BANGABANDHU SHEIKH MUJIBUR RAHMAN SCIENCE AND TECHNOLOGY UNIVERSITY

Name of the Assignment

Scalars and vectors, Equality of vectors, Addition and subtraction of vectors, Dot, Cross and multiple product of vectors, Vector differentiation, Gradient, Divergence and Curl.

Course Code: MAT 205
Course Title: Vector, Matrix and Fourier Analysis
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chapter - 1
Vectors and Scalars

Vector: A vector is a quantity having magnitude and direction, such as displacement, velocity, force, and acceleration.

Analytically a vector is represented by a letter with an arrow over it as $\vec{A}$ and its magnitude is denoted by $|\vec{A}|$ or A.

Scalar: A Scalar is a quantity having magnitude but no direction. Ex: mass, length, time, temperature, and any real number. Scalars are indicated by letters in ordinary type as in elementary algebra. operations with scalars follow the same rules as in elementary algebra.

Vector Algebra: The operations of addition, subtraction and multiplication familiar in the algebra of numbers or scalars are, with suitable definition, capable of extension to an algebra of vectors. The following definitions are fundamental.

1. Two vectors A and B are equal if they have the same magnitude and direction regardless of the position of their initial points. Thus A=B

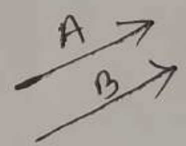
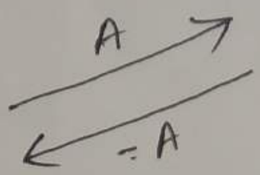


Fig-1

2. A vector having direction opposite of that to vector A but having the same magnitude is denoted by -A



3. The sum or resultant of vectors $\vec{A}$ and B is a vector C formed by placing the initial point of B on the terminal point of A and then joining the initial point of A to the terminal point of B. The sum is written A + B. C = A+B The definition here is equivalent to the parallelogram law for vector addition. Extensions to sums of more than two vectors are immediate

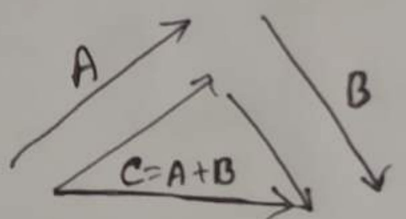


Fig-3

4. The difference of vectors A and B, represented by A − B, is that vector C which added B yields vector A. Equivalently, A − B can be defined as the sum A + (B).

if A = B then A-B is defined as the null or zero vectors and represented by the symbol 0 or simply 0. It has magnitude and no specific direction. A vector which is not null is a proper vector. All vectors will be assumed proper unless otherwise stated.

5. The product of a vector A by a scalar m is a vector mA with magnitude |m| times the magnitude of A and with direction the same as or opposite to that of A, according as m is positive or negative. If m = 0, mA is a null vector.

Laws of vector Algebra: If A, B and C are vectors and m and n are scalars, then

1. $A + B = B + A$ Commutative Law for Addition
2. $A + (B + C) = (A + B) + C$ Associative law for additi

3. $mA = Am$ commutative law for multiplication

4. $m(nA) = (mn)A$ Associative law for multiplication

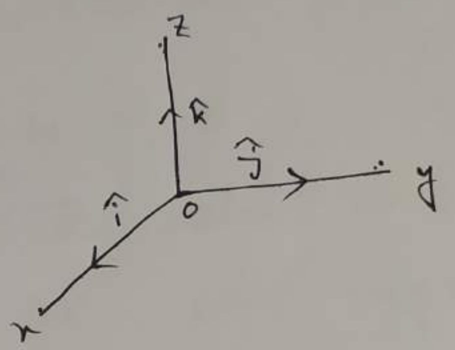
5. $(m+n)A = mA + nA$ Distributive law

6. $m(A+B) = mA + mB$ Distributive law

Unit vector: A unit vector is a vector having unit magnitude, if A is a vector with magnitude $A \neq 0$ then A/A is a unit vector having the same direction as A.

any vector A can be represented by a unit vector a in the direction of A multiplied by the magnitude of A. In symbol, $A = Aa$.

Rectangular unit vector: The rectangular unit vectors $\vec{i}, \vec{j}, \vec{k}$. An important set of unit vectors are those ~~tha~~ having the directions of the positive x, y, and z axes of a three dimonsional rectangular coordinate system, and are denoted respectively by $\hat{i}, \hat{j}, \hat{k}$



Components of A vector: Any vector $\vec{A}$ in 3 dimensions can be represented with initial point at the origin O of a rectangular coordinate system. Let (A_1, A_2, A_3) be the rectangular coordinates of the terminal point of vector $\vec{A}$ with initial point at O. The vectors $A_1 i$, $A_2 j$, and $A_3 k$ are called the rectangular component vectors or simply component vectors of A in the x, y and z directions respectively. A_1, A_2 and A_3 are called the rectangular components or simply components of A in the x, y and z directions respectively.

The sum or resultant of $A_1 i$, $A_2 j$ and $A_3 k$ is the vector **A** so that we can write

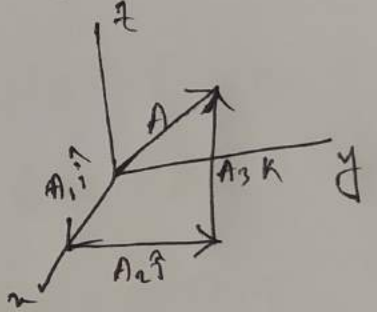
$$A = A_1 i + A_2 j + A_3 k.$$

The magnitude of A is $A = |A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$

In particular, the position vector or radius vector r from O to the point (x, y, z) is written

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

and has magnitude $r = |r| = \sqrt{x^2 + y^2 + z^2}$



Scalar Field: If to each point (x, y, z) of a region R in space there corresponds a number or scalar $\phi(x, y, z)$ then ϕ is called a scalar function of position or scalar point function and we say that scalar field ϕ has been defined in R.

Ex. $\phi(x, y, z) = x^3 y - z^2$ defines a scalar field.

Vector Field: If to each point (x, y, z) of a region R in space there corresponds a vector $v(x, y, z)$, then v is called a vector function of position or vector point function and we say that a vector field v has been defined in R.

Ex. $v(x, y, z) = xy^2\hat{i} - 2yz^3\hat{j} + x^2z\hat{k}$ defines a vector field.

Problem

22. $r_1 = 3\hat{i} - 2\hat{j} + \hat{k}$, $r_2 = 2\hat{i} - 4\hat{j} - 3\hat{k}$, $r_3 = -\hat{i} + 2\hat{j} + 2\hat{k}$, find the magnitudes of

(a) r_3, (b) $r_1 + r_2 + r_3$, (c) $2r_1 - 3r_2 - 5r_3$

(a) $|r_3| = |-\hat{i} + 2\hat{j} + 2\hat{k}| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$

(b) $r_1 + r_2 + r_3 = (3\hat{i} - 2\hat{j} + \hat{k}) + (2\hat{i} - 4\hat{j} + 3\hat{k}) + (-\hat{i} + 2\hat{j} + 2\hat{k}) = 4\hat{i} - 4\hat{j} + 0\hat{k} = 4\hat{i} - 4\hat{j}$

then $|r_1 + r_2 + r_3| = |4\hat{i} - 4\hat{j} + 0\hat{k}| = \sqrt{4^2 + (-4)^2 + 0^2}$

$= 4\sqrt{2}$

(c) $2r_1 - 3r_2 - 5r_3 = 2(3\hat{i} - 2\hat{j} + k) + 3(2\hat{i} - 4\hat{j} + 3\hat{k})$

$- 5(-\hat{i} + 2\hat{j} + 2\hat{k})$

$= 6\hat{i} - 4\hat{j} + 2\hat{k} - 6\hat{i} + 12\hat{j} + 9\hat{k} +$

$5\hat{i} - 10\hat{j} - 10\hat{k}$

$= 5\hat{i} - 2\hat{j} + \hat{k}.$

then $|2r_1 - 3r_2 - 5r_3| = |5\hat{i} - 2\hat{j} + \hat{k}|$

$= \sqrt{5^2 + (-2)^2 + 1^2}$

$= \sqrt{30}$

A

The Dot or Scalar Product:

The Dot or scalar product of two vectors $\bar{A}$ and $\bar{B}$, denoted by $\bar{A}\cdot\bar{B}$ (read $\bar{A}$ dot $\bar{B}$) is defined as the product of the magnitudes of A and B and the cosine of the angle θ between them. In symbols,

$$\bar{A}\cdot\bar{B} = AB\cos\theta \qquad 0 \leqq \theta \leqq \pi$$

Some valid laws:

1. $\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$ Commutative law for dot product
2. $\bar{A}\cdot(\bar{B}+\bar{C}) = \bar{A}\cdot\bar{B} + \bar{A}\cdot\bar{C}$ Distributive law
3. $m(\bar{A}\cdot\bar{B}) = (m\bar{A})\cdot\bar{B} = \bar{A}\cdot(m\bar{B}) = (\bar{A}\cdot\bar{B})m$ where m is scalar
4. If $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\bar{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$ then,

$$\bar{A}\cdot\bar{B} = A_1B_1 + A_2B_2 + A_3B_3$$
$$\bar{A}\cdot\bar{A} = A^2 = A_1^2 + A_2^2 + A_3^2$$
$$\bar{B}\cdot\bar{B} = B^2 = B_1^2 + B_2^2 + B_3^2$$

5. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1, \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
6. If $\bar{A}\cdot\bar{B} = 0$ and $\bar{A}$ and $\bar{B}$ are not null vectors then $\bar{A}$ and $\bar{B}$ are perpendicular.

The cross or Vector Product :

The cross or vector product of $\bar{A}$ and $\bar{B}$ is a vector $\bar{C} = \bar{A} \times \bar{B}$ (read $\bar{A}$ cross $\bar{B}$). The magnitude of $\bar{A} \times \bar{B}$ is defined as the product of the magnitudes of $\bar{A}$ and $\bar{B}$ and the sine of the angle θ between them. The direction of the vector $\bar{C} = \bar{A} \times \bar{B}$ is perpendicular to the plane of $\bar{A}$ and $\bar{B}$ and such that $\bar{A}, \bar{B}$ and $\bar{C}$ form a right handed system. In symbols,

$$\bar{A} \times \bar{B} = AB \sin\theta\, u, \quad 0 \leqq \theta \leqq \pi$$

where u is a unit vector indicating the direction of $A \times B$. If $\bar{A} = \bar{B}$ or if $\bar{A}$ is parallel to $\bar{B}$, then $\sin\theta = 0$ and we define $\bar{A} \times \bar{B} = 0$.

Some Valid laws:

1. $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$
2. $\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}$ Distributive law
3. $m(\bar{A} \times \bar{B}) = (m\bar{A}) \times \bar{B} = \bar{A} \times (m\bar{B}) = (\bar{A} \times \bar{B})m$ [m is scalar]
4. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0, \quad \hat{i} \times \hat{j} = \hat{k}, \; \hat{j} \times \hat{k} = \hat{i}, \; \hat{k} \times \hat{i} = \hat{j}$
5. If $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\bar{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

then,

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

6. The magnitude of $\bar{A} \times \bar{B}$ is the same as the area of a parallelogram with sides $\bar{A}$ and $\bar{B}$.

7. If $\bar{A} \times \bar{B} = 0$ and $\bar{A}$ and $\bar{B}$ are not null vector then $\bar{A}$ and $\bar{B}$ are parallel.

Triple Products:

Dot and Cross multiplication of three vectors $\bar{A}, \bar{B}$ and $\bar{C}$ may produce meaningful products of the form - $(\bar{A} \cdot \bar{B})\bar{C}$, $\bar{A} \cdot (\bar{B} \times \bar{C})$ and $\bar{A} \times (\bar{B} \times \bar{C})$, these are called triple products.

* Some valid laws:

1. $(\bar{A} \cdot \bar{B})\bar{C} \neq \bar{A}(\bar{B} \cdot \bar{C})$

2. $\bar{A} \cdot (\bar{B} \times \bar{C}) = \bar{B} \cdot (\bar{C} \times \bar{A}) = \bar{C} \cdot (\bar{A} \times \bar{B})$

If $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\bar{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$ and $\bar{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$ then

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_2 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

3. $\bar{A} \times (\bar{B} \times \bar{C}) \neq (\bar{A} \times \bar{B}) \times \bar{C}$

4. $\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$

$(\bar{A} \times \bar{B}) \times \bar{C} = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{B} \cdot \bar{C})\bar{A}$

Reciprocal Sets of Vectors:

The sets of vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{a}', \bar{b}', \bar{c}'$ are called reciprocal sets or systems of vectors if,

$$\bar{a}\cdot\bar{a}' = \bar{b}\cdot\bar{b}' = \bar{c}\cdot\bar{c}' = 1$$

$$\bar{a}'\cdot\bar{b} = \bar{a}'\cdot\bar{c} = \bar{b}'\cdot\bar{a} = \bar{b}'\cdot\bar{c} = \bar{c}'\cdot\bar{a} = \bar{c}'\cdot\bar{b} = 0$$

The sets of vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{a}', \bar{b}', \bar{c}'$ are reciprocal if and only if -

$$\bar{a}' = \frac{\bar{b}\times\bar{c}}{\bar{a}\cdot\bar{b}\times\bar{c}}, \quad \bar{b}' = \frac{\bar{c}\times\bar{a}}{\bar{a}\cdot\bar{b}\times\bar{c}},$$

$$\bar{c}' = \frac{\bar{a}\times\bar{b}}{\bar{a}\cdot\bar{b}\times\bar{c}}$$

where, $\bar{a}\cdot\bar{b}\times\bar{c} \neq 0$

Problems:

1. Prove $\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$

$$\bar{A}\cdot\bar{B} = AB\cos\theta$$
$$= BA\cos\theta$$
$$= \bar{B}\cdot\bar{A}$$

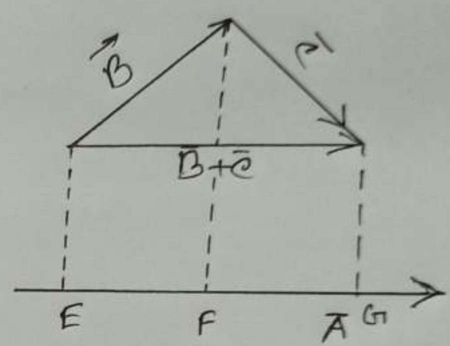
Then the commutative law for dot products is valid.

3. Prove $\bar{A}\cdot(\bar{B}+\bar{C}) = \bar{A}\cdot\bar{B} + \bar{A}\cdot\bar{C}$

Let, $\hat{a}$ be a unit vector in the direction of $\bar{A}$, then,

Projection of $(\bar{B}+\bar{C})$ on $\bar{A}$ = proj. of $\bar{B}$ on $\bar{A}$ + Proj. of $\bar{C}$ on $\bar{A}$

$$\Rightarrow (\bar{B}+\bar{C})\cdot\hat{a} = \bar{B}\cdot\hat{a} + \bar{C}\cdot\hat{a}$$



Multiplying by A,

$$(\bar{B}+\bar{C})\cdot A\hat{a} = \bar{B}\cdot A\hat{a} + \bar{C}\cdot A\hat{a}$$

$$\Rightarrow (\bar{B}+\bar{C})\cdot\bar{A} = \bar{B}\cdot\bar{A} + \bar{C}\cdot\bar{A}$$

Then by commutative law for dot products-

$$\bar{A}\cdot(\bar{B}+\bar{C}) = \bar{A}\cdot\bar{B} + \bar{A}\cdot\bar{C}$$

So, the distributive law is valid.

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* Evaluate each of the following:

a) $\hat{i} \cdot \hat{i}$

$\Rightarrow |i|\,|i| \cos 0°$

$= (1)(1)(1)$

$= 1$

b) $\hat{i} \cdot \hat{k}$

$= |i|\,|k| \cos 90°$

$= (1)(1)(0)$

$= 0$

c) $\hat{k} \cdot \hat{j}$

$= |k|\,|j| \cos 90°$

$= (1)(1)(0)$

$= 0$

d) $\hat{j} \cdot (2\hat{i} - 3\hat{j} + \hat{k})$

$= 2\hat{j} \cdot \hat{i} - 3\hat{j} \cdot \hat{j} + \hat{j} \cdot \hat{k}$

$= 0 - 3 + 0$

$= -3$

e) $(2\hat{i} - \hat{j}) \cdot (3\hat{i} + \hat{k})$

$= 2\hat{i} \cdot (3\hat{i} + \hat{k}) - \hat{j} \cdot (3\hat{i} + \hat{k})$

$= 6\hat{i} \cdot \hat{i} + 6\hat{i} \cdot \hat{k} - 3\hat{j} \cdot \hat{i} + \hat{j} \cdot \hat{k}$

$= 6 + 0 - 0$

$= 6$

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$\checkmark$

* If $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\bar{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

prove that $\bar{A}\cdot\bar{B} = A_1B_1 + A_2B_2 + A_3B_3$

$$\Rightarrow \bar{A}\cdot\bar{B} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$= A_1\hat{i}(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) + A_2\hat{j}(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) + A_3\hat{k}(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$= A_1B_1\,\hat{i}\cdot\hat{i} + A_1B_2\,\hat{i}\cdot\hat{j} + A_1B_3\,\hat{i}\cdot\hat{k} + A_2B_1\,\hat{j}\cdot\hat{i} + A_2B_2\,\hat{j}\cdot\hat{j} + A_2B_3\,\hat{j}\cdot\hat{k} + A_3B_1\,\hat{k}\cdot\hat{i} + A_3B_2\,\hat{k}\cdot\hat{j} + A_3B_3\,\hat{k}\cdot\hat{k}$$

$$= A_1B_1 + 0 + 0 + 0 + A_2B_2 + 0 + 0 + 0 + A_3B_3$$

$$\left[\because \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{i} = 0\right]$$

$$= A_1B_1 + A_2B_2 + A_3B_3$$

(proved)

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Question No - 07: If $A = A_1 i + A_2 j + A_3 k$, show that

$$A = \sqrt{A \cdot A} = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

Solve: $A \cdot A = (A)(A) \cos 0^\circ = A^2$

Then, $A = \sqrt{A \cdot A}$

Also, $A \cdot A = (A_1 i + A_2 j + A_3 k) \cdot (A_1 i + A_2 j + A_3 k)$

$= (A_1)(A_1) + (A_2)(A_2) + (A_3)(A_3)$

$= A_1^2 + A_2^2 + A_3^2$

By taking $B = A$.

Then $A = \sqrt{A \cdot A} = \sqrt{A_1^2 + A_2^2 + A_3^2}$ is the magnitude of A. sometimes $A \cdot A$ is written A^2.

Question No-08: Find the angle between $A = 2i + 2j - k$ and $B = 6i - 3j + 2k$.

Solve:

$$A \cdot B = AB \cos\theta.$$

$$A = \sqrt{(2)^2 + (2)^2 + (-1)^2}$$

$$= 3$$

$$B = \sqrt{(6)^2 + (-3)^2 + (2)^2}$$

$$= 7$$

$$\therefore A \cdot B = (2)\cdot(6) + (2)(-3) + (-1)(2)$$

$$= 12 - 6 - 2$$

$$= 4$$

$$\text{Then } \cos\theta = \frac{A \cdot B}{AB}$$

$$= \frac{4}{(3)(7)}$$

$$= \frac{4}{21}$$

$$= 0.1905$$

and $\theta = 79^\circ$ (Approximately).

Question no-09:

If $A \cdot B = 0$ and if A and B are not zero, show that A is perpendicular to B.

Answer:

If $A \cdot B = AB \cos\theta$
$= 0$

Then $\cos\theta = 0$ or $\theta = 90°$

conversely, If $\theta = 90°$, $A \cdot B = 0$.

Question no-10: Determine the value of a so that $A = 2i + aj + k$ and $B = 4i - 2j - 2k$ are perpendicular

Answer: From problem 9, A and B are perpendicular

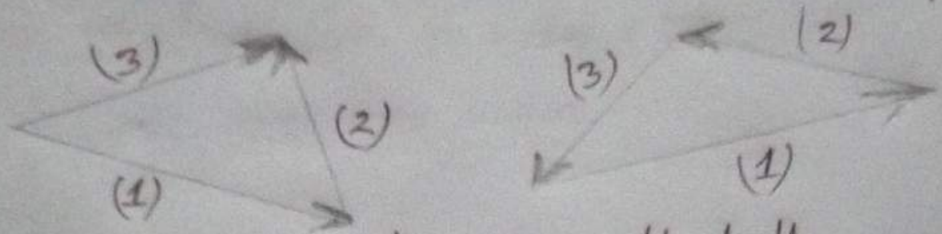
If $A \cdot B = 0$

Then, $A \cdot B = (2)(4) + (a) \cdot (-2) + (1)(-2)$
$= 8 - 2a - 2$
$= 0$ for $a = 3$

Question no - 11:

show that the vectors $A = 3i - 2j + k$, $B = i - 3j + 5k$, $C = 2i + j - 4k$ form a right triangle.

We first have to show that the vectors form a triangle.



From the figures it is seen that the vectors will form a triangle if

(a) One of the vectors, say (3), is the resultant of sum of (1) and (2).

(b) The sum of resultant of the vectors (1)+(2)+(3) is zero according as (a) two vectors has a common terminal point or (b) none of the vectors have a common terminal point. By trial we find $A = B + C$ so that the vectors do form a triangle.

since $A \cdot B = (3)(1) + (-2)(-3) + (1)(5) = 14$,

$A \cdot C = (3)(2) + (-2)(1) + (1)(-4) = 0$

and $B \cdot C = (1)(2) + (-3)(1) + (5)(-4) = -21$

it follows that a and c are perpendicular and the Triangle is a right triangle.

Question no-13:

Find the projection of the vector $A = i - 2j + k$ on the vector $B = 4i - 4j + 7k$.

sallution:

A unit vector on the direction ~~is~~ B ~~is~~ $b = \frac{B}{B}$

$$= \frac{4i - 4j + 7k}{\sqrt{(4)^2 + (-4)^2 + (7)^2}}$$

$$= \frac{4}{9} i - \frac{4}{9} j + \frac{7}{9} k$$

Projection of A on the vector $B = A \cdot b = (i - 2j + k) \cdot \left(\frac{4}{9} i - \frac{4}{9} j + \frac{7}{9} k\right)$

$$= (1)\left(\frac{4}{9}\right) + (-2)\left(-\frac{4}{9}\right) + 1\left(\frac{7}{9}\right)$$

$$= \frac{19}{9}.$$

Question no-17:

Find the work done in moving an object along a vector $r = 3i + 2j - 5k$ if the applied force is $F = 2i - j - k$. Refer to fig.(a) below.

work done = (magnitude of force in direction of motion) (distanced moved)

$= (F \cos \theta)(r) = F \cdot r$

$= (2i - j - k) \cdot (3i + 2j - 5k) = 6 - 2 + 5$

$= 9.$

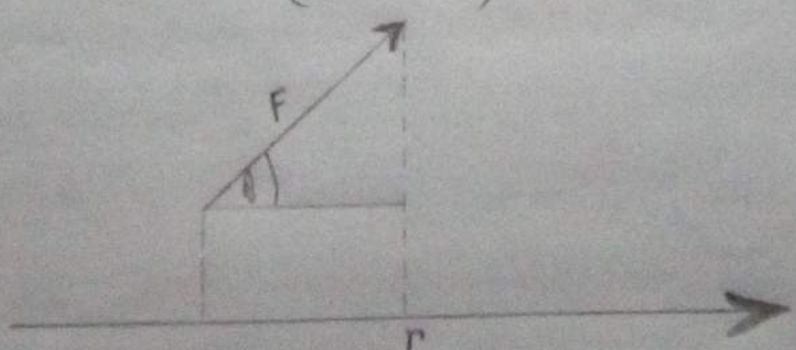


Fig (a)

(The dot and cross product)

Triple Products

Problem-37: Show that $\vec{A}\cdot(\vec{B}\times\vec{C})$ is in absolute value equal to the volume of a parallelepiped with sides $\vec{A},\vec{B}$ and $\vec{C}$.

Solution: Let $\hat{n}$ be a unit normal to parallelogram I, having the direction of $\vec{B}\times\vec{C}$ and let h be the height of the terminal point of $\vec{A}$ above the parallelogram I.

Volume of parallelogram = (height h)(area of parallelogram I)

$$= (\vec{A}\cdot\hat{n})\,(|\vec{B}\times\vec{C}|)$$

$$= \vec{A}\cdot\{|\vec{B}\times\vec{C}|\hat{n}\}$$

$$= \vec{A}\cdot(\vec{B}\times\vec{C})$$

If $\vec{A},\vec{B}$ and $\vec{C}$ do not form a right-handed system, $A\cdot n<0$ and the volume $=|\vec{A}\cdot(\vec{B}\times\vec{C})|$

[Showed]

(The dot and cross product)

Problem-38: If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, $\vec{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$. Show that.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Solution:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{A}\cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot[(B_2C_3 - B_3C_2)\hat{i} + (B_3C_1 - B_1C_3)\hat{j} + (B_1C_2 - B_2C_1)\hat{k}]$$

$$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$ [Showed]

Problem-39: Evaluate $(2\hat{i}-3\hat{j})\cdot[(\hat{i}+\hat{j}-\hat{k})\times(3\hat{i}-k)]$

Solution:

$$(2\hat{i}-3\hat{j})\cdot[(\hat{i}+\hat{j}-\hat{k})\times(3\hat{i}-\hat{k})]$$

$$=(2\hat{i}-3\hat{j})\cdot[\hat{i}\times(3\hat{i}-\hat{k})+\hat{j}\times(3\hat{i}-\hat{k})-\hat{k}(3\hat{i}-\hat{k})]$$

$$=(2\hat{i}-3\hat{j})\cdot[3\hat{i}\times\hat{i}-\hat{i}\times\hat{k}+3\hat{j}\times\hat{i}-\hat{j}\times\hat{k}-3\hat{k}\times\hat{i}+\hat{k}\times\hat{k}]$$

$$=(2\hat{i}-3\hat{j})\cdot(0+\hat{j}-3\hat{k}-\hat{i}-3\hat{j}+0)$$

$$=(2\hat{i}-3\hat{j})\cdot(-\hat{i}-2\hat{j}-3\hat{k})$$

$$=(2)(-1)+(-3)(-2)+(0)(-3)$$

$$=-2+6$$

$$=4$$ Ans:

Problem-40: Prove that, $\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}(\vec{A}\times\vec{B})$

Solution:

From Problem 38, $\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

By a theorem of determinats which states that interchange of two rows of a determinats changes its sign, we have

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = - \begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{B} \cdot (\vec{C} \times \vec{A})$$

Again,

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = - \begin{vmatrix} B_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\therefore \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \text{ [Proven]}$$

Problem-41: Show that $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$

Solution: From Problem 40,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$= (\vec{A} \times \vec{B}) \cdot \vec{C}$$

Occasionally $A \cdot (\vec{B} \times \vec{C})$ is written without Parentheses as $\vec{A} \cdot \vec{B} \times \vec{C}$. In such case there cannot be any ambiguity since the only possible interpretations are $\vec{A}\cdot(\vec{B}\times\vec{C})$ and $(\vec{A}\cdot\vec{B})\times\vec{C}$. The latter however has no meaning since the cross product of a scalar with a vector is undefined. The result $\vec{A}\cdot\vec{B}\times\vec{C} = \vec{A}\times\vec{B}\cdot\vec{C}$ is sometimes summarized in the statement that the dot and cross can be interchanged without affecting the value.

$$\therefore \vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C} \quad [\text{Showed}]$$

<u>Problem-42:</u> Prove that $\vec{A}\cdot(\vec{A}\times\vec{C}) = 0$

<u>Solution:</u>

From Problem 41, $\vec{A}\cdot(\vec{A}\times\vec{C}) = (\vec{A}\times\vec{A})\cdot\vec{C}$

$$= 0 \quad [\text{Proven}]$$

Problem-43: Prove that a necessary and sufficient condition for the vectors $\vec{A}, \vec{B}$ and $\vec{C}$ tobe coplanar is that $\vec{A} \cdot \vec{B} \times \vec{C} = 0$

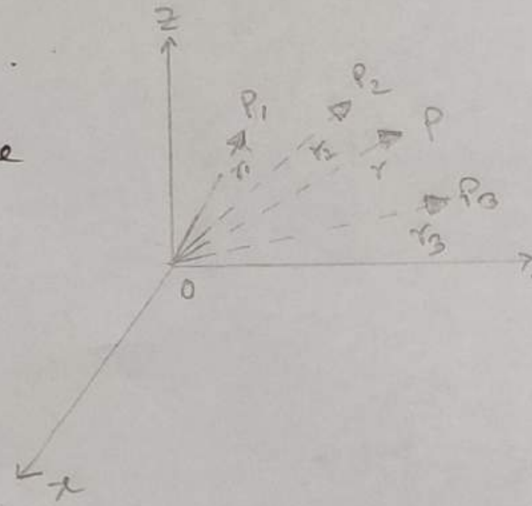
Solution: Note that $\vec{A} \cdot \vec{B} \times \vec{C}$ can have no meaning other than $\vec{A} \cdot (\vec{B} \times \vec{C})$

If $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar the volume of the parallelepiped formed by them is zero. Then by Problem 37,

$$\vec{A} \cdot \vec{B} \times \vec{C} = 0$$

Conversely, if $\vec{A} \cdot \vec{B} \times \vec{C} = 0$ the volume of the parallelepiped formed by vectors, $\vec{A}, \vec{B}$ and $\vec{C}$ is zero, and so the vectors must lie in a plane. [Proven]

Problem: 44 - Let $r_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $r_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $r_3 = x_3\hat{i} + y_3\hat{j} + z_3 k$ be the position vectors of points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$. Find and equation for the plane passing through P_1, P_2 and P_3.



⇒ we assume that P_1, P_2 and P_3 do not lie in the same straight line; hence they determine a plane.

Let $r = x\hat{i} + y\hat{j} + z\hat{k}$ denote the position vector of any point $P(x, y, z)$. In the plane.

Consider vectors $P_1P_2 = (r_1 - r_2)$, $P_1P_3 = (r_3 - r_1)$ and $P_1P = (r - r_1)$ which all lie in the plane.

we know that, $P_1P \cdot P_1P_2 \times P_1P_3 = 0$

$$\text{or, } (r - r_1) \cdot (r_2 - r_1) \times (r_3 - r_1) = 0$$

In terms of rectangular coordinates this becomes

$$[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}] \cdot [(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$
$$\times [(x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}] = 0$$

$$\text{or, } \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

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Problem: 45 - Find the equation for the plane determined by the points $P_1(2,-1,1)$, $P_2(3,2,-1)$ and $P_3(-1,3,2)$.

$\Rightarrow$ The position vectors of P_1, P_2, P_3 and any point $P(x,y,z)$ are respectively $r_1 = 2\hat{i} - \hat{j} + k$, $r_2 = 3\hat{i} + 2\hat{j} - \hat{k}$
$r_3 = -\hat{i} + 3\hat{j} + 2k$ and $r = x\hat{i} + y\hat{j} + z\hat{k}$

Then $PP_1 = r - r_1$, $P_2P_1 = r_2 - r_1$, $P_3P_1 = r_3 - r_1$ all lie in the required plane

so that, $(r - r_1)\cdot(r_2 - r_1)\times(r_3 - r_1) = 0$

i.e. $[(x-2)\hat{i} + (y+1)\hat{j} + (z-1)\hat{k}]\cdot[\hat{i} + 3\hat{j} - 2\hat{k}]\times[-3\hat{i} + 4\hat{j} + \hat{k}] = 0$

$\Rightarrow [(x-2)\hat{i} + (y+1)\hat{j} + (z-1)\hat{k}]\cdot[11\hat{i} + 5\hat{j} + 13\hat{k}] = 0$

$\Rightarrow 11(x-2) + 5(y+1) + 13(x-1) = 0$

$\Rightarrow 11x + 5y + 13z = 0$

Ans.

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Problem: 46 - If the points P, Q and R, not all lying on the same straight line have position vectors a, b and c relative to a given origin. Show that, $a \times b + b \times c + c \times a$ is a vector perpendicular to the plane P, Q and R.

⇒ Let r be the position vector of any point in the plane of P, Q and R. Then the vectors $r-a$, $b-a$ and $c-a$ are coplanar. So that

$$(r-a) \cdot (b-a) \times (c-a) = 0$$

or, $(r-a) \cdot (a \times b + b \times c + c \times a) = 0$

Thus $a \times b + b \times c + c \times a$ is perpendicular to $(r-a)$ and is therefore perpendicular to the plane of P, Q and R

(Showed)

Problem: 47 - Prove: a) $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

b) $(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{B}\cdot\vec{C})$

$\Rightarrow$ Let, $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$\vec{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$

Then, $A\times(B\times C) = (A_1\hat{i}+A_2\hat{j}+A_3\hat{k}) \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

$= (A_2\hat{i}+A_2\hat{j}+A_3\hat{k}) \times [B_2C_3 - B_3C_2]\hat{i} +$

$[B_3C_1 - B_1C_3]\hat{j} + [B_1C_2 - B_2C_1]\hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_2C_3 - B_3C_2 & B_3C_1 - B_1C_3 & B_1C_2 - B_2C_1 \end{vmatrix}$$

Also,

$\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$= (B_1\hat{i}+B_2\hat{j}+B_3\hat{k})(A_1C_1 + A_2C_2 + A_3C_3) - (C_1\hat{i}+C_2\hat{j}+C_3\hat{k})$

$(A_1B_1 + A_2B_2 + A_3B_3)$

$= (A_2B_1C_2 + A_3B_1C_3 - A_2C_1B_2 - A_3C_1B_3)\hat{i} + (B_2A_1C_1 + B_2C_3A_3 - C_2A_1B_1$

$- C_2A_3B_3)\hat{j} + (B_3A_1C_1 + B_3A_2C_2 - C_3A_1B_1 - C_3A_2B_2)\hat{k}$

The Dot and Cross product

48. prove: $(A\times B)\cdot(C\times D) = (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$

we know that $X\cdot(C\times D) = (X\times C)\cdot D$

let $X = A\times B$

$(A\times B)\cdot(C\times D) = \{(A\times B)\times C\}\cdot D$

$= \{B(A\cdot C) - A(B\cdot C)\}\cdot D$

$= (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$

(proved)

The Dot and Cross product

50. prove : $(A\times B)\times(C\times D) = B(A\cdot C\times D) - A(B\cdot C\times D)$

$= C(A\cdot B\times D) - D(A\cdot B\times C)$

$X\times(C\times D) = C(X\cdot D) - D(X\cdot C)$ let $X = A\times B$

$(A\times B)\times(C\times D) = C\ (A\times B\cdot D) - D(A\times B\cdot C)$

$= C\ (A\cdot B\times D) - D(A\cdot B\times C)$

$(A\times B)\times Y = B(A\cdot Y) - A\ (B\cdot Y)$

let $Y = C\times D$ then

$(A\times B)\times(C\times D) = B(A\cdot C\times D) - A(B\cdot C\times D$

(proved)

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49.

prove : $A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0$

we know that

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B) \quad \text{--- (i)}$$

$$B \times (C \times A) = C(B \cdot A) - A(B \cdot C) \quad \text{--- (ii)}$$

$$C \times (A \times B) = A(C \cdot B) - B(C \cdot A) \quad \text{--- (iii)}$$

(i) + (ii) + (iii)

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = B(A \cdot C) - C(A \cdot B) + A(B \cdot C) - B(A \cdot C) + C(A \cdot B) - A(B \cdot C)$$

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0$$

(proved)

The Dot and cross product

51. P,Q,R be a spherical triangle whose sides p, q, r are arcs of great circles. prove that

$$\frac{\sin P}{\sin p} = \frac{\sin Q}{\sin q} = \frac{\sin R}{\sin r}$$

Suppose that the sphere has unit radius, and let unit vectors A, B and C be the centre O of the sphere to P, Q and R respectively.

$$(A \times B) \times (A \times C) = (A \cdot B \times C) A$$

A unit vector perpendicular to $A \times B$ and $A \times C$ is A, so that (1) becomes

$$\sin r \; \sin q \; \sin P \; A = (A \cdot B \times C) A$$

$$\sin r \; \sin q \; \sin P = A \cdot B \times C$$

By cyclic permutation of P, q, r, P, Q, R and A, B, C we obtain

$$\sin p \, \sin r \, \sin Q = B \cdot C \times A$$
$$\sin q \, \sin p \, \sin R = C \cdot A \times B$$

then since the right hand sides of (3), (4) (5) are equal

$$\sin r \, \sin q \, \sin P = \sin p \, \sin r \, \sin Q = \sin q \, \sin p \, \sin R$$

from which we find $\frac{\sin P}{\sin p} = \frac{\sin Q}{\sin q} = \frac{\sin R}{\sin r}$

(proved)

this is called the law of sines for spherical triangles.

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52. Prove: $(A\times B)\cdot(B\times C)\times(C\times A) = (A\cdot B\times C)^2$

By problem (47) a $x\times(C\times A) = C(x\cdot A) - A(x\cdot C)$.

Let $x = B\times C$ then

$$(B\times C)\times(C\times A) = C(B\times C\cdot A) - A(B\times C\cdot C)$$
$$= C(A\cdot B\times C) - A(B\cdot C\times C) = C(A\cdot B\times C)$$

Thus, $(A\times B)\cdot(B\times C)\times(C\times A) = (A\times B)\cdot C(A\cdot B\times C)$
$$=(A\times B\cdot C)(A\cdot B\times C) = (A\cdot B\times C)^2$$

53] Given the vectors $a' = \frac{b\times c}{a\cdot b\times c}$, $b' = \frac{c\times a}{a\cdot b\times c}$ and $c' = \frac{a\times b}{a\cdot b\times c}$

show that if $a\cdot b\times c \neq 0$,

a) $a'\cdot a = b'\cdot b = c'\cdot c = 1$

b) $a'\cdot b = a'\cdot c = 0$, $b'\cdot a = b'\cdot c = 0$, $c'\cdot a = c'\cdot b = 0$

c) if $a\cdot b\times c = V$ then $a'\cdot b'\times c' = \frac{1}{V}$

d) a', b', and c' are non-coplanar if a, b and c are non-coplanar.

a) $a'\cdot a = a\cdot a' = a\cdot\frac{b\times c}{a\cdot b\times c} = \frac{a\cdot b\times c}{a\cdot b\times c} = 1$

$b'\cdot b = b\cdot b' = b\frac{c\times a}{a\cdot b\times c} = \frac{b\cdot c\times a}{a\cdot b\times c} = \frac{a\cdot b\times c}{a\cdot b\times c} = 1$

$c'\cdot c = c\cdot c' = c\cdot\frac{a\times b}{a\cdot b\times c} = \frac{c\cdot a\times b}{a\cdot b\times c} = \frac{a\cdot b\times c}{a\cdot b\times c} = 1$

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b) $a' \cdot b = b \cdot a' = b \cdot \frac{b \times c}{a \cdot b \times c} = \frac{b \cdot b \times c}{a \cdot b \times c} = \frac{b \times b \cdot c}{a \cdot b \times c} = 0$

Similarly the other results follow. The results can also be seen by nothing, for example, that a' has the direction of $b \times c$ and so much be perpendicular to both b and c, from which $a' \cdot b = 0$ and $a' \cdot c = 0$. from (a) and (b) we see that the sets of vectors a, b, c and a', b', c' are reciprocal vectors. see also supplementary problems 104 and 106

c) $a' = \frac{b \times c}{V}, b' = \frac{c \times a}{V}, c' = \frac{a \times b}{V}$

then $$a' \cdot b' \times c' = \frac{(b \times c) \cdot (c \times a) \cdot (a \times b)}{V^3}$$

$$= \frac{(a \times b) \cdot (b \times c) \times (c \times a)}{V^3}$$

$$= \frac{(a \cdot b \times c)^2}{V^3}$$

$$= \frac{V^2}{V^3}$$

$$= \frac{1}{V}$$

using problem 52

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d) By problem 43, if a, b and c are non-coplanar $a \cdot b \times c \neq 0$. Then from part (c) it follows that $a' \cdot b' \times c' \neq 0$, so that a', b' and c' are also non-coplanar.

54/ Show that any vector r can be expressed in term of the reciprocal vector of problem 53 as $r = (r \cdot a')a + (r \cdot b')b + (r \cdot c') \cdot c$

from problem 50,

$$B(A \cdot C \times D) - A(B \cdot C \times D) = C(A \cdot B \times D) - D(A \cdot B \times C)$$

$$D = \frac{A(B \cdot C \times D)}{A \cdot B \times C} - \frac{B(A \cdot C \times D)}{A \cdot B \times C} + \frac{C(A \cdot B \times D)}{A \cdot B \times C}$$

Let $A = a$, $B = b$, $C = c$ and $D = r$. Then

$$r = \frac{r \cdot b \times c}{a \cdot b \times c} a + \frac{r \cdot c \times a}{a \cdot b \times c} b + \frac{r \cdot a \times b}{a \cdot b \times c} c$$

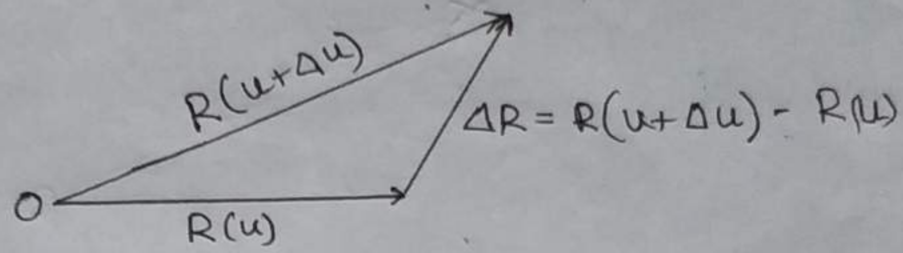
$$= r \cdot \left(\frac{b \times c}{a \cdot b \times c}\right) a + r \cdot \left(\frac{c \times a}{a \cdot b \times c}\right) b + r \cdot \left(\frac{a \times b}{a \cdot b \times c}\right) c$$

$$= (r \cdot a')a + (r \cdot b')b + (r \cdot c')c$$

M.R. 18ICTCSE056

VECTOR DIFFRENTIATION

ORDINARY DERIVATIVES OF VECTORS:



Let $R(u)$ be a vector depending on a single scalar variable u.

Then, $\dfrac{\Delta R}{\Delta u} = \dfrac{R(u+\Delta u) - R(u)}{\Delta u}$; where Δu denotes an increment in u

The ordinary derivative of the vector $R(u)$ with respect to the scalar u is given by

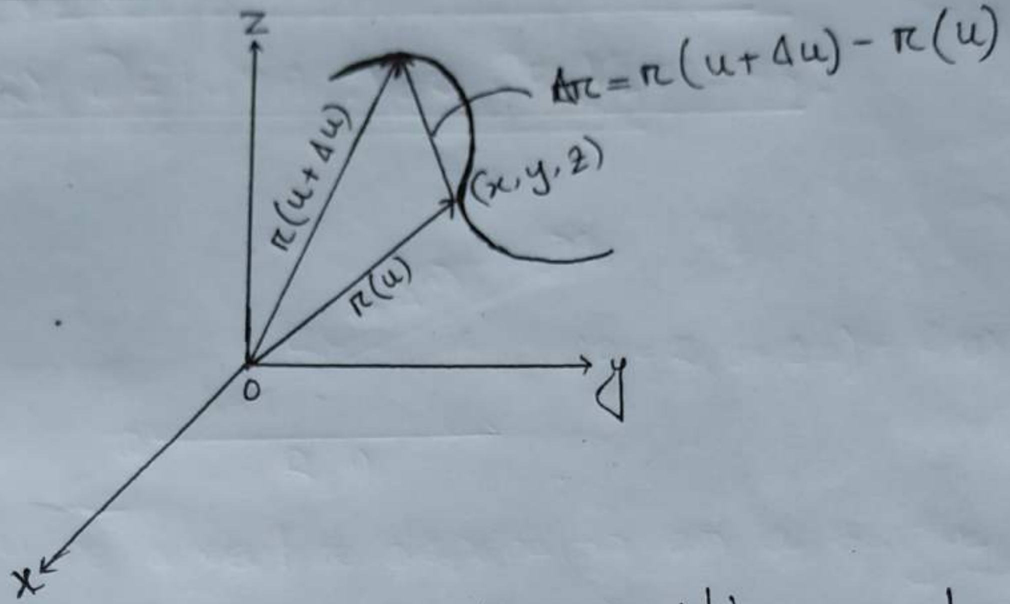
$$\frac{dR}{du} = \lim_{\Delta u \to 0} \frac{\Delta R}{\Delta u} = \lim_{\Delta u \to 0} \frac{R(u+\Delta u) - R(u)}{\Delta u}$$

If the limit exists.

Since $\dfrac{dR}{du}$ is itself a vector depending on u, we can consider its derivative with respect to u.

If this derivative exists it is denoted by $\dfrac{d^2R}{du^2}$.

SPACE CURVES :



If in particular R(u) is the position vector r(u) joining the origin O of a co-ordinate system and point (x, y, z), then,

$$r(u) = x(u)i + y(u)j + z(u)k$$

and specification of the vector function $r(u)$ defines x, y and z as functions of u.

As u changes, the terminal point of r describes a space curve having parametric equations.

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

Then $\frac{\Delta r}{\Delta u} = \frac{r(u+\Delta u) - r(u)}{\Delta u}$ is a vector in the direction of Δr. If $\lim_{\Delta u \to 0} \frac{\Delta r}{\Delta u} = \frac{dr}{du}$ exists, the limit will be a vector in the direction of the tangent to the space curve at (x, y, z) and is given by,

$$\frac{dr}{du} = \frac{dx}{du} i + \frac{dy}{du} j + \frac{dz}{du} k$$

If u is the time t, $\frac{dr}{dt}$ represents the velocity v with which the terminal point of r describes the curve.

Similarly, $\frac{dv}{dt} = \frac{d^2 r}{dt^2}$ represents its acceleration a along the curve.

CONTINUITY AND DIFFERENTIABILITY:

A scalar function $\phi(u)$ is called continuous at u if $\lim_{\Delta u \to 0} \phi(u+\Delta u) = \phi(u)$.

Equivalently. $\phi(u)$ is continuous at u if for each positive number ϵ we can find some positive number δ such that

$$|\phi(u+\Delta u) - \phi(u)| < \epsilon \text{ whenever } |\Delta u| < \delta.$$

A vector function $R(u) = R_1(u)i + R_2(u)j + R_3(u)k$ is called continuous at u if the three scalar functions $R_1(u)$, $R_2(u)$ and R_3 are continuous at u or if $\lim_{\Delta u \to 0} R(u+\Delta u) = R(u)$.

Equivalently. $R(u)$ is continuous at u if for each positive number ϵ we can find some positive numbers δ such that

$$|R(u+\Delta u) - R(u)| < \epsilon \text{ whenever } |\Delta u| < \delta.$$

- A scalar or vector function of u is called "differentiable of order n" if its "nth" derivative exists.

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DIFFERENTIATION FORMULAS:

If A, B and C are differentiable vector functions of a scalar u, and ϕ is a differentiable scalar function of u then

1. $\frac{d}{du}(A+B) = \frac{dA}{du} + \frac{dB}{du}$

2. $\frac{d}{du}(A \cdot B) = A \cdot \frac{dB}{du} + \frac{dA}{du} \cdot B$

3. $\frac{d}{du}(A \times B) = A \times \frac{dB}{du} + \frac{dA}{du} \times B$

4. $\frac{d}{du}(\phi A) = \phi \frac{dA}{du} + \frac{d\phi}{du} A$

5. $\frac{d}{du}(A \cdot B \times C) = A \cdot B \times \frac{dC}{du} + A \cdot \frac{dB}{du} \times C + \frac{dA}{du} \cdot B \times C$

6. $\frac{d}{du}\{A \times (B \times C)\} = A \times \left(B \times \frac{dC}{du}\right) + A \times \left(\frac{dB}{du} \times C\right) + \frac{dA}{du} \times (B \times C)$

PARTIAL DERIVATIVES OF VECTORS:

If A is a vector depending on more than one scalar variable, say x, y, z for example, we write $A = A(x, y, z)$.

The partial derivative of A with respect to x is defined as.

$$\frac{\partial A}{\partial x} = \lim_{\Delta x \to 0} \frac{A(x+\Delta x, y, z) - A(x, y, z)}{\Delta x}$$

if this limit exists. Similarly,

$$\frac{\partial A}{\partial y} = \lim_{\Delta y \to 0} \frac{A(x, y+\Delta y, z) - A(x, y, z)}{\Delta y}$$

$$\frac{\partial A}{\partial z} = \lim_{\Delta z \to 0} \frac{A(x, y, z+\Delta z) - A(x, y, z)}{\Delta z}$$

are the partial derivatives of A with respect to y and z respectively if these limits exist.

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PARTIAL DIFFENTIATION FORMULAS :

Here, A and B are functions of x, y, z

1. $$\frac{\partial}{\partial x}(A \cdot B) = A \cdot \frac{\partial B}{\partial x} + \frac{\partial A}{\partial x} \cdot B$$

2. $$\frac{\partial}{\partial x}(A \times B) = A \times \frac{\partial B}{\partial x} + \frac{\partial A}{\partial x} \times B$$

3. $$\frac{\partial^2}{\partial y \partial x}(A \cdot B) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(A \cdot B)\right\}$$

$$= \frac{\partial}{\partial y}\left\{A \cdot \frac{\partial B}{\partial x} + \frac{\partial A}{\partial x} \cdot B\right\}$$

$$= A \cdot \frac{\partial^2 B}{\partial y \cdot \partial x} + \frac{\partial A}{\partial y} \cdot \frac{\partial B}{\partial x} + \frac{\partial A}{\partial x} \cdot \frac{\partial B}{\partial y} + \frac{\partial^2 A}{\partial y \partial x} \cdot B$$

DIFFERENTIALS OF VECTORS:

1. If $A = A_i i + A_2 j + A_3 k$

   Then, $dA = dA_1 i + dA_2 j + dA_3 k$

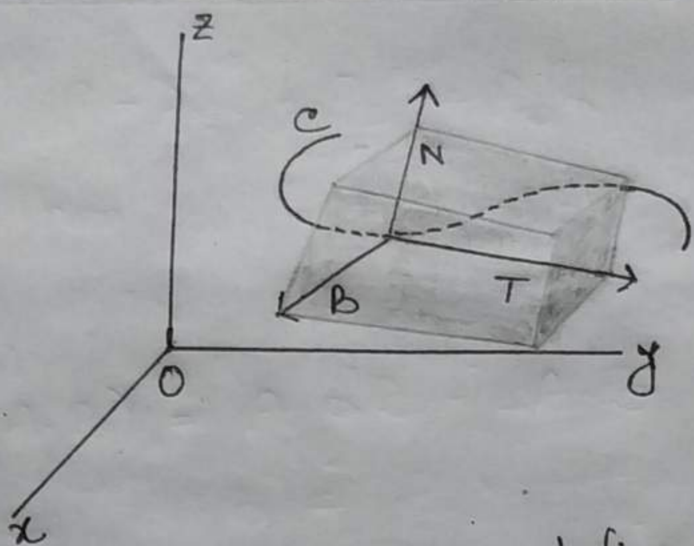
2. $d(A \cdot B) = A \cdot dB + dA \cdot B$

3. $d(A \times B) = A \times dB + dA \times B$

4. If $A = A(x, y, z)$,

   Then, $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz$

DIFFENTIAL GEOMETRY :



If C is a space curve defined by the function $r(u)$, then we have seen that $\frac{dr}{du}$ is a vector in the direction of the tangent to C. If the scalar u is taken as the arc length s measured from some fixed point on C, then $\frac{dr}{ds}$ is a unit tangent vector to C and is denoted by T.

The rate at which T changes with respect to s is a measure of the curvature of C and is given by $\frac{dT}{ds}$.

The direction of $\frac{dT}{ds}$ at any given point on C is normal to the curve at that point.

If N is a unit vector in this normal direction, it is called the "principle normal" to the curve.

Then, $\frac{dT}{ds} = KN$, where K is called the curvature of C at the specified point.

The quantity $\rho = \frac{1}{K}$ is called the radius of curvature.

A unit vector B perpendicular to the plane of T and N and such that $B = T \times N$, is called the "bi-normal" to the curve. It follows that directions T, N, B form a localized right handed rectangular co-ordinate system at the specified point of C.

A set of relations involving derivatives of the fundamental vectors T, N and B is known collectively as the "Frenet - Serret Formulas" given by

$$\frac{dT}{ds} = kN$$

$$\frac{dN}{ds} = \tau B - kT$$

$$\frac{dB}{ds} = -\tau N$$

where τ is a scalar called "Torsion".

The quantity $\sigma = \frac{1}{\tau}$ is called the radius of Torsion.

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Vector Differentiation.

MECHANICS often includes a study of the motion of Particles along Curves, this study being known as kinematics. In this Connection some of the results of differential geometry Can be of value.

A study of force on moving objects is Considered in dynamics. Fundamental to this study is Newton's famous law which states that if F is the net force acting on an object of mass m moving with velocity v, then

$$F = \frac{d}{dt}(mv)$$

Where mv is the momentum of the objects. If m is Constant this becomes $F = m\frac{dv}{dt} = ma$ where a is the accelation of the object.

Problem Solve

1. If $R(u) = x(u)i + y(u)j + z(u)k$, Where x, y and z are differentiable function of a scalar u, Prove that $\frac{dR}{du} = \frac{dx}{du}i + \frac{dy}{du}j + \frac{dz}{du}k$.

$$\frac{dR}{du} = \lim_{\Delta u \to 0} \frac{R(u+\Delta u) - R(u)}{\Delta u}$$

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$$= \lim_{\Delta u \to 0} \frac{[x(u+\Delta u)i + y(u+\Delta u)j + z(u+\Delta u)k] - [x(u)i + y(u)j + z(u)k]}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{x(u+\Delta u) - x(u)}{\Delta u} i + \frac{y(u+\Delta u) - y(u)}{\Delta u} j + \frac{z(u+\Delta u) - z(u)}{\Delta u} k$$

$$= \frac{dx}{du} i + \frac{dy}{du} j + \frac{dz}{du} k.$$

17. Let F depend on x, y, z, t where $x, y,$ and z depend on t, Prove that

$$\frac{dF}{dt} = \frac{dF}{dt} + \frac{dF}{dx}\frac{dx}{dt} + \frac{dF}{dy}\frac{dy}{dt} + \frac{dF}{dz}\frac{dz}{dt}$$

under suitable assumptions of differentiability

Suppose that

$$F = F_1(x, y, z, t) i + F_2(x, y, z, t) j + F_3(x, y, z, t) k$$

$$\therefore dF = dF_1 i + dF_2 j + dF_3 k$$

$$= \left[\frac{dF_1}{dt} dt + \frac{dF_1}{dx} dx + \frac{dF_1}{dy} dy + \frac{dF_1}{dz} dz\right] i$$

$$+ \left[\frac{dF_2}{dt} dt + \frac{dF_2}{dx} dx + \frac{dF_2}{dy} dy + \frac{dF_2}{dz} dz\right] j + \left[\frac{dF_3}{dt} dt + \frac{dF_3}{dx} dx + \frac{dF_3}{dy} dy + \frac{dF_3}{dz} dz\right] \cancel{dF_3}$$

$$= \left(\frac{dF_1}{dt} i + \frac{dF_2}{dt} j + \frac{dF_3}{dt} k\right) dt + \left(\frac{dF_1}{dx} i + \frac{dF_2}{dx} j + \frac{dF_3}{dx} k\right) dx$$

$$+ \left(\frac{dF_1}{dy} i + \frac{dF_2}{dy} j + \frac{dF_3}{dy} k\right) dy + \left(\frac{dF_1}{dz} i + \frac{dF_2}{dz} j + \frac{dF_3}{dz} k\right) dz$$

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$$= \frac{dF}{dt}dt + \frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz$$

and so $\frac{dF}{dt} = \frac{dF}{dt} + \frac{dF}{dx}\frac{dx}{dt} + \frac{dF}{dy}\frac{dy}{dt} + \frac{dF}{dz}\frac{dz}{dt}$.

Differential Geometry

18. Prove the Frenet - Serret formulas (a) $\frac{dT}{ds} = kN$ (b) $\frac{dB}{ds} = -\tau N$. (c) $\frac{dN}{ds} = \tau B - kT$

(a) Since $T \cdot T = 1$, it follows from Problem 9 that $T \cdot \frac{dT}{ds} = 0$ i.e $\frac{dT}{ds}$ is perpendicular to T.

If N is a unit vector in the direction $\frac{dT}{ds}$, then $\frac{dT}{ds} = kN$. We call N the Principal normal, k the Curvature and $\rho = 1/k$ the radius of Curvature.

(b). Let $B = T \times N$, so that $\frac{dB}{ds} = T \times \frac{dN}{ds} + \frac{dT}{ds} \times N$

$$= T \times \frac{dN}{ds} + kN \times N = T \times \frac{dN}{ds}.$$

Then $T \cdot \frac{dB}{ds} = T \cdot T \times \frac{dN}{ds} = 0$, so that T is Perpendicular to $\frac{dB}{ds}$.

But from $B \cdot B = 1$ it follows that $B \cdot \frac{dB}{ds} = 0$ so that $\frac{dB}{ds}$ is Perpendicular to B and is thus in the plane of T and N.

Since $\frac{dB}{ds}$ is the plane of T and N and is perpendicular to T, it must be parallel to N; then $\frac{dB}{ds} = -\tau N$.

We call B the binormal, τ the torsion and $\sigma = 1/\tau$

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(c) Since T, N, B form a right-handed system so do N, B and T. i.e $N = B \times T$.

Then $\frac{dN}{ds} = B \times \frac{dT}{ds} + \frac{dB}{ds} \times T = B \times KN - \tau N \times T$

$= -KT + \tau B = \tau B - KT$

22. Given the space curve $x = t$, $y = t^2$, $z = \frac{2}{3}t^3$ find (a) the curvature κ, (b) the torsion τ.

(a) The Position vector is $r = ti + t^2 j + \frac{2}{3}t^3 k$

Then $\frac{dr}{dt} = i + 2tj + 2t^2 k$

$$\frac{ds}{dt} = \left|\frac{dr}{dt}\right| = \sqrt{\frac{dr}{dt} \cdot \frac{dr}{dt}} = \sqrt{(1)^2 + (2t)^2 + (2t^2)^2}$$

$$= 1 + 2t^2$$

and $$T = \frac{dr}{ds} = \frac{dr/dt}{ds/dt} = \frac{i + 2tj + 2t^2 k}{1 + 2t^2}$$

$$\frac{dT}{dt} = \frac{(1+2t^2)(2j + 4tk) - (i + 2tj + 2t^2 k)(4t)}{(1+2t^2)^2}$$

$$= \frac{-4ti + (2 - 4t^2)j + 4tk}{(1+2t^2)^2}$$

Then $$\frac{dT}{ds} = \frac{dT/dt}{ds/dt} = \frac{-4ti + (2-4t^2)j + 4tk}{(1+2t^2)^3}$$

Since $\frac{dT}{ds} = KN$, $K = \left|\frac{dT}{ds}\right| = \frac{\sqrt{(-4t)^2 + (2-4t^2)^2 + (4t)^2}}{(1+2t^2)^3}$

$$= \frac{2}{(1+2t^2)^2}$$

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(b) From (a), $N = \frac{1}{k}\frac{dT}{ds} = \frac{-2ti + (1-2t^2)j + 2tk}{1+2t^2}$

Then $B = T\times N = \begin{vmatrix} i & j & k \\ \frac{1}{1+2t^2} & \frac{2t}{1+2t^2} & \frac{2t^2}{1+2t^2} \\ \frac{-2t}{1+2t^2} & \frac{1-2t^2}{1+2t^2} & \frac{2t}{1+2t^2} \end{vmatrix}$

$= \frac{2t^2 i - 2tj + k}{1+2t^2}$

Now $\frac{dB}{dt} = \frac{4ti + (4t^2-2)j - 4tk}{(1+2t^2)^2}$

and $\frac{dB}{ds} = \frac{dB/dt}{ds/dt} = \frac{4ti + (4t^2-2)j - 4tk}{(1+2t^2)^3}$

Also, $-\tau N = -\tau\left[\frac{-2ti + (1-2t^2)j + 2tk}{(1+2t^2)}\right]$

Since $\frac{dB}{dt} = -\tau N$, we find $\tau = \frac{2}{(1+4t^2)}$,

Note that $k = \tau$ for this curve.

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27. Show that the accelaration a of a particle which travels along a space curve with velocity v is given by

$$a = \frac{dv}{dt} T + \frac{v^2}{\rho} N$$

where T is the unit tangent vector to the space curve, N is its principal normal and ρ is the radius of curvature

Velocity v = magnitude of v multiplied by unit tangent vector T

or, $\quad v = vT$

Differentiating, $a = \frac{dv}{dt} = \frac{d}{dt}(vT) = \frac{dv}{dt} T + v\frac{dT}{dt}$

But by $\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \kappa N \frac{ds}{dt} = \kappa v N = \frac{vN}{\rho}$

Then, $\quad a = \frac{dv}{dt} T + v\left(\frac{vN}{\rho}\right) = \frac{dv}{dt} T + \frac{v^2}{\rho} N$

This shows that the component of the acceleration is dv/dt In a direction tangent to the path and v^2/ρ in a direction of the principal normal to the path. The letter acceleration is often called the centripetal accel-eration. ~~For~~

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28. If r is the position vector of a particle of mass m relative to point O and F is the external force on the particle, Then $r \times F = M$ is the torque or moment of F about O. show that $M = dH/dt$. where $H = r \times mv$ and v is the velocity of the particle.

$$M = r \times F = r \times \frac{d}{dt}(mv) \text{ by Newton's law.}$$

$$\text{But } \frac{d}{dt}(r \times mv) = r \times \frac{d}{dt}(mv) + \frac{dr}{dt} \times mv$$

$$= r \times \frac{d}{dt}(mv) + v \times mv$$

$$= r \times \frac{d}{dt}(mv) + 0$$

$$M = \frac{d}{dt}(r \times mv) = \frac{dH}{dt}$$

Note: that the result holds whether m is constant or not. H is called angular momentum. The result states that the torque is equal to the time rate of change of angular momentum.

This result is easily extended to a system of n particles having respective masses $m_1, m_2, \dots m_n$ and position vectors $r_1, r_2, \dots r_n$ with external forces $F_1, F_2, \dots F_n$. For this case $H = \sum_{k=1}^{n} r_k \times v_k$ is the total angular momentum, $M = \sum_{k=1}^{n} r_k \times F_k$ is the total torque, and the result is $M = \frac{dH}{dt}$ as before.

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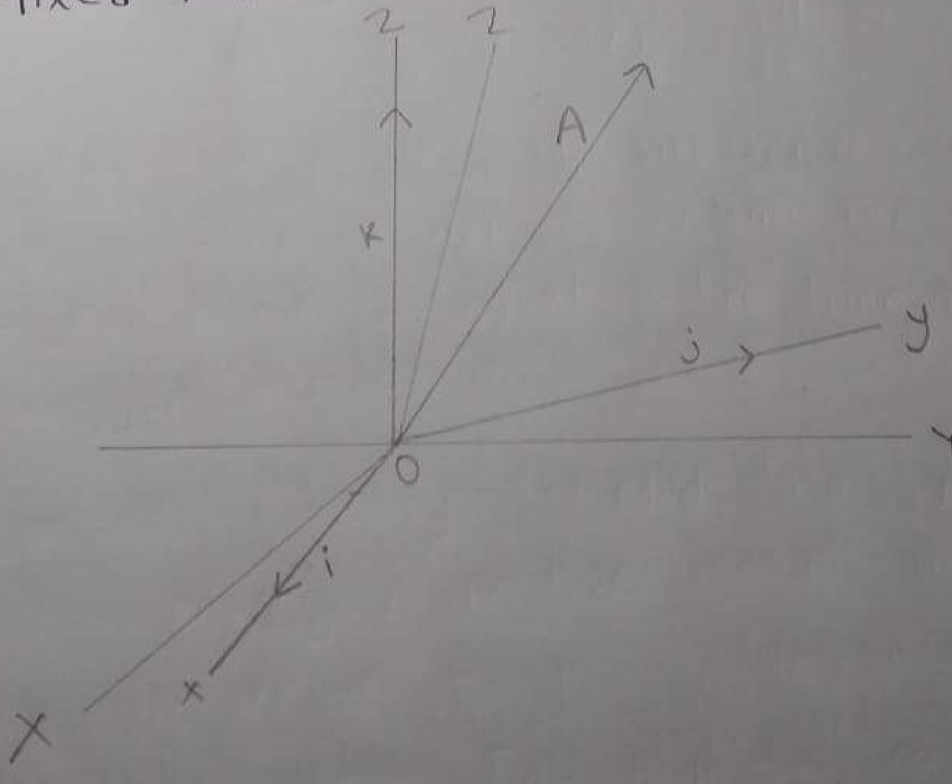
29. An observer stationed at a point which is fixed relative to an xyz coordinate system with origin O, as shown in the adjoining diagram. observes a vector $A = A_1 i + A_2 j + A_3 k$ and calculates its time derivate to be $\frac{dA_1}{dt} i + \frac{dA_2}{dt} j + \frac{dA_3}{dt} k$. Letar, he finds out that he and his coordinate system are actually rotating with respect to an XYZ coordinate system taked as fixed in space and having origin also at O. He asks what would be the time derivative of A for an observer who is fixed relative to the XYZ coordinate system?



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(a) If $\frac{dA}{dt}\Big|_f$ and $\frac{dA}{dt}\Big|_m$ denote respectively the time derivatives of A with respect to the fixed and moving systems, show that there exists a vector quantity w such that

$$\frac{dA}{dt}\Big|_f = \frac{dA}{dt}\Big|_m + w \times A$$

b) Let Df and Dm be symbolic time derivative operators in the fixed and moving systems respectively, Demonstrate the operator equivalence

$$D_f = D_m + w\times$$

(a) To the fixed observer the unit vectors i, j, k actually chang with time. Hence such an observer would compute the time derivative of A as

$$1. \quad \frac{dA}{dt} = \frac{dA_1}{dt} i + \frac{dA_2}{dt} j + \frac{dA_3}{dt} k + A_1\frac{di}{dt} + A_2\frac{dj}{dt} + A_3\frac{dk}{dt}$$

$$2: \quad \frac{dA}{dt}\Big|_f = \frac{dA}{dt}\Big|_m + A_1\frac{di}{dt} + A_2\frac{dj}{dt} + A_3\frac{dk}{dt}$$

Since i is a unit vector. di/dt is perpendicular to i and must therefore lie in the plane of j and k. Then

3. $\frac{di}{dt} = \alpha_1 j + \alpha_2 k$

4. $\frac{dj}{dt} = \alpha_3 k + \alpha_4 i$

5. $\frac{dk}{dt} = \alpha_5 i + \alpha_6 j$

From. $i \cdot j = 0$, diffenentiation yields $i \cdot \frac{dj}{dt} \cdot j = 0$, But $i \cdot \frac{dj}{dt} = \alpha_4$ from(4) and $\frac{di}{dt} \cdot j = \alpha_1$ from(3) then $\alpha_4 = -\alpha_1$

Similarly from $i \cdot k = 0$, $i \cdot \frac{dk}{dt} + \frac{di}{dt} k = 0$ and $\alpha_5 = -\alpha_2$

from $j \cdot k = 0$, $j \cdot \frac{dk}{dt} + \frac{dj}{dt} k = 0$ and $\alpha_5 = -\alpha_3$

Then $\frac{di}{dt} = \alpha_1 j + \alpha_2 k$, $\frac{dj}{dt} = \alpha_3 k - \alpha_1 i$, $\frac{dk}{dt} = -\alpha_2 i - \alpha_3 j$ and

$$A_1 \frac{di}{dt} + A_2 \frac{dj}{dt} + A_3 \frac{dk}{dt} = (-\alpha_1 A_2 - \alpha_2 A_3) i + (\alpha_1 A_1 - \alpha_3 A_3) j + (\alpha_2 A_1 + \alpha_3 A_2) k$$

which can be written as
$$\begin{vmatrix} i & j & k \\ \alpha_3 & -\alpha_2 & \alpha_1 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Then if we choose $\alpha_3 = \omega_1, -\alpha_2 = \omega_2, \alpha_1 = \omega_3$ the determinant becomes
$$\begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \omega \times A$$

where $\omega = \omega_1 i + \omega_2 j + \omega_3 k$. The quantity ω is the angular velocity of the moving system with respect to the fixed system.

b) By definition $D_f A = \frac{dA}{dt}\Big|_f$ = derivative in fixed system.

$D_m A = \frac{dA}{dt}\Big|_m$ = derivative in moving system.

From (a), $D_f A = D_m A + \omega \times A = (D_m + \omega\times)A$

and shows the equivalence of the operators $D_f = D_m + \omega\times$.

30. Determine the (a) velocity and (b) acceleration of a moving particle as seen by the two observers.

a) Let vector A in Problem be the position vector r of the particle. Using the operator notation of we have,

$$D_f r = (D_m \times \omega\times) r = D_m r + \omega \times r$$

But $D_f r = v_{p|f}$ = velocity of particle relative to fixed system.

$D_m r = V_{p/m}$ = velocity of particle relative to moving system.

$\omega \times r = V_{m|f}$ = Velocity of moving system relative to fixed system.

Then (1) can be written as

(2) $$V_{p|f} = V_{p|m} + \omega \times r$$

or in the suggestive notation

$$V_{p|f} = V_{p|m} + V_{m|f}$$

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Note that the roles of fixed and moving observes can, of course be interchanged. Thus the fixed observer can thik of himself as really moving with respect to the other. For this case we must inter-change subscripts m and f and also change w to -w since the relative rotation is reseved. If this is done(2) becomes,

$$v_{p|m} = v_{p|f} - w \times r \text{ or, } v_{p|f} = v_{p|m} + w \times r$$

so that the result is valid for each observer.

(b) The accelaration of the particle as detemined by the fixed observer at O is $D_f^2 r = D_f(D_f r)$. Take D_f of both sides(1) using the operators equivilance established then,

$$D_f(D_f r) = D_f(D_m r + w \times r)$$
$$= (D_m + w\times)(D_m r + w \times r)$$
$$= D_m(D_m r + w \times r) + w\times(D_m r + w \times r)$$
$$= D_m^2 r + D_m(w \times r) + w \times D_m r + w \times (w \times r)$$

or, $D_f^2 r = D_m^2 r + 2w \times D_m r + (D_m w) r + w(w \times r)$

or, Let, $a_{p|f} = D_f^2 r$ = acceleration of particle relative of fixed system.

$a_{p|m} = D_m^2 r$ = acceration of particle pelative to moving system.

Then, $a_{m|f} = 2w \times D_m r + (D_m w) \times r + w \times (w \times r)$
= acceleration of moving system relative to fixed

and we write.

$$a_{p|f} = a_{p|m} + a_{m|f}$$

For many cases of importance ω is a constant vector. i.e. rotation proceeds with constant angular velocity

Then, $D_m \omega = 0$ and

$$a_{m|f} = 2\omega \times D_m r + \omega \times (\omega \times r) = 2\omega \times V_m + \omega \times (\omega \times r)$$

The quantity $2\omega \times V_m$ is called the coriolis acceleration and $\omega \times (\omega \times r)$ is called the centripetal accelerdtion.

Newton's laws are strictly valid only inertial systems systems which move with constant velocity relative to a fixed system.

(4) $$M D_m^2 r = F - 2M(\omega \times D_m r) - M[\omega \times (\omega \times r)]$$

where D_m denotes d/dt as computed by an observer on the earth. F is Resultant of all real forces as measured by observer.

The theory of relatively due to Einistein has modified quite radically the concepts of absulate motion which are implied by Newton and has led to revision of concepts Newton's Laws.

Chapter - 4

The vector differential operator DEL, Written ∇, is defined by

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

This vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining three quantities which arise in practical applications and are known as the gradient, the divergence and the curl. The operator ∇ is known as nobla.

The Gradient:

Let $\phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then the gradient of ϕ, written $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$
$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

The Divergence:

Let $V(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space. Then the divergence of V, written $\nabla . V$ or div V, is defined by

$$\nabla . V = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

The curl:

If $V(x,y,z)$ is a differentiable vector field then the curl or rotation of V, written $\nabla \times V$, curl V or rot V, is defined by

$$\nabla \times V = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

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$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\hat{j}$$

$$+ \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

Formulas:

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

2. $\nabla \cdot (A + B) = \nabla \cdot A + \nabla \cdot B$

3. $\nabla \times (A + B) = \nabla \times A + \nabla \times B$

4. $\nabla \cdot (\phi A) = (\nabla\phi) \cdot A + \phi(\nabla \cdot A)$

5. $\nabla \times (\phi A) = (\nabla\phi) \times A + \phi(\nabla \times A)$

6. $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

~~7.~~ $\cancel{\nabla \times (A \times B) = B \cdot \nabla A)}$

7. $\nabla \times (A \times B) = (B \cdot \nabla)A - B(\nabla \cdot A) - (A \cdot \nabla)B + A(\nabla \cdot B)$

8. $\nabla \cdot (\nabla\phi) = \nabla^2\phi$

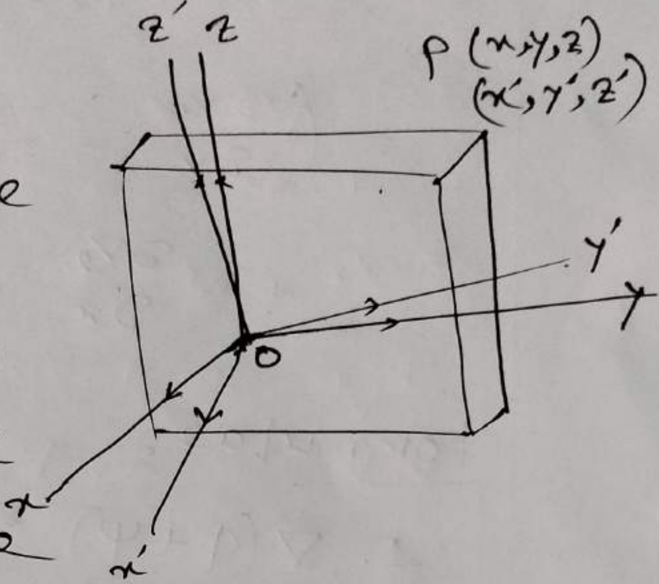
$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

9. $\nabla \times (\nabla\phi) = 0$

10. $\nabla \cdot (\nabla \times A) = 0$

11. $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A - \nabla^2 A$

Invariance:



A point P in space has ~~consi coor~~ coordinates (x, y, z) or (x', y', z') relative to these coordinate systems. The equations of transformation between coordinate transformations are given by

① $$\begin{cases} x' = l_{11}x + l_{12}y + l_{13}z \\ y' = l_{21}x + l_{22}y + l_{23}z \\ z' = l_{31}x + l_{32}y + l_{33}z \end{cases}$$

② $$\begin{cases} x' = l_{11}x + l_{12}y + l_{13}z + a_1' \\ y' = l_{21}x + l_{22}y + l_{23}z + a_2' \\ z' = l_{31}x + l_{32}y + l_{33}z + a_3' \end{cases}$$

where origin O of the xyz ~~cooridad~~ coordinate system is located at (a_1', a_2', a_3') relative to the $x'y'z'$ coordinate system. The trasformation equation (1) define a pure relation, while equations (2)' define a rotation plus ~~transfor~~ translation. Any rigid body motion has the effect of a ~~transfor~~ translation followed by a rotation. The transformation (1) is also called an orthogonal trans-formation. A general linear transfor-formation is called an affine transfor-mation is called an affine transfor-mation.

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The Gradient

(1) If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad ϕ) at the point $(1,-2,-1)$.

Ans: $\nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(3x^2y - y^3z^2)$

$$= i\frac{\partial}{\partial x}(3x^2y - y^3z^2) + j\frac{\partial}{\partial y}(3x^2y - y^3z^2) + k\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xyi + (3x^2 - 3y^2z^2)j + (-2y^3zk)$$

$$= 6\cdot1\cdot(-2)i + (3\cdot1^2 - 3\cdot(-2)^2\cdot(-1)^2)j - 2\cdot(-2)^3\cdot(-1)k$$

$$= -12i - 9j - 16k$$

(2) Prove (a) $\nabla(F+G) = \nabla F + \nabla G$, (b) $\nabla(FG) = F\nabla G + \nabla F$ where F and G are differentiable scalar functions of x, y and z.

(a) $\nabla(F+G) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(F+G)$

$$= i\frac{\partial}{\partial x}(F+G) + j\frac{\partial}{\partial y}(F+G) + k\frac{\partial}{\partial z}(F+G)$$

$$= i\frac{\partial F}{\partial x} + i\frac{\partial G}{\partial x} + j\frac{\partial F}{\partial y} + j\frac{\partial G}{\partial y} + k\frac{\partial F}{\partial z} + k\frac{\partial G}{\partial z}$$

$$= i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z} + i\frac{\partial G}{\partial x} + j\frac{\partial G}{\partial y} + k\frac{\partial G}{\partial z}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)F + \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)G$$

$$= \nabla F + \nabla G$$

(b) $\nabla(FG)$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(FG)$$

$$= \frac{\partial}{\partial x}(FG)i + \frac{\partial}{\partial y}(FG)j + \frac{\partial}{\partial z}(FG)k$$

$$= \left(F\frac{\partial G}{\partial x} + G\frac{\partial F}{\partial x}\right)i + \left(F\frac{\partial G}{\partial y} + G\frac{\partial F}{\partial y}\right)j + \left(F\frac{\partial G}{\partial z} + G\frac{\partial F}{\partial z}\right)k$$

$$= F\left(\frac{\partial G}{\partial x}i + \frac{\partial G}{\partial y}j + \frac{\partial G}{\partial z}k\right) + G\left(\frac{\partial F}{\partial x}i + \frac{\partial F}{\partial y}j + \frac{\partial F}{\partial z}k\right)$$

$$= F\nabla G + G\nabla F$$

3 Find $\nabla\phi$ if (a) $\phi = \ln|r|$; (b) $\phi = \frac{1}{r}$

(a) $r = xi + yj + zk$

Then $|r| = \sqrt{x^2+y^2+z^2}$ and $\phi = \ln|r| = \frac{1}{2}\ln(x^2+y^2+z^2)$

$$\nabla\phi = \frac{1}{2}\nabla\ln(x^2+y^2+z^2)$$

$$= \frac{1}{2}\left\{i\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + j\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + k\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\}$$

$$= \frac{1}{2}\left\{i\frac{2x}{x^2+y^2+z^2} + j\frac{2y}{x^2+y^2+z^2} + k\frac{2z}{x^2+y^2+z^2}\right\}$$

$$= \frac{xi+yi+zk}{x^2+y^2+z^2}$$

$$= \frac{r}{r^2}$$

(b) $\nabla\phi = \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla\left\{(x^2+y^2+z^2)^{-1/2}\right\}$

$$= i\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + j\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}$$
$$+ k\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= i\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x\right\} + j\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2y\right\}$$
$$+ k\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2z\right\}$$

$$= \frac{-xi-yj-zk}{(x^2+y^2+z^2)^{3/2}} = -\frac{r}{r^3}$$

(4) Show that $\nabla r^n = n r^{n-2} r$

$$\nabla r^n = \nabla\left(\sqrt{x^2+y^2+z^2}\right)^n = \nabla\left(x^2+y^2+z^2\right)^{n/2}$$

$$= i\,\frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2\right)^{n/2}\right\} + j\,\frac{\partial}{\partial y}\left\{\left(x^2+y^2+z^2\right)^{n/2}\right\}$$
$$+ k\,\frac{\partial}{\partial z}\left\{\left(x^2+y^2+z^2\right)^{n/2}\right\}$$

$$= i\left\{\frac{n}{2}\left(x^2+y^2+z^2\right)^{n/2-1} 2x\right\} + j\left\{\frac{n}{2}\left(x^2+y^2+z^2\right)^{n/2-1} 2y\right\}$$
$$+ k\left\{\frac{n}{2}\left(x^2+y^2+z^2\right)^{n/2-1} 2z\right\}$$

$$= n\left(x^2+y^2+z^2\right)^{n/2-1}\left(xi + yj + zk\right)$$

$$= n\left(r^2\right)^{n/2-1} r = n r^{n-2} r$$

5. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$ where c is a constant.

Let $r = xi + yj + zk$ be the position vector of any point $P(x,y,z)$ on the surface. Then $dr = dxi + dyj + dzk$ lies in the tangent plane to the surface at P.

But $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$

or, $\left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)\cdot(dxi + dyj + dzk) = 0$

$\nabla\phi \cdot dr = 0$ so that $\nabla\phi$ is perpendicular to dr and therefore to the surface.

6. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

$\nabla(x^2y + 2xz) = (2xy + 2z)i + x^2j + 2xk = -2i + 4j + 4k$

at the point $(2,-2,3)$

Then a unit normal to the surface $= \frac{-2i + 4j + 4k}{\sqrt{(-2)^2 + 4^2 + 4^2}}$

$= -\frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k$

Another unit normal is $\frac{1}{3}i - \frac{2}{3}j - \frac{2}{3}k$ having direction opposit to that above.

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(7) Find an equation for the tangent plane to the surface $2xz^2-3xy-4x=7$, at the point $(1,-1,2)$.

Soln:
$$\nabla(2xz^2-3xy-4x)=(2z^2-3y-4)i-3xj+4xzk$$

Then a normal to the surface at the point $(1,-1,2)$ is $7i-3j+8k$

The equation of a plane passing through a point whose position vector is r_0 and which is perpendicular to the normal N is $(r-r_0).N=0$. then the required equation is

$$[(xi+yj+zk)-(i-j+2k)].(7i-3j+8k)=0$$
$$\Rightarrow 7(x-1)-3(y+1)+8(z-2)=0$$

(10) Find the directional derivate of $\phi=x^2yz+4xz^2$ at $(1,-2,-1)$ in the direction $2i-j-2k$.

Soln: $\nabla\phi=\nabla(x^2yz+4xz^2)$
$$=(2xyz+4z^2)i+x^2zj+(x^2y+8xz)k$$
$$=8i-j-10k \text{ at } (1,-2,-1)$$

The unit vector in the direction of $2i-j-2k$ is

$$a=\frac{2i-j-2k}{\sqrt{(2)^2+(-1)^2+(-2)^2}}$$
$$=\frac{2}{3}i-\frac{1}{3}j-\frac{2}{3}k$$

Then the required directional derivative is

$$\nabla\phi \cdot a = (8i - j + 10k) \cdot (\tfrac{2}{3}i - \tfrac{1}{3}j - \tfrac{2}{3}k)$$

$$= \tfrac{16}{3} + \tfrac{1}{3} + \tfrac{20}{3}$$

$$= \tfrac{37}{3}$$

Since this is positive, ϕ is increasing in this direction

(11) a. In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum?

b. What is the magnitude of this maximum?

Solⁿ: $\nabla\phi = \nabla(x^2yz^3)$

$$= 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$$

$$= -4i - 4j + 12k \text{ at } (2,1,-1)$$

(a) The directional derivative is a maximum in the direction $\nabla\phi = -4i - 4j + 12k$.

(b) The magnitude of this maximum is

$$|\nabla\phi| = \sqrt{(-4)^2 + (-4)^2 + (12)^2}$$

$$= \sqrt{176}$$

$$= 4\sqrt{11}$$

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(12) Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Soln: The angle between the surface at the point is the angle between the normals to the surface at the point.

A normal to $x^2+y^2+z^2=9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2+y^2+z^2)$$
$$= 2xi+2yj+2zk$$
$$= 4i-2j+4k$$

A normal to $z=x^2+y^2-3$ or $x^2+y^2-z=3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2+y^2-z)$$
$$= 2xi+2yj-k$$
$$= 4i-2j-k$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1||\nabla\phi_2|\cos\theta$ where θ is the required angle. Then $(4i-2j+4k)\cdot(4i-2j-k) = |4i-2j+4k|\,|4i-2j-k|\cos\theta$

$$\Rightarrow 16+4-4 = \sqrt{(4)^2+(-2)^2+(4)^2}\cdot\sqrt{(4)^2+(-2)^2+(-1)^2}\cos\theta$$

and $\cos\theta = \frac{16}{6\sqrt{21}} = \frac{8\sqrt{21}}{63} = 0.5819$; Thus the acute angle is $\theta = \operatorname{arc}\cos(0.5819)$

$$= 54^\circ 25'.$$

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(13) Let R be the distance from a fixed point $A(a,b,c)$ to any point $P(x,y,z)$. Show that ∇R is a unit vector in the direction $AP = R$.

Soln: If r_A and r_B are the position vectors $ai+bj+ck$ and $xi+yj+zk$ of A and P respectively, then

$$R = r_B - r_A$$
$$= (x-a)i + (y-b)j + (z-c)k$$

So that $R = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$. Then,

$$\nabla R = \nabla\left(\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}\right)$$
$$= \frac{(x-a)i+(y-b)j+(z-c)k}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$
$$= \frac{R}{R}$$

is a unit vector in the direction R.

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15) If $A = x^2 z i - 2y^3 z^2 j + xy^2 z k$, find $\nabla . A$ (or div A) at the point (1, -1, 1)

Sol: $\nabla . A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) . (x^2 z i - 2y^3 z^2 j + x^2 y^2 z k)$

$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(x y^2 z)$

$= 2xz - 6y^2 z^2 + x^2 y^2$

$= 2(1)(1) - 6(-1)^2 (1)^2 + (1)(-1)^2$

$= 3$ at $(1, -1, 1)$

16) Given $\phi = 2x^3 y^2 z^4$

(a) Find $\nabla . \nabla \phi$ (or div grad ϕ)

(b) Show that $\nabla . \nabla \phi = \nabla^2 \phi$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian operator

a) ~~an~~ sol: $\nabla \phi = i \frac{\partial}{\partial x}(2x^3 y^2 z^4) + j \frac{\partial}{\partial y}(2x^3 y^2 z^4) + k \frac{\partial}{\partial z}(2x^3 y^2 z^4)$

$= 6x^2 y^2 z^4 i + 4x^3 y z^4 j + 8x^3 y^2 z^3 k$

Then $\nabla \cdot \nabla \phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(6x^2y^2z^4 i + 4x^3yz^4 j + 8x^3y^2z^3 k\right)$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

b) sol: $\nabla \cdot \nabla \phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial^2}\right)\phi$$

$$= \nabla^2 \phi$$

17 Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

similarly

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$$

and $$\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=0$

The equation $\nabla^2\phi=0$ is called Laplace's equation

It follows that $\phi=1/r$ is a solution of this equation.

18. Prove a) $\nabla\cdot(A+B)=\nabla\cdot A+\nabla\cdot B$

b) $\nabla\cdot(\phi A)=(\nabla\phi)\cdot A+\phi(\nabla\cdot A)$

a) Sol: Let $A=A_1 i+A_2 j+A_3 k$, $B=B_1 i+B_2 j+B_3 k$

Then $\nabla\cdot(A+B)=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\cdot\left[(A_1+B_1)i+(A_2+B_2)j+(A_3+B_3)k\right]$

$$=\frac{\partial}{\partial x}(A_1+B_1)+\frac{\partial}{\partial y}(A_2+B_2)+\frac{\partial}{\partial z}(A_3+B_3)$$

$$=\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}$$

$$=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\cdot(A_1 i+A_2 j+A_3 k)$$
$$+\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\cdot(B_1 i+B_2 j+B_3 k)$$

$=\nabla\cdot A+\nabla\cdot B$

b) Sol: $\nabla \cdot (\phi A) = \nabla \cdot (\phi A_1 i + \phi A_2 j + \phi A_3 k)$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial \phi}{\partial x} A_1 + \phi \frac{\partial A_1}{\partial x} + \frac{\partial \phi}{\partial y} A_2 + \phi \frac{\partial A_2}{\partial y} + \frac{\partial \phi}{\partial z} A_3 + \phi \frac{\partial A_3}{\partial z}$$

$$= \frac{\partial \phi}{\partial x} A_1 + \frac{\partial \phi}{\partial y} A_2 + \frac{\partial \phi}{\partial z} A_3 + \phi \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right)$$

$$= \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \right) \cdot (A_1 i + A_2 j + A_3 k)$$

$$+ \phi \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (A_1 i + A_2 j + A_3 k)$$

$$= (\nabla \phi) \cdot A + \phi (\nabla \cdot A)$$

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(1)

(23) If $A = xz^3 i - 2x^2yz j + 2yz^4 k$

define the curl at $(1, -1, 1)$

$$\nabla \times A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times \left(xz^3 i - 2x^2yz j + 2yz^4 k\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] i +$$

$$\left[\frac{\partial}{\partial z}(xz^3 - \frac{\partial}{\partial x}(2yz^4)\right] j +$$

$$\left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right] k$$

$$= (2z^4 + 2x^2y) i + 3xz^2 j - 4xyz k = 3j + 4k$$

(Ans)

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(2)

(24) If $A = x^2 y i - 2xz j + 2yz k$

find curl of curl A ?

$= \nabla \times (\nabla \times A)$

$= \nabla \times \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$

$= \nabla \times \left[(2x + 2z) i - (x^2 + 2z) k \right]$

$= \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2x+2z & 0 & -x^2-2z \end{vmatrix} = (2x + 2) j$

(25)

(a) let $A = A_1 i + A_2 j + A_3 k$

$B = B_1 i + B_2 j + B_3 k$

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(3)

$$\nabla \times (A+B) = \left(\frac{\delta}{\delta x} i + \frac{\delta}{\delta y} j + \frac{\delta}{\delta z} k\right) \times \left[(A_1+B_1) i + (A_2+B_2) j + (A_3+B_3) k\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_1+B_1 & A_2+B_2 & A_3+B_3 \end{vmatrix}$$

$$= \left[\frac{\delta}{\delta y}(A_3+B_3) - \frac{\delta}{\delta z}(A_2+B_2)\right] i + \left[\frac{\delta}{\delta z}(A_1+B_1) - \frac{\delta}{\delta x}(A_3+B_3)\right] j + \left[\frac{\delta}{\delta x}(A_2+B_2) - \frac{\delta}{\delta y}(A_1+B_1)\right] k$$

$$= \left[\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right] i + \left[\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}\right] j + \left[\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right] k$$

$$+ \left[\frac{\delta B_3}{\delta y} - \frac{\delta B_2}{\delta z}\right] i + \left[\frac{\delta B_1}{\delta z} - \frac{\delta B_3}{\delta x}\right] j + \left[\frac{\delta B_2}{\delta x} - \frac{\delta B_1}{\delta y}\right] k$$

$$= \nabla \times A + \nabla \times B$$

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(b) $\nabla \times (\phi A) = \nabla \times (\phi A_1 i + \phi A_2 j + \phi A_3 k)$

$$= \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left[\frac{\delta}{\delta y}(\phi A_3) - \frac{\delta}{\delta z}(\phi A_2)\right] i + \left[\frac{\delta}{\delta z}(\phi A_1) - \frac{\delta}{\delta x}(\phi A_3)\right] j + \left[\frac{\delta}{\delta x}(\phi A_2) - \frac{\delta}{\delta y}(\phi A_1)\right] k$$

$$= \phi\left[\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right) i + \left(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}\right) j + \left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right) k\right] + \left[\left(\frac{\delta \phi}{\delta y} A_3 - \frac{\delta \phi}{\delta z} A_2\right) i + \left(\frac{\delta \phi}{\delta z} A_1 - \frac{\delta \phi}{\delta x} A_3\right) j + \left(\frac{\delta \phi}{\delta x} A_2 - \frac{\delta \phi}{\delta y} A_1\right) k\right]$$

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(5)

$$= \phi(\nabla \times A) + \begin{vmatrix} i & j & k \\ \frac{\delta\phi}{\delta x} & \frac{\delta\phi}{\delta y} & \frac{\delta\phi}{\delta z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \phi(\nabla \times A) + (\nabla\phi) \times A$$

(26) Let $A = A_1 i + A_2 j + A_3 k$.

$$r = xi + yj + zk$$

Then $A \times r = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$

$$= (zA_2 - yA_3)i + (xA_3 - zA_1)j + (yA_1 - xA_2)k$$

and $\nabla \cdot (A \times r) = \frac{\delta}{\delta x}(zA_2 - yA_3) + \frac{\delta}{\delta y}(xA_3 - zA_1)$

$$+ \frac{\delta}{\delta z}(yA_1 - xA_2)$$

$$= z\frac{\delta A_2}{\delta x} - y\frac{\delta A_3}{\delta x} + x\frac{\delta A_3}{\delta y} - z\frac{\delta A_1}{\delta y} + y\frac{\delta A_1}{\delta z} -$$

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$-x\frac{\delta A_2}{\delta z}$ (6)

$$= x\left(\frac{\delta A_3}{\delta y} - \frac{\delta A}{\delta z}\right) + y\left(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}\right) + z\left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right)$$

$$= \left[xi + yj + zk\right] \cdot \left[\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right)i + \frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}\right)j + \left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right)k\right]$$

$$= r \cdot (\nabla \times A) = r \cdot \text{curl } A$$

if $\nabla \times A = 0$ this reduces to zero.

(27) (a) $\nabla \times (\nabla \phi) = 0$ curl grad $\phi = 0$

$$= \nabla \times \left(\frac{\delta \phi}{\delta x} i + \frac{\delta \phi}{\delta y} j + \frac{\delta \phi}{\delta z} k\right)$$

$$= \left(\frac{\delta^2 \phi}{\delta y \delta z} - \frac{\delta^2 \phi}{\delta z \delta y}\right) i + \left(\frac{\delta^2 \phi}{\delta z \delta y} - \frac{\delta^2 \phi}{\delta x \delta z}\right) j + \left(\frac{\delta^2 \phi}{\delta x \delta y} - \frac{\delta^2 \phi}{\delta y \delta x}\right) k = 0$$

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provided we assume that ϕ has continious second partial derivatives so that the order of differentiation is immaterial.

(b) $\nabla \cdot (\nabla \times A) = \nabla \cdot \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$= \nabla \cdot \left[\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z} \right) i + \left(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x} \right) j + \left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y} \right) k \right]$$

$$= \frac{\delta}{\delta x} \left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z} \right) + \frac{\delta}{\delta y} \left(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x} \right) + \frac{\delta}{\delta z} \left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y} \right)$$

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$$= \frac{\delta^2 A_3}{\delta x \delta y} - \frac{\delta^2 A_2}{\delta x \delta z} + \frac{\delta^2 A_1}{\delta y \delta z} - \frac{\delta^2 A_3}{\delta y \delta x} + \frac{\delta^2 A_2}{\delta z \delta x} - \frac{\delta^2 A_1}{\delta z \delta y} = 0$$

assuming A has continuous second partial derivatives.

Note the similarity between the above results and the results $(c \times cm) = (c \times c) m = 0$ where m is a scalar and $c \cdot (c \times A) = (c \times c) \cdot A = 0$

(2.8) Find curl $(r f(r))$ where $f(r)$ is differentiable.

$$\text{curl}\,(r f(r)) = \nabla \times (r f(r))$$

$$= \nabla \times (x f(r) i + y f(r) j + z f(r) k$$

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$$= \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$$= \left(z\frac{\delta f}{\delta y} - y\frac{\delta f}{\delta z} \right) i + \left(x\frac{\delta f}{\delta z} - z\frac{\delta f}{\delta x} \right) j$$

$$+ \left(y\frac{\delta f}{\delta x} - x\frac{\delta f}{\delta y} \right) k$$

But, $\frac{\delta f}{\delta x} = \left(\frac{\delta f}{\delta r}\right)\left(\frac{\delta r}{\delta x}\right) = \frac{\delta f}{\delta r} \cdot \frac{\delta}{\delta x}\left(\sqrt{x^2+y^2+z^2}\right)$

$$= \frac{f'(r)\, x}{\sqrt{x^2+y^2+z^2}} = \frac{f' x}{r}$$

similarly, $\frac{\delta f}{\delta y} = \frac{f' y}{r}$ and $\frac{\delta f}{\delta z} = \frac{f' z}{r}$

Then the result =

$$\left(z\frac{f' y}{r} - y\frac{f' z}{r} \right) i + \left(x\frac{f' z}{r} - z\frac{f' x}{r} \right) j$$

$$+ \left(y\frac{f' x}{r} - x\frac{f' y}{r} \right) k = 0$$

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30 $\text{curl}\ v = \nabla \times v = \nabla \times (\omega \times r)$

$$= \nabla \times \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[(\omega_2 z - \omega_3 y) i + (\omega_3 x - \omega_1 z) j + (\omega_1 y - \omega_2 x) k \right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= 2(\omega_1 i + \omega_2 j + \omega_3 k)$$

$$= 2\omega$$

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Miscellaneous problems

(1)(a) A vector V is called irrotational If curl V = 0. Find constant a, b, c so that
$V = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$
is irrotational.

(b) Show that V can be expressed as the gradient of a scalar function.

(a)

→ Sol^n: $\text{curl } V = \nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$= (c+1)i + (a-4)j + (b-2)k$

This equals zero when $a=4$, $b=2$, $c=-1$

and $V = (x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)k$

(b)

→ Sol^n: Assume $V = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

Then (1) $\frac{\partial\phi}{\partial x} = x+2y+4z$, (2) $\frac{\partial\phi}{\partial y} = 2x-3y-z$,
(3) $\frac{\partial\phi}{\partial z} = 4x-y+2z$.

Integrating (1) partially with respect to x, keeping y & z constant.

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(4) $\phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z)$

where $f(y,z)$ is an arbitrary function of y & z.

Similarly (2) & (3).

(5) $\phi = 2xy - \frac{3y^2}{2} - yz + g(x,z)$

(6) $\phi = 4xz - yz + z^2 + h(x,y)$

comparison of (4), (5) and (6) shows that there will be a common value of ϕ if we choose

$f(y,z) = -\frac{3y^2}{2} + z^2$, $\quad g(x,z) = \frac{x^2}{2} + z^2$

$h(x,y) = \frac{x^2}{2} - \frac{3y^2}{2}$

so that, $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz$

we can also add any constant to ϕ.

In general if $\nabla \times V = 0$, then we can find ϕ so that $V = \nabla\phi$. A vector field V which can be derived from a scalar field ϕ so that $V = \nabla\phi$ is called a consevative vector field and ϕ is called the scalar potential. conversely if $V = \nabla\phi$. then $\nabla \times V = 0$.

② If $A = 2yz\,i - x^2y\,j + xz^2\,k$

$B = x^2\,i + yz\,j - xy\,k$

and $\phi = 2x^2yz^3$, find

ⓐ $(A\cdot\nabla)\phi$

Soln: $(A\cdot\nabla)\phi = \left[(2yz\,i - x^2y\,j + xz^2\,k)\cdot\left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\right]\phi$

$= \left(2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial x}\right)(2x^2yz^3)$

$= 2yz\frac{\partial}{\partial x}(2x^2yz^3) - x^2y\frac{\partial}{\partial y}(2x^2yz^3) + xz^2\frac{\partial}{\partial x}(2x^2yz^3)$

$= (2yz)(4xyz^3) - (x^2y)(2x^2z^3) + (xz^2)(6x^2yz^2)$

$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$

ⓑ $A\cdot\nabla\phi = (2yz\,i - x^2y\,j + xz^2\,k)\cdot\left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)$

$= (2yz\,i - x^2y\,j + xz^2\,k)\cdot(4xyz^3\,i + 2x^2z^3\,j + 6x^2yz^2\,k)$

$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$

comparison with ⓐ illustrates the result

$(A\cdot\nabla)\phi = A\cdot\nabla\phi.$

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(c) $(B\cdot\nabla)A = \left[(x^2 i + yz\, j - xy\, k)\cdot\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\right]A$

$= \left(x^2\frac{\partial}{\partial x} + yz\frac{\partial}{\partial x} - xy\frac{\partial}{\partial x}\right)A$

$= x^2\frac{\partial A}{\partial x} + yz\frac{\partial A}{\partial y} - xy\frac{\partial A}{\partial z}$

$= x^2(-2xy\, j + z^2 k) + yz(2z\, i - x^2 j) - xy(2y\, i + 2xz\, k)$

$= (2yz^2 - 2xy^2)\, i - (2x^3y + x^2yz)\, j + (x^2z^2 - 2x^2yz)\, k$

(d) $(A\times\nabla)\phi = \left[(2yz\, i - x^2y\, j + xz^2 k)\times\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\right]\phi$

$$\begin{vmatrix} i & j & k \\ 2yz & -x^2y & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$= \left[i\left(-x^2y\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial y}\right) + j\left(xz^2\frac{\partial}{\partial x} - 2yz\frac{\partial}{\partial z}\right) + k\left(2yz\frac{\partial}{\partial y} + x^2y\frac{\partial}{\partial x}\right)\right]\phi$

$= -\left(x^2y\frac{\partial\phi}{\partial z} + xz^2\frac{\partial\phi}{\partial y}\right)i + \left(xz^2\frac{\partial\phi}{\partial x} - 2yz\frac{\partial\phi}{\partial z}\right)j + \left(2yz\frac{\partial\phi}{\partial y} + x^2y\frac{\partial\phi}{\partial x}\right)k$

$= -(6x^4y^2z^2 + 2x^3z^5)\, i + (4x^2yz^5 - 12x^2y^2z^3)\, j + (4x^2yz^4 + 4x^3y^2z^3)\, k$

(e) $A \times \nabla\phi = (2yz\,i - x^2y\,j + xz^2\,k) \times \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)$

$$\begin{vmatrix} i & j & k \\ 2yz & -x^2y & xz^2 \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left(-x^2y\frac{\partial\phi}{\partial z} - xz^2\frac{\partial\phi}{\partial y}\right)i + \left(xz^2\frac{\partial\phi}{\partial x} - 2yz\frac{\partial\phi}{\partial z}\right)j$$

$$+ \left(2yz\frac{\partial\phi}{\partial y} + x^2y\frac{\partial\phi}{\partial x}\right)k$$

$$= -(6x^4y^2z^2 + 2x^3z^5)i + (4x^2yz^5 - 12x^2y^2z^3)j +$$

$$(4x^2yz^4 + 4x^3y^2z^3)k$$

comparison with (d) illustrates the result $(A \times \nabla)\phi = A \times \nabla\phi$.