

$$= \left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx$$

$$= -\frac{\pi \cos n\pi}{n} + 0 + \frac{1}{n^2}[\sin nx]_0^\pi$$

$$= \frac{-\pi \cos n\pi}{n}$$

Taking inverse finite Fourier sine transform,

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty} -\frac{\pi \cos n\pi}{n} \sin nx$$

Therefore, $\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-3\cos n\pi}{n} \sin nx$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n} \sin nx + \frac{2}{\pi}\sum_{n=1}^{\infty} -\frac{2\cos n\pi}{n} \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n} \sin nx + \frac{2}{\pi}\cdot\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{-\pi \cos n\pi}{n} \sin nx$$

$$= 1 + \frac{2}{\pi}\cdot x$$

Thus $\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-3\cos n\pi}{n} \sin nx = 1 + \frac{2x}{\pi}$

hence, from (:), we have,

$$U(x,t) = 1 + \frac{2x}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} e^{-n^2 t} \sin nx$$

$$= 1 + \frac{2x}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin nx$$

yeasin 03

Important note :

(A) Fourier's integral theorem can be written in the form,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(x-u)}\cdot du\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\, d\alpha \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du$$

Then $F(\alpha) = \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha v}\, du \quad -(I)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha \quad -(II)$$

the function $F(\alpha)$ is called the Fourier transform of $f(x)$ and is some times written as $F(\alpha) = \mathcal{F}\{f(x)\}$

The function $f(x)$ is called the Inverse Fourier transform of $F(\alpha)$ & is written as $f(x) = \mathcal{F}^{-1}\{F(\alpha)\}$

18ICTCSE009 (Roksana)

Parseval's Formula

One of the most important properties of fourier series is Parseval's Formula or the completeness relation which gives a relation between the average of the square (or absolute square) of the function f(x) and the co-efficients in fourier series of f(x).

A. Particular Case

B. General Case

Particular Case :

Let f(x) be a real valued Function of period 2π whose Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

Where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$, $\quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$

and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Now average of the square of $f(x)$ over $(-\pi,\pi)$ is $\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx$

Thus we have average of $[f(x)]^2$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right]^2 dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} a_0^2\,dx + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n^2 \cos^2 nx\,dx$$

$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n^2 \sin^2 nx\,dx + \text{other}$$

terms (which vanish when average is taken).

$$= \frac{1}{2\pi}a_0^2\left[x\right]_{-\pi}^{\pi} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}a_n^2\int_{-\pi}^{\pi}(1+\cos 2nx)\,dx$$

$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}b_n^2\int_{-\pi}^{\pi}(1-\cos 2nx)\,dx + 0$$

$$= a_0^2 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}a_n^2 \cdot 2\pi + \frac{1}{2\pi}\sum_{n=1}^{\infty} b_n^2 \cdot 2\pi$$

$$= a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Hence $\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$

This is one form of Parseval's Formula. One can esily verify that the formula is unchanged if f(x) has period 2c in place of 2π and its square is averaged over any period of length 2c. Then we have

$$\frac{1}{2c}\int_{-c}^{c}[f(x)]^2\,dx = a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

Let f(x) be a complex-valued function of period 2π, whose Fourier series is

$$\sum_{n=-\infty}^{\infty} C_n e^{inx}$$

Where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}$ $(n = 0, \pm1, \pm2, \cdots)$

Then the average square of f(x)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \sum_{n=-\infty}^{\infty}|C_n|^2$$

18ICT CSE 009 (Roksana)

General Case :

A general form of the Parseval's Formula state that if $F(x)$ and $g(x)$ are two real valued functions of period 2π, whose fourier series are

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ and}$$

$$a_0' + \sum_{n=1}^{\infty} (a_n' \cdot \cos nx + b_n' \sin nx) \text{ respectively}$$

Where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx$$

Similarly a_0', a_n' and b_n' are defined in terms of

Then we have

$$\frac{1}{2\pi}\int_{\pi}^{\pi} F(x)\, g(x)\, dx = a_0 a_n' + \frac{1}{\cancel{2}} \sum_{n=1}^{\infty} (a_n a_n' + b_n b_n')$$

When $f(x)$ and $g(x)$ are two complex valued function of period 2π, whose fourier series are

$$\sum_{n=-\alpha}^{\alpha} c_n e^{inx} \text{ and } \sum_{n=-\alpha}^{\alpha} c_n' e^{inx}$$ respectively

$$\text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \quad (n=0,\pm1,\pm2 \cdots)$$

$$\text{and } c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x) e^{-inx}\, dx \quad (n=0,\pm1,\pm2 \cdots)$$

Then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \sum_{n-\alpha}^{\alpha} c_n c_n'.$$

Example

[E] Find the Fourier Cosine transform of e^{-x}, $x \geq 0$

Solution:

By difinition of Fourier cosine transform of $f(x)$ for $0 < x < \infty$, we have

$$f_c(n) = \int_0^\infty F(x) \cdot \cos nx \, dx \quad \cdots \text{(1)}$$

$$\therefore f_c(n) = \int_0^\infty e^{-x} \cos nx \, dx \quad \text{since } F(x) = e^{-x}$$

$$= \left[\frac{e^{-x} \sin nx}{n}\right]_0^\infty + \int_0^\infty e^{-x} \frac{\sin nx}{n} \, dx$$

$$= 0 - \frac{1}{n^2}\left[e^{-x} \cos nx\right]_0^\infty - \frac{1}{n^2}\int_0^\infty e^{-x} \cos nx \, dx$$

$$= 0 + \frac{1}{n^2} - \frac{1}{n^2} f_c(n)$$

$$\text{or, } f_c(n) + \frac{1}{n^2} f_c(n) = \frac{1}{n^2}$$

$$\text{or, } \left(\frac{n^2+1}{n^2}\right) f_c(n) = \frac{1}{n^2}$$

$$\text{or, } f_c(n) = \frac{1}{n^2+1}$$

Hence the Fourier Cosine transform of e^{-x} is $\frac{1}{n^2+1}$

Find the inverse Fourier Cosine transform of $f_c(n) = \frac{1}{1+n^2}$

Solution:

By diffinition of the inverse Fourier cosine transform, we have.

$$F(x) = \frac{2}{\pi}\int_0^\infty f_c(n) \cdot \cos nx \, dn$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{1+n^2} \cos nx \, dn$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos nx}{1+n^2} \, dn \quad \cdots (1)$$

From the Fourier integral Formula of an even Function, we have

$$f(x) = \frac{2}{\pi}\int_0^\alpha du \int_0^\alpha f(t)\cos ut \cos ux \, dt$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\alpha dn \int_0^\alpha f(t)\cos nx \, dt \quad \cdots (2)$$

Taking $f(x) = e^{-x}$ in (2) we get

$$e^{x} = \frac{2}{\pi}\int_0^\alpha dn \cdot \int_0^\alpha e^{-t} \cos nt \cos nx \, dt$$

$$= \frac{2}{\pi}\int_0^\alpha \cos nx \left\{ \int_0^\alpha e^{-t} \cos nt \, dt \right\} dn$$

$$= \frac{2}{\pi}\int_0^\alpha \frac{\cos nx}{1+n^2} dn$$

$$\therefore \int_0^\alpha \frac{\cos nx}{1+n^2} dn = \frac{\pi}{2} e^{x} \quad \cdots (3)$$

Combining (1) and (3) we get

$$F(x) = \frac{2}{\pi}, \frac{\pi}{2} e^{-x} = e^{-x}$$

Hence $F(x) = e^{-x}$ which is the required inverse Fourier cosine transform of $F_c(n)$

$$= \frac{1}{1+n^2}.$$

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Finite Fourier sine and cosine transforms

The finite Fourier sine transforms of $f(x)$, $0<x<l$, is define as

$$f_s(n) = \int_0^l F(x) \sin\frac{n\pi x}{l}\, dx \quad \cdots (1)$$

where n is an integer

The Function $f(x)$ is then called the interse finite Fourier sine transform of $f_s(n)$ and given by

$$F(x) = \frac{2}{l}\sum_{n=1}^{\alpha} f_s(n) \sin\frac{n\pi x}{l}$$

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The Finite Fourier cosine transform of $f(x)$, $0<x<l$, is defined as

$$f_c(n) = \int_0^l F(x) \cos\frac{n\pi x}{l}\, dx \quad \text{--- (3)}$$

where n is an integer

∴ The function $F(x)$ is then called the inverse finite Fourier cosine transform of $f_c(n)$ and given by

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(n) \cos\frac{n\pi x}{l} \quad \text{------ (4)}$$

(Infinite) Fourier sine and cosine translation:

The (Infinite) Fourier sine and cosine transform of a function $F(x)$ of x such that $0 < x < \alpha$ is denoted by $f_s(n)$ and is defined as

$$f_s(n) = \int_0^{\alpha} F(x) \sin nx \, dx \quad \text{--- (1)}$$

The function $F(x)$ is then called the inverse Fourier sine transform of $f_{(s)}(n)$ and is given by

$$F(x) = \frac{2}{\pi} \int_0^{\alpha} f_{(s)}(n) \sin nx \, dx \quad \text{--- (}$$

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The (infinite) Fourier cosine transform of ~~an~~ function $F(x)$ ~~of~~ x for $0<x<\alpha$ is denoted by $f_c(n)$ and define as

$$f_c(n) = \int_0^{\alpha} F(x) \cos nx \, dx \quad \text{--- (3)}$$

The function $F(x)$ is then called the ~~interval~~ inverse Fourier cosine transform of $f_{(c)}(n)$ is given by

$$F(x) = \frac{2}{\pi} \int_0^{\alpha} f_c(n) \cos nx \, dx$$

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Note 1. The infinite Fourier sine transform and the infinite Fourier Cosine transform are generally known as Fourier sine transform and Fourier cosine tranform respectily

Note 2. Some athors also define Fourier cosine transform and Fourier sine transform in following ways respectively

(i) $F_{(s)}\{f(x)\} = f_s(n) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin nx \, dx$

(ii) $F_c\{f(x)\} = f_c(n) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos nx \, dx$

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<u>Note-3:</u> Some authors also define inverse Fourier sine transform and inverse Fourier cosine transform in the following way respectively:

(i) $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_{(s)}(n) \sin nx \, d$

(ii) $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(n) \cos nx \, dn$

Complex form of the Fourier integral and Fourier transforms

Form the definition of fourier intergal

We have

$$f(x) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} f(t)\, dt \int_{-\alpha}^{\alpha} \cos u(x-t)\, du \quad \text{--- (i)}$$

$$= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \left[\int_{-\alpha}^{\alpha} f(t) \cos\{u(x-t)\, dt \right] du \quad \text{---- (ii)}$$

Adding (2) and (3), we get

$$f(x) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \left[\int_{-\alpha}^{\alpha} f(t) \{\cos u(x-t) + i \sin u(x-t)\}\, dt \right] du$$

$$= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \left[\int_{-\alpha}^{\alpha} f(t)\, e^{iu(x-t)}\, dt \right] du \quad \text{--- (4)}$$

which is called the complex from of the Fourier integral

Also (4) can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} \left[\frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(t) e^{-iut} dt \right] e^{iux} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} c(u) e^{iux} du \quad \cancel{\text{where}}$$

where $c(u) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(t) e^{iut} dt$

$$\cancel{= \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\infty} c}$$

$f(x)$ is called the Fourier transform of $c(u)$ and $c(u)$ is called the inverse Fourier transform of $f(x)$

Ans to the Q.N. 4(A) 11

Furier cosine function of series f(x)

$$f(x) = \sum_{n=1}^{\alpha} a_n \cos nx$$

$$a_n = \int_a^b f(x) \cos nx \, dx$$

$$= \left[\int_0^{\frac{\pi}{2}} (1) \cos nx \, dx + \int_{\pi/2}^{\pi} (-1) \cos nx \, dx \right]$$

$$= \left[\left[\frac{\sin nx}{n} \right]_0^{\pi/2} + (-1) \left[\frac{\sin nx}{n} \right]_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{n} \left\{ \sin n \frac{\pi}{2} - \sin 0 \right\}$$

$$+ \left(\frac{-1}{n} \right) \left\{ \sin n\pi - \sin \frac{n\pi}{2} \right\}$$

$$= \frac{1}{n} \sin \frac{n\pi}{2} + \frac{1}{n} \sin \frac{n\pi}{2}$$

$$= \sin \frac{n\pi}{2} \left(\frac{1}{n} + \frac{1}{n} \right)$$

$$= \sin \frac{n\pi}{2} \cdot \frac{2}{n}$$

$$= \frac{2}{n} \sin \frac{n\pi}{2}$$ Ans.

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Ans to the Q.N. (4A) 12

For sine Fourier series;

$$a_0 = a_n = 0.$$

$$F(x) = \sum_{n=1}^{\alpha} b_n \sin nx, \quad 0 < x < \pi$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin x \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) \sin nx \, dx + \int_{\pi/2}^{\pi} f(x) \sin nx \, dx \right]$$

$$= \frac{\pi}{2} \left[\int_0^{\pi/2} x \sin x \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right]$$

$$b_n = \left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi x) \sin nx \, dx \right]$$

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$$= \left[\left(-\frac{x\cos nx}{n} \right)_0^{\pi/2} - \int_0^{\pi/2} -\frac{\cos nx}{n}\,dx \right.$$
$$\left. \left(-\frac{(\pi-x)\cos nx}{n} \right)_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} (-1)\, \frac{-\cos nx}{n}\,dx \right]$$

$$= \frac{2}{n} \left[\frac{-\frac{\pi}{2}\cos\frac{n\pi}{2}}{n} + \left(\frac{\sin nx}{n^2} \right)_0^{\pi/2} + \frac{\frac{\pi}{2}\cos\frac{n\pi}{2}}{n} - \left. \frac{\sin nx}{n^2} \right)_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{n^2} \sin\left(\frac{n\pi}{2}\right)$$ Ans

Complex from of the Fourier integral and Fourier transform

From the defination of Fourier integral, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \cos u(x-t)\,du \quad \text{———— (1)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}\,dt\right]du \quad \text{———— (2)}$$

One Can easily show that

$$\frac{i}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\sin\{u(x-t)\}\,dt\right]du = 0 \quad \text{———— (3)}$$

Adding (2) and (3) we get

$$= f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\{\cos u(x-t) + i\sin u(x-t)\}\,dt\right]du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\,e^{iu(x-t)}\,dt\right]du \quad \text{———— (4)}$$

which is called the Complex from of the Fourier integral
Also (4) Can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-iut}\, dt\right] e^{iux}\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(u)\, e^{iux}\, du \quad \text{where } C(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-iut}\, dt$$

$f(x)$ is called the Fourier transform of $C(u)$ and $C(u)$ called the inverse Fourier transform of $f(x)$.

Alternative form of Fourier transform

Ⓐ if $F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$ ——— ①

Then, $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} du$ ——— ②

The function $F(u)$ is also called the Fourier transform of $f(x)$ and is sometimes written as $F(u) = F\{f(x)\}$

The function $f(x)$ is called the inverse Fourier transform of $F(u)$ and is sometimes written as

$f(x) = F^{-1}\{F(u)\}$.

Ⓑ if $F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$ ——— ①

Then $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{-iux} du$ — ②

The function $F(u)$ is called the Fourier transform of $f(x)$ is sometimes written as $F(u) = F\{f(x)\}$

The function $f(x)$ is called the inverse Fourier transform of $F(u)$ and is sometimes written as

$f(x) = F^{-1}\{F(u)\}$

(c) If $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(u) e^{-iux} dx$ ——— (1)

then $C(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{iut} dt$ ——— (2)

The function $f(x)$ is called the Fourier transform of $C(u)$ and $C(u)$ is called the inverse Fourier transform of $f(x)$.

Fourier Integral: The Fourier Integral is very useful in the field of electrical Communication and forms of basis of Cauchy's method for the solution of partial differential equation.

General Fourier Series of a periodic function $f(x)$ in the interval $(-c,c)$ is given by

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt \sin\frac{n\pi x}{c} \quad \text{—(i)}$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\left[\sum_{n=1}^{\infty}\left\{\cos\frac{n\pi x}{c}\cos\frac{n\pi t}{c} + \sin\frac{\pi n x}{c}\sin\frac{n\pi t}{c}\right\}\right]dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{2c}\int_{-c}^{c} f(t)\left[2\sum_{n=1}^{\infty}\cos\left\{\frac{n\pi(x-t)}{c}\right\}\right]dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\left[1+2\sum_{n=1}^{\infty}\cos\left\{\frac{n\pi(x-t)}{c}\right\}\right]dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\frac{\pi}{c}+\sum_{n=1}^{\infty}2\cdot\frac{\pi}{c}\cos\left\{\frac{n\pi(x-t)}{c}\right\}\right]dt.$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\frac{\pi}{c}\cos\left\{0\cdot\frac{\pi}{c}(x-t)\right\}+\sum_{n=1}^{\infty}\frac{\pi}{c}\cos\left\{\frac{n\pi(x-t)}{c}\right\}+\sum_{n=1}^{\infty}\frac{\pi}{c}\cos\left\{-\frac{n\pi(x-t)}{c}\right\}\right]dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\,\frac{\pi}{c}\sum_{n=0}\left[\cos\left\{\frac{n\pi(x-t)}{c}\right] + \cos\left[\left\{\frac{-n\pi(x-t)}{c}\right.\right.$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{n=-n}^{n}\frac{\frac{1}{c}}{\frac{c}{\pi}}\cos\left\{\frac{n\pi}{c}(x-t)\right\}\right]dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{n}\frac{1}{\frac{c}{\pi}}\cos\left\{\frac{r}{\frac{c}{\pi}}(x-t)\right\}\right]dt \text{ --- (ii)}$$

Now when $c\to\infty$, $\frac{c}{\pi}\to\infty$ and we have

$$\lim_{c\to\infty}\sum_{r=-\infty}^{\infty}\frac{1}{\frac{c}{\pi}}\cos\left\{\frac{r}{\frac{c}{\pi}}(x-t)\right\}$$

$$= \lim_{\Delta u\to 0}\sum_{r=-\infty}^{\infty}\cos r\,\Delta u\,(x-t)\,\Delta u \text{ where } \Delta u = \frac{1}{\frac{c}{\pi}}$$

$$= \int_{-\infty}^{\infty}\cos\{u(x-t)\}\,du \left\{\begin{array}{l}\text{writing } r\,\Delta u = u \text{ and}\\ \Delta u = du\end{array}\right\}$$

By the definition of the integral as the limit of a sum.

Substituting this value of the sum in the equation – (ii)

We get, $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, dt . \int_{-\infty}^{\infty} \cos\{u(x-t)\} du$ ——— (iii)

This double integral is known as Fourier integral and holds if x is a point of continuity of $f(x)$.

The Second integral in the equation - (iii) can be written as:-

$$\int_{-\infty}^{\infty} \cos\{u(x-t)\} du = \int_{-\infty}^{0} \cos\{u(x-t)\} du +$$

$$\int_{0}^{\infty} \cos\{u(x-t)\} du = 2 \int_{0}^{\infty} \cos\{u(x-t)\} du$$

Thus equation (iii) can also written as

f

Thus equation (iii) can also written as

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt.\int_{0}^{\infty} \cos\{u(x-t)\,du$$

$$= \frac{1}{\pi}\int_{0}^{\infty} du \int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}\,du \quad \text{—(iv)}$$

which gives another form of the Fourier integral.

Fourier Integral for an even function:

When ~~f(x)~~ $f(x)$ is an even function of x, that is, $f(-x) = f(x)$,

we have $\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}\,dt$

$$= \int_{-\infty}^{0} f(t)\cos\{u(x-t)\,dt + \int_{0}^{\infty} f(t)\cos\{u(x-t)\}\,dt$$

[replacing t by (-t) in the first integral]

$$= \int_{-\infty}^{0} f(-t) \cos\{u(x-t)\} dt + \int_{0}^{\infty} f(t) \cos\{u(x-t)\} dt$$

$$= \int_{0}^{\infty} f(t) \cos\{u(x+t)\} dt + \int_{0}^{\infty} f(t) \cos\{u(x-t)\} dt$$

$$= 2\int_{0}^{\infty} f(t) \cos ut \cos ux dt$$

Substituting this result in the equation (iv)

we get, $f(x) = \frac{2}{\pi} \int_{0}^{\infty} du \int_{0}^{\infty} f(t) \cos ut \cos ux dt$

$$= \frac{2}{\pi} \int_{0}^{\infty} f(t) dt \int_{0}^{\infty} \cos ut \cos ux dt \quad \text{— (v)}$$

which gives Fourier integral of an even function.

For an odd Function:

When $f(x)$ is an odd function of x, that is,

$f(-x) = -f(x)$; We have

$$\int_{-\infty}^{\infty} f(t) \cos\{u(x-t)\} \, dt$$

$$= \int_{-\infty}^{0} f(t) \cos\{u(x-t)\} \, dt + \int_{0}^{\infty} f(t) \cos\{u(x-t)\} \, dt$$

On replacing t by $(-t)$ in the first integral on the right hand side, we have

$$\int_{-\infty}^{0} f(t) \cos\{u(x-t)\} \, dt = \int_{\infty}^{0} f(-t) \cos\{u(x+t)\} \, -dt$$

$$= -\int_{0}^{\infty} f(t) \{u(x+t)\} \, dt$$

Thus $\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)dt$

$$= \int_{0}^{\infty} f(t)\left[\cos\{u(x-t) - \cos\{u(x+t)\}\right] dt$$

$$= 2\int_{0}^{\infty} f(t)\sin ux \sin ut\, dt \text{ —— (vi)}$$

Substituting this relation in the equation —(iv)

we get $f(x) = \frac{2}{\pi}\int_{0}^{\infty} du \int_{0}^{\infty} \sin ut \sin ux\, dt$

$$= \frac{2}{\pi}\int_{0}^{\infty} f(t)\, dt \int_{0}^{\infty} \sin ut \sin ux\, du$$

which is the Fourier integral of an odd function

Sumon ROY-008

Related Math of this topics:

Example: 1. Show that $\int_0^\infty \frac{\cos ux}{u^2+1} du = \frac{\pi}{2} e^{-x}, x>0$.

proof: Let $f(x) = \begin{cases} e^{x}, x<0 \\ e^{-x}, x>0 \end{cases}$

Therefore, $f(x)=f(-x)$. Thus $f(x)$ is an even function. Now by defination of Fourier integral of an even function for $x>0$, we have $f(x) = \frac{2}{\pi}\int_0^\infty du \int_0^\infty f(t) \cos ut \cos ux dx$ — (i)

putting $f(x)=e^{-x}$ in (i) we get

$$e^{-x} = \frac{2}{\pi}\int_0^\infty du \int_0^\infty e^{-t} \cos ut \cos ux dt.$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\,du \int_0^\infty e^{-t}\cos ut\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\,du \left[\frac{e^{-t}}{1+u^2}(-\cos ut+\sin ut)\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\left\{0+\frac{1}{1+u^2}\right\}du =$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos ux}{u^2+1}\,du$$

$$\therefore \int_0^\infty \frac{\cos ux}{u^2+1}\,du = \frac{\pi}{2}e^{-x},\quad x>0$$

<u>Note:</u> $\int_0^\infty e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$.

Example: 2

prove that $\int_0^\infty \frac{x \sin mx}{x^2+1}\,dx = \frac{\pi}{2} e^{-m}, m>0$

proof: Let $f(x) = \begin{cases} e^{-m}, & m>0 \\ -e^{m}, & m<0 \end{cases}$

then $f(-m) = \begin{cases} e^{m}, & m<0 \\ -e^{-m}, & m>0 \end{cases}$

Again $-f(m) = \begin{cases} -e^{-m}, & m>0 \\ e^{m}, & m<0 \end{cases}$

$\therefore f(-m) = -f(m)$

Thus $f(x)$ is an odd function. Now by definition of Fourier integral of an odd function for $m>0$, we have

$$f(m) = \frac{2}{\pi}\int_0^\infty dx \int_0^\infty f(t) \sin xt \sin mx\,dt \quad \text{(i)}$$

putting $f(m) = e^{-m}$ in $-$ (i), we get

$$e^{-m} = \frac{2}{\pi}\int_0^\infty dx \int_0^\infty e^{-t} \sin xt \sin mx \, dt$$

$$= \frac{2}{\pi}\int_0^\infty \sin mx \, dx \int_0^\infty e^{-t} \sin xt \, dt.$$

$$= \frac{2}{\pi}\int_0^\infty \sin mx \, dx \left[\frac{e^{-t}}{1+x^2}(-\sin xt - x\cos xt)\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \sin mx \, dx \left[0 + \frac{x}{1+x^2}\right]$$

$$= \frac{2}{\pi}\int_0^\infty \frac{x \sin mx}{x^2+1} dx$$

or $\displaystyle\int_0^\infty \frac{x \sin mx}{x^2+1} dx = \frac{\pi}{2} e^{-m}, m>0.$

Example: 3. Find the Fourier integral of the function $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = f(x)$ for $k > 0$, and hence prove that,

$$\int_0^\infty \frac{\cos ux}{k^2+u^2}\,du = \frac{\pi}{2k} e^{-kx}.$$

Solution: Since $f(-x) = f(x)$, so $f(x)$ is an even and for the even function of Fourier integral, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty f(t)\,dt \int_0^\infty \cos ut \cos ux\,du$$

$$= \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(t)\cos ut\,dt\right]\cos ux\,du \quad \text{(i)}$$

Now $\int_0^{\infty} f(t) \cos ut \, dt = \int_0^{\infty} e^{-kt} \cos ut \, dt$

$$= \left[\frac{e^{-kt}}{k^2+u^2} (-k\cos ut + u \sin ut) \right]_0^{\infty}$$

$$= 0 + \frac{k}{k^2+u^2} = \frac{k}{k^2+u^2}$$

Thus, form (1) we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{k}{k^2+u^2} \cos ux \, du$$

$$= \frac{2k}{\pi} \int_0^{\infty} \frac{\cos ux}{k^2+u^2} \, du \; (x>0, k>0) \quad \text{(ii)}$$

Which is required Fourier integral of the function $f(x) = e^{-kx}$.

Again putting $f(x) = e^{-kx}$ in (ii) equation (ii), we get, $e^{-kx} = \frac{2k}{\pi}\int_0^{\infty} \frac{\cos ux}{k^2+u^2}\,dx$.

$$\therefore \int_0^{\infty} \frac{\cos ux}{k^2+u^2}\,du = \frac{\pi}{2k} e^{-kx}.$$

Example: 4 Find the Fourier integral of the function $f(x) = \begin{cases} 0 & \text{when } x<0 \\ \frac{1}{2} & \text{when } x=0 \\ e^{-x} & \text{when } x>0 \end{cases}$

Solution: By the definition of the Fourier integral, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \cos u(x-t)\,du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_0^{\infty} \cos u(x-t)\,du$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\,dt \int_0^{\infty} (\cos ux \cos ut + \sin ux \sin ut)\,du$$

$$= \frac{1}{\pi}\left[\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(t) \cos ut\,dt\right\} \cos ux\,du + \right.$$

$$\left. + \int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(t) \sin ut\,dt\right\} \sin ux\,du\right] \text{ --- (i)}$$

Now $\int_{-\infty}^{\infty} f(t) \cos ut\,dt = \int_{-\infty}^{0} f(t) \cos ut\,dt$

$$+ \int_0^{\infty} f(t) \cos ut\,dt$$

$$= \int_{-\infty}^{0} 0 \cos ut\,dt \int_0^{\infty} e^{-t} \cos ut\,dt$$

$$= 0 + \left[\frac{e^{-t}}{1+u^2}(-\cos ut + \sin ut)\right]_0^{\infty}$$

$$= \frac{1}{1+u^2} + 0 = \frac{1}{1+u^2}$$

Similarly, $\int_{-\infty}^{\infty} f(t) \sin ut \, dt = \int_{-\infty}^{0} f(t) \sin ut \, dt$

$+ \int_{0}^{\infty} f(t) \sin ut \, dt$

$= \int_{-\infty}^{0} 0 \sin ut \, dt + \int_{0}^{\infty} e^{-t} \sin ut \, dt$

$= 0 + \int_{0}^{\infty} \sin ut \, dt$

$= \left[\frac{e^{-t}}{1+u^2} (-\sin ut - u \cos ut) \right]_{0}^{\infty}$

$= 0 + \frac{u}{1+u^2} = \frac{u}{1+u^2}$

putting these value in equation - (i)

we get, $f(x) = \frac{1}{\pi} \left[\int_{0}^{\infty} \frac{\cos}{1+u^2} du + \int_{0}^{\infty} \frac{u \sin ux}{1+u^2} du \right]$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\frac{\cos ux + u\sin ux}{1+u^2}\right)du \quad \text{—— (ii)}$$

putting $x=0$ in equation — (ii), we get.

$$f(0) = \frac{1}{\pi}\int_0^{\infty}\frac{du}{1+u^2} = \frac{1}{\pi}\left[\tan^{-1}u\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{\pi}\cdot\left(\frac{\pi}{2}-0\right)$$

$$= \frac{1}{2}$$

So, $f(x) = \frac{1}{2}$ for $x=0$ is satisfied.

Hence, $f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{(\cos ux + u\sin ux)}{1+u^2}\,du$

Which is required Fourier integral of the given function.

18ICTCSE009

Q. The function x^2 is periodic with period $2L$ on the interval $[-L, L]$ find its fourier series

Solution: $f(x) = x^2$, $f(-x) = (-x)^2 = (-x)^2 = x^2 = f(x)$

So $f(x)$ is an even function and hence sine terms will vanish $b_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \;(n=0)$

and $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \;(n = 1, 2, 3 \ldots)$

Since, $f(x)$ is even

$a_0 = \frac{1}{L}\int_0^L f(v)\,dv$. $a_n = \frac{2}{L}\int_0^L f(v)\cos\frac{n\pi v}{L}\,dv$

$a_0 = \frac{1}{L}\int_0^L x^2\,dx = \frac{1}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{1}{3}L^2$

$a_n = \frac{2}{L} \boxed{\cancel{x \frac{1}{nx}}} \int_0^L x^2 \cos\frac{n\pi x}{L}\,dx.$

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$$= \frac{2}{c}\left[x^2 \frac{1}{n\pi} \sin\frac{n\pi x}{c}\right]_0^1 - \frac{2}{c}\cdot 2\int_0^1 x \frac{1}{n\pi} \sin\frac{n\pi x}{c}\, dx$$

$$= 0 - \frac{4}{n\pi}\int_0^1 x \sin\frac{n\pi x}{c}\, dx$$

$$= \frac{4c}{n^2\pi^2}\left[c\cos n\pi - 0\right] - \frac{4c^2}{n^3\pi^3}\left[\sin\frac{n\pi x}{c}\right]_0^c$$

$$= \frac{4c^2}{n^2\pi^2}(-1)^n - 0 = \frac{4c^2}{n^2\pi^2}(-1)^n \quad \therefore a_n = \frac{4c^2}{n^2\pi^2}(-1)^n$$

therefore,

$$f(x) = \frac{c^2}{3} + \sum_{n=1}^{\infty} \frac{4c^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{c}$$

$$= \frac{c^2}{3} + \frac{4c^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cos\frac{n\pi x}{c}$$

$$= \frac{c^2}{3} + \frac{4c^2}{\pi^2}\left[-\frac{1}{1^2}\cos\frac{\pi x}{c} + \frac{1}{2^2}\cos\frac{2\pi x}{c} - \frac{1}{3^2}\cos\frac{3\pi x}{c} + \ldots\ldots\right]$$

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$\boxed{\text{18ICT } \cancel{009} \text{ CSE009}}$

$$= \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{L} - \frac{1}{2^2}\cos\frac{2\pi x}{L} + \cdots\right]$$

$\boxed{\text{Ans}}$

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18ICTCSE009

4A (16)

Find the fourier cosine transform of a function $f(x) = \begin{cases} 1 & ; 0 < x < a \\ 0 & ; x \geq a \end{cases}$

Ans:

We know from fourier cosine transform,

$$f_c(u) = \int_0^\infty f(x) \cos(ux)\, dx$$

here,

$$f_c(u) = \int_0^a 1 \cdot \cos(ux)\, dx$$

$$= \left[\frac{\sin ux}{u} \right]_0^a$$

$$= \frac{\sin ua}{u} \quad \text{[Answer]}$$

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4A (17)

17. What is the function whose fourier cosine transfrom $\frac{\sin au}{u}$

Ans

we know

$$f(x) = \frac{2}{\pi}\int_0^\infty f_c(u)\cos ux \, du$$

$$= \frac{2}{\pi}\int_0^\infty \sin\frac{au}{u} \cdot \cos ax \, du$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{2} \frac{\sin(a+x)u + \sin(a-x)}{u} \, du$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)}{u} \, du + \int_0^\infty \frac{\sin(a-x)}{u} \, du$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)u}{(a+x)u}(a+x) \, du + \int_0^\infty \frac{\sin(a-x)(a-x)}{u(a-x)} \, du$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)u}{u(a+x)} \, d\{(a+x)u\} + \int_0^\infty \frac{\sin(a-x)a}{au(a-x)} \, d\{u(a-x)\}$$

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$$\Rightarrow \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) \text{ if } (x < a)$$

$$= 1$$

else $\frac{1}{\pi}\left(\pi/2 - \pi/2\right)$

$$= 0$$

So function are

$$f(x) = \begin{cases} 1, & 0 \le x < a \\ 0, & x \ge a \end{cases}$$

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Exaple 2:

The complex from of the fourier series can be written as,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{c}}, \quad -c < x < c$$

Where;

$$C_n = \frac{1}{2c}\int_{-c}^{c} f(x) e^{-\frac{in\pi x}{c}} dx$$

and $n = 0, \pm 1, \pm 2 \ldots$

1st portion,

By def^m, we have $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ — ①

Where, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

now $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} 1\, dx\right]$$

$$= \frac{1}{2\pi}[0 + \pi] = \frac{1}{2} \quad \therefore \boxed{a_0 = \frac{1}{2}}$$

Again, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \; (n \neq 0)$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$

$= \frac{1}{\pi}[0] + \frac{1}{\pi n}\left[\sin nx\right]_{0}^{\pi} = 0 + 0 = 0 \therefore \boxed{a_n = 0}$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_{0}^{\pi} 1.\sin nx\,dx\right]$

$= \frac{1}{\pi}[0] + \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_{0}^{\pi}$

$= 0 - \frac{1}{n\pi}(\cos n\pi - \cos 0)$

$= -\frac{1}{n\pi}\left[(-1)^n - 1\right]$

$= \frac{1}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{2}{n\pi} & \text{when } n \text{ is odd} \end{cases}$

now putting the values of a_0, a_n and b_n in ① we get

$f(x) = \frac{1}{2} + 0 + \sum_{n=1}^{\infty} b_n \sin nx$

$= \frac{1}{2} + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \cdots$

$= \frac{1}{2} + \frac{2}{\pi}\sin x + 0 + \frac{2}{3\pi}\sin 3x + 0 + \frac{2}{5\pi}\sin 5x + \cdots$

$= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

second portion.

now we have to expand the function $f(x)$ in the complex fourier series By def^n, we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{c}}, \quad -c < x < c$$

where, $c_n = \frac{1}{2c}\int_{-c}^{c} f(x) e^{-\frac{in\pi x}{c}} dx$, $n = 0, \pm 1, \pm 2 \cdots$

Here in our given problem $c = \pi$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad -\pi < x < \pi$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$n = 0, \pm 1, \pm 2, \pm 3 \cdots$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\,dx\right]$$

$$= \frac{1}{2\pi}\left[0 + \pi\right] = \frac{1}{2} \quad \therefore \boxed{c_0 = \frac{1}{2}}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad (n \neq 0)$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x) e^{-inx} dx + \int_{0}^{\pi} f(x) e^{-inx} dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\, e^{-inx} dx + \int_{0}^{\pi} 1 \cdot e^{-inx} dx\right]$$

$$= \cancel{0} + \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_{0}^{\pi}$$

$$= -\frac{1}{2n\pi}\left[e^{-in\pi} - e^{0}\right]$$

$$= -\frac{1}{2n\pi i}\left[\cancel{e^{-in\pi}} \cos n\pi - i \sin n\pi - 1\right]$$

$$= -\frac{1}{2n\pi i}\left[(-1)^{n} - 1\right] \text{ since } \sin n\pi = 0$$

$$= \begin{cases} \frac{1}{n\pi i} & \text{when } n = \pm 1, \pm 3, \pm 5 \cdots \\ 0 & \text{when } n = \pm 2, \pm 4, \pm 6 \cdots \end{cases}$$

Thus $f(x) = \frac{1}{2} + \frac{1}{\pi i}\left(\frac{e^{ix}}{1} + 0 + \frac{e^{i3x}}{3} + 0 + \frac{e^{i5x}}{5} + \cdots\right) + \frac{1}{\pi i}\left(\frac{e^{-ix}}{-1} + 0 + \frac{e^{-i3x}}{-3} \cdots\right)$

$$= \frac{1}{2} + \frac{1}{\pi i}\left[(e^{ix} - e^{-ix}) + \frac{1}{3}(e^{i3x} - e^{-i3x}) + \frac{1}{5}(e^{i5x} - e^{-i5x}) + \cdots\right]$$

$$= \frac{1}{2} + \frac{1}{\pi i}\left[2i \sin x + \frac{1}{3} 2i \sin 3x + \frac{1}{5} 2i \sin 5x + \cdots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

Hence, $f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$

which is same as in the real form.

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$\boxed{\text{Example : 8}}$ Find the Fourier sine transfrom of e^{-x}, $x \geq 0$

Solⁿ: By defination of Furier sine transfrom of $f(x)$ for $0 < x < \infty$, we have

$$f_s(n) = \int_0^\infty F(x) \sin nx \, dx$$

$$\therefore f_s(n) = \int_0^\infty e^{-x} \sin nx \, dx \quad \text{Since } F(x) = e^{-x}.$$

$$\text{or, } f_s(n) = \left[-\frac{e^{-x}}{n} \cos nx \right]_0^\infty - \int_0^\infty \frac{e^{-x} \cos nx}{n} dx$$

$$\text{or, } f_s(n) = 0 + \frac{1}{n} - \left[\frac{e^{-x} \sin nx}{n^2} \right]_0^\infty - \int_0^\infty \frac{e^{-x} \sin nx}{n^2} dx$$

$$\text{or, } f_s(n) = 0 + \frac{1}{n} - \frac{1}{n^2} f_s(n)$$

$$\text{or, } f_s(n) = \frac{1}{n} - \frac{1}{n^2} f_s(n)$$

$$\text{or. } f_s(n) + \frac{1}{n^2} f_s(n) = \frac{1}{n}$$

$$\text{or, } \left(1 + \frac{1}{n^2}\right) f_s(n) = \frac{1}{n}$$

$$\text{or, } \left(\frac{n^2+1}{n^2}\right) f_s(n) = \frac{1}{n}$$

$$\text{or, } f_s(n) = \frac{1}{n} \cdot \frac{n^2}{n^2+1}$$

$$\therefore f_s(n) = \frac{n}{n^2+1}.$$

Hence the Fourier sine transfrom of e^{-x} is $\frac{n}{n^2+1}$.

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$\boxed{\text{Example : 12}}$ Find the (a) finite Fourier sine transform (b) Finite Fourier cosine transtrom of the function.

$$F(x) = 2x, \quad 0<x<4$$

Sol^n: (a) Since $l = 4$, we have

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx$$

$$= \int_0^4 F(x) \sin \frac{n\pi x}{4} dx$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4} dx$$

$$= \left[-2x \cdot \frac{4}{n\pi} \cos \frac{n\pi x}{4}\right]_0^4 + \frac{8}{n\pi}\int_0^4 \cos \frac{n\pi x}{4} dx.$$

$$= -\frac{32}{n\pi} \cos n\pi + 0 + \frac{32}{n^2\pi^2}\left[\sin \frac{n\pi x}{4}\right]_0^4$$

$$= -\frac{32}{n\pi} \cos n\pi + \frac{32}{n^2\pi^2}(0-0)$$

$= -\frac{32}{n\pi} \cos n\pi$ which is the finite Fourier sine transtrom of $F(x) = 2x$.

(b) If $n>0$. $f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx$

$$= \int_0^4 2x \cos \frac{n\pi x}{4} dx$$

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$$= \left[2x \cdot \frac{4}{n\pi} \sin \frac{n\pi x}{4}\right]_0^4 - \frac{8}{n\pi}\int_0^4 \sin \frac{n\pi x}{4}\, dx$$

$$= 0 - \frac{8}{n\pi}\left(-\frac{4}{n\pi}\right)\left[\cos \frac{n\pi x}{4}\right]_0^4$$

$= \frac{32}{n^2\pi^2}(\cos n\pi - 1)$ which is the finite Fourier Cosine transfrom of $F(x) = 2x$.

If $n=0$. $f_c(n) = f_c(0) = \int_0^4 2x\, dx$

$$= 2\left[\frac{x^2}{2}\right]_0^4$$

$$= (4^2 - 0)$$

$$= 16.$$

$\therefore f_c(n) = f_c(0) = 16.$

$\boxed{\text{Exercise:21}}$ Find the Fourier Sine transform of $\frac{e^{-ax}}{x}$.

Sol^n: By defination of Fourier sine transfrom of $f(x)$, wehave,

$$f_s(n) = \int_0^\infty F(x) \sin nx\, dx$$

$$\therefore f_s(n) = \int_0^\infty \frac{e^{-ax}}{x} \sin nx\, dx \quad \text{---} (1)$$

$$\therefore F(x) = \frac{e^{-ax}}{x}.$$

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Differentiating both side w.r.t n

$$\frac{d}{dn}[F_s(n)] = \int_0^\infty \frac{e^{-ax}}{x} \cdot x \cos nx \, dx$$

$$= \int_0^\infty e^{-ax} \cos nx \, dx$$

$$= \left[\frac{e^{-ax}}{a^2+n^2} (-a\cos nx + n \sin nx) \right]_0^\infty$$

$$= \frac{1}{a^2+n^2} [0 - 1(-a+0)]$$

$$\therefore \frac{d}{dn}[F_s(n)] = \frac{a}{a^2+n^2}.$$

or, $F_s(n) = \int \frac{a}{a^2+n^2} dn$

or, $F_s(n) = a \int \frac{1}{a^2+n^2} dn$

or, $F_s(n) = a \cdot \frac{1}{a} \tan^{-1} \frac{n}{a} + c.$

$$F_s(n) = \tan^{-1} n/a + c.$$

Put, $n=0$, $F_s(n) = 0$ (From ①)

$$\Rightarrow c = 0.$$

$\therefore F_s(n) = \tan^{-1} n/a$.

Hence the Fourier sine transfrom of $\frac{e^{-ax}}{x}$ is $\tan^{-1} n/a$.

ID: 014 (Sakib Ahmed)

Example: Use finite Fourier transforms to solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $U(0,t) = 0$; $U(\pi, t) = 0$, $U(x, 0) = 2x$ where $0 < x < \pi$, $t > 0$

Soln: The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad \text{-----} (1)$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial U}{\partial t} \sin nx\,dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx\,dx \quad \text{-----}(2)$$

Let, $u = u(n,t) = \int_0^\pi U(x,t) \sin nx\,dx$

then $\frac{du}{dt} = \int_0^\pi \frac{\partial U}{\partial t} \sin nx\,dx$

$= \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx\,dx$ [using (2)]

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ID: 014 (Sakib Ahamed)

(on intergrating by parts)

$$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^{\pi} - n\int_0^{\pi} \cos nx \frac{\partial U}{\partial x} dx$$

$$= 0 - n\int_0^{\pi} \cos nx \frac{\partial U}{\partial x} dx$$

$$= -n\left[\cos nx\, U(x,t)\right]_0^{\pi} - n^2\int_0^{\pi} \sin nx\, U(x,t) dx$$

$$= 0 - n^2\int_0^{\pi} U(x,t) \sin nx\, dx, \text{ since } U(x,t)=0$$

and $U(0,t)=0$.

$$= -n^2 u, \text{ since } u = \int_0^{\pi} U(x,t)\sin nx\, dx$$

$$\therefore \frac{du}{dt} = -n^2 u$$

or, $\frac{du}{u} = -n^2 dt$

Intergrating both sides, we get

$\log u = -n^2 t + \log A$, A being some constant of integration.

or, $\log u = \log e^{-n^2 t} + \log A$

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ID: 014 (Sakib Ahamed)

$= \log Ae^{-n^2t}$

$\therefore u = Ae^{-n^2t}$ ..... (3)

Now, $u = u(n,t) = \int_0^\pi U(x,t)\sin nx\,dx$

$\therefore u(n,0) = \int_0^\pi U(x,0)\sin nx\,dx$

$= \int_0^\pi 2x\sin nx\,dx$ [since $U(x,0) = 2x$]

$= 2\left[-\frac{x\cos nx}{n}\right]_0^\pi + \frac{2}{n}\int_0^\pi \cos nx\,dx$

$= -\frac{2\pi}{n}\cos n\pi + 0 + \frac{2}{n^2}\left[\sin nx\right]_0^\pi$

$= -\frac{2\pi}{n}\cos n\pi$

when $t = 0$, $u(n,0) = Ae^0 = A$

$\therefore A = -\frac{2\pi}{n}\cos n\pi$

Putting the value of A in (3), we get

$u(n,t) = u = -\frac{2\pi}{n}\cos n\pi\, e^{-n^2t}$

2021/6/27 00:32

ID: 014 (Sakib Ahamed)

Applying the inversion formula for finite Fourier sine transform, we get $U(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(-\frac{2\pi}{n} \cos n\pi e^{-n^2 t}\right) \sin nx$

For physical interpretation, $U(x,t)$ may be regarded as the temperature at any point. x at an instant of time t in a solid bounded by the planes $x=0$ and $x=\pi$. The boundary conditions $U(0,t)=0$ and $U(\pi,t)=0$ give the zero temperature at the ends while $U(x,0) = 2x$ represents that the initial temperature is a function of x.

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Ismail Hosen

ID : 15

* Double Fourier series:

The idea of a fourier series expansion for a function of a single variable x can be extended to the case of functions of two variables x and y i.e. $f(x,y)$. As for example, we can explain $f(x,y)$ into a double Fourier.

sine series $f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{c_1} \sin \frac{n\pi y}{c_2}$

where, $B_{mn} = \frac{4}{c_1 c_2} \int_0^{c_1} \int_0^{c_2} f(x,y) \sin \frac{m\pi x}{c_1} \sin \frac{n\pi y}{c_2} \, dx dy$

* Similarly, we can expand $f(x,y)$ into a double Fourier cosine series,

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos \frac{m\pi x}{c_1} \cos \frac{n\pi y}{c_2}$$

where, $B_{mn} = \frac{4}{c_1 c_2} \int_0^{c_1} \int_0^{c_2} f(x,y) \cos \frac{m\pi x}{c_1} \cos \frac{n\pi y}{c_2} \, dx dy$

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* The Fourier cosine and sine series :

→ Difinition Even function :

A function $f(x)$ is called even if $f(-x) = f(x)$

Graphically, an even function is symmetrical about the y axis.

If $f(x)$ is an even function, then,

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= \int_{\pi}^{0} f(-x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= -\int_{\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= \int_{0}^{\pi} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= 2\int_{0}^{\pi} f(x)\,dx$$

Thus if $f(x)$ is even, we have

$$a_0 = \frac{1}{2\pi} \cdot 2\int_{0}^{\pi} f(x)\,dx$$

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$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(v)\,dv.$$

Also if (x) is even i;e $f(-x) = f(x)$

then, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n = 1, 2, 3 \cdots)$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\cos n(-x)\,d(-x)$$

$$= -\frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(v)\cos nv\,dv.$$

but, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\sin n(-x)\,d(-x)$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cdot -\sin nx \cdot -dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -b_n$$

$\therefore 2b_n = 0$ or, $b_n = 0$

Therefore, if $f(x)$ is even, then we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

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$$= \frac{1}{\pi}\int_0^\pi f(v)\,dv + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^\pi f(v)\cos nv\,dv\right\}\cos nx$$

which represent the function $f(x)$ in a series of cosine and therefore it is known as Fourier cosine series in the interval $(0,\pi)$

* Odd function :

A function $f(x)$ is called odd if $f(-x) = -f(x)$

Graphically, an odd function is symmetrical about the origin when $f(x)$ is odd, we have,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^\pi f(v)\sin nv\,dv\right\}\sin nx.$$

which represent the function $f(x)$ in a series of sines in the interval $(0,\pi)$ and therefore it is known as Fourier sine serice in the interval $(0,\pi)$.

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* Parseval's Formula:

$$\rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) \rightarrow \text{Particular case}$$

$$\rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = a_0 a_0' + \frac{1}{2}\sum_{n=1}^{\infty}(a_n a_n' + b_n b_n') \rightarrow \text{General case.}$$

$$\longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) g(x) dx = \sum_{n=\infty} c_n c_n' \rightarrow \text{Complex case.}$$

Worked out examples:

① The function x^2 is periodic with period $2l$ on the interval $[-l, l]$. Find its Fourier series.

Solⁿ: $f(x) = x^2; f(-x) = (-x)^2 = x^2 = f(x)$ so $f(x)$ is an even function and hence sine terms will vanish i.e. $b_n = 0$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{l}{2l}\int_{-l}^{l} f(x)\,dx. \quad (n=0)$

and $a_n = \frac{l}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad (n = 1,2,3\cdots)$

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Since $f(x)$ is even,

$$a_0 = \frac{1}{l}\int_0^l f(v)\,dv\,, \quad a_n = \frac{2}{l}\int_0^l f(v)\cos\frac{n\pi v}{l}\,dv$$

$$a_0 = \frac{1}{l}\int_0^l x^2\,dx = \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{1}{3}l^2$$

$$a_n = \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,dx \quad \text{[Integrating by part]}$$

$$= \frac{2}{l}\left[x^2\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right]_0^l - \frac{2}{l}\cdot 2\int_0^l x\frac{l}{n\pi}\sin\frac{n\pi x}{l}\,dx$$

$$= 0 - \frac{4}{n\pi}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4}{n\pi}\cdot\frac{l}{n\pi}\left[x\cos\frac{n\pi x}{l}\right]_0^l - \frac{4}{n\pi}\cdot\frac{l}{n\pi}\int_0^l \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{4l}{n^2\pi^2}\left[l\cos n\pi - 0\right] - \frac{4l^2}{n^3\pi^3}\left[\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n - 0 = \frac{4l^2}{n^2\pi^2}(-1)^n \quad \therefore a_n = \frac{4l^2}{n^2\pi^2}(-1)^n$$

Therefore, $f(x) = \dfrac{l^2}{3} + \displaystyle\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{l}$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n\cdot\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[-\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{2^2}\cos\frac{2\pi x}{l} - \cdots\right.$$

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exerceises:

Q. (a) Fourier sine transform:

Sol. $f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l}\, dx$; $F(x)=1;\ 0<x<l$

$= \int_0^l 1 \cdot \sin \frac{n\pi x}{l}\, dx$

$= \left[\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right]_0^l$

$= \frac{-\cos \frac{n\pi l}{l}}{\frac{n\pi}{l}} + \frac{1}{\frac{n\pi}{l}}$

$= \frac{-\cos n\pi + 1}{\frac{n\pi}{l}}$

$= \frac{l(1-\cos n\pi)}{n\pi}$ (Ans).

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(b) Fourier cosine transform:

Sol Fourier cosine tranform of $F(x)=1$ where $0<x<l$.

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \int_0^l \cos \frac{n\pi x}{l}\, dx$$

$$= \left[\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right]_0^l$$

$$= \frac{\sin \frac{n\pi l}{l}}{\frac{n\pi}{l}} - 0$$

$$= \frac{\sin n\pi}{\frac{n\pi}{l}}$$

$= 0$ [if $n=1,2,3\cdots$; l if $n=0$.] (Ans)

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Exercise: 9

(a) Find Fourier sine transform:

Q: Find the Fourier sine transform of $F(x) = x^2$ where $0 < x < b$.

Soln:

$$f_s(n) = \int_0^b F(x) \sin\frac{n\pi x}{b}\,dx$$

$$= \left[x^2\left(\frac{-\cos\frac{n\pi x}{b}}{\frac{n\pi}{b}}\right) - 2x\left(\frac{-\sin\frac{n\pi x}{b}}{\frac{n^2\pi^2}{b^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{b}}{\frac{n^3\pi^3}{b^3}}\right)\right]_0^b$$

$$= -\frac{b^3}{n\pi}\cos n\pi + \frac{2b^3}{n^3\pi^3}\cos n\pi - \frac{2b^3}{n^3\pi^3}$$

$$= \frac{2b^3}{n^3\pi^3}(\cos n\pi - 1) - \frac{b^3}{n\pi}\cos n\pi.$$

(Ans)

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9 (b)

Fourier cosine transform of $F(x) = x^2$

where, $0 < x < b$

$$f_c(n) = \int_0^b F(x) \cos \frac{n\pi x}{b} \, dx$$

$$= \int_0^b x^2 \cos \frac{n\pi x}{b} \, dx$$

$$= \left[x^2 \left(\frac{\sin \frac{n\pi x}{b}}{\frac{n\pi}{b}} \right) - 2x \left(\frac{-\cos \frac{n\pi x}{b}}{\frac{n^2\pi^2}{b^2}} \right) + 2 \left(\frac{-\sin \frac{n\pi x}{b}}{\frac{n^3\pi^3}{b^3}} \right) \right]_0^b$$

$$= \left[b^2 \left(\frac{\sin n\pi}{\frac{n\pi}{b}} \right) - 2b \left(\frac{-\cos n\pi}{\frac{n^2\pi^2}{b^2}} \right) + 2 \left(\frac{-\sin n\pi}{\frac{n^3\pi^3}{b^3}} \right) \right.$$

$$= \frac{2b^3}{n^2\pi^2} (\cos n\pi - 1).$$

(Ans)

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Example 10: Use the complex from of the fourier transform to solve the boundary value problem

$$\frac{du}{dt} = k\frac{d^2u}{dx^2}, \; U(x,0) = f(x), \; U(x,t) < M,$$

where $-\alpha < x < \alpha, \; t > 0$

Solution:
The given partial differential equation is

$$\frac{du}{dt} = k\frac{d^2u}{dx^2} \text{ ——— (1)}$$

Taking the complex from the Fourier transform with respect to x of both sides of (1),
we get,

$$\int_{-\alpha}^{\alpha} \frac{du}{dt} e^{i\alpha x}\, dx = \int_{-\alpha}^{\alpha} k\frac{d^2u}{dx^2} e^{i\alpha x}\, dx \text{ ——— (2)}$$

let $\mathcal{F}(U) = \int_{-\alpha}^{\alpha} U(x,t) e^{i\alpha x}\, dx$

$$\therefore \frac{df(U)}{dt} = \int_{-\alpha}^{\alpha} \frac{du}{dt} e^{i\alpha x}\, dx$$

$$= \int_{-\alpha}^{\alpha} k\frac{d^2u}{dx^2} e^{i\alpha x}\, dx \text{ using (2)}$$

$$= k\left[e^{i\alpha x}\frac{du}{dx}\right]_{-\alpha}^{\alpha} - ki\alpha\int_{-\alpha}^{\alpha} e^{i\alpha x}\frac{du}{dt}\, dx.$$

$$= 0 - ki\alpha\int_{-\alpha}^{\alpha} e^{i\alpha x}\frac{du}{dt}\, dx$$

$$\left[\text{Since } \frac{du}{dt} \to 0 \text{ as } x \to \pm\alpha\right]$$

$$= -ki\alpha\left[e^{i\alpha x} U(x,t)\right]_{-\alpha}^{\alpha} + ki^2\alpha^2\int_{-\alpha}^{\alpha} e^{i\alpha x} U(x,t)\, dx$$

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$$= 0 - ka^2 \int_{-\alpha}^{\alpha} e^{iax} u(x,t)\, dx$$

[since $u \to 0$ as $x \to \alpha$ and $i^2 = -1$]

$$= -ka^2 \int_{-\alpha}^{\alpha} u(x,t)\, e^{iax}\, dx$$

$$= -ka^2 F(u)$$

$$\therefore \frac{dF(u)}{dt} = -ka^2 F(u)$$

$$\text{or, } \frac{dF(u)}{dt} = -ka^2 dt \quad \text{——— (3)}$$

Integrating both sides of (3) we get, $\log F(u) = -ka^2 t + \log c$, c being some constant of integration

$$\text{or, } \log F(u) = \log e^{-ka^2 t} + \log c = \log c\, e^{-ka^2 t}$$

$$\therefore F(u) = c e^{-ka^2 t} \quad \text{——— (4)}$$

That is,

$$F\{u(x,t)\} = F\{f(x)\} = ce^{0} = c \quad \text{——— (5)}$$

Therefore,

$$c = F\{f(x)\} = \int_{-\alpha}^{\alpha} u(x,0)\, e^{iax}\, dx$$

$$= \int_{-\alpha}^{\alpha} f(x)\, e^{iax}\, dx$$

$$= \int_{-\alpha}^{\alpha} f(u)\, e^{iax}\, du.$$

Thus from (4) we have

$$\mathcal{F}\{u(x,t)\} = \mathcal{F}\{f(x)\} e^{-k\alpha^2 t} \quad \text{———} (6)$$

Taking the inverse Fourier transform, we get

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(f) e^{-k\alpha^2 t} e^{-i\alpha x} d\alpha$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} f(u) e^{i\alpha u} du\right] e^{-k\alpha^2 t} e^{-i\alpha x} d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-k\alpha^2 t - i\alpha(x-u)} d\alpha\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{k}t}\right]^2 - \frac{(x-u)^2}{4kt}}\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-\left[\sqrt{t}\left\{\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{k}t}\right\}\right]^2} du\right] e^{-\frac{(x-u)^2}{4kt}} du$$

$$\text{———} (7)$$

putting $\sqrt{t}\left\{\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{k}t}\right\} = y$

So that $\sqrt{kt}\, d\alpha = dy \;\therefore\; d\alpha = \frac{1}{\sqrt{kt}} dy$

Now, $\int_{-\infty}^{\infty} e^{-\left[\sqrt{t}\left\{\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{k}t}\right\}\right]^2} d\alpha$

$$= 2\int_{0}^{\infty} e^{-\left[\sqrt{t}\left\{\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{k}t}\right\}\right]^2} d\alpha$$

$$= 2\int_{0}^{\infty} e^{-y^2} \frac{dy}{\sqrt{kt}}$$

$$=\frac{2}{\sqrt{kt}}\int_0^\alpha e^{-y^2}dy$$

$$=\frac{2}{\sqrt{kt}}\cdot\frac{\sqrt{\pi}}{2}$$

$$=\frac{\sqrt{\pi}}{\sqrt{kt}}$$

Therefore, from (7) we get

$$u(x,t)=\frac{1}{2\pi}\int_{-\alpha}^{\alpha} f(u)\cdot\frac{\sqrt{\pi}}{\sqrt{kt}}\, e^{\frac{-(x-u)^2}{4kt}} du$$

$$=\frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{kt}}\int_{-\alpha}^{\alpha} f(u) e^{-\frac{(x-u)^2}{4kt}} du$$

$$=\frac{1}{\sqrt{4\pi kt}}\int_{-\alpha}^{\alpha} f(u)\, e^{-\frac{(x-u)^2}{4kt}} du \quad\text{———} (8)$$

Now if we change variable from u to z according to the transformation.

$$\frac{(x-u)^2}{4kt}=z^2 \quad \text{or,}\ \frac{x-u}{2\sqrt{kt}}=z$$

$$u=((x-2)\sqrt{kt}z) \quad \text{limits}$$

$$du=-2\sqrt{kt}\,dz \qquad \left.\begin{matrix}u=\alpha\\ z=-\alpha\end{matrix}\right\}\left\{\begin{matrix}u=-\alpha\\ z=\alpha\end{matrix}\right\}$$

Thus form (8) we ~~get~~ have

$$u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int^{\alpha} f(x-2z\sqrt{kt})\,e^{-z^2}\cdot 2\sqrt{kt}\,dz$$

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$$= -\frac{2\sqrt{kt}}{2\sqrt{\pi}\sqrt{kt}} \int_{\alpha}^{-\alpha} f(x - 2z\sqrt{kt})\, e^{-z^2}\, dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\alpha}^{\alpha} f(x - 2z\sqrt{kt})\, e^{-z^2}\, dz$$

Note: $-k\alpha^2 t - i\alpha(x-u)$

$$= -t\left[\sqrt{k}\,\alpha + \frac{i(x-u)}{2\sqrt{kt}}\right]^2 - \frac{(x-u)^2}{4kt}$$

$$= -t\left[k\alpha^2 + \frac{i\alpha}{t}(x-u) + i^2\frac{(x-u)^2}{4kt^2}\right] - \frac{(x-u)^2}{4kt}$$

$$= -k\alpha^2 t - i\alpha(x-u) + \frac{(x-u)^2}{4kt} - \frac{(x-u)^2}{4kt}$$

$$= -k\alpha^2 t - i\alpha(x-u)$$

$$\therefore -k\alpha^2 t - i\alpha(x-u) = -t\left[\sqrt{k}\,\alpha + \frac{i(x-u)}{2\sqrt{kt}}\right]^2 - \frac{(x-u)^2}{4kt}$$

Example 11: Solve $\frac{du}{dt} = \frac{d^2u}{dx^2}$, $0<x<6$, $t>0$

$u_x(0,t)=0$; $u_x(6,t)=0$; $u(x,0)=2x$;

Soalution:

The given partial differential equation is

$$\frac{du}{dt} = \frac{d^2u}{dx^2} \quad \text{——— (1)}$$

According to the given boundary conditions, here the finite fourier cosine transform is more useful Taking the finite Fourier cosine transform ($l=6$) of both sides of (1), we get,

$$\int_0^6 \frac{du}{dt}\cos\frac{n\pi x}{6}\,dx = \int_0^6 \frac{d^2u}{dx^2}\cos\frac{n\pi x}{6}\,dx \quad \text{——— (2)}$$

Let, $V = V(n,t) = \int_0^6 u(x,t)\cos\frac{n\pi x}{6}\,dx = f_c(u)$

$$\therefore \frac{dV}{dt} = \int_0^6 \frac{du}{dt}\cos\frac{n\pi x}{6}\,dx$$

$$= \int_0^6 \frac{d^2u}{dx^2}\cos\frac{n\pi x}{6}\,dx$$

Note: $f_c\left\{\frac{d^2u}{dx^2}\right\} = \int_0^l \frac{d^2u}{dx^2}\cos\frac{n\pi x}{l}\,dx$

$$= \frac{du(l,t)}{dx}\cos n\pi - \frac{du(0,t)}{dx} - \frac{n^2\pi^2}{l^2} f_c(u)$$

$$\therefore \int_0^6 \frac{d^2U}{dx^2} \cos\frac{n\pi x}{6}\,dx = \frac{dU(6,t)}{dx}\cos n\pi - \frac{dU(0,t)}{dx} - \frac{n^2\pi^2}{6^2} f_c(U)$$

$$= U_x(6,t)\cos n\pi - U_x(0,t) - \frac{n^2\pi^2}{6^2} f_c(U)$$

$$= U_x(6,t)\cos n\pi - U_x(0,t) - \frac{n^2\pi^2}{6^2} V$$

Since $f_c(U) = \int_0^6 U(x,t)\cos\frac{n\pi x}{6}\,dx$

$$\therefore \boxed{f_c(U) = V}$$

Therefore,

$$\frac{dV}{dt} = U_x(6,t) - U_x(0,t) - \frac{n^2\pi^2}{6^2} V$$

or, $\frac{dV}{dt} = 0 - 0 - \frac{n^2\pi^2}{6^2} V$ [Since $U_x(0,t) = U_x(6,t) = 0$]

$$\frac{dV}{dt} = \frac{n^2\pi^2}{6^2} V \text{ ——— } (3)$$

Integration both sides, we get $\log V = -\frac{n^2\pi^2}{6^2}t + \log A$, A being some constant of integation.

Or, $\log V = \log e^{-\frac{n^2\pi^2 t}{6^2}} + \log A = \log Ae^{-\frac{n^2\pi^2 t}{6^2}}$

$$\therefore V = Ae^{-\frac{n^2\pi^2 t}{6^2}} \text{ ——— } (4)$$

When $t=0$

R.H.S of (4) $= Ae^{0} = A$

When $t=0$

L.H.S of (4) $= \int_{-\infty}^{\infty} u(x,0)\cos\frac{n\pi x}{6}\,dx$

$$= \int_{0}^{6} 2x\cos\frac{n\pi x}{6}\,dx$$

$$= \left[2x\frac{6}{n\pi}\sin\frac{n\pi x}{6}\right]_0^6 - 2\frac{6}{n\pi}\int_0^6 \sin\frac{n\pi x}{6}\,dx$$

$$= 0 - \frac{12}{n\pi}\int_0^6 \sin\frac{n\pi x}{6}\,dx$$

$$= \frac{12}{n\pi}\cdot\frac{6}{n\pi}\left[\cos\frac{n\pi x}{6}\right]_0^6$$

$$= \frac{72}{n^2\pi^2}(\cos n\pi - 1)$$

Therefore, from (4), we get $A = \frac{72}{n^2\pi^2}(\cos n\pi - 1)$

Putting the value of A in (4), we get

$$V = \frac{72}{n^2\pi^2}(\cos n\pi - 1)\, e^{-\frac{n^2\pi^2 t}{6^2}}$$

Taking the inverse fourier cosine transform, we get,

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(n)\cos\frac{n\pi x}{l}$$

That is, $u(x,t) = \frac{1}{6} f_c(0) + \frac{2}{6}\sum_{n=1}^{\infty} V\cos\frac{n\pi x}{6}$

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or, $u(x,t) = \frac{1}{6} f_c(0) + \frac{1}{3} \sum_{n=1}^{\infty} \frac{72}{n^2\pi^2} (\cos n\pi - 1) \cdot e^{-\frac{n^2\pi^2 t}{6^2}} \cdot \cos \frac{n\pi x}{6}$ —— (5)

$f_{c}(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx = \int_0^6 F(x) \cos \frac{n\pi x}{6} dx$

$\therefore f_c(0) = \int_0^6 F(x) dx = \int_0^6 u(x,0) dx$

$= \int_0^6 2x \, dx = [x^2]_0^6 = 36$

Thus from (5), we get,

$u(x,t) = \frac{36}{6} + \frac{24}{\pi^2} \sum_{n=1}^{\infty} \frac{(\cos n\pi - 1)}{n^2} \cdot e^{-\frac{n^2\pi^2 t}{36}} \cos \cdot \frac{n\pi x}{6}$

or, $u(x,t) = 6 + \frac{24}{\pi^2} \sum_{n=1}^{\infty} \frac{(\cos n\pi - 1)}{n^2} e^{-\frac{n^2\pi^2 t}{36}} \cdot \cos \frac{n\pi x}{6}$.

[Solved]

Example 12:- Use the method of Fourier transform to determine the deplacement $y(x,t)$ of an infinite string, given that the string is initialy at rest and that the initial displacement is $f(x)$, $-\alpha < x < \alpha$. Also show that the solution can be put in the form

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right].$$

Solution:- Here we have to one-dimensional wave equation $\frac{\delta^2 y}{\delta x^2} = c^2 \frac{\delta^2 y}{\delta x^2}$ $-\alpha < x < \alpha$.

Subject to the following initial conditions.

$y(x,0)$ = initial displacement $f(x)$

and $y_t(x,0)$ = initial valocity $= 0$

the given partial differential equation is.

$$\frac{\delta^2 y}{d t^2} = c \frac{\delta^2 y}{\delta x^2} \quad \text{—— (1)}$$

Taking the complex fourier franform of both sides of (1), we have.

$$\int_{-\alpha}^{\alpha} \frac{\delta^2 y}{\delta t^2} e^{-iu\alpha} dx = c^2 \int_{-\alpha}^{\alpha} \frac{\delta^2 y}{\delta x^2} e^{-iu\alpha} dx \quad \text{—— (11)}$$

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By the fourier transform of the derivative of a function we have if $F^n(x)$ is the nth-derivative of $F(x)$ and the first $(n-1)$ — derivatives of $F(x)$ vanish as $x \to \pm\alpha$, then

$$F\{F^n(x)\} = (-iu)^n F\{F(x)\}$$

Thus from (11) we have,

$$\frac{d^2}{dt^2}\int_{-\alpha}^{\alpha} y e^{-iux} = c^2 \int (-iu)^2 F\{y(x,t)\}$$

or, $$\frac{d^2}{dt^2}\int_{-\alpha}^{\alpha} y(x,t) e^{-iux} dx = c^2(-u^2)\{F(y\ x,t)\}$$

$$\Rightarrow \frac{d^2}{dt^2}\{F\ y(x,t)\} = -c^2u^2 F\{y(x,t)\}$$

where, $\bar{y} = \bar{y}(u,t) = F\{y(x,t)\}$

$$= \int_{-\alpha}^{\alpha} y(x,t) e^{-iux} dx,$$

Or, $$\frac{d^2\bar{y}}{dt^2} + c^2u^2\bar{y} = 0 \quad \text{——— (13)}$$

which is ordinary second order differential equation whose solution, is.

$$\bar{y} = \bar{y}(u,t) = A\cos(cut) + B\sin(cut) \quad \text{——— (14)}$$

Differentiating both sides with respect to, t,

we get $\bar{y}_t(u,t) = Acu\sin(cut) + Bcu\cos(cut)$ — (v)

Also from the initial given conditions we have,

$y(x,0) = f(x)$ — (vi)

$y_t(x,0) = 0$ — (vii)

Taking the fourier transform of (vi) and (vii)

we get

$$\bar{y}(u,0) = \int_{-\alpha}^{\alpha} y(x,0)e^{-iux}dx = \int_{-\alpha}^{\alpha} f(x)e^{-iux}dx$$

$$= \bar{f}(u) \text{ (say)}$$

$\therefore \bar{y}(u,0) = \bar{f}(u)$ — (viii)

and $\bar{y}_t(u,0) = \int_{-\alpha}^{\alpha} y_t(x,0)e^{-iux}dx$

$$= \int_{-\alpha}^{\alpha} 0\, e^{-iux}dx$$

$\therefore \bar{y}(u,0) = 0$ — (ix)

putting $t=0$ in (v) we have $\bar{y}_t(u,0) = Bcu$

or $Bcu = 0$ using (ix)

again putting $t=0$ in (iv), we get

$\bar{y}(u,0) = A$

$\therefore \bar{f}(u) = A.$ using (viii)

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Putting the values of A and B (iv) we get.

$\bar{y} = \bar{y}(u,t) = \bar{f}(u)\cos(cut)$ ———— (x)

Taking the inverse fourier transform of (x)

$$y(x,t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \bar{f}(u)\cos(cut)\, e^{-iux}\, du$$

$$\Rightarrow y(x,t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \bar{f}(u)\left(\frac{e^{icut} + e^{-icut}}{2}\right) e^{iux} du$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\alpha}^{\alpha} \bar{f}(u)\, e^{iu(x+ct)} du\right.$$

$$\left. + \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \bar{f}(u)\, e^{iu(x-ct)} du\right]$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

[using the definition of inverse fourier transform]

$$\therefore y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

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Example 13:- A thin membrane of great extent is released from rest in the position $z = f(x,y)$, show that the displacement at any subsequent time is given by,

$$z(y,t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} F(u,v)\cos ct\sqrt{u^2+v^2}\, e^{-i(ux+vy)}\, du\, dv$$

where $F(u,v)$ is the double Fourier transform of $f(x,y)$.

Proof: Here the displacement of the membrane is governed by two dimensional wave equation.

$$\frac{\delta^2 z}{\delta x^2} = c^2\left(\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2}\right) \cdots\cdots (1) \quad c^2 = \frac{T}{\rho}$$

Taking the double Fourier transform of both sides of (1) we get.

$$\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{\delta^2 z}{\delta t^2} e^{i(ux+vy)}\, dx\, dy.$$

$$= \frac{c^2}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\left(\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{dy^2}\right) e^{i(ux+vy)}\, dx\, dy.$$

Or, $$\frac{d^2}{dt^2}\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} z\, e^{i(ux+vy)}\, dx\, dy$$

$$= \frac{c^2}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\left(\frac{\delta^2 z}{dx^2} + \frac{\delta^2 z}{\delta y^2}\right) e^{i(ux+vy)}\, dx\, dy$$

$$\Rightarrow \frac{d^2\bar{z}}{dt^2} = c^2 \left\{ \left\{ (-iu)^2 + (-iv)^2 \right\} F\left\{ z(x,y,t) \right\} \right\}$$

where, $\bar{z} = \bar{z}(u,v,t) = F\{z(x,y,t)\}$

$$= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} z e^{i(ux+vy)} dx dy$$

$$\Rightarrow \frac{d^2\bar{z}}{dt^2} = -c^2(u^2+v^2)\bar{z}$$

$$\Rightarrow \frac{d^2\bar{z}}{dt^2} + c^2(u^2+v^2)\bar{z} = 0 \quad \text{——— (2)}$$

which is an ordinary differential equation whose solution is.

$$\bar{z} = A \cos c\sqrt{u^2+v^2} + B \sin\left\{c\sqrt{u^2+u^2}\right\} \quad \text{——— (3)}$$

the given initial conditions are,

$z = f(x,y)$ and $\frac{\delta z}{dt} = 0$ at $t = 0$

Taking the fourier transform of these initial conditions, we get.

$$\bar{z} = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} f(x,y)\, e^{i(ux+vy)} dx dy$$

$$= F(u,v) \quad \text{——— (4)}$$

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$$\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{\delta z}{\delta t} e^{-i(ux+vy)}\, dx\, dy$$

$$= \frac{d}{dt}\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} z e^{i(ux+vy)}\, dx\, dy.$$

$$= \frac{d\bar{z}}{dt} \text{ since } \bar{z} = \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} z e^{i(ux+vy)}\, dx\, dy$$

$\therefore \frac{d\bar{z}}{dt} = 0$ at $t=0$

when $t=0$, combining (3) and (4), we get,

$A = F(u,v)$

Also, $\frac{d\bar{z}}{dt} = A c\sqrt{u^2+v^2}\sin c\sqrt{u^2+v^2} + Bc\sqrt{u^2+v^2}\cos c\sqrt{u^2+v^2}\, t.$

$\therefore 0 = \frac{d\bar{z}}{dt}\, t=0 = Bc\sqrt{(u^2+v^2)}$

$\Rightarrow B = 0$

putting the values of A B in (3), we get,

$\bar{z} = F(u,v)\cos(c\sqrt{u^2+v^2}) + \quad$ —— (5)

$\therefore$ Inversion formula of double fourier transform,

$$z(x,y,t) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} F(u,v)\cos(c\sqrt{u^2+v^2}\,t)\, e^{i(ux+vy)}\, dv\, du$$

which is the required displacement at any subsequent time t.