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CHAPTER FOUR
RANK OF A MATRIX

CHAPTER FIVE
VECTORS IN IR^n AND C^n

CHAPTER FOUR

RANK OF A MATRIX

4.1 Necessary definitions applied in Rank

Echelon Matrix

An **echelon matrix** is a matrix which have the property that if in any of its rows the first element distinct from zero is in the kth position, then in all the following rows there are zeros in the first k positions. or, equivalently

A matrix $A = [a_{ij}]$ is an **echelon matrix** if the number of zeros preceding the first non-zero entry of a row increases row by row until only zero rows remain. or, equivalently

A matrix is said to be in **echelon form** if it satisfies the following two properties :

(i) The first k rows are non-zero; the other rows are zero

(ii) The first non-zero element in each non-zero row is 1 and it appears in a column to the right of the first non-zero element of any preceding row. A matrix which is in echelon form and the first non-zero element in each non-zero row is the only non-zero element in its column is said to be in **reduced echelon form.**

Examples of echelon matrices and matrices of reduced echelon form are given below :

(i) $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (Echelon matrix)

(ii) $\begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (Echelon matrix)

(iii) $\begin{bmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$ (Reduced echelon form)

(iv) $\begin{bmatrix} 0 & 1 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (Reduced echelon form)

where each * denotes some scalar.

Canonical Matrix

A canonical matrix is one in which all terms not of the principal diagonal are zero, all terms on the principle diagonal are zero or one, and all ones precedes all zeros.

For instance $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a canonical matrix.

Elementary Transformation (or operation)

An **elementary transformation** of a matrix is any one of the following operations :

(i) The multiplication of each element of a row or a column by some non-zero constant,

(ii) The interchange of two rows or two columns,

(iii) The addition of any multiple of the elements of one row or one column to the corresponding elements of another row or column respectively.

Equivalent Matrix

Two matrices A and B are called **equivalent** and is written as $A \sim B$ if one can be obtained from the other by means of a finite number of elementary transformations.

If a matrix A is reduced to B by the use of elementary row transformations alone; then B is said to be **row equivalent** to A and conversely.

4.2 Trace of a matrix and Properties of trace

Definition. Let A be a square matrix of order n

$$\text{say } A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Then the sum of the diagonal elements of A is called the **trace** of A and is written as $tr(A) = a_{11} + a_{22} + \dots + a_{nn} = \sum_{i=1}^{n} a_{ii}$

Properties of trace :

(1) If $A = [a_{ij}]_{n \times n}$ and λ is a scalar (real number) then $tr(\lambda A) = \lambda\, tr(A)$

Proof : $tr(\lambda A) = \sum_{i=1}^{n} \lambda a_{ii} = \lambda \sum_{i=1}^{n} a_{ii} = \lambda\, tr(A)$.

(2) Let A and B be n xn matrices. Then
$tr(\alpha A + \beta B) = \alpha tr(A) + \alpha tr(B)$ where α and β are scalars.

Proof : $\alpha tr(A) = \alpha \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \alpha a_{ii}$

$$\beta tr(B) = \beta \sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n} \beta b_{ii}$$

Now since multiplying a matrix by a scalar number is equivalent to multiplying every element of the matrix by that number, we have

$$\alpha tr(A) + \beta tr(B) = \sum_{i=1}^{n} \alpha a_{ii} + \sum_{i=1}^{n} \beta b_{ii}$$

$$= \sum_{i=1}^{n} (\alpha a_{ii} + \beta b_{ii}) = tr(\alpha A + \beta B).$$

For the special case where $\alpha = \beta = 1$, we have

$tr(A + B) = tr(A) + tr(B)$.

Similarly, we have $tr(\alpha A - \beta B) = \alpha tr(A) - \beta tr(B)$

(3) If A and B are matrices such that AB and BA co-exist then $tr(AB) = tr(BA)$.

Proof : Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times m}$

such that AB and BA co-exist

$$L.H.S = tr(AB) = tr\{[a_{ij}]_{m\times n} [b_{ij}]_{n\times m}\}$$

$$= tr\{\sum_{k=1}^{n} [a_{ik} b_{kj}]_{m\times m}\}$$

$$= tr[c_{ij}]_{m\times m} \text{ where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{i=1}^{m} c_{ii} = \sum_{i=1}^{m} [\sum_{k=1}^{n} a_{ik} b_{ki}]$$

$$= \sum_{i=1}^{m} [a_{i1} b_{1i} + a_{i2} b_{2i} + \dots + a_{in} b_{ni}]$$

$$\text{R. H. S} = \operatorname{tr}(BA) = \operatorname{tr} [b_{ij}]_{n \times m} [a_{ij}]_{m \times n}$$

$$= \operatorname{tr} \{ \sum_{k=1}^{m} [b_{ik} a_{kj}]_{n \times n} \}$$

$$= \operatorname{tr} [d_{ij}]_{n \times n} \text{ where } d_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$$

$$= \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} [\sum_{k=1}^{m} b_{ik} a_{ki}]$$

$$= \sum_{i=1}^{n} [b_{i1} a_{1i} + b_{i2} a_{2i} + \dots + b_{im} a_{mi}]$$

$$\begin{aligned}
&= b_{11} a_{11} + b_{21} a_{12} + \dots + b_{n1} a_{1n} \\
&+ b_{12} a_{21} + b_{22} a_{22} + \dots + b_{n2} a_{2n} \\
&+ \dots \quad \dots \quad \dots \quad \dots \\
&+ b_{1m} a_{m1} + b_{2m} a_{m2} + \dots + b_{nm} a_{mn} \\
&= b_{11} a_{11} + b_{12} a_{21} + \dots + b_{1m} a_{m1} \\
&+ b_{21} a_{12} + b_{22} a_{22} + \dots + b_{2m} a_{m2} \\
&+ \dots \quad \dots \quad \dots \quad \dots \\
&+ b_{n1} a_{1n} + b_{n2} a_{2n} + \dots + b_{nm} a_{mn} \\
&= a_{11} b_{11} + a_{21} b_{12} + \dots + a_{m1} b_{1m} \\
&+ a_{12} b_{21} + a_{22} b_{22} + \dots + a_{m2} b_{2m} \\
&+ \dots \quad \dots \quad \dots \quad \dots \\
&+ a_{1n} b_{n1} + a_{2n} b_{n2} + \dots + a_{mn} b_{nm}
\end{aligned}$$

$$= \sum_{i=1}^{m} [a_{i1} b_{1i} + a_{i2} b_{2i} + \dots + a_{in} b_{ni}]$$

$$= \sum_{i=1}^{n} [\sum_{k=1}^{n} a_{ik} b_{ki}] = \text{L. H. S.}$$

Hence tr (AB) = tr (BA).

Especially, when $B = A^T$ (i. e A is a symmatric matrix of order n) then the trace of the Grammian matrix is given by

$$tr(AA^T) = tr(A^TA) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}^2 > 0.$$

More generally, when A is of order $n \times 1$, we have.

$$tr(A^TA) = \sum_{j=1}^{1} \sum_{i=1}^{n} a_{ij}^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{1} a_{ij}^2 = tr(AA^T)$$

Evidently $tr(A^TA) = tr(AA^T) = 0$ if and only if $A = 0$.

Also, Since transposing a square matrix leaves diagonal elements unchanged, we have $tr(A) = tr(A^T)$.

In general one can easily show that

$tr(ABC) = tr(BCA) = tr(CAB)$.

(4) Traces of similar matrices are equal i, e, if A and B are two similar matrices, then $tr(A) = tr(B)$

Proof : let $B = P^{-1} AP$ where P is a non-singular matrix.

$\therefore tr(B) = tr(P^{-1} AP) = tr(P^{-1} (AP))$ since $tr(AB) = tr(BA)$

$= tr((AP) P^{-1}) = tr(A(PP^{-1})) = tr(AI) = tr(A)$ Since $PP^{-1} = I$

Hence $tr(B) = tr(A)$.

(5) If C is an orthogonal matrix, then $tr(C^TAC) = tr(A)$

Proof : $tr(C^T AC) = tr(C^T (AC)) = tr((AC) C^T)$

$= tr(A(CC^T)) = tr(AI) = tr(A)$

Since $tr(AB) = tr(BA)$ and $CC^T = I$, where C is orthogonal.

(6) If A is symmetric matrix and B is a skew-symmetri[c] matrix, then $tr(AB) = 0$.

Proof : $\operatorname{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j a_{ij} b_{ji}$

$= \sum_i \sum_j b_{ji} a_{ij} = -\sum_i \sum_j b_{ij} a_{ij}$ since $b_{ji} = -b_{ij}$

$= -\sum_i \sum_j b_{ij} a_{ji}$ since $a_{ij} = a_{ji}$

$= -\sum_i (BA)_{ii} = -\operatorname{tr}(BA) = -\operatorname{tr}(AB)$

Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$\therefore 2\operatorname{tr}(AB) = 0$ or, $\operatorname{tr}(AB) = 0$ since $2 \neq 0$.

4.3 Definition of rank of a matrix

Once of the most important characteristics of a matrix is its rank. The rank of a matrix can be defined in several equivalent ways. We use the following definitions :

(i) Let A be an arbitrary $m \times n$ matrix over a field F. The **rank** of the matrix A is the largest value of r for which there exists an $r \times r$ submatrix of A with non-vanishing determinant.

or, equivalently

(ii) A matrix A is said to have **rank** r if it contains at least one r-rowed square submatrix with a non-zero determinant whereas the determinant of any square submatrix having (r+1) or more rows, possibly contained in A, is zero.

or, equivalently

(iii) A non-zero matrix A is said to have **rank** if at least one of its r-square minors is different from zero while every (r + 1) –square minor, if any, is zero.

(iv) The **rank** of a matrix A is the maximum number of linearly independent rows or columns in the matrix,

or, equivalently

(v) Let A be an $m \times n$ matrix and let A_R be the row echelon form of A. Then the **rank r** of the matrix A is the number of non-zero rows of A_R.

The rank of a matrix A is denoted by rank (A) or $\rho(A)$.

The rank of a null matrix (zero matrix) is zero and the rank of a matrix of order $m \times n$ can not be greater than m or n.

An n-rowed square matrix A has rank $r<n$ if and only if $|A| = 0$. In this case A is called a **singular matrix**. The matrix A has rank $r = n$ if and only if $|A| \neq 0$ and is then called a **non-singular matrix.**

The rank of a matrix can be determined by the following processes :

(i) First Process

Let A be an $m \times n$ ordered matrix. When $m = n$ evaluate the value of the determinant of the matrix; if $|A| \neq 0$, the rank of the matrix is n. But if $|A| = 0$, we have to find out the n^2 minors of the matrix A. When the value of any of the minors is non-zero, the rank of the matrix is $(n-1)$. If every $(n-1) \times (n-1)$ ordered minors is zero, to find the rank of the matrix A, we have to proceed as before until we get the non-singular square submatrix whose order is the rank of the given matrix.

Again when $m<n$. the rank of the matrix can not be greater than m and it can be determined taking the largest square submatrices of order m and evaluating the determinants of them. If the value of any determinant of them is non-zero, the rank of the matrix is m. But when the value of every determinant is zero, we will find the values of the determinants of all square submatrices of order $(m-1)$. If the value of any determinant is non-zero, the rank of the matrix is $(m-1)$ and if the value of every determinant is zero, we have to continue the process until we get a non-singular square submatrix whose order is the rank of the given matrix.

Second Process

Reduce the given matrix A to echelon form using elementary row operations (transformations). Since the non-zero rows of a matrix in echelon form are linearly independent, the number of non-zero rows of the echelon matrix is the rank of the given matrix.

It is to be noted that the rank of a matrix is unaltered by the elementary row transformations and equivalent matrices have the same order and the same rank.

Theorm 4.1 The rank of the transpose of a matrix is the same as that of the original matrix.

Proof : Let A be the matrix and rank (A) = r. Let R be an r-rowed square submatrix of A with det $R \neq 0$. Clearly R^T is a submatrix of A^T. Now Since the value of the determinant is not altered if its rows are written as columns in the same order. det R^T = det R. Thus rank $(A^T) \geq r$. On the other hand, if A contains (r + 1)-rowed square submatrix S, then, by the definition of rank, det.S = 0. Since S corresponds to S^T in A^T and det $S^T = 0$. It follows that A^T can not contain an (r + 1)-rowed square submatrix with a non-zero determinant. Consequently rank $(A^T) \leq r$. Altogether, rank $(A^T) = r$ and the proof is complete.

Theorem 4.2 Let A be an n-square matrix. Then A is invertible if and only if rank (A) = n.

Proof : The rows of the n-square identity matrix I_n are linearly independent, since I_n is in echelon form; hence rank $(I_n) = n$. Now if A is invertible, then A is row equlvalent to I_n hence rank (A) = n. But if A is not invertible then A is row equivalent to a matrix with a zero row; hence rank (A) < n. That is; A is invertible if and only if rank (A) = n.

Definition : The maximum number of linearly independent rows of matrix A is called the **row rank** of A and the maximum number of linearly independent columns of a matrix A is called the **column rank** of A.

Theorem 4.3 The row and column ranks of a matrix are equal.

Proof : Let a be an arbitrary m ×n matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let $R_1, R_2, R_3, \ldots, R_m$ denote its rows.

$R_1 = (a_{11}, a_{12}, \ldots, a_{1n}), \ldots, R_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$

Suppose the row rank is r and the following r vectors form a basis for the row space :

$$v_1 = (b_{11}, b_{12}, \ldots, b_{1n}), v_2 = (b_{21}, b_{22}, \ldots, b_{2n}), \ldots, v_r = (b_{r1}, b_{r2}, \ldots, b_{rn})$$

Then each of the row vectors is a linear combination of the v_i, $i = 1, 2, \ldots, r$.

$$R_1 = \alpha_{11}v_1 + \alpha_{12}v_2 + \ldots + \alpha_{1r}v_r$$
$$R_2 = \alpha_{21}v_1 + \alpha_{22}v_2 + \ldots, + \alpha_{2r}v_r$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$R_m = \alpha_{m1}v_1 + \alpha_{m2}v_2 + \ldots + \alpha_{mr}v_r$$

where α_{ij} are scalars, Setting the i th components of each the above vector equations equal to each other, we obtain following system of equations, each valid for $i = 1, 2, \ldots, n$.

$$a_{1i} = \alpha_{11}b_{1i} + \alpha_{12}b_{2i} + \ldots + \alpha_{1r}b_{ri}$$
$$a_{2i} = \alpha_{21}b_{1i} + \alpha_{22}b_{2i} + \ldots + \alpha_{2r}b_{ri}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{mi} = \alpha_{m1}b_{1i} + \alpha_{m2}b_{2i} + \ldots + \alpha_{mr}b_{ri}$$

hus for $i = 1, 2, \ldots, n$ we have

$$\begin{bmatrix} a_{1i} \\ a_{2i} \\ \ldots \\ a_{mi} \end{bmatrix} = b_{1i}\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \ldots \\ \alpha_{m1} \end{bmatrix} + b_{2i}\begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \ldots \\ \alpha_{m2} \end{bmatrix} + \ldots + b_{ri}\begin{bmatrix} \alpha_{1r} \\ \alpha_{2r} \\ \ldots \\ \alpha_{mr} \end{bmatrix}$$

other words, each of the columns of A is linear mbination of the r vectors.

$$\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \ldots \\ \alpha_{m1} \end{bmatrix} \begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \ldots \\ \alpha_{m2} \end{bmatrix} \ldots, \begin{bmatrix} \alpha_{1r} \\ \alpha_{2r} \\ \ldots \\ \alpha_{mr} \end{bmatrix}$$

Linear Algebra-11

Thus the column space of the matrix A has dimension at most r. i. e. column rank $\leq r$. Similarly (considering the transpose matrix A^T) we obtain row rank $\leq$ column rank. Thus the row rank and the column rank of the matrix A are equal.

4.4 Reduction of a matrix to the normal form.

Theorem 4.4 Every non-zero matrix of rank r can be reduced to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary transformations, where I_r is an r-rowed unit matrix.

Proof : Let A be a given matrix of rank r.

Since $A \neq 0$, it has at least one non-zero element.

Let $a_{ij} = k \neq 0$.

By interchanging the i th row with the first row and the j th column with the first column, we obtain an equivalent matrix B such that $b_{11} = k \neq 0$. By elementary transformation $R_1\left(\frac{1}{k}\right)$ on B we obtain an equivalent matrix C such that $C_{11} = 1$.

Subtracting from the elements of the j th column of C, the products of the corresponding elements of the first column by c_{1j}, we get a matrix D whose all the elements in the first row except the first element are zero.

Again subtracting from the elements of i th row of D, the products of the corresponding elements of the first row by d_{i1} we obtain an equivalent matrix E such that all the elements in the first row and first column are zero except the first element which is unity. thus E is of the form $\begin{bmatrix} I & 0 \\ 0 & A_1 \end{bmatrix}$

Now if $A_1 \neq 0$, we can deal with it as we did with A, without affecting the first row and the first column.

Thus proceeding in this way, we shall obtain a diagonal matrix of the given form. Since the elementary transformations do not alter the rank of the matrix, the finally obtained diagonal matrix, whose rank is r, must have r and only r non-zero elements. Hence the theorem is proved.

Definition : Two matrices A and B are said to be **equivalent** if there exist non-singular matrices P and Q such that $PAQ = B$.

Theorem 4.6 If A is an $m \times n$ matrix of rank r then there exists a non-singular matrix P of order m such that $PA = \begin{bmatrix} B \\ 0 \end{bmatrix}$ where B is an $r \times n$ matrix of rank r and 0 is the null matrix of order $(m-r) \times n$.

Proof : Since A is an $m \times n$ matrix with $\rho(A) = r$. there exist non-singular matrices P and Q of orders m and n respectively such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (1)

Further, since every non-singular matrix can be expressed as the product of elementary matrices, let $Q = Q_1 . Q_2 .... Q_t$ where Q_i, $(i = 1, 2, ..., t)$ are elementary matrices. Moreover, since every elementary matrix is non-singular, Q is also non-singular and hence has an inverse given by

$$Q^{-1} = (Q_1 Q_2 \cdots Q_t)^{-1} = Q_t^{-1} Q_{t-1}^{-1} ...., Q_2^{-1} .. Q_1^{-1} \quad (2)$$

Post multiplying (1) by Q^{-1} and using (2) we get

$$PA = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q_t^{-1} Q_{t-1}^{-1} \cdots Q_2^{-1} Q_1^{-1} \quad (3)$$

Each Q_i^{-1} $(i = 1, 2, ..., t)$ being the inverse of an elementary matrix is also an elementary matrix. Since every elementary column transformation of a matrix is equivalent to post-multiplication with the corresponding elementary matrix, the matrix on the right hand side of (3) is the matrix obtained from $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (*) by subjecting it to column transformation. But the (*) has the last (m–r) rows consisting of zeros only and since no column transformation can affect the last (m–r) zero rows, we shall finally obtain on the right hand side of (3) the matrix of the form $\begin{bmatrix} B \\ 0 \end{bmatrix}$ where B is r×n matrix and 0 is (m − r) × n matrix. Thus from (3) we have $PA = \begin{bmatrix} B \\ 0 \end{bmatrix}$. Hence the theorem is proved.

Corollary : If A is an m ×n matrix of rank r, then there exists non-singular matrix Q of order n such that $AQ = [C\ O]$ where C is an m ×r matrix of rank r and O is the null matrix of order $m \times (n - r)$.

Theorem 4.7 The rank of the product of two matrices can not exceed the rank of either matrix.

i. e. $\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$.

Proof : Let A be an m ×n matrix such that $\rho(A) = r$. Then there exists two non-singular matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Now if B is an n ×p matrix, then

$$AB = p^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}B$$

Now since P is a non-singular matrix, P^{-1} will be also a non-singular matrix. Then the rank of AB will be equal to the rank of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}(Q^{-1}B)$. But the components of the last m−r rows of the matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}(Q^{-1}B)$ will be all zero. Therefore, the rank of AB can not be greater than r, that is, $\rho(AB) \leq r$. which implies that $\rho(AB) \leq \rho(A)$.

Similarly, by the same procedure, we can show that $\rho(AB) \leq \rho(B)$.

Remarks 1. The above theorem can also be expressed as $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$.

Corollary 1. If A, B and C are three matrices conformable for multiplication, then $\rho(ABC) \leq \min\{\rho(A), \rho(B), \rho(C)\}$.

Theorem 4.8 If A is an idempotent matrix (i. e. $A^2 = A$) then rank (A) = trace of A i. e. $\rho(A) = tr(A)$.

Proof : If $\rho(A) = r$, then there exists non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (1)

Multiplying on the left of (1) by P^{-1} and on the right of (1) by Q^{-1}, we get

$$A = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

$$\therefore \quad A^2 = AA = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

$$= P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = A \text{ since } A^2 = A.$$

$$\text{or, } P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} - P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = 0$$

or, $P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\left\{Q^{-1}P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} - I_r\right\} Q^{-1} = 0$ which implies $Q^{-1}P^{-1} = I \Rightarrow Q^{-1} = P$. Therefore, the normal form of an idempotent matrix A is given by $A = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P$ where $\rho(A) = r$.

$$\text{Consider } tr(A) = tr\left[P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P\right]$$

$$= tr\left[PP^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right] \text{ Since } tr(AB) = tr(BA)$$

$$= tr\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = r = \rho(A).$$

Hence $\rho(A) = tr(A)$ if A is an idempotent matrix.

4.5 SYLVESTER'S LAW

Theorem 4.9 If A and B are square matrices of order n and $\rho(A) = r$, $\rho(B) = s$, then $\rho(AB) \geq r + s - n$

i. e $\rho(AB) \geq \rho(A) + \rho(B) - n$ where ρ denotes the rank of the matrix.

Proof : The equality holds when A and B are non-singular matrices. Since $\rho(A) = r$, there exists two non-singular matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}.$$

Let $C = P^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^{-1}$, then we have

$$A + C = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^{-1}$$

Now $\rho(A + C) = n$, $\rho(A) = r$ and $\rho(C) = n - r$.

Again since $A + C$ is non-singular, we have

$\rho\{(A + C)B\} = \rho(B)$.

Therefore, $\rho(B) = \rho(AB + CB) \leq \rho(AB) + \rho(CB)$ (1)

[since for any two matrices $(A)_{m \times n}$ and $(B)_{m \times n}$

$\rho(A + B) \leq \rho(A) + \rho(B)$]

Also $\rho(CB) \leq \rho(C)$

[since for any two matrices $(A)_{m \times n}$ $B_{n \times r}$

we have $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$]

Thus from (1), we get $\rho(B) \leq \rho(AB) + \rho(C) = \rho(AB) + n - r$

or, $\rho(B) \leq \rho(AB) + n - \rho(A)$

or, $\rho(AB) \geq \rho(A) + \rho(B) - n$ i, e $\rho(AB) \geq r + s - n$.

4.6 Condition for consistency of a system of linear equations

Consider the system of linear equations $Ax = b$, where A is an $m \times n$ matrix. Then the $m \times (n + 1)$ matrix $(A \mid b)$, obtained by adjoining the column vector b to the matrix A on the right, is called the **augmented matrix** of the system $Ax = b$.

Therem 4.10 The system of linear equations $Ax = b$ is **consistent** if and only if the rank of the augmented matrix $(A \mid b)$ is equal to the rank of the matrix A.

Proof : Let A be an $m \times n$ matrix and let rank $(A) = r$.

Then A has r linearly independent columns. Now we can take the first r columns $A^1, A^2, \ldots, A^r$ to be linearly independent. Then for each $j = r+1, r+2, \ldots, n$, A_j is a linear combination of $A^1, A^2, \ldots, A^r$. Now the augmented matrix $(A \mid b)$ has just one column, namely b, in addition to the n columns of A. Hence the maximum numbers of linearly independent columns in $(A \mid b)$ is either r or $r+1$. Therefore, rank $(A \mid b)$ is either r or $r+1$.

Suppose that rank $(A \mid b) = r$. Then the column vectors $A^1, A^2, \ldots, A^r$, b are linearly dependent.

Therefore, there exit scalars $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ (not all zero) such that $\alpha_1 A^1 + \alpha_2 A^2 + \ldots + \alpha_r A^r + \alpha_{r+1} b = 0$.

Now $\alpha_{r+1} \neq 0$, otherwise the column vectors $A^1, A^2, \ldots, A^r$ would be linearly dependent, contrary to our hypothesis.

$$\text{Hence } b = -\frac{\alpha_1}{\alpha_{r+1}} A^1 - \frac{\alpha_2}{\alpha_{r+1}} A^2 - \ldots - \frac{\alpha_r}{\alpha_{r+1}} A^r = A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\beta_j = -\frac{\alpha_j}{\alpha_{r+1}}$ $j = 1, 2, \ldots, r$

Thus $x = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution of the given system which is therefore consistent.

Conversely, suppose that the system $Ax = b$ is consistent and let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a solution.

Then $b = A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1A^1 + x_2A^2 + \ldots + x_nA^n$

Now A^j, $j = r+1, r+2, \ldots, n$ is a linear combination of $A^1, A^2, \ldots, A^r$, Hence b is also a linear combination of $A^1, A^2, \ldots, A^r$. Therefore, $(r + 1)$ column vectors $A^1, A^2, \ldots, A^r$, b are linearly dependent. Consequently, $(A | b)$ has only r linearly independent columns.

Thus rank $(A | b) = r$

Corollary : If A is an $m \times n$ matrix of rank m, the system $Ax = b$ is consistent.

WORKED OUT EXAMPLES.

Example 1. Find the rank of each of the following matrices :

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Solution : (i) Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ Then

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0.$$

So the rank of the matrix A is 1 (one) since $|A| = 0$, but not every element of A is zero. say $|2| \neq 0$

(ii) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ then

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$$

$= 1(21 - 20) - 2(14 - 12) + 3(10 - 9)$
$= 1 - 4 + 3 = 0.$

So the rank of the matrix A is less than 3. Now let us take two-rowed minor of A, say.

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3-4 = -1 \neq 0.$$

Since $|A| = 0$, but $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \neq 0$,

therefore, the rank of the given matrix is 2.

Example 2. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Solution : $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 2 \times 3 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$

$= 6 \times 0 = 0$ (Since three rows are equal)

So the rank of the matrix A is less than 3. Now let us consider the two-rowed minors of A.

Say, $\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 36-36 = 0$, $\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 18-18 = 0$, $\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12-12 = 0$

$\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} = 18-18 = 0$, $\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 9-9 = 0$, $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6-6 = 0$,

$\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 12-12 = 0$, $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6-6 = 0$, $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4 = 0$.

Thus every two-rowed minor of A is zero. So the rank of A is less than 2. But $|4| = 4 \neq 0$. Hence the rank of A is 1 (one).

Example 3. Find the rank of the matrix

$$A = \begin{bmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{bmatrix}$$ [R. U. P. 1969]

Solution : Since the given matrix A is of order 3×4, the rank of the given matrix A can not be greater than 3. Now we observe that the matrix A has the following largest square submatrices of order 3×3 :

$$A_1 = \begin{bmatrix} 6 & 2 & 0 \\ -2 & -1 & 3 \\ -1 & -1 & 6 \end{bmatrix},\ A_2 = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -1 & 4 \\ -1 & -1 & 10 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 6 & 0 & 4 \\ -2 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix},\ A_4 = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix}$$

Now $|A_1| = \begin{bmatrix} 6 & 2 & 0 \\ -2 & -1 & 3 \\ -1 & -1 & 6 \end{bmatrix} \begin{array}{l} = 6(-6+3) - 2(-12+3) + 0 \\ = -18 + 18 = 0. \end{array}$

$$|A_2| = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -1 & 4 \\ -1 & -1 & 10 \end{bmatrix} \begin{array}{l} = 6(-10+4) - 2(-20+4) + 4(2-1) \\ = -36 + 32 + 4 = 0. \end{array}$$

$$|A_3| = \begin{bmatrix} 6 & 0 & 4 \\ -2 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix} \begin{array}{l} = 6(30-24) + 0 + 4(-12+3) \\ = 36 - 36 = 0. \end{array}$$

$$|A_4| = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix} \begin{array}{l} = 2(30-24) + 0 + 4(-6+3) \\ = 12 - 12 = 0. \end{array}$$

So the rank of the matrix A can not be 3. Let us consider the square submatrices of order 2×2. Then we can at once show that $\begin{vmatrix} 6 & 2 \\ -2 & -1 \end{vmatrix} = -6 + 4 = -2 \neq 0.$

Therefore, the rank of the given matrix is 2.

Example 4. Find the echelon form and the row reduced echelon form of the following matrix :

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$ [R. U. S. 1989]

Solution : First let us reduce the matrix A to echelon form by the elementary row operations. We multiply 1st row by 2 and 3 and then subtract from 2nd & 3rd rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix}$$

We multiply 3rd row by 3.

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 3 & -6 & 1\\ 0 & 0 & 15 & -36 & 6\end{bmatrix}$$

We multiply 2nd row by 5 and then subtract from the 3rd row.

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 3 & -6 & 1\\ 0 & 0 & 0 & -6 & 1\end{bmatrix}$$

This matrix is in row echelon form.

We subtract 3rd row from the 2nd row. Also we multiply 3rd row by $-\frac{1}{6}$.

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{bmatrix}$$

We multiply 2nd row by $\frac{1}{3}$

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{bmatrix}$$

We add 2nd row with the 1st row

$$\sim\begin{bmatrix}1 & 2 & 0 & 2 & 1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{bmatrix}$$

We multiply 3rd row by 2 and then subtract from the 1st row.

$$\sim\begin{bmatrix}1 & 2 & 0 & 0 & \frac{4}{3}\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{bmatrix}$$

This matrix is in row reduced echelon form.

Example 5. Reduce the matrix A to the normal (or canonical form and hence obtain its rank where

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

Solution : We will apply both elementary column and row operations to the matrix A for reducing it to the normal form.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

We replace C_2 and C_4 by $C_2 - 2C_1$ and $C_4 + C_1$ respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix}$$

We replace C_2 and C_4 by $C_2 + 2C_3$ and $C_4 - 5C_3$ respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 11 & 2 & -7 \end{bmatrix}$$

We replace C_1 by $C_1 + C_3$ and C_4 by $C_4 + \frac{7}{11} C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$$

We replace R_2 by $R_2 - 4R_1$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$$

We replace R_3 by $R_3 - 2R_2$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 0 \end{bmatrix}$$

We interchange C_2 and C_3.

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 11 & 0\end{bmatrix}$$

We replace C_3 by $\frac{1}{11}C_3$.

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}=[I_3\ \ O]\text{ where } I_3=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\text{ and } O=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Hence the rank of A is 3.

Example 6. Find the rank of the matrix

$$A=\begin{bmatrix}2 & -1 & 3 & 4\\0 & 3 & 4 & 1\\2 & 3 & 7 & 5\\2 & 5 & 11 & 6\end{bmatrix}$$ by reducing it to the normal (or canonical) form.

Solution : We will apply both elementary row and column operations to the matrix A for reducing it to the normal form and will join successive matrices by the equivalence sign ~.

$$A=\begin{bmatrix}2 & -1 & 3 & 4\\0 & 3 & 4 & 1\\2 & 3 & 7 & 5\\2 & 5 & 11 & 6\end{bmatrix}$$

We replace R_3 and R_4 by R_3-R_1 and R_4-R_3 respectively.

$$\begin{bmatrix}2 & -1 & 3 & 4\\0 & 3 & 4 & 1\\0 & 4 & 4 & 1\\0 & 2 & 4 & 1\end{bmatrix}$$

We replace R_2 and R_3 by R_2-R_4 and R_3-R_4 respectively.

$$\sim\begin{bmatrix}2 & -1 & 3 & 4\\0 & 1 & 0 & 0\\0 & 2 & 0 & 0\\0 & 2 & 4 & 1\end{bmatrix}$$

We replace C_1 by $\frac{1}{2}C_1$ and R_3 by $\frac{1}{2}R_3$.

$$\sim\begin{bmatrix}1 & -1 & 3 & 4\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 2 & 4 & 1\end{bmatrix}$$

We replace R_3 by $R_3 - R_2$.

$$\sim\begin{bmatrix}1 & -1 & 3 & 4\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 2 & 4 & 1\end{bmatrix}$$

We interchange R_3 and R_4.

$$\sim\begin{bmatrix}1 & -1 & 3 & 4\\ 0 & 1 & 0 & 0\\ 0 & 2 & 4 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

We replace C_2, C_3 and C_4 by $C_2 + C_1$, $C_3 - 3C_1$ and $C_4 - 4C_1$ respectively.

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 2 & 4 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

We replace C_2 and C_3 by $C_2 - 2C_4$ and $C_3 - 4C_4$ respectively.

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

We interchange C_3 and C_4.

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\sim\begin{bmatrix}I_3 & 0\\ 0 & 0\end{bmatrix}.$$ Hence the rank of A is 3.

Example 7. Determine the rank of the matrix A

$$\text{where } A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

[D. U. S. 1987]

Solution : Reduce the given matrix to echelon form by means of elementary row transformations and join successive matrices by the equivalence sign ~. Interchange first row and fourth row.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

We subtract first row from the second row and multiply the first row by 3 and then subtract from the third row.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

We multiply second row by 2 and then sutract from the third row. Also we add second row with the fourth row.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Second row is multiplied by -1.

This matrix is row equivalent to the given matrix A and is in the row echelon form. Since the echelon matrix has two non-zero rows, the rank of the given matrix A is 2.

Example 8. Determine the rank of the matrix

$$\begin{bmatrix} 2 & 3 & 5 & -3 & -2 \\ 3 & 4 & 3 & -1 & -3 \\ 5 & 6 & -1 & 3 & -5 \end{bmatrix}$$

Solution : Reduce the given matrix to row echelon form by means of elementary row transformations and join successive matrices by the equivalence sign $\sim$. We subtract the first row from the second and interchange these rows.

$$\sim \begin{bmatrix} 2 & 3 & 5 & -3 & -2 \\ 1 & 1 & -2 & 2 & -1 \\ 5 & 6 & -1 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 2 & 3 & 5 & -3 & -2 \\ 5 & 6 & -1 & 3 & -5 \end{bmatrix}$$

We multiply the first row by 2 and by 5 and then subtract from the second and third rows respectively.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 0 & 1 & 9 & -7 & 0 \\ 0 & 1 & 9 & -7 & 0 \end{bmatrix}$$

We subtract the second row from the third row.

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & 1 \\ 0 & 1 & 9 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is row equivalent to the given matrix and is in the row echelon form. Since the echelon matrix has two non-zero rows, the rank of the given matrix is 2.

Example 9. Find the rank of the matrix A

$$\text{where } A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$ **[D. U. H. 1987]**

Solution : Reduce the given matrix to row echelon form by means of the elementary row transformations and join successive matrices by the equivalence sign $\sim$. We subtract the first row from the second row and also multiply the first row by 2 and by 3 and then subtract from the third row and the fourth row respectively.

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix}$$

We multiply second row by 3 and add with third row. Also we add second row with the fourth row.

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is row equivalent to the given matrix and is in the row echelon form. Since the echelon matrix has two non-zero rows; the rank of the given matrix is 2.

Example 10. Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 & 0 & 4 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 3 & 0 & 2 \end{bmatrix}$$

Solution : Reduce the given matrix to row echelon form by means of elementary row transformations and join successive matrices by the equivalence sign $\sim$. Interchange first and third rows and also fourth and fifth rows.

$$\sim \begin{bmatrix} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 2 & 2 & 0 & 4 & 0 & 4 \\ -1 & 1 & -2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We multiply first row by 2 and then subtract from the third row. Also we add first row with fourth row.

$$\sim \begin{bmatrix} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 0 & 0 & -12 & 2 & 0 & 4 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We subtract second row from the fourth row.

$$\sim \begin{bmatrix} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2 & 4 & 4 & 0 & 2 \\ 0 & 0 & -12 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now multiply second and third rows by $\frac{1}{2}$ and by $\left(-\frac{1}{12}\right)$ respectively.

$$\sim\begin{bmatrix} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is row equivalent to the given matrix and is in the row echelon form. Since the echelon matrix has three non-zero rows, the rank of the given matrix is 3.

Example 11. Prove that the following system of linear equations is inconsistent :

$$\left.\begin{aligned} 3x_1 + 4x_2 - x_3 + 2x_4 &= 1 \\ x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\ 3x_1 + 14x_2 - 11x_3 + x_4 &= 3 \end{aligned}\right\}$$

Proof : Form the augmented matrix

$$(A\,|b) = \begin{bmatrix} 3 & 4 & -1 & 2 & : & 1 \\ 1 & -2 & 3 & 1 & : & 2 \\ 3 & 14 & -11 & 1 & : & 3 \end{bmatrix}$$

Reduce the augmented matrix to row echelon form by the elementary row operations and join successive matrices by the equivalence sign $\sim$. Interchange first and second rows.

$$\sim\begin{bmatrix} 1 & -2 & 3 & 1 & : & 2 \\ 3 & 4 & -1 & 2 & : & 1 \\ 3 & 14 & -11 & 1 & : & 3 \end{bmatrix}$$

We multiply first row by 3 and then subtract from the second and third rows.

$$\sim\begin{bmatrix} 1 & -2 & 3 & 1 & : & 2 \\ 0 & 10 & -10 & -1 & : & -5 \\ 0 & 20 & -20 & -2 & : & -3 \end{bmatrix}$$

We multiply second row by 2 and then subtract from the third row.

$$\sim\begin{bmatrix} 1 & -2 & 3 & 1 & : & 2 \\ 0 & 10 & -10 & -1 & : & -5 \\ 0 & 0 & 0 & 0 & : & 7 \end{bmatrix}$$

Thus the augmented matrix is reduced to row-echelon form. Here we observe that rank (A) = 2 and rank (A | b) = 3. Therefore, rank A $\neq$ rank (A | b). Hence the given system of linear equations is inconsistent.

Example 12. Solve the following system of linear equations :

$$\left.\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 1 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 &= -1 \\ 3x_1 + 5x_2 + 6x_3 + 7x_4 + 4x_5 &= 2 \\ 4x_1 + 7x_2 + 10x_3 + 13x_4 + 16x_5 &= 1 \\ 5x_1 + 8x_2 + 9x_3 + 10x_4 + 3x_5 &= 3 \end{aligned}\right\}$$

Solution : Form the augmented matrix

$$(A \mid b) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & : & 1 \\ 2 & 3 & 4 & 5 & 6 & : & -1 \\ 3 & 5 & 6 & 7 & 4 & : & 2 \\ 4 & 7 & 10 & 13 & 16 & : & 1 \\ 5 & 8 & 9 & 10 & 3 & : & 3 \end{bmatrix}$$

Reduce the augmented matrix to row-echelon form by elementary row operations and join successive matrices by the equivalence sign '~'. We multiply first row by 2, 3, 4, and 5 and then subtract from the 2nd, 3rd, 4th and 5th rows respectively.

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & : & 1 \\ 0 & -1 & -2 & -3 & -4 & : & -3 \\ 0 & -1 & -3 & -5 & -11 & : & -1 \\ 0 & -1 & -2 & -3 & -4 & : & -3 \\ 0 & -2 & -6 & -10 & -22 & : & -2 \end{bmatrix}$$

We subtract 2nd row from the 4th row. and also we multiply 3rd row by 2 and then subtract from the 5th row.

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & : & 1 \\ 0 & -1 & -2 & -3 & -4 & : & -3 \\ 0 & -1 & -3 & -5 & -11 & : & -1 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We subtract 2nd row from the 3rd row.

$$\sim\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & : & 1 \\ 0 & -1 & -2 & -3 & -4 & : & -3 \\ 0 & 0 & -1 & -2 & -7 & : & 2 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We multiply 3rd row by 2 and then subtract from the 2nd row.

$$\sim\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & : & 1 \\ 0 & -1 & 0 & 1 & 10 & : & -7 \\ 0 & 0 & -1 & -2 & -7 & : & 2 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We multiply both 2nd and 3rd rows by (–1).

$$\sim\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & : & 1 \\ 0 & 1 & 0 & -1 & -10 & : & 7 \\ 0 & 0 & 1 & 2 & 7 & : & -2 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We multiply 2nd row by 2 and then subtract from the 1st row.

$$\sim\begin{bmatrix} 1 & 0 & 3 & 6 & 25 & : & -13 \\ 0 & 1 & 0 & -1 & -10 & : & 7 \\ 0 & 0 & 1 & 2 & 7 & : & -2 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We multiply 3rd row by 3 and then subtract from the 1st row.

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 4 & : & -7 \\ 0 & 1 & 0 & -1 & -10 & : & 7 \\ 0 & 0 & 1 & 2 & 7 & : & -2 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

This matrix is in row-echelon form. Therefore, rank (A) = rank (A|b) = 3. Hence the given system is consistent and transforms into the linear system.

$$\left.\begin{array}{lllll} x_1 & & & + 4x_5 & = -7 \\ & x_2 & - x_4 & -10x_5 & = 7 \\ & & x_3 + 2x_4 & + 7x_5 & = -2 \end{array}\right\}$$

This system is in echelon form and it has three equations in five unknowns. Therefore, the system has 5–3 = 2 free variables which are x_4 and x_5. Hence the system has infinite number of non-zero solutions.

Let $x_4 = a$ and $x_5 = b$ where a and b are arbitrary constants (real numbers).

Then $x_3 = -2 - 2a - 7b$

$x_2 = 7 + a + 10b$

$x_1 = -7 - 4b$

Thus the general solution is $x_1 = -7 - 4b$

$x_2 = 7 + a + 10b$

$x_3 = -2 - 2a - 7b$

$x_4 = a$

$x_5 = b$

where a and b are arbitrary constants.

4.7 Sweep out Method and its application

The method is explained and illustrated with the help of different worked out examples.

Example 13. Test the consistency and hence solve the following system of linear equations by using **sweep out method :**

$$\left.\begin{array}{l} x + y + z = 1 \\ x + 3y + 6z = 10 \\ x + 2y + 3z = 4 \\ x + 4y + 10z = 19 \end{array}\right\}$$

Solution :

x	y	z	
1	1	1	$1 \rightarrow R_1$
1	3	6	$10 \rightarrow R_2$
1	2	3	$4 \rightarrow R_3$
1	4	10	$19 \rightarrow R_4$
1	1	1	$1 \rightarrow R_5 = \frac{R_1}{1} \rightarrow$ 1st pivotal row
0	2	5	$9 \rightarrow R_6 = R_2 - R_5$
0	1	2	$3 \rightarrow R_7 = R_3 - R_5$
0	3	9	$18 \rightarrow R_8 = R_4 - R_5$
	1	2·5	$4{\cdot}5 \rightarrow R_9 = \frac{R_6}{2} \rightarrow$ 2nd pivotal row
	0	−0·5	$-1{\cdot}5 \rightarrow R_{10} = R_7 - R_9$
	0	1·5	$4{\cdot}5 \rightarrow R_{11} = R_8 - 3R_9$
		1	$3 \rightarrow R_{12} = \frac{R_{10}}{-0{\cdot}5} \rightarrow$ 3rd pivotal row
		0	$0 \rightarrow R_{13} = R_{11} - 1{\cdot}5 R_{12}$

Here the rank of the coefficient matrix A = 3 and the rank of the augmented matrix $(A \mid B) = 3$.

Hence the system of linear equations is consistent.

From the Ist pivotal row $x + y + z = 1$

" " 2nd " " $y + 2{\cdot}5z = 4{\cdot}5$

" " 3rd " " $z = 3$

Putting $z = 3$ in the second pivotal row, we have

$y + 7{\cdot}5 = 4{\cdot}5$ or, $y = -3$.

Also putting $z = 3$ and $y = -3$ in the equation $x + y + z = 1$ for the 1st pivotal row, we get $x = 1$.

Hence $x = 1$, $y = -3$, $z = 3$ is the unique solution of the given system of linear equations.

Example 14. Find rank of the matrix

$$\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}$$ [D. U. S. 1983]

by using **Sweep out method.**

Solution :

$$
\begin{array}{ccccc|l}
1 & 2 & -3 & -2 & -3 & \to R_1 \\
1 & 3 & -2 & 0 & -4 & \to R_2 \\
3 & 8 & -7 & -2 & -11 & \to R_3 \\
2 & 1 & -9 & -10 & -3 & \to R_4 \\
\hline
1 & 2 & -3 & -2 & -3 & \to R_5 = \frac{R_1}{1} \to \text{1st pivotal row} \\
\hline
0 & 1 & 1 & 2 & -1 & \to R_6 = R_2 - R_5 \\
0 & 2 & 2 & 4 & -2 & \to R_7 = R_3 - 3R_5 \\
0 & -3 & -3 & -6 & 3 & \to R_8 = R_4 - 2R_5 \\
\hline
 & 1 & 1 & 2 & -1 & \to R_9 = \frac{R_6}{1} \text{ 2nd pivotal row} \\
\hline
 & 0 & 0 & 0 & 0 & \to R_{10} = R_7 - 2R_9 \\
 & 0 & 0 & 0 & 0 & \to R_{11} = R_8 + 3R_9
\end{array}
$$

Now the rank of the given matrix is equal to the number of pivotal rows = 2.

EXERCISES - 4

1. Find the echelon form and the row reduced echelon form of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

Answers : $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 & \frac{4}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. Reduce the following matrix to echelon form and then to its row canonical form :

$$A = \begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix}$$

Answers : $\begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 0 & -11 & 10 & -15 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & \frac{4}{11} & \frac{5}{11} & \frac{13}{11} \\ 0 & 1 & -\frac{10}{11} & \frac{15}{11} & -\frac{5}{11} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

We write the rank of a matrix $A = \rho(A)$

3. Determine the ranks of the following matrices :

(i) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & 2 & -1 & 3 \\ 0 & 5 & -1 & 2 \\ 12 & -4 & -1 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 2 & 4 & 6 \\ 3 & -1 & 4 & -2 \\ 6 & -1 & 10 & -1 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 1 & -1 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 4 \end{bmatrix}$ **[J. U. 1972]**

Answer : (i) rank is 1 (one)
(ii) rank is 2 (iii) rank is 2
(iv) rank is 2 (v) rank is 3.

4. (a) Find the rank of the matrix

$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ by any method. **[D. U. P. 1983]**

(b) Find the rank of the matrix

$B = \begin{bmatrix} 1 & 3 & -2 & -1 \\ 2 & 6 & -4 & -2 \\ 1 & 3 & -2 & 1 \\ 2 & 6 & 1 & -1 \end{bmatrix}$ **[C. U. P. 1977]**

Answers : (a) $\rho(A) = 2$ (b) $\rho(B) = 3$.

5. (a) Determine the rank of the matrix

$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & -3 & 0 & -2 \end{bmatrix}$ **[D. U. P. 1982]**

(b) Determine the rank of the matrix

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Answers : (a) $\rho(A) = 2$

(b) rank is 2.

6. Compute the ranks of the following matrices :

(i) $\begin{bmatrix} -1 & 1 & 3 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 2 & -1 & 1 & 2 \\ 3 & 2 & -2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}$

[D. U. S. 1985] **[D. U. S. 1983]**

Answers: (i) rank is 4

(ii) rank is 2.

7. Find the rank of each of the following matrices :

(i) $\begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 4 & 2 & 5 \\ -1 & 0 & 3 & -2 \\ 4 & 1 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 1 & 0 & 2 & 3 \\ 1 & 1 & -1 & -1 & 2 \\ 3 & 0 & -1 & 1 & -2 \\ 2 & -1 & 0 & 2 & -1 \end{bmatrix}$

[D. U. Prel. 1984] **[D. U. Prel 1984]**

Answers : (i) rank is 4

(ii) rank is 4.

8. (a) Find the rank of the matrix A

$$\text{where } A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

(b) Determine the rank of the matrix A.

$$\text{where } A = \begin{bmatrix} -6 & 1 & 0 & 3 & 2 \\ 2 & -4 & 3 & -7 & 0 \\ 0 & 1 & -2 & -1 & 5 \\ -4 & -1 & -1 & -6 & 12 \end{bmatrix}$$

(c) Compute the rank of the matrix

$$\begin{bmatrix} 2 & 6 & -5 & 8 \\ 4 & 3 & -1 & 7 \\ -1 & 1 & 6 & 0 \\ 5 & 1 & 4 & 7 \end{bmatrix}$$

Answers : (a) $\rho(A) = 2$ (b) $\rho(A) = 3$ (c) rank is 3

9. For the matrix $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -2 & -3 & -1 & 1 \\ 0 & 1 & 7 & 1 \\ -2 & -2 & 6 & 2 \\ -3 & -6 & -12 & 0 \end{bmatrix}$

show the followings :

(i) All minors of A of order 3 or more are zero.

(ii) The rank of A is 2. i. e $\rho(A) = 2$.

10. Find the ranks of the following matrices :

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 2 \\ 5 & 9 & 12 & 14 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$

Answers : (i) rank is 2. (ii) rank is 3

11. (i) Find the rank of matrix

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 1 & -2 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

[C. U. P. 1979]

(ii) Find the rank of the matrix A

where $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$

Answers : (i) rank is 3 (ii) $\rho(A) = 3$.

12. (i) Find the rank of the matrix A

$$\text{where } A = \begin{bmatrix} 3 & 6 & 1 & 2 \\ 1 & 0 & \frac{1}{6} & 1 \\ 1 & -7 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ 3 & -6 & 0 & 4 \end{bmatrix}$$

(ii) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

Answers : (i) $\rho(A) = 3$

(ii) $\rho(A) = 1$.

13. (i) Find the rank of the matrix A,

$$\text{where } A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

(ii) Determine the rank of the matrix

$$\begin{bmatrix} 0 & 1 & 3 & -2 & -1 & 2 \\ 0 & 2 & 6 & -4 & -2 & 4 \\ 0 & 1 & 3 & -2 & 1 & 4 \\ 0 & 2 & 6 & 1 & -1 & 0 \end{bmatrix}$$ [J. U. 1978]

Answers : (i) $\rho(A) = 2$. (ii) rank is 3

14. (i) Find the rank of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 9 & 2 & -1 & 6 & 4 \\ 7 & -4 & -5 & 6 & 2 \\ 17 & 1 & -4 & 12 & 7 \end{bmatrix}$$

(ii) Determine the rank of the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 3 & 2 \\ 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & -4 & -5 & -2 \end{bmatrix}$$

Answers : (i) rank is 2. (ii) rank is 3.

15. Find the ranks of the following matrices :

(i) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$

Answers.: (i) rank is 2. (ii) rank is 3.

16. Find the ranks of the following matrices :

(i) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$

[D. U. S. 1980] **[C. U. P. 1973]**

Answers : (i) rank is 3 (ii) rank is 3.

17. (i) Determine the rank of the matrix :

$$A = \begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 5 & 8 \\ 3 & 1 & 2 & 3 \\ 1 & 0 & 0 & 9 \end{bmatrix}$$

(ii) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$ **[D. U. P. 1982]**

Answers : (i) rank is 4. (ii) rank is 3.

18. Find the ranks of the following matrices :

(i) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 2 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 4 & 2 & 4 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & -1 & -2 \end{bmatrix}$

Answers : (i) rank is 2 (ii) rank is 2.

19. Determine the rank of the matrix A.

$$\text{where } A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 2 & 2 & 3 \\ 3 & 4 & 1 & 3 \end{bmatrix}$$

(ii) Find the rank of the matrix

$$\begin{bmatrix} 2 & 1 & 0 & -1 & 3 \\ 1 & 2 & 1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ -1 & -5 & -3 & -7 & 3 \end{bmatrix}$$

[D. U. P. 1984]

Answers : (i) $\rho(A) = 2$. (ii) rank is 3.

20. Find the ranks of the following matrices :

(i) $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 13 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{bmatrix}$

Answers : (i) $\rho(A) = 2$. (ii) $\rho(B) = 3$

21. Find the rank of the following matrix :

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$$

Answer : $\rho(A) = 2$.

22. Find the rank of the following matrix :

$$A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -2 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -3 & 6 & 6 & 3 \\ 5 & -3 & 10 & 10 & 5 \end{bmatrix}$$

Answer : $\rho(A) = 3$.

23. Find the rank of the matrix

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 11 & 12 \\ 12 & 13 & 14 & 15 & 16 \end{bmatrix}$$

Answer : $\rho(A) = 2.$

24. Find the rank of the matrix

$$A = \begin{bmatrix} 0 & c & -b & a' \\ -c & 0 & a & b' \\ b & -a & 0 & c' \\ -a' & -b' & -c' & 0 \end{bmatrix}$$ where $aa' + bb' + cc' = 0$ and a, b, c are all positive numbers.

Answer : $\rho(A) = 2.$

25. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

find the non-singular matrices P and Q such that PAQ is in normal form. and hence determine the rank of the matrix A.

Answer : $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

26. Reduce the following matrix to normal form and hence find its rank : $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Answer : $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \rho(A) = 2.$

27. Reduce the following matrix to normal form and hence find its rank : $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Answer : $P = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\rho(A) = 2$, since $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

28. Find the rank of the following matrix by reducing it to the normal form :

$$A = \begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$$

Answer : $\rho(A) = 3$.

29. Reduce the following matrix to normal form and hence obtain its rank :

$$A = \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$$

Answer : $\rho(A) = 2$.

30. Reduce the following matrix to normal form and hence obtain its rank :

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Answer : $\rho(A) = 2$.

31. If A is a square matrix of order n such that $A^2 = A$, then prove that $\rho(A) + \rho(I - A) = n$ where I is a unit matrix of order n.

32. If A, B, C are any three matrices such that their products considered are defined then prove that

$$\rho(AB) + \rho(BC) \le \rho(ABC) + \rho(B).$$

33. If a, b, c are all unequal, find the rank of the following matrix, using only row-operations :

$$A = \begin{bmatrix} 0 & b-a & c-a & b+c \\ a-b & 0 & c-b & c+a \\ a-c & b-c & 0 & a+b \\ b+c & c+a & a+b & 0 \end{bmatrix}$$

Answer : $\rho(A) = 2$.

34. For all values of a, find the rank of the following matrix : $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & a \\ 2 & 2a-2 & -a-2 & 3a-1 \\ 3 & a+2 & -3 & 2a+1 \end{bmatrix}$

Answers : $\rho(A) = 4$ unless $a = 1$ or $a = 5$

If $a = 1$, $\rho(A) = 3$ and if $a = 5$, $\rho(A) = 2$.

35. Solve the following systems of linear equations :

(i) $\left.\begin{array}{r} x + y + z = 6 \\ 2x + 3y + 4z = 20 \\ 3x - 2y + z = 2 \end{array}\right\}$ (ii) $\left.\begin{array}{r} x + 2y + 3z = 5 \\ 2x - y + z = 5 \\ 4x + 2y - 3z = 5 \end{array}\right\}$

Answers : $\left.\begin{array}{ll} \text{(i)} & x = 1, y = 2, z = 3 \\ \text{(ii)} & x = 2, y = 0, z = 1 \end{array}\right\}$

36. Solve the following systems of linear equations :

(i) $\left.\begin{array}{r} x+y+z+t=2 \\ 2x+3y+4z+5t=9 \\ 3x-2y+z+2t=9 \\ x+2y-2z+3t=5 \end{array}\right\}$ (ii) $\left.\begin{array}{r} x+y+z-2t=-4 \\ x-2y+3z+4t=10 \\ 2x+3y-z+2t=9 \\ 4x-y+2z-t=-7 \end{array}\right\}$

Answers : (i) $x=1, y=-1, z=0, t=2.$

(ii) $x=-1, y=2, z=1, t=3.$

37. Solve the following system of linear equations :

$$\left.\begin{array}{r} x_1+x_2+x_3+x_4+x_5=15 \\ x_1-x_2+x_3-x_4+x_5=3 \\ x_1+2x_2+3x_3+x_4-2x_5=8 \\ 2x_1+3x_2-x_3-2x_4+x_5=2 \\ 3x_1-2x_2+3x_3-2x_4+x_5=5 \end{array}\right\}$$

Answer : $x_1=1, x_2=2, x_3=3, x_4=4, x_5=5.$