BANGABANDHU SHEIKH MUJIBUR RAHMAN SCIENCE AND TECHNOLOGY UNIVERSITY

Name of the Assignment

Vector Integration(Line, Surface, Volume integral),Divergence Theorem of Gauss, Stoke’s Theorem, Green’s Theorem

Course Code: MAT205

Course Title: Vector, Matrix and Fourier Analysis

Date of Submission: 27/06/2021

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CONTENTS

Line integrals:

Let $r(u) = x(u)i + y(u)j + z(u)K$, where $r(u)$ is the position vector of (x, y, z), define a curve c joining points P_1 and P_2, where $u = u_1$ and $u = u_2$ respectively. We assume that c is composed of a finite number of curve for each of which $r(u)$ has a continuous derivative. Let $A(x, y, z) = A_1 i + A_2 j + A_3 K$ be a vector function of position defined and continuous along c. Then the integral of the tangential component of A along c from P_1 to P_2 written us

$$\int_{P_1}^{P_2} A \cdot dr = \int_c A \cdot dr$$

$$= \int_c A_1 dx + A_2 dy + A_3 dz$$

Is an example of a line integral. If A is the force F on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve (which we shall suppose is a simple close -d curve, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by

$$\oint A \cdot dr = \oint A_1 \, dx + A_2 \, dy + A_3 \, dz$$

In aerodynamics and fluid mechanics this integral is called the circulation of A about C, where A represents the velo -city of a fluid.

In general, any integral which is to be evaluated along a curve is called a line integral. Such integrals can be defined in terms of limits of sums as are the integrals of elementary calculus.

For methods of evaluation of line integrals, see the solved problems.

The following theorem is important.

THEOREM

If $A = \nabla \phi$ everywhere in a region R of space, defined by $a_1 \leqq x \leqq a_2$, $b_1 \leqq y \leqq b_2$, $c_1 \leqq z \leqq c_2$, where $\phi(x, y, z)$ is single-valued and has continuous derivatives in R,

then

1. $\int_{P_1}^{P_2} A.dr$ is independent of the path C in R joining P_1 and P_2.

2. $\oint A \cdot dr = 0$ around any closed curve C in R.

In such case A is called a conservative vector field and ϕ is its scala

potential.
A vector field A is conservation if and only if $\nabla \times A = 0$, or equivalently $A = \nabla \phi$. In such case $A \cdot dr = A_1\, dx + A_2\, dy + A_3\, dz = d\phi$, an exat differential.

Vector Integration

Problem NO:01

If $R(u) = (u-u^2)\,i + 2u^3 j - 3k$

(a) $\int R(u)\,du$ and (b) $\int_1^2 R(u)\,du$.

Sol^n: (a) $\int R(u)\,du$

$$= \int \left[(u-u^2)\,i + 2u^3 j - 3k\right] du$$

$$= i\int (u-u^2)\,du + j\int 2u^3\,du + k\int -3\,du$$

$$= i\left(\frac{u^2}{2} - \frac{u^3}{3} + c_1\right) + j\left(\frac{u^4}{2} + c_2\right) + K(-3u+c_3)$$

$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right) i + \frac{u^4}{2} j - 3uK + c_1 i + c_2 j + c_3 K$$

$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right) i + \frac{u^4}{2} j - 3uK + c$$

Where c is the constant vector $c_1 i + c_3 j + c_3 K$

Mamlot-034
page-07

(b) From (a), $\int_1^2 R(u)\,du = \left(\frac{u^2}{2} - \frac{u^3}{3}\right)i + \frac{u^4}{2}j - 3uk + c \Big|_1^2$

$$= \left[\left(\frac{2^2}{2} - \frac{2^3}{3}\right)i + \frac{2^4}{2}j - 3(2)k + c\right] - \left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right)i + \frac{1^4}{2}j - 3(1)k + c\right]$$

$$= -\frac{5}{6}i + \frac{15}{2}j - 3k$$

Another method.

$$\int_1^2 R(u)\,du = i\int_1^2 (u - u^2)\,du + j\int_1^2 2u^3\,du + k\int_1^2 -3\,du$$

$$= i\left(\frac{u^2}{2} - \frac{u^3}{3}\right)\Big|_1^2 + j\left(\frac{u^4}{2}\right)\Big|_1^2 + k(-3u)\Big|_1^2$$

$$= -\frac{5}{6}i + \frac{15}{2}j - 3k$$

Mamlot-034
page - 08

Problem no: 02

The acceleration of a particle at any time $t \geqq 0$ is given by

$$a = \frac{dv}{dt} = 12\cos 2t\, i - 8\sin 2t\, j + 16t\, k$$

If the velocity v and displacement r are zero at $t=0$, find v and r at any time.

Integration,

$$v = i\int 12\cos 2t\, dt + j\int -8\sin 2t\, dt + k\int 16t\, dt$$

$$= 6\sin 2t\, i + 4\cos 2t\, j + 8t^2 k + c_1$$

putting $v=0$ when $t=0$,

we find $0 = 0i + 4j + 0k + c_1$ and $c_1 = -4j$

Then $v = 6\sin 2t\, i + (4\cos 2t - 4)\, j + 8t^2 k$

so that $\frac{dr}{dt} = 6\sin 2t\, i + (4\cos 2t - 4)\, j + 8t^2 k$

Mamlot-034
page - 09

Integrating,

$$r = i\int 6 \sin 2t \, dt + j\int 4(\cos 2t - 4)\, dt + k\int 8 t^2 \, dt$$

$$= -3\cos 2t \, i + (2\sin 2t - 4t)\, j + \frac{8}{3}t^3 k + c_2$$

putting $r = 0$ when $t = 0$,

$$0 = -3i + 0j + 0k + c_2 \text{ and } c_2 = 3i.$$

Then

$$r = (3 - 3\cos 2t)\, i + (2\sin 2t - 4t)\, j + \frac{8}{3}t^3 K.$$

Mamlat-034
page-10

line integrals

problem no: 06

If $A = (3x^2+6y)i - 14yzj + 20xz^2K$,

evaluate $\int_C A \cdot dr$ from $(0,0,0)$ to $(1,1,1)$

along the following paths C:

(a) $x = t$, $y = t^2$, $z = t^3$.

(b) the straight lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$, and then to $(1,1,1)$.

(c) the straight line joining $(0,0,0)$ and $(1,1,1)$

$$\int_C A \cdot dr = \int_C (3x^2+6y)i - 14yzj + 20xz^2K \cdot (dxi + dyj + dzK)$$

$$= \int_C (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$$

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Mamlat-034
page - 11

(a) If $x = t$, $y = t^2$, $z = t^3$, points $(0,0,0)$ and $(1,1,1)$ correspond to $t = 0$ and $t = 1$ respectively. Then

$$\int_C A \cdot dr = \int_{t=0}^{1} (3t^2 + 6t^2)\,dt - 14(t^3)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)$$

$$= \int_{t=0}^{1} 9t^2 dt - 28t^6 dt + 60t^9 dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= 3t^3 - 4t^7 + 6t^{10} \Big|_0^1$$

$$= 5$$

Another method.

Along C, $A = 9t^2 i - 14t^5 j + 20t^7 K$

and $r = xi + yj + zK$

$= ti + t^2 j + t^3 K$

and $dr = (i + 2tj + 3t^2 K)\, dt$

Then $\int_C A \cdot dr = \int_{t=0}^{1} (9t^2 i - 14t^5 j + 20t^7 K)\, dt$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\, dt$

$= 5$

(b) Along the straight line from $(0,0,0)$ to $(1,0,0)$ $y = 0$, $z = 0$, $dz = 0$ while x varies from 0 to 1. Then the integral over this part of the path is

$\int_{x=0}^{1} (3x^2 + 6(0))\, dx - 14(0)(0)(0) + 20x(0)^2(0)$

$$= \int_{x=0}^{1} 3x^2 dx = x^3 \Big|_0^1 = 1$$

Along the straight line from (1, 0, 0) to (0,0,
to (1, 1, 0) $x = 1$, $z = 0$, $dx = 0$, $dz = 0$
while y varies from 0 to 1.
Then the integral over this part of the path is

$$\int_{y=0}^{1} (3(1)^2 + 6y)0 - 14y(0)dy + 20(1)(0)^2 0$$

$$= 0$$

Along the straight line from (1, 1, 0) to (1, 1, 1) $x = 1$, $y = 1$, $dx = 0$, $dy = 0$ while z varies from 0 to 1.
Then the integral over this part of the part is

$$\int_{z=0}^{1} (3(1)^2 + 6(1))0 - 14(1)z(0) + 20(1)z^2 dz$$

$$= \int_{z=0}^{1} 20 z^2 dz = \frac{20z^3}{3}\Big|_0^1 = \frac{20}{3}$$

Adding $\int_C A \cdot dr = 1 + 0 + \frac{20}{3}$

$$= \frac{23}{3}$$

(c) The straight line joining (0,0,0) and (1, 1, 1) is given in parametic from by $x = t$, $y = t$, $z = t$.

Then $$\int_C A \cdot dr = \int_{t=0}^{1} (3t^2+6t)\, dt - 14(t)(t)\, dt + 20(t)(t)^2\, dt$$

$$= \int_{t=0}^{1} (3t^2 + 6t - 14t^2 + 20t^3)\, dt$$

$$= \int_{t=0}^{1} (6t - 11t^2 + 20t^3)\, dt$$

$$= \frac{13}{3}$$

problem no: 07

Find the total work done in moving a particle in a force field given by $F = 3xy\,i - 5z\,j + 10x\,K$ along the curve $x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t=1$ to $t=2$

Sol^n:

$$\text{Total work} = \int_C F \cdot dr$$

$$= \int_C (3xy\,i - 5z\,j + 10x\,K) \cdot (dx\,i + dy\,j + dz\,K)$$

$$= \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

$$= \int_{t=1}^{2} 3(t^2+1)(2t^2)\,d(t^2+1) - (5(t^3)\,d(2t^2) + 10(t^2+1)\,d(t^3)$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 3t^2)\,dt$$

$$\cancel{= 330} = 303$$

Supplementary problems

problem no: 28

If $R(t) = (3t^2 - t)i + \cancel{2}(2-6t)j - 4tK$

find (a) $\int R(t)\,dt$ and (b) $\int_2^4 R(t)\,dt$

Sol^n: (a) $\int R(t)\,dt = i\int(3t^2 - t)\,dt + j\int(2-6t)\,dt - K\int 4t\,dt$

$= i\left[t^3 - \frac{t^2}{2}\right] + j\,(2t - 3t^2) - K\,2t^2 + c$

Sol^n: (b) $\int_2^4 R(t)\,dt = i\left[4^3 - \frac{4^2}{2} - 2^3 + \frac{2^2}{2}\right] + j\,(8 - 48 - 4 + 12) - K\,(32 - 8)$

$= i\,(64 - 8 - 8 + 2) + j\,(20 - 52) - K\,(24)$

$= 50i - 32j - 24K$

Mamlot-034
page-17

problem no: 29

Evaluate $\int_0^{\pi/2} (3\sin u\, i + 2\cos u\, j)\, du$

Sol^n: $\left[-3\cos u\, i + 2\sin u\, j\right]_0^{\pi/2}$

$= -3i(\cos \pi/2 - \cos 0) + 2j(\sin \pi/2 - \sin 0)$

$= 3i + 2j$

mamlot-034
page-18

problem no: 30

If $A(t) = ti - t^2 j + (t-1)K$ and
$B(t) = 2t^2 i + 6tK$, evaluate
(a) $\int_0^2 A \cdot B \, dt$, (b) $\int_0^2 A \times B \, dt$

Sol^n: (a) $\vec{A} \cdot \vec{B} = t \cdot 2t^2 - t^2 \cdot 0 + (t-1) \cdot 6t$

$= 2t^3 + 6t^2 - 6t$

$$\int_0^2 (\vec{A} \cdot \vec{B}) \, dt = \int_0^2 (2t^3 + 6t^2 - 6t) \, dt$$

$$= \left[\frac{t^4}{2} + 2t^3 - 3t^2 \right]_0^2$$

$= 8 + 16 - 12$

$= 24 - 12$

$= 12$

mamlot-034
page-19

Sol^n: (b)

$$\vec{A}\times\vec{B} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -t^2 & (t-1) \\ 2t^2 & 0 & 6t \end{vmatrix}$$

$$= \hat{i}(-6t^3) + \hat{j}(2t^3 - 2t^2 - 6t^2) + \hat{k}(0 + t^2 \cdot 2t)$$

$$= -6t^3\hat{i} + (2t^3 - 8t^2)\hat{j} + 2t^4\hat{k}$$

$$\int_0^2 \vec{A}\times\vec{B}\, dt = \left[-\frac{3}{2}t^4\hat{i} + \left(\frac{t^4}{2} - \frac{8}{3}t^3\right)\hat{j} + \frac{2}{5}t^5 \right]_0^2$$

$$= -3\times 2^3\hat{i} + \left(2^3 - \frac{8}{3}\times 2^3\right)\hat{j} + \frac{2}{5}\cdot 2^5$$

$$= -24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}$$

Mamlot-034
page-20

problem no: 31

Let $A = ti - 3j + 2tK$, $B = i - 2j + 2K$,

$C = 3i + tj - K$

Evaluate (a) $\int_0^2 A \cdot B \, dt$.

(b) $\int_0^2 A \times B \, dt$

Sol^n: (a) $\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & t & -1 \end{vmatrix}$

$= \hat{i}\,(2 - 2t) + \hat{j}\,(6 + 1) + \hat{k}\,(t + 6)$

$= (2 - 2t)\,\hat{i} + 7\hat{j} + (t + 6)\,\hat{k}$

$\vec{A} \cdot (\vec{B} \times \vec{C}) = (2 - 2t)\,t - 21 + 2t\,(t + 6)$

$= (2t - 2t^2) - 21 + 2t^2 + 12t$

$= 14t - 21$

mambt-034
page - 21

$$\int_1^2 \vec{A}\cdot(\vec{B}\times c)\,dt = \int_1^2 (14t-21)\,dt$$

$$= \left[7t^2 - 21t\right]_1^2$$

$$= 7(3) - 21(2-1)$$

$$= 21 - 21$$

$$= 0$$

Sol^n: (b) $\vec{A}\times(\vec{B}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -3 & 2t \\ 2-2t & 7 & t+6 \end{vmatrix}$

$$= \hat{i}(-3t - 18 - 14t) + \hat{j}(4t - 4t^2 - t^2 - 6t) + \hat{k}(7t + 6 - 6t)$$

$$= -(17t + 18)\hat{i} - (5t^2 + 2t)\hat{j} + (t+6)\hat{k}$$

$$\int_1^2 \vec{A}\times(\vec{B}\times\vec{c})\,dt = \left[\frac{17t^2}{2} + 18t\right]_1^2 \hat{i} - \hat{j}\left[\frac{5t^3}{3} + t^2\right]_1^2 + \hat{k}\left[\frac{t^2}{2} + 6t\right]_1^2$$

mamlat-034
page-22

$$= -\hat{i}\left(\frac{51}{2} + 18\right) - \left(\frac{35}{3} + 3\right)\hat{j} + \left(14 - \frac{1}{2} - 6\right)\hat{k}$$

$$= -\frac{87}{2}\hat{i} - \frac{44}{3}\hat{j} + \frac{15}{2}\hat{k}$$

problem no: 32

The acceleration a of a particle at any time $t \geq 0$ is given by $a = e^{-t} i - 6(t+1) j + 3 \sin t\, k$. If the velocity v and displacement r are zero at $t = 0$, find v and r at any time.

sol^n: $\vec{v} = \int \vec{a}\, dt = -e^{-t}\hat{i} - 6\left(\frac{t^2}{2} + t\right)\hat{j} - 3\cos t\,\hat{k} + \vec{A} \rightarrow$ constant vector

at $t = 0$, $\vec{v} = 0$ $\therefore 0 = -\hat{i} - 3\hat{k} + \vec{A}$

$\therefore \vec{A} = \hat{i} + 3\hat{k}$

$$\vec{v} = -e^{-t}\hat{i} - (3t^2+6t)\hat{j} - 3\cos t\,\hat{k} + \hat{i} + 3\hat{k}$$

$$v = (1-e^{-t})\,\hat{i} - (3t^2+6t)\,\hat{j} + 3(1-\cos t)\hat{k}$$

$\Rightarrow$ velocity

$$\vec{r} = \int \vec{v}\,dt = (t + e^{-t})\,\hat{i} - (t^3+3t^2)\,\hat{j} + 3(t-\sin t)\hat{k} + \vec{B} \rightarrow \text{constant vector}$$

at $t=0$, $r=0$

$$\therefore\ 0 = \hat{i} + \vec{B}$$

$$\Rightarrow\ \vec{B} = -\hat{i}$$

$$\vec{r} = (t+e^{-t}-1)\,\hat{i} - (t^3+3t^2)\,\hat{j} + 3(t-\sin t)\hat{k}$$

$\rightarrow$ (position vector)

Mamlot-034
page-24

problem no : 33

The acceleration a of an object at any time t is given by $a = -g j$. Where g is a constant. At $t=0$, the velocity given by $v = v_0 \cos\theta_0 \, i + v_0 \sin\theta_0 \, j$ ant the displace-ment $r = 0$. Find v and r at any time $t > 0$. This describes the motion of a projectile fired from a cannon inclind at angle θ_0 with the positive x-axis with initial velocity of magnitude v_0.

Sol^n: $\vec{v} = \int \vec{a}\, dt = -\int g \hat{j}\, dt = -g t \hat{j} + \vec{A}$ → constant vector

at $t=0$, $v_0 \cos\theta_0 \hat{i} + v_0 \sin\theta_0 \hat{j} = \vec{A}$

$$\vec{v} = -g t \hat{j} + v_0 \cos\theta_0 \hat{i} + v_0 \sin\theta_0 \hat{j}$$

$$= v_0 \cos\theta_0 \hat{i} + v_0(\sin\theta_0 - g t) \hat{j}$$

Mamlot- 034
page - 25

→ velocity

$$\vec{r} = \int \vec{v}\, dt$$

$$= v_0 \cos\theta_0 \cdot t \,.\, \hat{i} + \left(v_0 \sin\theta_0\, t - \frac{gt^2}{2}\right)\hat{j}$$

$+ B \rightarrow$ constant vector

at $t = 0$, $r = 0$, $0 = B$

$$\vec{r} = v_0 \cos\theta_0\, t\, \hat{i} + \left(v_0 \sin\theta_0 \cdot t - \frac{gt^2}{2}\right)\hat{j}$$

→ possition vector

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8/ If $F = 3xyi - y^2j$, evaluate $\int_C F\cdot dr$ Where C is the curve in the xy plane, $y = 2x^2$, from (0,0) to (1,2).

Since the integration is performed in the xy plane (z=0), we can take $r = xi + yj$. Then

$$\int_C F\cdot dr = \int_C (3xyi - y^2j)(dxi + dyj)$$

$$= \int_C 3xy\,dx - y^2 dy$$

Let $x = t$ in $y = 2x^2$. Then the parametric equations of C are $x = t$, $y = 2t^2$. points (0,0) and (1,2) correspond to $t=0$ and $t=1$ respectively. Then

$$\int_C F\cdot dr = \int_{t=0}^{1} 3(t)(2t^2)dt - (2t^2)^2 d(2t^2)$$

$$= \int_{t=0}^{1} (6t^3 - 16t^5)\,dt$$

$$= -\frac{7}{6}$$

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ID : 35

9/ Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by

$$F = (2x - y + z)i + (x + y - z^2)j + (3x - 2y + 4z)k$$

In the plane $z=0$, $F = (2x-y)i + (x+y)j + (3x-2y)k$ and $dr = dx\,i + dy\,j$ So that the work done is

$$\int_C F \cdot dr = \int_C [(2x-y)i + (x+y)j + (3x-2y)k][dx\,i + dy\,j]$$

$$= \int_C (2x-y)\,dx + (x+y)\,dy$$

Choose the parametric equations of the circle as $x = 3\cos t$, $y = 3\sin t$. where t varies from 0 to 2π. Then the line integral

$$\int_{t=0}^{2\pi} [2(3\cos t) - 3\sin t][-3\sin t]\,dt + [3\cos t + 3\sin t][3\cos t]\,dt$$

$$= \int_0^{2\pi} (9 - 9\sin t \cdot \cos t)\,dt$$

$$= 9t - \frac{9}{2}\sin^2 t \Big|_0^{2\pi} = 18\pi$$

ID : 35

10/ (a) If $F = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in this field to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.

$$\text{Work done} = \int_{P_1}^{P_2} F \, dr = \int_{P_1}^{P_2} \nabla\phi \cdot dr$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)(dx\,i + dy\,j + dz\,k)$$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1)$$

$$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Then the integral depends only on points P_1 and P_2 and not on the path joining them. This is true of course only if $\phi(x, y, z)$ is single-valued at all points P_1 and P_2.

10/ (b) Conversely, if $\int_C F \cdot dr$ is independent of the path C joining any two points, show that there exists a function ϕ such that $F = \nabla \phi$

Let $F = F_1 i + F_2 j + F_3 k$. By hypothesis $\int_C F \cdot dr$ is independent of the path C joining any two points, which we take as (x_1, y_1, z_1) and (x, y, z) respectively. Then,

$$\phi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y, z)} F \cdot dr = \int_{(x_1, y_1, z_1)}^{(x, y, z)} F_1 dx + F_2 dy + F_3 dz$$

is independent of the path joining (x_1, y_1, z_1) and (x, y, z). Thus,

$$\phi(x+\Delta x, y, z) - \phi(x, y, z)$$

$$= \int_{(x_1, y_1, z_1)}^{(x+\Delta x, y, z)} F \cdot dr - \int_{(x_1, y_1, z_1)}^{(x, y, z)} F \cdot dr$$

$$= \int_{(x, y, z)}^{(x_1, y_1, z_1)} F \cdot dr + \int_{(x, y, z_1)}^{(x+\Delta x, y, z)} F \cdot dr$$

ID: 35

$$= \int_{(x,y,z)}^{(x+\Delta x,y,z)} F\cdot dr = \int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1 dx + F_2 dy + F_3 dz$$

Since the last integral must be independent of the path joining (x,y,z) and $(x+\Delta x, y, z)$ we may choose the path to be a straight line joining those points so that dy and dz are zero.

Then, $$\frac{\phi(x+\Delta x, y, z) - \phi(x,y,z)}{\Delta x} = \frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1 dx$$

Taking the limit of both side as $\Delta x \to 0$, we have $\frac{\partial \phi}{\partial x} = F_1$.

Similarly we can show that $\frac{\partial \phi}{\partial y} = F_2$, $\frac{\partial \phi}{\partial z} = F_3$

Then, $F = F_1 i + F_2 j + F_3 k = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = \nabla \phi$

If $\int_{P_1}^{P_2} F\cdot dr$ is independent of the path C joining P_1 and P_2, then F is called a conservative field.

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ID : 35

It follows that if $F = \nabla\phi$ then F is conservative, and conversely,

proof using vector. If the line integral is independent of the path, then,

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} F.dr = \int_{(x_1,y_1,z_1)}^{(x,y,z)} F\frac{dr}{ds}.ds$$

By differentiation $\frac{d\phi}{ds} = F.\frac{dr}{ds}$

But $\frac{d\phi}{ds} = \nabla\phi\frac{dr}{ds}$

So that $(\nabla\phi - F).\frac{dr}{ds} = 0$

Since this must hold irrespective of $\frac{dr}{ds}$, we have $F = \nabla\phi$.

ID: 35

11/ (b) Conversely, if $\nabla \times F = 0$, prove that F is conservative.

If $\nabla \times F = 0$, then $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$ and thus

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

We must prove that $F = \nabla \phi$ follows as a consequence of this.

The work done in moving a particle from (x_1, y_1, z_1) to (x, y, z) in the force field F is

$$\int_e F_1(x, y, z_1)\, dx + \int_{y_1}^{y} F_2(x, y, z_1) + \int_{z_1}^{z} F_3(x, y, z)\, dz$$

It follows that,

$$\frac{\partial \phi}{\partial z} = F_3(x, y, z)$$

$$\frac{\partial \phi}{\partial y} = F_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial F_3}{\partial y}(x, y, z)\, dz$$

ID: 35

$$= F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_2}{\partial z}(x,y,z)\,dz$$

$$= F_2(x,y,z_1) + F_2(x,y,z)\Big|_{z_1}^{z}$$

$$= F_2(x,y,z_1) + F_2(x,y,z) - F_2(x,y,z_1)$$

$$= F_2(x,y,z)$$

$$\frac{\partial \phi}{\partial x} = F_1(x,y,z_1) + \int_{y_1}^{y} \frac{\partial F_2}{\partial x}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_3}{\partial x}(x,y,z)\,dz$$

$$= F_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial F_1}{\partial y}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_1}{\partial z}(x,y,z)\,dz$$

$$= F_1(x,y_1,z_1) + F_1(x,y,z_1)\Big|_{y_1}^{y} + F_1(x,y,z)\Big|_{z_1}^{z}$$

$$= F_1(x,y_1,z_1) + F_1(x,y,z_1) - F_1(x,y_1,z_1) + F_1(x,y,z) - F_1(x,y,z_1)$$

$$= F_1(x,y,z)$$

Then $F = F_1 i + F_2 j + F_3 k = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = \nabla \phi$

Thus a necessary and sufficient condition that a field F be conservative is that curl $F = \nabla \times F = 0$

ID : 35

12/ (a) Show that $F = (2xy + z^3)i + x^2 j + 3xz^2 k$ is a conservative force field.

If $\nabla \times F = 0$, then $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$

~~and~~. Thus F is a conservative force field.

(b) Find the sca-lar potential

$F = \nabla\phi$. or $\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$= (2xy + z^3) i + x^2 j + 3xz^2 k$. Then

(1) $\frac{\partial\phi}{\partial x} = 2xy + z^3$ (2) $\frac{\partial\phi}{\partial y} = x^2$ (3) $\frac{\partial\phi}{\partial z} = 3$

Integrating, we find from (1), (2), (3) respectively

$\phi = x^2y + xz^3 + f(y, z)$

$\phi = x^2y + g(x, z)$

$\phi = xz^3 + h(x, y)$

ID: 35

These agree if we choose $f(y,z)=0$, $g(x,z)=xz^3$, $h(x,y)=x^2y$ so that $\phi=x^2y+xz^3$ to which may be added any constant.

ⓒ Find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

Work done $=\int_{P_1}^{P_2} F.dr$

$$=\int_{P_1}^{P_2} (2xy+z^3)dx+x^2dy+3xz^2dz$$

$$=\int_{P_1}^{P_2} d(x^2y+xz^3)$$

$$=x^2y+xz^3\Big|_{P_1}^{P_2}$$

$$=x^2y+xz^3\Big|_{(1,-2,1)}^{(3,1,4)}$$

$$=202$$

ID : 35

34/ Evaluate $\int_2^3 A \cdot \frac{dA}{dt}\, dt$. If $A(3) = 2i - j + 2k$ and $A(3) = 4i - 2j + 3k$.

$$\frac{d}{dt}(\vec{A} \cdot \vec{A}) = \frac{d\vec{A}}{dt} \cdot \vec{A} + \vec{A}\,\frac{d\vec{A}}{dt}$$

$$= 2\vec{A} \cdot \frac{d\vec{A}}{dt}$$

$$\int_2^3 \vec{A} \cdot \frac{d\vec{A}}{dt} \cdot dt = \frac{1}{2}\int_2^3 \frac{d}{dt}(\vec{A} \cdot \vec{A})\, dt$$

$$= \frac{1}{2}\int_2^3 d(\vec{A} \cdot \vec{A})$$

$$= \frac{1}{2}[\vec{A} \cdot \vec{A}]_2^3$$

$$= \frac{1}{2}(A^2)_2^3$$

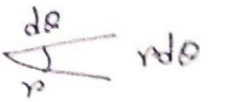
$$= \frac{1}{2}[\{A(3)\}^2 - \{A(2)\}^2]$$

$$= \frac{1}{2}\{(4^2 + 2^2 + 3^2) - (2^2 - 1^2 + 2^2)\}$$

$$= 10$$

ID : 35

35/ Find the areal velocity of a particle which move along the path $r = a\cos\omega t\,\hat{i} + b\sin\omega t\,\hat{j}$ where a, b, ω are constants and t is time.



$$dA \quad \text{Area} = \frac{1}{2} \times \vec{r} \times r\,d\theta$$

$$\text{Area velocity} \rightarrow \frac{dA}{dt} = \frac{1}{2} \times \vec{r} \times r\frac{d\vec{\theta}}{dt}$$

$$= \frac{1}{2} \times \vec{r} \times r\vec{\omega}$$

$$= \frac{1}{2} \times \vec{r} \times \vec{v}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -\omega a \sin\omega t\,\hat{i} + b\omega\cos\omega t\,\hat{j}$$

$$\hat{r} \times \hat{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\omega t & b\sin\omega t & 0 \\ -\omega a\sin\omega t & b\omega\cos\omega t & 0 \end{vmatrix}$$

ID: 35

$= \hat{k}\left[ab\omega\cos^2\omega t + ab\omega\sin^2\omega t\right]$

$= ab\omega\,\hat{k}$

$\therefore \dfrac{d\vec{A}}{dt} = \dfrac{1}{2} \times ab\omega\,\hat{k}$

Sayad 036

Sayad-036

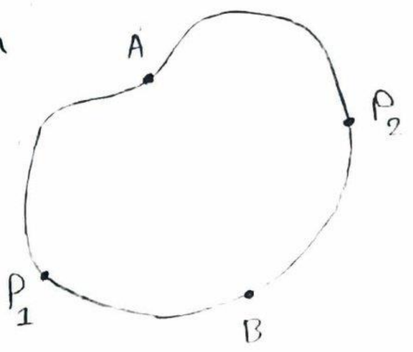
13. Prove that if $\int_{P_1}^{P_2} F.dr$ is independent of the path joining any points P_1 and P_2 is a given region, then $\oint F.dr = 0$ for all closed paths in the region and conversely.

* Let, $P_1AP_2BP_1$ (see adjacent figure) be a closed curve. Then,

$$\oint F.dr = \int_{P_1AP_2BP_1} F.dr = \int_{P_1AP_2} F.dr + \int_{P_2BP_1} F.dr$$

$$= \int_{P_1AP_2} F.dr - \int_{P_1BP_2} F.dr = 0$$

~~So~~ Since the integral from P_1 to P_2 along a path through A is the same as that along a path through B, by hypothesis.



· Sayad-036

Conversely if $\oint F.dr = 0$ then,

$$\int_{P_1AP_2BP_1} F.dr = \int_{P_1AP_2} F.dr + \int_{P_2BP_1} F.dr$$

$$= \int_{P_1AP_2} F.dr - \int_{P_1BP_2} F.dr = 0$$

So that, $\displaystyle\int_{P_1AP_2} F.dr = \int_{P_1BP_2} F.dr$ [Proved]

14. (a) Show that a necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ be an exact differential is that $\nabla \times F = 0$ where $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

* Suppose, $F_1dx + F_2dy + F_3dz = \cancel{d\phi}\ d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$ an exact differential. Then since x, y and z are independent variables.

Sayad-036

$$F_1 = \frac{\partial \phi}{\partial x}, \quad F_2 = \frac{\partial \phi}{\partial y}, \quad F_3 = \frac{\partial \phi}{\partial z}$$

and so, $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$$= \nabla \phi$$

Thus,
$$\nabla \times F = \nabla \times \nabla \phi = 0.$$

conversely, if $\nabla \times F = 0$ then by problem 11,

$F = \nabla \phi$ and so $F \cdot dr = \nabla \phi \cdot dr = d\phi$

i.e., $F_1 dx + F_2 dy + F_3 dz = d\phi$, an exact differential.

14. (b) Show that, $(y^2 z^3 \cos x - 4x^3 z)\,dx + (2z^3 y \sin x)\,dy + (3y^2 z^2 \sin x - x^4)\,dz$ is and exact differential of a function ϕ and find ϕ.

$*$ $F = (y^2 z^3 \cos x - 4x^3 z)\hat{i} + 2z^3 y \sin x\,\hat{j} + (3y^2 z^2 \sin x - x^4)\hat{k}$

and $\nabla \times F$ is computed to be zero, so that by part (a)

Sayad-036

$$(y^2z^3 \cos x - 4x^3z)\,dx + 2z^3y \sin x\,dy + (3y^2z^2 \sin x - x^4)\,dz = d\varphi$$

By any of the methods of Problem 12 we find

$$\varphi = y^2x^3 \sin x - x^4 z + \text{constant}.$$

15. Let F be a conservative force field such that $F = -\nabla\varphi$. Suppose a particle of constant mass m to move in this field. If A and B are ~~very~~ any two points in space, prove that

$$\varphi(A)\ \tfrac{1}{2}mv_A^2 = \varphi(B) + \tfrac{1}{2}mv_B^2$$

where v_A and v_B are the magnitudes of the velocities of the particle at A and B respectively.

* $F = ma = m\dfrac{d^2r}{dt^2}$, Then $F.\dfrac{dr}{dt} = m\dfrac{dr}{dr}.\dfrac{d^2r}{dt^2}$

$$= m\frac{d}{dt}\left(\frac{dr}{dt}\right)^2$$

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Integrating, $\int_A^B F.\,dr = \frac{m}{2} v^2 \Big|_A^B = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$

if $F = -\nabla\varphi$, $\int_A^B F.\,dr = -\int_A^B \nabla\varphi.\,dr = -\int_A^B d\varphi$

$= \varphi(A) - \varphi(B)$

Then, $\varphi(A) - \varphi(B) = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2$ and the result follows.

$\varphi(A)$ is called the potential energy at A and $\frac{1}{2} m v^2$ is the kinetic energy at A. The result states that the total energy at A equals the total energy at B (conservation of energy). Note the use of the minus sign in $F = -\nabla\varphi$

16. If $\varphi = 2xyz^2$, $F = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t=0$ to $t=1$, evaluate the line integrals

(a) $\int_C \varphi\, dr$ (b) $\int_C F \times dr$

Sayad-036

(a) Along C, $\varphi = 2xyz^2 = 2(t^2)(2t)(t^3)^2 = 4t^9$

$$r = x\hat{i} + y\hat{i} + z\hat{k} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

and, $dr = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$

Then,

$$\int_C \varphi\, dr = \int_{t=0}^{1} 4t^9 (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$$

$$= \hat{i}\int_0^1 8t^{10}\,dt + \hat{j}\int_0^1 8t^9\,dt + \hat{k}\int_0^1 12t^{11}\,dt$$

$$= \frac{8}{11}\hat{i} + \frac{4}{5}\hat{j} + k$$ (Ans)

b) Along C, $F = xy\hat{i} - z\hat{j} + x^2k = 2t^3\hat{i} + t^4\hat{k} - t^3\hat{j}$

Then, $F \times dr = (2t^3\hat{i} - t^3\hat{j} + t^4k) \times (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt$$

$$= \left[(-3t^5 - 2t^4)\hat{i} + (2t^5 - 6t^5)\hat{j} + (4t^3 + 2t^4)k\right] dt$$

Sayad-036

And,

$$\int_C F \times dr = \hat{i}\int_0^1 (-3t^5 - 2t^4)\,dt - \hat{j}\int_0^1 (-4t^5)\,dt + k\int_0^1 (4t^3 + 2t^4)\,dt$$

$$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$$

Ans

Ex: 40 (a)

Along the straight line from (0,0,0) to (2,1,3) is given in parametric form by,

$x = 2t$, $y = t$ and $z = 3t$

$\Rightarrow dx = 2dt$. $\Rightarrow dy = dt$ $\Rightarrow dz = 3dt$

$$W = \int \vec{F}\cdot d\vec{r} = \int_0^1 \left[3x^2\,dx + (2xz - y)\,dy + z\,dz\right]$$

$$= \int_0^1 \left[3.(2t)^2.\,2dt + (2.\,2t\cdot 3t - t)\,dt + 3t.\,3dt\right]$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\,dt$$

$$= \int_0^1 (36t^2 + 8t)\,dt$$

Sayad-036

$$= \left[12t^3 + 4t^2\right]_0^1$$

$$= 12 + 4 = 16 \quad \underline{\text{Ans}}$$

<u>Ex: 40 (b)</u>

Given that,

$$x = 2t^2, \quad y = t \quad \text{and} \quad z = 4t^2 - t$$

$$\Rightarrow dx = 4t\,dt, \Rightarrow dy = dt \quad \Rightarrow dz = (8t-1)\,dt$$

$$\int \vec{F}\cdot d\vec{r} = \int_0^1 \left[3x^2\,dx + (2xz - y)\,dy + z\,dz\right]$$

$$= \int_0^1 \left\{3.4t^4.4t\,dt + \left[2.2t^2.(4t^2-t) - t\right]dt + \left[(4t^2-t)(8t-1)\right]dt\right\}$$

$$= \int_0^1 \left(48t^5 + 16t^4 - 4t^3 - \cancel{t} + 32t^3 - 4t^2 - 8t^2 + \cancel{t}\right)dt$$

$$= \int_0^1 \left(48t^5 + \cancel{16t}\ 16t^4 + 28t^3 - 12t^2\right)dt$$

$$= \left[8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3\right]_0^1$$

$$= 8 + \frac{16}{5} + 7 - 4 = 14.2 \quad \underline{\text{Ans}}$$

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Ex: 40 (c)

Given that,

$x^2 = 4y$ And, $3x^3 = 8z$

$\Rightarrow y = \frac{1}{4}x^2$ $\Rightarrow z = \frac{3}{8}x^3$

$\Rightarrow dy = \frac{1}{2}x\,dx$ $\Rightarrow dz = \frac{9}{8}x^2 dx$

$$\int \vec{F}\cdot d\vec{r} = \int_0^2 \left[3x^2 dx + (2xz - y)\,dy + z\,dz\right]$$

$$= \int_0^2 \left[3x^2 dx + \left(2x\cdot\frac{3}{8}x^3 - \frac{1}{4}x^2\right)\frac{1}{2}x\,dx + \frac{3}{8}x^3\cdot\frac{9}{8}x^2 dx\right]$$

$$= \int_0^2 \left[3x^2 + \frac{3}{8}x^5 - \frac{1}{8}x^3 + \frac{27}{64}x^5\right]dx$$

$$= \int_0^2 \left[\frac{51}{64}x^5 - \frac{1}{8}x^3 + 3x^2\right]dx$$

$$= \left[\frac{51}{64}\cdot\frac{x^6}{6} - \frac{x^4}{32} + x^3\right]_0^2$$

$$= \frac{51}{6} - \frac{1}{2} + 8$$

$$= 16$$

Ans

Sayad-036

Ex: 41 Given that,

$$x = 2\cos t \quad \text{And} \quad y = 3\sin t$$

$$\Rightarrow dx = -2\sin t\, dt \quad \Rightarrow y = 3\cos t\, dt$$

$$\oint \vec{F}.d\vec{r} = \int_0^{2\pi} [(x-3y)\,dx + (y-2x)\,dy]$$

$$= \int_0^{2\pi} \left[(2\cos t - 3.3\sin t)(-2\sin t\, dt) + (3\sin t - 2.2\cos t) . 3\cos t\, dt \right]$$

$$= \int_0^{2\pi} \left[-4\cos t \sin t + 18\sin^2 t + 9\sin t\cos t - 12\cos^2 t \right] dt$$

$$= \int_0^{2\pi} \left[5\sin t\cos t + 18\sin^2 t - 12(1-\sin^2 t) \right] dt$$

$$= \frac{1}{2}\int_0^{2\pi} 5\sin 2t + 2.30\sin^2 t - 2.12] \, dt$$

$$= \frac{1}{2}\int_0^{2\pi} [5\sin 2t + 30 - 30\cos 2t - 24] \, dt$$

$$= \frac{1}{2}\int_0^{2\pi} (5\sin 2t - 30\cos 2x + 6)\, dt$$

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$$= \frac{1}{2}\left[-\frac{5}{2}\cos 2t - \frac{30}{2}\sin 2t + 6t\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[-5/2 - 0 + 6 \times 2\pi + 5/2\right]$$

$$= 6\pi + 0$$

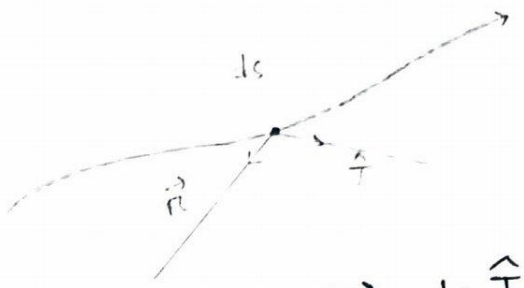
$$= 6\pi$$

Ans

Ex: 42

Given that,

$\hat{T}$ = tangent unit vector



$$d\vec{s} = ds\,\hat{T}$$

Work done, $W = \int \vec{F} \cdot d\vec{s}$

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$$= \int \vec{F} . ds \hat{T}$$

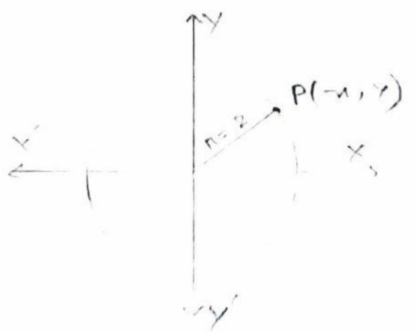
$$= \int \vec{F} . \hat{T} \, ds$$

[Proved]

<u>Ex: 45</u> Given that,

$$A = (y - 2x)\hat{i} + (3x + 2y)\hat{j}$$

$$r = 2$$



$$x = r\cos\theta = 2\cos\theta$$

$$\Rightarrow dx = -2\sin\theta \, d\theta$$

And, $y = r\sin\theta = 2\sin\theta$

$$\Rightarrow dy = 2\cos\theta \, d\theta$$

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$$\int \vec{A}\cdot d\vec{r} = \int_0^{2\pi} \left[(y-2x)\,dx + (3x+2y)\,dz \right]$$

$$= \int_0^{2\pi} \left[(2\sin\theta - 2.2\cos\theta)(-2\sin\theta\, d\theta) + (3.2\cos\theta + 2.2\sin\theta)\, 2\cos\theta\, d\theta \right]$$

$$= \int_0^{2\pi} (-4\sin^2\theta + 8\sin\theta\cos\theta + 12\cos^2\theta + 8\sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} (16\cos^2\theta + 16\sin\theta\cos\theta - 4)\, d\theta$$

$$= \int_0^{2\pi} \left[8(1-\cos 2\theta) + 8\sin 2\theta - 4 \right] d\theta$$

$$= \left[8\left(\theta - \frac{\sin 2\theta}{2}\right) + 8\,\frac{-\cos 2\theta}{2} - 4\theta \right]_0^{2\pi}$$

$$= 4.2\pi$$

$$= 8\pi$$

Ans

Sayad-036

Ex: 49

Given that,

$$F = 2xz\,\hat{i} + (x^2 - y)\,\hat{j} + (2z - x^2)\,\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz & x^2 - y & 2z - x^2 \end{vmatrix}$$

$$= \hat{i}\,(0 - 0) + \hat{j}\,(2x + 2x) + \hat{k}\,(2x - 0)$$

$$= 4x\,\hat{j} + 2x\,\hat{k} \neq 0$$

$\therefore$ $\vec{\nabla} \times \vec{F} \neq 0$ Then $\vec{F}$ is non-conservative

Ans

Sayad-036

Ex: 51 Given that,

$$A = (yz + 2x)\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz+2x & xz & xy+2z \end{vmatrix}$$

$$= \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z)$$

$$= 0$$

$\vec{A} \rightarrow$ conservative Then $\vec{A} = \vec{\nabla}\varphi$

$$\int \vec{A} \cdot \vec{dr} = \int \vec{\nabla}\varphi \cdot d\vec{r}$$

$$= \int \frac{d\varphi}{dr}\, dr = [\varphi]_{0,1,1}^{1,0,1}$$

If we compare with A

$$\frac{\partial \varphi}{\partial x} = yz + 2x$$

$$\Rightarrow \varphi = xyz + x^2 + c_1$$

Sayad-036

$$\frac{\partial \varphi}{\partial y} = xz$$

$$\Rightarrow \varphi = xyz + c_2$$

And $\frac{\partial \varphi}{\partial z} = xy + 2z$

$$\Rightarrow \varphi = xyz + z^2 + c_3$$

$$\therefore \varphi = xyz + x^2 + z^2 + c$$

Then, $\varphi_{(1,0,1)} = 0 + 1 + 1 + c$

$= 2 + c$

$\varphi_{(0,1,1)} = 0 + 0 + 1 + c$

$= 1 + c$

$\therefore \varphi_{(1,0,1)} - \varphi_{(0,1,1)} = 2 + c - 1 - c$

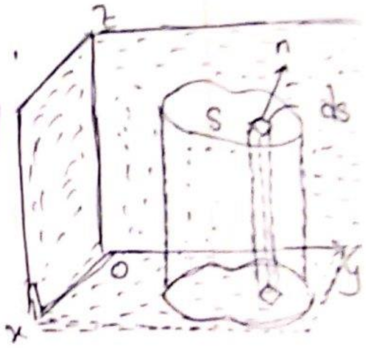
$= 1$ (Ans)

Khirul 037

khairul-037

Definition of surface integral :-

Let s be a two sided-Surface. Such as known in the figure below. Let side of s be considered arbitarilly as the possitive side (If s is a closed surface this is taken as the outer side). A



Unit normal n to any point of possitive side of s is called a possitive or outward drawn unit normal. Associate with the differential of surface area ds a vector ds whose magnitude is ds and whose direction is that of n. Then ds = n ds. The integral

$$\iint_S A \cdot ds = \iint_S A \cdot n \, ds$$

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is an example of surface integral called the flux of A over s. Other surface integrals are

$$\iint_s \phi \, ds. \iint_s \phi \, n \, ds, \iint_s A \times ds$$

where ϕ is a scalar function. Such integrals can be defined in terms of limits of sum as in elementary calculus.

The notation $\oiint_s$ is sometimes in used to indicate integration over the closed Surface s. Where no confusion can arise the notation $\oint_s$ may also be used.

To evalute surface integrals. It is convenient to express them as double integral taken over the projected area of surface S on one of the co-ordinate planes. This is possible if any line perpendicular to the co-ordinate plane. chosen meets the surface is no more than one point.

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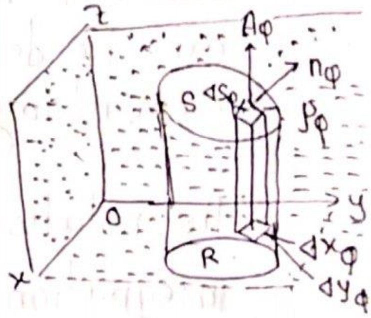
MATH NO-17 :

Give a definition of $\iint_S A \cdot n \, ds$ over a surface S in terms of Limit of a Sum.

Solution:-



Subdivide the are S into M elements of area Δs_ϕ where $p = 1, 2, 3, \cdots M$. choose any point P_ϕ within ΔS_ϕ whose co-ordinate are (x_ϕ, y_ϕ, z_ϕ). Define $A(x_\phi, y_\phi, z_\phi) = A_\phi$. Let n_ϕ be the possitive unit normal to ΔS_ϕ at p. put the Sum.

$$\sum_{\phi=1}^{N} A_\phi \cdot n_\phi \Delta S_\phi$$

where $A_\phi \cdot n_\phi$ is the normal component of A_ϕ at P_ϕ.

Now take the limit of this sum as $m \to \infty$ in such a way that the largest dimension of each ΔS_q approaches zero. This limit if it exists is called the Surface integral of the normal component of A over S is donated by

$$\iint_S A \cdot n \, ds.$$

MATH NO:- 18

Suppose that the Surface S has projection R on the xy plane. show that

$$\iint_S A \cdot n \, ds = \iint_R A \cdot n \frac{dx . dy}{|n \cdot k|}$$

Solution :

We know the Surface integral is the limit of the Sum.

(1)
$$\sum_{q=1}^{N} A_q \cdot n_q \Delta S_q$$

The projection of ΔS_q on the xy plane is $|(n_q \Delta S_q) \cdot k|$ or $|n_q \cdot k| \Delta S_q$.

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which is equal to $\Delta x_\phi \Delta y_\phi$ so that.

$\Delta S_\phi = \frac{\Delta x_\phi \Delta y_\phi}{|n_\phi \cdot k|}$. Thus the Sum ① becomes.

②

$$\sum_{\phi=1}^{N} A_\phi \cdot n_\phi \frac{\Delta x_\phi \Delta y_\phi}{|n_\phi \cdot k|}$$

By the fundamental theorem of integral Calculus the limit of the sum as $m \to \infty$ is a Such a manner that the largest Δx_ϕ and Δy_ϕ approach zero in

$$\iint A \cdot n \frac{dx . dy}{|n . k|}$$

so the required result follows.

strictly speaking the result $\Delta S_\phi = \frac{\Delta x_\phi \Delta y_\phi}{|n_\phi \cdot k|}$ is only approximately true but it can be show on the closer examination that they differ from each other by infinitesimals of higher than $\Delta x_\phi \Delta y_\phi$ and using the limit of ① and ② can in fact be shown in equal.

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MATH NO: 19

$\iint_S A\cdot n\, ds$, where $A = 18z\, i - 12\, j + 3y\, k$ and s is that part of the plane $2x+3y+6z=12$ which is located in the first octant.

solution :-

From problem 17.

$$\iint_S A\cdot n\, ds = \iint_R A\cdot n \frac{dx\, dy}{|n\cdot k|}$$

To obtain n note that a vector perpendicular to the surface $2x+3y+6z=12$ is given by $\nabla(2x+3y+6z) = 2i+3j+6k$. Then a unit normal to any point of s is,

$$n = \frac{2i+3j+6k}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}i + \frac{3}{7}j + \frac{6}{7}k$$

Thus,

$$n\cdot k = \left(\frac{2}{7}i + \frac{3}{7}j + \frac{6}{7}k\right)\cdot k$$

$$= \frac{6}{7} \text{ and so } \frac{dx\, dy}{|n\cdot k|} = \frac{7}{6}\, dx\, dy$$

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Also.

$$A\cdot n = (18z\,i - 12\hat{j} + 3y\,k)\cdot\left(\frac{2}{7}i + \frac{3}{7}j + \frac{6}{7}k\right)$$

$$= \frac{36z - 36 + 18y}{7}$$

$$= \frac{36 - 12x}{7}$$

Using the fact that.

$$z = \frac{12 - 2x - 3y}{6}$$ from the equal of-s.

Then,

$$\iint_S A\cdot n\, ds = \iint_R A\cdot n \frac{dx\,dy}{|n\cdot k|}$$

$$= \iint_R \left(\frac{36-12x}{7}\right)\frac{7}{6}\,dx\,dy$$

$$= \iint_R (6-2x)\,dx\,dy.$$

To evalute the double integral over R. keep x fixed and integrate with respect to y from $y=0$ to $y = \frac{12-2x}{3}$ then integrate with respect to x from $x=0$ to $x=6$.

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In this manner R is completely covered. The integral becomes,

$$\int_{x=0}^{6}\int_{y=0}^{\left(\frac{12-2x}{3}\right)}(6-2x)\,dy\,dx$$

$$=\int_{x=0}^{6}\left(24-12x+\frac{4x^2}{3}\right)dx$$

$$=24$$

If we have chosen the positive unit normal n opposite to that in the figure above. we would have obtained the result -24.

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MATH NO: 20

$\iint_S A \cdot n \, ds$, where $A = zi + xj - 3y^2 z k$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

Solution:-

$$\iint_S A \cdot n \, ds = \iint_R A \cdot n \frac{dx\,dy}{|n \cdot j|}$$

A normal to $x^2+y^2=16$ is $\nabla(x^2+y^2) = 2xi + 2yj$

Thus the unit normal to S as shown in the adjointing figure is

$$n = \frac{2x\,i + 2y\,j}{\sqrt{(2x)^2+(2y)^2}}$$

$$= \frac{xi + yj}{4}$$

Since $x^2+y^2=16$ on S

$$A \cdot n = (zi + xj - 3y^2 z k) \cdot \left(\frac{xi+yj}{4}\right)$$

$$= \frac{1}{4}(xz + xy)$$

$$n \cdot j = \frac{xi+yj}{4} \cdot j$$

$$= \frac{y}{4}$$

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Then the Surface integral equals

$$\iint_R \frac{xy + xz}{y}\, dx\, dz = \int_{z=0}^{5} \int_0^4 \left(\frac{xz}{\sqrt{16 - x^2}} + x \right) dx\, dz$$

$$= \int_{z=0}^{5} (4z + 8)\, dz$$

$$= 90$$

MATH NO : 58

Evalute $\iint_S A \cdot n \, ds$ for each of the following cases.

(a) $A = yi + 2xj - zk$ and S is the Surface of the plane $2x + y = 6$ in the first octant Cutt off by the plane $z = 4$.

Solution :-

Given that, $2x + y = 6$.

$$\therefore \frac{x}{3} + \frac{y}{6} = 1$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{1}{\sqrt{5}}(2\hat{i} + \hat{j})$$

$$\phi = 2x + y - 6$$

$$\nabla \phi = 2\hat{i} + \hat{j}$$

$$|\nabla \phi| = \sqrt{4+1} = \sqrt{5}$$

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$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint \frac{1}{\sqrt{5}}(\cancel{z}y + 2x)\cdot\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$= \iint (x+y)\, dy\, dz$$

$$= \iint \left(3 - \frac{y}{2} + y\right) dy\, dz$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$$

and,

$$2x = 6 - y$$

$$\therefore x = 3 - \frac{y}{2}$$

Now,

$$\iint \hat{A}\cdot\hat{n}\, ds = \iint \left(3 + \frac{y}{2}\right) dy\, dz$$

$$= \int_{y=0}^{6} \left(3 + \frac{y}{2}\right) dy \int_{z=0}^{4} dz$$

$$= \left[3y + \frac{y^2}{4}\right]_0^6 \cdot \left[z\right]_0^4$$

$$= (18+9)(4-0)$$

$$= 27\times 4$$

$$= 108$$

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ⓑ $A = (x+y^2)i - 2xj + 2yzk$ and S is the surface of the plane $2x+y+2z=6$ in the first octant

Solution:

Given that, $2x+y+2z=6$

$$\therefore \frac{x}{3}+\frac{y}{6}+\frac{z}{3}=1$$

$$\phi = 2x+y+2z-6$$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= 2\hat{i}+\hat{j}+2\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{4+1+4} = 3$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\nabla\phi|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

$$\therefore \vec{A}\cdot\hat{n} = \frac{1}{3}\left[2(x+y^2) - 2x + 2.2yz\right]$$

$$= \frac{2}{3}\left[x^2+y^2-x+2yz\right]$$

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$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{2/3}$$

$$\therefore \iint_S \vec{A}\cdot n\, ds = \iint \frac{2}{3}\left[x^2+y^2-x+2yz\right] \frac{dz\,dy}{2/3}$$

$$= \iint \left[x^2+y^2-x+2yz\right] dx\,dy$$

$$\iint \vec{A}\cdot\hat{n}\, ds = \iint x^2+y^2-x+y\,(6-2x-y)]\,dx\,dy$$

$$= \iint (6y-2xy)\,dx\,dy$$

$$= \int_{y=0}^{6}\int_{x=0}^{3-y/2}(6y-2xy)\,dx\,dy$$

$$= \int_{y=0}^{6}\left[6xy-x^2y\right]_{x=0}^{3-y/2} dy$$

$$= \int_0^6 \left[6y\,(3-y/2)-(3-y/2)^2 y\right] dy$$

$$= \int_0^6 \left[18y-3y^2-9y+3y^2-\frac{y^3}{4}\right] dy$$

here
$2x+y+2z=6$
$2x+y=6$
$2x=6-y$
$\therefore x=3-\frac{y}{2}$
$x: 0\to 3-y/2$
$y: 0\to 6$

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$$= \int_{y=0}^{6} \left(9y - \frac{y^3}{4}\right) dy$$

$$= \left(\frac{9}{2}y^2 - \frac{y^4}{16}\right)_0^6$$

$$= \frac{9}{2} \times 36 - \frac{y^4}{16}$$

$$= 162 - 81$$

$$= 81$$

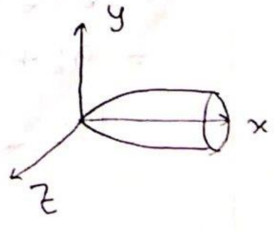
MATH NO: 59

Suppose $F = 2yi - z\hat{j} + x^2k$ and S is the Surface of the parabolic Cylinder $y^2 = 8x$ in the first octant bounded by the planes $y=4$ and $z=6$. Evalute $\iint_S F.n\, ds$.

Solution:-

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{dy} + \hat{k}\frac{\partial\phi}{dz}$$

$$= -8\hat{i} + 2y\,\hat{j}$$



$$|\vec{\nabla}.\phi| = \sqrt{8^2 + 4y^2}$$

$$= \sqrt{64 + 4y^2}$$

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$$\hat{n} = \frac{\nabla\phi}{|\nabla\cdot\phi|} = \frac{1}{\sqrt{64+4y^2}}\left(-8\hat{i}+2y\hat{j}\right)$$

$$\vec{F}\cdot\hat{n} = \frac{-16y-2yz}{\sqrt{64+4y^2}}$$

$$\iint_s \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$= -\iint \frac{16y+2yz}{\sqrt{64+4y^2}}\,\frac{dy\,dz}{\frac{-8}{\sqrt{64+4y^2}}}$$

$$= \iint \left(2y+\frac{1}{4}yz\right)dy\,dz$$

$$= \int \left[y^2+\frac{y^2}{8}z\right]_0^4 dz$$

$$= \int_{z=0}^{6} (16+2z)\,dz$$

$$= \left[16z+z^2\right]_0^4$$

$$= 16\times 6+6^2$$

$$= 96+36$$

$$= 132$$

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MATH NO : 60

Suppose $A = 6z\,i + (2x+y)\,j - x\,k)$ Evalute $\iint_S A\cdot n\ ds$ over the entire surface s of region bounded by the Cylinder $x^2+z^2=9$, $x=0$, $y=0$, $z=0$ and $y=8$.

Solution:-

$$\vec{\nabla}\cdot\phi = \hat{i}\frac{d\phi}{dx} + \hat{j}\frac{d\phi}{dy} + \hat{k}\frac{d\phi}{dz}$$

$$= \hat{i}\,2x + \hat{j}\cdot 0 + \hat{k}\cdot 2z$$

$$= 2x\hat{i} + 2z\hat{k}$$

$$|\vec{\nabla}\cdot\phi| = \sqrt{4x^2+4z^2} = 2\sqrt{x^2+z^2} = 2\cdot 3 = 6$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\cdot\phi|} = \frac{1}{6}(2x\hat{i}+2z\hat{k}) = \frac{1}{3}(x\hat{i}+z\hat{k}).$$

$$\vec{A}\cdot\hat{n} = \frac{1}{3}(6xz - xz) = \frac{5x^3}{3}$$

$$\iint \vec{A}\cdot\hat{n}\ ds = \iint \frac{5xz}{3}\cdot\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint \frac{5xz}{3}\ \frac{dx\,dy}{z/3}.$$

$$= \iint 5x\ dx\ dy$$

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$$= \int \int_{x=0}^{3} 5x \, dx \, dy$$

$$= \left[5 \frac{x^2}{2} \cdot y \right]_{x=0}^{3} {}_{y=0}^{8}$$

$$= 5 \, {}^9\!/_2 \cdot 8$$

$$= 5 \times 9 \times 4$$

$$= 180$$

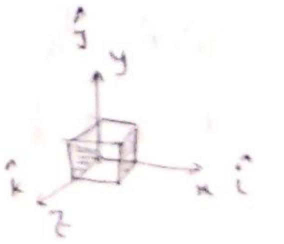
MATH NO : 61

Evalute $\iint_S r \cdot n \, ds$ over (a) The surface of S of the unit cube bounded by the co-ordinate planes and the planes $x=1$, $y=1$, $z=1$ (b) The Surface of the sphere of radius a with Center at $(0,0,0)$.

Solution:

(a)

Along x axis



$$\vec{r} \cdot \hat{n} = (\hat{i} x + \hat{j} y + \hat{k} z) \hat{n}$$

$$I_1 = \iint x \, dy \, dz$$

$$= \int_{y=0}^{1} \int_{z=0}^{1} dy \, dz = 1$$

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$I_2 = \iint x \, dy \, dz = 0$

Along y – axis.

$I_3 = \iint y \, dz \, dx$

$= \int_0^1 \int_0^1 dz \, dx = 1$

$I_4 = \iint y \, dz \, dx = 0$

Along z axis

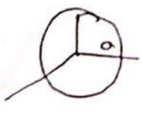
$I_5 = \iint z \, dx \, dy = 1$

$I_6 = \iint z \, dx \, dy = 0$

So that. $\iint \vec{F} \cdot \hat{n} \, ds = 3$.

(b) equation of the sphere

$x^2 + y^2 + z^2 = a^2$



$\phi = x^2 + y^2 + z^2 - a^2$

$\nabla \phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$|\nabla \cdot \phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2a$.

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$$\hat{n} = \frac{1}{2a}\left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right)$$

$$= \frac{1}{a}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\vec{F}\cdot\hat{n} = \frac{1}{a}\left(x^2 + y^2 + z^2\right)$$

$$= a$$

So, $$\iint_S \vec{F}\cdot\hat{n}\, ds$$

$$= a\iint \frac{dx\, dy}{|\hat{n}\cdot\hat{k}|}$$

$$= a\iint \frac{dx\, dy}{z/a}$$

$$= a^2\iint \frac{dx\, dy}{\sqrt{a^2-(x^2+y^2)}}$$

$$= a^2\iint \frac{dx}{\sqrt{a^2-y^2-x^2}}$$

$$= a^2\cdot\left[\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_{\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy$$

$$= a^2\int_a^a \pi\, dy$$

$$= \pi a^2 (a+a)$$

$$= 2\pi a^3.$$

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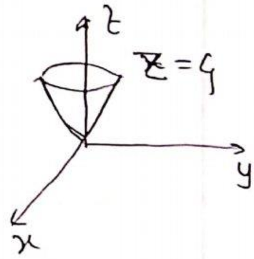
MATH-62:

Suppose $A = 4xz\,i + xyz^2\,j + 3z\,k$. Evalute $\iint_S A.n\,ds$ over the entire surface of region above the xy plane bounded by the cone $z^2 = x^2+y^2$ and the plane $z=4$.

Solution:

Now, $\phi = x^2+y^2-z^2$



$$\hat{n} = \frac{\nabla\phi}{|\nabla\cdot\phi|}$$

$$= \frac{2x\hat{i}+2y\hat{j}-2z\hat{k}}{\sqrt{4x^2+4y^2+z^2}}$$

$$= \frac{x\hat{i}+y\hat{j}\cancel{+}z\hat{k}}{\sqrt{2z^2}}$$

$$= \frac{1}{\sqrt{z}}\ (\cancel{4x^2+xy^2z-3z})\ (x\hat{i}+y\hat{j}-z\hat{k})$$

$$\vec{A}\cdot\hat{n} = \frac{1}{\sqrt{z}\,z}\cancel{z}(4x^2z+xy^2z^2-3z^2)$$

$$= \frac{1}{\sqrt{z}}\ (4x^2+xy^2z-3z)$$

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Now,

$$\iint \vec{A}\cdot\hat{n}\, ds = \iint \frac{1}{\sqrt{z}}\left(4x^2+xy^2z-3z\right)\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$$

$$= \iint \frac{1}{\sqrt{z}}\left(4x^2+xy^2z-3z\right)\frac{dy\,dz}{\frac{1}{\sqrt{z}}\frac{x}{z}}$$

$$= \iint \left(4xz+y^2z^2-\frac{3z}{x}\right) dy\,dz$$

$$= \iint \left(4z\sqrt{z^2-y^2}+y^2z^2-\frac{3z}{\sqrt{z^2-y^2}}\right) dy\,dz$$

$$= \int \left[4z\left\{\frac{y}{2}\sqrt{z^2-y^2}+\frac{z^2}{2}\sin^{-1}\frac{y}{z}\right\}+\frac{y^3}{3}z^2-3z\sin^{-1}\left(\frac{y}{z}\right)\right]_{y=-z}^{z} dz$$

$$= \int \left[\left\{4z\cdot\frac{z^2}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right\}\frac{z^2}{3}(2z^3)-3z\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right]dz$$

$$= \left[2\pi\frac{z^4}{4}+\frac{2z^6}{3\cdot 6}-3\cdot\frac{z^3}{3}\pi\right]_0^4$$

$$= 2\pi\times 64+\frac{1}{9}4^6-\pi\cdot 4^3$$

$$= 64\pi+\frac{4096}{9}$$

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MATH NO: 63

(a) Let, R be the projection of surface s on the xy plane. prove that the Surface area of S is given by $\iint_R \sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\, dx\, dy$ if the equation for s it $z = f(x,y)$

Solution:-

$z = f(x,y)$

$\phi = z - f(x,y)$

$\vec{\nabla}\phi = -\frac{df}{dx}\hat{i} - \frac{df}{dy}\hat{j} + \hat{k}$

$|\vec{\nabla}\phi| = \sqrt{1+\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2}$

$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{1+\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2}}$

Now, $\iint ds$

$= \iint \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$= \iint_S \sqrt{1+\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2}\, dx\, dy$

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(b) what is the Surface area if s has the equation $F(x,y,z)=0$?

Solution :-

$$\phi = F(x,y,z)$$

$$\vec{\nabla}.\phi = \frac{dF}{dx}\hat{i} + \frac{dF}{dy}\hat{j} + \frac{dF}{dz}\hat{k}$$

$$|\vec{\nabla}.\phi| = \sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\hat{n}.\hat{k} = \frac{dF/dz}{\sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2}}$$

Here, $\iint ds$

$$= \iint \frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

$$= \iint \frac{\sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2}}{\frac{dF}{dz}}\,dx\,dy$$

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P-1

21: Evaluate $\iint_S \phi n\, ds$ where $\phi = \frac{3}{8}xyz$ and S is the surface of problem-20.

Sol: We have $\iint_S \phi n\, ds = \iint_R \phi n \frac{dx\,dz}{|n \cdot j|}$

using $n = \frac{xi + yj}{4}$ $\therefore n \cdot j = \frac{y}{4}$ as in problem-20

this last integral becomes $\iint_R \frac{3}{8} xz (xi + yj)\, dx\, dz$

$$\Rightarrow \frac{3}{8}\int_{z=0}^{5}\int_{x=0}^{4} x^2 z i + xyz j\, dx\, dz$$

let, $x^2 + y^2 = 16$
$y^2 = 16 - x^2$
$y = \sqrt{16 - x^2}$

$$\Rightarrow \frac{3}{8}\int_{z=0}^{5}\int_{x=0}^{4} x^2 z i + xz(\sqrt{16-x^2}\, j)\, dx\, dz$$

$$\Rightarrow \frac{3}{8}\int_{z=0}^{5} \left(\frac{64}{3} z i + \frac{64}{3} z j\right) dz$$

$$\Rightarrow \frac{3}{8}\int_{z=0} \left[\frac{64}{3}\cdot\frac{z^2}{2} i + \frac{64}{3}\cdot\frac{z^2}{2} j\right]_0^5$$

$$\Rightarrow \frac{3}{8}\left\{\left(\frac{64}{3}\cdot\frac{25}{2} i + \frac{64}{3}\cdot\frac{25}{2}\right)\right\} - 0$$

$$\Rightarrow \frac{3}{8}\left(\frac{800}{3} i + \frac{800}{3} j\right) = 100 i + 100 j$$

ID - 038

P-2

22) If $F = y\,i + (x - 2xz)\,j - xy\,k$, evaluate $\iint_S (\nabla \times F) \cdot n\, ds$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy plane.

Soln: $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y & x-2xz & -xy \end{vmatrix}$

$\Rightarrow x\,i + y\,j - 2z\,\hat{k}$

A normal to $x^2+y^2+z^2=a^2$ is

$\nabla (x^2+y^2+z^2) = 2x\,i + 2y\,j + 2z\,k$

Then the unit normal n of the figure above is given by:-

$$n = \frac{2x\,i + 2y\,j + 2z\,k}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(x\,i + y\,j + z\,k)}{2\sqrt{a^2}}$$

$$\Rightarrow \frac{x\,i + y\,j + z\,k}{a}$$

ID-038

P-3

Since $x^2+y^2+z^2=a^2$

The projection of S on the xy plane is the region R bounded by the circle $x^2+y^2=a^2$, $z=0$. Then,

$$\iint_S (\Delta\times F)\,n\cdot ds = \iint_R (\Delta\times F)\,n\cdot \frac{dx\,dy}{|n\cdot k|}$$

$$\Rightarrow \iint_R (xi+yj-2zk)\cdot\left(\frac{xi+yj+zk}{a}\right)\cdot\frac{dx\,dy}{z/a}$$

$$\Rightarrow \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{x^2+y^2-2z^2}{a}\cdot\frac{a}{z}\,dx\,dy$$

$\because x^2+y^2+z^2=a^2$
$z^2=a^2-x^2-y^2$

$$\Rightarrow \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\,dy\,dx$$

using the fact that $z=\sqrt{a^2-x^2-y^2}$. To evaluate the double integral. transform to polar coordinates where $x=\rho\cos\phi$, $y=\rho\sin\phi$ and $dy\,dx$ is replaced by

ID-038

P-4

$\rho d\rho d\phi$. Then double integral becomes.

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{3\rho^2 - 2a^2}{\sqrt{a^2-\rho^2}}\, \rho\, d\rho\, d\phi$$

$$\Rightarrow \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{3(\rho^2-a^2)+a^2}{\sqrt{a^2-\rho^2}}\, \rho\, d\rho\, d\phi$$

$$\Rightarrow \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \left(-3\rho\sqrt{a^2-\rho^2} + \frac{a^2\rho}{\sqrt{a^2-\rho^2}}\right) d\rho\, d\phi$$

$$\Rightarrow \int_{\phi=0}^{2\pi} \left[(a^2-\rho^2)^{3/2} - a^2\sqrt{a^2-\rho^2}\right]_0^a d\phi$$

$$\Rightarrow \int_{\phi=0}^{2\pi} (a^3 - a^3)\, d\phi = 0$$

ID-038

P-5

23| If $F = 4xz\,i - y^2 j + yz\,k$, evaluate $\iint_S F.n\,ds$ where S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

Soln: Face DEFG: $n = i,\ x = 1$. Then,

$$\iint_{DEFG} F.n\,ds = \int_0^1 \int_0^1 (4zi - y^2 j + yzk) \cdot i\,dy\,dz$$

$$= \int_0^1 \int_0^1 4z\,dy\,dz$$

$$\Rightarrow 4 \cdot \frac{z^2}{2} = 2 \cdot 1^2$$

$$= 2$$

Face ABCO: $n = -i,\ x = 0$. Then

$$\iint_{ABCO} F.n\,ds = \int_0^1 \int_0^1 (-y^2 j + yzk)(-i)\,dy\,dz$$

$$\Rightarrow 0$$

ID-038

P-6

Face ABEF: $n = j$, $y = 1$. Then

$$\iint_{ABEF} F.n\,ds = \int_0^1 \int_0^1 (4xz\,i - j + z\,k)\,j\,dx\,dz$$

$$= \int_0^1 \int_0^1 -dx\,dz = -1$$

Face OGDC: $n = -j$, $y = 0$ Then

$$\iint_{OGDC} F.n\,ds = \int_0^1 \int_0^1 (4xz\,i).(-j)\,dx\,dz$$

$$= 0$$

Face BCDE. $n = k$, $z = 1$ Then

$$\iint_{BCDE} F.n\,ds = \int_0^1 \int_0^1 (4x\,i - y^2 j + y\,k).k\,dx\,dy$$

$$\Rightarrow \int_0^1 \int_0^1 y\,dx\,dy = \frac{1}{2}$$

ID-038

P-7

Face AFGO : $n = -k$, $z = 0$ Then.

$$\iint_{AFGO} F.n\,ds = \int_0^1 \int_0^1 (-y^2 j).(-k)\,dx\,dy$$

$$= 0$$

Adding, $\iint_S F.n\,ds = 2+0+(-1)+0+\frac{1}{2}+0$

$$= \frac{3}{2}.$$

Ans: $\frac{3}{2}$

ID-038

P-8

24) In dealing with surface integrals we have restricted ourselves to surface which are two sided. Given an example of a surface which is not two-sided.

Soln: Take a strip of paper such as ABCD as shown in the adjoining figure. Twist the strip so that points A and B fall on D and C respectively, as in the adjoining figure If n is the positive normal at point P of the surface, we find that as n moves around the surface its reverses its original direction when it reaches P again. If we tried to colour only one side of the surface we would find the whole thing colored.

ID-038.

P-9

This surface, called a Moebius strip, is an example of a one-sided surface. This is ~~Somethings~~ times called a non-orientable surface. A two-sided surface is orientable.

ID-038

P-10:

Exercise

63 Let R be the projection of a surface S on the xy-plane. prove that the surface area of S is given by $\iint_R \sqrt{1+\left(\frac{\delta z}{\delta x}\right)^2+\left(\frac{\delta z}{\delta y}\right)^2}\,dx\,dy$ if the equation for S is $z=f(x,y)$.

Sol^n: we Know,

$$\text{the area of the surface} = \iint ds$$

$$= \iint \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Now,

$$z = f(x,y)$$

$$\Rightarrow \phi = z - f(x,y)$$

$$\Rightarrow \vec{\nabla}\phi = -\frac{\delta f}{\delta x}\hat{i} - \frac{\delta f}{\delta y}\hat{j} + \hat{k}$$

$$\Rightarrow |\vec{\nabla}\phi| = \sqrt{1+\left(\frac{\delta f}{\delta x}\right)^2+\left(\frac{\delta f}{\delta y}\right)^2}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\therefore \hat{n}\cdot\hat{k} = \frac{\Delta\phi}{|\Delta\phi|}\cdot\hat{k}$$

ID-038

P-11

$$\Rightarrow \frac{1}{\sqrt{1+\left(\frac{\delta f}{\delta x}\right)^2+\left(\frac{\delta f}{\delta y}\right)^2}}$$

$$\therefore \iint_R \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\Rightarrow \iint_R \sqrt{1+\left(\frac{\delta f}{\delta x}\right)^2+\left(\frac{\delta f}{\delta y}\right)^2}\;dx\,dy$$

Ans.

(b) what is the surface area if S has the equation $F(x,y,z)=0$

Sol^n!

$$\phi = F(x,y,z)$$

$$\vec{\nabla}\phi = \frac{\delta F}{\delta x}\hat{i} + \frac{\delta F}{\delta y}\hat{j} + \frac{\delta F}{\delta z}\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{\left(\frac{\delta F}{\delta x}\right)^2+\left(\frac{\delta F}{\delta y}\right)^2+\left(\frac{\delta F}{\delta z}\right)^2}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

ID-038

P-12

$$\iint ds$$

$$\Rightarrow \iint \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\hat{n}\cdot\hat{k} = \frac{\delta F/\delta z}{\sqrt{\left(\frac{\delta F}{\delta x}\right)^2 + \left(\frac{\delta F}{\delta y}\right)^2 + \left(\frac{\delta F}{\delta z}\right)^2}}$$

$$\Rightarrow \iint \frac{\sqrt{\left(\frac{\delta F}{\delta x}\right)^2 + \left(\frac{\delta F}{\delta y}\right)^2 + \left(\frac{\delta F}{\delta z}\right)^2}}{\frac{\delta F}{dz}}\, dx\, dy$$

Ans.

64) Find the surface area of the plane $x+2y+2z=12$ cut off by (a) $x=0,\ y=0,\ x=1,\ y=1$.

(b) $x=0,\ y=0,$ and $x^2+y^2=16$

Sol^n: (a) Given that,

$$x+2y+2z=12$$

$$\therefore \phi = x+2y+2z-12$$

$$\Rightarrow \vec{\nabla}\phi = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\Rightarrow |\vec{\nabla}\phi| = \sqrt{1+4+4} = 3$$

ID-038

P-13:

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\Rightarrow \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

$\therefore \hat{n}.\hat{k} = \frac{2}{3}$

Now, $\iint ds$

$$\Rightarrow \iint \frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

$$\Rightarrow \iint \frac{3}{2}\,dx\,dy$$

$$\Rightarrow \frac{3}{2}\int_{x=0}^{1} dx \int_{y=0}^{1} dy$$

$$\Rightarrow \frac{3}{2}.1.1 = \frac{3}{2}$$ Ans:

ID-038

P-14

(b) From '(a)' we got,

$$\hat{n}\cdot\hat{k} = \frac{2}{3}$$

now,

$$\iint \frac{3}{2}\frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$$

$$\Rightarrow \frac{3}{2}\int_{y=0}^{4}\int_{0}^{\sqrt{16-y^2}} dx\,dy$$

Here,
$x^2 = 16 - y^2$
$x = 0$ to $\sqrt{16-y^2}$
$y = 0$ to 4

$$\Rightarrow \frac{3}{2}\int_0^4 \sqrt{16-y^2}\,dy$$

$$\Rightarrow \frac{3}{2}\left[\frac{y}{2}\sqrt{16-y^2} + \frac{16}{2}\sin^{-1}\frac{y}{4}\right]_0^4$$

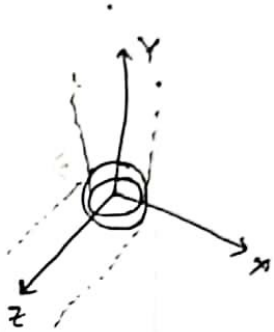
$$\Rightarrow \frac{3}{2}\left[\frac{16}{2}\cdot\frac{\pi}{2}\right] = 6\pi$$

Ans;

ID- 038

P-15

65: Find the surface area of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.



Sol^n: We know that,

There are 8 octed in a cylinder.

Now, $x^2+y^2=a^2$

$\phi = x^2+y^2-a^2$

$\vec{\nabla}\phi = 2x\hat{i}+2y\hat{j}$

$|\vec{\nabla}\phi| = \sqrt{4x^2+4y^2}$

$\Rightarrow 2a$

$\therefore \hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i}+y\hat{j})}{2a}$

$\Rightarrow \frac{1}{a}(x\hat{i}+y\hat{j})$

$\therefore \hat{n}\cdot\hat{i} = \frac{x}{a}$

ID-038

P-16

Now,

$$\iint ds$$

$$\Rightarrow \iint \frac{dy\,dz}{|\hat{n}.\hat{i}|}$$

$$\Rightarrow \int \int_{z=0}^{\sqrt{a^2-x^2}} \frac{a}{x}\, dy\, dz$$

$$\Rightarrow \int_0^a \frac{a}{x} \left[z\right]_0^{\sqrt{a^2-x^2}} dy$$

$$\Rightarrow 2a \int_0^a \frac{\sqrt{a^2-x^2}}{x}\, dy$$

$$\Rightarrow 2a \int_0^a \frac{y}{\sqrt{a^2-y^2}}\, dy$$

$$\Rightarrow -2a \int_0^a \frac{t\,dt}{t}$$

$x^2+y^2=a^2$

$y^2=a^2-x^2$

put,

$a^2-y^2=t^2$

$-2y\,dy=2t\,dt$

$-y\,dy=t\,dt$

$$\Rightarrow -2a\left[t\right]_0^a = -2a\left[\sqrt{a^2-y^2}\right]_0^a$$

$$\Rightarrow 2a^2$$

$\therefore$ total $= 8.2a^2 = 16a^2.$

ID-038

P-17

66/ Evaluate (a) $\iint_S (\nabla \times F) \cdot n \, ds$ and (b) $\iint_S \phi n \, ds$

if $F = (x+2y)\hat{i} - 3z\hat{j} + x\hat{k}$, $\phi = 4x + 3y - 2z$ and S

is the surface of $2x + y + 2z = 6$ bounded by $x=0, x=1$

$y=0$ and $y=2$.

Soln: (a) $\vec{\nabla} \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x+2y & -3z & x \end{vmatrix}$

$\Rightarrow \hat{i}\left(\frac{\delta}{\delta y} x - \frac{\delta}{\delta z}(-3z) \right) + \hat{j}\left(\frac{\delta}{\delta z}(x+2y) - \frac{\delta}{\delta x}(-x \right.$

$+ \hat{k}\left(\frac{\delta}{\delta x}(-3z) - \frac{\delta}{\delta y}(x+2y) \right)$

$\Rightarrow \hat{i}(0+3) + \hat{j}(0 - 1) + \hat{k}(0 - 2)$

$\Rightarrow 3\hat{i} - \hat{j} - 2\hat{k}$

Here, $\phi = 2x + y + 2z - 6$

$\Delta \phi = 2\hat{i} + \hat{j} + 2\hat{k}$

$|\vec{\nabla}\phi| = \sqrt{4+1+4} = \sqrt{9} = 3$

ID-038

P-18

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{1}{3}(2\hat{i}+\hat{j}+2\hat{k})$$

$$\hat{n}\cdot\hat{k} = \frac{2}{3}$$

Now,

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \frac{1}{3}(6-1-4)$$

$$= \frac{1}{3}$$

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds$$

$$\Rightarrow \frac{1}{3}\iint_S \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\Rightarrow \frac{1}{3}\cdot\frac{3}{2}\iint_S dx\,dy$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{1} dx\int_{y=0}^{2} dy$$

$$\Rightarrow \frac{1}{2}\cdot 1\cdot 2 =$$

$$= 1$$

Ans.

ID-038

P-19

(b) $\iint_S \phi \, \hat{n} \, ds$

$\Rightarrow \iint (4x+3y-2z) \cdot \frac{1}{3}(2\hat{i}+\hat{j}+2\hat{k}) \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$\Rightarrow \frac{1}{3} \cdot \frac{3}{2} \iint (2\hat{i}+\hat{j}+2\hat{k})(4x+3y-6+2x+y)\,dx\,dy$

$2x+y+2z=6$

$2z=6-2x-y$

$\Rightarrow (\hat{i}+\frac{1}{2}\hat{j}+\hat{k}) \iint (6x+4y-6)\,dx\,dy$

$\Rightarrow (\hat{i}+\frac{1}{2}\hat{j}+\hat{k}) \int \left[\frac{6\cdot x^2}{2}+4xy-6x\right]_0^1 dy$

$\Rightarrow (\hat{i}+\frac{1}{2}\hat{j}+\hat{k}) \int \left[3x^2+4xy-6x\right]_0^1 dy$

$\Rightarrow (\hat{i}+\frac{1}{2}\hat{j}+\hat{k}) \int_{y=0}^{2} (3+4y-6)\,dy$

$\Rightarrow (\hat{i}+\frac{1}{2}\hat{j}+\hat{k}) \left[-3y+2y^2\right]_0^2$

$\Rightarrow (\hat{i}+\frac{1}{2}\hat{j}+\hat{k})(-6+8)$

$\Rightarrow 2\hat{i}+\hat{j}+2\hat{k}$ Ans!

ID-038

P-20

67: Solve the preceding problem if S is the surface of $2x+y+2z=6$ bounded by $x=0$, $y=0$ and $z=0$.

Soln: (a) From '66' we get -

$$\iint (\vec{\nabla}\times\vec{F})\cdot\hat{n}\cdot ds$$

$$\Rightarrow \frac{1}{2}\iint dx\,dy$$

$$\Rightarrow \frac{1}{2}\int_{y=0}^{6}\left[x\right]_0^{3-\frac{y}{2}} dy$$

$$\Rightarrow \frac{1}{2}\int_{y=0}^{6}\left(3-\frac{y}{2}\right) dy$$

$$\Rightarrow \frac{1}{2}\left[3y-\frac{y^2}{4}\right]_0^6$$

$$\Rightarrow \frac{1}{2}\left[18-9\right]$$

$$\Rightarrow 9/2$$

For $z=0$

$2x+y=6$

$x=3-\frac{y}{2}$

For $x=0, z=0$

$y=6$

ID-038

P-21

(b) $\iint (\hat{i} + \frac{1}{2}\hat{j} + \hat{k})(6x + 4y - 6)\,dx\,dy$

$\Rightarrow (\hat{i} + \frac{1}{2}\hat{j} + \hat{k}) \int \left[3x^2 + (4y-6)x\right]_0^{3-y/2} dy \quad \Big| \; x = 3 - \frac{y}{2}$

$\Rightarrow [\hat{i} + \frac{1}{2}\hat{j} + \hat{k}] \int \left[3(3-\frac{y}{2})^2 + (4y-6)(3-\frac{y}{2})\right] dy$

$\Rightarrow (\hat{i} + \frac{1}{2}\hat{j} + \hat{k}) \int \left[3\{(9 - 3y + \frac{y^2}{4}) + 12y - 2y^2 - 18 + 3y\right] dy$

$\Rightarrow (\hat{i} + \frac{1}{2}\hat{j} + \hat{k}) \left[3(9y - \frac{3y^2}{2} + \frac{y^3}{12}) + (15\frac{y^2}{2} - \frac{2y^3}{3} - 18y)\right]_0^6$

$\Rightarrow (\hat{i} + \frac{1}{2}\hat{j} + \hat{k}) \times 72$

$\Rightarrow 72\hat{i} + 36\hat{j} + 72\hat{k}$ Ans:

*** End ***

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Volume Integral.

▣ What is volume integral ?

★ Answer : In multivariable calculus, a volume integral refers to an integral over a 3-dimensional domain ; that is, it is a special case of multiple integrals. Volume integral are especially important in physics for many application for exaple, to calculate flux densities

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□ Consider a closed surface in space enclosing a volume V. Then,

$$\iiint_V A \cdot dV \text{ and } \iiint_V \phi \, dV$$

are example of volume integral or space integral as they are sometimes called.

Solved Problem 25

Let $\phi = 45x^2y$ and let V denote the closed region bounded by the planes $4x+2y+z=8$, $x=0$, $y=0$, $z=0$

(a) Express $\iiint_V \phi \, dV$ as the limit of a sum.

(b) Evaluate the integral in (a).

Soln:

Subdivided region V into M cubes having volume $\Delta V_K = \Delta x_K \Delta y_K \Delta z_K$ $K = 1, 2, 3 \ldots M$ as indicated in the adjoining figure and let (x_K, y_K, z_K) be a point within this cube. Define $\phi(x_K, y_K, z_K) = \phi_K$; consider the sum —

(1) $\sum_{K=1}^{M} \phi_K \Delta V_K$.

taken over all possible cubes in the region. The limit of this sum when $M \to \infty$ in such a manner that the largest of the quantities ΔV_K will approach zero. if it exists is denoted by $\iiint_V \phi \, dV$. It can be shown that this limit is independent of the method of subdivision if ϕ is continuous throughout V.

In forming the sum (1) over all possible cubes in the region it is advisable to proceed in an order-ly fashion. One possibility is to add first all terms in (1) corresponding to volume

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elements contained in a column. This amounts to keeping x_K and y_K fixed and adding overr all z_K's. Next keep x_K fixed but sum overr all y_K's. This amount to adding all columns and consequently amounts to summing overr all cubes contained in such a slab. Finally varry x_K. This amounts to adding of all slabs.

In the prrocess outlined the summation is taken firrst overr z_K's then overr y_K's and finally overr x_K's.
Howeverr, the summation can clearly be taken in any otherr orrderr.

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(b)

The ideas involved in the method of summation outlined in (a) can be used in evaluating the integral. Keeping x and y constant integral from $z=0$ (base of column) to $z = 8-4x-2y$ (top to column). Next keep x constant and integral with respect to y. This amounts to addition of columns having base in the xy plane $(z=0)$ and the integral is from ~~yo~~ $y=0$ to $y=4-2x$. Finally we add all slabs parallel to the yz plane, which amount to integration form $x=0$ to $x=2$.

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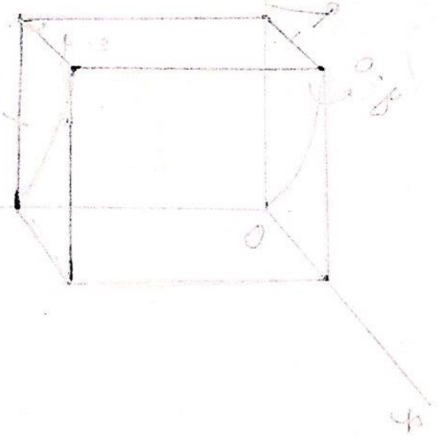
The integration can be written.

$$\int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45\, x^2 y \, dz\, dy\, dx.$$

$$= 45 \int_{x=0}^{2}\int_{y=0}^{4-2x} x^2 y \,(8-4x-2y)\, dy\, dx$$

$$= 45 \int_{x=0}^{2} \frac{1}{3}\, x^2 (4-2x)^3\, dx$$

$$= 128$$



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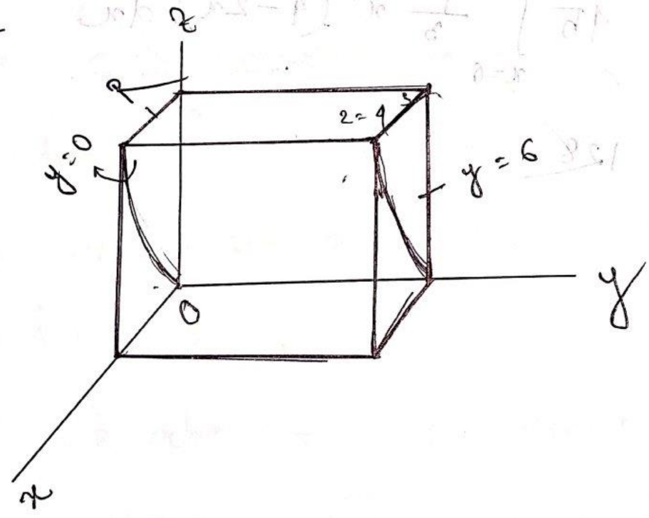
Solved Problem 26

Question: Let $F = 2xzi - xj + y^2K$.
Evaluate $\iiint_V F \cdot dv$ where V is the region bounded by the surface -
$x=0,\ y=0,\ y=6,\ z=x^2,\ z=4$.

Sol^n:



The region V is covered by keeping x and y fixed and integrating

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from $z = x^2$ to ~~$y = 6$~~ $z = 4$ then by keeping x fixed and integrating from $y = 0$ to $y = 6$, finally integrating from $x = 0$ to $x = 2$. Then the required integral is

$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (2xz\,i - x\,j + y^2 K)\, dz\, dy\, dx.$$

$$= i\int_0^2\int_0^6\int_{x^2}^4 2xz\, dz\, dy\, dx - j\int_0^2\int_0^6\int_{x^2}^4 x\, dz\, dy\, dx$$

$$+ K\int_0^2\int_0^6\int_{x^2}^4 y^2\, dz\, dy\, dx$$

$$= 128\, i - 24\, j + 384\, K.$$

Ans

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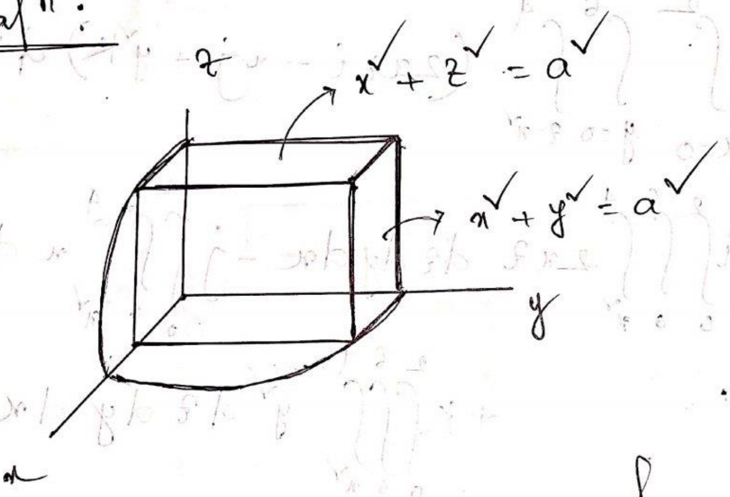
Solved problem 27

Question: Find the volume of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Sol^n:



Required volume = 8 times volume of region in above figure

$$= 8\int_{x=0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2}} dz\,dx\,dy$$

$$= 8\int_{x=0}^{a}(a^2-x^2)\,dx = \frac{16a^3}{3}$$

Ans

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Excercise Problem - 69

<u>Question</u>: Evaluate $\iiint_V (2x+y)\, dv$, where V is the closed region bounded by the cylinder $z = 4-x^2$ and the planes $x=0$, $y=0$, $y=2$ and $z=0$.

<u>Soln</u>:

Required value $= \int_0^2\int_0^2\int_0^{4-x^2} (2x+y)\, dz\, dy\, dx$.

$$= \int_0^2\int_0^2 (2x+y)(z)_0^{4-x^2}\, dy\, dx$$

$$= \int_0^2\int_0^2 (2x+y)(4-x^2)\, dy\, dx.$$

$$= \int_0^2\int_0^2 (8x-2x^3+4y-x^2y)\, dy\, dx.$$

$$= \int_0^2 \left[8xy - 2x^3y + 2y^2 - \frac{x^2y^2}{2}\right]_0^2 dx.$$

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$$= \int_0^2 \left(16x - 4x^3 + 8 - 2x^2\right) dx$$

$$= \left[8x^2 - x^4 + 8x - \frac{2x^3}{3}\right]_0^2$$

$$= 32 - \cancel{16} + \cancel{16} - \frac{16}{3}$$

$$= \frac{96 - 16}{3}$$

$$= \frac{80}{3} \quad \text{Ans}$$

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Exercise Problem : 20

Question: If $F = (2x^2 - 3z)i - 2xy\,j - 4x\,K$, evaluate (a) $\iiint_V \nabla \cdot F \, dv$ and (b) $\iiint_V \nabla \times F \, dv$ where V is the closed region bounded by the plane $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Soln: (a)

Here, $\vec{F} = (2x^2 - 3z)i - 2xy\,j - 4x\,K$.

Now,

$$\vec{\nabla} \cdot \vec{F} = 4x - 2x = 2x.$$

We get from the equation $2x + 2y + z = 4$

$$\therefore z = 4 - 2x - 2y \quad \cdots\cdots (i)$$

if the value of z is 0, then we can get $y = 2 - x \quad \cdots\cdots (ii)$

And if the value of y is 0, then we can get, $x = 2 \quad \cdots\cdots (iii)$

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Putting the limit of x, y, z from (i) (ii) and (iii) we can calculate the required volume.

Now,

$$\iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (2x) \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^{2-x} 2x(4-2x-2y) \, dy \, dx$$

$$= \int_0^2 \left[2x(4y - 2xy - y^2) \right]_0^{2-x} \cdot dx$$

$$= \int_0^2 2x \left[4(2-x) - 2x(2-x) - (2-x)^2 \right] dx$$

$$= \int_0^2 2x(8 - 4x - \cancel{4x} + 2x^2 - 4 + \cancel{4x} + x^2) dx$$

$$= \int_0^2 (8x - 8x^2 + 2x^3) \, dx$$

Turza-039

$$= \left[4x^2 - \frac{8}{3}x^3 + \frac{x^4}{2}\right]_0^2$$

$$= 16 - \frac{64}{3} + \frac{16}{2}$$

$$= 24 - \frac{64}{3}$$

$$= \frac{72-64}{3}$$

$$= \frac{8}{3}$$

(b)

Here, $F = (2x^2 - 3z)\,i - 2xy\,j + 4x\,K$

Now, $\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$

$$= i(0-0) + j(-3+4) + K(-2y-0)$$

$$= j - 2yK$$

We get from the equation -

$2x + 2y + z = 4$

$\therefore z = 4 - 2x - 2y$ ..... (i)

if the value of z is 0 then

$y = 2 - x$ ..... (ii)

And if the value of y is 0 then

$x = 2$ ..... (iii)

Putting the value of x, y, z from (i), (ii), (iii) we can calculate the required valume -

$$\therefore \iiint \vec{\nabla} \times \vec{F} \, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (j - 2y\,k) \, dz\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} (j - 2y\,k)(4 - 2x - 2y) \, dy \, dx$$

Turrza-039

$$= \int_0^2 \int_0^{2-x} \left[(4-2x-2y)\,j + 4(xy+y^2-2y)\,k\right] dy\,dx$$

$$= \int_0^2 \left[(4y-2xy-y^2)\,j + 4\left(\frac{xy^2}{2}+\frac{y^3}{3}-y^2\right)k\right]_0^{2-x} dx$$

$$= \int_0^2 \Big[(8-4x-4x+2x^2-4+4x-x^2)\,j + 4\Big\{\frac{x}{2}(4-4x+x^2) + \frac{1}{3}(8-12x+6x^2-x^3) - (4-4x+x^2)\Big\}k\Big] dx.$$

$$= \int_0^2 \left[(4-4x+x^2)\,j + 4(-x^2+2x-4)\,k\right] dx$$

$$= \left[\left(4x-2x^2+\frac{x^3}{3}\right)j + 4\left(\frac{-x^3}{3}+x^2-4x\right)k\right]_0^2$$

$$= \left(8-8+\frac{8}{3}\right)j + 4\left(-\frac{8}{3}+4-8\right)k$$

$$= \frac{8}{3}\,j - \frac{80}{3}\,k$$

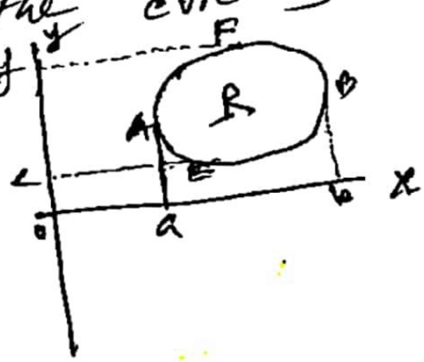
$$= \frac{8}{3}(j-10k)$$

Ans

rayhan 040

Rayhan-040 01
Date :

1. Prove Green's theorem in the plane if C a closed curve with which has the property that any straight line parallel to the co-ordinate axes cuts C in most two points.



let, the equation of the curves AEB and AFB be, $y = y_1(x)$ and $y = y_2(x)$ respectively. If R is the region bounded by C, we have, Raynam-040 02

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=y_1(x)}^{y_2(x)} \frac{\delta M}{\delta y}\,dy\right]dx =$$

$$\int_{x=a}^{b} M(x,y)\Big|_{y=y_2(x)}^{y_2(x)}\,dx = \int_a^b \left[M(x_2 y_2) - M(x_1 y_1)\right]dx$$

$$= -\int_a^b M(x_1 y_1)\,dx - \int_b^a M(x_1 y_2)\,dx = -\oint_C M\,dx$$

Then,

$$(1)\quad \oint_C M\,dx = -\iint_R \frac{\delta M}{\delta y}\,dx\,dy$$

similarly let the equations of the curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively, then,

$$\iint_R \frac{\delta N}{\delta x}\,dx\,dy = \int_{y=e}^{f}\left[\int_{x=X_1(y)}^{X_2(y)} \frac{\delta N}{\delta x}\,dx\right]dy =$$

$$\int_e^f \left[N(X_2, y) - N(X_1, y)\right]dy$$

Rayhan-040 3

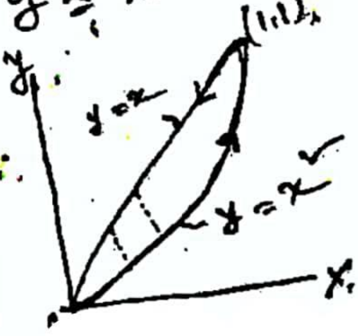
Date :

Then, ② $\oint_C N dy = \iint_R \frac{\delta N}{\delta x} dx dy$

Adding ① and ②,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} \right) dx dy.$$

2. Verify Green's theorem in the plane for $\oint_C (xy+y^2) dx + x^2 dy$ where c is the closed curve of the region bounded by $y=x$ and $y=x^2$



$y=x$ & $y=x^2$ intersection at $(0,0)$ and $(1,1)$ the positive direction in traversing C is as shown in the adjacent diagram.

Rayhan-040 4

Along $y=x^2$, the integral equals

$$\int_0^1 \left((x)(x^2)+x^4\right)dx+(x^2)(2x)\,dx=\int_0^1\left(3x^3+x^4\right)dx=\frac{19}{20}$$

Along $y=x$ from $(1,1)$ to $(0,0)$

$$\int_1^0 \left((x)(x)+x^2\right)dx+x^2\,dx=\int_1^0 3x^2\,dx=-1$$

Then the required line integral,

$$=\frac{19}{20}-1=-\frac{1}{20}.$$

$$\iint_R\left(\frac{\delta N}{\delta x}-\frac{\delta M}{\delta y}\right)dx\,dy=\iint_R\left[\frac{\delta}{\delta x}(x^2)-\frac{\delta}{\delta y}(xy+y^2)\right]dx\,dy$$

$$=\iint_R(x-2y)\,dx\,dy=\int_{x=0}^{1}\int_{y=x^2}^{x}(x-2y)\,dy\,dx$$

$$=\int_0^1\left[\int_{x^2}^{x}(x-2x)\,dy\right]dx=\int_0^1\left(xy-y^2\right)\Big|_{x^2}^{x}dx$$

$$=\int_0^1\left(x^4-x^3\right)dx=-\frac{1}{20}$$

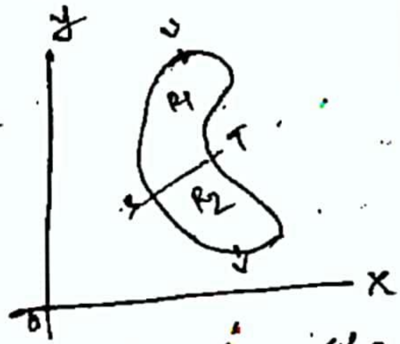
So that the theorem is verified

Kayram—090 5

Date : ....

3. Extend the proof of Green's theorem in the plane given in problem 1 to the curves c for which lines parallel to the co-ordinate axes my ~~~~ cut c in more than two points.

consider a closed curve c such as shown in the adjoining figure,



in which lines parallel to the axes may meet c in more than two points. By constructing line ST the region is divided into two regions R_1 & R_2 which are of the type considered in problem 1 and for which Green's theorem applies, i.e.

$$\text{(i)} \int_{STUS} M\,dx + N\,dy = \iint_{R_1} \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\,dy$$

$$\text{(ii)} \int_{SVTS} M\,dx + N\,dy = \iint_{R_2} \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\,dy$$

adding the left hands sides of (i) & (ii), we have, omitting the integral $M\,dx + N\,dx$ in each case

$$\int_{STUS} + \int_{STUS} = \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} = \int_{TUS} + \int_{SVT} =$$

using the fact that, $\int_{ST} = -\int_{TS}$.

Raytham-040 7

Date :

Adding the right hands sides
① & ⓘⓘ, omitting the integral,

$$\iint_{R_1} + \iint_{R_2} = \iint_{R}$$

when R consists of regions R_1 & R_2 —

then $\int_{TUSVT} M\,dx + N\,dx = \iint_{R} \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\,dy$ and

the theorem is proved.

For more completed simply connected regions it may be necessary to construct more lines, such as ST to establish the theorem.

Raynam-040

4. Express Green's theorem in the plane in vector notation.

We have $M\,dx + N\,dx = (Mi + Nj) \cdot (dx\,i + dy\,j)$

$= A \cdot dr$; where $A = Mi + Nj$ and $r = xi + yj$

so that $dr = dx\,i + dy\,j$.

Also, if $A = Mi + Nj$ then

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ M & N & 0 \end{vmatrix}$$

$$= -\frac{\delta N}{\delta z} i + \frac{\delta M}{\delta z} j + \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) k$$

so that $(\nabla \times A) \cdot k = \frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}$

Then Green's theorem in the plane,

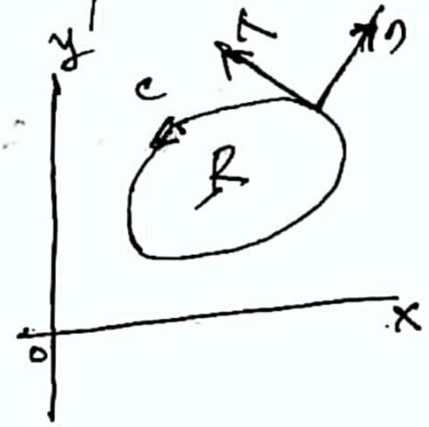
$$\oint A \cdot dr = \iint_R (\nabla \times A) \cdot k \, dR$$

where $dR = dx\,dy$.

Rayhan-040

Date : ....

A generalization of this surface S in space having a curve C as boundary leads quite naturally to Stokes theorem which is proved in problem 31.



Another method,

As above,

$M dx + N dy = A \cdot dr = A \cdot \frac{dr}{ds}$

$A \cdot \frac{dr}{ds} ds = A \cdot T \, ds$

where, $\frac{dr}{ds} = T$ unit tangent vector C. If a is outward drawn unit normal to C then $T = k \times a$, so that

Rayhan-046 10

$M\,dx + N\,dy = A \cdot T\,ds = A \cdot (k \times a)\,ds =$
$(A \times k) \cdot a\,ds$

Since $A = Mi + Nj$, $B = A \times K =$
$(Mi + Nj) = Ni - Mj$ and

$\frac{\delta N}{\delta x} = \frac{\delta M}{\delta y} = \nabla \cdot B$, Then,

$$\oint B \cdot a\,ds = \iint_R \nabla \cdot B\,dR$$

Israt 041

(5)

Q. Interpret physically the first result of problem 4.

Sol^n: If $\vec{A}$ denotes the force field acting on a particle, then $\oint_C \vec{A}\cdot d\vec{r}$ is the work done in moving the particle around a closed path C and is determined by the value of $\vec{\nabla}\times\vec{A}$. It follows in particular that if $\vec{\nabla}\times\vec{A}=\vec{0}$ or equivalently if $\vec{A}=\vec{\nabla}\phi$, then the integral around a closed path is zero. This amounts to saying that the work done in moving the particle from one point in the plane to another is independent of the path in the plane joining the points or that the force field is conservative. These results have already been demonstrated for force field eand curves in space

Conversely, if the integral is ~~idem~~ independent of the path joining any two points of a region, i.e. if the integral around any closed path is zero, then $\vec{\nabla}\times\vec{A}=\vec{0}$.

In the plane, the condition $\vec{\nabla}\times\vec{A}=\vec{0}$ is equivalent to the condition $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$ where, $\vec{A} = M\hat{i} + N\hat{j}$

Isrcat 041

(6)

Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$ along the path $x^4 - 6xy^3 = 4y^2$

Sol^n :

Here, $M = 10x^4 - 2xy^3$

$N = -3x^2y^2$

$\therefore \frac{\delta M}{\delta y} = -6xy^2$ and $\frac{\delta N}{\delta x} = -6xy^2$

It follows that the integral is independent of the path. Then we can use any path, for example the path consisting of straight line segment from $(0,0)$ to $(2,0)$ and then $(2,0)$ to $(2,1)$

Along the straight line path from $(0,0)$ to $(2,0)$, $y=0$, $dy=0$

and the integrals $= \int_{x=0}^{2} 10x^4\,dx = 64$

Along the straight line path from $(2,0)$ to $(2,1)$, $x=2$, $dx=0$

and the integrals $= \int_{y=0}^{1} -12y^2\,dy = -4$ $[x=2]$

$\therefore$ The reequired value $= 64 - 4 = 60$

Isrât 041

(7)

Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint x\,dy - y\,dx$.

Soln:

In Green's Theorem,

$$M = -y$$
$$N = x$$

$$\therefore \oint_C x\,dy - y\,dx = \iint_R \left[\frac{\delta}{\delta x}(x) - \frac{\delta}{\delta y}(y)\right] dx\,dy$$

$$= \iint_R (1+1)\,dy\,dx$$

$$= 2\iint_R dy\,dx$$

$$= 2A \quad \text{(where A is the area)}$$

$$\therefore A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

(showed)

Israt 041

(8)

Find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$

Solⁿ:

$dx = -a\sin\theta\, d\theta$
$dy = b\cos\theta\, d\theta$

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} a\cos\theta . b\cos\theta\, d\theta - b\sin\theta . (-a\sin\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\cos^2\theta\, d\theta + ab\sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\left(\cos^2\theta + \sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, d\theta$$

$$= \frac{1}{2} ab\theta \Big|_0^{2\pi}$$

$$= \frac{1}{2} ab\left(2\pi - 0\right)$$

$$= \frac{1}{2} ab . 2\pi$$

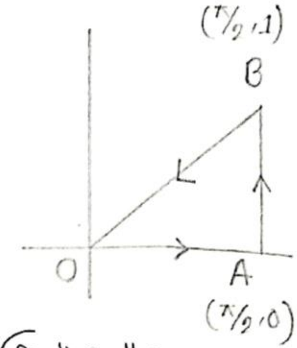
$$= \pi ab$$

Israt 041

(9) Evaluate $\oint_C (y-\sin x)\,dx + \cos x\,dy$ while C is the triangle of the adjoining figure:

(a) directly,

(b) by using Green's Theorem in the plane.



Soln:

(a) Along OA, $y=0$, $dy=0$

and the integral eqn

$$= \int_0^{\pi/2} (0-\sin x)\,dx + (\cos x)(0)$$

$$= \int_0^{\pi/2} -\sin x\,dx$$

$$= \cos x \Big|_0^{\pi/2}$$

$$= -1$$

finding the eqn of BO:

BO passing through $(\frac{\pi}{2}, 1)$ and $(0,0)$ point:

$$\therefore \frac{x-\pi/2}{\pi/2-0} = \frac{y-1}{1-0}$$

$$\Rightarrow \frac{2x-\pi}{2} = \frac{\pi}{2}(y-1)$$

$$\Rightarrow 2x-\pi = \pi y - \pi$$

$$\Rightarrow \pi y = 2x$$

$$\Rightarrow y = \frac{2x}{\pi}$$

Along AB, $x = \pi/2$, $dx = 0$

$$\therefore \text{Integrals} = \int_0^1 (y-1).0 + 0\,dy = 0$$

Along BO, $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi}dx$

$$\therefore \text{Integrals} = \int_{\pi/2}^{0} \left(\frac{2x}{\pi} - \sin x\right)dx + \frac{2}{\pi}\cos x\,dx$$

$$= \left(\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right)\Big|_{\pi/2}^{0}$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

Israt 041

Then the integral along $C = -1+0+1-\frac{\pi}{4}-\frac{2}{\pi} = -\frac{\pi}{4}-\frac{2}{\pi}$

(b) Here, $M = y - \sin x \qquad \frac{\delta M}{\delta y} = 1$

$N = \cos x \qquad \frac{\delta N}{\delta x} = -\sin x$

$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\,dy$$

$$= \iint_R (-\sin x - 1)\,dy\,dx$$

$$= \int_{x=0}^{\pi/2}\int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\,dx$$

$$= \int_{x=0}^{\pi/2} (-y\sin x - y)\Big|_0^{2x/\pi}\,dx$$

$$= \int_{x=0}^{\pi/2} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx$$

$$= -\frac{2}{\pi}\int_{x=0}^{\pi/2} (x\sin x + x)\,dx$$

$$= -\frac{2}{\pi}\int_{x=0}^{\pi/2} \left(-x\cos x + \sin x + \frac{x^2}{2}\right) dx$$

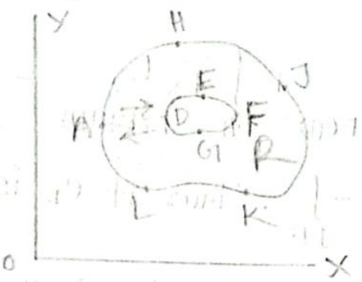
$$= -\frac{2}{\pi}\left(-x\cos x + \sin x + \frac{x^2}{2}\right)\Big|_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left(1 + \frac{\pi^2}{8}\right)$$

$$= -\frac{2}{\pi} - \frac{\pi}{4}$$

Isrcat 041

10. Show that Green's theorem in the plane is also valid for a multiply-connected region R such as shown in the figure below.



Solⁿ:

The shaded region R, shown in the figure, is multiply-connected since not every closed curve lying in R can be shrunk to a point without leaving R, as is observed by considering a curve sorrounding DEFGD for example. The boundary of R, which consists of the exterior boundary AHJKLA and the interior boundary DEFGD, is to be traversed in the positive direction, so that a person travelling in this direction always has the region on his left. It is seen that the positive directions are those indicated in the adjoining figure.

In order to establish the theorem, construct a line, such as AD, called a cross-cut, connecting the exterior and interior boundaries. The region bounded by ADEFGDALKJHA is simple connected, and so green's theorem is valid.

$$\oint_{ADEFGDALKJHA} M\,dx + N\,dy = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\,dy$$

Israt041

But the integral on the left, leaving out the integrals is equal to

$$\int_{AD} + \int_{DEFGD} + \int_{DA} + \int_{ALKJHA} = \int_{DEFGD} + \int_{ALKJHA}$$

since $\int_{AD} = -\int_{DA}$. Thus if C_1 is the curve ALKJHA, C_2 is the curve DEFGD and C is the boundary of R consisting of C_2 (traversed in the positive directions), then $\int_{C_1} + \int_{C_2} = \int_{C}$ and so

$$\oint_C M dx + N dy = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y} \right) dx\, dy$$

Israt 041

41. Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy-y^2)dx+(3x^2-2xy)dy$ along the cycloid $x=\theta-\sin\theta$, $y=1-\cos\theta$

Sol^n: $M=6xy-y^2$; $N=(3x^2-2xy)$

$\therefore \frac{\delta M}{\delta y}=6x-2y \qquad \frac{\delta N}{\delta x}=6x-2y$

$\therefore \frac{\delta M}{\delta y}=\frac{\delta N}{\delta x}$, exact differential.

$$\int_{(0,0)}^{(\pi,2)} 6xy\,dx-y^2dx+3x^2dy-2xy\,dy$$

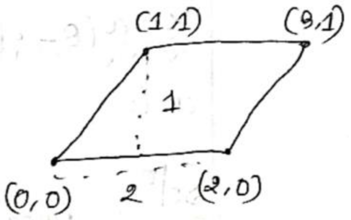
$$=\int_{(0,0)}^{(\pi,2)} d(3x^2y-xy^2)$$

$$=3x^2y-xy^2\Big|_{(0,0)}^{(\pi,2)}$$

$$=6\pi^2-4\pi$$

Israt 041

(42) Evaluate $\oint (3x^2+2y)dx - (x+3\cos y)dy$ around the parallelogram having vertices at $(0,0)$, $(2,0)$, $(3,1)$ and $(1,1)$



Sol^n :

Here, $M = 3x^2 + 2$

$\therefore \frac{\delta M}{\delta y} = 2$

and $N = -(x + 3\cos y)$

$\frac{\delta N}{\delta x} = -1$

By Green's theorem,

$$\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\,dy = \iint_R (-1-2)\,dy\,dx$$

$$= \iint_R (-3)\,dy\,dx$$

$$= -3\iint_R dy\,dx$$

$$= -3 \times A$$

$$= -3 \times \text{base} \times \text{hight}$$

$$= -3 \times 2 \times 1$$

$$= -6$$

Israt 041

(43)

Find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, $a>0$, and th x axis.

Sol^n:

$$\text{Area} = \int y\,dx$$

$$= \int_0^{2\pi} a(1-\cos\theta) \cdot a(1-\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} a^2(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

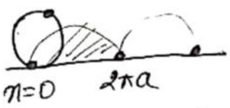
$$= \frac{a^2}{2}\int_0^{2\pi} (2 - 4\cos\theta + 1 + \cos 2\theta)\,d\theta$$

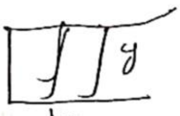
$$= \frac{a^2}{2}\left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\cdot 3\cdot 2\pi$$

$$= 3\pi a^2$$

cycloid: path due to rotation of circle on a line



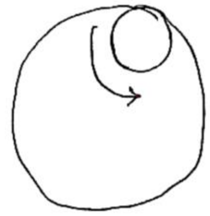


$\int y\,dx$

Isrcat 041

(44)

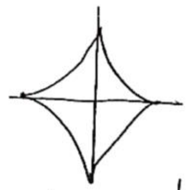
Find the area bounded by the hypercycloid $x^{2/3}+y^{2/3}=a^{2/3}$, $a>0$, Hint: Parametric equations are: $x=a\cos^3\theta$, $y=a\sin^3\theta$



Soln:

$dx = -3a\cos^2\theta \sin\theta \, d\theta$

$dy = a\, 3\sin^2\theta \cos\theta \, d\theta$



hypercycloid

$$\therefore \frac{1}{2}\oint (x\,dy - y\,dx)$$

$$= \frac{1}{2}\oint a\cos^3\theta \cdot a\,3\sin^2\theta \cos\theta \, d\theta - $$

$$+ a\sin^3\theta \cdot 3a\cos^2\theta \sin\theta \, d\theta$$

$$= \frac{3a^2}{2}\oint (\sin^2\theta\cos^4\theta + \sin^4\theta\cos^2\theta)\,d\theta$$

$$= \frac{3a^2}{2}\oint \sin^2\theta\cos^2\theta \, d\theta$$

$$= \frac{3a^2}{8}\oint \sin^2 2\theta \, d\theta$$

$$= \frac{3a^2}{8}\oint \cancel{2}(1+\cos 4\theta)\,d\theta$$

$$= \frac{3a^2}{16}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{3a^2}{16}\cdot 2\pi$$

$$= \frac{3\pi a^2}{16}$$

ID: 043

6.46: Find the area of a loop of the four-leafed rose $\rho = 3\sin 2\varphi$

Soln:

$$\text{Area} = \frac{1}{2}\int \rho^2 \, d\varphi$$

$$= \frac{1}{2}\int_0^{\pi/2} 9\sin^2 2\varphi \, d\varphi$$

at $\varphi = 0$, $\rho = 0$

at $\varphi = \pi/2$, $\rho = 0$

$$= \frac{9}{4}\int_0^{\pi/2} (1-\cos 4\varphi)\, d\varphi$$

$$= \frac{9}{4}\left[\varphi - \frac{\sin 4\varphi}{4}\right]_0^{\pi/2}$$

$$= \frac{9}{4} \times \frac{\pi}{2}$$

$$= \frac{9\pi}{8}$$

$$\text{Total} = 4 \times \frac{9\pi}{8}$$

$$= \frac{9\pi}{2} \quad \text{(Answer)}$$

ID:043

6.47: Find the area of both loops of the Lemniscate $\rho^2 = a^2\cos 2\varphi$

Soln:

$$\text{Area} = \frac{1}{2}\int \rho^2 d\varphi$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\pi/4} a^2\cos 2\varphi \cdot d\varphi$$

$$= \frac{a^2}{2}\left[\frac{\sin 2\varphi}{2}\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{a^2}{2}\left(1-(-1)\right)$$

$$= \frac{a^2}{2} \quad [\text{for one loop}]$$

at $\varphi = 0$, $\rho^2 = a^2$

at, $\varphi = \pi/4$, $\rho^2 = 0$

$$\text{Total} = 2 \times \frac{a^2}{2} = a^2 \quad (\text{Answer})$$

ID:043

6.49: verify Green's theorem in the plane for $\oint_C (2x-y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

Direct method:

Take,
$x = r\cos\theta,\ dx = -r\sin\theta\,d\theta$
$y = r\sin\theta,\ dy = r\cos\theta\,d\theta$

$\oint (2x-y^3)\,dx - xy\,dy$

$= \int (2 \cdot r\cos\theta - r^3\sin^3\theta)(-r\sin\theta)\,d\theta - r\cos\theta \cdot r\sin\theta \cdot r\cos\theta\,d\theta$

$= \int (-2r^2 \cdot \sin\theta\cos\theta + r^4\sin^4\theta - r^4\sin\theta\cos^2\theta)\,d\theta$

$= \int \left[-r^2\sin 2\theta + \frac{r^4}{4}(1+\cos 2\theta)^2 - r^3\sin\cos^2\theta\right]d\theta$

$= \int \left[-r^2\sin 2\theta + \frac{r^4}{4}(1+2\cos 2\theta + \cos^2 2\theta - r^3 \sin\theta\cos\theta\right]d\theta$

ID:043

$$\int_{\theta=0}^{2\pi}\left[r\sin 2\theta+\frac{r^4}{8}\left(2+4\cos 2\theta+1+\cos 4\theta\right)\right]$$

$$d\theta+\int_{\theta=0}^{2\pi} r^3\cos^2\theta\, d(\cos\theta)$$

$$=\left[\frac{3r^4}{8}\theta\right]_0^{2\pi}+r^3\cdot\left[\frac{\cos^3\theta}{3}\right]_0^{2\pi}$$

$$=\frac{3}{8}r^4\cdot 2\pi$$

$$=\frac{3}{4}r^4\cdot\pi$$

$$\text{Total}=\left[\frac{3\pi}{4}r^4\pi\right]_1^3$$

$$=\frac{3\pi}{4}\left(3^4-1\right)=60\pi$$

Green's Theorem method:

given that

$$\oint(2x-y^3)\,dx-xy\,dy$$

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ID: 043

$M = x - y^3, \frac{\partial M}{\partial y} = -3y^2$

$N = -xy, \frac{\partial N}{\partial x} = -y$

$x = r\cos\theta$

$y = r\sin\theta$

$r \rightarrow 1 \rightarrow 3$

$\theta \rightarrow 0 \rightarrow 2\pi$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3y^2 - y$$

$$= \iint (3y^2 - y)\, dx\,dy$$

$$= \iint (3r^2\sin^2\theta - r\sin\theta)\, r.d\theta\, dr$$

$$= \iint \left[\frac{3}{2} r^3 (1+\cos 2\theta) - r^2\sin\theta\right] d\theta\, dr$$

$$= \int \left[\frac{3}{2} r^3 \left(\theta + \frac{\sin 2\theta}{2}\right) + r^2\cos\theta\right]_{\theta=0}^{2\pi} dr$$

$$= \int_{r=1}^{3} \left(\frac{3}{2} r^3 . 2\pi\right) dr$$

$$= 3\pi \left[\frac{r^4}{4}\right]_1^3$$

$$= \frac{3\pi}{4}(81-1)$$

$$= 60\pi \quad \text{(verified)}$$

ID:043

6.11: Show that Green's theorem in the plane holds for the region R, of the figure below, bounded by simple closed curves $C_1(ABDEFGA)$, $C_2(HKLPH)$, $C_3(QSTUQ)$ and $C_4(VWXYV)$.

Solution: The integral over this boundary is equal to

$$\int_{AH} + \int_{HKL} + \int_{LQ} + \int_{QST} + \int_{TV} + \int_{VWXYV} + \int_{VT} + \int_{TUQ} +$$

$$\int_{QL} + \int_{LPH} + \int_{HA} + \int_{ABDEFGA}$$

Since the integrals along AH and HA, LQ and QL, TV and VT cancel out in pairs, this becomes

$$\int_{HKL} + \int_{QST} + \int_{VWXYV} + \int_{TUQ} + \int_{LPH} + \int_{ABDEFGA}$$

ID:043

$$+\left(\int_{HKL} + \int_{LPH}\right) + \left(\int_{QST} + \int_{TUQ}\right) + \int_{VWXYZ} + \int_{ABDEFGA}$$

$$= \int_{HKLPH} + \int_{QSTUQ} + \int_{VWXYZ} + \int_{ABDEFGA}$$

$$= \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_1} = \int_{C}$$

where C is the boundary consisting of C_1, C_2, C_3 and C_4. Then

$$\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

as required.

ID:043

6.12: Prove that $\oint M\,dx + N\,dy = 0$ around every closed curve C in a simply-connected region if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ everywhere in the region.

Solution: Assume that M and N are continous and have continous partial derivatives everywhere in the region R bounded by C, so that Green's theorem is applicable. Then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R, then clearly $\oint_C M\,dx + N\,dy = 0$.

Conversely, suppose $\oint_C M\,dx + N\,dy = 0$ for all curves C.

ID:04-3

If $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} > 0$ at a point P, then from the continuity of the derivatives it follows that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} > 0$ in some region A surrounding P. If Γ is the boundary of A then

$$\oint_{\Gamma} M dx + N dy = \iint_{A} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy > 0$$

which contradicts the assumption that the line intergal is zero around every closed curve. Similarly the assumption $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} < 0$ leads to a contradiction. Thus $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ at all points.

Note that the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is equivalent to the condition $\nabla \times A = 0$ where $A = Mi + Nj$

ID: 043

6.13: Let $F = \frac{-yi + xj}{x^2+y^2}$. (a) calculate $\nabla \times F$. (b) Evaluate $\oint F \cdot dr$ around any closed path and explain the results.

Solution (a): $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = 0$

in any region excluding $(0,0)$.

(b) $\oint F.dr = \oint \frac{-y\,dx + x\,dy}{x^2+y^2}$. Let $x = \rho \cos\phi$, $y = \rho \sin\phi$, where (ρ, ϕ) are polar coordinates;

Then

$dx = -\rho \sin\phi \, d\phi + d\rho \cos\phi$, $dy = \rho \cos\phi \, d\phi + d\rho \sin\phi$

ID:043

and so $\frac{-y\,dx + x\,dy}{x^2+y^2} = d\phi = d\left(\arctan \frac{y}{x}\right)$

For a closed curve ABCDA surrounding the origin, $\phi = 0$ at A and $\phi = 2\pi$ after a complete circuit back to A. In this case the line intergal equals $\int_0^{2\pi} d\phi = 2\pi$.

For a closed curve PQRSP not surrounding the origin, $\phi = \phi_0$ at P and $\phi = \phi_0$ after a complete circuit back to P. In this case the line intergal equals $\int_{\phi_0}^{\phi_0} d\phi = 0$.

ID:045

Since $F = Mi + Nj$, $\nabla \times F = 0$ is equivalent to $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the results would seem to contradict those of problem 12. However, no contradiction exists since $M = \frac{-y}{x^2+y^2}$ and $N = \frac{x}{x^2+y^2}$ do not have continous derivatives throughout any region including $(0,0)$ and this was assumed in prob. 12.

masud 044

Page: 1

ID. 044 (Masud)

The Divergence Theorem of Gauss: states that if V is the volume bounded by a closed surface S and A is a vector function of position with continuous then,

$$\iiint_V \nabla . A dV = \iint_S A \cdot n \, ds = \oiint_S A \cdot ds$$

where n is the positive (outward drawn) normal to S.

(14)

a) Express the divergence theorem in words and
b) write it in rectangular form.

a) Sol^n: The surface integral of the normal component of a vector A taken over a closed surface is equal to the integral of the divergence of A taken ~~ore~~ over the volume enclosed by the surface

(2')

ID.044 (Masud)

b) Sol^n: let $A = A_1 i + A_2 j + A_3 k$. Then div A =

$$\nabla \cdot A = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

The unit normal to S is $n = n_1 i + n_2 j + n_3 k$. Then $n_1 = n \cdot i = \cos\alpha$, $n_2 = n \cdot j = \cos\beta$ and $n_3 = n \cdot k = \cos\gamma$ where α, β, γ are the angles which n makes with the positive x, y, z axes or i, j, k directions respectively. The quantities $\cos\alpha, \cos\beta, \cos\gamma$ are the direction ~~co~~ cosines of n. Then

$$A \cdot n = (A_1 i + A_2 j + A_3 k) \cdot (\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k)$$
$$= A_1 \cos\alpha + A_2 \cos\beta + A_3 \cos\gamma$$

and the divergence ~~thero~~ theorem can be written

$$\iiint_V \left(\frac{\partial A}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dx\,dy\,dz = \iint_S (A_1 \cos\alpha + A_2 \cos\beta + A_3 \cos\gamma)\, ds$$

3

ID.044 (Masud)

(15)

Demonstrate the divergence theorem physically.

Sol^n: Let A = velocity v at any point of moving fluid

volume of fluid crossing ds in Δt seconds

$= $ volume contained in cylinder of base ds and slant height $v\Delta t$

$= (v\Delta t) \cdot n\, ds$

$= v \cdot n\, ds\, \Delta t$

Then, volume per second of fluid crossing ds.

$ds = v \cdot n\, ds$

Total volume per second of fluid emerging from closed surface $S = \iint\limits_S v \cdot n\, ds$

From problem 21 of chapter 4, $\nabla \cdot v\, dV$ is the volume per second of fluid emerging from a

④ ID.044 (Masud)

volume element dV. Then

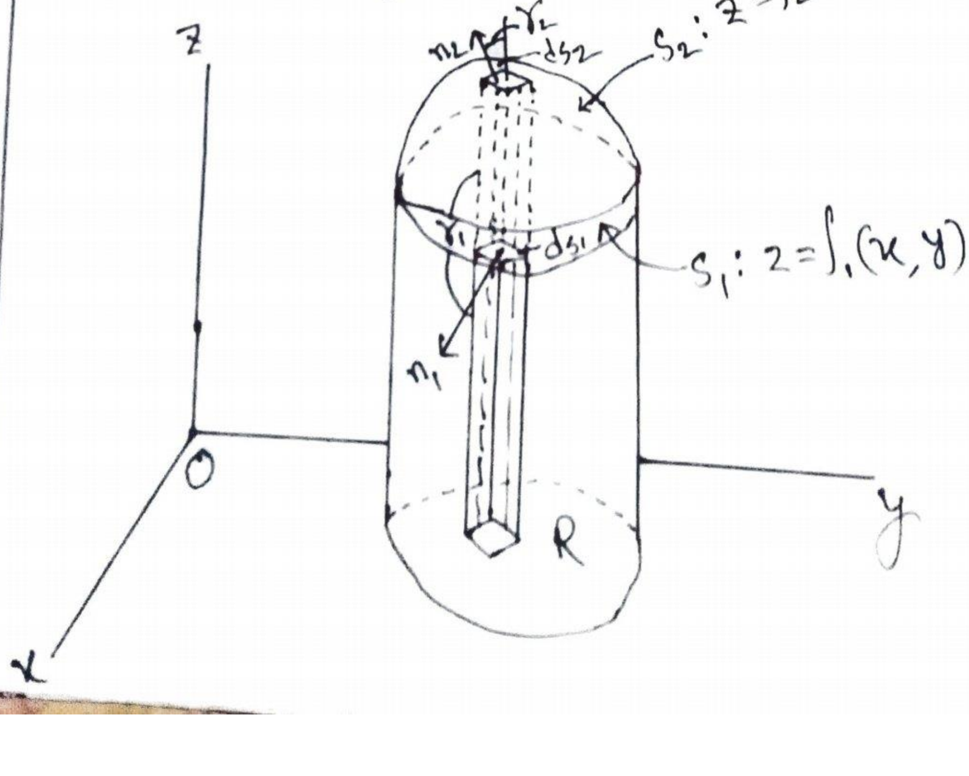
Total volume per second of fluid emerging from all volume elements in S

$$= \iiint_V \nabla \cdot v \, dV$$

Thus $$\iint_S v \cdot n \, dS = \iiint_V \nabla \cdot v \, \cancel{dS} \, dV$$

(16)

Prove the divergence theorem.



5

ID.044 (Masud)

Let S be a closed surface which is such that any line parallel to the coordinate axes cuts S in at most two points. Assume the equations of the lower and upper portions, S_1 and S_2, to be $z=f_1(x,y)$ and $z=f_2(x,y)$ respectively. Denote the projection of the surface on the xy plane by R. Consider

$$\iiint_V \frac{\partial A_3}{\partial z}dV = \iiint_V \frac{\partial A_3}{\partial z}\, dz\,dy\,dx = \iint_R \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z}dz\right]dy\,dx$$

$$= \iint_R A_3(x,y,z)\Big|_{z=f_1}^{f_2} dy\,dx = \iint_R \left[A_3(x,y,f_2) - A_3(x,y,f_1)\right] dy\,dx$$

For the upper portion S_2, $dy\,dx = \cos\gamma_2\, dS_2 = k\cdot n_2\, dS_2$ since the normal n_2 to S_2 makes an acute angle γ_2 with R.

For the lower portion S_1, $dy\,dx = -\cos\gamma_1\, dS_1 = -k\cdot n_1\, dS_1$

(6) ID. 044 (Masud)

Since the normal n_1 to S_2 makes an obtuse angle γ_1 with k.

Then

$$\iint_R A_3(x,y,f_2)\,dy\,dx = \iint_{S_2} A_3\, k\cdot n_2\, dS_2$$

$$\iint_R A_3(x,y,f_1)\,dy\,dx = -\iint_{S_1} A_3\, k\cdot n_1\, dS_1$$

and

$$\iint_R A_3(x,y,f_2)\,dy\,dx - \iint_R A_3(x,y,f_1)\,dy\,dx$$

$$= \iint_{S_2} A_3 k\cdot n_2 dS_2 + \iint_{S_1} A_3 k\cdot n_1 dS_1$$

$$= \iint_S A_3 k\cdot n\, dS$$

So that (1) $\displaystyle\iiint_V \frac{\partial A_3}{\partial z}\,dv = \iint_S A_3 k\cdot n\, dS$

Similarly, by projection·s on the other coordinate planes, ④ ID.044 (Masud)

(2) $$\iiint_V \frac{\partial A_1}{\partial x}\, dv = \iint_S A_1 i \cdot n\, ds$$

(3) $$\iiint_V \frac{\partial A_2}{\partial y}\, dv = \iint_S A_2 j \cdot n\, ds$$

Adding (1), (2) and (3),

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dv = \iint_S (A_1 i + A_2 j + A_3 k) \cdot n\, ds$$

$$\Rightarrow \iiint_V \nabla \cdot A\, dv = \iint_S A \cdot n\, ds$$

The theorem can be extended surface which are such the ~~liear~~ lines parallel to the coordinate axes meet them in more than two points. To establish this extension, subdivide the region bounded by S into subregions whose surface do satisfy this condition. The procedure is analogous to that used in Green's theorem ~~of~~ for the plane.

(8) ID.044 (Masud)

(17)

Evaluate $\iint_S F.n\, dS$, where $F = 4xzi - y^2 j + yzk$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Sol^n:

By the divergence theorem the required integral is equal to

$$\iiint_V \nabla . F\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - y)\, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4x - y)\, dz\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz \Big|_{z=0}^{1} dy\, dx = \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\, dy\, dx = \frac{3}{2}$$

The surface is integral may also be evaluated directly as in problem 23, chapter 5.

(9)

ID.044 (Masud)

(18)

Verify the divergence theorem for $A = 4xi - 2y^2j + z^2k$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$

Soln: Volume integral

$$= \iiint_V \nabla \cdot A\,dV = \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dv$$

$$= \iiint_V (4-4y+2z)\,dv = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dy\,dz\,dx$$

$$= 84\pi$$

The surface s of the cylinder consists of a base s_1 $(z=0)$; the top s_2 $(z=3)$ and the convex portion s_3 $(x^2+y^2=4)$.

(10) ID.044 (Masud)

Then

surface integral

$$= \iint_S A\cdot n\, dS = \iint_{S_1} A\cdot n\, dS_1 + \iint_{S_2} A\cdot n\, dS_2 + \iint_{S_3} A\cdot n\, dS_3$$

On $S_1 (z=0)$. $n = -k$, $A = 4xi - 2y^2 j$ and $A\cdot n = 0$ so that

$$\iint_{S_1} A\cdot n\, dS_1 = 0$$

On $S_2 (z=3)$, $n = k$, $A = 4xi - 2y^2 j + 9k$ and $A\cdot n = 9$

so that,

$$\iint_{S_2} A\cdot n\, dS_2 = 9 \iint_{S_2} dS_2 = 36\pi$$

since area of $S_2 = 4\pi$

On $S_3 (x^2+y^2=4)$. A perpendicular to $x^2+y^2=4$ has the direction $\nabla(x^2+y^2) = 2xi + 2yj$.

(11) ID. 044 (Masud)

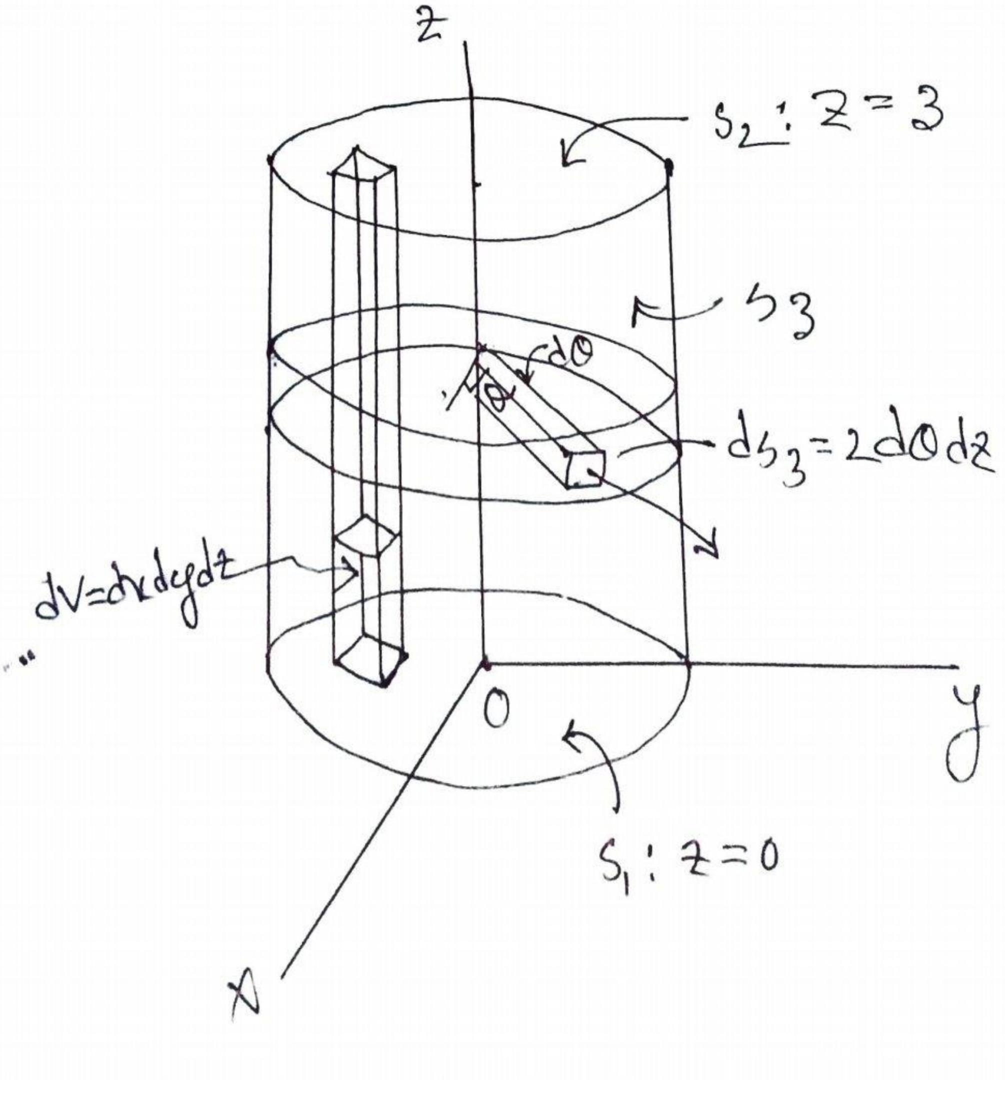
Then a ~~unit~~ unit normal is $n = \frac{2xi + 2yj}{\sqrt{4x^2+4y^2}}$

$= \frac{xi+yj}{2}$

since $x^2+y^2=4$

$A \cdot n = (4xi - 2y^2 j + z^2 k) \cdot \left(\frac{xi+yj}{2}\right)$

$= 2x^2 - y^3$



(12)

ID.044 (Masud)

From the figure above, $x = 2\cos\theta, y = 2\sin\theta$, $ds_3 = 2\,d\theta\,dz$ and so,

$$\iint_{S_3} A \cdot n\, ds_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} \left[2(2\cos\theta)^2 - (2\sin\theta)^3\right] 2\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi} (48\cos^2\theta - 48\sin^3\theta)\,d\theta$$

$$= \int_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi$$

Then the surface integral $= 0 + 36\pi + 48\pi = 84\pi$ agreeing with the volume integral and verifying the divergence theorem.

Note that evaluation of the surface integral over S_3 could also have been done by projection of S_3 on the xz or yz coordinate planes.

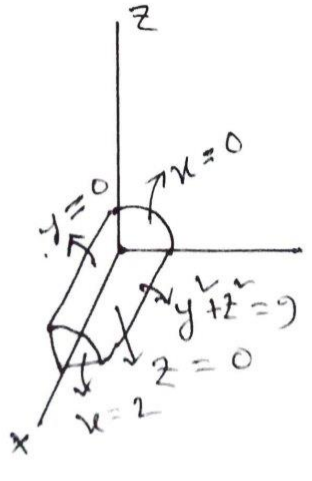
13 ID. 044 (Masud)

(53)
Verify the divergence theorem for $A = 2x^2y\,i - y^2j + 4xz^2k$ taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$

Sol^n:

Divergence theorem;

$$\oiint_S A\cdot ds = \iiint_V (\nabla\cdot A)\,dv$$



curve surface:

$$y^2+x^2=9$$

$$\therefore \phi = y^2+z^2-9$$

$$\nabla\phi = j2y + 2zk$$

$$|\nabla\phi| = \sqrt{4y^2+4z^2}$$

$$= 2\sqrt{y^2+z^2}$$

$$= 2\times 3 = 6$$

(14) ID. 044 (Masud)

$$n = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{1}{6}(2yj + 2zk$$

$$= \frac{1}{3}(yj + zk)$$

$$A\cdot n = (2x^2 y i - y^2 j + 4xz^2 k)\cdot\frac{1}{3}(yj + zk)$$

$$= \frac{1}{3}(-y^3 + 4xz^3)$$

$$\iint_S A\cdot n\, ds = \iint_S \frac{1}{3}(-y^3 + 4xz^3)\frac{dxdy}{|n\cdot k|}$$

$$= \iint \frac{1}{3}(-y^3 + 4xz^3)\cdot\frac{dxdy}{\frac{z}{3}}$$

$$= -\iint \frac{y^3}{z}dxdy + 4\iint xz^2 dxdy$$

$$= -\iint \frac{y^3}{\sqrt{9-y^2}}dxdy + 4\iint x(9-y^2)dxdy \qquad y^2+z^2=9$$

$$= -\int \frac{2y^3}{\sqrt{9-y^2}} + 4\int (9-y^2)\cdot\frac{2^2}{2}dy \qquad z^2 = 9-y^2$$

$$z = \sqrt{9-y^2}$$

$$= -2\int_0^3 \frac{y^3}{\sqrt{9-y^2}}dy + 8\int_0^3 (9-y^2)dy$$

$$= 2\int_3^0 \frac{(9-t^2)t}{t}dt + 8\int_0^3 (9-y^2)dy \qquad 9-y^2=z^2$$

$$-ydy = tdt$$

$$= 2\left[\frac{t^3}{3} - 9t\right]_0^3 + 8\left[9y - \frac{y^3}{3}\right]_0^3$$

15 ID.044 (Masud)

$$2(9-27)+8\times(27-9)$$
$$= 2\times(-18)+8\times 18$$
$$= 18(-2+8)$$
$$= 108$$

plane surface

(i) $n=-i,\ x=0$

$A\cdot n = -2x^2y = 0$

$\iint A\cdot n\,ds = 0$

(ii) $n = i,\ x=2$

$A\cdot n = 2x^2y = 8y$

$$\iint A\cdot n\,ds = \iint 8y\,\frac{dy\,dz}{|n\cdot i|}$$
$$= \int 8y\,[z]_0^{\sqrt{9-y^2}}\,dy$$
$$= 8\int_0^3 y\sqrt{9-y^2}\,dy$$
$$= -8\int_3^0 t\cdot t\,dt$$
$$= \frac{8}{3}\times 3^3$$
$$= 72$$

$9-y^2=t^2$

$-y\,dy = t\,dt$

(16) ID.044 (Masud)

(iii) $n = j$, $y = 0$

$$A \cdot n = y^2 = 0$$
$$\iint A \cdot n\, dS = 0$$

(iv) $n = k$, $z = 0$

$$A \cdot n = -4xz^2 = 0$$
$$\iint \vec{A} \cdot n\, dS = 0$$

Total surface integral $\iint_S A \cdot dS = 108 + 72 = 180$

RHS

$$\nabla \cdot A = \frac{\partial}{\partial x}(2x^2 y) + \frac{\partial}{\partial y}(4xz^2) = 4xy - 2y + 8xz$$

volume integral

$$\iiint_V (\nabla \cdot A)\, dV = \iiint_V (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \iint \left[2x^2 y - 2xy + 4x^2 z\right]_{x=0}^{2} \quad \Big| x : 0 \to 2$$

$$= \iint (8y - 4y + 16z)\, dy\, dz$$

$$= \iint (4y + 16z)\, dy\, dz \quad \Big| \begin{array}{l} z : 0 \to \sqrt{9 - y^2} \\ y : 0 \to 3 \end{array}$$

$$= \int \left. 4yz + 8z^2 \right]_{z=0}^{\sqrt{9-y^2}} dy$$

$$= \int_{y=0}^{3} \left\{4y\sqrt{9 - y^2} + 8(9 - y^2)\right\} dy$$

$$= 4\int_0^3 y\sqrt{9 - y^2}\, dy + 8\int_0^3 (9 - y^2)\, dy$$

17 ID.044 (Masud)

$$= 4\int_0^3 t^2 dt + 8\int_0^3 (9-y^2)\,dy \qquad \left|\begin{array}{l} 9-y^2 = t^2 \\ -y\,dy = t\,dt \end{array}\right.$$

$$= 4\left[\frac{t^3}{3}\right]_0^3 + 8\left(9y - \frac{y^3}{3}\right]_0^3$$

$$= \frac{4}{3}(27) + 8(9\times 3 - 9)$$

$$= 4\times 9 + 8\times 9\times 2$$

$$= \cancel{18}\ 36 + 8\times 18$$

$$= 18(2+8)$$

$$= 18\times 10$$

$$= 180$$

so Divergence theorem is verifi

(54)

Evalute $\iint_S \vec{r}\cdot\hat{n}\,ds$ where (a) s is the sphere of radius 2 with center at $(0,0,0)$, (b) s is the surface of the cube bounded by $x=-1, y=-1, z=-1, x=1, y=1, z=1$ (c) S is the surface bounded by the paraboloid $z = 4-(x^2+y^2)$ and the xy plane

(18) ID. 044 (Masud)

Sol^n: a) $x^2+y^2+z^2=2^2$

$$\iint_S A\cdot ds = \iiint_V \nabla\cdot A \, dv$$
$$= 3\int_V dv$$
$$= 3V$$
$$= 3\times\frac{4}{3}\pi\times 2^3$$
$$= 32\pi$$

$A = r$
$\nabla\cdot r = 3$

b)

$$= 3\int dv$$
$$= 3V$$
$$= 3\times 8 = 24$$

$V = 2^3 = 8$

c) $$\iint A\cdot n\, ds = 3\int_V dv$$
$$= 3\iiint dx\,dy\,dz$$
$$= 3\iint \left[4-(x^2+y^2)\right] dx\,dy$$
$$= 3\times 2\int \left[(4-x^2)y-\frac{y^3}{3}\right]_0^{\sqrt{4-x^2}} dx$$
$$= 6\int (4-x^2)\sqrt{4-x^2}-\frac{(4-x^2)^{3/2}}{3}\, dx$$

$z: 0\to 4-(x^2+y^2)$
$y: \sqrt{4-x^2}\to\sqrt{4-x^2}$
$x: -2\to 2$

(19) ID. 044 (Masud)

$$= 6 \times \frac{2}{3} \int (4 - x^2)^{3/2} dx$$

$$= 8 \int_0^{\pi/2} (4 - 4\sin^2\theta)^{3/2} \cdot 2\cos\theta \, d\theta \quad \begin{array}{|l} x = 2\sin\theta \\ dx = 2\cos\theta \, d\theta \end{array}$$

$$= 16 \int_0^{\pi/2} 2^3 \cos^3\theta \cdot \cos\theta \, d\theta$$

$$= 16 \int_0^{\pi/2} 2 \, (2\cos^2\theta)^2 d\theta$$

$$= 16 \int_0^{\pi/2} 2 \, (1 + \cos 2\theta)^2 d\theta$$

$$= 16 \int_0^{\pi/2} 2 \, (1 + 2\cos 2\theta + \cos^2 2\theta) \, d\theta$$

$$= 16 \int_0^{\pi/2} (2 + 4\cos 2\theta + 2\cos^2 2\theta)$$

$$= 16 \int_0^{\pi/2} (2 + 4\cos 2\theta + 1 + \cos 4\theta) \, d\theta$$

$$= 16 \int_0^{\pi/2} (3 + 4\cos 2\theta + \cos 4\theta) \, d\theta$$

$$= 16 \left[3\theta + 4 \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{4} \right]_0^{\pi/2}$$

$$= 16 \, 3 \cdot \pi/2$$

$$= 24\pi$$

(20) ID. 044 (Masud)

(55)

If S is any closed surface enclosing a volume V and $A = axi + byj + czk$, prove that $\iint_S A \cdot n\, ds = (a+b+c)V$.

Soln:

By Gauss's theorems

$$\iint_S A \cdot ds = \iiint_V \nabla \cdot A\, dv$$

$$\iint_S A \cdot n\, ds = \iiint_V (\nabla \cdot A)\, dv$$

$\because \nabla \cdot A$

$= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$

$= a+b+c$

$= (a+b+c)\int_V dv$

$= a+b+c\, V$

Here proved

Rakibul 045

[Rakibul 045]

19. If div A denotes the divergence of a vector field A at a point P, show that

$$\operatorname{div} A = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} A \cdot n \, ds}{\Delta V}$$

where ΔV is volume enclosed by the surface ΔS and the limit is obtained by shrinking ΔV to the point P.

By the divergence theorem,

$$\iiint_{\Delta V} \operatorname{div} A \, dV = \iint_{\Delta S} A \cdot n \, ds$$

By the mean-value theorem for integrals, the left side can be written

$$\overline{\operatorname{div} A} \iiint_{\Delta V} dv = \overline{\operatorname{div} A} \; \Delta V$$

where $\overline{\operatorname{div} A}$ is some value intermediate between the maximum and minimum of div A throughout ΔV.

Then

$$\overline{\operatorname{div} A} = \frac{\iint_{\Delta S} A \cdot n \, ds}{\Delta V}$$

[Rakibul 045]

Taking the limit as $\Delta V \to 0$ such that P is always interior to ΔV, $\overline{\text{div } A}$ approaches the value div A at point P; hence

$$\text{div} A = \lim_{\Delta V \to 0} \frac{\iint\limits_{\Delta S} A \cdot n \, ds}{\Delta V}$$

This result can be taken as a starting point for defining the divergence of A, and from it all the properties may be derived including proof of the divergence theorem.

Physically,

$$\frac{\iint\limits_{\Delta S} A \cdot n \, ds}{\Delta V}$$

represents the flux or net outflow per unit volume of the vector A from the surface ΔS. If div A is positive in the neighborhood of a point P it means that the out flow from P is positive and we

[Rakibul 045]

can call P a source. Similarly, if div A is negative in the neighborhood of P the outflow is really an inflow and P is called a sink.
If in a region there are no sources or sinks, then div $A = 0$ and we call A a solenoidal vector field.

[Rakibul045]

20. Evaluate $\iint_S r\cdot n\, ds$, where S is a closed surface.

By the divergence theorem,

$$\iint_S r\cdot n\, ds = \iiint_V \nabla\cdot r\cdot dv$$
$$=\iiint_V \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(xi+yj+zk)dv$$
$$=\iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)dv$$
$$=3\iiint dv$$
$$=3V$$

where V is the value enclosed by S.

8

[Rakibul 045]

21.

Prove $\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dv = \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot ds$

Let $A = \phi \nabla \psi$ in the divergence theorem

Then

$$\iiint_V \nabla \cdot (\phi \nabla \psi) dv = \iint_S (\phi \nabla \psi) \cdot n ds = \iint_S (\phi \nabla \psi) \cdot ds$$

But $\nabla \cdot (\phi \nabla \psi) = \phi (\nabla \cdot \nabla \psi) + (\nabla \phi) \cdot (\nabla \psi) = \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)$.

Thus $\iiint_V \nabla \cdot (\phi \nabla \psi) dv = \iiint_V [\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)] dv$

or

$$(1) \iiint_V [\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)] dv = \iint_S (\phi \nabla \psi) \cdot ds$$

which proves Green's first identity. Interchanging ϕ and ψ in 1

$$(2) \iiint_V [\psi \nabla^2 \phi + (\nabla \psi) \cdot (\nabla \phi)] dv = \iint_S (\psi \nabla \phi) \cdot ds$$

Subtracting (2) from (1)

$$(3) \iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dv = \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d$$

[Rakibul 045]

which is Green's second identity or symmetrical theorem. At the proof we have assumed that ϕ and ψ are the scaler functions of position with continous derivatives of the second order at least

[Rakibul 045]

22.

Prove $\iiint_V \nabla\phi \, dv = \iint_S \phi \, n \, ds$

In the divergence theorem, let $A = \phi C$ where C is a constant vector. Then

$$\iiint_V \nabla \cdot (\phi C) \, dv = \iint_S \phi C \cdot n \, ds$$

Since $\nabla \cdot (\phi C) = (\nabla \phi) \cdot C = C \cdot \nabla \phi$ and $\phi C \cdot n = C \cdot (\phi n)$

$$\iiint_V C \cdot \nabla \phi \, dv = \iint_S C \cdot (\phi n) \, ds$$

Taking C outside the integrals,

$$C \cdot \iiint_V \nabla \phi \, dv = C \cdot \iint_S \phi n \, ds$$

and since C is an arbitrary constant vector.

$$\iiint_V \nabla \phi \, dv = \iint_S \phi n \, ds$$

[Rakibul 045]

23.

Prove $\iiint_V \nabla \times B\,dv = \iint_S n \times B\,ds$

In the divergence theorem, let $A = B \times C$ where C is constant vector. Then

$$\iiint_V \nabla (B \times C)\,dv = \iint_S (B \times C) \cdot n\,ds$$

Since $\nabla \cdot (B \times C) = C \cdot (\nabla \times B)$ and

$(B \times C) \cdot n = B \cdot (C \times n) = (C \times n) \cdot B = C \cdot (n \times B)$

$$\iiint_V C \cdot (\nabla \times B)\,dv = \iint_S C \cdot (n \times B)\,ds$$

Taking C outside the intergals,

$$C \iiint_V \nabla \times B\,dv = C \iint_S n \times B\,ds$$

and since C is arbitrary constant vector,

$$\iiint_V \nabla \times B\,dv = \iint_S n \times B\,ds$$

[Rakibul 045]

56. If $H = \text{curl}\, A$, prove that $\iint_S H \cdot n\, ds = 0$ for any closed surface S.

Sol:

$H = \nabla \times A$

$$L.H.S = \iint_S H \cdot n\, ds$$

$$= \iiint_V (\nabla \cdot H)\, dv \quad \left[\because \iint_S A\, ds = \iiint_V (\nabla \cdot A)\, dv\right]$$

$$= \iiint_V \nabla \cdot (\nabla \times A)\, dv$$

$$= \iiint_V 0\, dv \quad [\because \nabla \cdot (\nabla \times A) = 0]$$

$$= 0 \quad [R.H.S]$$

[Proved]

[Rakibul 045]

57. Suppose n is the unit outward drawn normal to any closed surface of area S. Show that $\iiint_V \operatorname{div} n \, dv = S$.

Sol:

$$L.H.S = \iiint_V (\nabla \cdot n) \, dv$$

$$= \iint_S n \cdot n \, ds \quad \left[\because \iiint_V \nabla \cdot A \, dv = \iint_S n \, A \, ds\right]$$

$$= \iint_S ds$$

$$= S \quad \text{[Showed]}$$

[Rakibul 045]

58. Prove $\iiint_V \frac{dv}{r^2} = \iint_S \frac{r.n}{r^2} ds$

<u>Sol:</u>

$$\vec{\nabla}(\phi\vec{A}) = \vec{\nabla}\phi.\vec{A} + \phi\,\vec{\nabla}.\vec{A}$$

$$\vec{\nabla}\left(\frac{1}{r^2}\vec{r}\right) = \vec{\nabla}\left(\frac{1}{r^2}\right)\vec{r} + \frac{1}{r^2}(\vec{\nabla}.\vec{r})$$

$$= \frac{-2}{r^3}.\frac{\vec{r}.\vec{r}}{r} + \frac{3}{r^2}$$

$$= -\frac{2}{r^2} + \frac{3}{r^2}$$

$$= \frac{1}{r^2}$$

L.H.S: $\iiint_V \frac{dv}{r^2}$

$$= \iiint_V \vec{\nabla}\left(\frac{1}{r^2}\vec{r}\right)dv$$

$$= \iint_S \frac{\vec{r}}{r^2}\hat{n}\,ds \quad \left[\because \iiint_V \vec{\nabla}\vec{A} = \iint_S \hat{n}\vec{A}\,ds\right]$$

$= R.H.S$

[Proved]

[Rakibul 045]

58. Prove $\iiint_V \frac{dv}{r^2} = \iint_S \frac{r.n}{r^2} ds$

Sol:

$$\vec{\nabla}(\phi\vec{A}) = \vec{\nabla}\phi.\vec{A} + \phi\,\vec{\nabla}.\vec{A}$$

$$\vec{\nabla}\left(\frac{1}{r^2}\vec{r}\right) = \vec{\nabla}\left(\frac{1}{r^2}\right)\vec{r} + \frac{1}{r^2}(\vec{\nabla}.\vec{r})$$

$$= \frac{-2}{r^3}\cdot\frac{\vec{r}.\vec{r}}{r} + \frac{3}{r^2}$$

$$= -\frac{2}{r^2} + \frac{3}{r^2}$$

$$= \frac{1}{r^2}$$

L.H.S: $\iiint_V \frac{dv}{r^2}$

$$= \iiint_V \vec{\nabla}\left(\frac{1}{r^2}\vec{r}\right) dv$$

$$= \iint_S \frac{\vec{r}}{r^2}\hat{n}\, ds \quad \left[\because \iiint_V \vec{\nabla}\vec{A} = \iint_S \hat{n}\vec{A}\, ds\right]$$

= R.H.S [Proved]

Sakin 046

Math - 205

Chapter - 06

Shakin 046

1

29. Show that at any point P

(a) $$\nabla \phi = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} \phi n \, dS}{\Delta V}$$

and (b) $$\nabla \times A = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} n \times A \, dS}{\Delta V}$$

where ΔV is the volume enclosed by the surface ΔS, and the limit is obtained by shrinking ΔV to the point P.

(a) from problem 22, $\iiint_{\Delta V} \nabla \phi \, dV = \iint_{\Delta S} \phi n \, dS$

Then $\iiint_{\Delta V} \nabla \phi . i dV = \iint_{\Delta S} \phi n . i ds$

using the same principle employed in problem 19,

we have $$\overline{\nabla \phi . i} = \frac{\iint_{\Delta S} \phi n . i ds}{\Delta V}$$

②

where $\overline{\nabla\phi . i}$ is some value intermediate between the maximum and minimum of $\nabla\phi . i$ throughout ΔV. Taking the limit as $\Delta V \to 0$ in such a way that P is always interior to ΔV. $\nabla\phi . i$ approaches the value

$$(1) \quad \nabla\phi . i = \lim_{\Delta V \to 0} \frac{\iint_S \phi n . i\, ds}{\Delta V}$$

Similarly we find

$$(2) \quad \nabla\phi . j = \lim_{\Delta V \to 0} \frac{\iint_S \phi n . j\, ds}{\Delta V}$$

$$(3) \quad \nabla\phi . k = \lim_{\Delta V \to 0} \frac{\iint_S \phi n . j\, ds}{\Delta V}$$

Multiplying (1), (2), (3) by i, j, k respectively, and adding, using

$$\nabla\phi = (\nabla\phi . i) i + (\nabla\phi . j) j + (\nabla\phi . k) k,$$

$$n = (n . i) i + (n . j) j + (n . k) k$$

③

(b) $$\iiint_{\Delta V} \nabla \times A \, dv = \iint_{\Delta S} n \times A \, ds$$

Then as in part (a), we can show that

$$(\nabla \times A) \cdot i = \lim_{\Delta V \to 0} \frac{\iint_{\Delta S} (n \times A) \cdot i \, ds}{\Delta V}$$

and similar results with j and k replacing i. Multiplying by i, j, k and adding, the result follows.

25. Establish the operator equivalence

$$\nabla \circ = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{\Delta S} ds \circ$$

where $\circ$ indicates a dot product, cross product or ordinary product.

To establish the equivalence, the results of the operation on a vector or scalar field must be consistent with already established results.

If $\circ$ is the dot product, then for a vector A,

$$\nabla \circ A = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} ds \circ A$$

or

$$\text{div } A = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} ds . A$$

$$= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} A . n \, ds$$

26. Let S be a closed surface and let r denote the position vector of any point (x, y, z) measured from an origin O. Prove that

$$\iint_S \frac{n . r}{r^3} \, dS$$

is equal to (a) zero if O lies outside S;
(b) 4π if O lies inside S.

This result is known as Gauss' theorem

(5)

(a) By the divergence theorem,

$$\iint_S \frac{n \cdot r}{r^3}\, ds = \iiint_V \nabla \cdot \frac{r}{r^3}\, dv.$$

But $\nabla \cdot \frac{r}{r^3} = 0$ everywhere within V provided

$r \neq 0$ in V,

i.e. provided O is outsid of V and thus

outside of S, Then $\iint_S \frac{n \cdot r}{r^3}\, ds = 0$.

(b) If O is insid S, surround O by a small

sphere s of radius a, Let τ denote

the region bounded by S and s. Then

by the divergence theorem.

$$\iint_{S+s} \frac{n \cdot r}{r^3}\, ds = \iint_S \frac{n \cdot r}{r^3}\, ds + \iint_s \frac{n \cdot r}{r^3}\, ds$$

$$= \iiint_\tau \nabla \cdot \frac{r}{r^3}\, dv = 0$$

6

Since $r \neq 0$ in T. Thus

$$\iint_S \frac{n.r}{r^3}\,ds = -\iint \frac{n.r}{r^3}\,ds$$

Now on S, $r = a$, $n = -\frac{r}{a}$ so that

$$\frac{n.r}{r^3} = \frac{(-r/a).r}{a^3} = -\frac{r.r}{a^4} = -\frac{a^2}{a^4} = \frac{1}{a^2} \quad \text{and}$$

$$\iint_S \frac{n.r}{r^3}\,ds = -\iint_S \frac{n.r}{r^3}\,ds = \iint_S \frac{1}{a^2}\,ds = \frac{1}{a^2}\iint ds$$

$$= \frac{4\pi a^2}{a^2} = 4\pi$$

7

28. A fluid of density $\rho(x, y, z, t)$ moves with velocity $v(x, y, z, t)$. If there are so sources or sinks prove that

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0 \quad \text{where } J = \rho v$$

Consider and arbitrary surface enclosing a volume v of the fluid. At any time the mass of fluid within v is

$$M = \iiint_V \rho \, dv$$

The time rate of increase of this mass is

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \iiint_V \rho \, dv = \iiint_V \frac{\partial \rho}{\partial t} \, dv$$

The mass of fluid per unit time leaving v is

$$\iint_S \rho v \cdot n \, ds$$

8

and the time rate of increase in mass is therefore

$$-\iint_S \rho V . n d = -\iiint_V \nabla . (\rho V) dv$$

by the divergence theorem. Then

$$\iiint_V \frac{\partial \rho}{\partial t} dv = -\iiint \nabla . (\rho V) dv$$

or,

$$\iiint_V \left(\nabla . (\rho V) + \frac{\partial \rho}{\partial t} \right) dv = 0$$

Since V is arbitrary, the integrand, assumed continuous, must be indentically zero, by reasoning similar to that used in problem 12. Then

$$\nabla . J + \frac{\partial \rho}{\partial t} = 0 \quad \text{where } J = \rho V$$

9

The equation is called continuous equation.

If ρ is a constant, the fluid is incompressible

and $\nabla \cdot V = 0$ i.e. v is solenoidal.

The continuity equation also arises in electromagnetic theory, where ρ is the charge density and $J = \rho v$ is the current density.

29. If the temperature at any ~~time~~ point (x, y, z) of a solid at time t is $U(x, y, z, t)$ and if k, ρ and c are respectively the thermal conductivity, density and specific heat of the solid, assumed constant show that

$$\frac{\partial U}{\partial t} = k \nabla^2 U \quad \text{where } k = K/\rho c$$

(18)

Let V be an arbitrary volume lying within the solid, and let S denote its surface. The total flux of heat across S, or the quantity of heat leaving S per unit time, is

$$\iint_S (-k\nabla U).n\, ds$$

Thus the quantity of heat entering S per unit time is

$$(1) \quad \iint_S (k\nabla U).n\, ds = \iiint_V \nabla\cdot(k\nabla U)\, dv$$

by the divergence theorem. The heat contained in a volume V is given by

$$\iiint_V c\rho U\, dv$$

Then the time rate of increase of heat is

$$(2) \quad \frac{\partial}{\partial t}\iiint_V c\rho U\, dv = \iiint_V c\rho \frac{\partial U}{\partial t}\, dv$$

Equating the right hand sides of (1) and (2)

$$\iiint_V \left[c\rho \frac{\partial U}{\partial t} - \nabla\cdot(K\nabla U)\right] dv = 0$$

and since V is arbitrary, the intergand assumed continuous, must be identically zero so that

$$c\rho\frac{\partial U}{\partial t} = \nabla\cdot(K\nabla U)$$

or if K, c, ρ are constants,

$$\frac{\partial U}{\partial t} = \frac{K}{c\rho}\nabla\cdot\nabla U = k\nabla^2 U$$

The quantity k is called the diffusivity.
For steady-state heat flow (i.e. $\frac{\partial U}{\partial t} = 0$ or U is independent of time) the equation reduces to Laplace's equation $\nabla^2 U = 0$.

Ashiq-047

(30)

(a) Express Stokes theorem in words and

(b) Write it in rectangular form.

Soln:

(a) The line integral of the tangential component of a vector A taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of A taken over any surface S having C as its boundary.

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(b) Let $A = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$n = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

Then,

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j}$$

$$+ \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

$$(\vec{\nabla}\times\vec{A})\cdot n = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta$$

$$+ \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma.$$

~~A dx~~

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$$\vec{A}\cdot d\vec{r} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= A_1 dx + A_2 dy + A_3 dz$$

and Stokes' theorem becomes

$$\iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_2}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma\right] ds$$

$$\oint_C A_1 dx + A_2 dy + A_3 dz \quad \text{(rectangular form)}$$

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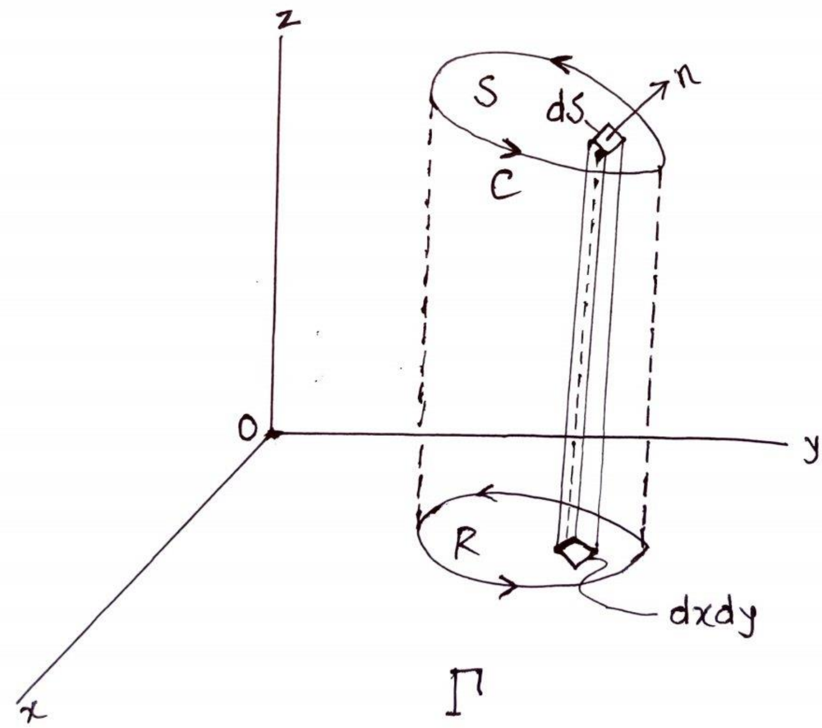
(31)

Prove Stokes' theorem

Proof:



Let S be the surface which is such that its projections on the xy, yz, and xz planes

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are regions bounded by simple closed curves, as indicated in the adjoining figure. Assume S to have representation $z = f(x, y)$ or $x = g(y, z)$ or $y = h(x, z)$, where f, g, h are single valued, continuous and differentiable function. We must show that

$$\iint_S (\vec{\nabla} \times \vec{A}) \cdot n \, dS = \iint_S \left[\vec{\nabla} \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\right] \cdot \hat{n} \, dS$$

$$= \oint_C \vec{A} \cdot d\vec{r}$$

Where C is the boundary of S.

Consider first $\iint_S \left[\vec{\nabla} \times (A_1\hat{i})\right] \cdot \hat{n} \, dS$.

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Since $\nabla\times(A_1 i) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z}\hat{j} - \frac{\partial A_1}{\partial y}\hat{k}$

(1) $\quad [\vec{\nabla}\times(A_2 i)].n\,ds = \left(\frac{\partial A_1}{\partial z} n.\hat{j} - \frac{\partial A_1}{\partial y} n.\hat{k}\right) ds$

If $z = f(x,y)$ is taken as the equation of S. Then the position vector to any point of S is $r = x\hat{i} + x\hat{j} + \cancel{x} z\hat{k} = x\hat{i} + y\hat{j} + f(x,y)k$ so that $\frac{\partial r}{\partial y}\cancel{k} = \hat{j} + \frac{\partial x}{\partial y}\hat{k} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$. But $\frac{\partial r}{\partial y}$ is a vector tangent to S and thus perpendicular to n. So that,

$$n.\frac{\partial r}{\partial y} = n.\hat{j} + \frac{\partial x}{\partial y}\check{n}.\hat{k} = 0$$

or, $\quad n.j = -\frac{\partial z}{\partial y}\,n.K$

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Substitute in (1) to obtain

$$\left(\frac{\partial A_1}{\partial z}\hat{n}\cdot\hat{j} - \frac{\partial A_1}{\partial y}\hat{n}\cdot\hat{k}\right)ds = \left(-\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}n\cdot k - \frac{\partial A_1}{\partial y}n\cdot k\right)ds$$

or,

(2) $$\left[\vec{\nabla}\times(A_1\hat{i})\right]\cdot n\,ds = -\left(\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\right)n\cdot k\,ds$$

Now on S, $A_1(x,y,z) = A_1(x,y,f(x,y)) = F(x,y)$;

Hence $\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$ and (2) become

$$\left[\vec{\nabla}\times(A_1\hat{i})\right]\cdot n\,ds = -\frac{\partial F}{\partial y}n\cdot k\,ds = -\frac{\partial F}{\partial y}dx\,dy$$

Then

$$\iint_S \left[\vec{\nabla}\times(A_1\hat{i})\right]\cdot\hat{n}\,ds = \iint_R -\frac{\partial F}{\partial y}dx\,dy$$

Where R is the projection of S on the xy plane. By Green's theorem for the plane the last

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integral equals $\oint F\,dx$ where Γ is the boundary of R. Since at each point (x,y) of Γ the value of F is the same as the value of A_1 at each point (x,y,z) of C, and since dx is the same for both curves, we must have

$$\oint_\Gamma F\,dx = \oint_C A_1\,dx$$

or,

$$\iint_S [\vec{\nabla}\times(A_1\hat{i})]\cdot\hat{n}\,ds = \oint_\Gamma A_1\,dx$$

Similarly, by projections on the other coordinate planes,

$$\iint_S [\vec{\nabla}\times(A_2 j)]\cdot\hat{n}\,ds = \oint_C A_2\,dy$$

$$\iint_S [\vec{\nabla}\times(A_3 K)]\cdot n\,ds = \oint_C A_3\,dz$$

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Thus by addition,

$$\iint_S (\vec{\nabla} \times \vec{A}) \cdot n \, ds = \oint_C \vec{A} \cdot d\vec{r}$$

The theorem is also valid for surfaces S which may not satisfy the restrictions imposed above. For assume that S can be subdivided into surfaces $S_1, S_2, \cdots S_K$ with boundaries $C_1, C_2, \cdots C_K$ which do satisfy the restrictions. Then Stoke's theorem holds for each such surface. Adding these surface integrals, the total surface integral over S is obtained. Adding the corresponding line integrals over $C_1, C_2, \cdots C_K$, the line integral over C is obtained.

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32

Verify Stoke's theorem for $A = (2x-y)i - yz^2 j - y^2 z k$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

Solⁿ:

The boundary C of S is a circle in the xy plane of radius one and center at the origin.

Let $x = \cos t$, $y = \sin t$, $z=0$, $0 \leqq t < 2\pi$ be parametric equations of C. Then

$$\oint_C \vec{A}\cdot d\vec{r} = \oint_C (2x-y)\,dx - yz^2 dy - y^2 z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \pi$$

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Also.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = K$$

Then,

$$\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, ds = \iint_S k \cdot n\, ds = \iint_R dx\, dy$$

Since $n \cdot k\, ds = dx dy$ and R is the projection of S on the xy plane. This last integral equals

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx = 4\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = 4\int_0^1 \sqrt{1-x^2}\, dx$$

$$= \pi$$

and Stoke's' theorem is verified.

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(63)

Verify Stokes' theorem for $A=(y-z+2)\hat{i}+(yz+4)\hat{j}-xz\hat{k}$, where S is the surface of the cube $x=0$, $y=0$, $z=0$, $x=2$, $y=2$, $z=2$ above the xy plane.

Soln:

Has given,

$$A=(y-z+2)\hat{i}+(yz+4)\hat{j}-xz\hat{k}.$$

from Stokes' theorem,

$$\oint_c \tilde{A}\cdot d\tilde{r} = \iint_S (\tilde{\nabla}\times\tilde{A})\cdot\hat{n}\,ds$$

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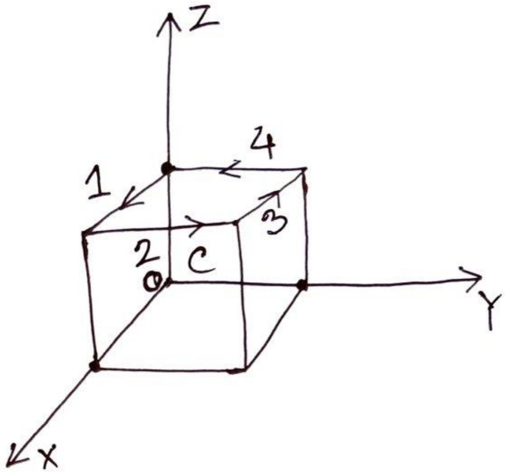

For Path (1)

$y = 0,\ dy = 0$

$x : 0 \rightarrow 1$

$z = 2,\ dz = 0$

Now,

$$\vec{A}\cdot d\vec{r} = (y-z+2)dx + (yz+4)dy - xz\,dz$$

$$= (y-z+2)dx$$

$$= (0-2+2)dx = 0$$

$$\therefore \int_1 \vec{A}\cdot d\vec{r} = 0$$

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For Path ②

$y: 0 \longrightarrow 2$

$x = 2, \ dx = 0$

$z = 2, \ dz = 0$

Since $dx = 0, dz = 0$.

$$\int (yz + 4)\, dy$$

$$= \int_0^2 (2y + 4)\, dy$$

$$= \left[y^2 + 4y \right]_0^2$$

$$= 4 + 8$$

$$= 12$$

$$\therefore \int_2 \tilde{A} \cdot d\tilde{r} = 12$$

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For Path 3

$x: 2 \longrightarrow 0$

$y = 2, \; dy = 0$

$z = 2, \; dz = 0$

Since, $dy = 0, \; dz = 0$

$$\int (y - z + 2)\, dx$$

$$= 2 \int_2^0 dx$$

$$= 2(-2)$$

$$\int_3 \vec{A} \cdot d\vec{r} = -4$$

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For Path 4

$y : 2 \to 0$

$x = 0, \ dx = 0$

$z = 2, \ dz = 0$

since $dx = 0, \ dz = 0,$

$$\int (yz+4)\, dy$$

$$= \int_2^0 (2y+4)\, dy$$

$$= \left[y^2 + 4y \right]_2^0$$

$$= -4 - 8$$

$$\int_4 \vec{A} \cdot d\vec{r} = -12$$

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So, the total integration of Paths are

$$\oint_C \tilde{A} \cdot d\tilde{r} = 0 + 12 - 4 - 12 = -4$$

and

$$\tilde{\nabla} \times \tilde{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= \hat{i}\left(0-y\right) + \hat{j}\left(-1+z\right) + \hat{k}\left(0-1\right)$$

$$= -y\hat{i} + (z-1)\hat{j} - \hat{k}$$

So,

$$(\vec{\nabla} \times \vec{A}) \cdot \hat{n} = (\vec{\nabla} \times \vec{A}) \cdot \hat{k} = -1$$

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$$\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} \, ds = -1 \iint \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= -\iint dx\,dy$$

$$= -\int 2dy$$

$$= -2 \times 2$$

$$= -4$$

Since, $\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \hat{n} \cdot ds$

So, Stokes' theorem theorem is valid.

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64

Verify stokes' theorem for $F = xzi - yj + x^2yk$ where S is the surface of the region bounded by $x=0$, $y=0$, $z=0$, $2x+y+2z=8$ which is not included in the xz-plane.

Soln:

Has given,

$$F = xz\hat{i} - y\hat{j} + x^2y\hat{k}$$

∵ Stokes' theorem,

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$$

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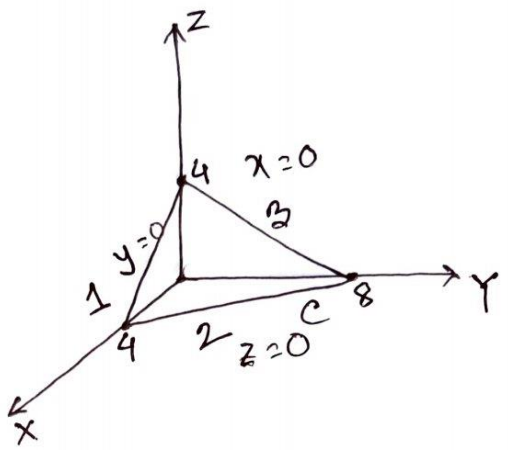

For Path (1)

$y=0,\ dy=0$

$x=0 \to 4$

$z: 4 \to 0$

$2x+y+2z=8$

$\frac{x}{4}+\frac{y}{8}+\frac{z}{4}=1$

$$\vec{A}\cdot d\vec{r} = xz\,dx - y\,dy + x^2y\,dz$$

$\frac{x}{4}+\frac{z}{4}=1$

$x+z=4$

$$\int_1 \vec{A}\cdot d\vec{r} = \int xz\,dx$$

$$= \int_0^4 x(4-z)\,dx$$

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$$= \int_0^4 x(4-x)\,dx$$

$$= \int_0^4 (4x - x^2)\,dx$$

$$= 2x^2 - \frac{x^3}{3}\Big|_0^4$$

$$= 32 - \frac{64}{3} = \frac{32}{3}$$

For Path ②

$z = 0, \; dz = 0$

$y : 0 \rightarrow 8$

Since, $z = 0$

$$\frac{x}{4} + \frac{y}{8} = 1$$

$$2x + y = 8$$

$$\int_2 \vec{A} \cdot d\vec{r} = \int xz\,dx - y\,dy$$

$$= -\int_0^8 y\,dy = -\frac{8^2}{2} = -32$$

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For Path (3)

$x=0, \quad dx=0$

$y: 8 \to 0$

Since, $x=0$.

$\frac{z}{4}+\frac{y}{8}=1$

$2z+y=8$

$$\int_3 \tilde{A}\cdot d\tilde{r} = \int xz\,dx - y\,dy$$

$$= -\int_8^0 y\,dy$$

$$= -\left(\frac{-8^2}{2}\right) = 32$$

So, the total of Paths are,

$$\oint \tilde{A}\cdot d\tilde{r} = \frac{32}{3} - 32 + 32 = \frac{32}{3}$$

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Now,

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y & x^2 y \end{vmatrix}$$

$$= \hat{i}\,(x^2 - 0) + \hat{j}\,(x - 2xy) + \hat{k}\,(0-0)$$

$$= x^2\hat{i} + (x - 2xy)\hat{j}$$

$$\hat{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|}$$

$$\hat{n} = \frac{1}{3}\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$$

$$2x + y + 2z = 8$$

$$\varphi = 2x + y + 2z - 8$$

$$\vec{\nabla}\varphi = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{\nabla}\varphi = \sqrt{4+1+4} = 3$$

$$\therefore (\vec{\nabla} \times \vec{A}) \cdot \hat{n}$$

$$= \frac{1}{3}\left(2x^2 + x - 2xy\right)$$

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So,

$$\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} \frac{dx\,dy}{(\hat{n} \cdot \hat{k})} = \iint \frac{1}{3}\left(2x^2 + x - 2xy\right) \frac{dx\,dy}{2/3}$$

$$= \iint \left(x^2 + \frac{x}{2} - xy\right) dx\,dy$$

$$\therefore \iint \left(x^2 + \frac{x}{2} - xy\right) dx\,dy$$

$$= \int_{x=0}^{4} \left[x^2 y + \frac{x}{2} y - \frac{xy^2}{2}\right]_{y=0}^{8-2x} dx$$

$2x + y + 2z = 8$

$2x + y = 8$

$y = 8 - 2x$

$x = 4$

$$= \int_{x=0}^{4} \left\{x^2(8-2x) + \frac{x}{2}(8-2x) - \frac{x}{2}(8-2x)^2\right\} dx$$

$$= \int_{x=0}^{4} \left\{8x^2 - 2x^2 + 4x - x^2 - \frac{x}{2}(64 - 32x + 4x^2)\right\} dx$$

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$$= \int_{x=0}^{4} \left(7x^2 - 2x^3 + 4x - 32x + 16x^2 - 2x^3\right) dx$$

integration with respect to x,

$$= \int_{x=0}^{4} \left(-4x^3 + 23x^2 - 28x\right) dx$$

$$= \left[-x^4 + \frac{23}{3}x^3 - 14x^2\right]_0^4$$

$$= -4^4 + \frac{23}{3} \times 4^3 - 14 \times 4^2$$

$$= \frac{32}{3}$$

Since,

$$\cancel{\oint \vec{A} \times d\vec{r}}$$

$$\oint \vec{A} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$$

So, Stokes' theorem is valid.

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65

Evaluate $\iint_S (\nabla \times A) \cdot n \, ds$, where $A = (x^2+y-4)i + 3xyj + (2xz+z^2)k$ and S is the surface of

(a) the hemisphere $x^2+y^2+z^2=16$ above the xy-plane

(b) the paraboloid $z=4-(x^2+y^2)$ above the xy-plane.

Sol^n:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0-2z) + \hat{k}(3y-1)$$

$$= -2z\hat{j} + (3y-1)\hat{k}$$

(a) $x^2+y^2+z^2=16$, $\varphi = x^2+y^2+z^2-16$

$$\vec{\nabla}\varphi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

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$$|\vec{\nabla}\varphi| = 2\sqrt{x^2+y^2+z^2}$$

$$= 2\sqrt{16} = 2\times 4$$

$$\hat{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|} = \frac{1}{4}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$(\vec{\nabla}\times\vec{A})\cdot\hat{n} = \frac{1}{4}\left(-2yz+z(3y-1)\right)$$

$$= \frac{1}{4}(yz-z)$$

Now,

$$\iint (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,ds = \iint \frac{1}{4}(yz-z)\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$x^2+y^2+\overset{\to 0}{z^2}=16$

$x^2+y^2=16$

$x^2=16-y^2$

$x=\sqrt{16-y^2}$

$y: -4\to 4$

$$= \iint \frac{1}{4}(yz-z)\frac{dx\,dy}{\frac{z}{4}}$$

$$= \iint (y-1)\,dx\,dy$$

$$= \int 2\int_0^{\sqrt{16-y^2}} (y-1)\,dx\,dy$$

$$= 2\int_{-4}^{4} (y-1)\sqrt{16-y^2}\,dy$$

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$$= 2\int_{-4}^{4}\left(\underbrace{y\sqrt{16-y^2}}_{\text{odd}=0} - \underbrace{\sqrt{16-y^2}}_{\text{even}}\right)dy$$

$$= -2\cdot 2\int_0^4 \sqrt{16-y^2}\,dy$$

$$= -4\left[\frac{y}{2}\sqrt{16-y^2} + \frac{16}{2}\sin^{-1}\frac{y}{4}\right]_0^4$$

$$= -4 \times \frac{16}{2} \times \frac{\pi}{2}$$

$$= -16\pi \text{ (Ans)}$$

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(b) $\varphi = x^2+y^2+z-4$, $\cancel{\Delta}\ \Delta\varphi$

$$\nabla\varphi = 2x\hat{i} + 2y\hat{j} + \hat{k}$$

$$|\vec{\nabla}\varphi| = \sqrt{4(x^2+y^2)+1}$$

$$= \sqrt{4(4-z)+1} = \sqrt{17-4z}$$

$x^2+y^2+z=4$

$x^2+y^2=4-z$

$$\hat{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|} = \frac{1}{\sqrt{17-4z}}\left(2x\hat{i} + 2y\hat{j} + \hat{k}\right)$$

$$(\vec{\nabla}\times\vec{A})\cdot\hat{n} = \left\{-2z\hat{j} + (3y-1)\hat{k}\right\}\cdot\left\{\frac{1}{\sqrt{17-4z}}\left(2x\hat{i}+2y\hat{j}+\hat{k}\right)\right\}$$

$$= \frac{1}{\sqrt{17-4z}}\left(-4yz+3y-1\right)$$

Now,

$$\iint(\vec{\nabla}\times\vec{A})\cdot\hat{n}\,ds = \iint\frac{1}{\sqrt{17-4z}}\left(-4yz+3y-1\right)\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\frac{1}{\sqrt{17-4z}}$$

$$= \iint\left(-4yz+3y-1\right)dx\,dy$$

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$$= \iint \left\{-4y\left(4-x^2-y^2\right)+3y-1\right\} dx\,dy$$

$$= \iint \left(-16y+4x^2y+4y^3+3y-1\right) dx\,dy$$

$$= \iint \left(4y^3+4x^2y-13y-1\right) dx\,dy \longleftarrow (1)$$

put the value of $x^2+y^2=4$, $x=r\cos\theta$, $y=r\sin\theta$, $dx\,dy = r\,d\theta\,dr$. in equation (1)

$$= \iint \left(4\cdot r^3\sin^3\theta + 4\cdot r^2\cos^2\theta\cdot r\sin\theta - 13\cdot r\sin\theta - 1\right) r\,dr\,d\theta$$

$$= \iint \left\{ r^4\left(\underset{0}{3\sin\theta} - \underset{0}{\sin 3\theta}\right) - 4r^4 \cos^2\theta\, d(\cos\theta) - 13r^2\underset{0}{\sin\theta\, d\theta} - r\right\} dr\,d\theta$$

$\theta : 0 \to 2\pi$

$r : 0 \to 2$

$$= -\iint r\,dr\,d\theta = -\frac{2^2}{2}\cdot 2\pi$$

$$= -4\pi \text{ (Ans)}$$

NAHID 048

33. Prove that a necessary and sufficient condition that $\oint_C A\cdot dr = 0$ for every closed curve C is that $\nabla \times A = 0$ identically.

Solⁿ:

Sufficiency. Suppose $\nabla \times A = 0$. Then by Stokes' theorem

$$\oint_C A\cdot dr = \iint_S (\nabla\times A)\cdot n \, dS = 0$$

Necessity. Suppose $\oint_C A\cdot dr = 0$ around every closed path C, and assume $\nabla\times A \neq 0$ at some point P. Then assuming $\nabla\times A$ is continuous there will be a region with P as an interior point, where $\nabla\times A \neq 0$. Let S be a surface contained in this region whose normal n at each point has the same direction as $\nabla\times A$, i.e. $\nabla\times A = \alpha n$ where α is a positive constant. Let C be the boundary of S. Then by Stoke's theorem

$$\oint_C A\cdot dr = \iint_S (\nabla\times A)\cdot n \, dS = \alpha \iint_S n\cdot n \, dS > 0$$

Which contradicts the hypothesis that $\oint_C A\cdot dr = 0$ and show that $\nabla\times A = 0$.

It follows that $\nabla\times A = 0$ is also a necessary and sufficient condition for line integral $\int_{P_1}^{P_2} A\cdot dr$ to be independent of the path joining points P_1 and P_2.

NAHID 048

34. Prove $\oint dr \times B = \iint_S (n \times \nabla) \times B \, dS$.

Soln:

In Stokes' theorem, Let $A = B \times C$ Where C is a constant vector, then

$$\oint dr \cdot (B \times C) = \iint_S [\nabla \times (B \times C)] \cdot n \, dS$$

$$\oint C \cdot (dr \times B) = \iint_S [(C \cdot \nabla) B - C(\nabla \cdot B)] \cdot n \, dS$$

$$C \cdot \oint dr \times B = \iint_S [(C \cdot \nabla) B] \cdot n \, dS - \iint_S [C(\nabla \cdot B)] \cdot n \, dS$$

$$= \iint_S C \cdot [\nabla (B \cdot n) \, dS - \iint_S C \cdot [n(\nabla \cdot B)] \, dS$$

$$= C \cdot \iint_S [\nabla (B \cdot n) - n(\nabla \cdot B)] \, dS = C \cdot \iint_S (n \times \nabla) \times B \, dS$$

Since C is an arbitrary constant vector $\oint dr \times B = \iint_S (n \times \nabla) \times B \, dS$

NAHID 048

35. If ΔS is a surface bounded by a simple closed C, P is any point of ΔS not on C and n is a unit normal to ΔS at P. Show that at P

$$(\text{curl } A) \cdot n = \lim_{\Delta S \to 0} \frac{\oint_C A \cdot dr}{\Delta S}$$

Where the limit is taken in such a way that ΔS shrinks to P.

Soln: By Stokes' theorem, $\iint_{\Delta S} (\text{curl } A) \cdot n \, dS = \oint_C A \cdot dr$.

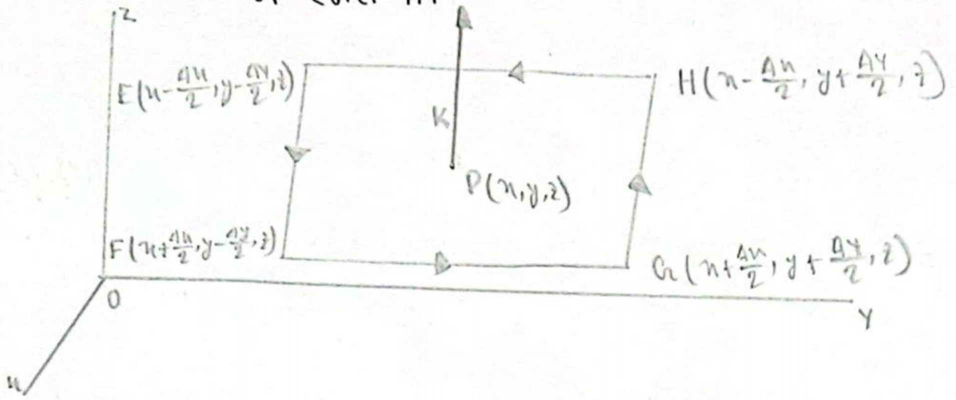
Using the mean value theorem for integrals, this can be written

$$\overline{(\text{curl } A) \cdot n} = \frac{\oint_C A \cdot dr}{\Delta S}$$

and the required result follows upon taking the limit as $\Delta S \to 0$. This can be used as a starting point for defining curl A and is useful in obtaining curl A in coordinate systems other than rectangular. Since $\oint_C A \cdot dr$ is called the circulation of A about C, the normal component of the curl can be interpreted physically as the limit of the circulation per unit area, thus accounting for the synonym rotation of A (rot A) instead of curl of A.

NAHID 048

36. If Curl A is defined according to the limiting, find the z component of curl A.



Let EFGH be a rectangle parallel to the xy plane with interior point $P(x,y,z)$ taken as midpoint, as shown in the figure above. Let A_1 and A_2 be the components of A at P in the positive x and directions respectively.

If c is the boundary of the rectangle, then

$$\oint_C A\cdot dr = \int_{EF} A\cdot dr + \int_{FG} A\cdot dr + \int_{GH} A\cdot dr + \int_{HE} A\cdot dr$$

But,

$$\int_{EF} A\cdot dr = \left(A_1 - \frac{1}{2}\frac{\partial A_1}{\partial y}\Delta y\right)\Delta x \qquad \int_{GH} A\cdot dr = -\left(A_1 + \frac{1}{2}\frac{\partial A_1}{\partial y}\Delta y\right)\Delta x$$

$$\int_{FG} A\cdot dr = \left(A_2 + \frac{1}{2}\frac{\partial A_2}{\partial x}\Delta x\right)\Delta y \qquad \int_{HE} A\cdot dr = -\left(A_2 - \frac{1}{2}\frac{\partial A_2}{\partial x}\Delta x\right)\Delta y$$

except for infinitiesimals of higher order than $\Delta x \Delta y$.

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NAHID 048

Adding, We have approximately $\oint_C A \cdot dr = \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \Delta x \Delta y$

Then, Since $\Delta S = \Delta x \Delta y$.

z component of curl A $= (\text{curl} A) \cdot K = \lim_{\Delta S \to 0} \frac{\oint A \cdot dr}{\Delta S}$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \Delta x \Delta y}{\Delta x \Delta y}$$

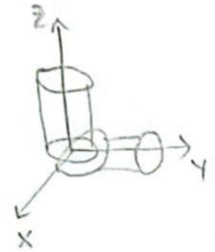
$$= \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$

NAHID048

66. If $A = 2yz\,i - (x+3y-2)\,j + (x^2+z)\,K$, evaluate $\iint_S (\nabla \times A)\cdot n \, ds$ over the Surface of intersection of the cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$ which is included in the first octant.

Sol^n: Given that,

$$A = 2yz\,i - (x+3y-2)\,j + (x^2+z)\,K$$

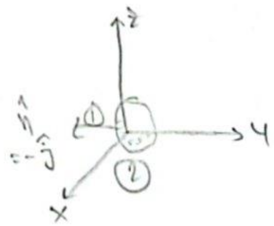


Now,

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{K} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2yz & -(x+3y-2) & x^2+z \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(2y-2x) + \hat{K}(-1-2z)$$

$$= 2(y-x)\hat{j} - (1+2z)\hat{K}$$



For Surface ① $\hat{n} = -\hat{j}$

$$(\vec{\nabla}\times\vec{A})\cdot\hat{n} = -2(y-x)$$

then,

$$\iint (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, ds$$

$$= -\iint 2(y-x)\,\frac{dx\,dz}{(\hat{n}\cdot\hat{j})}$$

$$= +\iint 2x \, dx\, dz$$

$$= \int 2x \,[z]_0^{\sqrt{a^2-x^2}}\, dx$$

$x^2+z^2=a^2$

$z=\sqrt{a^2-x^2}$

$$= \int_0^a 2x\sqrt{a^2-x^2}\, dx$$

NAHID 048

$$= -2\int_a^0 t \cdot t \, dt$$

$$= -2\left(0 - \frac{a^3}{3}\right)$$

$$= \frac{2}{3}a^3$$

put

$a^2 - x^2 = t^2$

$-x\,dx = t\,dt$

x	0	a
t	a	0

For region ②

$\hat{n} = -\hat{k}$

$(\vec{\nabla} \times \vec{A}) \cdot \hat{n} = 1 + 2z$

$$\iint (1 + 2z)\, dx\, dy$$

$$= \iint dx\, dy$$

$$= \int [y]_0^{\sqrt{a^2 - x^2}}\, dx$$

$x^2 + y^2 = a^2$

$y = \sqrt{a^2 - x^2}$

$$= \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1} x/a\right]_0^a$$

$$= \frac{a^2}{2} \cdot \sin^{-1} 1 = \frac{\pi a^2}{4}$$

NAHID 048

67. A Vector B is always normal to a given closed Surface S. Show that $\iiint_V \text{curl } B \, dv = 0$, Where V is the region bounded by S.

Soln:

$$\iiint_V (\vec{\nabla} \times \vec{A})\, dv = \iint_S (\hat{n} \times \vec{B})\, dS$$

$$\iiint_V (\vec{\nabla} \times \vec{B})\, dv = \iint_S (\hat{n} \times \vec{B})\, dS$$

$\hat{n} \parallel \hat{B}$

$\because \hat{n} \times \vec{B} = 0$

$$= \iint 0$$

$$= 0$$

(Show that)

NAHID 048

68. If $\oint_C E \cdot dr = -\frac{1}{c} \frac{\partial}{\partial t} \iint_S H \cdot dS$, where S is any surface bounded by the curve C, show that $\nabla \times E = -\frac{1}{c} \frac{\partial H}{\partial t}$.

Soln: In Stock's theorem $\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S}$

Given, $\iint_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\frac{1}{c} \frac{\partial}{\partial t} \iint_S \vec{H} \cdot d\vec{S}$

$\Rightarrow \iint_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = \iint_S \left(-\frac{1}{c} \frac{\partial \vec{H}}{\partial t}\right) \cdot d\vec{S}$

$\Rightarrow (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t} \cdot d\vec{S}$

$\therefore \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$

(Show that)