TITAS
MATH SERIES

Complex Analysis

জটিল বিশ্লেষণ

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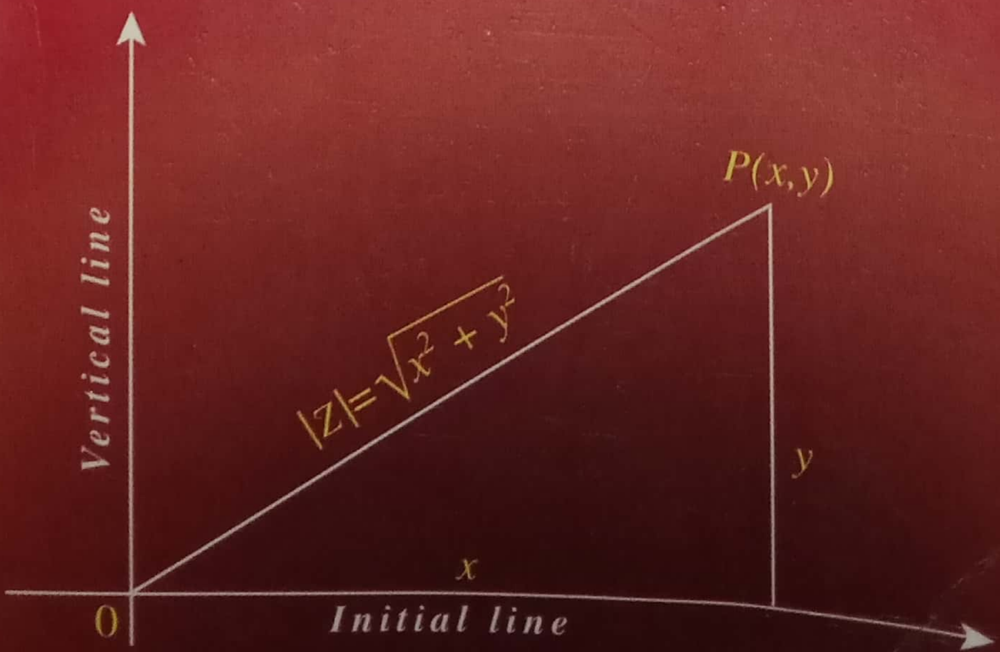

NATIONAL UNIVERSITY HONOUR'S SYLLAB
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CHAPTER–1
COMPLEX NUMBER

1.1. Complex Number system :

The equation $x^2 + 3 = 0$ or $x^2 = -3$ has no solution in real number system. This prompted the way to enlarge the real number system. For solving such systems the real number system was enlarged to complex number system. The term complex number was introduced by C. F. Gauss, a German mathematician. Later A. L. Cauchy, B. Riemann, K. Weierstrass and others enriched the subject.

Consider the set of all ordered pair of real numbers (x, y) defined by

$$\mathbf{R}^2 = \{(x, y) : x \in \mathbf{R}, y \in \mathbf{R}\}.$$

Here 'ordered' means (x, y) and (y, x) are different unless x = y.

We define the operations of addition (+) and multiplication (•) in $\mathbf{R}^2$ as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

Definition–1. A complex number is defined as the ordered pair (x, y) of real numbers, z = (x, y) satisfying the following rules for addition and multiplication [একটি জটিল সংখ্যা z = (x, y) কে বাস্তব সংখ্যার ক্রম জোড় (x, y) হিসাবে বর্ণনা করা হয় যাহা নিম্নের যোগ ও গুণের নিয়ম সিদ্ধ করে]

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$
$$z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2),$$

where [এখানে] $z_1 = (x_1, y_1)$ and [এবং] $z_2 = (x_2, y_2)$.

It is trivial to prove that for any real numbers a and b, we have

$$(a, 0) + (b, 0) = (a + b, 0)$$
$$(a, 0) \cdot (b, 0) = (ab, 0)$$

This motivates the idea that complex numbers of the form (a, 0) have the same arithmetic properties as the corresponding real numbers a. Thus we can identify the ordered pair (a, 0) by the real number a.

Definition–2. The imaginary unit i (iota) is defined as i = (0, 1).

Proposition–1. $i^2 = -1$.

Proof : $i^2 = i \cdot i = (0, 1) \cdot (0, 1)$

$$= (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0)$$
$$= (-1, 0) \equiv -1$$

Proposition-2. Every complex number $z = (x, y)$ can be written as $z = x + iy$ or $x + yi$.

Proof : $z = (x, y) = (x, 0) + (0, y)$
$= (x, 0) + (0, 1)(y, 0)$
$= x + iy$

$x + iy = (x, 0) + (0, 1)(y, 0)$
$= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y)$
$= (x, 0) + (0, y)$
$= (x, y) = z$

Similarly, $z = (x, y) = (x, 0) + (0, y)$
$= (x, 0) + (y, 0)(0, 1)$
$= x + yi$

Thus, definition-1 can be modify as, "any number of the form $x + iy$ is called a complex number where $x, y \in \mathbb{R}$." [$x + iy$ আকারের যে কোন সংখ্যাকে জটিল সংখ্যা বলে যেখানে $x, y \in \mathbb{R}$.]

With the introduction of i, Definition-1 for addition and multiplication of complex numbers can be translated as

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

The term imaginary number does not mean that such a number does not exist. The letter i is a symbol which denotes imaginary unit just as 1 denotes the unit of real numbers. Thus the imaginary number iy means y unit of imaginary numbers just as x means x units of real numbers. The expression $x + iy$ is not an imaginary number, it is a complex number. If $z = x + iy$, then x is called the real part of z and y is called the imaginary part of z, written as

$$\text{Re}(z) = x, \text{Im}(z) = y.$$

[কাল্পনিক সংখ্যা পদটি বুঝায় না যে এরূপ একটি সংখ্যার অস্তিত্ব নাই। i অক্ষরটি একটি প্রতীক যাহা কাল্পনিক একক নির্দেশ করে ঠিক যেমনটা বাস্তব সংখ্যার একক 1 নির্দেশ করে। অতএব কাল্পনিক সংখ্যা iy এর অর্থ কাল্পনিক সংখ্যার y একক ঠিক যেমনটা x অর্থ বাস্তব সংখ্যার x একক। $x + iy$ রাশিটি একটি কাল্পনিক সংখ্যা নয়, ইহা একটি জটিল সংখ্যা। যদি $z = x + iy$ হয় তখন x কে z এর বাস্তব অংশ বলে এবং y কে z এর কাল্পনিক অংশ বলে, যাহা লেখা হয়

$$\text{Re}(z) = x, \text{Im}(z) = y.]$$

1.1.1. Complex number : Any number of the form $x + iy$ is called a complex number, where $x, y \in \mathbb{R}$. If is denoted by Z.

[NUH-2013]

$\therefore z = x + iy$

[জটিল সংখ্যা ঃ $x + iy$ আকারের যে কোন সংখ্যাকে জটিল সংখ্যা বলে, যেখানে $x, y \in \mathbb{R}$. ইহাকে z দ্বারা প্রকাশ করা হয়।

$\therefore z = x + iy$

1.1.2. Real and Imaginary parts of Z : [NUH-2013]

If $z = x + iy$ be a complex number, then

Real part of $z = \text{Re}(z) = x$

Imaginary part of $z = \text{Im}(z) = y$

[z এর বাস্তব ও কাল্পনিক অংশ ঃ যদি $z = x + iy$ একটি জটিল সংখ্যা হয়, তখন

z এর বাস্তব অংশ $= \text{Re}(z) = x$

z এর কাল্পনিক অংশ $= \text{Im}(z) = y$]

1.2. Equality of complex numbers : Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal $\Leftrightarrow x_1 = x_2$ and $y_1 = y_2$, that is, the real part of the one is equal to the real part of the other, and the imaginary part of the one is equal to the imaginary part of the other.

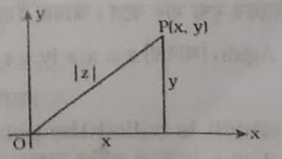
[জটিল সংখ্যার সমতা ঃ দুইটি জটিল সংখ্যা $z_1 = (x_1, y_1)$ এবং $z_2 = (x_2, y_2)$ সমান $\Leftrightarrow x_1 = x_2$ এবং $y_1 = y_2$ হয়, অর্থাৎ একটির বাস্তব অংশ অপরটির বাস্তব অংশের সমান, এবং একটির কাল্পনিক অংশ অপরটির কাল্পনিক অংশের সমান।]

Note : The phrases "greater than" or "less than" have no meaning in relation between two complex numbers. Inequalities can only occur in relation between the moduli of complex numbers.

[নোট ঃ দুইটি জটিল সংখ্যার সম্পর্কের ক্ষেত্রে 'হতে বৃহত্তর' অথবা 'হতে ক্ষুদ্রতর' প্রবাদের কোন অর্থ নাই। অসমতা শুধুমাত্র জটিল সংখ্যার মানাঙ্ক বা মডুলাসের ক্ষেত্রে হয়।]

1.3. Geometrical representation : Every complex number can be represented geometrically as a point in the xy-plane. We can identify the complex number $z = x + iy$ with the point P(x, y). The set of all real numbers (x, 0) corresponding to the x-axis, called real axis, and the set of all imaginary numbers (0, y) corresponding to the y-axis, called the imaginary axis. The origin identifies complex number $0 = 0 + i0$.

The distance of P from the origin is $\sqrt{x^2 + y^2}$. The nonnegative value of $\sqrt{x^2 + y^2}$ is denoted by $|z|$ and hence $|z| = \sqrt{x^2 + y^2}$. $|z|$ is called the modulas or absolute value of $z = (x, y)$.



[জ্যামিতিক উপস্থাপন ঃ প্রত্যেক জটিল সংখ্যাকে জ্যামিতিকভাবে xy তলে বিন্দু আকারে উপস্থাপন করা যায়। আমরা জটিল সংখ্যা $z = x + iy$ কে বিন্দু $P(x, y)$ দ্বারা চিহ্নিত করতে পারি। x অক্ষের অনুসঙ্গী সকল বাস্তব সংখ্যার সেট $(x, 0)$ কে বাস্তব অক্ষ বলে, এবং y অক্ষের অনুসঙ্গী সকল কাল্পনিক সংখ্যার সেট $(0, y)$ কে কাল্পনিক অক্ষ বলে। মূলবিন্দুটি $0 = 0 + i0$ জটিল সংখ্যা দ্বারা চিহ্নিত হয়।

মূলবিন্দু হতে P বিন্দুর দূরত্ব $\sqrt{x^2+y^2}$.

$\sqrt{x^2+y^2}$ এর অঋণাত্মক মানকে $|z|$ দ্বারা প্রকাশ করা হয় এবং অতএব $|z| = \sqrt{x^2+y^2}$.

$|z|$ কে $z = (x, y)$ এর মডুলাস বা পরম মান বা মানাঙ্ক হলে।]

We shall use the following inequalities [আমরা নিম্নলিখিত অসমতাগুলি ব্যবহার করব]

$$x \le |x| \le \sqrt{x^2+y^2} \Rightarrow \text{Re}(z) \le |\text{Re}(z)| \le |z|$$

$$y \le |y| \le \sqrt{x^2+y^2} \Rightarrow \text{Im}(z) \le |\text{Im}(z)| \le |z|$$

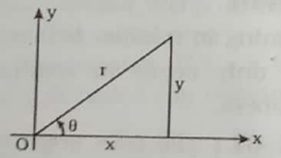
1.4. Polar and exponential form of complex numbers: Let r and θ be polar coordinates of a point $z = (x, y)$. For $z \neq 0$, let [ধরি একটি বিন্দু $z = (x, y)$ এর পোলার স্থানাংক r এবং θ. $z \neq 0$ এর জন্য ধরি]

$x = r\cos\theta,\ y = r\sin\theta$. Then [তখন]

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\therefore r = +\sqrt{x^2+y^2}$$

and [এবং] $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{r\sin\theta}{r\cos\theta} = \dfrac{y}{x}$



$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

r is called the modulus or the absolute value and θ is called the amplitude or argument of the complex number z. [r কে জটিল সংখ্যা z এর মডুলাস বা পরমমান বা মানাঙ্ক এবং θ কোণাঙ্ক বা আর্গুমেন্ট বলে]

$$|z| = r = +\sqrt{x^2+y^2} = 0 \Leftrightarrow x = 0 \text{ and } y = 0 \Leftrightarrow z = x + iy = 0.$$

The value of θ between $-\pi$ and π is called the principal value of the amplitude. We denote it by Arg z. [$-\pi$ এবং π এর মধ্যবর্তী θ এর মানকে কোণাঙ্কের মূখ্য মান বলে। আমরা ইহাকে Arg z দ্বারা প্রকাশ করব।]

Again [আবার] $z = x + iy = r\cos\theta + ir\sin\theta$

$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$

which is called the polar or exponential form of the complex number z. [যাহাকে জটিল সংখ্যা z এর পোলার বা সূচক আকার বলে।]

Note : $r = |z|$ is a unique nonnegative number, but $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is a multivalued function.

1.4.1. Modulas and argument of z : [NUH–2013]

If $z = x + iy$ be a complex number and $x = r\cos\theta$, $y = r\sin\theta$

$r = |z| = +\sqrt{x^2+y^2}$ is called the modulas of z and

$\theta = \tan^{-1}\frac{y}{x}$ is called the argument of z.

[যদি $z = x + iy$ একটি জটিল সংখ্যা হয় এবং $x = r\cos\theta$, $y = r\sin\theta$ হয় তখন

$r = |z| = +\sqrt{x^2+y^2}$ কে z এর পরম মান বলে এবং

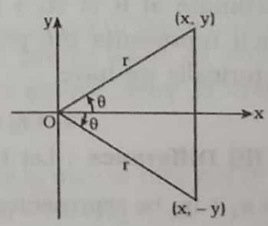
$\theta = \tan^{-1}\frac{y}{x}$ কে z এর আর্গুমেন্ট বা কোণাঙ্ক বলে।]

1.4.2. Argand plane or Argand diagram : When a complex number z is represented by a point $P(x, y)$ in the xy-plane, then this plane is called the Argand plane or Argand diagram or simply a complex plane.

[আর্গান্ড তল বা আর্গান্ড চিত্র ঃ যখন একটি জটিল সংখ্যাকে xy তলে একটি বিন্দু $P(x, y)$ দ্বারা উপস্থাপন করা হয়, তখন এই তলকে আর্গান্ড তল বা আর্গান্ড চিত্র বা সহজে একটি জটিল তল বলে।]

1.5. Conjugate complex number : If $z = x + iy$ is any complex number, then its complex conjugate denoted by $\bar{z}$ is defined as $\bar{z} = x - iy = (x, -y)$.

It is clear that $\bar{z}$ is the mirror image of z into the real axis. This indicates that $z = \bar{z} \Leftrightarrow z$ is purely a real number. Also, $\bar{\bar{z}} = z$.



[অনুবন্ধী জটিল সংখ্যা ঃ যদি $z = x + iy$ যে কোন জটিল সংখ্যা হয়, তখন ইহার জটিল অনুবন্ধীকে $\bar{z}$ দ্বারা প্রকাশ করা হয় যাহা $\bar{z} = x - iy = (x, -y)$ দ্বারা বর্ণিত।

ইহা স্পষ্ট যে $\bar{z}$ হলো x অক্ষের ভিতর z এর আয়না প্রতিবিম্ব। ইহা নির্দেশ করে যে $z = \bar{z} \Leftrightarrow z$ হয় একটি বিশুদ্ধ বাস্তব সংখ্যা। অধিকন্তু, $\bar{\bar{z}} = z$.]

Again [আবার] $\bar{z} = x - iy = r\cos\theta - ir\sin\theta$

$$= r(\cos\theta - i\sin\theta) = re^{-i\theta}$$

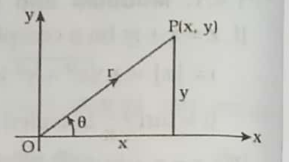
$\therefore |\bar{z}| = r = |z|$ and [এবং] Amp $\bar{z} = -\theta = -$Amp z.

Thus condition for two given numbers z_1 and z_2 to be conjugate [অতএব প্রদত্ত দুইটি সংখ্যা z_1 ও z_2 অনুবন্ধী হওয়ার শর্ত]

(i) $|z_1| = |z_2|$ and (ii) amp z_1 + amp $z_2 = 0$.

1.6. Vector representation of complex number : Let P(x, y) be a point in the complex plane corresponding to the complex number $z = x + iy$.

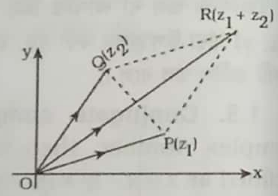
Then the modulas $r = \sqrt{x^2 + y^2}$ is represented by the magnitude of the vector $\overrightarrow{OP}$ and its amplitude θ is represented by the direction of the vector $\overrightarrow{OP}$.



Hence the complex number $z = x + iy$ is completely represents by the vector $\overrightarrow{OP}$.

(i) Sum : Let the complex numbers $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$ be represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Then the coordinates of P and Q are (x_1, y_1) and (x_2, y_2) respectively.

Complete the parallelogram OPRQ. Then the mid point of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ which is also the mid point of OR. Hence the coordinate of R is $(x_1 + x_2, y_1 + y_2)$ which represents the point $z_1 + z_2$. Vectorically we have

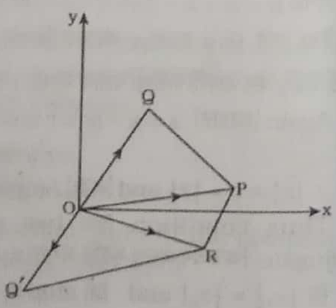


$$z_1 + z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$

(ii) Difference : Let the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be represented by the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Produce OQ backwards upto Q' such that $OQ = OQ'$.

Then Q' represents the point $-z_2$. Complete the parallelogram $OPRQ'$. Then R represents the sum of complex numbers represented by points P and Q', that is $R(z_1 - z_2)$.

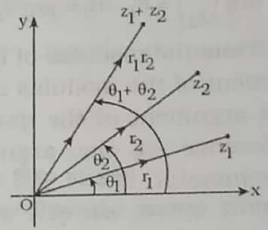
Vectorically, $z_1 - z_2 = \overrightarrow{OR} = \overrightarrow{QP}$



(iii) Product : In polar form, let the two complex numbers are [পোলার আকারে ধরি দুইটি জটিল সংখ্যা] $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. $|z_1| = |r_1 e^{i\theta_1}| = r_1 |\cos\theta_1 + i\sin\theta_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1 \cdot 1 = r_1$

Similarly [অনুরূপে] $|z_2| = r_2$.

Then their product is [তখন তাদের গুণফল]



$$z_1z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}$$
$$= r_1r_2\, e^{i(\theta_1+\theta_2)}$$
$$\therefore |z_1z_2| = |r_1r_2\, e^{i(\theta_1+\theta_2)}|$$
$$= |r_1|\ |r_2|\ |e^{i(\theta_1+\theta_2)}|$$
$$= r_1r_2, \text{ since [যেহেতু] } |e^{i(\theta_1+\theta_2)}| = |\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)|$$
$$= \sqrt{\cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2)}$$
$$= 1$$
$$= |z_1|\ |z_2|$$

and [এবং] $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$.

Thus, we have shown that the modulas of the product of two complex numbers is equal to the product of the moduli of these numbers and the argument of the product of two complex numbers is equal to the sum of the argument of these numbers. [অতএব, আমরা দেখালাম যে দুইটি জটিল সংখ্যার গুণফলের মড়ুলাস এই সংখ্যাগুলির মড়ুলাসের গুণফলের সমান এবং দুইটি জটিল সংখ্যার গুণফলের কোণাঙ্ক এই সংখ্যাগুলির কোণাঙ্কের যোগফলের সমান।]

The relation $\text{Arg}(z_1z_2) = \text{Arg } z_1 + \text{Arg } z_2$ may or may not be true for all z_1 and z_2. For example consider $z_1 = i$ and $z_2 = -1$.

Then $\text{Arg } z_1 = \frac{\pi}{2}$ and $\text{Arg } z_2 = \pi$.

Now, $z_1z_2 = -i$, $\text{Arg}(z_1z_2) = -\frac{\pi}{2}$.

Hence, $\text{Arg}(z_1z_2) \neq \text{Arg } z_1 + \text{Arg } z_2$ as $\frac{-\pi}{2} \neq \frac{\pi}{2} + \pi$.

(iv) Quotient : In polar form let the two complex numbers are [পোলার আকারে ধরি দুইটি জটিল সংখ্যা]

$z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

$$\therefore \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

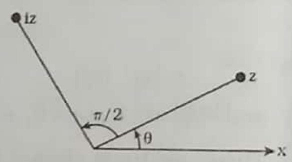
$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}\,|\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)| = \frac{r_1}{r_2}\cdot 1 = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

Thus the modulas of quotient of two complex numbers is the quotient of the modulas of the numerator and the denominator, and argument of the quotient of two complex numbers is the difference of the arguments of the numerator and the denominator. [অতএব দুইটি জটিল সংখ্যার ভাগফলের মডুলাস হলো লব ও হরের মডুলাসের ভাগফল, এবং দুইটি জটিল সংখ্যার ভাগফলের কোণাঙ্ক হলো লব ও হরের কোণাঙ্কের পার্থক্য।]

1.7. Geometrical interpretation of multiplication of a complex number by i :

Let z be a complex number whose polar form is



$$z = re^{i\theta}$$

$$\Rightarrow z = r(\cos\theta + i\sin\theta)$$

where r is the modulas and θ is the amplitude (argument) of z.

$$\text{Then } iz = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).\ r(\cos\theta + i\sin\theta)$$

$$= r\left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right]$$

Thus, the vector iz is one obtained by rotating the vector z through one right angle without changing its length.

[**একটি জটিল সংখ্যাকে i দ্বারা গুণের জ্যামিতিক ব্যাখ্যা ঃ**

মনে করি z একটি জটিল সংখ্যা যার পোলার আকার

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

যেখানে z এর মডুলাস r এবং কোণাঙ্ক θ, তখন

$$iz = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).\ r(\cos\theta + i\sin\theta)$$

$$= r\left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right]$$

অতএব, ভেক্টর z কে ইহার দৈর্ঘ্য পরিবর্তন না করে এক সমকোণ কোণে ঘুরাইয়া ভেক্টর iz পাওয়া গেল।]

Proposition : A complex number is purely real if the amplitude is 0 or π and purely imaginary if the amplitude is $\frac{\pi}{2}$ or $\frac{-\pi}{2}$. [একটি জটিল সংখ্যা বিশুদ্ধভাবে বাস্তব হবে যদি ইহার কোণাঙ্ক 0 অথবা π হয় এবং বিশুদ্ধভাবে কাল্পনিক হবে যদি ইহার কোণাঙ্ক $\frac{\pi}{2}$ অথবা $-\frac{\pi}{2}$ হয়।]

Proof : Let z be a complex number. Consider its polar form $z = r(\cos\theta + i\sin\theta)$

When $\theta = 0$ then $z = r(\cos 0 + i\sin 0) = r(1 + i\cdot 0) = r$

When $\theta = \pi$ then $z = r(\cos\pi + i\sin\pi) = r(-1 + i\cdot 0) = -r$

When $\theta = \frac{\pi}{2}$ then $z = r\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = r(0 + i\cdot 1) = ir.$

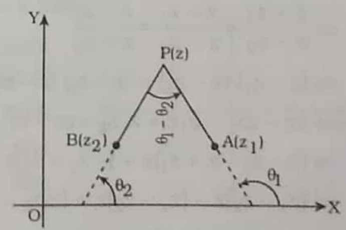
When $\theta = -\frac{\pi}{2}$ then $z = r\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right) = r(0 - i\cdot 1) = -ir.$

Thus, we have if z is purely real then amp $z = 0$ or π

and if z is purely imaginary then amp $z = \frac{\pi}{2}$ or $\frac{-\pi}{2}$.

1.8. Interpretation of arg$\frac{z - z_1}{z - z_2}$:

Let z, z_1, z_2 be the points P, A, B respectively on the Argands plane so that the complex numbers $z - z_1$ and $z - z_2$ represents the vector $\vec{AP}$ and $\vec{BP}$. Here arg $\vec{AP} = \theta_1$ and arg $\vec{BP} = \theta_2$.



$$\therefore \angle BPA = \theta_1 - \theta_2 = \arg(z - z_1) - \arg(z - z_2) = \arg\frac{z - z_1}{z - z_2}$$

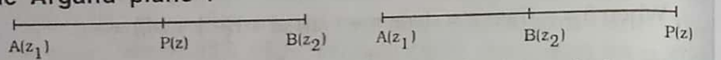
Thus arg $\frac{z - z_1}{z - z_2}$ represent the angle between the lines AP and BP in the positive sense. Similarly, arg $\frac{z - z_2}{z - z_1}$ gives the angle in the negative sense.

Condition for perpendicularity : If the lines are perpendicular then arg$\frac{z - z_1}{z - z_2} = \pm\frac{\pi}{2}$. We know that when the argument is $\pm\frac{\pi}{2}$, then the complex number is purely imaginary. Thus, in this case $\frac{z - z_1}{z - z_2}$ is purely imaginary.

Condition for parallelism or collinarity : If the lines are parallel or the point A, P, B are collinear, then arg $\frac{z-z_1}{z-z_2} = 0$ or π.

We know that when the argument is 0 or π then the complex number is purely real. Thus, in this case $\frac{z-z_1}{z-z_2}$ is purely real.

1.9. Equation of a straight line going the points z_1 and z_2 in the Argand plane :



Let $A(z_1)$ and $B(z_2)$ be any two points on the Argand plane, and P(z) be any point on the line AB.

Then arg $\frac{z-z_1}{z-z_2} = 0$ or π, according as P(z) lies outside or inside of AB. Therefore, $\frac{z-z_1}{z-z_2}$ is purely real. We know that the complex number z is purely real when $z = \bar{z}$. Thus,

$$\frac{z-z_1}{z-z_2} = \overline{\left(\frac{z-z_1}{z-z_2}\right)}$$

$$\Rightarrow \frac{z-z_1}{z-z_2} = \frac{\overline{z-z_1}}{\overline{z-z_2}} = \frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}$$

$$\Rightarrow (z-z_1)(\bar{z}-\bar{z}_2) = (z-z_2)(\bar{z}-\bar{z}_1)$$

$$\Rightarrow z\bar{z} - z\bar{z}_2 - z_1\bar{z} + z_1\bar{z}_2 = z\bar{z} - z\bar{z}_1 - z_2\bar{z} + \bar{z}_1 z_2$$

$$\Rightarrow (\bar{z} - \bar{z}_2 - \bar{z} + \bar{z}_1)z + (-z_1 + z_2)\bar{z} + (z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

$$\Rightarrow (\bar{z}_1 - \bar{z}_2)z - (z_1 - z_2)\bar{z} + (z_1\bar{z}_2 - z_2\bar{z}_1) = 0$$

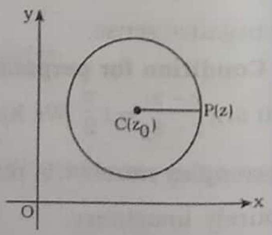
$$\Rightarrow az - \bar{a}\bar{z} + b = 0$$

where $a = \bar{z}_1 - \bar{z}_2$, $\bar{a} = z_1 - z_2$ and $b = z_1\bar{z}_2 - z_2\bar{z}_1$.

The above equation is the equation of a straight line passing through the points z_1 and z_2.

1.10. Equation of a circle :

Let $C(z_0)$ be the centre and r be the radius of the circle. Let P(z) be any point on its circumference.



Then $\overrightarrow{CP} = z - z_0$

$\therefore |\overrightarrow{CP}| = |z - z_0| = r$ is the equation of the required circle.

The above equation can be written as

$$|z - z_0|^2 = r^2$$

$$\Rightarrow (z - z_0)\overline{(z - z_0)} = r^2$$

$$\Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

$$\Rightarrow z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 = r^2$$

$$\Rightarrow (\bar{z} - \bar{z}_0)z - z_0\bar{z} + (z_0\bar{z}_0 - r^2) = 0$$

$$\Rightarrow z\bar{z} - z_0\bar{z} - \bar{z}_0 z + (|z_0|^2 - r^2) = 0$$

$$\Rightarrow z\bar{z} + b\bar{z} + \bar{b}z + c = 0, \text{ Where } b = -z_0,\ \bar{b} = -\bar{z}_0 \text{ and } c = |z_0|^2 - r^2$$

which is real. The above equation is the general equation of a circle. Its centre is

$$z_0 = -b \text{ and radius} = \sqrt{|z_0|^2 - c} = \sqrt{|b|^2 - c} = \sqrt{b\bar{b} - c}.$$

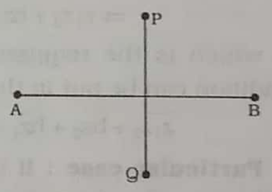
Note : If $z - z_0 = r\,e^{i\theta}$, $0 \le \theta \le 2\pi$, then $|z - z_0| = r|e^{i\theta}| = r\cdot 1 = r$. Thus, $|z - z_0| = r$ gives $z = z_0 + re^{i\theta}$.

Note : If z_0 is at the origin then the equation of the circle is $|z| = r$ and $z = re^{i\theta}$, $0 \le \theta \le 2\pi$.

1.11. Inverse points :

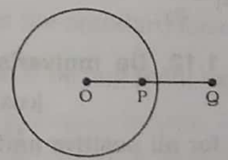
Inverse points with respect to a line.

Let AB be a line on the Argand plane. Then the two points P and Q are said to be inverse points with respect to the line AB if Q is the image of P in AB, that is, if the line AB is the right bisector of PQ.



Inverse point with respect to a circle :

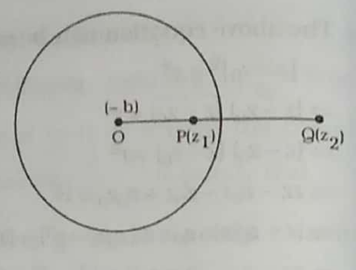
Let O be the centre and r be the radius of a circle. Then two points P and Q are said to be inverse points with respect to the circle if (i) O, P, Q are collinear, and (ii) $OP \cdot OQ = r^2$



Condition for inverse points : We know the general equation of a circle is

$$z\bar{z} + \bar{b}z + b\bar{z} + c = 0$$

where b is a complex constant and c is real. The centre of this circle is $-b$ and radius is $\sqrt{b\bar{b} - c}$.



$\vec{OP} = z_1 + b, \vec{OQ} = z_2 + b$

From the definition of inverse points we have

$$|\vec{OP}| \cdot |\vec{OQ}| = r^2$$
$$|z_1 + b| \cdot |z_2 + b| = r^2$$
$$= b\bar{b} - c \cdots (1)$$

Again, O, P, Q, are in collinear, so

$$\arg(z_1 + b) = \arg(z_2 + b) \cdots\cdots (2)$$

Also we know that $\arg z = -\arg \bar{z}$. Using this in (2) we have

$$\arg(z_1 + b) = \arg(z_2 + b) = -\arg\overline{(z_2 + b)} = -\arg(\bar{z}_2 + \bar{b})$$
$$\Rightarrow \arg(z_1 + b) + \arg(\bar{z}_2 + \bar{b}) = 0$$
$$\Rightarrow \arg((z_1 + b)(\bar{z}_2 + \bar{b})) = 0$$

$\Rightarrow (z_1 + b)(\bar{z}_2 + \bar{b})$ is purely real and positive

Again, since $|z_2 + b| = |\bar{z}_2 + \bar{b}|$ so from (1) we have

$$|z_1 + b|\;|\bar{z}_2 + \bar{b}| = b\bar{b} - c$$
$$\Rightarrow z_1\bar{z}_2 + b\bar{z}_2 + \bar{b}z_1 + b\bar{b} = b\bar{b} - c$$
$$\Rightarrow z_1\bar{z}_2 + b\bar{z}_2 + \bar{b}z_1 + c = 0$$

which is the required condition. On taking conjugate this condition can be put in the form

$$\bar{z}_1 z_2 + \bar{b}z_2 + b\bar{z}_1 + c = 0, \text{ since } c \text{ is real, so } \bar{c} = c.$$

Particular case : If the circle be a unit circle $|z| = 1$ then $|z|^2 = 1 \Rightarrow z\bar{z} = 1$. Writing the points z_1 and z_2 we have $z_1\bar{z}_2 = 1$ $\Rightarrow z_1 = \frac{1}{\bar{z}_2}$.

1.12. De moiver's theorem :

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all positive and negative integral or fractional values of n.

1.13. Complex polynomial and equations : Any expression of the form

$$a_0 z^n + a_1 z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_{n-1} z + a_n$$

is called a complex polynomial of degree n, where

$$a_0 \neq 0, a_1, a_2, \cdots, a_n \in \mathbf{C} \text{ and } n \in \mathbf{N}.$$

If $a_0 z^n + a_1 z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_{n-1} z + a_n = 0$ then it is called a complex equation of degree n, where $a_0 \neq 0, a_1, a_2, \cdots, a_n \in \mathbf{C}$ and $n \in \mathbf{N}$. If $z_1, z_2, \cdots, z_n$ are the roots of the above equation then

$$a_0(z - z_1)(z - z_2) \cdots (z - z_n) = 0$$

It is called the factor form of the above polynomial equation.

1.14. The roots of comlex numbers :

(i) Let z be a complex number.

Then w be an nth root of z if $w^n = z \Rightarrow w = (z)^{1/n}$.

In polar form if $z = r(\cos\theta + i\sin\theta)$, then

$$w = \{r(\cos\theta + i\sin\theta)\}^{1/n}$$
$$= r^{1/n}\{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}^{1/n}$$
$$= r^{1/n}\left(\cos\frac{2k\pi + \theta}{n} + i\sin\frac{2k\pi + \theta}{n}\right)$$

where $k = 0, 1, 2, \cdots (n-1)$.

(ii) The nth roots of unity.

Let $z^n = 1$. Then $z = (1)^{1/n} = (\cos 0 + i\sin 0)^{1/n}$

$$\Rightarrow z = (\cos 2k\pi + i\sin 2k\pi)^{1/n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

where $k = 0, 1, 2, \cdots, (n-1)$.

(iii) The nth roots of (– 1).

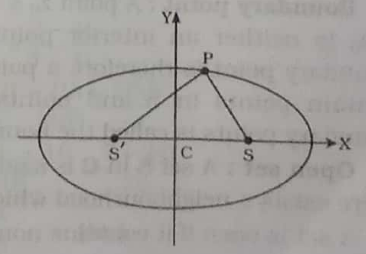
Let $z^n = -1$. Then $z = (-1)^{1/n} = (\cos\pi + i\sin\pi)^{1/n}$

$$\Rightarrow z = \{\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\}^{1/n}$$
$$= \cos\frac{(2k+1)\pi}{n} + i\sin\frac{(2k+1)\pi}{n}$$

where $k = 0, 1, 2, \cdots, (n-1)$

1.15. Relations in Ellipse and Hyperbola : From the coordinate geometry we know that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the standard form of an ellipse. Also, if S, S′ are the two focus and P be any point on the ellipse then

$$SP + S'P = 2a$$



that is, sum of the length of two focus from a point on the ellipse is equal to the length of the major axis.

In respect of complex variable, let $P(x, y) = P(z)$.

If $S = S(z_0)$, then $S' = S'(-z_0)$

$\therefore SP = |z - z_0|, S'P = |z + z_0|$

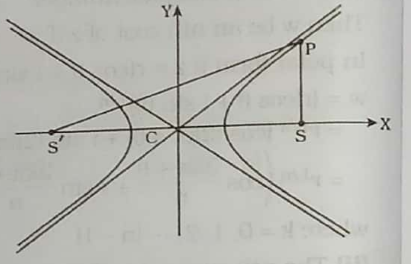
Therefore, the above relation becomes

$$|z - z_0| + |z + z_0| = 2a = \text{length of the major axis} \cdots\cdots (1)$$

Thus, the equation of the type (1) always hold in the case of an ellipse

Similarly, from any standard geometry book we have

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the standard equation of a hyperbola and



$$SP - S'P = 2a$$

that is, difference of the distance from any point on the hyperbola is equal to the length of the transverse axis. As above in complex variable this relation becomes

$$|z - z_0| - |z + z_0| = 2a = \text{length of the transverse axis.}$$

Regions in the complex plane :

Here we give some definitions only which are necessary for our discussion.

Neighbourhood : A neighbourhood (or circular neighbourhood) of a point z_0 in the complex plane C is a set of points given by

$$S = \{z \in \mathbf{C} : |z - z_0| < \delta, \delta > 0\}$$

Interior point : A point $z_0 \in S$ is said to be an interior point of S if there exists a neighbourhood of z_0 which is contained in S.

Exterior point : A point $z_0 \in S$ is said to be an exterior point of S if there exists a neighbourhood of z_0 which contains no point of S.

Boundary point : A point $z_0 \in S$ is called a boundary point of S if z_0 is neither an interior point nor a exterior point of S. A boundary point is therefore a point all of whose neighbourhoods contain points in S and points not in S. The totality of all boundary points is called the boundary of S.

Open set : A set S in **C** is said to be open if for each point of S there exists a neighbourhood which is contained in S.

A set is open if it contains none of its boundary points.

Closed set : A set S is said to be closed if it contains all its boundary points.

Some sets are neither open nor closed. For a set to be not open, there must be a boundary point that is contained in the set; and if a set is not closed, there exists a boundary point not contained in the set. Thus, an open set does not have any of its boundary points, while a closed set contains all its boundary points.

Connected set : An open set S is said to be connected if each pair of points z_1 and z_2 in it can be joined by a polygonal path, consisting of a finite number of line segments joined end to end, which lies entirely in S.

The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is open and also connected.

Domain : A non empty open connected set in **C** is called a domain. That is, an open set that is connected is called a domain.

Bounded set : A set S in complex plane is said to be bounded if every point of S lies inside some circle $|z| = R$.

Unbounded set : A set which is not bounded is called unbounded set.

Region : A domain together with some, none or all of its boundary points is referred to as a region.

SOLVED EXAMPLES

Example-1. Express $\frac{(1 + 2i)^2}{(2 + i)^2}$ in the form A + iB. Also find its modulas and argument. **[DUH-1983]**

Solution :

$$\frac{(1 + 2i)^2}{(2 + i)^2} = \frac{1 + 4i + 4i^2}{4 + 4i + i^2}$$

$$= \frac{1 + 4i - 4}{4 + 4i - 1} \quad [\because i^2 = -1]$$

$$= \frac{-3 + 4i}{3 + 4i} \times \frac{3 - 4i}{3 - 4i}$$

$$= \frac{-9 + 12i + 12i - 16i^2}{3^2 - (4i)^2}$$

$$= \frac{-9 + 24i + 16}{9 + 16} = \frac{7 + 24i}{25}$$

$$= \frac{7}{25} + i\frac{24}{25} \quad \textbf{(Ans)}$$

$$\text{Modulas [মডুলাস]} = \left|\frac{(1+2i)^2}{(2+i)^2}\right| = \left|\frac{7}{25} + i\frac{24}{25}\right|$$

$$= \sqrt{\left(\frac{7}{25}\right)^2 + \left(\frac{24}{25}\right)^2} = \sqrt{\frac{49+576}{625}}$$

$$= \sqrt{\frac{625}{625}} = 1$$

The principal argument [মুখ্য কোণাঙ্ক] $= \tan^{-1}\left(\frac{24/25}{7/25}\right) = \tan^{-1}\left(\frac{24}{7}\right)$

The general argument [সাধারণ কোণাঙ্ক] = $2n\pi$ + principal argumen

$$= 2n\pi + \tan^{-1}\left(\frac{24}{7}\right)$$

where [যেখানে] $n = 0, \pm 1, \pm 2, \cdots$ etc.

Example–2. Find the real and imaginary parts o $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}$ and also find its modulas. **[RUH–1986]**

Solution : $$\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi} = \frac{2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\phi}{2} + i\,2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$$

$$= \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\cos\frac{\phi}{2}\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \frac{e^{i\theta/2}}{e^{i\phi/2}} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} e^{i(\theta-\phi)/2}$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\left[\cos\left(\frac{\theta-\phi}{2}\right) + i\sin\left(\frac{\theta-\phi}{2}\right)\right]$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} e^{i(\theta-\phi)/2}$$

Real part of $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \cdot \cos\frac{\theta-\phi}{2}$

Imaginary part of $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \sin\frac{\theta-\phi}{2}$

$$\text{Modulas [মডুলাস]} = \left|\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}\right| = \left|\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} e^{i(\theta-\phi)/2}\right|$$

$$= \left|\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\right| \left|e^{i(\theta-\phi)/2}\right|$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \cdot 1 = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \quad \text{Ans.}$$

Example–3. Find the modulas and argument of the following complex numbers :

(i) $\frac{-2}{1+i\sqrt{3}}$, (ii) $\frac{1-i}{1+i}$ **[DUH–2005, JUH–1987]**

Solution : (i) $$\frac{-2}{1+i\sqrt{3}} = \frac{-2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{-2+i2\sqrt{3}}{1-i^2 3}$$

$$= \frac{-2+i2\sqrt{3}}{1+3} = \frac{-2}{4} + i\frac{2\sqrt{3}}{4} = -\frac{1}{2}\, i\frac{\sqrt{3}}{2}.$$

$$\therefore \text{Modulas [মডুলাস]} = \left|\frac{-2}{1+i\sqrt{3}}\right| = \left|-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

The principal argument [মুখ্য কোণাঙ্ক]

$$= \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

The general argument is $2n\pi + \frac{2\pi}{3}$, where $n = 0, \pm 1, \pm 2, \cdots$ etc.

(ii) $$\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{1+1} = \frac{-2i}{2} = -i$$

$$\text{Modulas} = \left|\frac{1-i}{1+i}\right| = |-i| = \sqrt{0+(-1)^2} = \sqrt{1} = 1$$

The principal argument $= \tan^{-1}\left(\frac{-1}{0}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}$

The general argument $= 2n\pi - \frac{\pi}{2}$, where $n = 0, \pm 1, \pm 2 \cdots$ etc.

Example–4. Find the modulas and argument of the complex number $\left(\frac{2+i}{3-i}\right)^2$. **[NUH–2011, DUHT–1991]**

Solution : $\left(\frac{2+i}{3-i}\right)^2 = \frac{4+4i+i^2}{9-6i+i^2} = \frac{4+4i-1}{9-6i-1} = \frac{3+4i}{8-6i}$

$$= \frac{3+4i}{8-6i} \times \frac{8+6i}{8+6i} = \frac{24+18i+32i+24i^2}{8^2-6^2i^2}$$

$$= \frac{24+50i-24}{64+36}$$

$$= \frac{i50}{100} = \frac{i}{2}$$

$\therefore$ Modulas [মডুলাস] $= \left|\left(\frac{2+i}{3-i}\right)^2\right| = \left|\frac{i}{2}\right| = \sqrt{0+\frac{1}{4}} = \frac{1}{2}$

The principal argument [মুখ্য কোণাঙ্ক] $= \tan^{-1}\left(\frac{1/2}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

The general argument [সাধারণ কোণাঙ্ক]

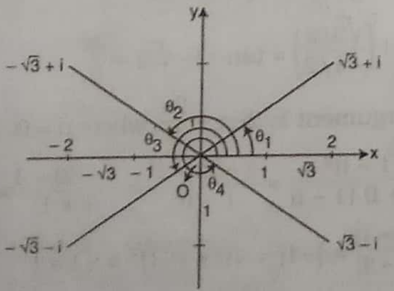
$$= 2n\pi + \frac{\pi}{2}, \text{ where } n = 0, \pm 1, \pm 2, \cdots \text{ etc.}$$

Example–4(i). Plot the four complex numbers $z = \pm\sqrt{3} \pm i$. Find the arguments and the modulas of these z [$z = \pm\sqrt{3} \pm i$ সংখ্যা চতুষ্টয় আরগান্ড ডায়াগ্রাম চিত্রায়িত কর। ইহাদের আর্গুমেন্ট ও পরম মান বাহির কর।] **[NUH–2013]**

Solution : Given [দেওয়া আছে] $z = \pm\sqrt{3} \pm i$. This gives [ইহা দেয়]

(i) $z = \sqrt{3} + i$, (ii) $z = -\sqrt{3} + i$, (iii) $z = \sqrt{3} - i$, (iv) $z = -\sqrt{3} - i$.

First we plot the four complex numbers [আমরা প্রথমে চারটি জটিল সংখ্যাকে চিত্রায়িত করব।]



In case (i) [(i) নং ক্ষেত্রে]

Argument [আর্গুমেন্ট] $\theta_1 = \tan^{-1}\frac{1}{\sqrt{3}} = \tan^{-1}\tan 30^0 = 30^0$

and Modulas of z [এবং z এর পরম মান]

$$|z| = +\sqrt{(\sqrt{3})^2 + 1^2} = +\sqrt{3+1} = +\sqrt{4} = 2$$

In case (ii) [(ii) নং ক্ষেত্রে]

Argument [আর্গুমেন্ট] $\theta_2 = \tan^{-1}\frac{1}{-\sqrt{3}} = \tan^{-1}(-\tan 30^0)$

$$= \tan^{-1}\tan(180^0 - 30^0) = 150^0$$

and Modulas of z [এবং z এর পরম মান]

$$|z| = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$$

In case (iii) [(iii) নং ক্ষেত্রে]

Argument [আর্গুমেন্ট] $\theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \tan^{-1}\tan 30^0$

$$= \tan^{-1}\tan(180^0 + 30^0) = 210^0$$

[x ও y ঋণাত্মক বলে z এর অবস্থান তৃতীয় চর্তুভাগে]

এবং Modulas of z [z এর পরমমান], $|z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

In case (iv) [(iv) নং ক্ষেত্রে]

Argument [আর্গুমেন্ট] $\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$$= \tan^{-1}(-\tan 30^0)$$

$$= \tan^{-1}\tan(360^0 - 30^0) = 330^0$$

and Modulas [এবং পরমমান] $|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$.

Example–5. Find the square roots of the complex number $5 - 12i$. **[RUH–2002]**

Solution : $5 - 12i = 9 - 12i - 4$

$$= 3^2 - 2 \cdot 3 \cdot 2i + (2i)^2 \quad [\because i^2 = -1]$$

$$= (3 - 2i)^2$$

$$\Rightarrow \sqrt{5 - 12i} = \sqrt{(3 - 2i)^2} = \pm(3 - 2i)$$

$\therefore$ The square roots of $5 - 12i$ are $3 - 2i$ and $-3 + 2i$. [$5 - 12i$ এর বর্গমূল হলো $3 - 2i$ এবং $-3 + 2i$.]

Example-6. Express $-5+5i$ in polar form. **[RUH-199**

Solution : Let $-5 = r\cos\theta$ and $5 = r\sin\theta$

Then by squaring and adding we get

$$(-5)^2 + 5^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$\Rightarrow 25 + 25 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow 50 = r^2 \Rightarrow r = \sqrt{50} = 5\sqrt{2}$$

Also $\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{5}{-5} = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$

$$\Rightarrow \theta = \frac{3\pi}{4}$$

Thus, $-5+5i = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

$$\Rightarrow -5+5i = 5\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

which is the required polar form.

Example-7. Show that the sum of the products of all the nth roots of unity taken 2, 3, 4, ⋯ , (n – 1) at a time is zero. **[RUH-1986]**

Solution : Let $z = (1)^{1/n}$. Then $z^n = 1 \Rightarrow z^n - 1 = 0$

$$\Rightarrow z^n + 0\cdot z^{n-1} + 0\cdot z^{n-2} + \cdots + 0\cdot z - 1 = 0$$

Let $z_1, z_2, z_3, \cdots, z_n$ be the roots of this equation.

Sum of the products of the roots

taken two at a time is $\sum z_1z_2 = 0$,

taken three at a time is $\sum z_1z_2z_2 = 0$

... ...

taken (n – 1) at a time is $\sum z_1z_2 \cdots z_{n-1} = 0$ (Proved)

Example-8. Find the fifth roots of unity. **[JUH-1987]**

Solution : Let [ধরি] $z = (1)^{1/5} = (\cos 0 + i\sin 0)^{1/5}$

$$= (\cos 2n\pi + i\sin 2n\pi)^{1/5}$$

$$= \cos\frac{2n\pi}{5} + i\sin\frac{2n\pi}{5} = e^{2n\pi i/5},$$

where [যেখানে] n = 0, 1, 2, 3, 4.

Thus, the required roots are [অতএব প্রয়োজনীয় মূলগুলি হল]

$$1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}.$$

Example-9. Find all values of $(1+i)^{1/4}$.

[NUH-2001, 2005, 2008, 2012(Old)]

Solution : Let [ধরি] $1 = r\cos\theta$ and [এবং] $1 = r\sin\theta$

$\therefore 1^2 + 1^2 = r^2(\cos^2\theta + \sin^2\theta) \Rightarrow r = \sqrt{1+1} = \sqrt{2}$

and [এবং] $\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{1}{1} = 1 = \tan\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore (1+i)^{1/4} = (r\cos\theta + ir\sin\theta)^{1/4}$

$$= r^{1/4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/4}$$

$$= 2^{1/8}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}^{1/4} ; n = 0, 1, 2, 3.$$

$$= 2^{1/8}\left(\cos\frac{8n\pi + \pi}{16} + i\sin\frac{8n\pi + \pi}{16}\right); n = 0, 1, 2, 3.$$

$$= 2^{1/8}\left(\cos\frac{8n+1}{16}\pi + i\sin\frac{8n+1}{16}\pi\right); n = 0, 1, 2, 3.$$

Example-10. Determine all real x and y which satisfy the given relation

(a) $x + iy = |x + iy|$ (b) $x + iy = (x + iy)^2$

Solution : (a) Given that $x + iy = |x + iy|$

$$\Rightarrow x + iy = \sqrt{x^2 + y^2}$$

Equating real and imaginary parts we get,

$$x = \sqrt{x^2 + y^2} \text{ ...... (1)}$$

and $y = 0$ ...... (2)

Using (2) in (1) we get, $x = \sqrt{x^2}$

This equation will be hold if $x \geq 0$.

Thus all real $x \geq 0$ and $y = 0$ satisfy the relation

$$x + iy = |x + iy|.$$

(b) Given that $x + iy = (x + iy)^2$

$$\Rightarrow x + iy = x^2 + 2ixy + i^2y^2$$

$$\Rightarrow x + iy = x^2 + i\,2xy - y^2 \quad [\because i^2 = -1]$$

Equating real and imaginary parts we get

$$x = x^2 - y^2 \text{ ...... (1)}$$

and $y = 2xy$ ...... (2)

From (2) we have $(2x-1)y=0$

$$\Rightarrow y=0 \text{ or } 2x-1=0$$

$$\Rightarrow y=0 \text{ or } x=\frac{1}{2}$$

When $y=0$ then (1) gives $x=x^2 \Rightarrow x(x-1)=0$

$$\therefore x=0 \text{ or } x=1$$

When $x=\frac{1}{2}$ then (1) gives $\frac{1}{2}=\frac{1}{4}-y^2$

$$\Rightarrow y^2=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$$

$$\Rightarrow y=\pm\sqrt{-\frac{1}{4}}=\pm\frac{1}{2}i$$

Thus the real $x=0$ or 1 and $y=0$ satisfy the relation

$$x+iy=|x+iy|.$$

Example–11. Find all the roots of the equation $\sinh z = i$. **[NUH–1995]**

Solution : Given that [দেওয়া আছে] $\sinh z = i$

$$\Rightarrow \frac{e^z-e^{-z}}{2}=i$$

$$\Rightarrow w-w^{-1}=2i \quad \text{where [যেখানে] } w=e^z$$

$$\Rightarrow w^2-1=2iw$$

$$\Rightarrow w^2-2iw-1=0$$

$$\Rightarrow w=\frac{2i\pm\sqrt{(-2i)^2-4\cdot1\cdot(-1)}}{2}$$

$$=\frac{2i\pm\sqrt{4i^2+4}}{2}=\frac{2i\pm\sqrt{-4+4}}{2}$$

$$\Rightarrow e^z=\frac{2i}{2}=i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$$

$$\Rightarrow e^z=\cos\left(2n\pi+\frac{\pi}{2}\right)+i\sin\left(2n\pi+\frac{\pi}{2}\right)$$

$$\Rightarrow e^z=e^{i(2n\pi+\pi/2)}$$

$$\Rightarrow z=i\left(2n+\frac{1}{2}\right)\pi=\frac{i(4n+1)\pi}{2}$$

where $n=0,\pm1,\pm2,\ldots$

Example–12. Find all solutions of the equation $\cosh z = 2$.

[NUH-2002, 2006, 2010, 2012, DUH-2001]

Solution : Given that [দেওয়া আছে] $\cosh z = 2$

$$\Rightarrow \frac{e^z+e^{-z}}{2}=2$$

$$\Rightarrow e^z+\frac{1}{e^z}=4$$

$$\Rightarrow (e^z)^2+1=4e^z$$

$$\Rightarrow (e^z)^2-4e^z+1=0$$

$$\Rightarrow e^z=\frac{4\pm\sqrt{(-4)^2-4}}{2}$$

$$=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm\sqrt{12}}{2}$$

$$=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

$$\therefore z=\ln(2\pm\sqrt{3}) \qquad \text{Ans.}$$

Example–13(a). Prove that $|z|^2=(\text{Re}(z))^2+(\text{Im}(z))^2$, where z is any complex number.

Solution : Let $z=x+iy$. Then $\text{Re}(z)=x$, $\text{Im}(z)=y$ and

$$|z|^2=(\sqrt{x^2+y^2})^2=x^2+y^2$$

$$=(\text{Re}(z))^2+(\text{Im}(z))^2. \qquad \text{(Proved)}$$

(b). If z be any complex number, then prove that

(i) $\text{Re}(z)=\dfrac{z+\bar{z}}{2}$, (ii) $\text{Im}(z)=\dfrac{z-\bar{z}}{2i}$

Solution : Let $z=x+iy$. Then $\bar{z}=\overline{x+iy}=x-iy$.

Also $\text{Re}(z)=x$, $\text{Im}(z)=y$.

(i) $z+\bar{z}=x+iy+x-iy=2x$

$$\Rightarrow x=\frac{z+\bar{z}}{2} \Rightarrow \text{Re}(z)=\frac{z+\bar{z}}{2}$$

(ii) $z-\bar{z}=x+iy-x+iy=2iy$

$$\Rightarrow y=\frac{z-\bar{z}}{2i} \Rightarrow \text{Im}(z)=\frac{z-\bar{z}}{2i} \qquad \text{(Proved)}$$

Example–14. If z be a complex number then prove that

(i) z is real if $z = \bar{z}$.

and (ii) z is purely imaginary if $z = -\bar{z}$.

Solution : (i) Let $z = x + iy$ be a complex number.

Then $\bar{z} = x - iy$.

Now $z = \bar{z}$ gives $x + iy = x - iy$

$$\Rightarrow 2iy = 0 \Rightarrow y = 0$$

$\therefore z = x + iy = x + i0 = x$, which is real. Thus, z is real if $z = \bar{z}$.

(ii) Let $z = x + iy$ be a complex number. Then $\bar{z} = x - iy$.

Now $z = -\bar{z}$ gives $x + iy = -(x - iy)$

$$\Rightarrow 2x = 0 \Rightarrow x = 0$$

$\therefore z = x + iy = 0 + iy = iy$, which is purely imaginary.

Thus z is purely imaginary if $z = -\bar{z}$.

Example–15. Prove that $|z|^2 = |-z|^2 = |\bar{z}|^2 = |-\bar{z}|^2 = z\bar{z}$.

Solution : Let [ধরি] $z = x + iy$. Then [তখন] $\bar{z} = \overline{x + iy} = x - iy$.

$$|z|^2 = |x + iy|^2 = (\sqrt{x^2 + y^2})^2 = x^2 + y^2$$

$$|-z|^2 = |-x - iy|^2 = (\sqrt{(-x)^2 + (-y)^2})^2 = x^2 + y^2$$

$$|\bar{z}|^2 = |x - iy|^2 = (\sqrt{x^2 + (-y)^2})^2 = x^2 + y^2$$

$$|-\bar{z}|^2 = |-x + iy|^2 = (\sqrt{(-x)^2 + y^2})^2 = x^2 + y^2$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2$$

Thus we see that [অতএব আমরা দেখি যে]

$$|z|^2 = |-z|^2 = |\bar{z}|^2 = |-\bar{z}|^2 = z\bar{z}.$$

Example–16. Prove that :

(i) $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$, (ii) $\bar{\bar{z}} = z$, and (iii) $\overline{z + \bar{z}} = z + \bar{z}$.

Solution : (i) Let [ধরি] $z_1 = x_1 + iy_1$ and [এবং] $z_2 = x_2 + iy_2$.

Then [তখন] $\bar{z_1} = x_1 - iy_1$ and [এবং] $\bar{z_2} = x_2 - iy_2$.

$\therefore z_1 + z_2 = x_1 + iy_1 + x_2 + iy = (x_1 + x_2) + i(y_1 + y_2)$

$\Rightarrow \overline{z_1 + z_2} = x_1 + x_2 - i(y_1 + y_2)$

$\Rightarrow \overline{z_1 + z_2} = x_1 - iy_1 + x_2 - iy_2$

$\Rightarrow \overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$

(ii) Let [ধরি] $z = x + iy$. Then [তখন] $\bar{z} = x - iy \Rightarrow \bar{\bar{z}} = x + iy = z$.

(iii) Let $z = x + iy$. Then $\bar{z} = x - iy$.

$\therefore z + \bar{z} = x + iy + x - iy = 2x$

$\overline{z + \bar{z}} = \overline{2x} = 2x$, since $2x$ is real.

$\Rightarrow \overline{z + \bar{z}} = 2x = (x + iy) + (x - iy) = z + \bar{z}$

Otherway : $\overline{z + \bar{z}} = \bar{z} + \bar{\bar{z}}$ [by (i)]

$= \bar{z} + z$ [by (ii)]

$\Rightarrow \overline{z + \bar{z}} = z + \bar{z}$

Example–17. Show that

$$\mathrm{Im}(iz) = \mathrm{Re}(z) \text{ and } \mathrm{Re}(iz) = |z|^2 \,\mathrm{Im}(z^{-1}).$$

Solution : Let $z = x + iy$. Then $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$

Now $\mathrm{Im}(iz) = \mathrm{Im}(ix + i^2y) = \mathrm{Im}(ix - y) = x = \mathrm{Re}(z)$

and $\mathrm{Re}(iz) = \mathrm{Re}(ix + i^2y) = \mathrm{Re}(ix - y) = -y$

On the other hand, $|z|^2 \,\mathrm{Im}(z^{-1}) = |x + iy|^2 \,\mathrm{Im}\left(\frac{1}{z}\right)$

$$= (x^2 + y^2).\ \mathrm{Im}\left(\frac{1}{x + iy} \cdot \frac{x - iy}{x - iy}\right)$$

$$= (x^2 + y^2).\ \mathrm{Im}\left(\frac{x - iy}{x^2 - i^2y^2}\right)$$

$$= (x^2 + y^2).\ \mathrm{Im}\left(\frac{x - iy}{x^2 + y^2}\right) \quad [\because i^2 = -1]$$

$$= (x^2 + y^2).\ \frac{-y}{x^2 + y^2} = -y$$

Thus, $\mathrm{Re}(iz) = -y = |z|^2 \,\mathrm{Im}(z^{-1})$. (Showed)

Example–18. Show that the modulas of the quotient of two conjugate complex numbers is 1. **[RUH–1985, JUH–1986]**

Solution : Let $z = x + iy$ be a complex number. Then its conjugate number $\bar{z} = \overline{x + iy} = x - iy$.

$$\therefore \left|\frac{z}{\bar{z}}\right| = \left|\frac{x + iy}{x - iy}\right| = \frac{|x + iy|}{|x - iy|} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + (-y)^2}} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = 1$$

Thus, the modulas of the quotient of two conjugate complex numbers is 1.

Other way : Let $z = r\,e^{i\theta}$. Then $\bar{z} = re^{-i\theta}$.

$$\therefore \left|\frac{z}{\bar{z}}\right| = \left|\frac{re^{i\theta}}{re^{-i\theta}}\right| = |e^{i2\theta}| = |\cos 2\theta + i \sin 2\theta|$$

$$= \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1. \quad \text{Hence proved.}$$

Example–19. Show that, $|z|\ \sqrt{2} \geq |Re(z)| + |Im(z)|$, where z any complex number. **[NUH–94, 13, NU(Pre)–08, DUH–88, 90, (**

Solution : Let [ধরি] $z = x + iy$.

Then [তখন] $Re(z) = x$ and [এবং] $Im(z) = y$.

For any two positive real numbers we know that [যে কোন দুই ধনাত্মক বাস্তব সংখ্যার জন্য আমরা জানি যে]

$$\frac{a^m + b^m}{2} \geq \left(\frac{a+b}{2}\right)^m, \text{ where m is any real number except } 0 < m <$$

Taking $a = |x|$, $b = |y|$ and $m = 2$ we have [$a = |x|$, $b = |y|$ ও $m = 2$ নিয়ে পাই]

$$\frac{|x|^2 + |y|^2}{2} \geq \left(\frac{|x| + |y|}{2}\right)^2$$

$$\Rightarrow x^2 + y^2 \geq \frac{(|x| + |y|)^2}{2}$$

$$\Rightarrow (\sqrt{x^2 + y^2})^2 \geq \left(\frac{|x| + |y|}{\sqrt{2}}\right)^2$$

$$\Rightarrow \sqrt{x^2 + y^2} \geq \frac{|x| + |y|}{\sqrt{2}}$$

$$\Rightarrow \sqrt{2}\ |z| \geq |x| + |y|$$

$$\Rightarrow \sqrt{2}\ |z| \geq |Re(z)| + |Im(z)| \qquad \text{(Showed)}$$

Example–20. Prove that $|x| + |y| \leq \sqrt{2}\ |z|$.

(a) If $z = x + iy$ then prove that $|x| + |y| \leq \sqrt{2}\ |x + iy|$

[NUH–1998, 2012(0

Solution : From the above problem we have [উপরের সমস্যা হ পাই]

$$\Rightarrow \sqrt{2}\ |z| \geq |x| + |y|$$

$$\Rightarrow |x| + |y| \leq \sqrt{2}\ |z|$$

$$\Rightarrow |x| + |y| \leq \sqrt{2}\ |x + iy| \qquad \text{(Proved)}$$

Example–21. If $z_1, z_2, \cdots, z_n$ are complex numbers then show that [যদি $z_1, z_2, \cdots, z_n$ জটিল সংখ্যা হয় তখন দেখাও যে]

(i) $|z_1 z_2| = |z_1|\ |z_2|$ **[DUM–1989]**

(ii) $|z_1 z_2 \cdots z_n| = |z_1|\ |z_2| \cdots |z_n|$ **[DUH–1985]**

Solution : (i) Let [ধরি] $z_1 = x_1 + iy_1$ and [এবং] $z_2 = x_2 + iy_2$.

Then [তখন] $|z_1| = |x_1 + iy_1| = \sqrt{x_1^2 + y_1^2}$ and [এবং]

$$|z_2| = |x_2 + iy_2| = \sqrt{x_2^2 + y_2^2}.$$

Now [এখন] $|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)|$

$$= |x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2|$$

$$= |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)|$$

$$= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2}$$

$$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2 + x_2^2y_1^2 + 2x_1x_2y_1y_2}$$

$$= \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}$$

$$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$= \sqrt{x_1^2 + y_1^2}\ \sqrt{x_2^2 + y_2^2}$$

$$= |z_1|\ |z_2|$$

$$\therefore |z_1 z_2| = |z_1|\ |z_2|$$

Other way [অন্যভাবে] **:** We know that [আমরা জানি যে]

$|z|^2 = z\bar{z}$ and [এবং] $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$.

$$\therefore |z_1 z_2|^2 = (z_1 z_2)\,\overline{(z_1 z_2)}$$

$$= z_1 z_2\, \bar{z}_1\, \bar{z}_2 = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2\, |z_2|^2$$

$$= (|z_1|\ |z_2|)^2$$

$$\Rightarrow |z_1 z_2| = |z_1|\ |z_2|$$

(ii) $|z_1 z_2 \cdots z_n|^2 = (z_1 z_2 \cdots z_n)\,\overline{(z_1 z_2 \cdots z_n)}$

$$= (z_1 z_2 \cdots z_n)(\bar{z}_1\, \bar{z}_2 \cdots \bar{z}_n)$$

$$= (z_1\bar{z}_1)(z_2\bar{z}_2) \cdots (z_n)(\bar{z}_n)$$

$$= |z_1|^2\ |z_2|^2 \cdots |z_n|^2$$

$$= (|z_1|\ |z_2| \cdots |z_n|)^2$$

$$\Rightarrow |z_1 z_2 \cdots z_n| = |z_1|\ |z_2| \cdots |z_n| \qquad \text{(proved)}$$

Example–22. For any complex number $z_1, z_2, \cdots, z_n$ pro[ve] that [যে কোন জটিল সংখ্যা $z_1, z_2, \cdots, z_n$ এর জন্য প্রমাণ কর যে]

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$

[NUH–02 (Old), 03, 04, 05, 07, 10, 12, NU(Pre)–08, DUH–98, 0

(ii) $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$

[RUH–1998, CUH–200

(iii) $|z_1 - z_2| \leq |z_1| + |z_2|$ **[NUH–200**

Solution : We know that [আমরা জানি] $|z|^2 = z\bar{z}$ and $\bar{\bar{z}} = z$.

$$\therefore |z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$$
$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$
$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$$
$$= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2$$
$$= |z_1|^2 + 2\text{Re}(z_1\overline{z_2}) + |z_2|^2 \quad [\because z + \bar{z} = 2\text{Re}(z)]$$
$$\Rightarrow |z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \quad [\because \text{Re}(z) \leq |z|]$$
$$\Rightarrow |z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1|\,|\overline{z_2}| + |z_2|^2 \quad [\because |z_1 z_2| = |z_1|\,|z_2|]$$
$$\Rightarrow |z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2 \quad [\because |z| = |\bar{z}|]$$
$$\Rightarrow |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$
$$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|.$$

Other way [অন্যভাবে] **:**

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then $|z_1| = |r_1 e^{i\theta_1}|$

$$= |r_1|\,|e^{i\theta_1}| = r_1 \cdot 1 = r_1 \text{ and } |z_2| = r_2$$

$$\therefore z_1 + z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2} = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$$
$$\Rightarrow |z_1 + z_2|^2 = (r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$$
$$= r_1^2\cos^2\theta_1 + 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\cos\theta_2^2$$
$$+ r_1^2\sin^2\theta_1 + 2r_1r_2\sin\theta_1\sin\theta_2 + r_2^2\sin^2\theta_2$$
$$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + 2r_1r_2(\cos\theta_1\cos\theta_2$$
$$+ \sin\theta_1\sin\theta_2) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$$
$$= r_1^2 \cdot 1 + 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2 \cdot 1$$
$$\Rightarrow |z_1 + z_2|^2 \leq r_1^2 + 2r_1r_2 + r_2^2 \quad [\because \cos(\theta_1 - \theta_2) \leq 1]$$

$$\Rightarrow |z_1 + z_2|^2 \leq (r_1 + r_2)^2$$
$$\Rightarrow |z_1 + z_2| \leq r_1 + r_2$$
$$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$$

(ii) $|z_1 + z_2 + \cdots + z_n| = |z_1 + (z_2 + \cdots + z_n)|$

$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2 + \cdots + z_n| \quad [\text{by (i)}]$$
$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2 + (z_3 + \cdots + z_n)|$$
$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3 + \cdots + z_n|$$

Proceding in the same way we get [একইভাবে অগ্রসর হয়ে আমরা পাই]

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

(iii) $|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)}$ $\quad [\because |z|^2 = z\bar{z}]$

$$= (z_1 - z_2)(\overline{z_1} - \overline{z_2}) \quad [\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}]$$
$$= z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2}$$
$$= |z_1|^2 - (z_1\overline{z_2} + z_2\overline{z_1}) + |z_2|^2 \quad [\because |z|^2 = z\bar{z}]$$
$$= |z_1|^2 - (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + |z_2|^2 \quad [\because \bar{\bar{z}} = z]$$
$$= |z_1|^2 - 2\text{Re}(z_1\overline{z_2}) + |z_2|^2$$
$$\Rightarrow |z_1 - z_2|^2 \leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \quad \left\{\begin{array}{l}\because -x \leq \sqrt{x^2 + y^2} \\ \Rightarrow -\text{Re}(z) \leq |z|\end{array}\right.$$
$$\Rightarrow |z_1 - z_2|^2 \leq (|z_1| + |z_2|)^2$$
$$\Rightarrow |z_1 - z_2| \leq |z_1| + |z_2| \quad \text{(Proved)}$$

Example–23. Let **C** be the set of all complex numbers. Consider $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in$ **C** with $x_1 < x_2$, $y_1 < y_2$. Do you agree that $z_1 < z_2$? What about $|z_1| < |z_2|$? Prove that $\left|\sum_{j=1}^{n} z_j\right| \leq \sum_{j=1}^{n} |z_j|$ and $\left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{n} |z_j|$, where $z_1, z_2, \cdots, z_n$ are complex numbers. **[NUH–1997]**

Solution : Given that $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$. That is, z_1 and z_2 are two points in the Argand (complex) plane. We know that greater than or less than have no meaning in relation between two complex numbers. So $z_1 < z_2$ has no meaning. Thus, I do not agree that $z_1 < z_2$. [দেওয়া আছে $z_1 = x_1 + iy_1 = (x_1, y_1)$ এবং $z_2 = x_2 + iy_2 = (x_2, y_2)$. অর্থাৎ, আর্গান্ড তলে z_1 ও z_2 দুইটি বিন্দু। আমরা জানি দুইটি জটিল সংখ্যা সর্ম্পকের ক্ষেত্রে হতে বৃহত্তর বা হতে ক্ষুদ্রতর-এর কোন অর্থ নাই। সুতরাং $z_1 < z_2$ ইহার কোন অর্থ নাই। অতএব আমি সম্মত নই যে $z_1 < z_2$.]

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2nd Part : Given that [দেওয়া আছে] $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

where [যেখানে] $x_1 < x_2$, $y_1 < y_2$.

$\therefore |z_1| = \sqrt{x_1^2 + y_1^2}$ and [এবং] $|z_2| = \sqrt{x_2^2 + y_2^2}$

Now [এখন] $x_1 < x_2 \Rightarrow x_1^2 < x_2^2$

and [এবং] $y_1 < y_2 \Rightarrow y_1^2 < y_2^2$

$\Rightarrow x_1^2 + y_1^2 < x_2^2 + y_2^2$ [by adding]

$\Rightarrow \sqrt{x_1^2 + y_1^2} < \sqrt{x_2^2 + y_2^2}$ [by taking squre ro

$\Rightarrow |z_1| < |z_2|$

But this inequality is not always true. We counter this by t following example. [কিন্তু এই অসমতা সব সময় সত্য নয়। আমরা নিম্নের উদা দ্বারা ইহার বিরোধিতা করি।]

Let [ধরি] $z_1 = x_1 + iy_1 = 1 + i(-7)$

and [এবং] $z_2 = x_2 + iy_2 = 3 + i2$

Here [এখানে] $x_1 = 1, x_2 = 3, y_1 = -7, y_2 = 2$

$1 < 3 \Rightarrow x_1 < x_2$ and [এবং] $-7 < 2 \Rightarrow y_1 < y_2$

$$|z_1| = \sqrt{x_1^2 + y_1^2} = \sqrt{1^2 + (-7)^2} = \sqrt{50}$$

$$|z_2| = \sqrt{x_2^2 + y_2^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

It is true that [ইহা সত্য যে] $\sqrt{50} > \sqrt{13} \Rightarrow |z_1| > |z_2|$

Thus, $|z_1| < |z_2|$ is not always true under the given conditi $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than point z_2 is. [$|z_1| < |z_2|$ সবসময় সত্য নয়। $|z_1| < |z_2|$ এর অর্থ হলো z_1 বিন্দু বিন্দু হতে মূলবিন্দুর অধিক নিকটে।]

3rd Part : Given that [দেওয়া আছে] $\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j|$

$\Rightarrow |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$

Now do as example-22.

4th Part : Given that $\left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{n} |z_j|$

that is, $|z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$

Now do as example-22.

Example-24. Prove that [প্রমাণ কর]

$$|z_1 - z_2| \ge \big||z_1| - |z_2|\big| \ge |z_1| - |z_2|$$

[NUH-1994, 2002 (Old), DUH-1998, 2005]

Solution : We know that [আমরা জানি]

$|z|^2 = z\bar{z}$ and [এবং] $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$$\begin{aligned}
\therefore |z_1 - z_2|^2 &= (z_1 - z_2)\,\overline{(z_1 - z_2)} \\
&= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\
&= z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2 \\
&= |z_1|^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2) + |z_2|^2 \\
&= |z_1|^2 - (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + |z_2|^2 \qquad [\because \bar{\bar{z}} = z] \\
&= |z_1|^2 - 2\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \qquad [\because z + \bar{z} = 2\mathrm{Re}(z)]
\end{aligned}$$

$\Rightarrow |z_1 - z_2|^2 \ge |z_1|^2 - 2|z_1\bar{z}_2| + |z_2|^2$ $\left[\begin{array}{l}\because x = \mathrm{Re}(z) \le |z| \\ \Rightarrow -\mathrm{Re}(z) \ge -|z|\end{array}\right]$

$\Rightarrow |z_1 - z_2|^2 \ge |z_1|^2 - 2|z_1|\,|\bar{z}_2| + |z_2|^2$

$\Rightarrow |z_1 - z_2|^2 \ge |z_1|^2 - 2|z_1|\,|z_2| + |z_2|^2$ $[\because |z| = |\bar{z}|]$

$\Rightarrow |z_1 - z_2|^2 \ge (|z_1| - |z|)^2 = (\big||z_1| - |z_2|\big|)^2$

$\Rightarrow |z_1 - z_2| \ge \big||z_1| - |z_2|\big|$ ...... (1)

Again, we have [আবার, আমাদের আছে]

$$\big||z_1| - |z_2|\big| \ge |z_1| - |z_2| \text{ ...... (2)}$$

From (1) and (2) we have [(1) ও (2) হতে পাই]

$$|z_1 - z_2| \ge \big||z_1| - |z_2|\big| \ge |z_1| - |z_2| \qquad \text{(Proved)}$$

Example-25. If z_1, z_2, z_3, z_4 are complex numbers then show that [যদি z_1, z_2, z_3, z_4 জটিল সংখ্যা হয় তখন দেখাও]

(i) $\left|\dfrac{z_1}{z_2 + z_3}\right| \le \dfrac{|z_1|}{\big||z_2| - |z_3|\big|}$, where [যেখানে] $|z_2| \ne |z_3|$

[NUH-2011, DUH-98]

(ii) $\left|\dfrac{z_1 + z_2}{z_3 + z_4}\right| \le \dfrac{|z_1| + |z_2|}{\big||z_3| - |z_4|\big|}$, where [যেখানে] $|z_3| \ne |z_4|$.

[CUH-2000, DUH-2001, 2003, 2006]

Solution : (i) We know that [আমরা জানি] $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

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Replacing z_2 by $-z_2$ we get [z_2 কে $-z_2$ দ্বারা প্রতিস্থাপন করে পাই]

$$|z_1+z_2| \geq \big||z_1| - |-z_2|\big|$$

$$\Rightarrow |z_1+z_2| \geq \big||z_1| - |z_2|\big| \qquad [\because |z| = |-z| \text{ ...... (1)}]$$

$$\Rightarrow \frac{1}{|z_1+z_2|} \leq \frac{1}{\big||z_1|-|z_2|\big|}$$

Multiplying both sides by $|z_1|$ we get [উভয় পক্ষকে $|z_1|$ দ্বারা গুণ করে পাই]

$$\frac{|z_1|}{|z_2+z_3|} \leq \frac{|z_1|}{\big||z_2|-|z_3|\big|}$$

(ii) We know that [আমরা জানি] $|z_1+z_2| \leq |z_1| + |z_2|$ ...... (2)

and by (1) we have [এবং (1) হতে পাই]

$$|z_3+z_4| \geq \big||z_3|-|z_4|\big|$$

$$\Rightarrow \frac{1}{|z_3+z_4|} \leq \frac{1}{\big||z_3|-|z_4|\big|} \text{ ...... (3)}$$

Combining (2) and (3) we get [(1) ও (2) কে একত্র করে পাই]

$$\frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|} \qquad \text{(Proved)}$$

Example–26. Find two complex numbers whose sum is 4 and whose product is 8. [দুইটি জটিল সংখ্যা বাহির কর যাদের যোগফল 4 ও গুণফল 8]

[NUH–2000, 2006(Old), 20...]

Solution : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. According to the question [ধরি $z_1 = x_1 + iy_1$ এবং $z_2 = x_2 + iy_2$ দুইটি জটিল সংখ্যা। প্রশ্নানুসারে]

$$z_1 + z_2 = 4 \text{ ...... (1)}$$

and [এবং] $z_1 z_2 = 8$ ...... (2)

From (1) [(1) হতে] $z_1 + z_2 = 4$

$$\Rightarrow x_1 + iy_1 + x_2 + iy_2 = 4$$

$$\Rightarrow (x_1 + x_2) + i(y_1 + y_2) = 4$$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$x_1 + x_2 = 4 \text{ ...... (3)}$$

$$y_1 + y_2 = 0 \text{ ...... (4)}$$

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Again, from (2) we get [আবার, (2) হতে পাই]

$$(x_1 + iy_1)(x_2 + iy_2) = 8$$

$$\Rightarrow x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 = 8$$

$$\Rightarrow (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = 8 \qquad [\because i^2 = -1]$$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$x_1x_2 - y_1y_2 = 8 \text{ ...... (5)}$$

$$x_1y_2 + x_2y_1 = 0 \text{ ...... (6)}$$

From (4) [(4) হতে] $y_1 = -y_2$

From (6) [(6) হতে] $x_1y_2 = -x_2y_1$

$$\Rightarrow \frac{x_1}{x_2} = \frac{-y_1}{y_2} = \frac{-(-y_2)}{y_2} = 1$$

$$\Rightarrow x_1 = x_2$$

From (3) [(3) হতে] $x_1 + x_2 = 4 \Rightarrow 2x_1 = 4 \Rightarrow x_1 = \frac{4}{2} = 2$

$$\therefore x_1 = x_2 = 2$$

From (5) [(5) হতে] $2 \cdot 2 - (-y_2)y_2 = 8$

$$\Rightarrow y_2^2 = 8 - 4 = 4$$

$$\therefore y_2 = \pm 2$$

$$\therefore y_1 = -y_2 = -(\pm 2) = \mp 2$$

That is [অর্থাৎ] $y_1 = -2,\ y_2 = 2$ and [এবং] $y_1 = 2,\ y_2 = -2$.

$$\therefore z_1 = x_1 + iy_1 = 2 - 2i \text{ or } 2 + 2i$$

$$z_2 = x_2 + iy_2 = 2 + 2i \text{ or } 2 - 2i$$

Thus, the required two complex numbers are $2 + 2i$ and $2 - 2i$. [অতএব প্রার্থিত (আবশ্যকীয়) দুইটি জটিল সংখ্যা হল $2 + 2i$ এবং $2 - 2i$]

Other way [অন্যভাবে] **:** Let z_1 and z_2 be two complex numbers. Then according to the question [ধরি z_1 ও z_2 দুইটি জটিল সংখ্যা, তখন প্রশ্ন অনুসারে]

$$z_1 + z_2 = 4 \text{ ...... (1)}$$

and [এবং] $z_1z_2 = 8$ ...... (2)

Now [এবং] $(z_1 - z_2)^2 = (z_1 + z_2)^2 - 4z_1z_2$

$$= 4^2 - 4 \cdot 8 = 16 - 32 = -16$$
$$= (4i)^2$$

$\therefore z_1 - z_2 = \pm 4i$

$\Rightarrow z_1 - z_2 = 4i \cdots\cdots (3)$

or $z_1 - z_2 = -4i \cdots\cdots (4)$

(1) + (3) gives [দেয়] $2z_1 = 4 + 4i \Rightarrow z_1 = \frac{4+4i}{2} = 2 + 2i$

(1) – (3) gives [দেয়] $2z_2 = 4 - 4i \Rightarrow z_2 = \frac{4-4i}{2} = 2 - 2i$

(1) + (4) gives [দেয়] $2z_1 = 4 - 4i \Rightarrow z_1 = \frac{4-4i}{2} = 2 - 2i$

(1) – (4) gives [দেয়] $2z_2 = 4 + 4i \Rightarrow z_2 = \frac{4+4i}{2} = 2 + 2i$

Thus the two complex numbers are $2 + 2i$, $2 - 2i$. [অতএব দু জটিল সংখ্যা হল $2 + 2i$, $2 - 2i$.]

Example–27. Prove that, if sum and product of two comple numbers are both real, then the two numbers must either be re or conjugate.

Solution : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two comple numbers. Then their sum $= z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x$ $+ i(y_1 + y_2)$ and product $= z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$$= x_1x_2 + i(x_1y_2 + x_2y_1) + i^2y_1y_2$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Sum will be real if $y_1 + y_2 = 0 \Rightarrow y_2 = -y_1 \cdots\cdots (1)$

or $y_1 = y_2 = 0 \cdots\cdots (2)$

Product will be real if $x_1y_2 + x_2y_1 = 0$

$$\Rightarrow x_1y_2 = -x_2y_1$$
$$\Rightarrow \frac{x_1}{x_2} = \frac{-y_1}{y_2} \cdots\cdots (3)$$

(1) and (3) gives, $\frac{x_1}{x_2} = \frac{-y_1}{-y_1} = 1 \Rightarrow x_1 = x_2$

Now, when $y_1 = y_2 = 0$ then $z_1 = x_1 + i0 = x_1$ and $z_2 = x_2 + i0 = x_2$

In this case, z_1 and z_2 are both real.

When $x_1 = x_2$ and $y_2 = -y_1$ then

$$z_1 = x_1 + iy_1 = x_2 - iy_2 = \overline{x_2 + iy_2} = \overline{z_2}$$
$$z_2 = x_2 + iy_2 = x_1 - iy_1 = \overline{x_1 + iy_1} = \overline{z_1}$$

Thus, if sum and product of two complex numbers are both real, then the two numbers must either be real or conjugate.

(Proved)

Example–28. Prove that [প্রমাণ কর]

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2.$$

[NUH–2000, 2006 (Old)]

Interpret the result geometrically and deduce that [ফলটি জ্যামিতিকভাবে ব্যাখ্যা কর এবং প্রতিষ্ঠিত কর]

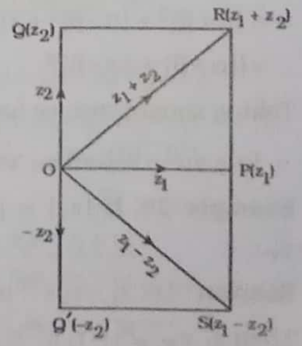
$$|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|.$$

[NUH–1993, 2004 (Old), 2006 (Old), 2012, NU(Pre)–2011]

Solution : $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$$= (z_1 + z_2)\,\overline{(z_1 + z_2)} + (z_1 - z_2)\,\overline{(z_1 - z_2)}$$
$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$
$$= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - \overline{z_1}z_2 + z_2\overline{z_2}$$
$$= |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2$$
$$= 2|z_1|^2 + 2|z_2|^2$$

2nd Part (Geometrical interpretation) :



Let z_1 and z_2 be two complex numbers represented by the points P and Q in the Argand diagram. Complete the parallelogram OPRQ. Produce OQ backward upto Q′ so that OQ′ = OQ. Complete the parallelogram OQ′SP. Then the diagonals OR represents $z_1 + z_2$ and OS represents $z_1 - z_2$.

Thus we have $|z_1| = OP$, $|z_2| = OQ = PR$, $|z_1 + z_2| = OR$, $|z_1 - z_2|$ = OS. Here P is the middle point of RS. Hence we have

$$OR^2 + OS^2 = 2OP^2 + 2PR^2$$

$$\Rightarrow |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,|z_1|^2 + 2|z_2|^2$$

[২য় অংশ (জ্যামিতিক ব্যাখ্যা) ঃ

মনে করি আর্গাণ্ড চিত্রে P ও Q বিন্দু দ্বারা নির্দেশিত দুইটি জটিল সংখ্যা z_1 ও z_2 OPQR সামন্তরিকটি সম্পূর্ণ করি। OQ কে পিছনের দিকে Q′ পর্যন্ত বর্ধিত করি যেন OQ′ = OQ হয়। OQ′SP সামন্তরিকটি সম্পূর্ণ করি। তখন কর্ণ OR, $z_1 + z_2$ নির্দেশ করে এবং OS, $z_1 - z_2$ নির্দেশ করে।

অতএব আমরা পাই $|z_1| = OP$, $|z_2| = OQ = PR$, $|z_1 + z_2| = OR$, $|z_1 - z_2|$ = OS. এখানে RS এর মধ্যবিন্দু হল P. অতএব আমরা পাই,

$$OR^2 + OS^2 = 2OP^2 + 2PR^2$$

$$\Rightarrow |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,|z_1|^2 + 2|z_2|^2]$$

3rd Part : We know that [আমরা জানি]

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2) \;\cdots\cdots (1)$$

Now [এখন] $(|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}|)^2$

$$= |\alpha + \sqrt{\alpha^2 - \beta^2}|^2 + |\alpha - \sqrt{\alpha^2 - \beta^2}|^2$$
$$+ 2|\alpha + \sqrt{\alpha^2 - \beta^2}|\;|\alpha - \sqrt{\alpha^2 - \beta^2}|$$

$$= 2|\alpha|^2 + 2|\sqrt{\alpha^2 - \beta^2}|^2 + 2|(\alpha + \sqrt{\alpha^2 - \beta^2})\,(\alpha - \sqrt{\alpha^2 - \beta^2})| \quad [\text{by (1)}]$$

$$= 2|\alpha|^2 + 2|\alpha^2 - \beta^2| + 2|\alpha^2 - \alpha^2 + \beta^2|$$

$$= 2|\alpha|^2 + 2|\alpha + \beta|\;|\alpha - \beta| + 2|\beta|^2$$

$$= |\alpha + \beta|^2 + |\alpha - \beta|^2 + 2|\alpha + \beta|\;|\alpha - \beta|$$

$$= (|\alpha + \beta| + |\alpha - \beta|)^2$$

Taking square root we have [বর্গমূল নিয়ে পাই]

$$|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta| \qquad \text{(Proved)}$$

Example–29. If $|z_1| = |z_2|$ and amp z_1 + amp z_2 = 0, then show that $z_2 = \bar{z}_1$.

Solution : Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

Then $\bar{z}_1 = \overline{r_1 e^{i\theta_1}} = r_1 e^{-i\theta_1}$

Given that $|z_1| = |z_2|$

$$\Rightarrow |r_1 e^{i\theta_1}| = |r_2 e^{i\theta_2}|$$

$$\Rightarrow |r_1|\;|e^{i\theta_1}| = |r_2|\;|e^{i\theta_2}| \qquad [\because |e^{i\theta}| = |\cos\theta + i\sin\theta|$$

$$\Rightarrow r_1 \cdot 1 = r_2 \cdot 1 \Rightarrow r_1 = r_2 \qquad = \sqrt{\cos^2\theta + \sin^2\theta} = 1]$$

Again, amp z_1 + amp z_2 = 0

$$\Rightarrow \text{amp}(z_1 z_2) = 0$$

$$\Rightarrow \theta_1 + \theta_2 = 2n\pi, \text{ where } n = 0, \pm 1, \pm 2, \cdots \text{ etc.}$$

$$\Rightarrow \theta_2 = 2n\pi - \theta_1$$

$$\therefore z_2 = r_2 e^{i\theta_2} = r_1 e^{i(2n\pi - \theta_1)}$$

$$= r_1 [\cos(2n\pi - \theta_1) + i\sin(2n\pi - \theta_1)]$$

$$= r_1(\cos\theta_1 - i\sin\theta_1)$$

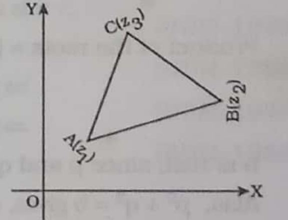
$$= r_1[\cos(-\theta_1) + i\sin(-\theta_1)] \qquad [\because \cos\theta = \cos(-\theta)]$$

$$= r_1 e^{-i\theta_1} = \bar{z}_1 .$$

Example–30. If z_1, z_2, z_3 are the vertices of an equilateral triangle in the argand plane, then show that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution : Let the triangle be ΔABC, where the points A, B, C are z_1, z_2, z_3 respectively. The triangle is equilateral, so



$$AB = BC = CA$$

$$\Rightarrow |z_2 - z_1| = |z_3 - z_2| = |z_1 - z_3|$$

$$\Rightarrow |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \qquad [\because |z| = |-z|]$$

$$\Rightarrow |z_1 - z_2|^2 = |z_2 - z_3|^2 = |z_3 - z_1|^2$$

$$\Rightarrow (z_1 - z_2)\,\overline{(z_1 - z_2)} = (z_2 - z_3)\,\overline{(z_2 - z_3)} = (z_3 - z_1)\,\overline{(z_3 - z_1)}$$

$$\Rightarrow (z_1 - z_2)\,(\bar{z}_1 - \bar{z}_2) = (z_2 - z_3)\,(\bar{z}_2 - \bar{z}_3) = (z_3 - z_1)\,(\bar{z}_3 - \bar{z}_1) \;\cdots\cdots (1)$$

From the first and second relation we have

$$(z_1 - z_2)\,(\bar{z}_1 - \bar{z}_2) = (z_2 - z_3)\,(\bar{z}_2 - \bar{z}_3)$$

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_2 - z_3}{\bar{z}_1 - \bar{z}_2} = \frac{(z_1 - z_2) + (z_2 - z_3)}{(\bar{z}_2 - \bar{z}_3) + (\bar{z}_1 - \bar{z}_2)} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$$

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3} \;\cdots\cdots (2)$$

From the last two relation of (1) we have

$$(z_2 - z_3)(\overline{z_2} - \overline{z_3}) = (z_3 - z_1)(\overline{z_3} - \overline{z_1}) \cdots\cdots (3)$$

Multiplying (2) and (3) we get

$$\frac{z_1 - z_2}{z_2 - z_3}(z_2 - z_3)(\overline{z_2} - \overline{z_3}) = \frac{z_1 - z_3}{z_1 - z_3}(z_3 - z_1)(\overline{z_3} - \overline{z_1})$$

$$\Rightarrow (z_1 - z_2)(z_2 - z_3) = (z_1 - z_3)(z_1 - z_3)$$

$$\Rightarrow z_1z_2 - z_1z_3 - z_2^2 + z_2z_3 = z_1^2 - z_3z_1 - z_3z_1 + z_3^2$$

$$\Rightarrow z_1z_2 + z_2z_3 + z_3z_1 = z_1^2 + z_2^2 + z_3^2$$

$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 \quad \text{(Showed)}$$

Example–31. If the equation $z^2 + az + b = 0$ has a pair of conjugate complex roots then prove that a, b are both real and $a^2 < 4b$.

Solution : Let the complex conjugate roots of $z^2 + az + b = 0$ are $p + iq$ and $p - iq$ where p and q are real.

sum of the roots $= p + iq + p - iq = -a$

$$\Rightarrow 2p = -a \Rightarrow p = \frac{-a}{2}.$$

$\therefore$ a is real, since p is real.

Product of the roots $= (p + iq)(p - iq) = b$

$$\Rightarrow p^2 - i^2q^2 = b$$

$$\Rightarrow p^2 + q^2 = b$$

b is real, since p and q are real.

Also, $p^2 + q^2 = b$ gives, $q^2 = b - p^2 > 0$

$$\Rightarrow b - p^2 > 0 \Rightarrow b - \left(\frac{-a}{2}\right)^2 > 0$$

$$\Rightarrow b - \frac{a^2}{4} > 0 \Rightarrow 4b - a^2 > 0$$

$$\Rightarrow 4b > a^2 \Rightarrow a^2 < 4b$$

Thus, we have a, b are both real and $a^2 < 4b$.

Example–32. Solve the equation $|z| - z = 2 + i$.

Solution : Let $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2}$

Given that $|z| - z = 2 + i$

$$\Rightarrow \sqrt{x^2 + y^2} - (x + iy) = 2 + i$$

Equating real and imaginary parts we get,

$$\sqrt{x^2 + y^2} - x = 2 \cdots\cdots (1)$$

$$-y = 1 \Rightarrow y = -1 \cdots\cdots (2)$$

Using (2) in (1) we get,

$$\sqrt{x^2 + 1} - x = 2$$

$$\Rightarrow \sqrt{x^2 + 1} = x + 2$$

$$\Rightarrow x^2 + 1 = x^2 + 4x + 4 \quad \text{by squaring}$$

$$\Rightarrow 4x = -3 \Rightarrow x = \frac{-3}{4}$$

$$\therefore z = x + iy = \frac{-3}{4} - i \quad \text{(Ans)}$$

Example–33. Describe geometrically the region of the following :

(i) $|z - 4| > |z|$ [NUH–1998, DUH–1988, 1998]

(ii) $|z - i| = |z + i|$ [NUH–03, 06, 10, 12, DUH–87, RUH–04]

(iii) $\text{Im}(z) > 1$ [DUH–1988]

(iv) $\text{Re}(\overline{z} - 1) = 2$

(v) $|z + 3i| > 4$ [DUH–1989]

(vi) $|z| > 4$ [DUH–1986]

(vii) $|z - 2 + i| \le 1$ [DUH–1988]

(viii) $|2z + 3| > 4$ [DUH–1988]

(ix) $\left|\frac{z-3}{z+3}\right| = 3$

(x) $\left|\frac{z-3}{z+3}\right| > 3$

(xi) $\left|\frac{z-3}{z+3}\right| < 3$ [DUH–1989]

(xii) $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$ [NU(Pre)–08, DUH–1988]

(xiii) $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$

[NUH–2000, 06 (Old), 10, 12(Old), DUH– 86, 88, 90,98, 03]

(xiv) $\text{Im}\left(\frac{1}{z}\right) < \frac{1}{2}$ [DUH–1987]

(xv) $1 < |z+i| \le 2$ [NUH-2007, 2012(Old), DUH-198

(xvi) $1 < |z+i| < 2$ [DUH-199

(xvii) $1 < |z-2i| \le 2$ [DUH-198

(xviii) $0 < \text{Re}(iz) < 1$ [DUH-199

(xix) $\frac{\pi}{3} \le \arg z \le \frac{\pi}{2}$

(xx) $-\pi < \arg z < \pi$ [DUH-198

(xxi) $-\pi < \arg z < \pi,\ z \ne 0$ [DUH-198

(xxii) $0 < \arg z < 2\pi,\ |z| > 0$ [DUH-198

(xxiii) $-\pi < \arg z < \pi,\ |z| > 2$ [DUH-198

(xxiv) $|z-1| + |z+1| \le 3$ [NUH-2005, CUH-200

(xxv) $|z+2-3i| + |z-2+3i| < 10$ [DUH-198

(xxvi) $|z-2| - |z+2| > 3$ [DUH-199

(xxvii) $\text{Re}(z^2) > 1$ [DUH-198

(xxviii) $\text{Im}(z^2) > 0$ [DUH-198

(xxix) $|z+i| + |z-i| \le 3$ [NUH-1995, 200

(xxx) $|z+2i| + |z-2i| = 6$ [NUH-2004 (Old

(xxxi) $1 \le |z+1+i| < 2$ [NUH-199

(xxxii) $\text{Re}\left(\frac{1}{z}\right) > \frac{1}{2}$ NUH-199

(xxxiii) $\text{Im}(z^2) > 2$ [NUH-199

(xxxiv) $|z+1+i| = |z-1+i|$ [NUH-199

Solution : (i) Let [ধরি] $z = x + iy$.

Then [এখন] $|z-4| > |z|$ gives [দেয়]

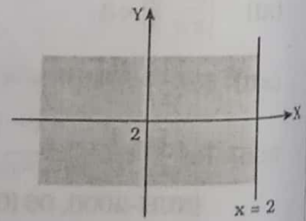
$$|x+iy-4| > |x+iy|$$

$$\Rightarrow \sqrt{(x-4)^2 + y^2} > \sqrt{x^2+y^2}$$

$$\Rightarrow x^2 - 8x + 16 + y^2 > x^2 + y^2$$

$$\Rightarrow -8x + 16 > 0$$

$$\Rightarrow 8x < 16 \Rightarrow x < 2$$



$\therefore$ The region is the set of all points (x, y) such that $x < 2$. That is, the set of all points (x, y) left hand side of the straight line $x = 2$. [এলাকাটি হয় সকল বিন্দুসমূহ (x, y) এর সেট যেন $x < 2$ হয়। অর্থাৎ $x = 2$ সরলরেখার বামদিকের সকল (x, y) বিন্দুর সেট।]

(ii) $|z-i| = |z+i|$

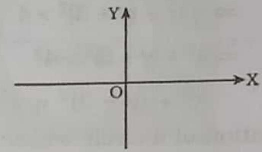
$$\Rightarrow |x+iy-i| = |x+iy+i|$$

$$\Rightarrow |x+i(y-1)| = |x+i(y+1)|$$

$$\Rightarrow \sqrt{x^2+(y-1)^2} = \sqrt{x^2+(y+1)^2}$$

$$\Rightarrow x^2 + y^2 - 2y + 1 = x^2 + y^2 + 2y + 1$$

$$\Rightarrow -2y = 2y$$

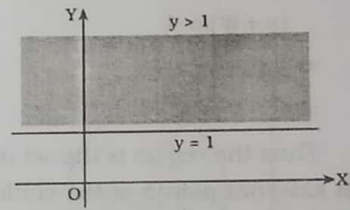


$\Rightarrow 4y = 0 \Rightarrow y = 0$, which is the equation of real axis (x – axis). Thus the region is the set of all points lie on the real axis.

[যাহা বাস্তব অক্ষ (x অক্ষ) এর সমীকরণ। অতএব এলাকাটি বাস্তব অক্ষের উপর সকল বিন্দুর সেট।]

(iii) $\text{Im}(z) > 1$

$$\Rightarrow \text{Im}(x+iy) > 1 \Rightarrow y > 1$$



The region is the set of all points (x, y) such that $y > 1$, that is, the set of all points (x, y) lie in the upper of the line $y = 1$.

[এলাকাটি সকল (x, y) বিন্দুর সেট যেন $y > 1$ হয়, অর্থাৎ $y = 1$ রেখার উপরের সকল (x, y) বিন্দুর সেট।]

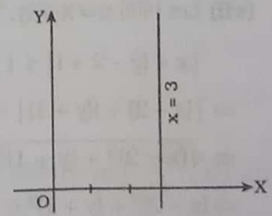
(iv) $\text{Re}(\bar{z} - 1) = 2$

$$\Rightarrow \text{Re}(\overline{x+iy} - 1) = 2, \text{ where } z = x + iy$$

$$\Rightarrow \text{Re}(x - iy - 1) = 2$$

$$\Rightarrow x - 1 = 2$$

$$\Rightarrow x = 3$$

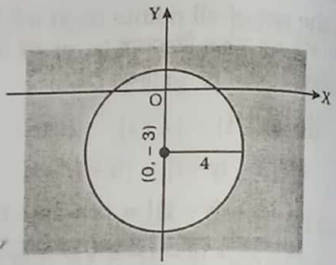


Which is the equation of a straight line parallel to y-axis. Thus the region is the set of all points $(3, y)$, where $y \in \mathbf{R}$.

[যাহা y অক্ষের সমান্তরাল একটি সরলরেখার সমীকরণ। অতএব এলাকাটি সকল $(3, y)$ বিন্দুর সেট যেখানে $y \in \mathbf{R}$.]

(v) Let [ধরি] $z = x + iy$. Then [তখন] $|z + 3i| > 4$ gives [দেয়]

$$|x + iy + 3i| > 4$$
$$\Rightarrow |x + i(y + 3)| > 4$$
$$\Rightarrow \sqrt{x^2 + (y+3)^2} > 4$$
$$\Rightarrow x^2 + (y+3)^2 > 4^2$$



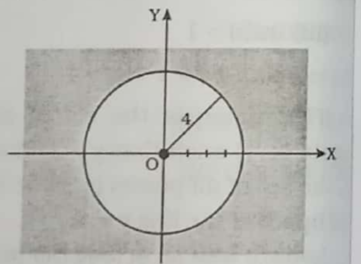
$x^2 + (y + 3)^2 = 4^2$ is the equation of a circle whose centre is $(0, -3)$ and radius is 4.

Thus, the region is the set of all external points of the circ whose centre is $(0, -3)$ and radius is 4.

[$x^2 + (y + 3)^2 = 4^2$ একটি বৃত্তের সমীকরণ যার কেন্দ্র $(0, -3)$ এবং ব্যাসার্ধ অতএব, এলাকাটি $(0, -3)$ কেন্দ্র ও 4 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট।]

(vi) Let [ধরি] $z = x + iy$. Then [তখন] $|z| > 4$ gives [দেয়]

$$|x + iy| > 4$$
$$\Rightarrow \sqrt{x^2 + y^2} > 4$$
$$\Rightarrow x^2 + y^2 > 4.$$

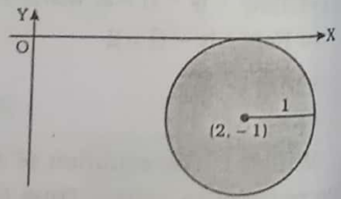


Thus the region is the set of all external points of the circle whose centre is $(0, 0)$ and radius is 4.

[অতএব এলাকাটি $(0, 0)$ কেন্দ্র ও 4 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট।]

(vii) Let [ধরি] $z = x + iy$. Then [তখন] $|z - 2 + i| \leq |$ gives [দেয়]

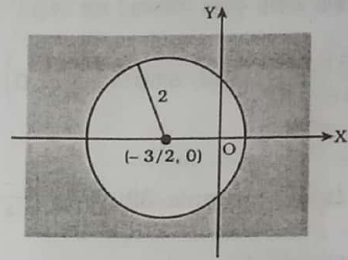
$$|x + iy - 2 + i| \leq 1$$
$$\Rightarrow |(x - 2) + i(y + 1)| \leq 1|$$
$$\Rightarrow \sqrt{(x-2)^2 + (y+1)^2} \leq 1$$
$$\Rightarrow (x - 2)^2 + (y + 1)^2 \leq 1$$



Thus, the region is the set of all internal points including th boundary points of a circle whose centre is $(2, -1)$ and radius is [অতএব, এলাকাটি $(2, -1)$ কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সীমানা বিন্দুসহ সকল অভ্য বিন্দুর সেট]

(viii) Let [ধরি] $z = x + iy$. Then [তখন] $|2z + 3| > 4$ gives [দেয়]

$$|2(x + iy) + 3| > 4$$
$$\Rightarrow \sqrt{(2x+3)^2 + (2y)^2} > 4$$
$$\Rightarrow 4x^2 + 12x + 9 + 4y^2 > 16$$
$$\Rightarrow 4x^2 + 4y^2 + 12x > 7$$
$$\Rightarrow x^2 + y^2 + 3x > \frac{7}{4}$$
$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 > \frac{7}{4} + \frac{9}{4}$$
$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 > 2^2$$

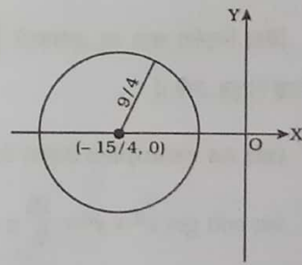


The region is the set of all external points of the circle whose centre is $\left(\frac{-3}{2}, 0\right)$ and radius is 2. [অতএব, এলাকাটি $\left(\frac{-3}{2}, 0\right)$ কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের বিহঃস্থ বিন্দুর সেট।]

(ix) Let [ধরি] $z = x + iy$.

Then [তখন] $\left|\frac{z-3}{z+3}\right| = 3$ becomes

$$\frac{|z-3|}{|z+3|} = 3$$
$$\Rightarrow 3|z + 3| = |z - 3|$$
$$\Rightarrow 3|x + iy + 3| = |x + iy - 3|$$
$$\Rightarrow 3\sqrt{(x+3)^2 + y^2} = \sqrt{(x-3)^2 + y^2}$$
$$\Rightarrow 9(x^2 + 6x + 9 + y^2) = x^2 - 6x + 9 + y^2, \text{ by squaring}$$
$$\Rightarrow 8x^2 + 8y^2 + 60x + 72 = 0$$
$$\Rightarrow x^2 + y^2 + \frac{15}{2}x + 9 = 0$$



This is the equation of a circle whose centre is $\left(\frac{-15}{4}, 0\right)$ and

$$\text{radius} = \sqrt{\left(\frac{15}{4}\right)^2 + 0 - 9} = \sqrt{\frac{81}{16}} = \frac{9}{4}$$

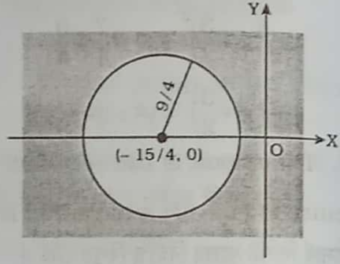
Thus, the region is the set of all boundary points of the circle whose centre is $\left(\frac{-15}{4}, 0\right)$ and radius $= \frac{9}{4}$.

ইহা একটি বৃত্তের সমীকরণ যার কেন্দ্র $\left(\frac{-15}{4}, 0\right)$ এবং ব্যাসার্ধ = $\sqrt{\left(\frac{15}{4}\right)^2 + 0 - 9}$ $= \sqrt{\frac{81}{16}} = \frac{9}{4}$. অতএব, এলাকাটি $\left(\frac{-15}{4}, 0\right)$ কেন্দ্র ও $\frac{9}{4}$ ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল সীমানা বিন্দুর সেট।]

(x) As example–33(ix) for $\left|\frac{z-3}{z+3}\right| > 3$

we will get

$$x^2 + y^2 + \frac{15}{2}x + 9 > 0$$

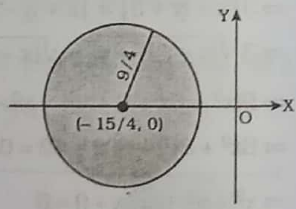


This represents the set of all external points of the circle whose centre is $\left(\frac{-15}{4}, 0\right)$ and radius is $\frac{9}{4}$.

[ইহা নির্দেশ করে যে এলাকাটি $\left(\frac{-15}{4}, 0\right)$ কেন্দ্র ও $\frac{9}{4}$ ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট।]

(xi) As examples 33(x) for $\left|\frac{z-3}{z+3}\right| < 3$

we will get $x^2 + y^2 + \frac{15}{2}x + 9 < 0$



This represents the set of all internal points of the circle whose centre is $\left(\frac{-15}{4}, 0\right)$ and radius is $\frac{9}{4}$.

[ইহা নির্দেশ করে যে এলাকাটি $\left(\frac{-15}{4}, 0\right)$ কেন্দ্র ও $\frac{9}{4}$ ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল অন্তঃস্থ বিন্দুর সেট।]

(xii) Let [ধরি] $z = x + iy$. Then [তখন] $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$ becomes

$$\text{Re}\left(\frac{1}{x + iy}\right) \le \frac{1}{2}$$

$$\Rightarrow \text{Re}\left(\frac{x - iy}{(x + iy)(x - iy)}\right) \le \frac{1}{2}$$

$$\Rightarrow \text{Re}\left(\frac{x - iy}{x^2 + y^2}\right) \le \frac{1}{2}$$

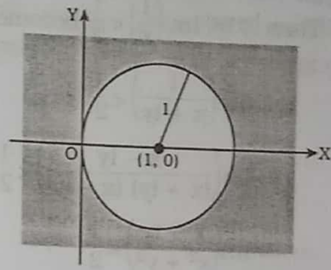
$$\Rightarrow \frac{x}{x^2 + y^2} \le \frac{1}{2}$$

$$\Rightarrow 2x \le x^2 + y^2$$

$$\Rightarrow 0 \le x^2 + y^2 - 2x$$

$$\Rightarrow x^2 + y^2 - 2x \ge 0$$

$$\Rightarrow (x - 1)^2 + y^2 \ge 1$$



Thus, the region is the set of all external points including the boundary points of the circle whose centre is (1, 0) and radius is 1. [অতএব, এলাকাটি (1, 0) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সীমানা বিন্দুসমূহসহ সকল বহিঃস্থ বিন্দুর সেট।]

(xiii) Let [ধরি] $z = x + iy$.

Then [তখন] $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$

becomes $\text{Re}\left(\frac{1}{x + iy}\right) < \frac{1}{2}$

$$\Rightarrow \text{Re}\left(\frac{x - iy}{(x + iy)(x - iy)}\right) < \frac{1}{2}$$

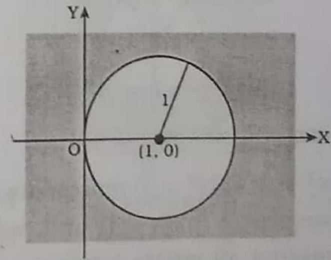
$$\Rightarrow \text{Re}\left(\frac{x - iy}{x^2 + y^2}\right) < \frac{1}{2}$$

$$\Rightarrow \frac{x}{x^2 + y^2} < \frac{1}{2}$$

$$\Rightarrow 2x < x^2 + y^2$$

$$\Rightarrow x^2 + y^2 - 2x > 0$$

$$\Rightarrow (x - 1)^2 + y^2 > 1$$



This represents the set of all external points of the circle whose centre is (1, 0) and radius is 1. [ইহা (1, 0) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট নির্দেশ করে।]

(xiv) Let [ধরি] $z = x + iy$.

Then [তখন] $\text{Im}\left(\frac{1}{z}\right) < \frac{1}{2}$ becomes,

$$\text{Im}\left(\frac{1}{x+iy}\right) < \frac{1}{2}$$

$$\Rightarrow \text{Im}\left(\frac{x-iy}{(x+iy)\,(x-iy)}\right) < \frac{1}{2}$$

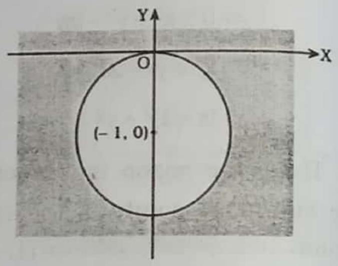
$$\Rightarrow \text{Im}\left(\frac{x-iy}{x^2+y^2}\right) < \frac{1}{2}$$

$$\Rightarrow \frac{-y}{x^2+y^2} < \frac{1}{2}$$

$$\Rightarrow -2y < x^2 + y^2$$

$$\Rightarrow x^2 + y^2 + 2y > 0$$

$$\Rightarrow x^2 + (y+1)^2 > 1$$



This represents the set of all external points of the circle whose centre is (– 1, 0) and radius is 1. [ইহা (– 1, 0) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুর সেট নির্দেশ করে।]

(xv) Let [ধরি] $z = x + iy$. Given that [দেওয়া আছে] $1 < |z + i| \leq 2$

$$\Rightarrow 1 < |x + iy + i| \leq 2$$

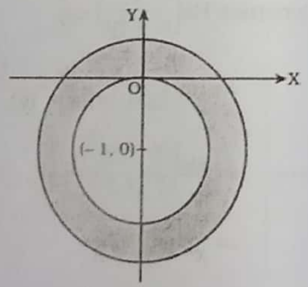
$$\Rightarrow 1 < |x + i(y+1)| \leq 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y+1)^2} \leq 2$$

$$\Rightarrow 1 < x^2 + (y+1)^2 \leq 2^2$$

$$\Rightarrow 1 < x^2 + (y+1)^2$$

and [এবং] $x^2 + (y+1)^2 \leq 2^2$



$x^2 + (y+1)^2 > 1$ represents all points external to the circle whose centre is (0, – 1) and radius is 1. Also, $x^2 + (y+1)^2 \leq 2^2$ represents all points internal and boundary of the circle whose centre is (0, – 1) and radius is 2. Thus the region is the set of all common points of external points of the circle $x^2 + (y+1)^2 = 1$ and internal points of the circle $x^2 + (y+1)^2 = 2^2$ including its boundary points.

[(0, – 1) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দু $x^2 + (y+1)^2 > 1$ নির্দেশ করে। $x^2 + (y+1)^2 \leq 2^2$, (0, – 1) কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল অন্তঃস্থ ও সীমানা বিন্দু সমূহ নির্দেশ করে। অতএব এলাকাটি, $x^2 + (y+1)^2 = 1$ বৃত্তের সকল বিহঃস্থ বিন্দুসমূহ এবং $x^2 + (y+1)^2 = 2^2$ বৃত্তের সীমানা বিন্দু সমূহসহ অন্তঃস্থ বিন্দু সমূহের সকল সাধারণ বিন্দুর সেট।]

(xvi) $1 < |z + i| < 2$

$$\Rightarrow 1 < |x + iy + i| < 2$$

$$\Rightarrow 1 < |x\, i(y+1)| < 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y+1)^2} < 2$$

$$\Rightarrow 1 < x^2 + (y+1)^2 < 2^2$$

$$\Rightarrow 1 < x^2 + (y+1)^2 \text{ and [এবং] } x^2 + (y+1)^2 < 2^2$$

$1 < |z + i| < 2$ represents the common points of the external of the circle $x^2 + (y+1)^2 = 1$ and internal of the circle $x^2 + (y+1)^2 = 2^2$. Thus the region is the set of all common points of external points of the circle $x^2 + (y+1)^2 = 1$ and internal points of the circle $x^2 + (y+1)^2 = 2^2$. [$1 < |z + i| < 2$ নির্দেশ করে $x^2 + (y+1)^2 = 1$ বৃত্তের বহিঃস্থ ও $x^2 + (y+1)^2 = 2^2$ বৃত্তের অন্তঃস্থ বিন্দু সমূহের সাধারণ বিন্দু সমূহ। অতএব, এলাকাটি হল $x^2 + (y+1)^2 = 1$ বৃত্তের বহিঃস্থ এবং $x^2 + (y+1)^2 = 2^2$ বৃত্তের অন্তঃস্থ বিন্দুসমূহের সাধারণ বিন্দুগুলির সেট।]

(xvii) Let [ধরি] $z = x + iy$. Given that [দেওয়া আছে] $1 < |z - 2i| \leq 2$

$$\Rightarrow 1 < x + iy - 2i| \leq 2$$

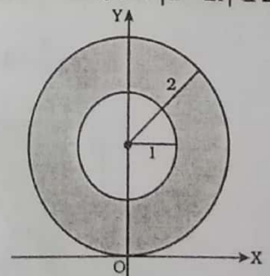
$$\Rightarrow 1 < |x + i(y-2)| \leq 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y-2)^2} \leq 2$$

$$\Rightarrow 1 < x^2 + (y-2)^2 \leq 2^2$$

$$\Rightarrow 1 < x^2 + (y-2)^2$$

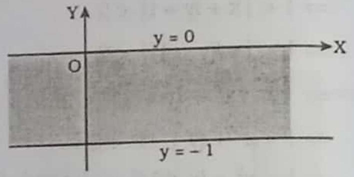
and [এবং] $x^2 + (y-2)^2 \leq 2^2$.



$1 < x^2 + (y-2)^2$ represents all points external to the circle whose centre is (0, 2) and radius is 1. Also, $x^2 + (y-2)^2 \leq 2^2$ represents all points internal and boundary of the circle whose centre is (0, 2) and radius is 2. Thus the region is the set of all common points of external points of the circle $x^2 + (y-2)^2 = 1$ and internal points of the circle $x^2 + (y-2)^2 = 2^2$ including its boundary points.

[$1 < x^2 + (y-2)^2$ নির্দেশ করে (0, 2) কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল বহিঃস্থ বিন্দুসমূহ। আবার, $x^2 + (y-2)^2 \le 2^2$ নির্দেশ করে (0, 2) কেন্দ্র ও 2 ব্যাসার্ধ বিশিষ্ট বৃত্তের সীমানা বিন্দুসহ সকল অন্তঃস্থ বিন্দু। অতএব, এলাকাটি হল $x^2 + (y-2)^2 = 1$ বৃত্তের বহিঃস্থ বিন্দু সমূহের এবং $x^2 + (y-2)^2 = 2^2$ বৃত্তের সীমানা বিন্দুসহ অন্তঃস্থ বিন্দুসমূহের সাধারণ সকল বিন্দুর সেট।]

(xviii) Let [ধরি] $z = x + iy$. Then [তখন] $0 < \text{Re}(iz) < 1$ becomes



$0 < \text{Re}(ix + i^2y) < 1$

$\Rightarrow 0 < \text{Re}(ix - y) < 1$

$\Rightarrow 0 < -y < 1$

$\Rightarrow 0 < -y$ and [এবং] $-y < 1$

$\Rightarrow 0 > y$ and [এবং] $y > -1$

$\Rightarrow y < 0$ and [এবং] $y > -1$

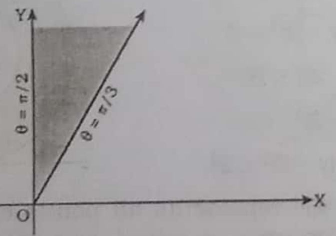
Thus the region lies between the lines $y = 0$ and $y = -1$. [অতএব এলাকাটি $y = 0$ ও $y = -1$ রেখার মধ্যে অবস্থিত।]

(xix) Let [ধরি] $z = x + iy$.

Then [তখন] $\arg z = \theta = \tan^{-1}\frac{y}{x}$.

Given that [দেওয়া আছে] $\frac{\pi}{3} \le \arg z \le \frac{\pi}{2}$

$$\Rightarrow \frac{\pi}{3} \le \theta \le \frac{\pi}{2}$$



Hence the region is the set of all infinite points bounded by the lines $\theta = \arg z = \frac{\pi}{3}$ and $\theta = \arg z = \frac{\pi}{2}$ including the points on these lines. [অতএব এলাকাটি $\theta = \frac{\pi}{3}$ এবং $\theta = \frac{\pi}{2}$ রেখাদ্বয়ের উপরস্থ বিন্দুসহ উহাদের দ্বারা সীমাবদ্ধ অসীম সংখ্যক বিন্দুর সেট।]

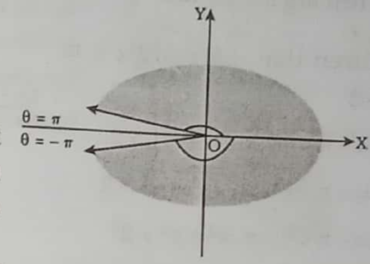
(xx) Let [ধরি] $z = x + iy$.

Then [তখন] $\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Given that [দেওয়া আছে]

$-\pi < \arg z < \pi$

$\Rightarrow -\pi < \theta < \pi$



Thus, the region is the set of all infinite points between the lines $\theta = \pi$ and $\theta = -\pi$ as shown in the figure.

[অতএব চিত্রে প্রদর্শন অনুসারে এলাকাটি $\theta = \pi$ এবং $\theta = -\pi$ রেখা দ্বারা সীমাবদ্ধ সকল অসীম সংখ্যক বিন্দুর সেট]

(xxi) Let $z = x + iy$.

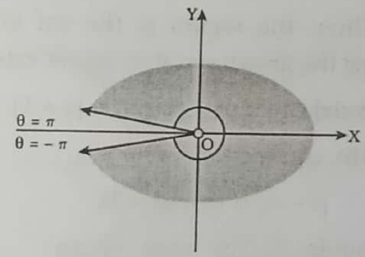
Then $\arg z = \theta = \tan^{-1}\frac{y}{x}$.

Given that

$-\pi < \arg z < \pi,\ z \ne 0$

$\Rightarrow -\pi < \theta < \pi,\ x + iy \ne 0$

$\Rightarrow -\pi < \theta < \pi,\ x \ne 0,\ y \ne 0$



Thus, the region is the set of all infinite points between the lines $\theta = \pi$ and $\theta = -\pi$ excluding the origin.

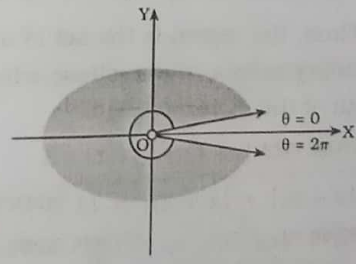
(xxii) Let $z = x + iy$. Then $\arg z = \theta = \tan^{-1}\frac{y}{x}$.

Given that $0 < \arg z < 2\pi$, $|z| > 0$

$\Rightarrow 0 < \theta < 2\pi,\ |x + iy| > 0$

$\Rightarrow 0 < \theta < 2\pi,\ \sqrt{x^2 + y^2} > 0$

$\Rightarrow 0 < \theta < 2\pi,\ x^2 + y^2 > 0$



Here $x^2 + y^2 = 0$ is the equation of the point circle which is the origin.

Hence the region is the set of all infinite points between the lines $\theta = 0$ and $\theta = 2\pi$ excluding the origin.

(xxiii) Let $z = x + iy$.

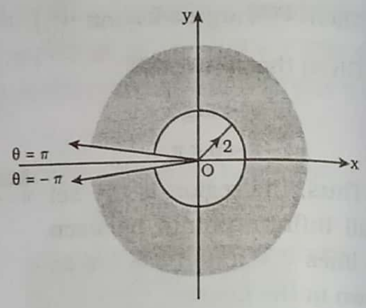
Then $\arg z = \theta = \tan^{-1}\frac{y}{x}$.

Given that $-\pi < \arg z < \pi$, $|z| > 2$

$\Rightarrow -\pi < \theta < \pi,\ |x + iy| > 2$

$\Rightarrow -\pi < \theta < \pi,\ \sqrt{x^2 + y^2} > 2$

$\Rightarrow -\pi < \theta < \pi,\ x^2 + y^2 > 2^2$



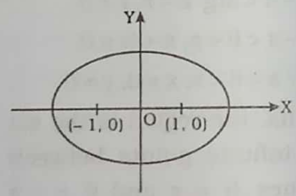
Here $x^2 + y^2 = 2^2$ is the equation of a circle whose centre is (0, 0) and radius is 2.

Thus, the region is the set of all infinite common points among the lines $\theta = \pi$, $\theta = -\pi$ and external of the circle $x^2 + y^2 = 2^2$.

(xxiv) Given that $|z - 1| + |z + 1| \le 3$

The equation of the type

$$|z - z_0| + |z + z_0| = 2a$$



holds in the case of an ellipse whose foci are $S(z_0)$, $S'(-z_0)$ and length of the major axis is 2a.

Here $z_0 = 1 = (1, 0)$, $-z_0 = -1 = (-1, 0)$ and $2a = 3$

Thus, the region is the set of all interior points including the boundary points of the ellipse whose foci are (1, 0), (− 1, 0) and length of the major axis is 3.

[দেওয়া আছে $|z - 1| + |z + 1| \le 3$

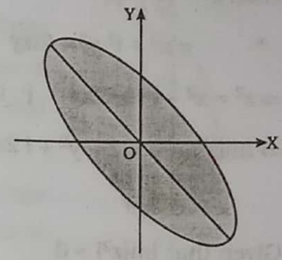
$|z - z_0| + |z + z_0| = 2a$ আকারের সমীকরণ উপবৃত্তের ক্ষেত্রে খাটে যার উপকেন্দ্রদ্বয় $S(z_0)$, $S'(-z_0)$ এবং বৃহৎ অক্ষের দৈর্ঘ্য 2a.

এখানে $z_0 = 1 = (1, 0)$, $-z_0 = -1 = (-1, 0)$ এবং $2a = 3$.

অতএব, এলাকাটি হবে একটি উপবৃত্তের সীমানা বিন্দুসহ অন্তঃস্থ বিন্দু সমূহের সেট যার উপকেন্দ্রদ্বয় S(1, 0), S′(− 1, 0) এবং বৃহৎ অক্ষের দৈর্ঘ্য 3.]

(xxv) Given that $|z + 2 - 3i| + |z - 2 + 3i| < 10$

$\Rightarrow |z + (2 - 3i)| + |z - (2 - 3i)| < 10$



We know that the equation of the type $|z - z_0| + |z + z_0| = 2a$ holds in the case of an ellipse whose foci are $S(z_0)$, $S'(-z_0)$ and length of the major axis is 2a.

Here $z_0 = 2 - 3i = (2, -3)$

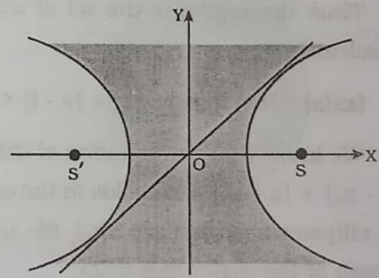
$\Rightarrow -z_0 = (-2, 3)$ and $2a = 10$.

Thus, the region is the set of interior points of an ellipse whose foci are S(2, − 3), S′(− 2, 3) and length of the major axis is 10

(xxvi) Given that $|z - 2| - |z + 2| > 3$

We know that the equation of the type

$$|z - z_0| - |z + z_0| = 2a$$



holds in the case of a hyperbola whose foci are $S(z_0)$, $S'(-z_0)$ and length of the transverse axis is 2a.

Here $z_0 = 2 \Rightarrow -z_0 = -2$ and $2a = 3$

Thus the region is the set of external points of a hyperbola whose foci are (2, 0), (− 2, 0) and length of the transverse axis is 3.

(xxvii) Let $z = x + iy$.

Then $z^2 = (x + iy)$

$= x^2 + 2ixy + i^2y^2$

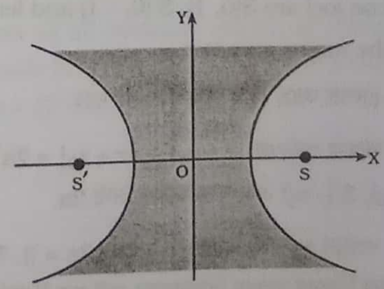
$\Rightarrow z^2 = x^2 - y^2 + 2ixy$,

where $i^2 = -1$.

$\Rightarrow \text{Re}(z^2) = x^2 - y^2$

Given that $\text{Re}(z^2) > 1$

$\Rightarrow x^2 - y^2 > 1$



Thus the region is the set of all external points of the rectangular hyperbola $x^2 - y^2 = 1$.

(xxviii) Let $z = x + iy$.

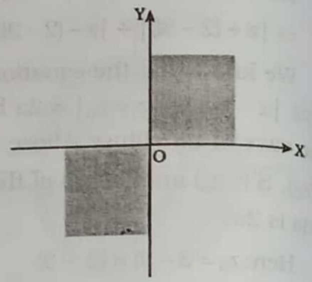
Then $z^2 = (x + iy)^2$

$= x^2 + i^2 y^2 + 2ixy$

$\Rightarrow z^2 = x^2 - y^2 + i2xy \quad [\because i^2 = -1]$

$\Rightarrow Im(z^2) = Im(x^2 - y^2 + i\,2xy)$

$= 2xy$



Given that $Im(z^2) > 0$

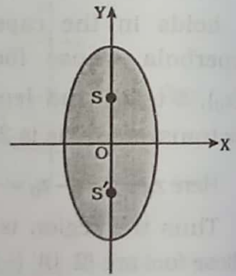
$\Rightarrow 2xy > 0 \Rightarrow xy > 0$

$\Rightarrow (-x)(-y) > 0$

Thus the region is the set of all points in the first and third quadrants.

(xxix) Given that $|z + i| + |z - i| \le 3$

We know that the equation of the type $|z - z_0| + |z + z_0| = 2a$ holds in the case of an ellipse whose foci are $S(z_0)$, $S(-z_0)$ and length of the major axis is 2a.



Here $z_0 = i$, $-z_0 = -i$ and $2a = 3$. Thus the region is the set of interior points including boundary points of an ellipse whose foci are $S(0, 1)$, $S'(0, -1)$ and length of the major axis is z.

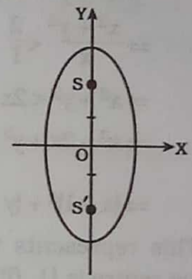
[দেওয়া আছে $|z + i| + |z - i| \le 3$

আমরা জানি যে $|z - z_0| + |z + z_0| = 2a$ উপবৃত্ত এর ক্ষেত্রে খাটে যার উপকেন্দ্রদ্বয় $S(z_0)$, $S'(-z_0)$ এবং বৃহৎ অক্ষের দৈর্ঘ্য 2a.

এখানে $z_0 = i$, $-z_0 = -i$ এবং $2a = 3$. অতএব এলাকাটি হবে একটি উপবৃত্তের সীমানা বিন্দুসহ অন্তঃস্থ বিন্দুসমূহের সেট যার উপকেন্দ্রদ্বয় $S(0, 1)$, $S'(0, -1)$ এবং বৃহৎ অক্ষের দৈর্ঘ্য 3.]

(xxx) Given that $|z + 2i| + |z - 2i| = 6$

We know that the equation of the type $|z - z_0| + |z + z_0| = 2a$ holds in the case of an ellipse whose foci are $S(z_0)$, $S'(-z_0)$ and length of the major axis is 2a.



Here $z_0 = 2i$, $-z_0 = -2i$ and $2a = 6$. Thus the region is the set of boundary points of an ellipse whose foci are $S(0, 2)$, $S'(0, -2)$ and length of the major axis is 6.

(xxxi) Let $z = x + iy$. Given that

$1 \le |z + 1 + i| < 2$

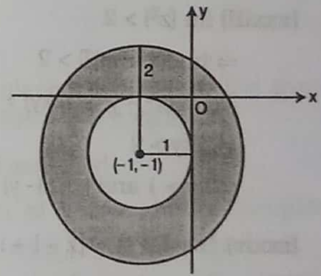
$\Rightarrow 1 \le |x + iy + 1 + i| < 2$

$\Rightarrow 1 \le |(x + 1) + i(y + 1)| < 2$

$\Rightarrow 1 \le \sqrt{(x + 1)^2 + (y + 1)^2} < 2$

$\Rightarrow 1 \le (x + 1)^2 + (y + 1)^2 < 2^2$

$\Rightarrow 1 \le (x + 1)^2 + (y + 1)^2$ and $(x + 1)^2 + (y + 1)^2 < 2^2$



$(x + 1)^2 + (y + 1)^2 \ge 1$ represents all points external and boundary of the circle whose centre is $(-1, -1)$ and radius is 1.

Also, $(x + 1)^2 + (y + 1)^2 < 2^2$ represents all internal points of the circle whose centre is $(-1, -1)$ and radius is 2.

Thus the region is the set of all common points of external and boundary points of the circle $(x + 1)^2 + (y + 1)^2 = 1$ and internal points of the circle $(x + 1)^2 + (y + 1)^2 = 2$.

(xxxii) $Re\left(\frac{1}{z}\right) > \frac{1}{2}$

$\Rightarrow Re\left(\frac{1}{x + iy}\right) > \frac{1}{2}$

$\Rightarrow Re\left(\frac{x - iy}{x^2 - i^2 y^2}\right) > \frac{1}{2}$

$\Rightarrow Re\left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}\right) > \frac{1}{2}$

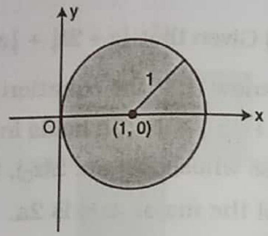
$$\Rightarrow \frac{x}{x^2+y^2} > \frac{1}{2}$$

$$\Rightarrow \frac{x^2+y^2}{x} < \frac{2}{1}$$

$$\Rightarrow x^2+y^2 < 2x$$

$$\Rightarrow x^2 - 2x + y^2 < 0$$

$$\Rightarrow (x-1)^2 + (y-0)^2 < 1$$



This represents the set of all internal points of the circle whose centre is (1, 0) and radius is 1. [ইহা (1, 0) কেন্দ্র ও 1 ব্যাসার্ধ বিশিষ্ট বৃত্তের সকল অন্তঃস্থ বিন্দু সমূহের সেট নির্দেশ করে।]

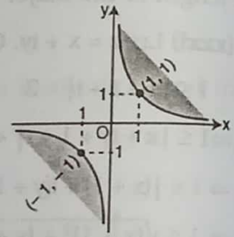
(xxxiii) $\text{Im}\,(z^2) > 2$

$$\Rightarrow \text{Im}\,\{(x+iy)^2\} > 2$$

$$\Rightarrow \text{Im}\,(x^2 - y^2 + i2xy) > 2$$

$$\Rightarrow 2ky > 2$$

$$\Rightarrow xy > 1 \text{ and } (-x)(-y) > 1$$



(xxxiv) $|z+1+i| = |z-1+i|$

$$\Rightarrow |x+iy+1+i| = |x+iy-1+i|$$

$$\Rightarrow |(x+1)+i(y+1)| = |(x-1)+i(y+1)|$$

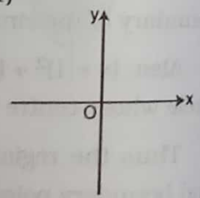
$$\Rightarrow \sqrt{(x+1)^2+(y+1)^2} = \sqrt{(x-1)^2+(y+1)^2}$$

$$\Rightarrow (x+1)^2+(y+1)^2 = (x-1)^2+(y+1)^2$$

$$\Rightarrow x^2+2x+1 = x^2-2x+1$$

$$\Rightarrow 4x = 0$$

$$\Rightarrow x = 0,$$



which is the equation of y-axis. [যাহা y অক্ষের সমীকরণ]

Example–34. Determine the set of points in the complex plane which satisfy the inequality $|z+1-i| \le |z-1+i|$, and sketch it. **[NUH–2002]**

Solution : Let [ধরি] $z = x + iy$.

Given that [দেওয়া আছে] $|z+1-i| \le |z-1+i|$

$$\Rightarrow |x+iy+1-i| \le |x+iy-1+i|$$

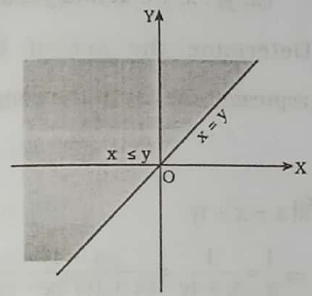
$$\Rightarrow |(x+1)+i(y-1)| \le |(x-1)+i(y+1)|$$

$$\Rightarrow \sqrt{(x+1)^2+(y-1)^2} \le \sqrt{(x-1)^2+(y+1)^2}$$

$$\Rightarrow x^2+2x+1+y^2-2y+1 \le x^2-2x+1+y^2+2y+1$$

$$\Rightarrow 4x \le 4y$$

$$\Rightarrow x \le y$$



The set of points in the complex plane which satisfy the given inequality is [প্রদত্ত অসমতা সিদ্ধ করে জটিল তলে এমন বিন্দু সমূহের সেট]

$$\{(x, y) : x, y \in \mathbf{R} \text{ and } x \le y\}.$$

Example–35. Determine the set of points in the complex plane which satisfy the inequality $|z-i| \le |z+i|$, and sketch it. **[NUH–2002(Old), NU(Pre)–2002]**

Solution : Let [ধরি] $z = x + iy$.

Given that [দেওয়া আছে] $|z-i| \le |z+i|$.

$$\Rightarrow |x+iy-i| \le |x+iy+i|$$

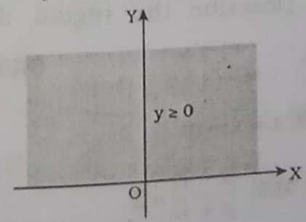
$$\Rightarrow |x+i(y-1)| \le |x+i(y+1)|$$

$$\Rightarrow \sqrt{x^2+(y-1)^2} \le \sqrt{x^2+(y+1)^2}$$

$$\Rightarrow x^2+y^2-2y+1 \le x^2+y^2+2y+1$$

$$\Rightarrow -4y \le 0$$

$$\Rightarrow y \ge 0$$



Thus the set of points in the complex plane which satisfy the given inequality is [অতএব জটিল তলে বিন্দুসমূহের সেট যাহা প্রদত্ত অসমতাকে সিদ্ধ করে তা হল]

$$\{(x, y\} : x, y \in \mathbf{R} \text{ and } y \ge 0\}$$

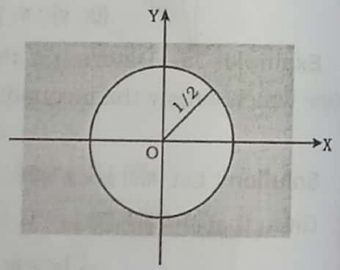
Example-36. Determine the set of complex numbers such that $\text{Re}\left(\frac{1}{z}\right) < 1$ represent the set in the complex plane. **[NUH-2001]**

Solution : Let [ধরি] $z = x + iy$.

$$\Rightarrow \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)\ (x - iy)} = \frac{x - iy}{x^2 + y^2}$$

$$\Rightarrow \text{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2 + y^2}$$

Given that [দেওয়া আছে] $\text{Re}\left(\frac{1}{z}\right) < 1$



$$\Rightarrow \frac{x}{x^2 + y^2} < 1$$

$$\Rightarrow x < x^2 + y^2$$

$$\Rightarrow x^2 + y^2 - x > 0$$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 > \left(\frac{1}{2}\right)^2$$

Thus the required set is the external points of the circle whose centre is $\left(\frac{1}{2}, 0\right)$ and radius is $\frac{1}{2}$. [অতএব আবশ্যকীয় সেটটি হল একটি বৃত্তের বহিঃস্থ বিন্দুসমূহ যার কেন্দ্র $\left(\frac{1}{2}, 0\right)$ এবং ব্যাসার্ধ $\frac{1}{2}$.]

Example-37. Describe the region determined by the relations $\left|\frac{z - 1}{z + 1}\right| = 2$. **[NUH-1995, 2005, 2008]**

Solution : Let [ধরি] $z = x + iy$.

Given that [দেওয়া আছে] $\left|\frac{z - 1}{z + 1}\right| = 2$

$$\Rightarrow \frac{|z - 1|}{|z + 1|} = 2$$

$$\Rightarrow 2|z + 1| = |z - 1|$$

$$\Rightarrow 2|x + iy + 1| = |x + iy - 1|$$

$$\Rightarrow 2\sqrt{(x + 1)^2 + y^2} = \sqrt{(x - 1)^2 + y^2}$$

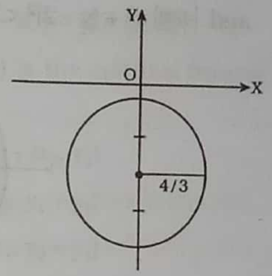
$$\Rightarrow 4(x^2 + 2x + 1 + y^2) = x^2 - 2x + 1 + y^2$$

$$\Rightarrow 3x^2 + 3y^2 + 10x + 3 = 0$$

$$\Rightarrow x^2 + y^2 + \frac{10}{3}x + 1 = 0$$

$$\Rightarrow x^2 + 2 \cdot \frac{5}{3} \cdot x + \left(\frac{5}{3}\right)^2 + y^2 = \frac{25}{9} - 1$$

$$\Rightarrow \left(x + \frac{5}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2$$



which is the equation of a circle whose centre is $\left(\frac{-5}{3}, 0\right)$ and radius is $\frac{4}{3}$. [যাহা একটি বৃত্তের সমীকরণ যার কেন্দ্র $\left(\frac{-5}{3}, 0\right)$ এবং ব্যাসার্ধ $\frac{4}{3}$.]

Thus the region is the set of boundary points of the circle whose centre is $\left(\frac{-5}{3}, 0\right)$ and radius $= \frac{4}{3}$. [অতএব এলাকাটি হল একটি বৃত্তের সীমানা বিন্দুসমূহের সেট যার কেন্দ্র $\left(\frac{-5}{3}, 0\right)$ এবং ব্যাসার্ধ $= \frac{4}{3}$.]

Example-38. Draw the sketch of the following region :

$1 < |z - 2i| < 2$ **[NUH-1998]**

Solution : Let [ধরি] $z = x + iy$.

Given that [দেওয়া আছে] $1 < |z - 2i| < 2$

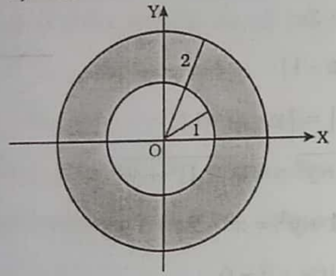
$$\Rightarrow 1 < |x + iy - 2i| < 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y - 2)^2} < 2$$

$$\Rightarrow 1 < x^2 + (y - 2)^2 < 2^2$$

$$\Rightarrow 1 < x^2 + (y - 2)^2$$

and [এবং] $x^2 + (y-2)^2 < 2^2$



Thus the region is the set of all common points of external points of the circle $x^2 + (y-2)^2 = 1$ and internal points of the circle $x^2 + (y-2)^2 = 2^2$. [অতএব এলাকাটি হল $x^2 + (y-2)^2 = 1$ বৃত্তের বহিঃস্থ এবং $x^2 + (y-2)^2 = 2^2$ বৃত্তের অন্তঃস্থ বিন্দুসমূহের সকল সাধারণ বিন্দু সমূহের সেট।]

Example–39. Prove that the set of complex numbers form an abelian group. **[RUH–2002]**

Solution : Let **C** be the set of complex numbers and

$$\mathbf{C} = \{z : z \text{ is a complex number } (x, y)\}.$$

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3) \in \mathbf{C}$.

(i) Closure law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$$= (x_1 + x_2, y_1 + y_2) \in \mathbf{C}$$

Thus, $z_1, z_2 \in \mathbf{C} \Rightarrow z_1 + z_2 \in \mathbf{C}$

(ii) Associative law : $(z_1 + z_2) + z_3 = ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$
$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$
$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$
$$= z_1 + ((x_2, y_2) + (x_3, y_3))$$
$$= z_1 + (z_2 + z_3)$$
$$\Rightarrow (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(iii) Identity law : $z + 0 = (x, y) + (0, 0) = (x + 0, y + 0) = (x, y) = z$

The complex number $(0, 0)$ is the identity called zero of the set **C**.

(iv) Inverse law : $(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$

Thus the complex number $(-x, -y)$ is the additive inverse of the complex number (x, y).

(v) Commutative law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$$= (x_1 + x_2, y_1 + y_2)$$
$$= (x_2 + x_1, y_2 + y_1)$$
$$= (x_2, y_2) + (x_1, y_1)$$
$$= z_2 + z_1$$
$$\Rightarrow z_1 + z_2 = z_2 + z_1$$

The set **C** satisfied all the laws of a an abelian group. Hence the set of all complex numbers form an abelian group.

Example–40. Define addition and multiplication in **C**, the set of complex numbers such that with these definitions **C** is a field. **[RUH–2003]**

Solution : Definition : Let $z_1, z_2 \in \mathbf{C}$, where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then their addition and multiplication are defined as

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and $z_1 z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

For addition we have the following :

(i) Closure law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbf{C}$

Thus, $z_1, z_2 \in \mathbf{C} \Rightarrow z_1 + z_2 \in \mathbf{C}$.

(ii) Associative law : $(z_1 + z_2) + z_3 = ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$
$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$
$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$
$$= z_1 + ((x_2, y_2) + (x_3, y_3))$$
$$= z_1 + (z_2 + z_3)$$
$$\therefore (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(iii) Identity law : $z + 0 = (x, y) + (0, 0) = (x + 0, y + 0) = (x, y) = z$

$\therefore z + 0 = z$

The complex number $0 = (0, 0)$ is the identity called zero of the set **C**.

(iv) Inverse law : $(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$

The complex number $(-x, -y)$ is the additive inverse of the complex number (x, y).

(v) Commutative law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$$\begin{aligned} &= (x_1 + x_2, y_1 + y_2) \\ &= (x_2 + x_1, y_2 + y_1) \\ &= (x_2, y_2) + (x_1, y_1) \\ &= z_2 + z_1 \end{aligned}$$

$$\Rightarrow z_1 + z_2 = z_2 + z_1$$

Hence the set **C** of all complex numbers form an abelian group.

For multiplication we have the following :

(vi) Closure law : $z_1z_2 = (x_1, y_1) \cdot (x_2, y_2)$

$$= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \in \mathbf{C}$$

since $x_1x_2 - y_1y_2$ and $x_1y_2 + x_2y_1 \in \mathbf{R}$.

(viii) Associative law : $(z_1z_2)z_3 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)(x_3, y_3)$

$$\begin{aligned} &= ((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + x_2y_1)y_3, \\ &\quad (x_1x_2 - y_1y_2)y_3 + (x_1y_2 + x_2y_1)y_3) \\ &= (x_1, y_1)(x_2x_3 - y_2y_3, x_2y_3 + x_3y_2) \\ &= z_1((x_2, y_2)(x_3, y_3)) \\ &= z_1(z_2z_3) \end{aligned}$$

$$\Rightarrow (z_1z_2)z_3 = z_1(z_2z_3)$$

(viii) Identity law : $(x, y)(1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + 1 \cdot y) = (x, y)$

The complex number $(1, 0)$ is the multiplicative identy of the set **C**.

(ix) Inverse law : z_2 will be the multiplicative inverse of z_1 if

$z_1z_2 = (1, 0)$

$\Rightarrow (x_1, y_1)(x_2, y_2) = (1, 0)$

$\Rightarrow (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) = (1, 0)$

$\Rightarrow x_1x_2 - y_1y_2 = 1$ ...... (1)

and $x_1y_2 + x_2y_1 = 0$ ...... (2)

From (2), $y_2 = \dfrac{-x_2y_1}{x_1}$ ...... (3)

By (3), (1) becomes, $x_1x_2 - y_1 \cdot \dfrac{-x_2y_1}{x_1} = 1$

$$\Rightarrow x_1^2x_2 + x_2y_1^2 = x_1$$

$$\Rightarrow x_2 = \frac{x_1}{x_1^2 + y_1^2}$$

From (3), $y_2 = \dfrac{-y_1}{x_1} \cdot \dfrac{x_1}{x_1^2 + y_1^2} = \dfrac{-y_1}{x_1^2 + y_1^2}$

If $(x_1, y_1) \neq (0, 0)$, that is, $x_1^2 + y_1^2 \neq 0$ then (x_2, y_2) exists.

Thus every nonzero complex number (x_1, y_1) has a unique inverse.

(x) Commutative law : $z_1z_2 = (x_1, y_1)(x_2, y_2)$

$$\begin{aligned} &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \\ &= (x_2x_1 - y_2y_1, x_2y_1 + x_1y_2) \\ &= (x_2, y_2)(x_1y_1) \\ &= z_2z_1 \end{aligned}$$

Thus the set **C** is an abelian multiplicative group.

(xi) Right distributive law :

$z_1(z_2 + z_3) = (z_1, y_1)((x_2, y_2) + (x_3, y_3))$

$$\begin{aligned} &= (x_1, y_1)(x_2 + x_3, y_2 + y_3) \\ &= (x_1(x_2 + x_3) - y_1(y_2 + y_3), x_1(y_2 + y_3) + (x_2 + x_3)y_1) \\ &= (x_1x_2 - y_1y_2 + x_1x_3 - y_1y_3, x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) + (x_1x_3 - y_1y_3, x_1y_3 + x_3y_1) \\ &= (x_1, y_1)(x_2, y_2) + (x_1, y_1)(x_3, y_3) \\ &= z_1z_2 + z_1z_3 \end{aligned}$$

Similarly, we can show that left distribution law

$$(z_2 + z_3)z_1 = z_2z_1 + z_3z_1$$

Hence according to the definition of a ring we can say that th set **C** form a ring.

In above we have seen that the multiplication is commutati and unit element (1, 0) exists and in **C** all nonzero elements hav multiplicative inverse. Thus the set **C** is a commutative ring wit unity such that every nonzero element has a multiplicati inverse. Hence the set **C** of all complex numbers is a field.

Example–41. Describe the locus represented by

$\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$ where z_1 and z_2 are two given points.

[NUH–1996, RUH–199

Solution : Let [ধরি] $z = x + iy$, $z_1 = x_1 + iy_1$ and [এবং] $z_2 = x_2 + iy_2$

$$\therefore z - z_1 = (x + iy) - (x_1 + iy_1) = x - x_1 + i(y - y_1)$$

$$z_2 - z_1 = x_2 + iy_2 - (x_1 + iy_1) = x_2 - x_1 + i(y_2 - y_1)$$

Given that [দেওয়া আছে] $\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$

$$\Rightarrow \arg(z - z_1) - \arg(z_2 - z_1) = 0 \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\right]$$

$$\Rightarrow \arg[x - x_1 + i(y - y_1)] - \arg[x_2 - x_1 + i(y_2 - y_1)] = 0$$

$$\Rightarrow \tan^{-1}\left(\frac{y - y_1}{x - x_1}\right) - \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = 0$$

$$\Rightarrow \tan^{-1}\frac{\frac{y - y_1}{x - x_1} - \frac{y_2 - y_1}{x_2 - x_1}}{1 + \frac{y - y_1}{x - x_1}\cdot\frac{y_2 - y_1}{x_2 - x_1}} = 0$$

$$\Rightarrow \frac{\frac{y - y_1}{x - x_1} - \frac{y_2 - y_1}{x_2 - x_1}}{1 + \frac{y - y_1}{x - x_1}\,\frac{y_2 - y_1}{x_2 - x_1}} = \tan 0 = 0$$

$$\Rightarrow \frac{y - y_1}{x - x_1} - \frac{y_2 - y_1}{x_2 - x_1} = 0$$

$$\Rightarrow \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{x}{x_1 - x_2} + \frac{y}{y_2 - y_1} = \frac{y_1}{y_2 - y_1} + \frac{x_1}{x_1 - x_2}$$

$$= \frac{x_1y_1 - x_2y_1 + x_1y_2 - x_1y_1}{(x_1 - x_2)(y_2 - y_1)}$$

$$= \frac{x_1y_2 - x_2y_1}{(x_1 - x_2)(y_2 - y_1)}$$

$$\Rightarrow \frac{x}{\frac{x_1y_2 - x_2y_1}{y_2 - y_1}} + \frac{y}{\frac{x_1y_2 - x_2y_1}{x_1 - x_2}} = 1$$

which is the equation of a straight line. [যাহা একটি সরলরেখার সমীকরণ]

Thus, the locus represented by $\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$ is a straight line. [অতএব $\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$ দ্বারা নির্দেশিত সঞ্চার পথ একটি সরলরেখা।]

Example–42. Prove that the equation of any circle in the z-plane $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$, where α, γ are real constants, while β may be a complex constant. **[NUH–2004]**

Solution : Let [ধরি] $z = x + iy$ and [এবং] $\beta = a + ib$. Then the given equation becomes [তখন প্রদত্ত সমীকরণটি দাঁড়ায়]

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$$

$$\Rightarrow \alpha(x + iy)(x - iy) + (a + ib)(x + iy) + (a - ib)(x - iy) + \gamma = 0$$

$$\Rightarrow \alpha(x^2 - i^2y^2) + (ax + iay + ibx + i^2by)$$
$$+ (ax - iay - ibx + i^2by) + \gamma = 0$$

$$\Rightarrow \alpha(x^2 + y^2) + 2ax - 2by + \gamma = 0$$

$$\Rightarrow x^2 + y^2 + 2\frac{a}{\alpha}x + 2\frac{-b}{\alpha} + \frac{\gamma}{\alpha} = 0$$

$$\Rightarrow x^2 + y^2 + 2gx + 2fy + c = 0, \text{ where } g = \frac{a}{\alpha},\ f = \frac{-b}{\alpha},\ c = \frac{\gamma}{\alpha}$$

which is the equation of a circle in the xy plane.

Thus, the equation $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ represents a circle in the z-plane. [যাহা xy তলে একটি বৃত্তের সমীকরণ। অতএব, $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$, সমীকরণ z তলে একটি বৃত্ত নির্দেশ করে।]

Example-43. z একটি জটিল সংখ্যা হলে $\left|\frac{z-1}{z+1}\right|$ = ধ্রুব এবং amp $\left(\frac{z-1}{z+1}\right)$ = ধ্রুব এর সঞ্চার পথসমূহ নির্ণয় কর এবং দেখাও যে, উহারা পরস্পরকে লম্বভাবে ছেদ করে। [If z is a complex number find the locuses of $\left|\frac{z-1}{z+1}\right|$ = constant and amp $\left(\frac{z-1}{z+1}\right)$ = constant. Show that they cut orthogonally each other.] **[NUH-2011]**

Solution : Given that [দেওয়া আছে]

$$\left|\frac{z-1}{z+1}\right| = \text{constant [ধ্রুবক]}$$

$$\Rightarrow \left|\frac{z-1}{z+1}\right| = c, \text{ say [ধরি]}$$

$$\Rightarrow \frac{|x+iy-1|}{|x+iy+1|} = c$$

$$\Rightarrow \frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}} = c$$

$$\Rightarrow (x-1)^2 + y^2 = c^2\,[(x+1)^2 + y^2]$$

$$\Rightarrow x^2 - 2x + 1 - y^2 + c^2x^2 + 2c^2x + c^2 + c^2y^2$$

$$\Rightarrow (1-c^2)\,x^2 - 2x - 2c^2x + (1-c^2)y^2 + 1 - c^2 = 0$$

$$\Rightarrow x^2 - \frac{2(1+c^2)}{1-c^2}\,x + y^2 + 1 = 0$$

$$\Rightarrow x^2 + 2\cdot\frac{c^2+1}{c^2-1}\,x + y^2 + 1 = 0 \;\cdots\cdots (1)$$

which is the equation of a circle [যাহা একটি বৃত্তের সমীকরণ]

2nd Part [২য় অংশ] ঃ Again, given that [আবার দেওয়া আছে]

$$\text{amp}\left(\frac{z-1}{z+1}\right) = \text{Constant [ধ্রুবক]}$$

$$\Rightarrow \text{amp}(z-1) - \text{amp}(z+1) = k_1, \text{ (ধরি)}$$

$$\Rightarrow \text{amp}(x-1+iy) - \text{amp}(x+1+iy) = k_1$$

$$\Rightarrow \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = k_1$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y}{x-1}\cdot\frac{y}{x+1}}\right) = k_1$$

$$\Rightarrow \tan^{-1}\frac{xy + y - xy + y}{x^2 - 1 + y^2} = k_1$$

$$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan k_1 = k, \text{ say [ধরি]}$$

$$\Rightarrow x^2 + y^2 - 1 = \frac{2}{k}\,y$$

$$\Rightarrow x^2 + y^2 - 2\cdot\frac{1}{k}\cdot y - 1 = 0 \;\cdots\cdots (2)$$

which is the equation of a circle. [যাহা একটি বৃত্তের সমীকরণ]

3rd part [৩য় অংশ] ঃ Comparing (1) with the circle $x^2 + y^2 + 2g_1x + 2f_1 + c_1 = 0$ and (2) with the circle $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ we get [(1) কে $x^2 + y^2 + 2g_1x + 2f_1 + c_1 = 0$ বৃত্তের সাথে এবং (2) নং কে $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ বৃত্তের সাথে তুলনা করে পাই]

$$g_1 = \frac{c^2+1}{c^2-1},\; f_1 = 0,\; c_1 = 1$$

and [এবং] $g_2 = 0,\; f_2 = -\frac{1}{k},\; c_2 = -1$

Now [এখন] $2g_1g_2 + 2f_1f_2 = 2\cdot\frac{c^2+1}{c^2-1}\cdot 0 + 2\cdot 0\cdot\left(\frac{-1}{k}\right) = 0$

and [এবং] $c_1 + c_2 = 1 - 1 = 0$

$\therefore 2g_1g_2 + 2f_1f_2 = c_1 + c_2$

Hence the two circles cut orthogonally. [অতএব বৃত্ত দুইটি পরস্পর লম্বভাবে ছেদ করে]

Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. Define complex number.
Ans : Any number of the form x + iy is called a complex number where x, y $\in \mathbb{R}$.

2. If z = x + iy, then what is $|z|$?
Ans : $|z| = \sqrt{x^2+y^2}$.

3. What is geometrical meaning of $|z|$.
Ans : $|z| = \sqrt{x^2+y^2}$ is the distance of z = x + iy = (x, y) form the origin.

4. What is the argument of the complex number z = x + iy.
Ans : If θ be the argument of z then $\theta = \tan^{-1}\frac{y}{x}$.

5. Is $|z|$ unique?

Ans : Yes, $|z|$ is a unique non-negative number.

6. What type of function $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is?

Ans : $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is a multivalued function.

7. What is Argand plane?

Ans : When a complex number z is represented by a point P(x, y) in the xy plane, then this plane is called the Argand plane.

8. Write the complex conjugate number of $z = x + iy$.

Ans : The complex conjugate number of z is $\bar{z} = x - iy$.

9. If $z = x + iy$ and $\bar{z} = x - iy$, then do you agree that $z > \bar{z}$?

Ans : Greater than or less than have no meaning in relation between two complex numbers. So $z > \bar{z}$ has no meaning and there fore I do not agree that $z > \bar{z}$.

10. If $z = x + iy$ and $\bar{z} = x - iy$ then comment about $|z|$ and $|\bar{z}|$.

Ans : $|z| = \sqrt{x^2 + y^2}$ and $|\bar{z}| = \sqrt{x^2 + y^2}$. This shows that the two complex numbers $z = (x, y)$ and $\bar{z} = (x, -y)$ are at same (equal) distance from the origin.

11. What is the amplitude when a complex number is purely real?

Ans : The amplitude is 0 or π when a complex number is purely real.

12. What is the amplitude when a complex number is purely imaginary?

Ans : The amplitude is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ when a complex numbers is purely imaginary.

13. Define imaginary unit i (iota).

Ans : The imaginary unit i(iota) is defined as $i = (0, 1)$.

14. Is the expression $x + iy$ is an imaginary number?

Ans : The expression $x + iy$ is not an imaginary number. It is a complex number.

15. Define modulus of a complex number [একটি জটিল সংখ্যার মডুলাসের সংজ্ঞা দাও।] **[NUH–2012]**

Ans : The modulus of a complex number $z = x + iy$ is $\sqrt{x^2 + y^2}$.

16. If $z = x + iy$ then comment about $|z|$ and $|\bar{z}|$. **[NUH–2012]**

[$z = x + iy$ হলে $|z|$ এবং $|\bar{z}|$ সম্বন্ধে মন্তব্য কর।]

Ans : If $z = x + iy$ then $\bar{z} = x - iy$

$\therefore |z| = |x + iy| = \sqrt{x^2 + y^2}$

and $|\bar{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$

$\therefore |z| = |\bar{z}|$, that is, modulus of z and $\bar{z}$ are equal.

17. Show that $|z|^2 = z\bar{z}$. **[NUH–2013]**

Ans : Let $z = x + iy$. Then $\bar{z} = \overline{x + iy} = x - iy$

$\therefore |z|^2 = (\sqrt{x^2 + y^2})^2 = x^2 + y^2$

and $z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2$

$\therefore |z|^2 = z\bar{z}$ (Showed)

18. Show that $|z|^2 = |\bar{z}|^2$

Ans : Let $z = x + iy$. Then $\bar{z} = \overline{x + iy} = x - iy$

$\therefore |z|^2 = (\sqrt{x^2 + y^2})^2 = (\sqrt{x^2 + (-y)^2})^2 = |\bar{z}|^2$ (Showed)

19. Find the argument of $-2 - i$. **[NUH–2013]**

Ans : Argument of $-2 - i$ is $\theta = \tan^{-1}\left(\frac{-1}{-2}\right) = \tan^{-1}\left(\frac{1}{2}\right)$.

20. What do you mean by equality of two complex number? **[NUH–2013]**

Ans : Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

That is, the real part of the one is equal to the real part of the other, and the imaginary part of the one is equal to the imaginary part of the other.

EXERCISE–1

Part–A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. What is the mirror image of a complex number z?
2. Under what condition a complex number z and its conjugate $\bar{z}$ are equal.
3. What are the conditions for two given numbers z_1 and z_2 to be conjugate?
4. What is the effect when a complex number z is multiplied by i.
5. Every real number is a complex number. Is the converse true?

Part–B : Short Questions (সংক্ষিপ্ত প্রশ্ন)

1. Give the geometrical representation of product of two complex numbers. **[CUH–2002]**

   Ans : See art–1.4
2. Give the geometrical representation of quotient of two complex numbers. **[CUH–2002]**

   Ans : See art–1.6
3. Show that the triangle inequality holds in $\mathbb{C}$. **[RUH–2003, 2006]**

   Ans : See Solved problem–22(i).
4. If z be a complex number, then represent iz geometrically.

   Ans : See art–1.7
5. Prove that a complex number is purely real if the amplitude is 0 or π.

   Ans : See proposition of art–1.7
6. Prove that a complex number is purely imaginary if the amplitude is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

   Ans : See proposition of art–1.7

7. Discuss the locus represented by $\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$, where z_1 and z_2 are two given points. **[NUH–1996, RUH–1996]**

   Ans : See Solved problem–41.
8. Prove that, if sum and product of two complex numbers are both real, then the two numbers must either be real or conjugate.

   Ans : See solved example–27.
9. If z_1 and z_2 are two non-zero complex numbers, then prove that the modulas of their difference is always greater than or equal to the difference of their moduli. **[CUH–2000]**

   Ans : See Solved problem–24.
10. Show that $|z|\sqrt{2} \geq |\text{Re}(z)| + |\text{Im}(z)|$, where z is any complex number. **[DUH–1988, 1990]**

    Ans : See Solved example–19.
11. If $z = x + iy$, then prove that $|x| + |y| \leq \sqrt{2}\,|x + iy|$. **[NUH–1998, RUH–1997]**

    Ans : See Solve example–20.
12. If z is any complex number than show that

    $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}\,|z|$ **[RUH–2004]**

    Ans : See Solved example–19.
13. Find two comple numbers whose sum is 4 and whose product is 8. **[NUH–2000, 2006(Old)]**

    Ans : See Solved example–26.
14. If z_1, z_2 are complex numbers then prove that

    $|z_1 z_2| = |z_1|\,|z_2|$ **[DUH–1989, RUH–1998]**

    Ans : See Solved example–21.
15. If $z_1, z_2, \cdots, z_n$ are complex numbers then prove that

    $|z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$. **[DUH–1985]**

    Ans : See Solved example–21.

16. For any two complex numbers z_1 and z_2 prove that

$|z_1 + z_2| \le |z_1| + |z_2|$ **[NUH-03, 04, 05, 07, RUH-1997]**

Ans : See Solved example-22(i)

17. For any two complex numbers z_1 and z_2 prove that

$|z_1 - z_2| \ge |z_1| - |z_2|$. **[RUH-1998, CUH-2004]**

Ans : See Solved example-24.

18. For any two complex numbers z_1 and z_2 prove that

$|z_1 - z_2| \le |z_1| + |z_2|$ **[NUH-2006]**

Ans : See Solved example-22(iii).

19. If $z_1, z_2, \cdots, z_n$ are complex numbers then prove that

$|z_1 + z_2 + .. + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$.

[RUH-1998, CUH-2004]

Ans : See Solved example-22(ii).

20. If z_1, z_2, z_3 are complex numbers then show that

$\left|\dfrac{z_1}{z_2 + z_3}\right| \le \dfrac{|z_1|}{\left||z_2| - |z_3|\right|}$, where $|z_2| \ne |z_3|$. **[NUH-2001]**

Ans : See Solved example-25(i).

21. If z_1, z_2, z_3, z_4 are complex numbers then show that

$\left|\dfrac{z_1 + z_2}{z_3 + z_4}\right| \le \dfrac{|z_1| + |z_2|}{\left||z_3| - |z_4|\right|}$, where $|z_3| \ne |z_4|$. **[CUH-2000]**

Ans : See solved example-25.

22. Prove that $|z_1 - z_2| \ge \left||z_1| - |z_2|\right| \ge |z_1| - |z_2|$, where z_1, z_2 are complex numbers. **[NUH-1994, 2002(Old), DUH-1998, 2005]**

Ans : See Solved example-24.

23. If z_1 and z_2 are two complex numbers, then prove that

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

[NUH-2000, 2006(Old), RUH-2002]

Ans : See Solved example-28.

24. Show that $|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|$

[RUH-2001]

Ans : See 2nd part of Solved example-28.

25. Show that $|a + \sqrt{a^2 - b^2}| + |a - \sqrt{a^2 - b^2}| = |a + b| + |a - b|$

[CUH-2004]

Ans : Write $\alpha = a$ and $\beta = b$ in Solved Example 28(2nd Part).

26. Solve $\sinh z = i$ **[NUH-1995]**

Ans : See Solved example-11.

27. Solve $\cosh z = 2$ **[NUH-2002, 2006, 2010, DUH-2001]**

Ans : See Solved example-12.

28. Prove that $|z| \le |Re(z)| + |Im(z)| \le \sqrt{2}\,|z|$

Ans : See Solved example-19.

29. Prove that $|z|^2 = |-z|^2 = |\bar{z}|^2 = |-\bar{z}|^2 = z\bar{z}$

Ans : See Solved example-15.

30. Show that $Im(iz) = Re(z)$ and $Re(iz) = |z|^2 Im(z^{-1})$.

Ans : See Solved example-17.

31. If the equation $z^2 + az + b = 0$ has a pair of conjugate complex roots then prove that a, b are both real and $a^2 < 4b$.

Ans : See Solved example-31.

32. Solve the equation $|z| - z = 2 + i$

Ans : See Solved example-32.

33. Determine all real x and y which satify the relation $x + iy = |z + iy|$.

Ans : See Solved example-10(a).

34. Determine all real x and y which satify the relation $x + iy = (x + iy)^2$.

Ans : See Solved example-10(b).

35. Find all values of the following :

(i) $(-8i)^{1/3}$, (ii) $(-8 - i8\sqrt{3})^{1/4}$ (iii) $(1 - i\sqrt{3})^{1/2}$ **[DUH-2006]**

Ans : (i) $\sqrt{3} - i,\ 2i,\ -\sqrt{3} - i$

(ii) $1 + \sqrt{3}i,\ -\sqrt{3} + i,\ -1 - i\sqrt{3},\ \sqrt{3} - i$

(iii) $\sqrt{3} - i,\ -\sqrt{3} + i$

Part-C (Broad Questions) (বড় প্রশ্ন)

1. Give the geometrical representation of product and quotient of two complex numbers. [CUH-200
   Ans : See art-1.4 and art-1.6.
2. Prove that the set of complex numbers form an abelian group. [RUH-200
   Ans : See Solved problem 39 or 40.
3. Define addition and multiplication in $\mathbb{C}$, the set of complex numbers such that with these definitions $\mathbb{C}$ is a field. [RUH-200
   Ans : See Definition-1 and Solved problem-40.
4. Prove that a complex number is purely real if the amplitude is 0 or π and purely imaginary if the amplitude is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.
   Ans : See Proposition of art-1.7.
5. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$. Interpret the result geometrically and deduce that
   $$|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|,$$
   all the numbers involved being complex. [NUH-2004(Old)]
   Ans : See Solved problem-28.
6. Prove that $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$ and interpret the identity geometrically. [NUH-2002(Old)]
   Ans : See Solved problem-28 [Write $z_1 = z$ and $z_2 = w$].
7. Prove that $|z + w| \le |z| + |w|$ and hence prove that $|z + w| \ge ||z| - |w||$.
   Ans : See Solved example-22 and 24.
8. If z_1, z_2, z_3 are the vertices of an equilateral triangle in the argand plane, then show that $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$.
   Ans : See Solved example-30.
9. If $z_1, z_2, \cdots, z_n$ are complex numbers then prove that
   $$\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j| \text{ and } \left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{n} |z_j|$$
   Ans : See Solved example-23(3rd and 4th part).

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CHAPTER-2
ANALYTIC FUNCTIONS

In this chapter we shall define functions of complex variable and discuss their limit, continuity and differentiability. Later we introduce analytic functions, which have wide range of applications in complex analysis.

2.1. Functions of a complex Variable : Let S be a set of complex numbers. A function f defined on S is a rule which assigns to each $z \in S$ a unique value $f(z) \in \mathbb{C}$. We write this as

$$w = f(z), z \in S \quad \text{or} \quad f : S \to \mathbb{C}.$$

We say f is a complex function of a complex variable. Here w is called the value of f at z. It is customary to locate $w = f(z) \in \mathbb{C}$ in another complex plane called w-plane. The set S is called the domain of f and $S_1 = \{f(z) : z \in S\}$, range of f.

When the domain of definition is not specified, we take the largest possible set as domain for f on which f is well defined.

Every functions f(z) can be written as

$$f(z) = u(x, y) + iv(x, y) \text{ or } w = f = u + iv \cdots\cdots (1)$$

For an example, let $w = f(z) = z^2$

$\Rightarrow w = f(z) = (x + iy)^2$

$\Rightarrow u + iv = x^2 + i^2y^2 + i2xy$

$= x^2 - y^2 + i2xy$

Hence $u = x^2 - y^2$ and $v = 2xy$.

On the other hand, if u(x, y) and v(x, y) are two given real-valued functions of the real variables x and y, equation (1) can be used to define a function of the complex variable $z = x + iy$.

If in equation (1) the number v(x, y) is always zero, then the number f(z) is always real.

Single valued function [NUH(Phy)-2005]

The function $w = f(z)$ is called a single valued function if for every value of z there is only one value of w.

[একমানী ফাংশন ঃ $w = f(z)$ ফাংশনকে একমানী ফাংশন বলে যদি z এর প্রত্যেক মানের জন্য w এর শুধুমাত্র একটি মান থাকে।]

Example : Let $w = f(z) = z^2$. Then for every value of z, there is only one value of w. Hence $w = f(z) = z^2$ is a single valued function of z.

Multivalued functions : [NUH–1999]

Generalization of a concept of a function is the multivalued function. These functions take more than one value at some or all points of the domain of definition.

The function $w = f(z)$ is called a many valued function if for every value of z, there are more value of w.

[বহুমানী ফাংশন ঃ $w = f(z)$ ফাংশনকে বহুমানী ফাংশন বলে যদি z এর প্রত্যেক মানের জন্য w এর অনেক (একাধিক) মান থাকে।]

Example : Let $w = f(z) = z^{1/2}$. Here for every nonzero value of z, there are two values of w. Hence $w = f(z) = z^{1/2}$ is a multivalued function of z.

Polynomial : The function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n;\ a_n \neq 0$$

is called a polynomial of degree n, where n is zero or a positive integer and $a_0, a_1, a_2, \cdots, a_n$ are complex constants.

Rational functions : If P(z) and Q(z) are two polynomials then $\frac{P(z)}{Q(z)}$ is called rational functions, which are defined at each point z except where $Q(z) = 0$.

2.2. Limits :

Definition : The function f(z) defined in some neighbourhood of z_0 is said to have limit w_0 at z_0 if, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

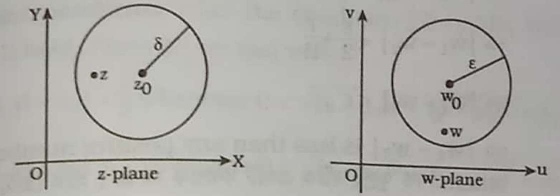
$$|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \quad \cdots\cdots (2)$$

Symbolically, we write $\lim\limits_{z \to z_0} f(z) = w_0$

This is also written as

$$f(z) = w_0 + \varepsilon(z)\ \forall z : 0 < |z - z_0| < \delta$$

with the condition that $\varepsilon(z) \to 0$ as $z \to z_0$.



Since (2) is true for all $z \in \{z : 0 < |z - z_0| < \delta\}$, $z \to z_0$ means z approaches to z_0 through any path. Hence limit w_0 is independent of path.

If $\lim f(z) \to w_0$ as $z \to z_0$ exists, then it is unique.

Theorem–1. If $\lim\limits_{z \to z_0} f(z)$ exists, then it must be unique.

[যদি $\lim\limits_{z \to z_0} f(z)$ বিদ্যমান থাকে, তবে ইহা অবশ্যই অনন্য]

[RUH–1994, CUH–2002]

Proof : Suppose $\lim\limits_{z \to z_0} f(z)$ exists and the limit is not unique. Let there exist two limits w_1 and w_2. Then by definition

$$\lim_{z \to z_0} f(z) = w_1 \text{ and } \lim_{z \to z_0} f(z) = w_2.$$

Also by hypothesis, for any given $\varepsilon > 0$, then exists a $\delta > 0$ such that

[প্রমাণ ঃ ধরি $\lim\limits_{z \to z_0} f(z)$ বিদ্যমান এবং সীমামান অনন্য না। ধরি দুইটি সীমা w_1 ও w_2 বিদ্যমান। তখন সংজ্ঞানুসারে

$$\lim_{z \to z_0} f(z) = w_1 \text{ এবং } \lim_{z \to z_0} f(z) = w_2.$$

অধিকন্তু কল্পনা অনুসারে, প্রদত্ত $\varepsilon > 0$ এর জন্য একটি $\delta > 0$ বিদ্যমান থাকবে যেন]

$$|f(z) - w_1| < \frac{\varepsilon}{2} \text{ when } 0 < |z - z_0| < \delta$$

$$|f(z) - w_2| < \frac{\varepsilon}{2} \text{ when } 0 < |z - z_0| < \delta$$

Now [যখন] $|w_1 - w_2| = |w_1 - f(z) + f(z) - w_2|$

$$\Rightarrow |w_1 - w_2| \leq |w_1 - f(z)| + |f(z) - w_2|$$

$\Rightarrow |w_1 - w_2| \le |f(z) - w_1| + |f(z) - w_2|$

$\Rightarrow |w_1 - w_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\Rightarrow |w_1 - w_2| < \varepsilon$

$\Rightarrow |w_1 - w_2|$ is less than any positive number ε however small. [যত ক্ষুদ্রই হউক একটি ধনাত্মক সংখ্যা ε হতে ছোট]

This means [এর অর্থ] $|w_1 - w_2| = 0 \Rightarrow w_1 = w_2$

Thus if $\lim\limits_{z \to z_0} f(z)$ exists, it must be unique. [অতএব যদি $\lim\limits_{z \to z_0} f(z)$ বিদ্যমান থাকে তবে ইহা অবশ্যই অনন্য ।]

Theorem–2. Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.

Then $\lim\limits_{z \to z_0} f(z) = w_0 \cdots\cdots (1)$

if and only if

$\lim\limits_{(x,y) \to (x_0, y_0)} u(x, y) = u_0$ and $\lim\limits_{(x,y) \to (x_0, y_0)} v(x, y) = v_0 \cdots\cdots (2)$

Proof : Necessary condition : First suppose that $\lim\limits_{z \to z_0} f(z) = w_0$.

Then, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

$\Rightarrow |u(x, y) + iv(x, y) - (u_0 + iv_0)| < \varepsilon$

whenever $0 < |x + iy - (x_0 + iy_0)| < \delta$

$\Rightarrow |u(x, y) - u_0 + i(v(x, y) - v_0)| < \varepsilon$

whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$

Since $|u - u_0| \le |(u - u_0) + i(v - v_0)|$ $[\because |\text{Re}(z)| \le |z|]$

and $|v - v_0| \le |(u - u_0) + i(v - v_0)|$ $[\because |\text{Im}(z)| \le |z|]$

it follows that

$|u(x, y) - u_0| < \varepsilon$ and $|v(x, y) - v_0| < \varepsilon$

whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$

$\Rightarrow \lim\limits_{(x,y) \to (x_0, y_0)} u(x, y) = u_0$ and $\lim\limits_{(x,y) \to (x_0, y_0)} v(x, y) = v_0$

Sufficient condition : Let the condition (2) holds. We shall prove that (1) hold. From (2) we can write

$|u(x, y) - u_0| < \frac{\varepsilon}{2}$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1$

and $|v(x, y) - v_0| < \frac{\varepsilon}{2}$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2$

Choosing $\delta = \min\{\delta_1, \delta_2\}$ we have

$|f(z) - w_0| = |u(x, y) + iv(x, y) - u_0 - iv_0|$

$\Rightarrow |f(z) - w_0| \le |u(x, y) - u_0| + |v(x, y) - v_0|$

$\Rightarrow |f(z) - w_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow |f(z) - w_0| < \varepsilon$, wherever $0 < |z - z_0| < \delta$

Thus, $f(z) \to w_0$ as $z \to z_0$

That is, $\lim\limits_{z \to z_0} f(z) = w_0$. Hence proved

2.3. Continuity :

Definition : A complex valued function $f(z)$ is said to be continuous at a point z_0 if for every (given) $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

Symbolically, we write $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Thus, in case of continuity at a point, the limiting value and functional value are equal.

[একটি জটিল মান ফাংশন $f(z)$ কে z_0 বিন্দুতে অবিচ্ছিন্ন বলে যদি প্রদত্ত $\varepsilon > 0$ এর জন্য একটি $\delta > 0$ বিদ্যমান থাকে যেন

$$|f(z) - f(z_0)| < \varepsilon \text{ যখন } |z - z_0| < \delta$$

প্রতীক আকারে আমরা লিখি $\lim\limits_{z \to z_0} f(z) = f(z_0)$

অতএব, একটি বিন্দুতে অবিচ্ছিন্নতার ক্ষেত্রে, সীমাস্থ মান ও ফাংশনাল মান সমান ।]

A function $f(z)$ is continuous on a set S if it is continuous at every point of S. If a function is not continuous at z_0, then it is discontinuous at z_0.

then $\lim\limits_{z \to z_0} \eta(z) = \eta(z_0)$. This shows that $\eta(z)$ is continuous at z_0 and $|\eta(z)| < \varepsilon$ whenever $|z - z_0| < \delta$.

Thus, we get an explicit expression for f(z) in the form

$$f(z) = f(z_0) + (z - z_0) f'(z) + (z - z_0) \eta(z) \quad \text{for } |z - z_0| < \delta.$$

Note–2. Writting $\Delta z = z - z_0$ in (1) we get

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(\Delta z + z_0) - f(z_0)}{\Delta z}$$

Droping the suffix from z_0 we get

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \ldots\ldots (2)$$

Note–3. If $w = f(z)$, $w + \Delta w = f(z + \Delta z)$ then from (2)

$$f'(z) = \lim_{\Delta z \to 0} \frac{w + \Delta w - w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}$$

$$\therefore \frac{dw}{dz} = f'(z).$$

Theorem–3. Every differentiable function is continuous. [প্রত্যেক অন্তরীকরণযোগ্য ফাংশন অবিচ্ছিন্ন]

Proof : Let the function f(z) is differentiable at z_0. [মনে করি f(z) ফাংশনটি z_0 এ অন্তরীকরণযোগ্য]

Now [এখন] $$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} (z - z_0)$$

$$\Rightarrow \lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$$

Hence f(z) is continuous at z_0. [অতএব z_0 এ f(z) অবিচ্ছিন্ন] Thus every differentiable function is continuous. [অতএব প্রত্যেক অন্তরীকরণযোগ্য ফাংশন অবিচ্ছিন্ন ।]

2.5. Analytic (or regular or holomorphic) functions :

[NUH–2000, 03, 05, 05(Old), 06(Old), 06, 12(Old), 12, NU(Pre)–2011, (Phy)–06, 10, DUH–05]

The concept of analytic function is the core heart of complex analysis. First we give the definition of an analytic function at a point and in a domain (region).

Definition : A complex function f(z) is said to be analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighbourhood of z_0.

[একটি জটিল ফাংশন f(z) কে একটি বিন্দু z_0 এ বৈশ্লেষিক বলে যদি ইহার অন্তরক শুধুমাত্র z_0 বিন্দুতে নয়, z_0 বিন্দুর নেইবারহুডের কিছু বিন্দু z বিন্দুতেও বিদ্যমান থাকে ।]

The above definition follows that if f(z) is analytic at z_0, it is actually analytic at each point in a neighbourhood of z_0.

Definition : A function f(z) is said to be analytic in a domain D (or region R) if it is analytic at each point of D (or R).

[একটি ফাংশন f(z) কে একটি ডোমেন D এ বৈশ্লেষিক বলে যদি ইহা D এর প্রত্যেক বিন্দুতে বৈশ্লেষিক হয় ।]

The terms holomorphic and regular are used with identical meaning in the literature.

Entire function : A complex function f(z) is said to be entire if it is analytic in the whole complex plane.

Singular point or Singularity : [JUH (Phy)–2000, 2003, 3005]

If a function f(z) fails to be analytic at a point z_0 but in every neighbourhood of z_0 there exist at least one point where the function is analytic, then z_0 is said to be a singular point or singularity of f(z).

[ব্যতিচার বিন্দু ঃ যদি একটি ফাংশন f(z) একটি বিন্দু z_0 এ বৈশ্লেষিক হতে ব্যর্থ হয় কিন্তু z_0 এর প্রত্যেক নেইবারহুডে কমপক্ষে একটি বিন্দু বিদ্যমান থাকে যেখানে ফাংশনটি বৈশ্লেষিক হয়, তখন z_0 কে f(z) এর ব্যতিচার বিন্দু বলে ।]

then $\lim\limits_{z\to z_0} \eta(z) = \eta(z_0)$. This shows that $\eta(z)$ is continuous at z_0 and $|\eta(z)| < \varepsilon$ whenever $|z - z_0| < \delta$.

Thus, we get an explicit expression for f(z) in the form

$$f(z) = f(z_0) + (z - z_0) f'(z) + (z - z_0)\,\eta(z) \quad \text{for } |z - z_0| < \delta.$$

Note–2. Writting $\Delta z = z - z_0$ in (1) we get

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(\Delta z + z_0) - f(z_0)}{\Delta z}$$

Droping the suffix from z_0 we get

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \ldots\ldots (2)$$

Note–3. If $w = f(z)$, $w + \Delta w = f(z + \Delta z)$ then from (2)

$$f'(z) = \lim_{\Delta z\to 0} \frac{w + \Delta w - w}{\Delta z} = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}$$

$$\therefore \frac{dw}{dz} = f'(z).$$

Theorem–3. Every differentiable function is continuous. [প্রত্যেক অন্তরীকরণযোগ্য ফাংশন অবিচ্ছিন্ন]

Proof : Let the function f(z) is differentiable at z_0. [মনে করি f(z) ফাংশনটি z_0 এ অন্তরীকরণযোগ্য]

Now [এখন] $\lim\limits_{z\to z_0} [f(z) - f(z_0)] = \lim\limits_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} (z - z_0)$

$$\Rightarrow \lim_{z\to z_0} f(z) - f(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z\to z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0$$

$$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0)$$

Hence f(z) is continuous at z_0. [অতএব z_0 এ f(z) অবিচ্ছিন্ন] Thus every differentiable function is continuous. [অতএব প্রত্যেক অন্তরীকরণযোগ্য ফাংশন অবিচ্ছিন্ন ।]

2.5. Analytic (or regular or holomorphic) functions :

[NUH-2000, 03, 05, 05(Old), 06(Old), 06, 12(Old), 12, NU(Pre)-2011, (Phy)-06, 10, DUH-05]

The concept of analytic function is the core heart of complex analysis. First we give the definition of an analytic function at a point and in a domain (region).

Definition : A complex function f(z) is said to be analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighbourhood of z_0.

[একটি জটিল ফাংশন f(z) কে একটি বিন্দু z_0 এ বৈশ্লেষিক বলে যদি ইহার অন্তরক শুধুমাত্র z_0 বিন্দুতে নয়, z_0 বিন্দুর নেইবারহুডের কিছু বিন্দু z বিন্দুতেও বিদ্যমান থাকে ।]

The above definition follows that if f(z) is analytic at z_0, it is actually analytic at each point in a neighbourhood of z_0.

Definition : A function f(z) is said to be analytic in a domain D (or region R) if it is analytic at each point of D (or R).

[একটি ফাংশন f(z) কে একটি ডোমেন D এ বৈশ্লেষিক বলে যদি ইহা D এর প্রত্যেক বিন্দুতে বৈশ্লেষিক হয় ।]

The terms holomorphic and regular are used with identical meaning in the literature.

Entire function : A complex function f(z) is said to be entire if it is analytic in the whole complex plane.

Singular point or Singularity : [JUH (Phy)-2000, 2003, 3005]

If a function f(z) fails to be analytic at a point z_0 but in every neighbourhood of z_0 there exist at least one point where the function is analytic, then z_0 is said to be a singular point or singularity of f(z).

[ব্যতিচার বিন্দু ঃ যদি একটি ফাংশন f(z) একটি বিন্দু z_0 এ বৈশ্লেষিক হতে ব্যর্থ হয় কিন্তু z_0 এর প্রত্যেক নেইবারহুডে কমপক্ষে একটি বিন্দু বিদ্যমান থাকে যেখানে ফাংশনটি বৈশ্লেষিক হয়, তখন z_0 কে f(z) এর ব্যতিচার বিন্দু বলে ।]

Cauchy-Riemann Partial Differential Equations :

[JUH (Phy)–2000, 2004]

Theorem–4. Necessary conditions for f(z) to be analytic. The necessary condition for $w = f(z) = u(x, y) + iv(x, y)$ to be analytic at a point $z = x + iy$ of its domain is that the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ should exist and satisfy the Cauchy-Riemann partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

[NUH–98, 2000, 02, 05 (Old), 06 (Old), 07, 10, 12, DUH–03, 05]

Proof : Given that $f(z) = u(x, y) + iv(x, y)$ is analytic (differentiable) at any point $z = x + iy$ of its domain.

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \quad \cdots (1)$$

Since f(z) is analytic, so $f'(z)$ exists, finite and unique independent of the path along which $\Delta z \to 0$.

Now there are two cases [এখন দুইটি ক্ষেত্র]

Case–1. Along real axis (x-axis) we have $\Delta y = 0$ and $\Delta x \to 0$, so from (1) we have [বাস্তব অক্ষের দিকে $\Delta y = 0$ এবং $\Delta x \to 0$, সুতরাং (1) হতে পাই]

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \cdots\cdots (2)$$

provided the partial derivatives exist. [আংশিক অন্তরীকরণ বিদ্যমান শর্তে]

Case–2. Along imaginary axis (y-axis) we have $\Delta x = 0$ and $\Delta y \to 0$, so from (1) we have [কাল্পনিক অক্ষের দিকে $\Delta x = 0$ এবং $\Delta y \to 0$, সুতরাং (1) হতে পাই]

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y}$$

$$\Rightarrow f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \cdots\cdots (2)$$

Since f(z) is analytic, so $f'(z)$ exists and the above two limits obtained in (2) and (3) should be unique (identical). [যেহেতু f(z) বৈশ্লেষিক, সুতরাং $f'(z)$ বিদ্যমান এবং উপরের (2) ও (3) প্রাপ্ত লিমিট অনন্য]

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts we get [বাস্তব ও অবাস্তব রাশি সমীকৃত করে পাই]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

That is [অর্থাৎ] $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and [এবং] $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. (Proved)

Theorem–5. Sufficient condition for f(z) to be analytic.

[NUH–1993, 1998, 2001, 2006]

The function $w = f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if

(i) u, v are differentiable in D and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(ii) The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ are all continuous in D. **[DUH–2006]**

[উপপাদ্য-৫ ঃ f(z) বৈশ্লেষিক হওয়ার জন্য যথেষ্ঠ শর্ত। একটি ডোমেন D এ $w = f(z) = u(x, y) + iv(x, y)$ বৈশ্লেষিক হবে যদি

(i) D এ u, v অন্তরীকরণযোগ্য হয় এবং $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(ii) আংশিক অন্তরীকরণ $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ এবং $\frac{\partial v}{\partial y}$ সকলেই D এর মধ্যে অবিচ্ছিন্ন হয়।]

Proof : Given that [দেওয়া আছে] $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are continuous and [অবিচ্ছিন্ন এবং]

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \;\cdots\cdots\; (1)$$

Now [এখন] $\Delta u = u(x+\Delta x, y+\Delta y) - u(x, y)$

$$\Rightarrow \Delta u = [u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)] + [u(x, y+\Delta y) - u(x, y)] \;\cdots\; (2)$$

From mean value theorem we know that if $f(x)$ is continuous in $a \le x \le b$ and differentiable in $a < x < b$ then

[গড়মান উপপাদ্য হতে আমরা জানি যে, যদি $a \le x \le b$ এ $f(x)$ অবিচ্ছিন্ন হয় ও $a < x < b$ এ অন্তরীকরণযোগ্য হয় তখন]

$$f(a+h) - f(a) = hf'(a+\theta h) \text{ where [যেখানে] } 0 < \theta < 1$$

Using this result in (2) we get [এইফল (2) এ ব্যবহার করে পাই]

$$\Delta u = \Delta x \cdot \frac{\partial}{\partial x} u(x+\theta\Delta x, y+\Delta y) + \Delta y \cdot \frac{\partial}{\partial y} u(x, y+\theta'\Delta y) \;\cdots\cdots\; (3)$$

where [যেখানে] $0 < \theta < 1$ and [এবং] $0 < \theta' < 1$.

Again, since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous, so [আবার যেহেতু $\frac{\partial u}{\partial x}$ এবং $\frac{\partial u}{\partial y}$ অবিচ্ছিন্ন, সুতরাং]

$$\left|\frac{\partial}{\partial x} u(x+\theta\Delta x, y+\Delta y) - \frac{\partial}{\partial x} u(x, y)\right| < \varepsilon$$

and [এবং] $$\left|\frac{\partial}{\partial y} u(x, y+\theta'\Delta y) - \frac{\partial}{\partial y} u(x, y)\right| < \eta$$

Choosing $\varepsilon_1 < \varepsilon$ and $\eta_1 < \eta$ we get [$\varepsilon_1 < \varepsilon$ এবং $\eta_1 < \eta$ পছন্দ করে পাই]

$$\frac{\partial}{\partial x} u(x+\theta\Delta x, y+\Delta y) - \frac{\partial}{\partial x} u(x, y) = \varepsilon_1$$

$$\Rightarrow \frac{\partial}{\partial x} u(x+\theta\Delta x, y+\Delta y) = \frac{\partial}{\partial x} u(x, y) + \varepsilon_1$$

$$= \frac{\partial u}{\partial x} + \varepsilon_1 \;\cdots\cdots\; (4)$$

and [এবং] $$\frac{\partial}{\partial y} u(x, y+\theta'\Delta y) - \frac{\partial}{\partial y} u(x, y) = \eta_1$$

$$\Rightarrow \frac{\partial}{\partial y} u(x, y+\theta'\Delta y) = \frac{\partial}{\partial y} u(x, y) + \eta_1 = \frac{\partial u}{\partial y} + \eta_1 \;\cdots\cdots\; (5)$$

By putting the values of (4) and (5) in (3) we get [(4) ও (5) এর মান (3) এ বসাইয়া পাই]

$$\Delta u = \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y \;\cdots\cdots\; (6)$$

where [যেখানে] $\varepsilon_1 \to 0$ and [এবং] $\eta_1 \to 0$ as $\Delta x \to 0$ and [এবং] $\Delta y \to 0$.

Proceeding in the same way we have [একইভাবে অগ্রসর হয়ে পাই]

$$\Delta v = \left(\frac{\partial v}{\partial x} + \varepsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y \;\cdots\cdots\; (7)$$

where [যেখানে] $\varepsilon_2 \to 0$ and [এবং] $\eta_2 \to 0$ as $\Delta x \to 0$ and [এবং] $\Delta y \to 0$

Now [এখন] $w = f(z) = u + iv$

$$\Rightarrow \Delta w = \Delta u + i\Delta v$$

$$= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y + i\left[\left(\frac{\partial v}{\partial x} + \varepsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y\right]$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + (\varepsilon_1 + i\varepsilon_2)\,\Delta x + (\eta_1 + i\eta_2)\,\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

where [যেখানে] $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0$ and [এবং] $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and [এবং] $\Delta y \to 0$

$$\Rightarrow \Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \varepsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta z + \varepsilon\Delta x + \eta\Delta y$$

$$\Rightarrow \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\varepsilon\Delta x + \eta\Delta y}{\Delta z}$$

Taking limit $\Delta z \to 0$ on both sides we get [উভয় পক্ষে $\Delta z \to 0$ লিমিট নিয়ে পাই]

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\varepsilon\Delta x + \eta\Delta y}{\Delta z}\right)$$

$$\Rightarrow \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + 0 + 0$$

$$\Rightarrow \frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

We see that the derivative exist and unique. Hence $w = f(z)$ is analytic. [আমরা দেখি যে অন্তরীকরণ বিদ্যমান এবং অনন্য। অতএব $w = f(z)$ বৈশ্লেষিক।]

Note-1. A necessary condition for a function f to be analytic in a domain D is the continuity of f throughout D. Satisfaction of the **Cauchy-Riemann (C-R)** equations is also necessary, but not sufficient.

Note-2. The real functions of complex variable are nowhere analytic unless these are constant valued. Hence, Re(z), Im(z), $|z|$, $|z|^2$, etc. are nowhere analytic, since these are real valued but not constant on any domain in complex plane.

2.6. Polar form of C-R equations :

[NUH-1997, 2003, 2004, 2008, 2010, 2012(Old), DUH-2004]

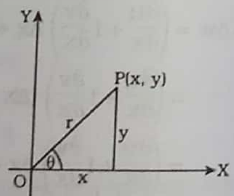
The relation between cartesian and polar form are [কার্তেসীয় ও পোলার স্থানাংকে সম্পর্ক হল]

$$x = r\cos\theta,\ y = r\sin\theta$$

$$r^2 = x^2 + y^2 \text{ and [এবং] } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$



Similarly [অনুরূপে]

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

Also [আরো] $$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$= \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

and [এবং] $$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$= \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Now u, v are functions of x, y; and x, y are functions of r, θ. So, u, v are functions of r, θ. [এখন u, v হল x, y এর ফাংশন এবং x, y হল r, θ এর ফাংশন। সুতরাং u, v হল r, θ এর ফাংশন]

That is [অর্থাৎ] $u = u(r, \theta)$ and [এবং] $v = v(r, \theta)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \quad \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \quad \cdots\cdots (2)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \quad \cdots\cdots (3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \quad \cdots\cdots (4)$$

From Cauchy-Riemann equations we know that [কচি রীম্যান সমীকরণ হতে আমরা জানি]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \quad \cdots\cdots (5)$$

$$\text{and } \sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \quad \cdots\cdots (6)$$

[By equations (1), (2), (3) and (4).]

Multiplying (5) by cos θ and (6) by sin θ, and then adding we get [(5) কে cos θ এবং (6) কে sin θ দ্বারা গুণ করে এবং অতপর যোগ করে পাই]

$$\cos^2\theta\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta} + \sin^2\theta\frac{\partial u}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$= \cos\theta\sin\theta\frac{\partial v}{\partial r} + \frac{\cos^2\theta}{r}\frac{\partial v}{\partial \theta} - \sin\theta\cos\theta\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow (\cos^2\theta + \sin^2\theta)\frac{\partial u}{\partial r} = \frac{1}{r}(\cos^2\theta + \sin^2\theta)\frac{\partial v}{\partial \theta}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \cdots\cdots (7)$$

Again, multiplying (5) by sin θ and (6) by cos θ, and then adding we get [আবার, (5) কে sin θ এবং (6) কে cos θ দ্বারা গুণ করে এবং অতপর যোগ করে পাই]

$$\cos\theta\sin\theta\frac{\partial u}{\partial r}-\frac{\sin^2\theta}{r}\frac{\partial u}{\partial \theta}-\sin\theta\cos\theta\frac{\partial u}{\partial r}-\frac{\cos^2\theta}{r}\frac{\partial u}{\partial \theta}$$

$$=\sin^2\theta\frac{\partial v}{\partial r}+\frac{\sin\theta\cos\theta}{r}\frac{\partial v}{\partial \theta}+\cos^2\theta\frac{\partial v}{\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow -\frac{1}{r}(\sin^2\theta+\cos^2\theta)\frac{\partial u}{\partial \theta}=(\sin^2\theta+\cos^2\theta)\frac{\partial v}{\partial r}$$

$$\Rightarrow -\frac{1}{r}\frac{\partial u}{\partial \theta}=\frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r} \quad \text{...... (8)}$$

Thus [অতএব] $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$ and [এবং] $\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$ are the required polar form of C–R equations. [পোলার আকারে আবশ্যকীয় C–R সমীকরণ]

2.7. Harmonic Functions : [NUH–93, 94, 01, 04, NU(Pre)–08]

Any real valued function of x and y is said to be harmonic in a domain of the xy plane if throughout the domain it has continuous partial derivatives of the first and second order and satisfy the Laplace equation. **[DUH–2005]**

Harmonic conjugate : [NUH–1993, 1994, 2004, DUH–2005]

The function v is said to be a harmonic conjugate of u if u and v are harmonic and u, v satisfy the C–R equations.

If $f(z) = u(x, y) + iv(x, y)$, then u, v are the component functions of f. For component functions we shall prove the following results.

Theorem–6. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D. **[NUH–2001]**

Or, The real and imaginary parts of an analytic function ore harmonic function. **[NUH–2001]**

Proof : Since f is analytic in D, so its component functions u and v satisfy the Cauchy-Riemann equations throughout D. [যেহেতু D এ f বৈশ্লেষিক, সুতরাং ইহার উপাংশ ফাংশন u এবং v, D এর সর্বত্র কচি-রীম্যান সমীকরণ সিদ্ধ করবে।]

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \text{...... (1)}$$

Differentiating partially w. r. to x both sides of (1) we get [(1) এর উভয় পক্ষে x এর সাপেক্ষে আংশিক অন্তরীকরণ করে পাই]

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\,\partial y} \text{ and [এবং] } \frac{\partial^2 u}{\partial x\,\partial y}=-\frac{\partial^2 v}{\partial x^2} \quad \text{...... (2)}$$

Again, differentiating partially w. r. to y both sides of (1) we get [আবার, (1) এর উভয় পক্ষে y এর সাপেক্ষে আংশিক অন্তরীকরণ করে পাই]

$$\frac{\partial^2 u}{\partial y\,\partial x}=\frac{\partial^2 v}{\partial y^2} \text{ and [এবং] } \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\,\partial x} \quad \text{...... (3)}$$

By calculas, the continuity of the partial derivatives ensures that [ক্যালকুলাস দ্বারা এ আংশিক অন্তরীকরণের অবিচ্ছিন্নতা নিশ্চিত করে যে]

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x} \text{ and [এবং] } \frac{\partial^2 v}{\partial x\,\partial y}=\frac{\partial^2 v}{\partial y\,\partial x}$$

$$\Rightarrow -\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{\partial y^2} \text{ and [এবং] } \frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}; \quad \text{[by (2) and (3)]}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \text{ and [এবং] } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ and [এবং] } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Thus, the both component functions u and v satisfy the Laplace equation. Hence u and v are harmonic in D. [অতএব, উভয় উপাংশ ফাংশন u এবং v ল্যাপলাস সমীকরণ সিদ্ধ করে। অতএব D এ u ও v হারমোনিক।]

That is, $u = \text{Re}\{f(z)\}$, $v = \text{Im}\{f(z)\}$ both are harmonic functions. [অর্থাৎ $u = \text{Re}\{f(z)\}$, $v = \text{Im}\{f(z)\}$ উভয়েই হারমনিক ফাংশন।]

2.8. Laplace equation in polar form :

Theorem–7. The Laplace equation in polar form is

$$\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}=0.$$

[DUH–1989, 2006, JUH–1991]

Solution : Let [ধরি] $z = x + iy$, $x = r\cos\theta$, $y = r\sin\theta$ and

$$w = f(z) = u + iv.$$

The Cauchy-Riemann equations in polar form are [পোলার আকারে কচি-রীম্যান সমীকরণ হল]

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{...... (1)}$$

$$\text{and [এবং] } \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{...... (2)}$$

If the second order partial derivatives of u and v w. r. to r and θ are continuous, then [r ও θ এর সাপেক্ষে u ও v এর যদি দ্বিতীয় ক্রমের আংশিক অন্তরজ থাকে, তখন]

$$\frac{\partial^2 u}{\partial r\,\partial\theta}=\frac{\partial^2 u}{\partial\theta\,\partial r} \quad \ldots\ldots (3)$$

and [এবং] $$\frac{\partial^2 v}{\partial r\,\partial\theta}=\frac{\partial^2 v}{\partial\theta\,\partial r} \quad \ldots\ldots (4)$$

From (3) we have [(3) হতে পাই]

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(-r\frac{\partial v}{\partial r}\right)=\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial v}{\partial\theta}\right), \quad \text{[by (1) and (2)]}$$

$$\Rightarrow -\frac{\partial v}{\partial r}-r\frac{\partial^2 v}{\partial r^2}=\frac{1}{r}\frac{\partial^2 v}{\partial\theta^2}$$

$$\Rightarrow -\frac{1}{r}\frac{\partial v}{\partial r}-\frac{\partial^2 v}{\partial r^2}=\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}, \text{ dividing by r both sides}$$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}=0 \quad \ldots\ldots (5)$$

In the same way from (4) we get [(4) হতে একইভাবে পাই]

$$\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left(\frac{\partial v}{\partial r}\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\frac{\partial u}{\partial\theta}\right), \quad \text{[by (1) and (2)]}$$

$$\Rightarrow \frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r^2}=-\frac{1}{r}\frac{\partial^2 u}{\partial\theta^2}$$

$$\Rightarrow \frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}=-\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}, \text{ diving both sides by r.}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0 \quad \ldots\ldots (6)$$

If we write u = v = ψ in (5) and (6) we get the same result which is [(5) ও (6) যদি আমরা u = v = ψ লিখি তাহলে আমরা একই ফল পাই যাহা]

$$\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial\theta^2}=0.$$

Theorem-8. A function f(z) = u + iv is analytic in a domain D if and only if v is harmonic conjugate of u.

Proof : Let f(z) = u + iv be analytic in D. Then the first order partial derivatives of u and v satisfy the Cauchy-Riemann equations. So u and v are harmonic functions. Hence v is harmonic conjugate of u.

Conversely, suppose that v is harmonic conjugate of u. Then u and v are harmonic functions and the first order partial derivatives of u and v satisfy the Cauchy-Riemann equations. Further, u and v are harmonic, means their second order partial derivatives and hence the first order partial derivatives are continuous. This implies that f(z) = u + iv is analytic.

From the above theorem we see that a necessary and sufficient condition for a function f(z) = u(x, y) + iv(x, y) to be analytic in a domain D is that v be a harmonic conjugate of u in D.

We note that if v is a harmonic conjugate of u in some domain D, then it is not in general true that u is a harmonic conjugate of v there. We show this by an example later.

2.9. Construction of an analytic function :

Method-1 : Let u be a harmonic function and v be its harmonic conjugate function. Then by C-R equations we have

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

The function v(x, y) is constructed in two steps.

(i) Integrate $\frac{\partial v}{\partial y}$ $\left(\text{which is equal to } \frac{\partial u}{\partial x}\right)$ with respect to y, treating x as constant :

$$v(x, y)=\int\frac{\partial u}{\partial x}\,dy+\phi(x) \quad \ldots\ldots (1)$$

where φ(x) is a function of x only, and hence the partial derivative of φ(x) with respect to y is zero.

(ii) Differentiating (1) w. r. to x and then replacing $\frac{\partial v}{\partial x}$ by $-\frac{\partial u}{\partial y}$ on the left side we get

$$-\frac{\partial u}{\partial y} = \frac{d}{dx}\int \frac{\partial u}{\partial x}\,dy + \phi'(x) \quad \text{...... (2)}$$

Since u is harmonic, all terms except those involving x in (2) cancel out, and a formula for $\phi'(x)$ will purely a function of x. Integrating $\phi'(x)$, we will get $\phi(x)$ and put it in (1) so that we get v(x, y). Thus we get our desired analytic function f(z) = u + iv.

Method–2. If v = v(x, y) then $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$$\Rightarrow dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{by C–R equations ...... (1)}$$

$$\Rightarrow dv = M\,dx + N\,dy$$

where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

$$\Rightarrow \frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow -\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \left[\because \text{ u is harmonic, so u satisfy Laplace equation } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right]$$

This shows that (1) is an exact equation. Therefore, by integrating (1), v can be found. Now both u and v are known and as such f(z) = u + iv can be determined.

Method–3. Milne Thomson method.

Let z = x + iy. Then $\bar{z} = x - iy$

$$\therefore z + \bar{z} = x + iy + x - iy = 2x \Rightarrow x = \frac{z+\bar{z}}{2} \quad \text{...... (1)}$$

$$\text{and } z - \bar{z} = x + iy - x + iy = 2iy \Rightarrow y = \frac{z-\bar{z}}{2i} \quad \text{...... (2)}$$

Now w = f(z) = u(x, y) + iv(x, y)

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \quad \text{...... (3)}$$

The above relation can be treated as a formal identity in two independent variables z and $\bar{z}$. Setting $\bar{z} = z$ in (1), (2) and (3) we get

$$x = \frac{z+z}{2} = z,\ y = \frac{z-z}{2i} = 0 \text{ and}$$

$$w = f(z) = u(z, 0) + iv(z, 0)$$

Now f(z) = w = u + iv gives

$$f'(z) = \frac{dw}{dz} = \frac{\partial u}{\partial x}\frac{dx}{dz} + i\frac{\partial v}{\partial x}\frac{dx}{dz} \qquad \left|\ x = z \Rightarrow \frac{dx}{dz} = 1\right.$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{...... (4)}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \text{ by C–R equation}$$

If $\frac{\partial u}{\partial x} = \phi_1(x, y) = \phi_1(z, 0)$ and $\frac{\partial u}{\partial y} = \phi_2(x, y) = \phi_2(z, 0)$

then the above relation becomes

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

Integrating this we get

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\,dz + c \quad \text{...... (5)}$$

This gives the construction of f(z) when u is given.

Similarly, when v is given and u is unknown then by replacing $\frac{\partial u}{\partial x}$ by $\frac{\partial v}{\partial y}$ in (4) by C–R equation we get

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

If we choose $\frac{\partial v}{\partial y} = \psi_1(x, y) = \psi_1(z, 0)$

and $\frac{\partial v}{\partial x} = \psi_2(x, y) = \psi_2(z, 0)$, then

$$f'(z) = \psi_1(z, 0) + i\psi_2(z, 0)$$

$$\Rightarrow f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]\,dz + c.$$

Note : Milne method is very nice and interesting method for getting f(z) rapidly. Some time other methods are very tedious and cumbersome.

2.10. Partial derivative in relation to z and $\bar{z}$:

We have $z = x + iy$ and $\bar{z} = x - iy$

$$\Rightarrow z + \bar{z} = 2x \text{ and } z - \bar{z} = 2iy$$

$$\Rightarrow x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i}$$

Every complex function $f(z) = u + iv$ can be written as

$$f(x + iy) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions of x and y. This means $f(x + iy)$ may be treated as a function of x and y, that is, $f = f(x, y)$.

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\frac{1}{2i} \qquad \because x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{...... (1)}$$

Again, $f = f(x, y)$ gives

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\frac{-1}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$

$$\Rightarrow \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{...... (2)}$$

(1) + (2) gives $\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} + \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$= \frac{1}{2}\cdot 2\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \quad \text{...... (3)}$$

(2) − (1) gives $\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$= \frac{1}{2}\cdot 2i\frac{\partial}{\partial y} = i\frac{\partial}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial y} = \frac{1}{i}\left(\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z}\right) = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \quad \text{...... (4)}$$

$$\therefore \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z} = 2\frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial z} = 2\frac{\partial}{\partial z}$$

Now $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} - i^2\frac{\partial^2}{\partial y^2}$

$$= \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$= 2\frac{\partial}{\partial \bar{z}}\left(2\frac{\partial}{\partial z}\right) = 4\frac{\partial^2}{\partial \bar{z}\,\partial z} \quad \text{...... (5)}$$

Again, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} - i^2\frac{\partial^2}{\partial y^2}$

$$= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$= 2\frac{\partial}{\partial z}\left(2\frac{\partial}{\partial \bar{z}}\right)$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}} \quad \text{...... (6)}$$

From (5) and (6) we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} = 4\frac{\partial^2}{\partial \bar{z}\,\partial z}.$$

SOLVED EXAMPLES

Example–1. If $f(z) = \frac{2z-1}{3z+2}$, prove that

$$\lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h} = \frac{7}{(3z_0+2)^2} \text{ provided } z_0 \neq \frac{-2}{3}.$$

[RUH–2000]

Solution : Given that [দেওয়া আছে] $f(z) = \frac{2z-1}{3z+2}$

$$\therefore \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{2(z_0+h)-1}{3(z_0+h)+2} - \frac{2z_0-1}{3z_0+2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(2z_0+2h-1)(3z_0+2)-(2z_0-1)(3z_0+3h+2)}{(3z_0+3h+2)(3z_0+2)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{6z_0^2+4z_0+6hz_0+4h-3z_0-2-6z_0^2-6hz_0-4z_0+3z_0+3h+2}{(3z_0+3h+2)(3z_0+2)}$$

$$= \lim_{h\to 0} \frac{7h}{h(3z_0+3h+2)(3z_0+2)}$$

$$= \lim_{h\to 0} \frac{7}{(3z_0+3h+2)(3z_0+2)}$$

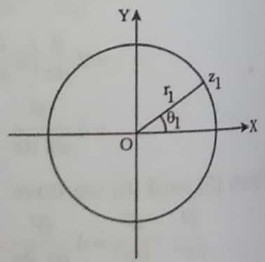
$$= \frac{7}{(3z_0+0+2)(3z_0+2)} = \frac{7}{(3z_0+2)^2}$$

Provided $3z_0+2 \neq 0 \Rightarrow z_0 \neq -\frac{2}{3}$.

Example–2. Prove that $f(z) = \ln z$ has a branch point at $z = 0$.

[NUH–2004, 200

Solution : Let [ধরি] $x = r\cos\theta$, $y = r\sin\theta$.



$$\therefore f(z) = \ln z = \ln(x+iy)$$
$$= \ln(r\cos\theta + ir\sin\theta)$$
$$= \ln\{r(\cos\theta + i\sin\theta)\}$$
$$= \ln(re^{i\theta})$$
$$= \ln r + \ln e^{i\theta}$$
$$= \ln r + i\theta$$

Suppose we start for [ধরি আমরা শুরু করছি]

$z = z_1 \neq 0$ and let [এবং ধরি] $r = r_1$, $\theta = \theta_1$

$\therefore \ln z_1 = \ln r_1 + i\theta_1$

After making one complete circuit about the origin in th positive or counterclockwise direction, on returning to z_1 we fin $r = r_1$, $\theta = \theta_1 + 2\pi$ so that

$$\ln z_1 = \ln r_1 + i(\theta_1 + 2\pi).$$

Thus we have another branch of f(z) and so $z = 0$ is a branc point.

[ধনাত্মক দিকে বা ঘড়ির কাটার বিপরীত দিকে মূল বিন্দুর চারিদিকে একটি পূর্ণ সার্কিট তৈরীর পর z_1 এর দিকে ফেরত আসতে পাই $r = r_1$, $\theta = \theta_1 + 2\pi$ যেন

$$\ln z_1 = \ln r_1 + i(\theta_1 + 2\pi).$$

অতএব আমরা f(z) এর অন্য একটি ব্রাঞ্চ পদ পাই এবং সেকারণে $z = 0$ একটি ব্রাঞ্চ বিন্দু।]

Example–3. For the function $f(z) = \frac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$, locate and name all the singularities in the finite z-plane and also determine where f(z) is analytic.

Solution : Given that $f(z) = \frac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$ ...... (1)

In the finite z-plane the singularities will be obtained by solving the equation

$$(z-1)^3(3z+2)^2 = 0$$

$$\Rightarrow (z-1)^3 = 0 \quad \text{or } (3z+2)^2 = 0$$

$$\Rightarrow z = 1, 1, 1 \quad \text{or } z = \frac{-2}{3}, \frac{-2}{3}$$

$\therefore$ The singularities in the finite z-plane are $z = 1$ of order 3 and $z = \frac{-2}{3}$ of order 2.

In the finite z-plane f(z) is analytic everywhere excepts the points $z = 1$ and $z = \frac{-2}{3}$.

Example–4. Determine the singular points of the function

$$f(z) = \frac{z^3+7}{(z^2-2z+2)(z-3)}.$$

Solution : The singular points are obtained by solving the equation

$$(z^2-2z+2)(z-3) = 0$$

$$\Rightarrow z-3 = 0 \text{ or } z^2-2z+2 = 0$$

$$\Rightarrow z = 3 \text{ or } z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1+i, 1-i$$

Thus the singular points are $z = 3$, $z = 1+i$, $z = 1-i$.

Example–10. Show that $f(z) = |z|$ is nowhere differentiab[le] but continuous every where. [DUH-200[?]]

Solution : For differentiability we have [অন্তরীকরণ যোগ্যতার জ[ন্য] পাই]

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z| - |z|}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z| - |z|}{\Delta z} \cdot \frac{|z+\Delta z| + |z|}{|z+\Delta z| + |z|}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\,(\overline{z+\Delta z}) - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\,(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{z\bar{z} + z\overline{\Delta z} + \bar{z}\,\Delta z + \Delta z\,\overline{\Delta z} - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{z\,\overline{\Delta z} + \bar{z}\,\Delta z + \Delta z\,\overline{\Delta z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \left(\frac{z}{|z+\Delta z| + |z|} \frac{\overline{\Delta z}}{\Delta z} + \frac{\bar{z}}{|z+\Delta z| + |z|} + \frac{\overline{\Delta z}}{|z+\Delta z| + |z|} \right)$$

Along the real axis [বাস্তব অক্ষের বরাবর]

$$\Delta x \to 0,\ \Delta y = 0 \Rightarrow \Delta z = \Delta x + i\Delta y = \Delta x \to 0$$

and [এবং] $\overline{\Delta z} = \overline{\Delta x} = \Delta x \to 0 \Rightarrow \Delta z = \overline{\Delta z}$

$$\therefore f'(z) = \frac{z}{|z+0| + |z|} \cdot 1 + \frac{\bar{z}}{|z| + |z|} + 0 = \frac{z}{2|z|} + \frac{\bar{z}}{2|z|}$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর]

$$\Delta x = 0,\ \Delta y \to 0$$

$$\Rightarrow \Delta z = \Delta x + i\Delta y = i\Delta y \to 0$$

and [এবং] $\overline{\Delta z} = \overline{i\Delta y} = -i\Delta y$. Also [অধিকন্তু] $\Delta z = -\overline{\Delta z}$

$$\therefore f'(z) = \frac{z}{|z| + |z|} \cdot \frac{-\Delta z}{\Delta z} + \frac{\bar{z}}{|z| + |z|} + 0 = \frac{-z}{2|z|} + \frac{\bar{z}}{2|z|}$$

Thus [অতএব]

$$f'(z) = \begin{cases} \dfrac{z}{2|z|} + \dfrac{\bar{z}}{2|z|}, & \text{along real axis [বাস্তব অক্ষের দিকে]} \\[2ex] \dfrac{-z}{2|z|} + \dfrac{\bar{z}}{2|z|}, & \text{along imaginary axis [কাল্পনিক অক্ষের দিকে]} \end{cases}$$

Hence the given function is nowhere differentiable in the complex plane. [অতএব প্রদত্ত ফাংশনটি জটিল তলে কোথাও অন্তরীকরণযোগ্য নয়]

For continuity, let z_0 be any arbitrary point in the complex plane. Then $f(z)$ will be continuous if [অবিচ্ছিন্নতার জন্য ধরি জটিল তলে ইচ্ছাধীন যে কোন বিন্দু z_0. তখন $f(z)$ অবিচ্ছিন্ন হবে যদি]

$$|f(z) - f(z_0)| < \varepsilon, \text{ whenever [যেখানে] } |z - z_0| < \delta$$

Let ε be given such that [ধরি ε প্রদত্ত যেন] $|f(z) - f(z_0)| < \varepsilon$

$$\Rightarrow \big|\,|z| - |z_0|\,\big| < \varepsilon$$

$$\Rightarrow \big|\,|z| - |z_0|\,\big| < |z - z_0| < \varepsilon$$

If we take $\delta = \varepsilon$ then [যদি আমরা $\delta = \varepsilon$ লই তখন]

$$\big|\,|z| - |z_0|\,\big| < \varepsilon \text{ whenrever } |z - z_0| < \varepsilon = \delta$$

$$\Rightarrow |f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

Thus the given function is continuous everywhere. [অতএব প্রদত্ত ফাংশনটি সর্বত্র অবিচ্ছিন্ন ।]

Example–11. For the function, $f(z)$ defined by

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

show that the C–R equations are satisfied at (0, 0) but the function is not differentiable at $0 + i0$. **[DUH–2003, 2005]**

Solution : Let [ধরি] $f(z) = u(x, y) + iv(x, y)$

At (0, 0) we have [(0, 0) এ পাই] $f(0) = 0 + i0$

$$\Rightarrow u(0, 0) + iv(0, 0) = 0 + i0$$

$$\Rightarrow u(0, 0) = 0,\ v(0, 0) = 0$$

Also [অধিকন্তু] $f(z) = \dfrac{(\bar{z})^2}{z} = \dfrac{(\bar{z})^3}{z\bar{z}} = \dfrac{(x - iy)^3}{(x + iy)\,(x - iy)}$

$$\Rightarrow f(z) = u(x, y) + iv(x, y) = \frac{x^3 - 3ix^2y + 3i^2xy^2 - i^3y^3}{x^2 - i^2y^2}$$

$$= \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{x^2 + y^2}$$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} + i\,\frac{y^3 - 3x^2y}{x^2 + y^2}.$$

Example–10. Show that $f(z) = |z|$ is nowhere differentiable but continuous every where. [DUH-2...]

Solution : For differentiability we have [অন্তরীকরণ যোগ্যতার পাই]

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z| - |z|}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z| - |z|}{\Delta z} \cdot \frac{|z+\Delta z| + |z|}{|z+\Delta z| + |z|}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\,(\overline{z+\Delta z}) - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\,(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{z\bar{z} + z\overline{\Delta z} + \bar{z}\,\Delta z + \Delta z\,\overline{\Delta z} - z\bar{z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \frac{z\,\overline{\Delta z} + \bar{z}\,\Delta z + \Delta z\,\overline{\Delta z}}{\Delta z(|z+\Delta z| + |z|)}$$

$$= \lim_{\Delta z \to 0} \left(\frac{z}{|z+\Delta z| + |z|} \frac{\overline{\Delta z}}{\Delta z} + \frac{\bar{z}}{|z+\Delta z| + |z|} + \frac{\overline{\Delta z}}{|z+\Delta z| + |z|} \right.$$

Along the real axis [বাস্তব অক্ষের বরাবর]

$$\Delta x \to 0,\ \Delta y = 0 \Rightarrow \Delta z = \Delta x + i\Delta y = \Delta x \to 0$$

and [এবং] $\overline{\Delta z} = \overline{\Delta x} = \Delta x \to 0 \Rightarrow \Delta z = \overline{\Delta z}$

$$\therefore f'(z) = \frac{z}{|z+0| + |z|} \cdot 1 + \frac{\bar{z}}{|z| + |z|} + 0 = \frac{z}{2|z|} + \frac{\bar{z}}{2|z|}$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর]

$$\Delta x = 0,\ \Delta y \to 0$$

$$\Rightarrow \Delta z = \Delta x + i\Delta y = i\Delta y \to 0$$

and [এবং] $\overline{\Delta z} = \overline{i\Delta y} = -i\Delta y$. Also [অধিকন্তু] $\Delta z = -\overline{\Delta z}$

$$\therefore f'(z) = \frac{z}{|z| + |z|} \cdot \frac{-\Delta z}{\Delta z} + \frac{\bar{z}}{|z| + |z|} + 0 = \frac{-z}{2|z|} + \frac{\bar{z}}{2|z|}$$

Thus [অতএব]

$$f'(z) = \begin{cases} \dfrac{z}{2|z|} + \dfrac{\bar{z}}{2|z|}, & \text{along real axis [বাস্তব অক্ষের দিকে]} \\[2mm] \dfrac{-z}{2|z|} + \dfrac{\bar{z}}{2|z|}, & \text{along imaginary axis [কাল্পনিক অক্ষের দিকে]} \end{cases}$$

Hence the given function is nowhere differentiable in the complex plane. [অতএব প্রদত্ত ফাংশনটি জটিল তলে কোথাও অন্তরীকরণযোগ্য নয়]

For continuity, let z_0 be any arbitrary point in the complex plane. Then $f(z)$ will be continuous if [অবিচ্ছিন্নতার জন্য ধরি জটিল তলে ইচ্ছাধীন যে কোন বিন্দু z_0. তখন $f(z)$ অবিচ্ছিন্ন হবে যদি]

$$|f(z) - f(z_0)| < \varepsilon, \text{ whenever [যেখানে] } |z - z_0| < \delta$$

Let ε be given such that [ধরি ε প্রদত্ত যেন] $|f(z) - f(z_0)| < \varepsilon$

$$\Rightarrow \big||z| - |z_0|\big| < \varepsilon$$

$$\Rightarrow \big||z| - |z_0|\big| < |z - z_0| < \varepsilon$$

If we take $\delta = \varepsilon$ then [যদি আমরা $\delta = \varepsilon$ লই তখন]

$$\big||z| - |z_0|\big| < \varepsilon \text{ whenrever } |z - z_0| < \varepsilon = \delta$$

$$\Rightarrow |f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

Thus the given function is continuous everywhere. [অতএব প্রদত্ত ফাংশনটি সর্বত্র অবিচ্ছিন্ন ।]

Example–11. For the function, $f(z)$ defined by

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

show that the C–R equations are satisfied at (0, 0) but the function is not differentiable at $0 + i0$. **[DUH-2003, 2005]**

Solution : Let [ধরি] $f(z) = u(x, y) + iv(x, y)$

At (0, 0) we have [(0, 0) এ পাই] $f(0) = 0 + i0$

$$\Rightarrow u(0, 0) + iv(0, 0) = 0 + i0$$

$$\Rightarrow u(0, 0) = 0,\ v(0, 0) = 0$$

Also [অধিকন্তু] $f(z) = \dfrac{(\bar{z})^2}{z} = \dfrac{(\bar{z})^3}{z\bar{z}} = \dfrac{(x - iy)^3}{(x + iy)\,(x - iy)}$

$$\Rightarrow f(z) = u(x, y) + iv(x, y) = \frac{x^3 - 3ix^2y + 3i^2xy^2 - i^3y^3}{x^2 - i^2y^2}$$

$$= \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{x^2 + y^2}$$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} + i\,\frac{y^3 - 3x^2y}{x^2 + y^2}.$$

$\therefore$ At the origin we have [$\therefore$ মূলবিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^3/(\Delta x)^2 -}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(\Delta y)^3/(\Delta y)^2 - 0}{\Delta y}$$

Thus at (0, 0) [অতএব (0, 0) এ] $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and [এবং] $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence C-R equations are satisfied at (0, 0). [অতএব (0, 0) এ সমীকরণ সিদ্ধ হয়]

2nd Part : For differentiability we have [অন্তরীকরণ যোগ্যতার পাই]

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{\frac{(\bar{z})^2}{z} - 0}{z} = \lim_{z \to 0} \frac{(\bar{z})^2}{z^2}$$

$$= \lim_{(x, y) \to (0, 0)} \left(\frac{x - iy}{x + iy}\right)^2$$

Along the real axis [বাস্তব অক্ষ বরাবর] $f'(0) = \lim_{x \to 0} \left(\frac{x}{x}\right)^2 = 1$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর]

$$f'(x) = \lim_{y \to 0} \left(\frac{-iy}{iy}\right)^2 = 1$$

Along the line $y = x$ [$y = x$ রেখা বরাবর]

$$f'(0) = \lim_{x \to 0} \left(\frac{x - ix}{x + ix}\right)^2 = \left(\frac{1 - i}{1 + i}\right)^2$$

$$= \frac{1 - 2i + i^2}{1 + 2i + i^2} = \frac{1 - 2i - 1}{1 + 2i - 1} = \frac{-2i}{2i} = -1$$

Since f′(0) have different values for different paths, f′(0) does not exist, that is, f is not differentiable at 0 + i0. [যেহেতু ভিন্ন ভিন্ন পথের জন্য f′(0) এর ভিন্ন ভিন্ন মান, সুতরাং f′(0) বিদ্যমান নাই, অর্থাৎ 0 + i0 তে অন্তরীকরণযোগ্য না।]

Example-12. Let f denote the function whose values are

$$f(0) = 0 \text{ and } f(z) = \frac{(\bar{z})^2}{z} \text{ when } z \neq 0$$

Show that the Cauchy-Riemann equations are satisfied at the point $z = 0$ but that the derivative of f fails to exist there.

Solution : See soloved problem-11.

Example-13. Let $f(z) = u + iv = \frac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2 + y^2}$, when $z \neq 0$ and $f(z) = 0$, when $z = 0$. Show that f(z) is continuous and the Cauchy-Riemann equations are satisfied but f(z) is not differentiable at $z = 0$. **[NUH-2000, 2003, 2005, 2013]**

Solution : We have $f(z) = u + iv = \frac{x^3 - 3xy^2 + i(y^3 - 3x^2y)}{x^2 + y^2}$

$\Rightarrow u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2}$ and [এবং] $v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2}$

$\therefore$ At the origin we have [$\therefore$ মূলবিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^3/(\Delta x)^2 - 0}{\Delta x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(\Delta y)^3/(\Delta y)^2 - 0}{\Delta y} = 1$$

Thus at (0, 0) [অতএব (0, 0) এ] $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and [এবং] $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence C-R equations are satisfied at (0, 0). [অতএব (0, 0) এ C-R সমীকরণ সিদ্ধ হয়]

2nd Part : For differentiability we have [অন্তরীকরণ যোগ্যতার জন্য পাই]

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{\frac{(\bar{z})^2}{z} - 0}{z} = \lim_{z \to 0} \frac{(\bar{z})^2}{z^2}$$

$$= \lim_{(x, y) \to (0, 0)} \left(\frac{x - iy}{x + iy}\right)^2$$

Along the real axis [বাস্তব অক্ষ বরাবর] $f'(0) = \lim_{x\to 0}\left(\frac{x}{x}\right)^2 = 1$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর]

$$f'(x) = \lim_{y\to 0}\left(\frac{-iy}{iy}\right)^2 = 1$$

Along the line $y = x$ [$y = x$ রেখা বরাবর]

$$f'(0) = \lim_{x\to 0}\left(\frac{x - ix}{x + ix}\right)^2 = \left(\frac{1-i}{1+i}\right)^2$$

$$= \frac{1 - 2i + i^2}{1 + 2i + i^2} = \frac{1 - 2i - 1}{1 + 2i - 1} = \frac{-2i}{2i} = -1$$

Since $f'(0)$ have different values for different paths, $f'(0)$ does not exist, that is, f is not differentiable at $0 + i0$. [যেহেতু ভিন্ন ভিন্ন পথের জন্য $f'(0)$ এর ভিন্ন ভিন্ন মান, সুতরাং $f'(0)$ বিদ্যমান নাই, অর্থাৎ $0 + i0$ তে অন্তরীকরণযোগ্য না।]

Example–14. Show that the function

$$f(z) = u + iv = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{if } z \neq 0$$

and $f(0) = 0$ if $z = 0$,

is continuous and that the Cauchy-Riemann equations are satisfied at the Origin, yet $f'(0)$ does not exist.

[NUH–1993, 2011, DUH–1993, 2001, 20..]

Solution : First part (Continuity at origin)

Given [দেওয়া আছে] $f(z) = u + iv = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2}$, if $z \neq 0$

$$= \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2}$$

Equating real and imaginary parts [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$u = u(x, y) = \frac{x^3 - y^3}{x^2 + y^2} \text{ and [এবং] } v = v(x, y) = \frac{x^3 + y^3}{x^2 + y^3}$$

Along the real axis [বাস্তব অক্ষ বরাবর] $y = 0$ and [এবং] $x \to 0$

$$\therefore \lim_{z\to 0} u = \lim_{\substack{x\to 0\\ y\to 0}} u(x, y) = \lim_{x\to 0} \frac{x^3 - 0}{x^2} = \lim_{x\to 0} x = 0$$

$$\lim_{z\to 0} v = \lim_{\substack{x\to 0\\ y\to 0}} v = \lim_{x\to 0} \frac{x^3 + 0}{x^2 + 0} = \lim_{x\to 0} x = 0$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর] $x = 0$, $y \to 0$

$$\therefore \lim_{z\to 0} u = \lim_{\substack{x\to 0\\ y\to 0}} u = \lim_{y\to 0} \frac{0 - y^3}{0 + y^2} = \lim_{y\to 0} (-y) = 0$$

$$\lim_{z\to 0} v = \lim_{\substack{x\to 0\\ y\to 0}} v = \lim_{y\to 0} \frac{0 + y^3}{0 + y^2} = \lim_{y\to 0} y = 0$$

Along the line $y = x$ we have [$y = x$ রেখা বরাবর পাই]

$$\lim_{z\to 0} u = \lim_{\substack{x\to 0\\ y=x}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{x\to 0} \frac{x^3 - x^3}{x^2 + x^2} = 0$$

$$\lim_{z\to 0} v = \lim_{\substack{x\to 0\\ y=x}} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{x\to 0} \frac{x^3 + x^3}{x^2 + x^2} = \lim_{x\to 0} \frac{2x^3}{2x^2} = 0$$

Thus in all cases [অতএব, সকল ক্ষেত্রে] $\lim_{z\to 0} u = 0$, $\lim_{z\to 0} v = 0$

$$\therefore \lim_{z\to 0} f(z) = \lim_{z\to 0} (u + iv) = \lim_{z\to 0} u + i \lim_{z\to 0} v = 0 + i0 = 0$$

$$\Rightarrow \lim_{z\to 0} f(z) = f(0)$$

Hence $f(z)$ is continuous at the origin. [অতএব মূল বিন্দুতে $f(z)$ অবিচ্ছিন্ন।]

2nd part (C–R equations) :

Given [দেওয়া আছে] $f(0) = 0 \Rightarrow u + iv = 0$

$$\Rightarrow u(0, 0) + iv(0, 0) = 0 + i0$$

$$\Rightarrow u(0, 0) = 0 \text{ and [এবং] } v(0, 0) = 0$$

Also, given [আরো দেওয়া আছে]

$$u(x, y) = \frac{x^3 - y^3}{x^2 + y^2} \text{ and [এবং] } v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

$$\therefore u(0 + \Delta x, 0) = \frac{(\Delta x)^3 - 0}{(\Delta x)^2 + 0} = \Delta x$$

$$u(0, 0 + \Delta y) = \frac{0 - (\Delta y)^3}{0 + (\Delta y)^2} = -\Delta y$$

$$v(0 + \Delta x, 0) = \frac{(\Delta x)^3 + 0}{(\Delta x)^2 + 0} = \Delta x$$

$$v(0, 0 + \Delta y) = \frac{0 + (\Delta y)^3}{0 + (\Delta y)^2} = \Delta y$$

At the origin we have [মূলবিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-\Delta y - 0}{\Delta y} = \ldots$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta y \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y - 0}{\Delta y} = 1$$

Thus [অতএব] $\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}$ and [এবং] $\frac{\partial u}{\partial y} = -1 = -\frac{\partial v}{\partial x}$

Hence Cauchy-Riemann equations are satisfied at the o... [অতএব মূল বিন্দুতে কচি-রীম্যান সমীকরণ সিদ্ধ হয়।]

3rd Part (Differentiability অন্তরীকরণযোগ্যতা) :

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z) - 0}{z}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(1 + i)x^3 - (1 - i)y^3}{x^2 + y^2} \cdot \frac{1}{x + iy}$$

Now along the real axis [এখন বাস্তব অক্ষ বরাবর] $y = 0$ and [এবং] x...

$$\therefore f'(0) = \lim_{x \to 0} \frac{(1 + i)x^3 - 0}{(x^2 + 0)(x + 0)} = \lim_{x \to 0} \frac{(1 + i)x^3}{x^3} = 1 + i \ldots\ldots (1)$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর] $x = 0$ and [এবং] y...

$$\therefore f'(0) = \lim_{y \to 0} \frac{0 - (1 - i)y^3}{(0 + y^2)(0 + iy)} = \lim_{y \to 0} \frac{(i - 1)y^3}{iy^3} = 1 + i \ldots\ldots (2)$$

Along the line $y = x$ we have [$y = x$ রেখা বরাবর পাই]

$$f'(0) = \lim_{x \to 0} \frac{(1 + i)x^3 - (1 - i)x^3}{(x^2 + x^2)(x + ix)}$$

$$= \frac{1 + i - 1 + i}{2(1 + i)} = \frac{2i}{2(1 + i)} = \frac{i}{1 + i}$$

$$= \frac{i(1 - i)}{1 - i^2} = \frac{i + 1}{1 + 1} = \frac{1}{2}(1 + i) \ldots\ldots (3)$$

From (1), (2) and (3) we see that the limit is not unique. H... $f'(0)$ does not exist. [(1), (2) ও (3) হতে দেখি লিমিট অনন্য নয়। অতএ... বিদ্যমান নাই।]

Example-15. Prove that $f(z) = |z|^2$ is continuous every where but not differentiable except at the origin.

[NUH-2004 (Old), 2006 (Old), 2007, DUH-2003, RUH-1994]

Solution : Given that [দেওয়া আছে] $f(z) = |z|^2 = z\bar{z} = x^2 + y^2$

Which shows that $f(z)$ is a function of non zero denominators for all x and y in the z-plane. Thus $f(z) = |z|^2$ is continuous every where. [যাহা দেখায় যে $f(z)$ ফাংশনটি z তলে x এবং y এর যে কোন মানের জন্য অশূন্য হর বিশিষ্ট। অতএব $f(z) = |z|^2$ সর্বত্র অবিচ্ছিন্ন।]

For differentiability we have [অন্তরীকরণযোগ্যতার জন্য পাই]

At $z = 0$ [$z = 0$ এ] $f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - |0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z\, \overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$$

At $z \neq 0$, say $z = z_0$ [$z \neq 0$, ধরি $z = z_0$ এ]

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)\overline{(z_0 + \Delta z)} - z_0 \bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \overline{\Delta z}) - z_0 \bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 \bar{z}_0 + z_0 \overline{\Delta z} + \bar{z}_0 \Delta z + \Delta z\, \overline{\Delta z} - z_0 \bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0 \overline{\Delta z} + \bar{z}_0 \Delta z + \Delta z\, \overline{\Delta z}}{\Delta z}$$

$$= z_0 \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} + \bar{z}_0 + \lim_{\Delta z \to 0} \overline{\Delta z}$$

$$= z_0 \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} + \bar{z}_0 + 0$$

$$= \bar{z}_0 + z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$=\begin{cases} \overline{z_0}+z_0 & \text{along real axis [বাস্তব অক্ষ বরাবর]} \\ \overline{z_0}-z_0 & \text{along imaginary axis [কাল্পনিক অক্ষ বরাবর]} \end{cases}$$

Since the two limits are not equal, so $f'(z_0)$ does not exist.

$f(z) = |z|^2$ is continuous every where but not differentiable at the origin. [যেহেতু দুইটি লিমিট সমান না, সুতরাং $f'(z_0)$ বিদ্যমান না। $f(z) = |z|^2$ সর্বত্র অবিচ্ছিন্ন কিন্তু মূলবিন্দু ব্যতীত অন্তরীকরণ যোগ্য না।]

Example–16. If $f(z)$ is analytic in a region R, then f is constant if Re $f(z)$ is constant. [RUH-]

Solution : Let [ধরি] $f(z) = u + iv$. Then [তখন] Re $f(z) = u$

According to the question, Re $f(z)$ = constant $\Rightarrow u$ = constant [প্রশ্নমতে, Re $f(z)$ = ধ্রুবক $\Rightarrow u$ = ধ্রুবক]

$$\therefore \frac{\partial u}{\partial x}=0, \frac{\partial u}{\partial y}=0 \ \ldots\ldots (1)$$

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ পাই]

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=0 \text{ and [এবং] } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=0 \ \ldots\ldots (2)$$

Now [এবং] $f(z) = u + iv$

$$\Rightarrow f'(z)=\frac{\partial u}{\partial x}\frac{dx}{dz}+i\frac{\partial v}{\partial x}\frac{dx}{dz}$$
$$=\frac{\partial u}{\partial x}\cdot 1+i\frac{\partial v}{\partial x}\cdot 1$$
$$=0+0=0 \text{ by (1) and (2)}$$

$$z=x+iy \Rightarrow \frac{dz}{dx}=1 \Rightarrow \frac{dx}{dz}=1$$

$\Rightarrow f(z) = c$ = constant, by integrating.

Example–17. If $f(z)$ is analytic in a region R, then f is constant if Im $f(z)$ is constant.

Solution : Let $f(z) = u + iv$. Then Im $f(z) = v$

According to the question Im $f(z) = v$ = constant.

$$\therefore \frac{\partial v}{\partial x}=0 \text{ and } \frac{\partial v}{\partial y}=0 \ \ldots\ldots (1)$$

By Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0 \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0 \ \ldots\ldots (2)$$

$$f(z)=u+iv$$
$$\Rightarrow f'(z)=\frac{\partial u}{\partial x}\frac{dx}{dz}+i\frac{\partial v}{\partial x}\frac{dx}{dz}$$
$$=0\cdot 1+i0\cdot 1=0$$

$$z=x+iy \Rightarrow \frac{dz}{dx}=1 \Rightarrow \frac{dx}{dz}=1$$

$\Rightarrow f(z) = c$, where c is a constant.

Example–18. Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic there.

[NUH-2002, DUH-1977, 1985, 2004, RUH-1984, 1999]

Solution : Given that [দেওয়া আছে] $f(z) = |z|^2$

$$\therefore f'(0)=\lim_{\Delta z\to 0}\frac{f(0+\Delta z)-f(0)}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{|0+\Delta z|^2-|0|^2}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{\Delta z\,\overline{\Delta z}}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\overline{\Delta z}=\overline{0}=0$$

$\Rightarrow f(z) = |z|^2$ is differentiable at $z = 0$. [$f(z) = |z|^2$, $z = 0$ এ অন্তরীকরণযোগ্য]

Again [আবার] $f(z) = |z|^2$

$$\Rightarrow f(z)=u+iv=|x+iy|^2$$
$$\Rightarrow u+iv=x^2+y^2$$
$$\Rightarrow u=x^2+y^2 \text{ and [এবং] } v=0$$

$$\therefore \frac{\partial u}{\partial x}=2x, \frac{\partial u}{\partial y}=2y, \frac{\partial v}{\partial x}=0 \text{ and [এবং] } \frac{\partial v}{\partial y}=0$$

At $z = 0$ [$z = 0$ এ] $\frac{\partial u}{\partial x}=0, \frac{\partial u}{\partial y}=0, \frac{\partial v}{\partial x}=0$ and [এবং] $\frac{\partial v}{\partial y}=0$

Thus the Cauchy-Riemann equations are satisfied at $z = 0$ but not in the neighbourhood $|z-0|<\delta$.

Thus $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic there.

[$z = 0$ এ কচি-রীম্যান সমীকরণ সিদ্ধ হয় কিন্তু $|z-0|<\delta$ নেইবারহুডে না। অতএব $f(z) = |z|^2$, $z = 0$ এ অন্তরীকরণযোগ্য কিন্তু সেখানে বৈশ্লেষিক না।]

Example–19. Show that $f(z) = 2x + ixy^2$ is no where analytic.

[RUH-1996]

Solution : Let $f(z) = u(x, y) + iv(x, y)$

$$\Rightarrow 2x+ixy^2=u(x,y)+iv(x,y)$$
$$\Rightarrow u(x,y)=2x \text{ and } v(x,y)=xy^2$$

$$\therefore \frac{\partial u}{\partial x}=2, \frac{\partial u}{\partial y}=0, \frac{\partial v}{\partial x}=y^2 \text{ and } \frac{\partial v}{\partial y}=2xy$$

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Thus we see that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$.

That is, Cauchy-Riemann equations are not sat... anywhere. Hence f(z) is not analytic at any point, that is, f(z)... where analytic.

Example–20. If p and q are functions of x and y satis... Laplace's equation, then show that (u + iv) is analytic w... $u = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}$ and $v = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$. **[RUH-19..]**

Solution : Given that p and q are functions of x a... satisfying Laplace's equation.

$$\therefore \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \;\;\cdots\cdots (1)$$

$$\text{and } \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} = 0 \;\;\cdots\cdots (2)$$

Again, given that $u = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}$ and $v = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 q}{\partial x^2} \;\;\cdots\cdots (3)$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 q}{\partial y \partial x} \;\;\cdots\cdots (4)$$

$$\frac{\partial v}{\partial x} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 q}{\partial x \partial y} \;\;\cdots\cdots (5)$$

$$\frac{\partial v}{\partial y} = \frac{\partial^2 p}{\partial y \partial x} + \frac{\partial^2 q}{\partial y^2} \;\;\cdots\cdots (6)$$

(3) – (6) gives, $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right) = 0$ [by (2)]

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

(4) + (5) gives, $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$ [by (1)]

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, u and v satisfied Cauchy-Riemann equations. He... (u + iv) is an analytic function.

Example–21. Prove that, if a function f(z) is differentiable... a point, then f(z) is continuous at that point, but the converse... not true.

Solution : First Part : See art-2.4, Theorem-3.

2nd Part (Converse part) :

The converse of the given statement is not true. We shall prove this by the following counter example.

Let $f(z) = \bar{z} = x - iy$. $\quad \therefore f(0) = \bar{0} = 0$

At $z = 0$ we have

$|f(z) - f(0)| = |\bar{z} - 0| = |\bar{z}| < \varepsilon$ when $|z - 0| = |z| < \varepsilon$

$\therefore$ f(z) is continuous at $z = 0$

Again, let $h = \Delta z = p + iq$. Then $\bar{h} = p - iq$

$$\therefore \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\bar{h} - 0}{h} = \lim_{\substack{p \to 0 \\ q \to 0}} \frac{p - iq}{p + iq}$$

Now along the real axis $q = 0$ and $p \to 0$

$$\therefore \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{p \to 0} \frac{p - 0}{p + 0} = 1$$

Along the imaginary axis, $p = 0$ and $q \to 0$

$$\therefore \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{q \to 0} \frac{0 - iq}{0 + iq} = -1$$

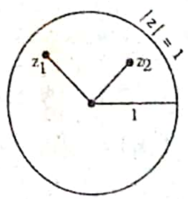
Since the limit is not unique, so $f(z) = \bar{z}$ is not differentiable at $z = 0$ even though it is continuous.

Example–22. Prove that a function which is analytic at a point, is continuous there but the converse is not necessarily true. **[RUH–1996]**

Solution : See example no 21.

Example–23. Show that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$, but the function $g(z) = \frac{1}{z}$ is not uniformly continuous in this region.

Solution : Let z_1 and z_2 are two points in $|z| < 1$.



Then $|f(z_1) - f(z_2)| = |z_1^2 - z_2^2|$

$$= |z_1 + z_2|\,|z_1 - z_2|$$

$$\Rightarrow |f(z_1) - f(z_2)| \le (|z_1| + |z_2|)\,|z_1 - z_2|$$

$$\Rightarrow |f(z_1) - f(z_2)| \le (1 + 1)\,|z_1 - z_2|$$

$$\Rightarrow |f(z_1) - f(z_2)| \le 2|z_1 - z_2|$$

If we choose $|z_1 - z_2| < \delta$ and $\delta = \frac{\varepsilon}{2}$, then $|f(z_1) - f(z_2)|$ ... whenever $|z_1 - z_2| < \delta$ and δ depends only on ε. This ensures $f(z) = z^2$ is uniformly continuous in $|z| < 1$.

For second case we fix $z_1 = \delta$ where $0 < \delta < 1$ and $z_2 = \frac{\delta}{1+\varepsilon}$

$$\therefore |z_1 - z_2| = \left|\delta - \frac{\delta}{1+\varepsilon}\right| = \frac{\varepsilon}{1+\varepsilon}\delta < \delta$$

and
$$|g(z_1) - g(z_2)| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$$
$$= \left|\frac{1}{\delta} - \frac{1}{\delta/1+\varepsilon}\right| = \left|\frac{1}{\delta} - \frac{1+\varepsilon}{\delta}\right|$$
$$= \left|\frac{1-1-\varepsilon}{\delta}\right| = \frac{\varepsilon}{\delta} > \varepsilon \text{ since } 0<\delta<1$$

$\therefore |g(z_1) - g(z_2)| > \varepsilon$ whenever $|z_1 - z_2| < \delta$

Hence $g(z) = \frac{1}{z}$ is not uniformly continuous in the region $|z| < 1$.

Example-24. Show that if a complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, then $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. [DUH-1...]

Solution : Given that $f(z)$ is differentiable at $z_0 = x_0 + iy_0$. Then by definition we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\Rightarrow f'(z_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{u(x, y) + iv(x, y) - \{u(x_0, y_0) + iv(x_0, y_0)\}}{x + iy - (x_0 + iy_0)} \quad \text{......}$$

Now taking the limit through the point (x_0, y_0) and parallel to the real axis (x-axis), we have $y = y_0$ and $x \to x_0$. So from (1) we get

$$f'(z_0) = \lim_{x \to x_0} \frac{u(x, y_0) + iv(x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x + iy_0 - x_0 - iy_0}$$

$$= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}$$

$$= \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial x} v(x_0, y_0)$$

$$\therefore f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Example-25. If $f'(z) = 0$ in a region R, then the function $f(z)$ must be constant in R. [DUH-1984]

Solution : Let $f(z) = u + iv$. Then

$$f'(z) = \frac{\partial}{\partial z}(u + iv)$$

$$\Rightarrow 0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial z} + i \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial z}$$

$$= \frac{\partial u}{\partial x} \cdot 1 + i \frac{\partial v}{\partial x} \cdot 1$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$z = x + iy \Rightarrow \frac{\partial z}{\partial x} = 1$$

Equating real and imaginary parts we get

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial x} = 0 \text{ ...... (1)}$$

By Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0, \quad \text{[by (1)]}$$

This shows that u and v both are independent of x and y.

Hence u = constant = c_1, say

and v = constant = c_2, say

$\therefore f(z) = u + iv = c_1 + ic_2$ = constant

Thus, if $f'(z) = 0$ then $f(z)$ must be constant.

Example-26. Show that an analytic function with constant modulas is constant. [NUH-2011, NU(Pre)-2008, DUH-1984, RUH-2006]

Solution : Let $f(z) = u + iv$ be an analytic function with constant modulas. Than $|f(z)|$ = constant = c, say [মনে করি $f(z) = u + iv$ ধ্রুব মানাংক বিশিষ্ট একটি বৈশ্লেষিক ফাংশন। তখন $|f(z)|$ = ধ্রুবক = c, ধরি]

$$\Rightarrow |u + iv| = c$$
$$\Rightarrow \sqrt{u^2 + v^2} = c$$
$$\Rightarrow u^2 + v^2 = c^2$$

Differentiating this partially w. r. to x and y separately we have [ইহাকে x ও y এর সাপেক্ষে আংশিক অন্তরীকরণ করে পাই]

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$$

and [এবং] $2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$

$$\Rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0 \;\ldots\ldots (1)$$

and [এবং] $u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0 \;\ldots\ldots (2)$

From Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ হতে পাই]

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \;\ldots\ldots (3)$$

By (3), (2) becomes[(3) দ্বারা (2) দাঁড়ায়]

$$-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0 \;\ldots\ldots (4)$$

Squaring (1) and (4) and then adding we get [(1) ও (2) কে বর্গ এবং অতপর যোগ করে পাই]

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)^2+\left(-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)^2=0$$

$$\Rightarrow u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u^2\left(\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2-2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}=0$$

$$\Rightarrow u^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}+v^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$$\Rightarrow (u^2+v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$$\Rightarrow c^2\cdot|f'(z)|^2=0$$

$$\Rightarrow |f'(z)|^2=0$$

$$\Rightarrow f'(z)=0$$

$\Rightarrow f(z)=$ constant [ধ্রুবক]

$\because f(z)=u+iv$

$$\Rightarrow f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\Rightarrow |f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

Example–27. Let $f(z)=u+iv$ be an analytic function where $u(x,y)=c_1=$ constant and $v(x,y)=c_2=$ constant represent two families of curves. Then these system of families are orthogonal.

[DUH–1975]

Solution : Given that $f(z)=u+iv$ be an analytic functions, so it must satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$\Rightarrow u_x=v_y \text{ and } u_y=-v_x \;\ldots\ldots (1)$$

Now $u(x,y)=c_1$

$$\Rightarrow du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$$

$$\Rightarrow u_x\,dx+u_y\,dy=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{u_x}{u_y}$$

$\therefore$ Slope of the first curve $m_1=\frac{dy}{dx}=-\frac{u_x}{u_y} \;\ldots\ldots (2)$

Similarly, from $v(x,y)=c_2$ we have slope of the second curve

$$m_2=-\frac{v_x}{v_y} \;\ldots\ldots (3)$$

Product of the slopes, $m_1\,m_2=\frac{-u_x}{u_y}\cdot\frac{-v_x}{v_y}$.

$$\Rightarrow m_1\,m_2=\frac{v_y}{-v_x}\cdot\frac{v_x}{v_y}=-1 \text{ by (1)}$$

Hence the given system of families of curves are orthogonal.

Example–28. If $f(z)=u+iv$ is analytic in a region R and if u and v have continuous second order partial derivatives in R, then u and v are harmonic in R.

[NU(Pre)–2008, DUH–1986, 1989, 1991, JUH–1986, 87]

Solution : Given $f(z)=u+iv$ is analytic in the region R. So, by Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \;\ldots\ldots (1)$$

and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \;\ldots\ldots (2)$

Again given u and v have continuous second order p[artial]
derivatives in R. So we have

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x} \quad \text{......} (3)$$

and $\dfrac{\partial^2 v}{\partial x\,\partial y}=\dfrac{\partial^2 v}{\partial y\,\partial x}$ ...... (4)

Now from (3) we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right). \quad \text{[by (1) and (2)]}$$

$$\Rightarrow -\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Thus, v satisfy Laplace equation and hence it is harm[onic].

Again, form (4) we get

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right). \quad \text{[by (1) and (2)]}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Thus, u satisfy **Laplace equation** and hence it is harm[onic].

Example–29. If f(z) = u + iv is a analytic function of z and ϕ is any function of x and y with differential coeff[icients] of first order, then show that

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left\{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right\}|f'(z)|^2$$

Solution : We have $\phi=\phi(x, y)$

$$\therefore \frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} \quad \text{......} (1)$$

$$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} \quad \text{......} (2)$$

From Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \text{......} (3)$$

By (3), (2) becomes

$$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial u}\left(-\frac{\partial v}{\partial x}\right)+\frac{\partial\phi}{\partial v}\cdot\frac{\partial u}{\partial x} \quad \text{......} (4)$$

Squaring and adding (1) and (4) we get

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+2\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$

$$+\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial u}{\partial x}\right)^2-2\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$

$$=\left(\frac{\partial\phi}{\partial u}\right)^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}+\left(\frac{\partial\phi}{\partial v}\right)^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$=\left\{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right\}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\} \quad \text{......} (5)$$

Now $f'(z)=\dfrac{dw}{dz}=\dfrac{\partial u}{\partial x}\dfrac{dx}{dz}+i\dfrac{\partial v}{\partial x}\dfrac{dx}{dz}$

$$\Rightarrow f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\therefore |f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

$$z = x + iy$$

$$\Rightarrow \frac{dz}{dx}=1 \Rightarrow \frac{dx}{dz}=1$$

Putting this value in (5) we get

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left\{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right\}|f'(z)|^2 \quad \text{(Showed)}$$

Example–30. If f(z) is analytic at a point z_0, then it must be continuous at z_0. Give an example to show that the converse of this theorem is not necessarily true. **[RUH–2001, 2004]**

Solution : Given f(z) is analytic at z_0. So it is differentiable there.

$$\therefore f'(z_0)=\lim_{z\to z_0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \text{ exist}$$

Now $f(z_0+\Delta z)-f(z_0)=\dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\cdot\Delta z$

$$\Rightarrow \lim_{\Delta z\to 0}[f(z_0+\Delta z)-f(z_0)]=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\cdot\lim_{\Delta z\to 0}\Delta z$$

$$\Rightarrow \lim_{\Delta z \to 0} f(z_0 + \Delta z) - f(z_0) = f'(z_0) \cdot 0$$

$$\Rightarrow \lim_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$$

which shows that f(z) must be continuous at z_0.

2nd Part : See 2nd part of example–21.

Example–31. For what values of z do the function defined by the following equations cease to be analytic.

(i) $z = e^{-v}(\cos u + i \sin u)$, $w = u + iv$

(ii) $z = \sinh u \cos v + i \cosh u \sin v$, $w = u + iv$.

Solution : (i) Given that $w = u + iv$

and $z = e^{-v}(\cos u + i \sin u)$

$$= e^{-v} e^{iu} = e^{iu} e^{i^2 v} = e^{i(u+iv)} = e^{iw}$$

$$\Rightarrow iw = \ln z$$

$$\Rightarrow i\frac{dw}{dz} = \frac{1}{z} \Rightarrow \frac{dw}{dz} = \frac{1}{iz}$$

when $z = 0$ then $\frac{dw}{dz} = \infty$

$\therefore$ w is not analytic at $z = 0$.

(ii) Given that $w = u + iv$

and $z = \sinh u \cos v + i \cosh u \sin v$

$= \frac{1}{i} \sin iu \cos v + i \cos iu \sin v$	$\cos iu = \cosh u$ $\sin iu = i \sinh u$

$$= -i \sin iu \cos v + i \cos iu \sin v$$

$$= -i[\sin iu \cos v - \cos iu \sin v]$$

$$= -i[\sin(iu - v)]$$

$$= -i[\sin(iu + i^2 v)]$$

$$= -i[\sin i(u + iv)]$$

$$= -i \sin iw = -ii \sinh w = \sinh w$$

$$\Rightarrow w = \sinh^{-1}(z)$$

$$\therefore \frac{dw}{dz} = \frac{1}{\sqrt{z^2 + 1}}$$

When $z^2 + 1 = 0 \Rightarrow z = \pm\sqrt{-1} = \pm\sqrt{i^2} = \pm i$

then $\frac{dw}{dz} = \infty$

Hence the function w is not analytic at $z = i, -i$.

Example–32. Show that the function

$$f(z) = \begin{cases} \frac{x^3 y^4 (x + iy)}{x^6 + y^8} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

the Cauchy-Riemann equations are satisfied at origin, but f(z) is not analytic there. **[RUH–1995]**

Solution : Let $f(z) = u(x, y) + iv(x, y)$

At $z = 0$, $f(0) = u(0, 0) + iv(0, 0)$

$$\Rightarrow 0 = u(0, 0) + iv(0, 0)$$

$$\Rightarrow u(0, 0) = 0 \text{ and } v(0, 0) = 0$$

At $z \neq 0$, $f(z) = \frac{x^3 y^4 (x + iy)}{x^6 + y^8}$

$$\Rightarrow u(x, y) + iv(x, y) = \frac{x^4 y^4}{x^6 + y^8} + i\frac{x^3 y^5}{x^6 + y^8}$$

$$\Rightarrow u(x, y) = \frac{x^4 y^4}{x^6 + y^8} \text{ and } v(x, y) = \frac{x^3 y^5}{x^6 + y^8}.$$

(i) For Cauchy-Riemann equations we have

$$u(0 + \Delta x, 0) = \frac{0}{(\Delta x)^6 + 0} = 0$$

$$u(0, 0 + \Delta y) = \frac{0}{0 + (\Delta y)^8} = 0$$

$$v(0 + \Delta x, 0) = \frac{0}{(\Delta x)^6 + 0} = 0$$

$$v(0, 0 + \Delta y) = \frac{0}{0 + (\Delta y)^8} = 0$$

$\therefore$ At the origin we have

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin.

(ii) For analyticity :

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$, Choosing z in place of Δz

$$\Rightarrow f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y^4 (x+iy)}{x^6 + y^8} \cdot \frac{1}{x+iy}$$

$$\Rightarrow f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y^4}{x^6 + y^8}$$

Now along the real axis, $y = 0$ and $x \to 0$

$$\therefore f'(0) = \lim_{x \to 0} \frac{0}{x^6 + 0} = 0$$

Along the imaginary axis, $x = 0$ and $y \to 0$

$$\therefore f'(0) = \lim_{y \to 0} \frac{0}{0 + y^8} = 0$$

Along the line $y = x$ we have

$$f'(0) = \lim_{\substack{x \to 0 \\ y=x}} \frac{x^3 y^4}{x^6 + y^8} = \lim_{x \to 0} \frac{x^7}{x^6 + x^8} = \lim_{x \to 0} \frac{x}{1 + x^2} = 0$$

Along the curve $y^4 = x^3$ we have

$$f'(0) = \lim_{\substack{x \to 0 \\ y^4 = x^3}} \frac{x^3 y^4}{x^6 + y^8} = \lim_{x \to 0} \frac{x^6}{2x^6} = \frac{1}{2}$$

The above limits along different paths are different, so f'(0) does not exist. Hence f(z) is not differentiable at $z = 0$. Hence f(z) is not analytic at the origin.

Example-33. Examine the nature of the function

$$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}; z \neq 0$$

$$f(0) = 0; z = 0$$

in the region including the origin. **[RUH-1998]**

Solution : Let $f(z) = u(x, y) + iv(x, y)$

At $z = 0$, $f(0) = u(0, 0) + iv(0, 0)$

$$\Rightarrow 0 + i0 = u(0, 0) + iv(0, 0)$$

$$\Rightarrow u(0, 0) = 0 \text{ and } v(0, 0) = 0.$$

At $z \neq 0$, $f(z) = \dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}}$

$$\Rightarrow u(x, y) + iv(x, y) = \frac{x^3 y^5}{x^4 + y^{10}} + i \frac{x^2 y^6}{x^4 + y^{10}}$$

$$\Rightarrow u(x, y) = \frac{x^3 y^5}{x^4 + y^{10}} \text{ and } v(x, y) = \frac{x^2 y^6}{x^4 + y^{10}}$$

(i) For continuity at the origin :

Along the real axis we have $x \to 0$ and $y = 0$

$$\therefore \lim_{z \to 0} u = \lim_{\substack{x \to 0 \\ y=0}} u = \frac{0}{x^4 + 0} = 0$$

$$\lim_{z \to 0} v = \lim_{x \to 0} v = \frac{0}{x^4 + 0} = 0$$

Along the imaginary axis $x = 0$, $y \to 0$

$$\therefore \lim_{z \to 0} u = \lim_{\substack{x=0 \\ y \to 0}} u = \frac{0}{0 + y^{10}} = 0$$

$$\lim_{z \to 0} v = \lim_{\substack{x=0 \\ y \to 0}} v = \frac{0}{0 + y^{10}} = 0$$

Along the line $y = x$, we have

$$\lim_{z \to 0} u = \lim_{\substack{x \to 0 \\ y=0}} u = \frac{x^3 \cdot x^5}{x^4 + x^{10}} = \lim_{x \to 0} \frac{x^4}{1 + x^6} = \frac{0}{1+0} = 0$$

$$\lim_{z \to 0} v = = \lim_{\substack{x \to 0 \\ y=x}} v = \frac{x^2 \cdot x^6}{x^4 + x^{10}} = \lim_{x \to 0} \frac{x^4}{1 + x^6} = \frac{0}{1+0} = 0$$

Thus, in all cases $\lim\limits_{z\to 0} f(z) = \lim\limits_{z\to 0} u + i \lim\limits_{z\to 0} v = 0 + i0 = 0$

Hence f(z) is continuous at the origin.

(ii) For Cauchy-Riemann equations :

$$u(0+\Delta x, 0) = \frac{0}{(\Delta x)^4 + 0} = 0$$

$$u(0, 0+\Delta y) = \frac{0}{0 + (\Delta y)^{10}} = 0$$

$$v(0+\Delta x, 0) = \frac{0}{(\Delta x)^4 + 0} = 0$$

$$v(0, 0+\Delta y) = \frac{0}{0 + (\Delta y)^{10}} = 0$$

$\therefore$ At the origin we have

$$\frac{\partial u}{\partial x} = \lim_{\Delta x\to 0} \frac{u(0+\Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0-0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y\to 0} \frac{u(0, 0+\Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0-0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x\to 0} \frac{v(0+\Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0-0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y\to 0} \frac{v(0, 0+\Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0-0}{\Delta y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin.

(iii) For differentiability :

$f'(x) = \lim\limits_{z\to 0} \frac{f(z) - f(0)}{z - 0}$. Choosing z in place of Δz

$$\Rightarrow f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}} \cdot \frac{1}{x+iy}$$

$$\Rightarrow f'(0) = = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Now along the real axis, $y = 0$ and $x \to 0$

$$\therefore f'(0) = \lim_{x\to 0} \frac{0}{x^4 + 0} = 0$$

Along the imaginary axis, $x = 0$ and $y \to 0$

$$\therefore f'(0) = \lim_{y\to 0} \frac{0}{0 + y^{10}} = 0$$

Along the line $y = x$ we have

$$f'(0) = \lim_{\substack{x\to 0\\ y=x}} \frac{x^2 y^5}{x^4 + y^{10}} = \lim_{x\to 0} \frac{x^7}{x^4 + x^{10}} = \lim_{x\to 0} \frac{x^3}{1 + x^6} = 0$$

Along the curve $y^5 = x^2$ we have

$$f'(0) = \lim_{\substack{x\to 0\\ y^5=x^2}} \frac{x^2 y^5}{x^4 + y^{10}} = \lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

The above limits along different paths are different, so $f'(0)$ does not exist. Thus the function is not differentiable at $z = 0$.

Example–33(a). Show that the function $f(z) = \sqrt{xy}$ is not regular at the origin although Cauchy–Riemann equations are satisfied at the point (0, 0). [দেখাও যে, $f(z) = \sqrt{xy}$ ফাংশনটি মূলবিন্দুতে regular নয়; যদিও কসি-রীম্যান সমীকরণগুলি (0, 0) বিন্দুতে সিদ্ধ হয়।]

[NU(Pre)–2011]

Solution : Let $f(z) = u(x, y) + iv(x, y)$

At $z = 0$, $f(0) = u(0, 0) + iv(0, 0)$

$$\Rightarrow 0 = u(0, 0) + iv(0, 0), \quad \because f(0) = \sqrt{0 \cdot 0} = 0$$

$$\Rightarrow u(0, 0) = 0, \; v(0, 0) = 0$$

At $z \neq 0$, $f(z) = u(x, y) + iv(x, y) = \sqrt{xy}$

$$\Rightarrow u(x, y) = \sqrt{xy} \text{ and } v(x, y) = 0$$

(i) For Cauchy-Riemann equations [কসি-রীম্যান সমীকরণের জন্য]

At the origin we have [মূল বিন্দুতে পাই]

$$\frac{\partial u}{\partial x} = \lim_{\Delta x\to 0} \frac{u(0+\Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0-0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0+\Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin.
[অতএব কসি-রীম্যান সমীকরণগুলি মূল বিন্দুতে সিদ্ধ হয় ।]

For regular (analytic) [regular এর জন্য] ঃ

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$\Rightarrow f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sqrt{xy}}{x + iy}$$

Now along the real axis [এখন বাস্তব অক্ষ বরাবর] $y = 0$ and $x \to 0$

$$\therefore f'(0) = \lim_{x \to 0} \frac{0}{x} = 0$$

Along the imaginary axis [কাল্পনিক অক্ষ বরাবর] $x = 0$ and $y \to 0$

$$\therefore f'(0) = \lim_{y \to 0} \frac{0}{iy} = 0$$

Along the line $y = mx$

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{x \cdot mx}}{x + imx} = \frac{\sqrt{m}}{1 + im}$$

This shows that for different values of m we get different values of $f'(0)$. Thus $f'(0)$ does not exist and so $f(z)$ is not regular (analytic) at (0, 0).

ইহা দেখায় যে m এর ভিন্ন ভিন্ন মানের জন্য $f'(0)$ এর ভিন্ন ভিন্ন মান পাব। অতএব $f'(0)$ বিদ্যমান নাই এবং সে কারণে $f(z)$, (0, 0) এ regular না] (Showed).

Example-34. Prove that the function $f(z) = \frac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0$
$f(0) = 0, z = 0$

is not analytic at origin although Cauchy-Riemann equations are satisfied there. **[RUH-2004]**

Solution : Let $f(z) = u(x, y) + iv(x, y)$

At $z = 0$, $f(0) = u(0, 0) + iv(0, 0)$

$$\Rightarrow 0 = u(0, 0) + iv(0, 0)$$

$$\Rightarrow u(0, 0) = 0 \text{ and } v(0, 0) = 0$$

At $z \neq 0$, $f(z) = \frac{xy^2(x + iy)}{x^2 + y^4}$

$$\Rightarrow u(x, y) + iv(x, y) = \frac{x^2 y^2}{x^2 + y^4} + i\frac{xy^3}{x^2 + y^4}$$

$$\Rightarrow u(x, y) = \frac{x^2 y^2}{x^2 + y^4} \text{ and } v(x, y) = \frac{xy^3}{x^2 + y^4}$$

(i) For Cauchy-Riemann equations :

$u(0 + \Delta x, 0) = 0$, $u(0, 0 + \Delta y) = 0$

$v(0 + \Delta x, 0) = 0$ and $v(0, 0 + \Delta y) = 0$

$$\therefore \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, 0+\Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin.

(ii) For analyticity :

$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$, Choosing z in place of Δz

$$\Rightarrow f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2(x+iy)}{x^2+y^4}\cdot\frac{1}{x+iy}$$

$$\Rightarrow f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$$

Along the real axis, $y = 0$ and $x \to 0$

$$\therefore f'(0) = \lim_{x\to 0} \frac{0}{x^2+0} = 0$$

Along the imaginary axis, $x = 0$ and $y \to 0$

$$f'(0) = \lim_{y\to 0} \frac{0}{0+y^4} = 0$$

Along the line $y = x$ we have

$$f'(0) = \lim_{\substack{x\to 0\\ y=x}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim_{x\to 0} \frac{x}{1+x^2} = 0$$

Along the curve $y^2 = x$ we have

$$f'(0) = \lim_{\substack{x\to 0\\ y^2=x}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

The above limits along different paths are different, so $f(z)$ is not differentiable at $z = 0$. Hence $f(z)$ is not analytic at the origin.

Example-35. Show that the function $f(z) = e^{-z^{-4}}$ $(z \neq 0)$ and $f(0) = 0$ is not analytic at $z = 0$ although Cauchy-Riemann equations are satisfied at the point.

Solution : Given that $f(z) = u + iv = e^{-z^{-4}}$ ...... (1)

Now $-z^{-4} = -\dfrac{1}{z^4} = -\dfrac{1}{(x+iy)^4} = -\dfrac{(x-iy)^4}{(x+iy)^4(x-iy)^4}$

$$= -\frac{x^4 - 4ix^3y + 6x^2i^2y^2 - 4xi^3y^3 + i^4y^4}{(x^2+y^2)^4}$$

$$= -\frac{x^4 - 4ix^3y - 6x^2y^2 + 4ixy^3 + y^4}{(x^2+y^2)^4}$$

$$= \frac{-(x^4 - 6x^2y^2 + y^4)}{(x^2+y^2)^4} + i\frac{4x^3y - 4xy^3}{(x^2+y^2)^4}$$

$= A + iB$, say

Where $A = \dfrac{6x^2y^2 - x^4 - y^4}{(x^2+y^2)^4}$ and $B = \dfrac{4x^3y - 4xy^3}{(x^2+y^2)^4}$

$\therefore f(z) = u + iv = e^{A+iB} = e^A[\cos B + i\sin B]$ ...... (2)

$\Rightarrow u = e^A\cos B$ and $v = e^A \sin B$ ...... (3)

Given that $f(0) = 0$

$\Rightarrow u(0, 0) + iv(0, 0) = 0 + i0$

$\Rightarrow u(0, 0) = 0$ and $v(0, 0) = 0$ ...... (4)

Also at $(x, 0)$ we have

$$A = \frac{-x^4}{x^8} = -\frac{1}{x^4},\ B = 0$$

At $(0, y)$, $A = \dfrac{-y^4}{y^8} = -\dfrac{1}{y^4},\ B = 0$

$\therefore u(x, 0) = e^{-1/x^4}\cos 0 = e^{-1/x^4}$

$u(0, y) = e^{-1/y^4}\cos 0 = e^{-1/y^4}$

$v(x, 0) = e^{-1/x^4}\cdot\sin 0 = 0$

$v(0, y) = e^{-1/y^4}\ \sin 0 = 0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{e^{-1/x^4}-0}{x}$$

$$= \lim_{x\to 0}\frac{1}{x\left(1 + \dfrac{1}{x^4} + \dfrac{1}{2!\,x^8} + \cdots\right)}$$

$$= \lim_{x\to 0}\frac{1}{x + \dfrac{1}{x^3} + \dfrac{1}{2!\,x^7} + \cdots} = \frac{1}{\infty} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{e^{-1/y^4}-0}{y} = 0, \text{ as above}$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

The above results show that Cauchy-Riemann equations are satisfied at $z = 0$

f(z) not analytic at z = 0

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}$$

$$= \lim_{z \to 0} \frac{f(z) - f(0)}{z}, \text{ Choosing } z \text{ for } \Delta z$$

$$= \lim_{z \to 0} \frac{e^{-z^{-4}} - 0}{z} \ \ldots\ldots (5)$$

Along the path $z = re^{i\frac{\pi}{4}}$ we have $r \to 0$ when $z \to 0$

$$\therefore -z^{-4} = \frac{-1}{z^4} = \frac{-1}{(re^{i\pi/4})^4} = \frac{-1}{r^4 e^{i\pi}}$$

$$= \frac{-1}{r^4(\cos\pi + i\sin\pi)} = \frac{-1}{r^4(-1+0)} = \frac{1}{r^4} = r^{-4}$$

Thus (5) becomes

$$\therefore f'(0) = \lim_{r \to 0} \frac{e^{r^{-4}} - 0}{re^{i\pi/4}}$$

$$= \lim_{r \to 0} \frac{1}{re^{i\pi/4}} \left[\frac{1}{r^4} + \frac{1}{2!\, r^8} + \cdots\right]$$

$$= \frac{1}{0} = \infty$$

Thus $f'(0)$ does not exist at $z = 0$ and hence $f(z)$ is not analy at $z = 0$.

N. B. For analyticity Cauchy-Riemann equations must satisfied and also the first order partial derivatives of u and should be continuous. Here the second condition is not satisf and hence f(z) is not analytic at $z = 0$.

Example–35(a). Prove that the function $f(z) = z^2 + 5iz + 3 -$ satisfy Cauchy-Riemann equations. **[NUH(Phy)-200**

Solution : Given that [দেওয়া আছে]

$$f(z) = z^2 + 5iz + 3 - i$$

$$\Rightarrow u + iv = (x + iy)^2 + 5i(x + iy) + 3 - i$$

$$= x^2 + 2ixy - y^2 + 5ix - 5y + 3 - i$$

$$= (x^2 - y^2 - 5y + 3) + i(2xy + 5x - 1)$$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$\Rightarrow u = x^2 - y^2 - 5y + 3 \text{ and [এবং] } v = 2xy + 5x - 1$$

$$\therefore \frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = -2y - 5,\ \frac{\partial v}{\partial x} = 2y + 5 \text{ and [এবং] } \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \ \ldots\ldots (1)$$

$$\text{and [এবং] } \frac{\partial u}{\partial y} = -(2y + 5) = -\frac{\partial v}{\partial x} \ \ldots\ldots (2)$$

From (1) and (2) we see that the given equation satisfy the Cauchy-Riemann equations. [সমীকরণ (1) ও (2) হতে দেখি যে প্রদত্ত সমীকরণ কচি-রীম্যান সমীকরণ সিদ্ধ করে।] **(Proved)**

Example–35(b). Test for analyticity of $W_1 = f_1(z) = |z|^2$ and $W_2 = f_2(z) = \frac{1}{2}$. **[NU(Pre)-2006]**

Solution : (i) Given that [দেওয়া আছে]

$$W_1 = f_1(z) = |z|^2$$

$$\Rightarrow u_1 + iv_1 = |x + iy|^2$$

$$\Rightarrow u_1 + iv_1 = x^2 + y^2$$

$$\Rightarrow u_1 = x^2 + y^2 \text{ and [এবং] } v_1 = 0$$

$$\therefore \frac{\partial u_1}{\partial x} = 2x,\ \frac{\partial u_1}{\partial y} = 2y,\ \frac{\partial v_1}{\partial x} = 0,\ \frac{\partial v_1}{\partial y} = 0$$

$$\Rightarrow \frac{\partial u_1}{\partial x} = 2x \neq \frac{\partial v_1}{\partial y} \text{ and [এবং] } \frac{\partial u_1}{\partial y} = 2y \neq -\frac{\partial u_1}{\partial x}$$

Therefore, $W_1 = f_1(z) = |z|^2$ does not satisfy Cauchy-Riemann equations and hence not analytic. [অতএব $W_1 = f_1(z) = |z|^2$ কচি-রীম্যান সমীকরণ সিদ্ধ করে না এবং সে কারণে বৈশ্লেষিক না]

(ii) Given that [দেওয়া আছে] $w_2 = f_2(z) = \frac{1}{2}$

$$\Rightarrow u_2 + iv_2 = \frac{1}{2}$$

$$\Rightarrow u_2 = \frac{1}{2} \text{ and [এবং] } v_2 = 0$$

$$\therefore \frac{\partial u_2}{\partial x}=0, \frac{\partial u_2}{\partial y}=0, \frac{\partial v_2}{\partial x}=0 \text{ and [এবং] } \frac{\partial v_2}{\partial y}=0$$

$$\therefore \frac{\partial u_2}{\partial x}=0=\frac{\partial v_2}{\partial y} \text{ and [এবং] } \frac{\partial u_2}{\partial y}=0=-\frac{\partial v_2}{\partial x}$$

Therefore, $W_2 = f_2(z) = \frac{1}{2}$ satisfy Cauchy-Riemann equa... and hence analytic. [অতএব $W_2 = f_2(z) = \frac{1}{2}$ কচি-রীম্যান সমীকরণ সি... এবং সে কারণে বৈশ্লেষিক।]

Example–35(c). Write down the Cauchy-Riemann p... differential equations. How are they related to ana... functions. [NU(Pre)-2...

Solution : If f(z) = u + iv then the Cauchy-Riemann p... differential equations are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Provided the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ ex...

2nd Part : The Cauchy-Riemann equations related to ana... functions in the following way :

The function f(z) = u + iv is analytic in a domain D if and... if the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

are satisfied in D.

Example–35(d). If for all z in the entire complex plane f... analytic and bounded, then prove that f(z) must be a constant... [NU(Pre)-...

Solution : Let [ধরি] f(z) = u + iv. Since f(z) is analytic... satisfies the Cauchy-Riemann equations [যেহেতু f(z) বৈশ্লেষি... ইহা কচি-রীম্যান সমীকরণ সিদ্ধ করে]

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and [এবং] } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \text{ ...... (1)}$$

Again, f(z) is bounded, so its value will reach to a finite constant c, that is [আবার, f(z) সীমাবদ্ধ, সুতরাং ইহার মান একটি সসীম ধ্রুবক c এ পৌছবে; অর্থা]

$$|f(z)| = c$$

$$\Rightarrow |u+iv| = c$$

$$\Rightarrow \sqrt{u^2+v^2} = c$$

$$\Rightarrow u^2+v^2 = c^2$$

Differentiating this partially w. r. to x and y seperately we have [ইহাকে x ও y এর সাপেক্ষে পৃথকভাবে আংশিক অন্তরীকরণ করে পাই]

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$$

$$\text{and [এবং] } 2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$$

$$\Rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0 \text{ ...... (2)}$$

$$\text{and [এবং] } u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0 \text{ ...... (3)}$$

By (1), (3) becomes [(1) এর সাহায্যে (3) দাঁড়ায়]

$$-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0 \text{ ...... (4)}$$

Squaring (2) and (4), and then adding we get [(2) ও (4) কে বর্গ করে এবং অতপর যোগ করে পাই]

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)^2+\left(-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)^2=0$$

$$\Rightarrow u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u^2\left(\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2-2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}=0$$

$$\Rightarrow u^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}+v^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$$\Rightarrow (u^2+v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}=0$$

$$\Rightarrow c^2\,|f'(z)|^2=0$$

$$\Rightarrow |f'(z)|^2=0$$

$$\Rightarrow f'(z)=0$$

$\Rightarrow$ f(z) = constant. **(Proved)**

$$\because f(z)=u+iv$$

$$\Rightarrow f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\Rightarrow |f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

Example–36. Find an analytic function $f(z)$ such that $\text{Im}\{f'(z)\} = 6xy + 4x$, $f'(0) = 0$ and $f(1 + i) = 0$. [RUH-19..]

Solution : Given that $\text{Im}\{f'(z)\} = 6xy + 4x$ ...... (1)

Let $w = f'(z) = \dfrac{df}{dz} = u + iv$

$\therefore \text{Im}\{f'(z)\} = v$

$\Rightarrow v = 6xy + 4x,$ by (1)

$\therefore \dfrac{\partial v}{\partial x} = 6y + 4$ and $\dfrac{\partial v}{\partial y} = 6x$ ...... (2)

By Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6x \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6y - 4 \text{ ...... (3)}$$

Now $u = u(x, y)$, so by calculas we have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$\Rightarrow du = 6x\,dx - (6y + 4)\,dy,$ which is an exact equation.

$\therefore u = \int 6x\,dx - \int (6y + 4)\,dy$

$\Rightarrow u = 3x^2 - 3y^2 - 4y + c$

$\therefore f'(z) = u + iv$

$$= 3x^2 - 3y^2 - 4y + c + i(6xy + 4x)$$

$$= 3(x^2 + 2ixy - y^2) + 4i\left(x - \frac{1}{i}y\right) + c$$

$$= 3(x^2 + 2ixy + i^2 y^2) + 4i(x + iy) + c$$

$\Rightarrow f'(z) = 3(x + iy)^2 + 4i(x + iy) + c$ ...... (4)

$\Rightarrow f'(z) = \dfrac{df}{dz} = 3z^2 + 4iz + c$

$\Rightarrow f = \int (3z^2 + 4iz + c)\,dz$

$\Rightarrow f(z) = z^3 + 2iz^2 + cz + D)$ ...... (5)

Given $f'(0) = 0$, so from (4)

$$f'(0) = 0 + 0 + c \Rightarrow c = 0$$

Given $f(1 + i) = 0$, so from (5)

$$f(1 + i) = (1 + i)^3 + 2i(1 + i)^2 + 0 + D \quad [\because c = 0]$$

$\Rightarrow 0 = 1 + 3i + 3i^2 + i^3 + 2i + 4i^2 + 2i^3 + D$

$= 1 + 3i - 3 - i + 2i - 4 - 2i + D$

$= -6 + 2i + D$

$\Rightarrow D = 6 - 2i$

By putting the value of c and D in (5) we get

$f(z) = z^3 + 2iz^2 + 0 + 6 - 2i$

$\Rightarrow f(z) = z^3 + 2iz^2 + 6 - 2i.$ (Ans)

Example–37. Find an analytic function $f(z)$ such that

$\text{Re}\{f'(z)\} = 3x^2 - 4y - 3y^2$ and $f(1 + i) = 0.$ **[RUH–1998]**

Solution : Given that $\text{Re}\{f'(z)\} = 3x^2 - 4y - 3y^2$

Let $w = f'(z) = \dfrac{df}{dz} = u + iv$

$\therefore \text{Re}\{f'(z)\} = u$

$\Rightarrow 3x^2 - 4y - 3y^2 = u$

$\therefore \dfrac{\partial u}{\partial x} = 6x$ and $\dfrac{\partial u}{\partial y} = -4 - 6y$

By Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6x \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -4 - 6y$$

Since $v = v(x, y)$, so

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$\Rightarrow dv = (6y + 4)\,dx + 6x\,dy$

Here $\dfrac{\partial}{\partial x}(6y + 4) = 0$ and $\dfrac{\partial}{\partial y}(6x) = 0$

So the above equation is an exact differential equation.

$\therefore v = \int (6y + 4)\,dx + \int 0\,dy$

$= 6xy + 4x$ omitting the integrating constant

$\therefore f'(z) = u + iv = 3x^2 - 4y - 3y^2 + i(6xy + 4x)$

$= 3(x^2 + 2ixy - y^2) + 4i(x + iy)$

$= 3(x + iy)^2 + 4i(x + iy)$

$\Rightarrow \frac{df}{dz} = 3z^2 + 4iz, \; z = x + iy$

$\therefore f(z) = \int (3z^2 + 4iz)\, dz$

$f(z) = z^3 + 2iz^2 + c$, where c is the integrating constant.

Given $f(1 + i) = 0$

$\Rightarrow (1 + i)^3 + 2i(1 + i)^2 + c = 0$

$\Rightarrow 1 + 3i + 3i^2 + i^3 + 2i + 4i^2 + 2i^3 + c = 0$

$\Rightarrow 1 + 3i - 3 - i + 2i - 4 - 2i + c = 0$

$\Rightarrow -6 + 2i + c = 0$

$\Rightarrow c = 6 - 2i$

Thus $f(z) = z^3 + 2iz^2 + 6 - 2i$. (Ans)

Example–38. If $\text{Im}\{f'(z)\} = 6x(2y - 1)$ and $f(0) = 3 - 2i$, $f(1) = 6 - 5i$ find $f(1 + i)$. [NUH-2000]

Solution : Given [দেওয়া আছে] $\text{Im}\{f'(z)\} = 6x(2y - 1)$ ...... (1)

Let $w = f'(z) = \frac{df}{dz} = u + iv$

$\therefore \text{Im}\{f'(z)\} = v$

$\Rightarrow 6x(2y - 1) = v$; [by (1)]

$\Rightarrow v = 12xy - 6x$

$\therefore \frac{\partial v}{\partial x} = 12y - 6$ and $\frac{\partial v}{\partial y} = 12x$ ...... (2)

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ পাই]

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 12x$ ...... (3)

and [এবং] $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -12y + 6$ ...... (4)

Since $u = u(x, y)$ so by calculas [যেহেতু $u = u(x, y)$, সুতরাং ক্যালকুলাস দ্বারা পাই]

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$\Rightarrow du = 12x\, dx + (-12y + 6)\, dy$, which is an exact differential equation. [যাহা একটি যথার্থ অন্তরক সমীকরণ]

By integrating we get [যোজিত ফল নিয়ে পাই]

$\int du = \int 12x\, dx + \int (-12y + 6)\, dy$

$\Rightarrow u = 12 \cdot \frac{x^2}{2} + \frac{-12y^2}{2} + 6y + c$, c is a integrating constant

$\Rightarrow u = 6x^2 - 6y^2 + 6y + c$

$\therefore f'(z) = u + iv$

$= 6x^2 - 6y^2 + 6y + c + i(12xy - 6x)$

$= 6(x^2 - y^2 + 2ixy) - 6i(x + iy) + c$

$= 6(x^2 + i^2 y^2 + 2ixy) - 6i(x + iy) + c$

$= 6(x + iy)^2 - 6i(x + iy) + c$

$\Rightarrow \frac{df}{dz} = 6z^2 - 6iz + c$

$\Rightarrow \int df = \int (6z^2 - 6iz + c)\, dz$

$\Rightarrow f = 6 \cdot \frac{z^3}{3} - 6i \frac{z^2}{2} + cz + D$, [D = integrating constant]

$\Rightarrow f(z) = 2z^3 - 3iz^2 + cz + D$ ...... (5)

Given [দেওয়া আছে] $f(0) = 3 - 2i$ and [এবং] $f(1) = 6 - 5i$ ...... (6)

Putting $z = 0$ in (5) we get [(5) এ $z = 0$ বসাইয়া পাই]

$f(0) = 0 - 0 + 0 + D$

$\Rightarrow 3 - 2i = D$ ...... (7) [by (6)]

Putting $z = 1$ in (5) we get [(5) এ $z = 1$ বসাইয়া পাই]

$f(1) = 2 - 3i + c + D$

$\Rightarrow 6 - 5i = 2 - 3i + c + 3 - 2i$; [by (6) and (7)]

$\Rightarrow 6 - 5i = 5 - 5i + c$

$\Rightarrow 6 - 5 = c \Rightarrow c = 1$

Putting the values of C and D in (5) we get [C ও D এর মান ... বসাইয়া পাই]

$$f(z) = 2z^3 - 3iz^2 + z + 3 - 2i$$

$$\therefore f(1+i) = 2(1+i)^3 - 3i(1+i)^2 + (1+i) + 3 - 2i; \text{ by putting } z = 1, ...$$

$$= 2(1 + 3i - 3 - i) - 3i(1 + 2i - 1) + 1 + i + 3 - 2i$$

$$= 2 + 6i - 6 - 2i - 3i + 6 + 3i + 4 - i = 6 + 3i. \quad \text{(Ans)}$$

Example–39. Prove that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find its harmonic conjugate v and express u + iv as an analytic function of z. **[RUH–1997, 2002, 2004, CUH-19...]**

Solution : Given that [দেওয়া আছে] $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = 6xy + 4x \quad \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y \quad \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \quad \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -6y - 4 \quad \cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$$

$\Rightarrow$ u satisfied Laplace equation. Hence u is harmonic. [u ল্যাপ্লাস সমীকরণ সিদ্ধ করে। অতএব u হারমোনিক।]

By Cauchy-Riemann equations we have [কচি-রীম্যান সমীকরণ ... পাই]

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -3x^2 + 3y^2 + 4y \quad \text{[by (2)]}$$

By integrating this w. r. to x keeping y as constant [y কে ... রেখে x এর সাপেক্ষে যোজিত করে পাই]

$$v = \int (-3x^2 + 3y^2 + 4y)\, dx$$

$$\Rightarrow v = -x^3 + 3xy^2 + 4xy + F(y) \quad \cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 6xy + 4x + F'(y); \quad \text{by C–R equation.}$$

$$\Rightarrow 6xy + 4x = 6xy + 4x + F'(y); \quad \text{[by (1)]}$$

$$\Rightarrow 0 = F'(y)$$

$\therefore F(y) = c_1$ by integrating

Putting this value in (5) we get [এই মান (5) এ বসাইয়া পাই]

$$v = -x^3 + 3xy^2 + 4xy + c_1$$

Let [ধরি] $f(z) = u + iv$

$$= 3x^2y + 2x^2 - y^3 - 2y^2 + i(-x^3 + 3xy^2 + 4xy + c_1)$$

$$= (-ix^3 + 3ixy^2 + 3x^2y - y^3) + 2(x^2 - y^2 + 2ixy) + ic_1$$

$$= -i(x^3 + 3ix^2y + 3i^2xy^2 + i^3y) + 2(x^2 + 2ixy + i^2y^2) + c$$

$$= -i(x + iy)^3 + 2(x + iy)^2 + c, \text{ where } c = ic_1$$

$$\Rightarrow f(z) = u + iv = -iz^3 + 2z^2 + c. \quad \text{(Ans)}$$

By Milne's methd :

Given that $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = 6xy + 4x = \phi_1(x, y), \text{ say} \quad \cdots\cdots (1)$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \quad \cdots\cdots (2)$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y = \phi_2(x, y), \text{ say} \quad \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -6y - 4 \quad \cdots\cdots (4)$$

(2) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$

$\Rightarrow$ u satisfy Laplace equation

$\Rightarrow$ u is harmonic.

Putting x = z and y = 0 in (1) and (3) we get

$$\phi_1(z, 0) = 0 + 4z = 4z$$

and $\phi_2(z, 0) = 3z^2 - 0 - 0 = 3z^2$

By Milne's theorem we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 4z - i3z^2$$

$\Rightarrow f(z) = 2z^2 - iz^3 + c$ by integrating

$$\Rightarrow u + iv = 2(x + iy)^2 - i(x + iy)^3 + c$$

$$= 2(x^2 + 2ixy + i^2y^2) - i(x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3) + c$$

$$= 2x^2 + 4ixy - 2y^2 - ix^3 + 3x^2y - 3ixy^2 - y^3 + c_1 + ic_2$$

where c ...

Equating imaginary parts we get,

$$v = 4xy - x^3 - 3xy^2 + c_2$$

Thus, $\left.\begin{array}{l} v = -x^3 - 3xy^2 + 4xy + c_2 \\ \text{and } u + iv = -iz^3 + 2z^2 + c \end{array}\right\}$ (Ans)

Example–40. Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic and also find the harmonic conjugate if $f(z) = u + iv$ is analytic. **[DUH–1991, 2003, CUH–1985, JUH–19..]**

Solution : Given that [দেওয়া আছে] $u = 2x - x^3 + 3xy^2$

$$\therefore \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = 6xy = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = -6x \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = 6x \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$

This shows that u satisfy Laplace equation and hence harmonic. [ইহা দেখায় যে u ল্যাপলাস সমীকরণ সিদ্ধ করে এবং ... হারমোনিক।]

To find the harmonic conjugate v, we have by putting x = z, y = 0 in (1) and (2) we get [হারমোনিক অনুবন্ধী v বাহির করতে আমরা ... x = z এবং y = 0 বসাইয়া পাই]

$$\phi_1(z, 0) = 2 - 3z^2$$

and [এবং] $\phi_2(z, 0) = 0$

By Milne's theorem we have [মিলনীর উপপাদ্য দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 2 - 3z^2 - 0$$

$$\therefore f(z) = \int (2 - 3z^2)\, dz$$

$$= 2z - z^3 + c$$

$$\Rightarrow u + iv = 2(x + iy) - (x + iy)^3 + c$$

$$= 2(x + iy) - (x^3 + 3ix^2y + 3i^2 xy^2 + i^3 y^3) + c$$

$$= 2(x + iy) - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$$

$$= (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3 + c_1); \text{ where } ic_1 = c$$

Equating imaginary parts we have [কাল্পনিক রাশি সমীকৃত করে পাই]

$$v = 2y - 3x^2y + y^3 + c_1 \quad \text{(Ans)}$$

Example–41. Find the harmonic conjugate of the function $u = x^3 + 6x^2y - 3xy^2 - 2y^3$. **[DUH–1987, 1991, JUH–1991]**

Solution : Given [দেওয়া আছে] $u = x^3 + 6x^2y - 3xy^2 - 2y^3$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 + 6x^2y - 3xy^2 - 2y^3)$$

$$= 3x^2 + 12xy - 3y^2 = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 + 6x^2y - 3xy^2 - 2y^3)$$

$$= 6x^2 - 6xy - 6y^2 = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 + 12xy - 3y^2)$$

$$= 6x + 12y \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(6x^2 - 6xy - 6y^2)$$

$$= -6x - 12y \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 12y - 6x - 12y$

Since u satisfies Laplace equation, so u is harmonic.

Let v be the harmonic conjugate of u, so that [যেহেতু u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক। ধরি u এর হারমোনিক অনুবন্ধী v.]

$f(z) = u + iv$ is analytic. [বিশ্লেষিক]

Now from (1), $\phi_1(z, 0) = 3z^2 + 0 - 0 = 3z^2$

and from (2), $\phi_2(z, 0) = 6z^2 - 0 - 0 = 6z^2$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= 3z^2 - 6iz^2$$
$$\Rightarrow f(z) = \int (3z^2 - 6iz^2)\, dz$$
$$\Rightarrow u + iv = \int (3 - 6i)\, z^2\, dz$$
$$= (3 - 6i)\frac{z^3}{3} + c_1 + ic_2, \text{ where } c_1 + ic_2 = \text{complex constant}$$
$$= (1 - 2i)\, z^3 + c_1 + ic_2$$
$$= (1 - 2i)\, (x + iy)^3 + c_1 + ic_2$$
$$= (1 - 2i)\, (x^3 + 3ix^2y - 3xy^2 - iy^3) + c_1 + ic_2$$
$$= x^3 + 3ix^2y - 3xy^2 - iy^3 - i2x^3 + 6x^2y + 6ixy^2 - 2y^3 + c_1 + i$$
$$= (x^3 - 3x\,y^2 + 6x^2y - 2y^3 + c_1) + i(3x^2y - y^3 - 2x^3 + 6xy^2 + $$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = 3x^2y - y^3 - 2x^3 + 6xy^2 + c_2. \quad \text{(Ans)}$$

Example–42. Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ harmonic fuunction. Find v such that u + iv is analytic.

[NUH–98, 06, 10, 12(

RUH–1986, CUH–1986, JUH (Phy)-19

Solution : Given [দেওয়া আছে] $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1)$$
$$= 3x^2 - 3y^2 + 6x = \phi_1(x, y), \text{ say } \ldots\ldots (1)$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1)$$
$$= -6xy - 6y = \phi_2(x, y), \text{ say } \ldots\ldots (2)$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3y^2 + 6x)$$
$$= 6x + 6 \ldots\ldots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-6xy - 6y)$$
$$= -6x - 6 \ldots\ldots (4)$$

(3) + (4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6$

$$\Rightarrow \nabla^2 u = 0$$

$\therefore$ u satisfies Laplace equation, so u is a harmonic function. Let v is the harmonic conjugate of u, so that f(z) = u + iv is analytic. [$\therefore$ u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u ফাংশন হারমোনিক। v, u এর হারমোনিক অনুবন্ধী হবে যেন f(z) = u + iv.]

2nd Part : Putting x = z, y = 0 in (1) and (2) we get [(1) ও (2) এ x = z, y = 0 বসাইয়া পাই]

$$\phi_1(z, 0) = 3z^2 + 6z$$

and [এবং] $\phi_2(z, 0) = -0 - 0 = 0$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= 3z^2 + 6z - 0i$$
$$\Rightarrow f(z) = \int (3z^2 + 6z)\, dz$$
$$\Rightarrow u + iv = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + c_1 + ic_2, \text{ where } c_1 + ic_2 \text{ is complex constant.}$$
$$= z^3 + 3z^2 + c_1 + ic_2$$
$$= (x + iy)^3 + 3(x + iy)^2 + c_1 + ic_2$$
$$= x^3 + 3ix^2y - 3xy^2 - iy^3 + 3x^2 + 6ixy - 3y^2 + c_1 + ic_2$$
$$= x^3 - 3xy^2 + 3x^2 - 3y^2 + c_1 + i(3x^2y - y^3 + 6xy + c_2)$$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = 3x^2y - y^3 + 6xy + c_2. \quad \text{(Ans)}$$

Example–43. Show that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic and hence find its harmonic conjugate v if f(z) = u + iv is analytic.

[NUH–2002]

Solution : Given [দেওয়া আছে] $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^2y + 2x^2 - y^3 - 2y^2)$$
$$= 6xy + 4x = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x^2y + 2x^2 - y^3 - 2y^2)$$
$$= 3x^2 - 3y^2 - 4y = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(6xy + 4x)$$
$$= 6y + 4 \cdots\cdots (3)$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(3x^2 - 3y^2 - 4y)$$
$$= -6y - 4 \cdots\cdots (4)$$
(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4$
$$\Rightarrow \nabla^2 u = 0$$

Since u satisfies Laplace equation, so u is harmonic. Let ... the harmonic conjugate of u, so that f(z) = u + iv is analytic. [... u ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক। ধরি u এর হারমোনিক অনু... যেন f(z) = u + iv বৈশ্লেষিক হয়।]

Putting x = z, y = 0 in (1) and (2) we get [(1) ও (2) এ x = z, y = ... বসাইয়া পাই]
$$\phi_1(z, 0) = 0 + 4z = 4z$$
and [এবং] $\phi_2(z, 0) = 3z^2 - 0 - 0 = 3z^2$

$\therefore$ By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]
$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$\Rightarrow f'(z) = 4z - i3z^2$$
$$\Rightarrow f(z) = \int (4z - i3z^2)\, dz$$
$\Rightarrow u + iv = 4 \cdot \frac{z^2}{2} - i3\frac{z^3}{3} + c_1 + ic_2$, where $c_1 + ic_2$ is com... const...
$$= 2z^2 - iz^3 + c_1 + ic_2$$
$$= 2(x + iy)^2 - i(x + iy)^3 + c_1 + ic_2$$
$$= 2x^2 + 4ixy - 2y^2 - ix^3 + 3x^2y + i3xy^2 - y^3 + c_1 + i\ldots$$
$$= (2x^2 - 2y^2 + 3x^2y - y^3 + c_1) + i(4xy - x^3 + 3xy^2 + c_2)$$

Equating imaginary parts we get [কাল্পনিক রাশি সমীকৃত করে পাই]
$$v = 4xy - x^3 + 3xy^2 + c_2. \text{ (Ans)}$$

Example–44. Show that $\psi(x, y) = \frac{1}{2}\log(x^2 + y^2)$ is a harmonic in the region C – {(0, 0)}. Find the harmonic conjugate of this function such that $f(z) = \psi + i\phi$ is analytic and also find f(z) interns of z. **[NUH–2004, DUH–1986]**

Solution : Given [দেওয়া আছে] $\psi = \psi(x, y) = \frac{1}{2}\log(x^2 + y^2)$
$$\therefore \frac{\partial \psi}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2} = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$
$$\frac{\partial \psi}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2} = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$
$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \cdots\cdots (3)$$
$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) = \frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \cdots\cdots (4)$$
(3) + (4) gives, $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2}$
$$= \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2}$$
$$= 0$$

$\therefore \psi = \psi(x, y)$ is harmonic.

Given, $f(z) = \psi + i\phi$ will be analytic

$\therefore \phi$ is the harmonic conjugate of ψ

Putting x = z and y = 0 in (1) and (2) we get

[$\therefore \psi = \psi(x, y)$ হারমোনিক। দেওয়া আছে $f(z) = \psi + i\phi$ বৈশ্লেষিক হবে।
$\therefore \psi$ এর হারমোনিক অনুবন্ধী ϕ। (1) ও (2) এ x = z, y = 0 বসাইয়া পাই]
$$\phi_1(z, 0) = \frac{z}{z^2 + 0} = \frac{1}{z}$$
and [এবং] $\phi_2(z, 0) = \frac{0}{z^2 + 0} = 0$

$\therefore$ By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]
$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= \frac{1}{z} + i0$$

$$\Rightarrow f(z) = \int \frac{1}{z} dz$$

$\Rightarrow \psi + i\phi = \log z + (c_1 + ic_2)$; $c_1 + ic_2$ is the imaginary constant

$$= \log(x + iy) + c_1 + ic_2$$

$$= \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\frac{y}{x} + c_1 + ic_2$$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\phi = \tan^{-1}\frac{y}{x} + c_2. \text{ (Ans)}$$

Example-45. Show that $u = \frac{1}{2}\log(x^2 + y^2)$ satisfies th laplace's equation and find v if $f(z) = u + iv$ is analytic.

[DUH-2006, RUH-197

Solution : Do as example 4 by writting $\psi = u$ and $\phi = v$

Example-46. Show that $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic in some domain and determine th corresponding analytic function.

[DUH-1984, 1986, 1987, 1988, 1990, JUH-198

Solution : Given $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$

$\therefore \frac{\partial u}{\partial x} = \cos x \cosh y - 2 \sin x \sinh y + 2x + 4y = \phi_1(x, y)$, say ...... (1)

$\frac{\partial u}{\partial y} = \sin x \sinh y + 2 \cos x \cosh y - 2y + 4x = \phi_2(x, y)$, say ...... (2)

$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y - 2 \cos x \sinh y + 2$ ...... (3)

$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y + 2 \cos x \sinh y - 2$ ...... (4)

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y - 2 \cos x \sinh y + 2$

$+ \sin x \cosh y + 2 \cos x \sinh y - 2$

$= 0$

$\therefore$ u satisfies Laplace equation, so u is harmonic

2nd part **:** Let v is the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic.

Putting $x = z$ and $y = 0$ in (1) and (2) we get

$\cos z \cdot \cos 0 - 2 \sin z \cdot \sin 0 + 2z + 4 \cdot 0 = \phi_1(z, 0)$

$\Rightarrow \cos z + 2z = \phi_1(z, 0)$ ...... (5)

and $\sin z \cdot \sin 0 + 2 \cos z \cdot \cos 0 - 0 + 4z = \phi_2(z, 0)$

$\Rightarrow 2 \cos z + 4z = \phi_2(z, 0)$ ...... (6)

By Milne's theorem we get

$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$

$= \cos z + 2z - i(2 \cos z + 4z)$

$\Rightarrow f(z) = \int (\cos z + 2z - 2i \cos z - 4iz)\, dz$

$\Rightarrow u + iv = \sin z + 2\frac{z^2}{2} - 2i \sin z - 4i \cdot \frac{z^2}{2} + c$

$= \sin z + z^2 - 2i \sin z - 2iz^2 + c$

$= (1 - 2i)z^2 + (1 - 2i) \sin z + c.$ (Ans)

Example-47. Show that $\phi = \log\{(x - 1)^2 + (y - 2)^2\}$ is harmonic in every region which does not include the point (1, 2). Find a function ψ such that $\phi + i\psi$ is analytic and express $\phi + i\psi$ as a function of z. **[DUH-2004]**

Solution : Given [দেওয়া আছে] $\phi = \log\{(x - 1)^2 + (y - 2)^2\}$

$\therefore \frac{\partial \phi}{\partial x} = \frac{1}{(x-1)^2 + (y-2)^2} \cdot 2(x - 1) = \phi_1(x, y)$, say [ধরি] ...... (1)

$\frac{\partial \phi}{\partial y} = \frac{2(y-2)}{(x-1)^2 + (y-2)^2} = \phi_2(x, y)$, say [ধরি] ...... (2)

$\frac{\partial^2 \phi}{\partial x^2} = \frac{\{(x-1)^2 + (y-2)^2\} \cdot 2 - 2(x-1) \cdot 2(x-1)}{\{(x-1)^2 + (y-2)^2\}^2}$

$= \frac{2(y-2)^2 - 2(x-1)^2}{\{(x-1)^2 + (y-2)^2\}^2}$ ...... (3)

$\frac{\partial^2 \phi}{\partial y^2} = \frac{\{(x-1)^2 + (y-2)^2\} \cdot 2 - 2(y-2) \cdot 2(y-2)}{\{(x-1)^2 + (y-2)^2\}^2}$

$= \frac{2(x-1)^2 - 2(y-2)^2}{\{(x-1)^2 + (y-2)^2\}^2}$ ...... (4)

(3) + (4) gives, $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{2(y-2)^2-2(x-1)^2}{\{(x-1)^2+(y-2)^2\}^2}+\frac{2(x-1)^2-2(y-2)^2}{\{(x-1)^2+(y-2)^2\}^2}$

$=\frac{2(y-2)^2-2(x-1)^2+2(x-1)^2-2(y-2)^2}{\{(x-1)^2+(y-2)^2\}^2}$

$=0$

Since ϕ satisfy Laplace equation, so ϕ is harmonic. Let ψ the harmonic conjugate ϕ, so that $\phi + i\psi$ is analytic. [যেহেতু ϕ ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং ϕ হারমোনিক। ধরি ϕ এর হারমোনিক অনুবন্ধী ψ যেন $\phi + i\psi$ বৈশ্লেষিক হয়।]

Putting $x = z$ and $y = 0$ in (1) and (2) we get [(1) এবং (2) এ $x = z$ ও $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0)=\frac{2(z-1)}{(z-1)^2+2^2}$$

and [এবং] $\phi_2(z, 0)=\frac{-4}{(z-1)^2+2^2}$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z)=\phi_1(z, 0)-i\phi_2(z, 0)$$

$$=\frac{2(z-1)}{(z-1)^2+2^2}-i\frac{-4}{(z-1)^2+2^2}$$

$$\Rightarrow f(z)=\int\left[\frac{2(z-1)}{(z-1)^2+2^2}+i4\frac{1}{(z-1)^2+2^2}\right]dz$$

$$\Rightarrow \phi+i\psi=\int\frac{2(z-1)}{(z-1)^2+2^2}d(z-1)+i4\int\frac{1}{(z-1)^2+2^2}dz$$

$$=\log\{(z-1)^2+2^2\}+i4\cdot\frac{1}{2}\tan^{-1}\frac{z-1}{2}+c$$

$$=\log\{(z-1)^2+4\}+i2\tan^{-1}\left(\frac{z-1}{2}\right)+c$$

$$=2\left[\frac{1}{2}\log\{(z-1)^2+2^2\}+i\tan^{-1}\left(\frac{z-1}{2}\right)\right]+c$$

$$=2\log\{2+i(z-1)\}+c$$

$$=2\log\{2+i(x+iy-1)\}+c$$

$$=2\log\{2+ix-y-i\}+c$$

$$=2\log\{2-y+i(x-1)\}+c$$

$$=2\left[\frac{1}{2}\log\{(2-y)^2+(x-1)^2\}+i\tan^{-1}\frac{x-1}{2-y}\right]+c$$

$$=\log\{(x-1)^2+(y-2)^2\}+2i\tan^{-1}\frac{x-1}{2-y}+c_1+ic_2,$$

where $c = c_1 + ic_2$

Equating imaginary parts [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\psi=2\tan^{-1}\left(\frac{x-1}{2-y}\right)+c_2. \text{ Ans.}$$

Example–48. Show that the function $u = 2x(1 - y)$ is harmonic and find a function v such that $f(z) = u + iv$ is analytic. Also find $f(z)$ in terms of z.

[NUH–1996, 2001, NU(Pre)–2008, DUH–1983]

Solution : Given $u = 2x(1 - y)$

$$\therefore \frac{\partial u}{\partial x}=2(1-y)=\phi_1(x, y), \text{ say ...... (1)}$$

$$\frac{\partial^2 u}{\partial x^2}=0 \text{ ...... (2)}$$

$$\frac{\partial u}{\partial y}=-2x=\phi_2(x, y), \text{ say ...... (3)}$$

$$\frac{\partial^2 u}{\partial y^2}=0 \text{ ...... (4)}$$

(2) + (4) gives, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0+0=0$

Since u satisfies Laplace equation, so u is harmonic. Let v is the harmonic conjugate of u such that $f(z) = u + iv$ is analytic.

Putting $x = z$ and $y = 0$ in (1) and (3) we get

$$\phi_1(z, 0) = 2(1 - 0) = 2$$

and $\phi_2(z, 0) = -2z$

By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 2 + i2z$$

$$\Rightarrow f(z)=\int(2+2iz)\,dz$$

$$\Rightarrow f(z)=2z+2i\frac{z^2}{2}+c$$

$$\Rightarrow f(z) = iz^2 + 2z + c \text{ ...... (5)}$$

$$\Rightarrow u + iv = i(x + iy)^2 + 2(x + iy) + c$$

$$= i(x^2 - y^2 + 2ixy) + 2x + 2iy + c$$

$$= ix^2 - iy^2 - 2xy + 2x + 2iy + c$$

$$= 2x - 2xy + i(x^2 - y^2 + 2y) + c_1 + ic_2, \text{ where } c = c_1 + ic_2$$

Equating imaginary parts we get

$$v = x^2 - y^2 + 2y + c_2$$

Thus, $\left.\begin{array}{l} v = x^2 - y^2 + 2y + c_2 \\ \text{and } f(z) = iz^2 + 2z + c \end{array}\right\}$ (Ans)

Example–49. Show that the function $u = e^x(x \cos y - y \sin y)$ is a harmonic function and find the corresponding analytic function $f(z) = u + iv$. From it find v.

[NUH–1994, DUH–1988, 2005, CUH–85, 88, JUH–91]

Solution : We have $u = e^x (x \cos y - y \sin y)$

$$\therefore \frac{\partial u}{\partial x} = e^x (x \cos y - y \sin y) + e^x \cos y = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x (x \cos y - y \sin y) + e^x \cos y + e^x \cos y \cdots\cdots (2)$$

$$\frac{\partial u}{\partial y} = e^x (- x \sin y - \sin y - y \cos y) = \phi_2(x, y), \text{ say } \cdots\cdots (3)$$

and $$\frac{\partial^2 u}{\partial y^2} = e^x (- x \cos y - \cos y - \cos y + y \sin y) \cdots\cdots (4)$$

(2) + (4) gives, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x (x \cos y - y \sin y) + e^x \cos y + e^x \cos y$$
$$+ e^x(- x \cos y - \cos y - \cos y + y \sin y)$$
$$= e^x(x \cos y - y \sin y + \cos y + \cos y$$
$$- x \cos y - \cos y - \cos y + y \sin y)$$
$$= e^x \times 0 = 0$$

$\therefore$ u satisfies Laplace equation.

$\Rightarrow$ u is a harmonic function.

Putting $x = z$, $y = 0$ in (1) and (3) we get

$$\phi_1(z, 0) = e^z (z \cdot 1 - 0) + e^z \cdot 1$$
$$= ze^z + e^z$$

and $$\phi_2(z, 0) = e^z (-0 - 0 - 0) = 0$$

By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$\Rightarrow f(z) = \int (ze^z + e^z - i0)\, dz$$

$$\Rightarrow u + iv = ze^z - \int 1 \cdot e^z + e^z$$
$$= ze^z - e^z + e^z + c, \text{ c is a complex constant.}$$
$$= ze^z + c$$
$$= (x + iy)\, e^{x+iy} + c$$
$$= (x + iy)\, e^x \cdot e^{iy} + c$$
$$= e^x(x + iy) (\cos y + i \sin y) + c$$
$$= e^x(x \cos y + ix \sin y + iy \cos y - y \sin y) + c_1 + ic_2$$

where $c = c_1 + ic_2$

$$= e^x(x \cos y - y \sin y) + ie^x(x \sin y + y \cos y) + c_1 + ic_2$$

Equating imaginary parts we get,

$$v = e^x (x \sin y + y \cos y) + c_2. \text{ (Ans)}$$

Example–50. Show that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic. Find v such that $f(z) = u + iv$ is analytic. **[NUH–2011, NUH (Phy)–2003, 2006, DUHT–1989, 2001, DUH–1981, 1982, DUMPT–1990, CUH–1982, RUMP–1985]**

Solution : We have [আমাদের আছে] $u = e^{-x} (x \sin y - y \cos y)$

$$\therefore \frac{\partial u}{\partial x} = - e^{-x} (x \sin y - y \cos y) + e^{-x} \sin y = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x} (x \sin y - y \cos y) - e^{-x} \sin y - e^{-x} \sin y$$
$$= e^{-x} (x \sin y - y \cos y) - 2e^{-x} \sin y \cdots\cdots (2)$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y - \cos y + y \sin y) = \phi_2(x, y), \text{ say } \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(- x \sin y + \sin y + \sin y + y \cos y) \cdots\cdots (4)$$

(2) + (4) gives, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}(x \sin y - y \cos y) - 2e^{-x} \sin y$$
$$+ e^{-x} (- x \sin y + \sin y + \sin y + y \cos y)$$
$$= e^{-x} (x \sin y - y \cos y - 2 \sin y - x \sin y$$
$$+ 2 \sin y + y \cos y)$$
$$= e^{-x} \times 0 = 0$$

Since u satisfies Laplace equation, so u is harmonic. Let v be the harmonic conjugate of u such that $f(z) = u + iv$ is analytic. [যেহেতু u ল্যাপলাসের সমীকরণ সিদ্ধ করে, সুতরাং u হারমোনিক। ধরি u এর হারমোনিক অনুবন্ধী v যেন $f(z) = u + iv$ বৈশ্লেষিক হয়।]

Putting $x = z$ and $y = 0$ in (1) and (3) we get [(1) ও (3) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = -e^{-z}(0-0) + 0 = 0$$

and [এবং] $\phi_2(z, 0) = e^{-z}(z \cdot 1 - 1 + 0) = e^{-z}(z-1) = ze^{-z} - e^{-z}$

By Milne's method we have [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$\Rightarrow f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz$$

$$\Rightarrow u + iv = \int [0 - i(ze^{-z} - e^{-z})]\, dz$$

$$= -i\int ze^{-z}\, dz + i\int e^{-z}\, dz$$

$$= -i\left[-ze^{-z} - \int -e^{-z}\, dz\right] + i(-1)\, e^{-z}$$

$$= ize^{-z} + ie^{-z} - ie^{-z} + c$$

$$= ize^{-z} + c$$

$$= i(x+iy)e^{-(x+iy)} + c$$

$$= (ix - y)\, e^{-x} \cdot e^{-iy} + c$$

$$= e^{-x}(ix - y)(\cos y - i\sin y) + c$$

$$= e^{-x}(ix\cos y + x\sin y - y\cos y + iy\sin y) + c_1 + ic_2$$

where $c = c_1 + ic_2$

$$= e^{-x}(x\sin y - y\cos y) + ie^{-x}(x\cos y + y\sin y) + c_1 + ic_2$$

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$v = e^{-x}(x\cos y + y\sin y) + c_2. \quad \text{(Ans)}$$

Example–50(i). Show that $u(x, y)$ is harmonic. Find its harmonic conjugate $v(x, y)$ and the corresponding analytic function $f(z) = u + iv$ where $u(x, y) = x^2 - y^2 + 2e^{-x}\sin y$. [দেখাও যে, $u(x, y)$ একটি হারমনিক ফাংশন এর হারমনিক কনজুগেট $v(x, y)$ এবং এদের অ্যানালেটিক ফাংশন $f(z)$ নির্ণয় কর, যেখানে $u(x, y) = x^2 - y^2 + 2e^{-x}\sin y$.] **[NUH-2012]**

Solution : Given that [দেওয়া আছে] $u(x, y) = x^2 - y^2 + 2e^{-x}\sin y$

$$\therefore \frac{\partial u}{\partial x} = 2x - 2e^{-x}\sin y = \phi_1(x, y), \text{ say ...... (1)}$$

$$\frac{\partial^2 u}{\partial x^2} = 2 + 2e^{-x}\sin y$$

$$\frac{\partial u}{\partial y} = -2y + 2e^{-x}\cos y = \phi_2(x, y), \text{ say ...... (2)}$$

$$\frac{\partial^2 u}{\partial y^2} = -2 - 2e^{-x}\sin y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2e^{-x}\sin y - 2 - 2e^{-x}\sin y = 0$$

$\Rightarrow$ u satisfy Laplace equation [u ল্যাপলাস সমীকরণ সিদ্ধ করে]

$\therefore$ u harmonic [u হারমনিক] 1st Part proved.

2nd Part : Putting $x = z$ and $y = 0$ in (1) and (2) we get [(1) ও (3) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\phi_1(z, 0) = 2z - 2e^{-z}\sin 0 = 2z$$

and $\phi_2(z, 0) = 0 + 2e^{-z}\cos 0 = 2e^{-z}$

By Milne's theorem we have [মিলনির উপপাদ্য দ্বারা পাই]

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 2z - i2e^{-z}$$

$$\Rightarrow f(z) = \int (2z - 2ie^{-z})\, dz$$

$$= 2 \cdot \frac{z^2}{2} - 2i \cdot \frac{e^{-z}}{-1} + c$$

$$= z^2 + 2ie^{-z} + c$$

which is the corresponding analytic function.

$$\Rightarrow f(z) = (x+iy)^2 + 2ie^{-(x+iy)} + c, \text{ where } z = x + iy$$

$$= x^2 + i^2y^2 + 2ixy + 2ie^{-x}e^{-iy} + c$$

$$= x^2 - y^2 + 2ixy + 2ie^{-x}[\cos(-y) + i\sin(-y)] + c$$

$$\Rightarrow u + iv = x^2 - y^2 + 2ixy + 2ie^{-x}\cos y + 2e^{-x}\sin y + c_1 + ic_2$$

where $c = c_1 + ic_2$

Equating real and imaginary parts we get [বাস্তব ও কাল্পনিক অংশ সমীকৃত করে পাই]

$$u = x^2 - y^2 + 2e^{-x}\sin y + c_1$$

$$v = 2xy + 2e^{-x}\cos y + c_2. \quad \text{(Ans)}$$

Example–51. If $u = x^2 - y^2$ and $v = \dfrac{-y}{x^2+y^2}$, then show that both u and v satisfy the Laplace's equation but u + iv is not an analytic function of z. **[DUH–1987, DUHT–199...]**

Solution : Given $u = x^2 - y^2$ and $v = \dfrac{-y}{x^2+y^2}$

$$\therefore \frac{\partial u}{\partial x} = 2x \;\cdots\cdots (1)$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \;\cdots\cdots (2)$$

$$\frac{\partial u}{\partial y} = -2y \;\cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \;\cdots\cdots (4)$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot 0 + y\cdot 2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \;\cdots\cdots (5)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{2xy}{(x^2+y^2)^2}\right]$$

$$= \frac{(x^2+y^2)^2\cdot 2y - 2xy\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\{(x^2+y^2)2y - 8x^2y\}}{(x^2+y^2)^4} = \frac{2y(x^2+y^2-4x^2)}{(x^2+y^2)^3}$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3} \;\cdots\cdots (6)$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot(-1) + y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \;\cdots\cdots (7)$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

$$= \frac{(x^2+y^2)^2\cdot 2y - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\{(x^2+y^2)\cdot 2y - (y^2-x^2)\cdot 4y\}}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2-2y^2+2x^2)}{(x^2+y^2)^3} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3} \;\cdots\cdots (8)$$

(2 + (4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$

and (6) + (8) gives, $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{2y(y^2-3x^2)}{(x^2+y^2)^3} + \dfrac{2y(3x^2-y^2)}{(x^2+y^2)^3}$

$$= \frac{2y^3 - 6x^2y + 6x^2y - 2y^3}{(x^2+y^2)^3} = 0$$

$\therefore$ u and v both satisfy Laplace's equation.

From (1) and (7) we have $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$

and from (3) and (5) we have $\dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$

$\therefore$ u + iv is not an analytic function of z. (Showel)

Example–52. Show that $u = \dfrac{y}{x^2+y^2}$ is harmonic and finds harmonic conjugate v and f(z) = u + iv if f(z) is analytic.

[DUH–1976]

Solution : Given that $u = \dfrac{y}{x^2+y^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)^2\cdot(-2y) - (-2xy)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{-2y(x^4+2x^2y^2+y^4) + 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{-2x^4y - 4x^2y^3 - 2y^5 + 8x^4y + 8x^2y^3}{(x^2+y^2)^4}$$

$$= \frac{6x^4y + 4x^2y^3 - 2y^5}{(x^2+y^2)^4} \;\cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)^2\cdot(-2y) - (x^2-y^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{-2x^4y - 4x^2y^3 - 2y^5 - 4x^4y + 4y^5}{(x^2+y^2)^4}$$

$$= \frac{-6x^4y - 4x^2y^3 + 2y^5}{(x^2+y^2)^4} \;\cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{6x^4y + 4x^2y^3 - 2y^5}{(x^2+y^2)^4} - \frac{6x^4y + 4x^2y^3 - 2y^5}{(x^2+y^2)^4}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow$ u satisfied Laplace equation and hence u is harmonic.

To find v, we put $x = z$ and $y = 0$ in (1) and (2)

$$\phi_1(z, 0) = 0 \text{ and } \phi_2(z, 0) = \frac{z^2 - 0}{(z^2+0)^2} = \frac{1}{z^2}$$

By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= 0 - i\frac{1}{z^2}$$

$$\Rightarrow f(z) = \frac{i}{z} + c$$

$$\Rightarrow u + iv = \frac{i}{x+iy} + c$$
$$= \frac{i(x-iy)}{(x+iy)(x-iy)} + c$$
$$= \frac{ix+y}{x^2+y^2} + c_1 + ic_2, \text{ where } c = c_1 + ic_2$$

Equating imaginary part we get,

$$v = \frac{x}{x^2+y^2} + c_2. \quad \text{(Ans)}$$

Example–53. Find the harmonic conjugate of the function $u = e^{x^2-y^2} \cos 2xy$ and the corresponding analytic function $f(z) = u + iv$. **[DUH–1978, CUH–2004, JUH–19..]**

Solution : Given that $u = e^{x^2-y^2} \cos 2xy$

$$\therefore \frac{\partial u}{\partial x} = 2xe^{x^2-y^2} \cos 2xy + e^{x^2-y^2}(-\sin 2xy) \cdot 2y$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy) = \phi_1(x, y), \text{ say} \ldots\ldots (1)$$

$$\frac{\partial u}{\partial y} = -2ye^{x^2-y^2} \cos 2xy + e^{x^2-y^2} \cdot (-\sin 2xy) \cdot 2x$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2e^{x^2-y^2}(y \cos 2xy + x \sin 2xy) = \phi_2(x, y), \text{ say} \ldots\ldots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 4xe^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$$
$$+ 2e^{x^2-y^2}(\cos 2xy - 2xy \sin 2xy - 2y^2 \cos 2xy)$$
$$= 2e^{x^2-y^2}(2x^2 \cos 2xy - 2xy \sin 2xy$$
$$+ \cos 2xy - 2xy \sin 2xy - 2y^2 \cos 2xy)$$

$$\frac{\partial^2 u}{\partial y^2} = 4ye^{x^2-y^2}(y \cos 2xy + x \sin 2xy)$$
$$- 2e^{x^2-y^2}(\cos 2xy - 2xy \sin 2xy + 2x^2 \cos 2xy)$$
$$= 2e^{x^2-y^2}(2y^2 \cos 2xy + 2xy \sin 2xy - \cos 2xy$$
$$+ 2xy \sin 2xy - 2x^2 \cos 2xy)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2e^{x^2-y^2}(2x^2 \cos 2xy - 2xy \sin 2xy$$
$$+ \cos 2xy - 2xy \sin 2xy - 2y^2 \cos 2xy + 2y^2 \cos 2xy$$
$$+ 2xy \sin 2xy - \cos 2xy + 2xy \sin 2xy - 2x^2 \cos 2xy)$$
$$= 2e^{x^2-y^2} \times 0 = 0$$

$\Rightarrow$ u is harmonic.

Now putting $x = z$ and $y = 0$ in (1) and (2) we get

$$\phi_1(z, 0) = 2e^{z^2}(z \cdot 1 - 0) = 2ze^{z^2}$$

and $\phi_2(z, 0) = -2z^2(0 + 0) = 0$

By Milne's theorem we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0) = 2ze^{z^2}$$

$$\Rightarrow f(z) = \int 2ze^{z^2}\, dz$$
$$= \int e^{z^2} \cdot d(z^2) = e^{z^2} + c$$

$$\Rightarrow u + iv = e^{(x+iy)^2} + c = e^{x^2-y^2} \cdot e^{i2xy} + c$$
$$= e^{x^2-y^2}(\cos 2xy + i \sin 2xy) + c_1 + ic_2, \; c = c_1 + ic_2$$

$$\left.\begin{aligned} &\Rightarrow v = e^{x^2-y^2} \sin 2xy + c_2 \\ &\text{and } f(z) = u + iv = e^{z^2} + c \end{aligned}\right\}. \quad \text{(Ans)}$$

Example–54. Show that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic and find the harmonic conjugate v. Also find $f(z) = u + iv$ if $f(z)$ is analytic. **[DUMP-199…]**

Solution : Given that $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\therefore \frac{\partial u}{\partial x} = 2x - 2y - 2 = \phi_1(x, y), \text{ say} \;\ldots\ldots\; (1)$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 = \phi_2(x, y), \text{ say} \;\ldots\ldots\; (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \;\ldots\ldots\; (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \;\ldots\ldots\; (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$

$\Rightarrow$ u is harmonic.

For finding v, 1st method :

By Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 2y - 2 \;\ldots\ldots\; (5)$$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y - 2x + 3 \;\ldots\ldots\; (6)$

From (5) $v = \int (2x - 2y - 2)\, dy$

$$v = 2xy - y^2 - 2y + F(x) \;\ldots\ldots\; (7)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 2y + F'(x)$$

$$\Rightarrow 2y + 2x - 3 = 2y + F'(x)$$

$$\Rightarrow F'(x) = 2x - 3$$

$$\Rightarrow F(x) = \int (2x - 3)\, dx = x^2 - 3x + c$$

Putting this value in (7) we get,

$$v = 2xy - y^2 - 2y + x^2 - 3x + c$$

$$\Rightarrow v = x^2 - y^2 + 2xy - 3x - 2y + c. \quad \text{(Ans)}$$

2nd method :

$$v = v(x, y)$$

$$\Rightarrow dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\Rightarrow dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy; \text{ by C–R equations}$$

$$\Rightarrow dv = (2y + 2x - 3)\, dx + (2x - 2y - 2)\, dy \;\ldots\ldots\; (8)$$

$$= M\, dx + N\, dy$$

where $M = 2y + 2x - 3$ and $N = 2x - 2y - 2$

$$\therefore \frac{\partial M}{\partial y} = 2 \text{ and } \frac{\partial N}{\partial x} = 2 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence equation (8) is an exact differential equation.

$$\therefore v = \int (2y + 2x - 3)\, dx + \int (-2y - 2)\, dy + c,$$

[In 1st integral y is constant]

$$\Rightarrow v = 2xy + x^2 - 3x - y^2 - 2y + c$$

$$\Rightarrow v = x^2 - y^2 + 2xy - 3x - 2y + c. \quad \text{(Ans)}$$

3rd method :

By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 2z - 0 - 2 - i(-0 - 2z + 3)$$

$$= 2z - 2 + 2iz - 3i$$

$$= (2 + 2i)\, z - 2 - 3i$$

$$\Rightarrow f(z) = \int \{(2 + 2i)\, z - 2 - 3i\}\, dz$$

$$= (2 + 2i)\frac{z^2}{2} - 2z - 3iz + c \;\ldots\ldots\; (9)$$

$$= (1 + i)\,(x + iy)^2 - 2(x + iy) - 3i(x + iy) + c$$

$$\Rightarrow u + iv = (1 + i)\,(x^2 - y^2 + 2ixy) - 2(x + iy) - 3i(x + iy) + c_1 + ic_2$$

where $c = c_1 + ic_2$

Equating imaginary parts

$$v = 2xy + x^2 - y^2 - 2y - 3x + c_2$$

$$\Rightarrow v = x^2 - y^2 + 2xy - 3x - 2y + c_2. \quad \text{(Ans)}$$

Third Part : Form (9), $f(z) = (1+i)z^2 - 2z - 3iz + c$

$$= (1+i)z^2 - (2+3i)z + c$$

OR, $f(z) = u + iv$

$$= x^2 - y^2 - 2xy - 2x + 3y + i(x^2 - y^2 + 2xy - 3x - 2y + c)$$

$$= (x^2 - y^2 + 2ixy) + i(x^2 - y^2 + 2ixy) - 2(x+iy) - 3i(x+iy) + c$$

$$= (x+iy)^2 + i(x+iy)^2 - 2(x+iy) - 3i(x+iy) + c$$

$$= z^2 + iz^2 - 2z - 3iz + c, \text{ where } z = x + iy$$

$$= (1+i)z^2 - (2+3i)z + c. \quad \text{(Ans)}$$

Example–55. Prove that $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic and find its harmonic conjugate. **[DUH–2000, RUH–2000]**

Solution : Give that $u = e^{-2xy}\sin(x^2 - y^2)$

$$\therefore \frac{\partial u}{\partial x} = -2ye^{-2xy}\sin(x^2 - y^2) + e^{-2xy}\cdot 2x \cdot \cos(x^2 - y^2)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2e^{-2xy}[-y\sin(x^2 - y^2) + x\cos(x^2 - y^2)] = \phi_1(x, y), \text{ say} \ldots (1)$$

$$\frac{\partial u}{\partial y} = -2xe^{-2xy}\sin(x^2 - y^2) - 2ye^{-2xy}\cos(x^2 - y^2)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2e^{-2xy}[x\sin(x^2 - y^2) + y\cos(x^2 - y^2)] = \phi_2(x, y), \text{ say} \ldots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = -4ye^{-2xy}[-y\sin(x^2 - y^2) + x\cos(x^2 - y^2)]$$

$$+ 2e^{-2xy}[-2xy\sin(x^2 - y^2) + \cos(x^2 - y^2) - 2x^2\sin(x^2 - y^2)]$$

$$= 4e^{-2xy}[y^2\sin(x^2 - y^2) - xy\cos(x^2 - y^2) - xy\cos(x^2 - y^2)$$

$$+ \frac{1}{2}\cos(x^2 - y^2) - x^2\sin(x^2 - y^2)] \ldots\ldots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = 4xe^{-2xy}[x\sin(x^2 - y^2) + y\cos(x^2 - y^2)]$$

$$- 2e^{-2xy}[-2xy\cos(x^2 - y^2) + \cos(x^2 - y^2) + 2y^2\sin(x^2 - y^2)]$$

$$= 4e^{-2xy}[x^2\sin(x^2 - y^2) + xy\cos(x^2 - y^2) + xy\cos(x^2 - y^2)$$

$$- \frac{1}{2}\cos(x^2 - y^2) - y^2\sin(x^2 - y^2)] \ldots\ldots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic.

To find v, the harmonic conjugate of u, we put $x = z$ and $y = 0$ in (1) and (2)

$\therefore \phi_1(z, 0) = 2(0 + z\cos z^2) = 2z\cos z^2$

and $\phi_2(z, 0) = -2(z\sin z^2 + 0) = -2z\sin z^2$

By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 2z\cos z^2 + i2z\sin z^2$$

$$= 2z(\cos z^2 + i\sin z^2) = 2ze^{iz^2}$$

$$\Rightarrow f(z) = \int 2ze^{iz^2}\,dz \qquad \text{Putting } iz^2 = t$$

$$= \int e^t \cdot \frac{1}{i}\,dt \qquad \Rightarrow 2iz\,dz = dt$$

$$= \frac{1}{i}e^t + c$$

$$= -ie^{iz^2} + c$$

$$= -ie^{i(x+iy)^2} + c$$

$$= -ie^{-2xy}\cdot e^{i(x^2-y^2)} + c$$

$$= -ie^{-2xy}[\cos(x^2 - y^2) + i\sin(x^2 - y^2)] + c$$

$$\Rightarrow u + iv = e^{-2xy}[-i\cos(x^2 - y^2) + \sin(x^2 - y^2)] + c_1 + ic_2$$

where $c = c_1 + ic_2$

Equating imaginary parts

$$v = -e^{-2xy}\cos(x^2 - y^2) + c_2. \quad \text{(Ans)}$$

Example–56. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region R and $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find f(z) subject to the condition $f\left(\frac{\pi}{2}\right) = \frac{1}{2}(3 - i)$. **[CUH–2003]**

Solution : Give that $u + iv = f(z) \ldots\ldots (1)$

and $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x} \ldots\ldots (2)$

$$\therefore (u - v) + i(u + v) = u + i^2v + iu + iv$$

$$= (u + iv) + i(u + iv)$$

$$= f(z) + if(z), \quad [\text{by } (1)]$$

$$= (1 + i)f(z) \ldots\ldots (3)$$

Let $u - v = U$, $u + v = V$ and $(1 + i)\, f(z) = F(z)$. Then from (3) we have

$U + iV = F(z)$ ...... (4) is an analytic function.

and $U = u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$

$$\therefore \frac{\partial U}{\partial x} = \frac{(\cosh y - \cos x)(\sin x + \cos x) - (e^y - \cos x + \sin x)\cdot \sin x}{(\cosh y - \cos x)^2}$$

$$= \frac{\sin x \cosh y - \sin x \cos x + \cos x \cosh y - \cos^2 x - e^y \sin x + \sin x \cos x - \sin^2 x}{(\cosh y - \cos x)^2}$$

$$= \frac{-1 + \sin x \cosh y + \cos x \cosh y - e^y \sin x}{(\cosh y - \cos x)^2} = \phi_1(x, y), \text{ say } \text{...... (5)}$$

$$\frac{\partial U}{\partial y} = \frac{(\cosh y - \cos x)(e^y) - (e^y - \cos x + \sin x)\sinh y}{(\cosh y - \cos x)^2}$$

$$= \frac{e^y \cosh y - e^y \cos x - e^y \sinh y + \cos x \sinh y - \sin x \sinh y}{(\cosh y - \cos x)^2}$$

$$= \phi_2(x, y), \text{ say } \text{...... (6)}$$

Putting $x = z$ and $y = 0$ in (5) and (6)

$$\phi_1(z, 0) = \frac{-1 + \sin z + \cos z - \sin z}{(1 - \cos z)^2} = \frac{-1}{1 - \cos z}$$

and $\phi_2(z, 0) = \dfrac{1 - \cos z - 0 + 0 - 0}{(1 - \cos z)^2} = \dfrac{1}{1 - \cos z}$

By Milne's method we have

$F'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$

$$= \frac{-1}{1 - \cos z} - i\frac{1}{1 - \cos z} = -(1 + i)\frac{1}{1 - \cos z} = \frac{-(1 + i)}{2 \sin^2 \frac{z}{2}}$$

$$= \frac{-(1 + i)}{2} \operatorname{cosec}^2 \frac{z}{2}$$

$$\Rightarrow F(z) = -\frac{1}{2}(1 + i) \int \operatorname{cosec}^2 \frac{z}{2}\, dz$$

$$= (1 + i) \cot \frac{z}{2} + c$$

$$\Rightarrow (1 + i)\, f(z) = (1 + i) \cot \frac{z}{2} + c$$

$$\Rightarrow f(z) = \cot \frac{z}{2} + A, \text{ where } A = \frac{c}{1 + i} \text{ ...... (7)}$$

By the given condition $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$

$$\Rightarrow \cot \frac{\pi}{4} + A = \frac{3 - i}{2}$$

$$\Rightarrow 1 + A = \frac{3 - i}{2}$$

$$\Rightarrow A = \frac{3 - i}{2} - 1 = \frac{3 - i - 2}{2} = \frac{1 - i}{2}$$

Putting this value in (7) we get,

$$f(z) = \cot \frac{z}{2} + \frac{1}{2}(1 - i). \quad \text{(Ans)}$$

Example–57. If $f(z)$ is analytic function of z, then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$$ **[RUH–1988]**

Also, from it show that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$

[NUH–1997, 2004(Old), 2012, DUH–1984, 1986, 1988,1990, RUH–1973, 1988]

and $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^3 = 9|f(z)|\ |f'(z)|^2$

Solution : $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^p = 4 \dfrac{\partial^2}{\partial z\, \partial \bar{z}} \{|f(z)|^2\}^{p/2}$

$$= 4 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \{f(z)\, f(\bar{z})\}^{p/2}$$

$$= 4 \frac{\partial}{\partial z} \{f(z)\}^{p/2} \cdot \frac{\partial}{\partial \bar{z}} \{f(\bar{z})\}^{p/2}$$

$$= 4 \cdot \frac{p}{2} \cdot \{f(z)\}^{\frac{p}{2} - 1} \cdot f'(z) \cdot \frac{p}{2} \{f(\bar{z})\}^{\frac{p}{2} - 1} \cdot f'(\bar{z})$$

$$= p^2 \{f(z)\, f(\bar{z})|^{\frac{p}{2} - 1} \cdot f'(z)\, f'(\bar{z})$$

$$= p^2 \{|f(z)|^2|^{\frac{p}{2} - 1} \cdot |f'(z)|^2$$

$$= p^2 |f(z)|^{p-2} \cdot |f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2 \text{ ...... (1)}$$

2nd Part : Putting $p = 2$ in (1) we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2^2|f(z)|^{2-2}\,|f'(z)|^2$$
$$= 4|f'(z)|^2. \quad \text{(Showed)}$$

3rd Part : Putting $p = 3$ in (1) we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^3 = 3^2|f(z)|^{3-2}\,|f'(z)|^2$$
$$= 9|f(z)|\;|f'(z)|^2 \quad \text{(Showed)}$$

Example–58. If $f(z) = u + iv$ is analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^7 = 49\,|f(z)|^5\,|f'(z)|^2$$ [RUH-1996]

Solution : $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^7 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\{|f(z)|^2\}^{7/2}$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\{f(z)\,f(\bar{z})\}^{7/2}$$
$$= 4\frac{\partial}{\partial z}\{f(z)\}^{7/2}\cdot\frac{\partial}{\partial \bar{z}}\{f(\bar{z})\}^{7/2}$$
$$= 4\cdot\frac{7}{2}\{f(z)\}^{5/2}\cdot f'(z)\cdot\frac{7}{2}\{f(\bar{z})\}^{5/2}\cdot f'(\bar{z})$$
$$= 49\{f(z)\,f(\bar{z})\}^{5/2}\cdot f'(z)\,f'(\bar{z})$$
$$= 49\{|f(z)|^2\}^{5/2}\cdot|f'(z)|^2$$
$$= 49\,|f(z)|^5\,|f'(z)|^2. \quad \text{(Proved)}$$

Example–59. If $f(z)$ is an analytic function of z, then show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$ [NUH-2006 (Old), DUH-1986, 1990, 1998, 2004, RUH-1973, 1984, 1998]

Solution : $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\,f(z)\,f(\bar{z})$

$$= 4\frac{\partial}{\partial z}\{f(z)\cdot f'(\bar{z})\}$$
$$= 4f'(z)\cdot f'(\bar{z})$$
$$= 4|f'(z)|^2 \quad \text{(Showed)}$$

Example–60. Show that $\psi = \log|f(z)|$ is harmonic in a region R if $f(z)$ is analytic in R and $f(z)\,f'(z) \neq 0$ in R.

Solution : We have $\nabla^2\psi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\log|f(z)|$$
$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\frac{1}{2}\log|f(z)|^2$$
$$= 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\log\{f(z)\cdot f(\bar{z})\}$$
$$= 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\{\log f(z) + \log f(\bar{z})\}$$
$$= 2\frac{\partial}{\partial z}\left\{0+\frac{f'(\bar{z})}{f(\bar{z})}\right\}$$
$$= 2\times 0 = 0$$

Here ψ satisfies Laplace equation.

$\therefore \psi = \log|f(z)|$ is harmonic.

Example–61. If $f(z)$ is analytic function such that $f'(z) \neq 0$, then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

If $|f'(z)|$ is the product of a function of x and a function of y, then show that $f'(z) = \exp(\alpha z^2 + \beta z + \gamma)$, where α is real and β and γ are complex constants. [JUH-1989]

Solution : 1st Part :

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|$$
$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\log|f'(z)|$$
$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\frac{1}{2}\log|f'(z)|^2$$
$$= 2\frac{\partial^2}{\partial z\,\partial \bar{z}}\log\{f'(z)\cdot f'(\bar{z})\}$$
$$= 2\frac{\partial^2}{\partial z\,\partial \bar{z}}\{\log f'(z) + \log f'(\bar{z})\}$$
$$= 2\frac{\partial}{\partial z}\left\{0+\frac{f''(\bar{z})}{f'(\bar{z})}\right\} = 2\times 0 = 0 \quad \text{(Proved)}$$

2nd Part : Given $|f'(z)|$ is the product of a function x and a function of y. Let $|f'(z)| = P(x) \cdot Q(y)$. Then from the first part,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{P(x)\cdot Q(y)\} = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\{\log P(x) + \log Q(y)\} = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\log P(x) + \frac{\partial^2}{\partial y^2}\log Q(y) = 0$$

$$\Rightarrow \frac{d^2}{dx^2}\log P(x) + \frac{d^2}{dy^2}\log Q(y) = 0,$$ since $P(x)$ and $Q(y)$ are functions of x and y alone respectively.

Let $\frac{d^2}{dx^2}\log P(x) = c$, then $\frac{d^2}{dy^2}\log Q(y) = -c$

$\Rightarrow \frac{d}{dx}\log P(x) = cx + d$	$\Rightarrow \frac{d}{dy}\log Q(y) = -cy + d'$
$\Rightarrow \log P(x) = \frac{1}{2}cx^2 + dx + e$	$\Rightarrow \log Q(y) = -\frac{1}{2}cy^2 + d'y + e'$
$\Rightarrow P(x) = \exp\left(\frac{1}{2}cx^2 + dx + e\right)$	$\Rightarrow Q(y) = \exp\left(-\frac{1}{2}cy^2 + d'y + e'\right)$

where d, e, d', e' are integrating real constants.

Now, $|f'(z)| = P(x)\cdot Q(y)$

$$= \exp\left(\frac{1}{2}cx^2 + dx + e\right)\cdot\exp\left(-\frac{1}{2}cy^2 + d'y + e'\right)$$

$$= e^{\frac{1}{2}cx^2 + dx + e}\cdot e^{-\frac{1}{2}cy^2 + d'y + e'}$$

$$= e^{\frac{1}{2}c(x^2-y^2) + dx + d'y + (e+e')} \quad \text{...... (1)}$$

Now putting $\beta = a_1 + ib_1,\ \gamma = c_1 + id_1,\ z = x + iy$ we have

$$|\exp(\alpha z^2 + \beta z + \gamma)| = |\exp\{\alpha(x+iy)^2 + (a_1 + ib_1)(x+iy) + c_1 + id_1\}|$$

$$= \exp\{\alpha(x^2 - y^2 + 2ixy) + (a_1x - b_1y) + i(b_1x + a_1y) + c_1 + id_1\}|$$

$$= \left|e^{\alpha(x^2-y^2) + (a_1x - b_1y) + c_1 + i(2\alpha xy + b_1x + a_1y + d_1)}\right|$$

$$= \left|e^{\alpha(x^2-y^2) + (a_1x - b_1y) + c_1}\right|\left|e^{i(2\alpha xy + b_1x + a_1y + d_1)}\right|$$

$$= \left|e^{\alpha(x^2-y^2) + a_1x - b_1y + c_1}\right|\cdot 1; \quad \because |e^{i\theta}| = 1$$

$$= \left|e^{\alpha(x^2-y^2) + a_1x - b_1y + c_1}\right| \quad \text{...... (1)}$$

From (1) and (2) we have if $f'(z) = \exp(\alpha z^2 + \beta z + \gamma)$, then

$$e^{\frac{1}{2}c(x^2-y^2) + dx + d'y + (e+e')} = e^{\alpha(x^2-y^2) + a_1x - b_1y + c_1}$$

$$\Rightarrow \frac{1}{2}c(x^2-y^2) + dx + d'y + (e+e') = \alpha(x^2-y^2) + a_1x - b_1y + c_1$$

Equating the coefficients of like terms

$$\frac{1}{2}c = \alpha,\ d = a_1,\ d' = -b_1,\ e + e' = c_1$$

$$\Rightarrow \alpha = \frac{1}{2}c = \text{real, since c is real.}$$

Thus, $f'(z) = \exp(\alpha z^2 + \beta z + \gamma)$ (Showed)

Example–62. Show that the function $\psi = ln[(x-1)^2 + (y-2)^2]$ is a harmonic other than the point (1, 2) and find a function ϕ such that $\phi + i\psi$ is analytic. [দেখাও যে (1, 2) বিন্দু ব্যতীত $\psi = ln[(x-1)^2 + (y-2)^2]$ ফাংশনটি harmonic এবং $\phi + i\psi$ analytic হলে ϕ ফাংশনটি নির্ণয় কর।]

Solution : Given [দেওয়া আছে] $\psi = ln[(x-1)^2 + (y-2)^2$

$$\therefore \frac{\partial\psi}{\partial x} = \frac{2(x-1)}{(x-1)^2 + (y-2)^2} = \psi_1(x, y), \text{ say [ধরি] ...... (1)}$$

$$\frac{\partial\psi}{\partial y} = \frac{2(y-2)}{(x-1)^2 + (y-2)^2} = \psi_2(x, y), \text{ say [ধরি] ...... (2)}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\{(x-1)^2 + (y-2)^2\}\cdot 2 - 2(x-1)\cdot 2(x-1)}{\{(x-1)^2 + (y-2)^2\}^2}$$

$$= \frac{2(y-2)^2 - 2(x-1)^2}{\{(x-1)^2 + (y-2)^2\}^2} \text{ ...... (3)}$$

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\{(x-1)^2 + (y-2)^2\}\cdot 2 - 2(y-2)\cdot 2(y-2)}{\{(x-1)^2 + (y-2)^2\}^2}$$

$$= \frac{2(x-1)^2 - 2(y-2)^2}{\{(x-1)^2 + (y-2)^2\}^2} \text{ ...... (4)}$$

(3) + (4) gives [(3) + (4) করে পাই]

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{2(y-2)^2 - 2(x-1)^2}{\{(x-1)^2 + (y-2)^2\}^2} + \frac{2(x-1)^2 - 2(y-2)^2}{\{(x-1)^2 + (y-2)^2\}^2}$$

$$= \frac{2(y-2)^2 - 2(x-1)^2 + 2(x-1)^2 - 2(y-2)^2}{\{(x-1)^2 + (y-2)^2\}^2}$$

$$= 0$$

Since ψ satisfy Laplace equation, so ψ is harmonic. Let ϕ is the harmonic conjugate of ψ, so that $\psi + i\phi$ is analytic.

[যেহেতু ψ ল্যাপলাস সমীকরণ সিদ্ধ করে, সুতরাং ψ হারমোনিক। ধরি ϕ হলো ψ এর হারমোনিক অনুবন্ধী, সুতরাং $\psi + i\phi$ বৈশ্লেষিক।]

Putting $x = z$ and $y = 0$ in (1) and (2) we get [(1) ও (2) এ $x = z$ এবং $y = 0$ বসাইয়া পাই]

$$\psi_1(z, 0) = \frac{2(z-1)}{(z-1)^2 + 2^2} \text{ and [এবং] } \psi_2(z, 0) = \frac{-4}{(z-1)^2 + 2^2}$$

By Milne's method we get [মিলনির পদ্ধতি দ্বারা পাই]

$$f'(z) = \psi_1(z-0) - i\psi_2(z, 0)$$

$$= \frac{2(z-1)}{(z-1)^2 + 2^2} - i\frac{-4}{(z-1)^2 + 2^2}$$

$$\Rightarrow f(z) = \int\left[\frac{2(z-1)}{(z-1)^2 + 2^2} + i4\frac{1}{(z-1)^2 + 2^2}\right]dz$$

$$\Rightarrow \psi + i\phi = \int\frac{2(z-1)}{(z-1)^2 + 2^2}d(z-1) + i4\int\frac{1}{(z-1)^2 + 2^2}dz$$

$$= ln\{(z-1)^2 + 2^2\} + i4 \cdot \frac{1}{2}\tan^{-1}\left(\frac{z-1}{2}\right) + c$$

$$= ln\{(z-1)^2 + 2^2\} + i2\tan^{-1}\left(\frac{z-1}{2}\right) + c$$

$$= 2\left[\frac{1}{2}ln\{(z-1)^2 + 2^2\} + i\tan^{-1}\left(\frac{z-1}{2}\right)\right] + c$$

$$= 2\,ln\{2 + i(z-1)\} + c$$

$$= 2\,ln\{2 + i(x + iy - 1)\} + c$$

$$= 2\,ln\{2 - y + i(x-1)\} + c$$

$$= 2\left[\frac{1}{2}ln\{(2-y)^2 + (x-1)^2\} + i\tan^{-1}\left(\frac{x-1}{2-y}\right)\right] + c$$

$$= ln\{(x-1)^2 + (y-2)^2\} + 2i\tan^{-1}\left(\frac{x-1}{2-y}\right) + c_1 + ic_2$$

[where [যেখানে] $c = c_1 + ic_2$]

Equating imaginary parts we get [কাল্পনিক অংশ সমীকৃত করে পাই]

$$\phi = 2\tan^{-1}\left(\frac{x-1}{2-y}\right) + c_2. \text{ Ans}$$

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Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. Define analytic function at a point. **[NUH-2012]**

Ans : A complex function f(z) is said to be analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighbourhood of z_0.

2. Define entire function.

Ans : A complex function f(z) is said to be entire if it is analytic in the whole complex plane.

3. What are Cauchy-Riemann partial differential equations?

Ans : If $w = f(z) = u(x, y) + iv(x, y)$ then the Cauchy-Riemann partial differential equations are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

4. Write the polar form of Cauchy-Riemann equations– **[NUH-2012]**

Ans : The polar of Cauchy-Rienann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

5. Write the polar form of Laplace equation.

OR, Write down the laplace equation in polar form of complex number. **[NUH-2013]**

Ans : The polar form of Laplace equation is

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

6. Can every complex function f(z) be written as

$$f(z) = u(x, y) + iv(x, y)?$$

Ans : Yes, every complex function f(z) be always written as

$$f(z) = u(x, y) + iv(x, y).$$

7. When a complex valued function is continuous at a point?

Ans : A complex valued function f(z) is said to be continuous at a point z_0 if for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$, whenever $|z - z_0| < \delta$.

8. When a maximum and minimum value of $|f(z)|$ exist?

Ans : If a function f(z) is continuous on a bounded and closed set $S \subset C$, then maximum and minimum value of $|f(z)|$ exist on S.

9. When a function f(z) is differentiable at a point?

Ans : A function f(z) is said to be differentiable at a point z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and finite.

10. When a function is differentiable on a set?

Ans : A function is differentiable on a set if it is differentiable at each point of the set.

11. Does every differentiable function is continuous?

Ans : Yes, every differentiable function is continuous.

12. Does every continuous function is differentiable?

Ans : No, every continuous function need not be differentiable.

13. Define harmonic function?

Ans : Any real valued function of x and y is said to be harmonic if it has continuous partial derivatives of the first and second order and satisfy the Laplace equation.

14. Define harmonic conjugate. **[NUH-2012]**

Ans : The function v is said to be a harmonic conjugate of u if u and v are harmonic and u, v satisfy the Cauchy-Riemann equations.

15. When is a function analytic? **[NUH-2013]**

Ans : When the derivative of the complex function f(z) exists not only at some point z_0 but also at each point z in some neighbourhood of z_0, then the function f(z) is said to be analytic.

16. Give an example of a harmonic function. **[NUH-2013]**

Ans : $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is an example of a harmonic function.

EXERCISE–2

Part–A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. Define a multivalued function with example.
2. What is the meaning of the symbol $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$?
3. When a real function of complex variable is analytic?
4. Why the real valued function Re(z), Im(z), $|z|$ and $|z|^2$ are not analytic?
5. Express the right hand side of $f(z) = x^2 - y^2 - 2y + i2x + i2xy$ interms of z.

Part–B : Short Questions (সংক্ষিপ্ত প্রশ্ন)

1. State and prove the sufficient condition for $f(z) = u(x, y) + iv(x, y)$ to be analytic. **[NUH-1993, 1998, 2001, 2006, DUH-1999, 2001, 2003, CUH-2002, 2004]**

Ans : See theorem–5.

2. Obtain a set of conditions sufficient for a complex function to be analytic in a domain. **[CUH-2001]**

Ans : See theorem–5.

3. State and derive the necessary condition for $f(z) = u(x, y) + iv(x, y)$ to be analytic in a region R. **[CUH-2003]**

Ans : See theorem–4.

4. Define limit, continuity and uniform continuity. **[CUH-2004]**

Ans : See the definitions of art–2.2 and 2.3.

5. If f(z) is analytic in a region R, then f(z) is constant if Re f(z) is constant. **[RUH-2001]**

Ans : See solved example–16.

6. State and prove the necessary condition for a function to be analytic. **[NUH-2005(Old), 2007, DUH-1998, 2003, 2005, RUH-1997, 2002, 2004]**

Ans : See theorem–4.

7. Let $W = f(z) = u(x, y) + iv(x, y)$ be defined in a region R. If in R the Cauchy Riemann equations are satisfied and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous, then f(z) is analytic in R. **[RUH-2001]**

Ans : See theorem–5.

8. State and prove the sufficient conditions for a function to be analytic in a region. **[NUH-2006, RUH-1995]**

Ans : See theorem–5.

9. If $\lim_{z \to z_0} f(z)$ exists, then it must be unique, prove it. **[RUH-1994, CUH-2002]**

Ans : See theorem–1.

10. Show that if the function $f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z = x + iy$ then the four partial derivatives u_x, v_x, u_y, v_y should exist and satisfy the equations $u_x = v_y, u_y = -v_x$. **[NUH-2004(Old)]**

Ans : See theorem–4.

11. Does $\lim_{z \to 0} \frac{\bar{z}}{z}$ exist? [NUH-1996]

Ans : See solved example-8.

12. Prove that $f(z) = \ln z$ has a branch point at $z = 0$. [NUH-2007]

Ans : See solved example-2.

Part–C (Broad Questions) (বড় প্রশ্ন)

1. State and prove the necessary and sufficient conditions for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic. [CUH-2004]

Ans : See theorem-4 and 5.

2. Prove that differentiability of $f(z)$ implies continuity but the converse is not true in general.

Ans : See theorem-3 and solved example-21.

3. Prove that an analytic function with constant modulus is constant. [RUH-2006]

Ans : See solve example-26.

4. If $f(z) = u + iv$ is analytic in a region R, prove that u and v are harmonic in R.

Ans : See Theorem of art-2.7.

5. If $f(z)$ is analytic at a point z_0, then it must be continuous at z_0. Give an example to show that the converse of this theorem is not necessarily true. [RUH-2001, 2004]

Ans : See theorem-3 and Solved example-21.

6. Prove that a function which is analytic at a point is continuous there but the converse is not necessarily true. [RUH-1996]

Ans : See theorem-3 and solve example-21.

7. If $w = f(z) = u + iv$ is an analytic function, show in polar form the Cauchy Riemann equations are
$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. [NUH-1997, 2003, 2004, 2008, 2010, DUH-2004, RUH-2006]

Ans : See art-2.6.

8. If $w = f(z) = u + iv$ is an analytic function, show in polar form the Cauchy-Riemann equations are
$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ [NUH-1997, 2003, DUH-2004, RUH-2006]

Ans : See art 2.6

9. Prove that the real and imaginary parts of an analytic function of a complex variable when expressed in polar form satisfy the equation
$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$ [RUH-2000, 2006]

Ans : See theorem of art 2.8

10. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and ϕ is any function of x and y with differential coefficient of first order, then show that
$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = \left\{\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right\} |f'(z)|^2.$$ [RUH-2001]

Ans : See solved example-29

11. If p and q are functions of x and y satisfying Laplace's equation, then show that $(u + iv)$ is analytic where $u = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}$ and $v = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$. [RUH-1999]

Ans : See solved example-20

12. What is mean by saying that a function of a complex variable $f(z)$ is analytic at z_0? Show that validity of the Cauchy-Riemann equation is necessary condition for analyticity, but not a sufficient condition. State a set of conditions sufficient for analyticity. [NUH-1995]

Ans : See definition, theorem-4, 5 and Notes of art 2.5

13. What is meant by the analyticity of a complex function at a point? Prove that the analyticity of a function at a point implies the continuity of the function at that points. Give an example to show that the converse is not necessarily true. [NUH-1996]

Ans : See definition of art 2.5, theorem-3 of art 2.4 and solved example-21

14. Find the necessary and sufficient conditions for $f(z) = u + iv$ to be regular, where u and v both real. [NUH-1998]

Ans : See theorem-4 and 5 of art 2.5.

15. Show that the real and imaginary parts of an analytic function are harmonic functions. [NUH-2001]

Ans : See theorem-6 of art 2.7

16. Find the polar form of Cauchy-Riemann equations. [CUH-2003]

Ans : See art 2.6

CHAPTER–4
SINGULARITIES, RESIDUE AND SOME THEOREMS

4.1. Zero or root of an analytic function : **[NUH–1993]**

A value of z for which the analytic function $f(z) = 0$ is called a zero of f(z).

If $f(z) = (z - z_0)^n g(z)$, where g(z) is analytic, $g(z_0) \neq 0$ and n is a positive integer, then $z = z_0$ is called a zero of order n of the function f(z).

Simple zero : If f(z) has a zero of order one at $z = z_0$, then f(z) is said to have a simple zero at $z = z_0$.

Singular point or critical point of an analytic function :
[NUH–1994, 2006 (Old), 2008]

A point at which an analytic function f(z) fails or ceases to be analytic is called a singular point.

4.2. Types of singularities : **[NUH–93, 96, 11, DUH–05]**

Isolated and non-isolated singularity : Let $z = z_0$ be a singularity of f(z). If there is no other singularity with in a small circle surrounding the point $z = z_0$, then this point is called an isolated singularity. If the singularity $z = z_0$ is not an isolated singularity then it is called a non-isolated singularity.

Ordinary point : If $z = z_0$ is not a singular point of f(z) and there can be found a small circle surrounding the point $z = z_0$ which encloses no singular point, then $z = z_0$ is called an ordinary point of f(z).

Pole : If there exists a positive integer n such that

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$$

then $z = z_0$ is called a pole of order n. **[NUH–02, 03, 06, 08]**

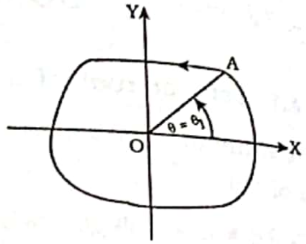
If $n = 1$ then z_0 is called a simple pole.

If $n = 2$ then z_0 is called a double pole and so on.

Branch points and branch lines [NUH–2004, 2007]

To define branch point and branch line we discuss the situation by the following example.

Let $w = z^{1/2}$. Suppose that z is allowed to make a complete circuit around the origin (z = 0) in the counterclockwise direction.



In polar form let $z = re^{i\theta}$.

Then $w = (re^{i\theta})^{1/2} = \sqrt{r}\, e^{i\theta/2}$.

If in this case $\theta = \theta_1$ then $w = \sqrt{r}\, e^{i\theta_1/2}$.

After a complete circuit back to A, $\theta = \theta_1 + 2\pi$ and

$$w = \sqrt{r}\, e^{i(\theta_1+2\pi)/2} = \sqrt{r}\, e^{i(\pi+\theta_1/2)}$$
$$= \sqrt{r}\left[\cos\left(\pi + \frac{\theta_1}{2}\right) + i\sin\left(\pi + \frac{\theta_1}{2}\right)\right]$$
$$= \sqrt{r}\left[-\cos\frac{\theta_1}{2} - i\sin\frac{\theta_1}{2}\right]$$
$$= -\sqrt{r}\, e^{i\theta_1/2}$$

Again, after a second complete circuit back to A, we have $\theta = \theta_1 + 4\pi$ and

$$w = \sqrt{r}\, e^{i(\theta_1+4\pi)/2} = \sqrt{r}\, e^{i(2\pi+\theta_1/2)}$$
$$= \sqrt{r}\left[\cos\left(2\pi + \frac{\theta_1}{2}\right) + i\sin\left(2\pi + \frac{\theta_1}{2}\right)\right]$$
$$= \sqrt{r}\left[\cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2}\right]$$
$$= \sqrt{r}\, e^{i\theta_1/2}$$

Thus we see that in the first complete circuit we do not obtain the same value of w but after second complete circuit we obtain the same value of w with which we started.

The above situation can be stated as, if $0 \le \theta \le 2\pi$ we are one branch of the multiple valued function $w = z^{1/2}$, while if $2\pi \le \theta \le 4\pi$ we are on the other branch of the function.

Again, let $w = f(z) = z^{1/2}$

$$\Rightarrow u(r, \theta) + iv(r, \theta) = \sqrt{r}\, e^{i\theta/2} = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \quad \text{...... (1)}$$

$$\Rightarrow u(r, \theta) = \sqrt{r}\cos\frac{\theta}{2} \text{ and } v(r, \theta) = \sqrt{r}\sin\frac{\theta}{2} \quad \text{...... (2)}$$

$$\Rightarrow \text{In short, } u = \sqrt{r}\cos\frac{\theta}{2} \text{ and } v = \sqrt{r}\sin\frac{\theta}{2}$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} \text{ and } \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}$$

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2} \text{ and } \frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2}$$

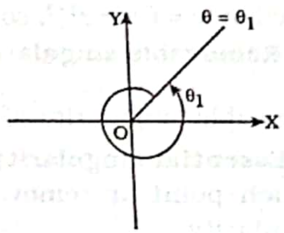
$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} = \frac{1}{r}\cdot\frac{\sqrt{r}}{2}\cos\frac{\theta}{2} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and $$\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2} = \frac{1}{r}\cdot\frac{\sqrt{r}}{2}\sin\frac{\theta}{2} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Thus, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ which are polar form of Cauchy-Riemann equations.

Also, $\frac{d}{dz}(z^{1/2}) = \frac{1}{2\sqrt{z}}$

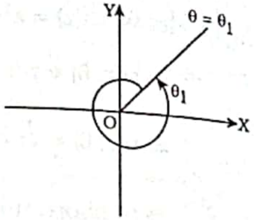
The function $z^{1/2} = \sqrt{r}\, e^{i\theta/2}$, $|z| > 0$, $-\pi < \theta < \pi$ is called the principal branch of the multivaled function $z^{1/2}$.



The function defined in (1) is single valued and continuous on $|z| > 0$, $\theta_1 < \arg z < \theta_1 + 2\pi$.

The function in (1) is not continuous on the line $\theta = \theta_1$ as there are points arbitrarily close to z at which the values of v(r, θ) are nearer to $\sqrt{r}\sin\frac{\theta_1}{2}$ and also points such that the values of v(r, θ) are nearer to $-\sqrt{r}\sin\frac{\theta_1}{2}$. The above discussion also shows that the function in (1) is not only continuous but also analytic. Now we define the following.

Branch line (Branch cut) : A branch line (cut) is a portion of a line or curve which is introduced in order to define the branch of a multivalued function.



Example : The origin and $\theta = \theta_1$ form branch cut for the function

$$w = f(z) = z^{1/2} = \sqrt{r}\, e^{i\theta/2},\ r > 0,\ \theta_1 < \theta < \theta_1 + 2\pi.$$

Branch point : **[NUH–2013]**

A multivalued function f(z) defined in some domain S is said to have a branch point at z_0 if, when z describes an arbitrary small circle about z_0, then for every branch F of f, F(z) does not return to its original value.

[ব্রাঞ্চ পয়েন্ট ঃ একটি ডোমেন S এ বর্ণিত একটি বহুমানী ফাংশন f(z) এর z_0 বিন্দুতে ব্রাঞ্চ পয়েন্ট আছে যদি, f এর প্রত্যেক ব্রাঞ্চF এর জন্য z_0 বিন্দুর চারিদিকে z একটি ইচ্ছাধীন ক্ষুদ্র বৃত্ত বর্ণনা করে যেখানে F(z) তার আদি মান দিতে পারে না।]

Example : Let $w = f(z) = z^{1/2}$.

In polar form $z = re^{i\theta}$, so $w = \sqrt{r}\, e^{i\theta/2}$

After complete circuit, θ becomes $\theta + 2\pi$ and w becomes,

$$w = \sqrt{r}\, e^{i(\theta+2\pi)/2} = -\sqrt{r}\, e^{i\theta/2}.$$

which shows that w has not returned to its original value. Since a complete circuit about z = 0 altered the branch of the function $w = f(z) = z^{1/2}$, so z = 0 is a branch point.

Removable singularity : If $\lim\limits_{z \to z_0} f(z)$ exists then z_0 is called a removable singularity of f(z).

Essential singularity : A singular point which is not a pole, branch point or removable singularity is called an essential singularity.

Singularities at infinity : The function f(z) has a singularity at $z = \infty$ if w = 0 is a singularity of $f\left(\frac{1}{w}\right)$.

N. B. If a function f(z) is single-valued and has a singularity, then this singularity is either a pole or an essential singularity. For this reason a pole is sometimes called a non-essential singularity.

Meromorphic function : A complex function f(z) which has poles as its only singularities in the finite part of the plane is called a meromorphic function.

Entire function : A complex function f(z) which has no singularities in the finite part of the plane is called an entire function.

4.3. Working rule for poles and singularies :

1. (i) If $\lim\limits_{z \to z_0} f(z) = \infty$ then $z = z_0$ is a pole of f(z).

   (ii) If there are only m terms in the negative powers of $z - z_0$ then $z = z_0$ is a pole of order m.

2. If $\lim\limits_{z \to z_0} f(z)$ exists finitely then $z = z_0$ is a removable singularity

3. If $\lim\limits_{z \to z_0} f(z)$ does not exist then $z = z_0$ is an essential singularity.

4. If the principal part of f(z) contains infinite number of terms then $z = z_0$ is an isolated essential singularity

[In the expansion $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n + \sum\limits_{n=1}^{\infty} b_n (z - z_0)^{-n}$, $0 < |z - z_0| < R$, the second term $\sum\limits_{n=1}^{\infty} b_n (z - z_0)^{-n}$ is called the principal part of f(z).]

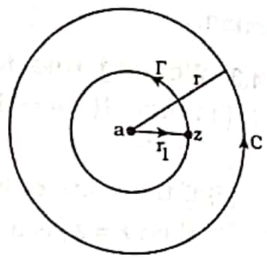
4.4. Taylor's theorem :

Theorem–1. If f(z) is analytic for all values of z inside a circle C with centre at a, then [যদি a কেন্দ্র বিশিষ্ট বৃত্ত C এর ভিতরে z এর সকল মানের জন্য f(z) বৈশ্লেষিক হয় তবে]

$$f(z) = f(a) + (z-a)\, f'(a) + \frac{(z-a)^2}{\underline{|2}} f''(a) + \frac{(z-a)^3}{\underline{|3}} f'''(a) + \cdots$$

[NUH–2005(Old), 2008, 2012(Old), NU(Pre)–2011, DUH–1975, 1983, 1988, 2005, DUM–1988, RUH–1973, 1982, 1984, 1988, CUH–1981]

Proof : Let a be the centre and r be the radius of the circle C. Let z be any point inside C such that $|z-a| = r_1 < r$. Then by Cauchy's integral formula we have [মনে করি C বৃত্তের কেন্দ্র a এবং ব্যাসার্ধ r, ধরি C এর ভিতর z যে কোন বিন্দু যেন $|z-a| = r_1 < r$. তখন কচির যোজিত সূত্র দ্বারা পাই]



$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}dw$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-a)-(z-a)}dw$$

$$= \frac{1}{2\pi i}\oint_C \left(1-\frac{z-a}{w-a}\right)^{-1}\frac{f(w)}{w-a}dw$$

$$= \frac{1}{2\pi i}\oint_C \left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^n + \left(\frac{z-a}{w-a}\right)^{n+1}+\left(\frac{z-a}{w-a}\right)^{n+2}+\cdots\infty\right]\frac{f(w)}{w-a}dw$$

$$= \frac{1}{2\pi i}\oint_C \left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^n\left\{1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots\right\}\right]\frac{f(w)\,dw}{w-a}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-a}dw + \frac{z-a}{2\pi i}\oint_C \frac{f(w)}{(w-a)^2}dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n} + \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)dw}{(w-a)^{n+1}}\cdot\frac{1}{1-\frac{z-a}{w-a}}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-a}dw + \frac{z-a}{2\pi i}\oint_C \frac{f(w)}{(w-a)^2}dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n} + \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n(w-z)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-a}dw + \frac{z-a}{2\pi i}\oint_C \frac{f(w)}{(w-a)^2}dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n} + U_n \;\ldots\ldots (1)$$

Where [যেখানে] $U_n = \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n(w-z)}$

Now [এখন] $|w-z| = |(w-a)-(z-a)| \geq |w-a| - |z-a| \geq r - r_1$

where $|w-a| = r$ for all points on C. [যেখানে C এর উপর সকল বিন্দুর জন্য $|w-a| = r$]

Since f(z) is analytic inside C, so there exist a constant M such that $|f(w)| \leq M$ and $\oint_C |dw| = 2\pi r$ = length of the circle. [যেহেতু C এর ভিতর f(z) বৈশ্লেষিক, সুতরাং একটি ধ্রুবক M থাকবে যেন $|f(w)| \leq M$ এবং $\oint_C |dw| = 2\pi r$ = বৃত্তের দৈর্ঘ্য।]

$$\therefore |U_n| = \left|\frac{(z-a)^n}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-a)^n(w-z)}\right| \leq \frac{1}{2\pi}\frac{|(z-a)^n|\;|f(w)|\oint_C |dw|}{|(w-a)^n|\;|w-z|}$$

$$\leq \frac{1}{2\pi}\frac{r_1^n\, M\, 2\pi r}{r^n\,(r-r_1)} = \frac{M}{1-\frac{r_1}{r}}\left(\frac{r_1}{r}\right)^n \to 0 \text{ when } n\to\infty$$

$$\left[\left|\frac{r_1}{r}\right| < 1 \Rightarrow \left|\frac{r_1}{r}\right|^n \to 0 \text{ as } n\to\infty\right]$$

Thus, when $n\to\infty$ we have [অতএব যখন $n\to\infty$ তখন পাই]

$$f(z) = f(a) + (z-a)\,f'(a) + \frac{(z-a)^2}{\lfloor 2}f''(a) + \frac{(z-a)^3}{\lfloor 3}f'''(a) + \cdots\infty$$

LAURENT'S THEOREM

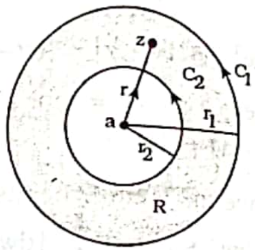
Theorem–2. If f(z) is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 and C_2 with centre at a and respective radii r_1 and r_2 ($r_2 < r_1$), then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty}\frac{a_{-n}}{(z-a)^n}$$

Where $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw,\; n = 0, 1, 2, \cdots$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{-n+1}}dw,\; n = 1, 2, 3, \cdots$$

[DUH–1982, 1985, 1988, 2003, RUH–1976, 1979, 1983, RUMP–1985, 1988, CUH–1982]



Proof : The ring shaped region $R = \{|z-a| = r,\ r_2 < r < r_1\}$. Then by Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\,dw \quad \ldots\ldots (1)$$

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^n+\cdots\right]$$

$$\Rightarrow \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \cdots$$

$$+ \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw + U_n$$

$$= a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + U_n \quad \ldots\ldots (2)$$

Where $a_0 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw$, $a_1 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw, \ldots$

$$a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw \text{ and } U_n = \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n\frac{f(w)}{w-z}\,dw$$

Again, $-\frac{1}{w-z} = \frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1}$

$$= \frac{1}{z-a}\left[1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\cdots+\left(\frac{w-a}{z-a}\right)^{n-1}+\left(\frac{w-a}{z-a}\right)^n\right]$$

$$\Rightarrow -\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{z-a}\,dw + \frac{1}{2\pi i}\oint_{C_2}\frac{w-a}{(z-a)^2}f(w)\,dw$$

$$+\cdots+\frac{1}{2\pi i}\oint_{C_2}\frac{(w-a)^{n-1}}{(z-a)^n}f(w)\,dw + V_n$$

$$= \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + V_n \quad \ldots\ldots (3)$$

Where $a_{-1} = \frac{1}{2\pi i}\oint_{C_2} f(w)\,dw$, $a_{-2} = \frac{1}{2\pi i}\oint_{C_2}(w-a)f(w)\,dw, \cdots,$

$a_{-n} = \oint_{C_2}(w-a)^{n-1}f(w)\,dw$ and $V_n = \frac{1}{2\pi i}\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n\frac{f(w)}{z-w}\,dw$

From (1), (2) and (3) we have

$$f(z) = a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1}$$
$$+\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\cdots+\frac{a_{-n}}{(z-a)^n}+U_n+V_n \quad \ldots\ldots (4)$$

Now, for all points w on C_1 we have

$$|w-z| = |(w-a)-(z-a)| \geq |w-a| - |z-a| \geq r_1 - r$$

and for all points w on C_2 we have

$$|z-w| = |(z-a)-(w-a)| \geq |z-a| - |w-a| \geq r - r_2$$

Length of the circle $c_1 = \oint_{C_1}|dw| = 2\pi r_1$

and Length of the circle $c_2 = \oint_{C_2}|dw| = 2\pi r_2$

$f(z)$ is continuous, so it is bounded. Then there exists a constant M_1 such that $|f(z)| \leq M_1$ on C_1 and a constant M_2 such that $|f(z)| \leq M_2$ on C_2.

Also, $r < r_1$ and $r_2 < r$

$$\Rightarrow \frac{r}{r_1} < 1 \text{ and } \frac{r_2}{r} < 1$$

$$\Rightarrow \left(\frac{r}{r_1}\right)^n < 1 \text{ and } \left(\frac{r_2}{r}\right)^n < 1$$

$$\Rightarrow \left(\frac{r}{r_1}\right)^n \to 0 \text{ and } \left(\frac{r_2}{r}\right)^n \to 0 \text{ as } n \to \infty$$

i. e. $\lim_{n\to\infty}\left(\frac{r}{r_1}\right)^n = 0$ and $\lim_{n\to\infty}\left(\frac{r_2}{r}\right)^n = 0$

$$\therefore |U_n| = \left|\frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n\frac{f(w)}{w-z}\,dw\right| \leq \frac{1}{2\pi}\left|\frac{z-a}{w-a}\right|^n\frac{|f(w)|\oint_{C_1}|dw|}{|w-z|}$$

$$\Rightarrow |U_n| \leq \frac{1}{2\pi}\left(\frac{r}{r_1}\right)^n\frac{M_1}{r_1-r}2\pi r_1 = \frac{M_1}{1-\frac{r}{r_1}}\left(\frac{r}{r_1}\right)^n = 0 \text{ when } n\to\infty$$

and $|V_n| = \left|\frac{1}{2\pi i}\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw\right| \le \frac{1}{2\pi}\left|\frac{w-a}{z-a}\right|^n \frac{|f(w)|\oint_{C_2}|dw|}{|z-w|}$

$$\Rightarrow |V_n| \le \frac{1}{2\pi}\left(\frac{r_2}{r}\right)^n M_2 \frac{2\pi r_2}{r-r_2} = \frac{M_2}{\frac{r}{r_2}-1}\left(\frac{r_2}{r}\right)^n = 0 \text{ when } n\to\infty$$

Thus, from (4) we have

$$f(z) = \{a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + \cdots\} + \left\{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + \cdots\right\}$$

$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty}\frac{a_{-n}}{(z-a)^n}, \quad \text{(Proved)}$$

Theorem–3. If f(z) has a pole at z = a, then $|f(z)| \to \infty$ as $z \to a$.

[NUH–1996]

Proof : Let z = a is a pole of order m. Then by Laurent's Theorem [z = a, m ক্রমের একটি পোল। তখন লরেন্টের উপপাদ্য দ্বারা পাই]

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} b_n(z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{1}{(z-a)^m}[b_m + b_1(z-a)^{m-1} + \cdots + b_{m-1}(z-a)]$$

When $z \to a$ then the square brakets term tends to b_m and the whole right hand expression tend to infinity. [যখন $z \to a$ তখন বর্গ বন্ধনীর পদ b_m হয় এবং ডানদিকের সম্পূর্ণ রাশি অসীম হয়।]

Hence [অতএব] $|f(z)| \to \infty$ as $z \to a$.

4.5. Residues and Residues theorem :

[NUH–2011, NU(Phy)–2004]

Definition : If the function f(z) is analytic within a circle C of radius r and centre a, except at z = a, then the coefficient a_{-1} of $\frac{1}{z-a}$ in the Laruent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty}\frac{a_{-1}}{(z-a)^n}$$

$$= \{a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\} + \left\{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\right\}$$

around z = a is called the residue of f(z) at z = a. It is denoted by Res (a) or a_{-1}. [z = a এর চারিদিকে থাকে z = a এ f(z) এর অবশেষ বলে। ইহাকে Res (a) বা a_{-1} দ্বারা নির্দেশ করা হয়]

[NUH–2000, 02, 06 (Old), NU(Pre)–06, 07, DUH–83, 89, 90]

Theorem–4. Let f(z) be analytic inside and on a simple closed curve C except at point a inside C, then

If z = a is a simple pole then $a_{-1} = \lim_{z\to a}(z-a)f(z)$

Proof : The corresponding Laurent series is [অনুসঙ্গী লরেন্ট ধারা]

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{a_{-1}}{z-a}, \text{ where } a_{-1} \ne 0$$

Multiplying this by (z – a) and then taking limit $z \to a$ we get [ইহাকে (z – a) দ্বারা গুণ করে এবং অতপর $z \to a$ লিমিট নিয়ে পাই]

$$\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\left[\sum_{n=0}^{\infty} a_n(z-a)^{n+1} + a_{-1}\right]$$

$$= 0 + a_{-1}$$

$$\Rightarrow a_{-1} = \lim_{z\to a}(z-a)f(z) \quad \text{(Proved)}$$

Residue at a multiple point.

Theorem–5. Let f(z) be analytic inside and on a simple closed curve C except at a pole a of order m inside C. Prove that the residue of f(z) at a is given by

$$a_{-1} = \lim_{z\to a}\frac{1}{\lfloor m-1}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

[NUH–93, 2000, 02, 05, 06(Old), 11, NU(Pre)–08, DUH–04]

Proof : If f(z) has a pole a of order m, then Laurent series of f(z) is [যদি f(z) এর m ক্রমের একটি পোল a থাকে, তখন f(z) এর লরেন্ট ধারা।]

$$f(z) = \{a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\} + \frac{a_{-1}}{(z-a)} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-m+1}}{(z-a)^{m-1}} + \frac{a_{-m}}{(z-a)^m}$$

$$\Rightarrow (z-a)^m f(z) = \{a_0(z-a)^m + a_1(z-a)^{m+1} + a_2(z-a)^{m+2} + \cdots\} + a_{-1}(z-a)^{m-1} + a_{-2}(z-a)^{m-2} + \cdots + a_{-m+1}(z-a) + a_{-m};$$

[Multiplying by $(z-a)^m$]

Differentiating this $(m-1)$ times w. r. to z we get [ইহাকে z এর সাথে $(m-1)$ বার অন্তরীকরণ করে পাই]

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = \left\{a_0 \underline{|m}(z-a) + \frac{a_1\underline{|m-1}}{\underline{|2}}(z-a)^2 + \cdots\right\} + a_{-1}\underline{|m-1}$$

Now taking limit $z \to a$ we get [এখন $z \to a$ লিমিট নিয়ে পাই]

$$\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = a_{-1}\underline{|m-1}$$

$$\therefore a_{-1} = \lim_{z\to a}\frac{1}{\underline{|m-1}}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}. \qquad \text{(Proved)}$$

CAUCHY'S RESIDUE THEOREM

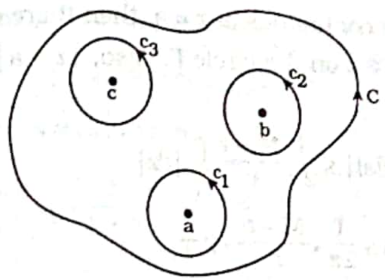
Theorem–6. If f(z) is analytic inside and on a simple closed curve C except at a finite number of points a, b, c, ··· inside C at which the residues are $a_{-1}, b_{-1}, c_{-1}, \cdots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

$$= 2\pi i \text{ (sum of the residues).}$$

[NUH-98, 02, 06, 11, NU(Phy)-03, 04, DUH-03]

Proof : Let us construct circles $c_1, c_2, c_3, \cdots$ with centre a, b, c, ··· which lie entirely within C. Then[a, b, c, ··· কেন্দ্র বিশিষ্ট c_1, c_2, c_3 বৃত্তগুলি অংকন করি যারা সম্পূর্ণভাবে C এর মধ্যে অবস্থিত। তখন]



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \cdots$$

$$= 2\pi i\, a_{-1} + 2\pi i\, b_{-1} + 2\pi i\, c_{-1} + \cdots$$

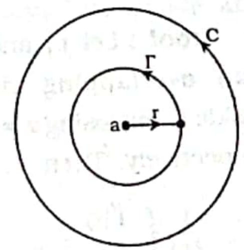
$$= 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots$$

$$= 2\pi i \text{ (sum of the residues)} \qquad \text{(Proved)}$$

MAXIMUM MODULUS THEOREM

Theorem–7. Let f(z) be analytic inside and on a simple closed curve C and is not identically equal to a constant. Then the maximum value of $|f(z)|$ occurs on C.

[DUH-75, 86, 89, 90, 02, 04, JUH-89, 91]



Proof : Given f(z) is analytic inside and on C, so f(z) is continuous inside and on C. Hence f(z) reaches its maximum value M (upper bound) with in or on C.

If possible let $|f(z)|$ have its maximum value M (say) at a point z = a lying with in C, and not on C.

i. e. $|f(a)| = M$, where a lies with in C.

If Γ be a small circle of radius r and centre at z = a, lying entriety with in C, then f(z) is analytic with in and on Γ. Hence by Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{z-a}$$

$$\Rightarrow |f(a)| = \left|\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{z-a}\right| \le \frac{1}{|2\pi i|}\frac{|f(z)|\oint_\Gamma |dz|}{|z-a|} \quad \cdots\cdots (1)$$

Since f(z) is continuous at z = a, then there exist $\varepsilon > 0$, such that $|f(z)| \le M - \varepsilon$, $\forall$ z on the circle Γ. Also, $|z - a| = r$. Thus, from (1) we get,

$$|f(a)| \le \frac{1}{2\pi}\frac{M-\varepsilon}{r}\oint_{\Gamma}|dz|$$

$$\Rightarrow M \le \frac{1}{2\pi}\cdot\frac{M-\varepsilon}{r}\cdot 2\pi r$$

$\Rightarrow M \le M - \varepsilon$, which is impossible, since M can not less than $M - \varepsilon$.

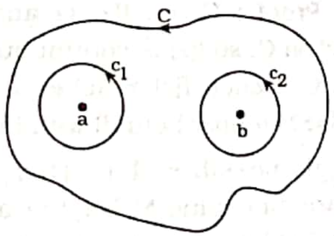
Hence the maxim of $|f(z)|$ occurs on C. (Proved)

THE ARGUMENT THEOREM

Theorem–8. Let f(z) be analytic inside and on a simple closed curve C except for a pole z = a of order p inside C. Suppose also that inside C f(z) has only one zero z = b of order n and no zeros on C. Prove that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = n - p$$ **[DUH-2002]**

Proof : Let c_1 and c_2 be two non overlapping circles lying inside C enclosing z = a and z = b respectively. Then



$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz$$

$$= \frac{1}{2\pi i}\oint_{c_1}\frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\oint_{c_2}\frac{f'(z)}{f(z)}dz \;\ldots\ldots (1)$$

Since f(z) has a pole of order p at z = a we have

$$f(z) = \frac{F(z)}{(z-a)^p} \;\ldots\ldots (2)$$

where F(z) is analytic and different from zero inside and on c_1.

Taking logarithm of (2) of both sides we get

$$\log f(z) = \log F(z) - p\log(z-a)$$

Differentiating this w. r. to z we get,

$$\frac{1}{f(z)}\cdot f'(z) = \frac{F'(z)}{F(z)} - p\cdot\frac{1}{z-a}$$

$$\Rightarrow \frac{1}{2\pi i}\oint_{c_1}\frac{f'(z)}{f(z)}dz$$

$$= \frac{1}{2\pi i}\oint_{c_1}\frac{F'(z)}{F(z)}dz - \frac{p}{2\pi i}\oint_{c_1}\frac{dz}{z-a}$$

$$= \frac{1}{2\pi i}\cdot 0 - \frac{p}{2\pi i}\cdot 2\pi i\cdot 1$$

$$= -p \;\ldots\ldots (3)$$

[F(z) is analytic, $\therefore$ F′(z) and $\frac{F'(z)}{F(z)}$ are also analytic on c_1

$\therefore \oint_C \frac{F'(z)}{F(z)}dz = 0$]

Again, since f(z) has a zero of order n, at z = b, we have

$$f(z) = (z-b)^n G(z)$$

where G(z) is analytic and different from zero inside and on c_2.

$$\Rightarrow \log f(z) = n\log(z-b) + \log G(z)$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{n}{z-b} + \frac{G'(z)}{G(z)};\text{ diff. w. r. to z.}$$

$$\Rightarrow \frac{1}{2\pi i}\oint_{c_2}\frac{f'(z)}{f(z)}dz = \frac{n}{2\pi i}\oint_{c_2}\frac{1}{z-b}dz + \frac{1}{2\pi i}\oint_{c_2}\frac{G'(z)}{G(z)}dz$$

$$= \frac{n}{2\pi i}\cdot 2\pi i + \frac{1}{2\pi i}\times 0$$

$$= n \;\ldots\ldots (4)$$

By (3) and (4), (1) becomes

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = n - p$$ (Proved)

THE GENERAL ARGUMENT THEOREM

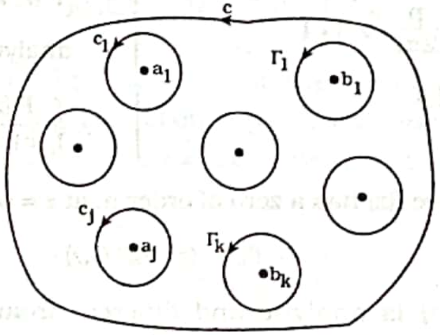
Theorem–9. Let f(z) be analytic inside and on a simple closed curve C apart from a finite number of poles inside C and no zeros on C. Then $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P$

Where N is the number of zeros

and P is the number of poles inside C counting multiplicities.

[NUH–1989, 1991, 1985, 2006, RUH–1978, 1986, CHU–1981]

Proof : Let $a_1, a_2, \cdots a_j$ and $b_1, b_2, \cdots, b_k$ be the respective poles and zeros of f(z) lying inside C and suppose their multiplicities are $p_1, p_2, \cdots, p_j$ and $n_1, n_2, \cdots, n_k$.



We now enclose each pole and zero by non-overlapping circles $c_1, c_2, \cdots, c_j$ and $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$.

This can always be done since the poles and zeros are isolated. Then we have

[প্রমাণ ঃ মনে করি $a_1, a_2, \cdots a_j$ এবং $b_1, b_2, \cdots, b_k$, C এর ভিতর অবস্থিত f(z) এর স্ব-স্ব পোল ও শূণ্যক সমূহ এবং তাদের সংখ্যাধিক্য $p_1, p_2, \cdots, p_j$ ও $n_1, n_2, \cdots, n_k$. আমরা এখন প্রত্যেক পোল ও শূন্যককে একসাথে মিলে না যায় এমন বৃত্ত $c_1, c_2, \cdots, c_j$ ও $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ দ্বারা আবৃত করি। যেহেতু পোল ও শুণ্যক বিচ্ছিন্ন তাই এরূপ সব সময় করা যায়। তখন আমরা পাই]

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \sum_{r=1}^{j} \frac{1}{2\pi i}\oint_{\Gamma_r} \frac{f'(z)}{f(z)}\,dz + \sum_{r=1}^{k} \frac{1}{2\pi i}\oint_{C_r} \frac{f'(z)}{f(z)}\,dz$$

$$= \sum_{r=1}^{j} n_r - \sum_{r=1}^{k} p_r$$

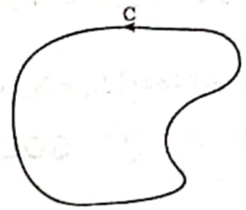
$$= N - P. \qquad \text{(Proved)}$$

ROUCHE'S THEOREM

Theorem–10. If f(z) and g(z) are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then f(z) + g(z) and f(z) have the same number of zeros inside C. [যদি f(z) ও g(z) একটি সাধারণ বদ্ধ বক্ররেখা C এর ভিতর ও উপর বৈশ্লেষিক হয় এবং C এর উপর যদি $|g(z)| < |f(z)|$ হয়, তবে C এর ভিতর f(z) + g(z) ও f(z) এর একই সংখ্যক শূন্যক (zeros) থাকবে।]

[NUH–1995, 2004, 2007, 2012, DUH–1975, 1978, 1984, 1987, 1989, 1990, 1995, 2002, 2003, RUH–1972, 1975, 1982, 1984, 1986, 1988, CUH–1987, 1989, 1993, JUH–1986, 1989,]

Proof : Let [ধরি] $F(z) = \frac{g(z)}{f(z)}$



$$\Rightarrow g(z) = f(z)\,F(z)$$

or in short, $g = fF$ ...... (1)

$$\Rightarrow g' = f'F + fF' \text{ ...... (2)}$$

[By differentiating w. r. to z]

If N_1 and N_2 are the number of zeros inside C of f + g and f respectively, then by the general argument theorem we have [যদি C এর ভিতর f + g ও f এর শূন্যের সংখ্যা যথাক্রমে N_1 ও N_2 হয়, তবে সাধারণ যুক্তি (আরগুমেন্ট) উপপাদ্য দ্বারা পাই]

$$N_1 = \frac{1}{2\pi i}\oint_C \frac{f' + g'}{f + g}\,dz \text{ ...... (3) and [এবং] } N_2 = \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz \text{ ...... (4)}$$

using the fact that these two functions have no poles inside C. [C এর ভিতর এই দুইটি ফাংশনের কোন পোল নাই এই ঘটনা প্রয়োগ করে]

Now [এখন] $N_1 - N_2 = \frac{1}{2\pi i}\oint_C \frac{f' + g'}{f + g}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$

$$= \frac{1}{2\pi i}\oint_C \frac{f' + f'F + fF'}{f + fF}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz; \quad \text{By (1) and (2)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f'(1+F) + fF'}{f(1+F)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \left(\frac{f'}{f} + \frac{F'}{1+F}\right)dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz + \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\oint_C F'(1+F)^{-1}\,dz$$

$$= \frac{1}{2\pi i}\oint_C F'(1 - F + F^2 - F^3 + \cdots)\,dz$$

$= 0$, using the given fact $|F| < 1$ one so that the given series is uniformly convergent on C and term by term integration gives the value zero. [$|F| < 1$ ঘটনাটি আরেকবার ব্যবহার করি যেন প্রদত্ত ধারাটি C এর উপর সুষমভাবে অভিসারি হয় এবং পদ ক্রম পদ যোজিত ফল শূন্য দেয়।]

Thus [অতএব] $N_1 = N_2$ and hence Proved [এবং প্রমাণিত]।

SOLVED PROBLEMS

Problem–1. $f(z) = \frac{1}{(z-1)^2}$ has a singularity at $z = 1$, because

$$f(1) = \frac{1}{(1-1)^2} = \infty$$

That is, f(z) fails to be analytic at $z = 1$.

Also, $\lim\limits_{z\to 1} (z-1)^2 f(z) = \lim\limits_{z\to 1} (z-1)^2 \cdot \frac{1}{(z-1)^2} = 1 \neq 0$.

Thus $z = 1$ is a pole of order 2.

Problem–2. $f(z) = \frac{\sin z}{z}$ has a singular point at $z = 0$.

But $\lim\limits_{z\to 0} \frac{\sin z}{z} = 1$, so the singular point $z = 0$ is a removal singularity of f(z).

Problem–2(a). Let $f(z) = e^{1/z-1}$. Then $f(1) = e^{\infty} = \infty$

$\therefore z = 1$ is a singular point of f(z).

But $\lim\limits_{z\to 1} f(z) = \lim\limits_{z\to 1} e^{1/z-1} = \infty$

This shows that $z = 1$ is not a removal singularity.

Also $z = 1$ is not a branch point or pole of f(z).

Hence $z = 1$ is an essential singularity.

Problem–3. Classify singularities. Give an example of each kind. [ব্যতিক্রমী বিন্দুগুলির শ্রেণীবিভাগ কর। প্রতিটি ধরণের একটি করে উদাহরণ দাও।] **[NUH-2008]**

OR, Discuss the different kinds of singularities [বিভিন্ন প্রকার ব্যতিক্রমী বিন্দুগুলির বর্ণনা দাও।] **[NUH-2011]**

Solution : A point at which an analytic function f(z) fails or ceases to be analytic is called a singular point.]

The singular points are of two types-isolated and non-isolated singular points.

Isolated singular point : Let $z = z_0$ be a singularity of f(z). If there is no other singularity with in a small circle surrounding the point $z = z_0$, that is, there exists a deleted neighbourhood of z_0 $\{z : 0 < |z - z_0| < \delta\}$ in which f(z) is analytic, then the point is called an isolated singularity.

Example : $f(z) = \frac{1}{z-1}$ has an isolated singularity at $z = 1$, since f(z) is analytic in $0 < |z - 1| < \delta, \delta > 0$.

Isolated singular points are of three types :

1. **Removable singularity**, 2. **Pole** and 3. **Essential singularity.**

Removable singularity. If $\lim\limits_{z\to z_0} f(z)$ exists then z_0 is called a removable singularity of f(z).

Example : $f(z) = \frac{\sin z}{z}$ has a singular point at $z = 0$.

But $\lim\limits_{z\to 0} \frac{\sin z}{z} = 1$, so the singular point $z = 0$ is a removable singularity of f(z).

Pole : If there exists a positive integer n such that

$$\lim_{z\to z_0} (z - z_0)^n f(z) = A \neq 0$$

then $z = z_0$ is called a pole of order n.

Example : $f(z) = \frac{1}{(z-1)^2}$ has a singularity at $z = 1$, because $f(1) = \frac{1}{(1-1)^2} = \infty$. That is, f(z) fails to be analytic at $z = 1$.

Also, $\lim\limits_{z \to 1} (z-1)^2 f(z) = \lim\limits_{z \to 1} (z-1)^2 \cdot \frac{1}{(z-1)^2} = 1 \neq 0.$

Thus $z = 1$ is a pole of order 2.

Essential singularity : A singular point which is not a pole, branch point or removable singularity is called an essential singularity.

Example : Let $f(z) = e^{\frac{1}{z-1}}$. Then $f(1) = e^{\infty} = \infty$.

$\therefore$ $z = 1$ is a singular point of f(z). But $\lim\limits_{z \to 1} f(z) = \lim\limits_{z \to 1} e^{\frac{1}{z-1}} = \infty$.

This shows that $z = 1$ is not a removal singularity.

Also $z = 1$ is not a branch point or pole of f(z).

Hence $z = 1$ is an essential singularity.

Singularity at infinity : The function f(z) has a singularity at $z = \infty$ if $w = 0$ is a singularity of $f\left(\frac{1}{w}\right)$.

Example : Let $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$.

Then putting $z = \frac{1}{w}$ we get

$$f\left(\frac{1}{w}\right) = \frac{\frac{1}{w^8} + \frac{1}{w^4} + 2}{\left(\frac{1}{w} - 1\right)^3 \left(\frac{3}{w} + 2\right)^2} = \frac{1 + w^4 + 2w^8}{w^3(1-w)^3 (3+2w)^2}$$

when $w = 0$ then $f\left(\frac{1}{w}\right) = \infty$

$\therefore w = 0 \Rightarrow z = \frac{1}{w} = \frac{1}{0} = \infty$ is a singularity of f(z).

Branch point is also a singular point.

A multivalued function f(z) defined in some domain S is said to have a branch point at z_0 if, when z describes an arbitrary small circle about z_0, then for every branch F of f, F(z) does not return to its original value.

Example : 1. $f(z) = z^{1/2}$ has a branch point at $z = 0$.

2. $f(z) = \ln(z^2 + z - 2)$ has branch point $z^2 + z - 2 = 0$

$\Rightarrow z = 1$ and $z = -2$.

Non-isolated singular point : If the singularity $z = z_0$ is not an isolated singularity then it is called a non-isolated singularity.

Example : Let $f(z) = \frac{1}{\tan\left(\frac{\pi}{z}\right)}$. Then $f(z) = \infty$ when

$$\tan\left(\frac{\pi}{z}\right) = 0 \Rightarrow \frac{\pi}{z} = n\pi \Rightarrow \frac{1}{z} = n \Rightarrow z = \frac{1}{n}$$

$$\Rightarrow z = 1, \frac{1}{2}, \frac{1}{3}, \cdots, 0$$

Here $z = 0$ is a non isolated singular point and all other are isolated singular points.

[সমাধান ঃ যে বিন্দুতে একটি ফাংশন f(z) বৈশ্লেষিক হতে ব্যর্থ হয় বা বৈশ্লেষিক হতে বিরত হয় বা থেমে যায়, সেই বিন্দুকে ব্যতিচার বিন্দু বলে। ব্যতিচার বিন্দু দুই প্রকার বিচ্ছিন্ন ও অবিচ্ছিন্ন ব্যতিচার বিন্দু।

Isolated [বিচ্ছিন্ন] ব্যতিচার বিন্দু ঃ মনে করি f(z) এর একটি ব্যতিচার বিন্দু $z = z_0$. যদি $z = z_0$ কে একটি ছোট বৃত্ত ঘিরে তার ভিতর আর কোন ব্যতিচারিতা না থাকে তখন এই বিন্দুকে বিচ্ছিন্ন ব্যতিচার বিন্দু বলে।

উদাহরণ ঃ $f(z) = \frac{1}{z-1}$ এর $z = 1$ এ একটি বিচ্ছিন্ন ব্যতিচার বিন্দু আছে, কারণ $0 < |z-1| < \delta, \delta > 0$ এ f(z) বৈশ্লেষিক।

বিচ্ছিন্ন ব্যতিচার বিন্দু তিন প্রকার ঃ

১। অপসারণযোগ্য ব্যতিচার বিন্দু, ২। পোল, ৩। অপরিহার্য ব্যতিচার বিন্দু।

অপসারণযোগ্য ব্যতিচার বিন্দু ঃ যদি $\lim\limits_{z \to z_0} f(z)$ বিদ্যমান থাকে তখন z_0 কে f(z) এর অপসারণযোগ্য ব্যতিচার বিন্দু বলে।

উদাহরণ ঃ $f(z) = \frac{\sin z}{z}$ এর $z = 0$ একটি ব্যতিচার বিন্দু।

কিন্তু $\lim\limits_{z \to 0} \frac{\sin z}{z} = 1$, সুতরাং $z = 0$ হল f(z) এর অপসারণযোগ্য ব্যতিচার বিন্দু।

পোল ঃ যদি একটি ধনাত্মক পূর্ণসংখ্যা n বিদ্যমান থাকে যেন

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$$

তখন $z = z_0$ কে n মাত্রার পোল বলে।

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উদাহরণ ঃ $f(z) = \frac{1}{(z-1)^2}$ এর $z = 1$ একটি ব্যতিচার বিন্দু, কারণ $f(1) = \frac{1}{(1-1)^2} = \infty$, অর্থাৎ $z = 1$ এ $f(z)$ বৈশ্লেষিক হতে ব্যর্থ হয়।

কিন্তু $\lim_{z\to 1} (z-1)^2 f(z) = \lim_{z\to 1} (z-1)^2 \cdot \frac{1}{(z-1)^2} = 1 \neq 0.$

অতএব $z = 1$ হল 2 মাত্রার একটি পোল।

অপরিহার্য ব্যতিচার বিন্দু ঃ একটি ব্যতিচার বিন্দু যা পোল, ব্রাঞ্চ বিন্দু বা অপসারণযোগ্য ব্যতিচার বিন্দু না তাকে অপরিহার্য ব্যতিচার বিন্দু বলে।

উদাহরণ ঃ ধরি $f(z) = e^{\frac{1}{z-1}}$. তখন $f(1) = e^{\infty} = \infty$

$\therefore z = 1$ হল $f(z)$ এর একটি ব্যতিচার বিন্দু।

কিন্তু $\lim_{z\to 1} f(z) = \lim_{z\to 1} e^{\frac{1}{z-1}} = \infty$

ইহা দেখায় যে $z = 1$ অপসারণযোগ্য ব্যতিচার বিন্দু না।

আরো, $z = 1$, $f(z)$ এর ব্রাঞ্চ বিন্দু বা পোল না।

অতএব $z = 1$ একটি অপরিহার্য ব্যতিচার বিন্দু।

ব্রাঞ্চ বিন্দু ও ব্যতিচার বিন্দু। কোন ডোমেন S এ সংজ্ঞায়িত একটি বহুমানী ফাংশন $f(z)$ এর z_0 বিন্দুতে ব্রাঞ্চ বিন্দু আছে যদি z_0 এর চারিদিকে একটি ইচ্ছাধীন ছোট বৃত্ত অংকিত হলে তখন f এর প্রত্যেক ব্রাঞ্চ F এর জন্য $F(z)$ ইহার আদি অবস্থায় ফিরে আসতে না পারে।

উদাহরণ ঃ ১। $f(z) = z^{1/2}$ ফাংশনের $z = 0$ একটি ব্রাঞ্চ বিন্দু।

২। $f(z) = ln(z^2 + z - 2)$ ফাংশনের $z^2 + z - 2 = 0$

অর্থাৎ $z = 1$ এবং $z = -2$ ব্রাঞ্চ বিন্দু।

অবিচ্ছিন্ন ব্যতিচার বিন্দু ঃ যে ব্যতিচার বিন্দু বিচ্ছিন্ন নয় তাকে অবিচ্ছিন্ন (non-isolated) ব্যতিচার বিন্দু বলে।

উদাহরণ ঃ ধরি $f(z) = \frac{1}{\tan\left(\frac{\pi}{z}\right)}$

$f(z) = \infty$ যখন $\tan\frac{\pi}{z} = 0$

$$\Rightarrow \frac{\pi}{z} = n\pi \Rightarrow z = \frac{1}{n}$$

$$\Rightarrow z = 1, \frac{1}{2}, \frac{1}{3}, \cdots, 0.$$

এখানে $z = 0$ অবিচ্ছিন্ন ব্যতিচার বিন্দু এবং অন্য সকল বিচ্ছিন্ন ব্যতিচার বিন্দু।।

Problem–4. $f(z) = ln(z^2 - 5z + 6)$ has branch points where

$$z^2 - 5z + 6 = 0$$
$$\Rightarrow (z-2)(z-3) = 0$$
$$\Rightarrow z = 2 \text{ and } z = 3.$$

Thus, $z = 2$ and $z = 3$ are branch points of $f(z) = ln(z^2 - 5z + 6)$.

Problem–5. Show that $f(z) = \frac{(z+5i)^3}{(z^2 - 2z + 5)^2}$ has double poles at $z = 1 \pm 2i$ and a simple pole at infinity.

Solution : Given that $f(z) = \frac{(z+5i)^3}{(z^2-2z+5)^2}$

$f(z)$ has singularity when $(z^2 - 2z + 5)^2 = 0$

$$\Rightarrow z^2 - 2z + 5 = 0$$

$$\Rightarrow z = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$\therefore f(z) = \frac{(z+5i)^3}{\{(z-1-2i)(z-1+2i)\}^2} = \frac{(z+5i)^3}{\{z-(1+2i)\}^2\{z-(1-2i)\}^2}$$

Now $\lim_{z\to 1+2i} \{z - (1+2i)\}^2 f(z)$

$$= \lim_{z\to 1+2i} \frac{(z+5i)^3}{\{z-(1-2i)\}^2}$$

$$= \frac{(1+2i+5i)^3}{(1+2i-1+2i)^2} = \frac{(1+7i)^3}{-4} \neq 0$$

and $\lim_{z\to 1-2i} \{z-(1-2i)\}^2 f(z)$

$$= \lim_{z\to 1-2i} \frac{(z+5i)^3}{\{z-(1+2i)\}^2}$$

$$= \frac{(1-2i+5i)^3}{(1-2i-1-2i)^2} = \frac{(1+3i)^3}{-4} \neq 0$$

$\therefore f(z)$ has double poles at $z = 1 \pm 2i$.

Again, $f\left(\frac{1}{w}\right) = \frac{\left(\frac{1}{w} + 5i\right)^3}{\left(\frac{1}{w^2} - \frac{2}{w} + 5\right)^2} = \frac{(1+5iw)^3}{w(1-2w+5w^2)}$

$$\therefore \lim_{w\to 0} wf\left(\frac{1}{w}\right) = \lim_{w\to 0} \frac{(1+5iw)^3}{1-2w+5w^2} = \frac{1}{1} = 1 \neq 0$$

$\therefore$ w = 0 is a simple pole of $f\left(\frac{1}{w}\right)$.

Hence $z = \infty$ is a simple pole of f(z).

Problem–6. Prove that $f(z) = e^{-1/z^2}$ has no singularities.

Solution : $f(z) = e^{-1/z^2} = \dfrac{1}{e^{1/z^2}}$

Poles are obtained when $e^{1/z^2} = 0$. This is not possible for any real or complex value of z. Hence f(z) has no poles.

Again $f(z) = e^{-1/z^2} = 0$ gives $e^{-1/z^2} = 0 = e^{-\infty}$

$$\Rightarrow \frac{1}{z^2} = \infty \Rightarrow z^2 = 0 \Rightarrow z = 0, 0.$$

Thus z = 0 is a zero of f(z) and hence no singularity.

Therefore, $f(z) = e^{-1/z^2}$ has no singularities.

Problem–7. Discuss the nature of the singularities of $f(z) = \dfrac{z-2}{z^2} \sin \dfrac{1}{z-2}$.

Solution : When $z^2 = 0 \Rightarrow z = 0, 0$ then $f(z) = \infty$.

$$\lim_{z\to 0} z^2 \cdot f(z) = \lim_{z\to 0} z^2 \cdot \frac{z-2}{z^2} \sin \frac{1}{z-2}$$

$$= \lim_{z\to 0} (z-2) \sin \frac{1}{z-2}$$

$$= -2 \sin\left(\frac{-1}{2}\right) = 2 \sin \frac{1}{2} \neq 0.$$

$\therefore$ z = 0 is a pole of order 2.

Problem–8. Find the singular points of the function $\dfrac{z^2}{(z+1)^2} \sin\left(\dfrac{1}{z-1}\right)$ and determine their nature.

[NUH–1995, 2000, NU(Pre)–2006]

Solution : Let [ধরি] $f(z) = \dfrac{z^2}{(z+1)^2} \sin\left(\dfrac{1}{z-1}\right)$

when [যখন] z = – 1 then [তখন] $f(z) = \infty$

$\therefore$ z = – 1, is the singular point of the given function. [প্রদত্ত ফাংশনের একটি ব্যতিচার বিন্দু।]

Now at z = – 1 we have [এখন z = – 1 এ পাই]

$$\lim_{z\to -1} (z+1)^2 \cdot f(z) = \lim_{z\to -1} (z+1)^2 \cdot \frac{z^2}{(z+1)^2} \sin\left(\frac{1}{z-1}\right)$$

$$= \lim_{z\to -1} z^2 \sin\left(\frac{1}{z-1}\right)$$

$$= (-1)^2 \sin\left(\frac{1}{-1-1}\right)$$

$$= -\sin \frac{1}{2}, \text{ which is finite.}$$ [যাহা সসীম]

$\therefore$ z = – 1 is a pole of order 2. [z = – 1, 2 ক্রমের একটি পোল]

Again, We have [আবার আমরা পাই]

$$\sin \frac{1}{z-1} = \frac{1}{z-1} - \frac{1}{\lfloor 3\,(z-1)^3} + \frac{1}{\lfloor 5\,(z-1)^5} + \cdots$$

In the above expansion, there are infinite number of terms in the negative powers of (z – 1). Hence z = 1 is an isolated essential singularity of $\sin \dfrac{1}{z-1}$ and hence of f(z). [উপরের বিস্তৃতিতে (z – 1) এর ঋণাত্মক ঘাতের অসীম সংখ্যক পদ আছে। অতএব z = 1, $\sin \dfrac{1}{z-1}$ এর একটি বিচ্ছিন্ন অপরিহার্য ব্যতিচার বিন্দু এবং অতএব f(z) এর]

Problem–8(i). Identify the singularities of $f(z) = \dfrac{\sin\left(\frac{1}{z}\right)}{(z^2-1)^2}$.

$\left[f(z) = \dfrac{\sin\left(\frac{1}{z}\right)}{(z^2-1)^2}\right.$ এর সিংগুলারিটি নির্ণয় কর।] **[NUH–2012]**

Solution : Given $f(z) = \dfrac{\sin\left(\frac{1}{z}\right)}{(z^2-1)^2} = \dfrac{\sin\left(\frac{1}{z}\right)}{(z-1)^2\,(z+1)^2}$

When [যখন] z = 1, – 1 তখন $f(z) = \infty$

$\therefore$ z = 1, – 1 are the singularities of f(z) [f(z) এর সিংগুলারিটি হল z = 1, z = – 1]

$$\text{Now } \lim_{z\to 1} (z-1)^2 \cdot f(z) = \lim_{z\to 1} (z-1)^2 \cdot \frac{\sin\left(\frac{1}{z}\right)}{(z-1)^2 (z+1)^2}$$

$$= \lim_{z\to 1} \frac{\sin\left(\frac{1}{z}\right)}{(z+1)^2} = \frac{1}{4}\sin(1)$$

Which is finite [যাহা সসীম]

$\therefore z = 1$ is a pole of order 2 [$z = 1$, 2 ক্রমের একটি পোল]

$$\text{Again [আবার]}, \lim_{z\to -1} (z+1)^2 f(z) = \lim_{z\to -1} (z+1)^2 \cdot \frac{\sin\left(\frac{1}{z}\right)}{(z-1)^2 (z+1)^2}$$

$$= \lim_{z\to -1} \frac{\sin\left(\frac{1}{z}\right)}{(z-1)^2} = \frac{1}{4}\sin(-1)$$

Which is finite [যাহা সসীম]

$\therefore z = -1$ is a pole of order 2 [$z = -1$ হল 2 ক্রমের একটি পোল]

Moreove [অধিকন্তু] $\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{\lfloor 3\, z^3} + \frac{1}{\lfloor 5\, z^5} - \cdots$

In the above expansion, there are infinite number of terms in the negative powers of z. Hence $z = 0$ is an isolated essential singularity of $\sin\frac{1}{z}$ and hence of f(z).

[উপরের বিস্তৃতিতে z এর ঋণাত্মক ঘাতের অসীম সংখ্যক পদ আছে। অতএব $z = 0$, $\sin\frac{1}{z}$ এর একটি বিচ্ছিন্ন অপরিহার্য ব্যতিচার বিন্দু (সিনগুলারিটি) এবং অতএব f(z) এর]

Problem–9. Find the nature and location of the singularities of the function $f(z) = \frac{1}{z(e^z - 1)}$. **[NUH–2005(Old)]**

OR, Determine and classify all the singularities of the function $f(z) = \frac{1}{z(e^z - 1)}$. **[NUH–2013]**

Solution : Given that [দেওয়া আছে] $f(z) = \frac{1}{z(e^z - 1)}$

In the finite z-plane the singularities will be obtained by solving the equation [সসীম z তলে নিম্নের সমীকরণ সমাধান করে ব্যাতিচার বিন্দু পাওয়া যাবে]

$$z(e^z - 1) = 0$$

$$\Rightarrow z = 0 \text{ or, } e^z - 1 = 0$$

Now [এখন] $e^z - 1 = 0$ gives [দেয়] $e^z = 1 = \cos 0 + i \sin 0$

$$= \cos 2n\pi + i \sin 2n\pi$$

$$= e^{i2n\pi}$$

$$\Rightarrow z = i2n\pi, \text{ where [যেখানে] } n = 0, \pm 1, \pm 2, \cdots$$

$\therefore$ The singularities are [ব্যতিচার বিন্দুগুলি হল] $z = 0$ and [এবং] $z = i2n\pi$, $n = 0, \pm 1, \pm 2, \cdots$

Problem–10. For the function $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$, locate and name all the singularities in the finite z-plane and also determine where f(z) is analytic.

[NUH–2003, 2006(Old), 2008, NU(Pre)–2011, DUH–1984]

Solution : Given [দেওয়া আছে] $f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$ ...... (1)

In the finite z-plane the singularities will be obtained by solving the equation $(z-1)^3 (3z+2)^2 = 0$ [সসীম z তলে ব্যতিচার বিন্দুগুলি পাওয়া যাবে $(z-1)^3 (3z+2)^2 = 0$ সমীকরণ সমাধান করে]

$$\Rightarrow (z-1)^3 = 0 \quad \text{or} \quad (3z+2)^2 = 0$$

$$\Rightarrow z = 1, 1, 1 \quad \text{or} \quad z = -\frac{2}{3}, -\frac{2}{3}$$

$\therefore$ The singularities in the finite z-plane are $z = 1$ and $z = -\frac{2}{3}$.

[সসীম z তলে ব্যতিচার বিন্দুগুলি হল $z = 1$ ও $z = -\frac{2}{3}$.]

To determine whether there is a singularity at $z = \infty$, let $z = \frac{1}{w}$.

Then from (1) we get [$z = \infty$ তে ব্যতিচার বিন্দু নির্ণয়ের জন্য ধরি $z = \frac{1}{w}$ তখন (1) হতে পাই]

$$f\left(\frac{1}{w}\right) = \frac{\frac{1}{w^8} + \frac{1}{w^4} + 2}{\left(\frac{1}{w} - 1\right)^3 \left(\frac{3}{w} + 2\right)^2}$$

$$= \frac{1 + w^4 + 2w^8}{w^8 (1-w)^3 (3+2w)^2 \cdot \frac{1}{w^5}}$$

$$= \frac{1 + w^4 + 2w^8}{w^3 (1-w)^3 (3+2w)^2}$$

when [যখন] $w = 0$, then [তখন] $f\left(\frac{1}{w}\right) = \infty$

$\therefore w = 0 \Rightarrow \frac{1}{z} = 0 \Rightarrow z = \frac{1}{0} = \infty$ is a singularity of f(z) [f(z) এর একটি ব্যতিচার বিন্দু]

Now [এখন] $\lim_{z\to 1} (z-1)^3 f(z) = \lim_{z\to 1} (z-1)^3 \cdot \frac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$

$$= \lim_{z\to 1} \frac{z^8+z^4+2}{(3z+2)^2} = \frac{1+1+2}{(3+2)^2} = \frac{4}{25} \neq 0$$

$$\lim_{z\to -2/3} \left(z+\frac{2}{3}\right)^2 f(z) = \lim_{z\to -2/3} \frac{(3z+2)^2}{9} \cdot \frac{z^8+z^4+1}{(z-1)^3(3z+2)^2}$$

$$= \lim_{z\to -2/3} \frac{z^8+z^4+1}{9(z-1)^3} = \frac{\left(\frac{2}{3}\right)^8 + \left(\frac{2}{3}\right)^4 + 1}{9\left(-\frac{2}{3}-1\right)^3} \neq 0$$

$$\lim_{w\to 0} (w-0)^3 f\left(\frac{1}{w}\right) = \lim_{w\to 0} w^3 \cdot \frac{1+w^4+2w^8}{w^3(1-w)^3(3+2w)^2}$$

$$= \lim_{w\to 0} \frac{1+w^4+2w^8}{(1-w)^3(3+2w)^2}$$

$$= \frac{1+0+0}{(1-0)^3(3+0)^2}$$

$$= \frac{1}{1\times 9} = \frac{1}{9} \neq 0$$

$\therefore z = 1$ is a pole of order 3 [3 ক্রমের একটি পোল]

$z = -\frac{2}{3}$ is a pole of order 2 [2 ক্রমের একটি পোল]

and $z = \infty$ is a pole of order 3 for the function f(z). [এবং $z = \infty$, f(z) এর 3 ক্রমের একটি পোল]

2nd Part : $z = 1$ and $z = -\frac{2}{3}$ are the singularities for the f(z) in the finite z-plane. Thus, in the finite z-plane f(z) is analytic every where excepts the points $z = 1$ and $z = -\frac{2}{3}$. [সসীম z তলে f(z) এর ব্যতিচার বিন্দু $z = 1$ এবং $z = -\frac{2}{3}$. অতএব, সসীম z তলে $z = 1$ ও $z = -\frac{2}{3}$ বিন্দুদ্বয় ব্যতীত f(z) সর্বত্র বৈশ্লেষিক।] Ans.

Problem-10(a). Calculate the Residue of the function $\frac{z^2}{z^2+a^2}$ (take only the positive value of z). **[NUH-2012, NU(Phy)-2003]**

Solution : Given that [দেওয়া আছে] $f(z) = \frac{z^2}{z^2+a^2}$

The poles of f(z) are obtained by solving the equations

$$z^2 + a^2 = 0$$

[$z^2 + a^2 = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$$\Rightarrow z = \pm\sqrt{-a^2} = \pm ai$$

The positive value of $z = ai$ which is a simple pole. [ধনাত্মক মান $z = ai$ যাহা একটি সরল পোল]

Residue at $z = ai$ is [$z = ai$ এ অবশেষ] $\lim_{z\to ai} \left\{(z-ai)\frac{z^2}{z^2+a^2}\right\}$

$$= \lim_{z\to ai} \frac{z^2(z-ai)}{(z+ai)(z-ai)}$$

$$= \lim_{z\to ai} \frac{z^2}{z+ai}$$

$$= \frac{(ai)^2}{ai+ai} = \frac{-a^2}{2ai} = \frac{ai}{2}. \quad \textbf{(Ans)}$$

Problem-11. Find the residues of $f(z) = \frac{z^2-2z}{(z+1)^2(z+4)}$ at all its poles in a finite plane. **[NUH-2013]**

Solution : Given that [দেওয়া আছে] $f(z) = \frac{z^2-2z}{(z+1)^2(z+4)}$

The poles of f(z) are obtained by solving the equation

$$(z+1)^2(z+4) = 0$$

[$(z+1)^2(z+4) = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$\Rightarrow (z+1)^2 = 0$ and $z+4=0$

$\Rightarrow z = -1, -1$ and $z = -4$

$\therefore z = -1$ is a double pole and $z = -4$ is a simple pole.

[$z = -1$ একটি দ্বিপোল এবং $z = -4$ একটি সরল পোল]

Residue at $z = -1$ is [$z = -1$ এ অবশেষ] $\lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\{(z+1)^2 f(z)\}$

$$= \lim_{z\to -1} \frac{d}{dz}\left\{(z+1)^2 \cdot \frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)$$

$$= \lim_{z\to -1} \frac{(z^2+4)(2z-2)-(z^2-2z)\cdot 2z}{(z^2+4)^2}$$

$$= \lim_{z\to -1} \frac{2z^3+8z-2z^2-8-2z^3+4z^2}{(z^2+4)^2}$$

$$= \lim_{z\to -1} \frac{2z^2+8z-8}{(z^2+4)^2}$$

$$= \frac{2-8-8}{(1+4)^2} = \frac{-14}{25}$$

Residue at $z = -4$ [$z = -4$ এ অবশেষ] $\lim\limits_{z \to -4} (z+4) \cdot f(z)$

$$= \lim_{z \to -4} \left\{ (z+4) \cdot \frac{z^2 - 2z}{(z+1)^2 (z+4)} \right\}$$

$$= \lim_{z \to -4} \frac{z^2 - 2z}{(z+1)^2} = \frac{16+8}{(-4+1)^2} = \frac{24}{9}$$

The residues at $z = -1$ is $\frac{-14}{25}$ and residue at $z = -4$ is $\frac{24}{9}$. **(Ans)**

Problem–11(a). Find the residues of the function

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}.$$

[DUH–1988, 1989]

Solution : Given that [দেওয়া আছে] $f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$

The poles of $f(z)$ are obtained by solving the equation

$$(z+1)^2 (z^2+4) = 0$$

[$(z+1)^2 (z^2+4) = 0$ সমীকরণ সমাধান করে $f(z)$ এর পোল পাওয়া যাবে]

$$\Rightarrow (z+1)^2 = 0 \text{ and } z^2 + 4 = 0$$

$$\Rightarrow z = -1, -1 \text{ and } z = \pm 2i$$

$\therefore$ $z = -1$ is a double pole and $z = 2i$, $z = -2i$ are simple poles.

[$z = -1$ একটি দ্বিপোল এবং $z = 2i$, $z = -2i$ সরল পোল]

Residue at $z = -1$ is [$z = -1$ এ অবশেষ] $\lim\limits_{z \to -1} \frac{1}{1!} \frac{d}{dz} \{(z+1)^2 f(z)\}$

$$= \lim_{z \to -1} \frac{d}{dz} \left\{ (z+1)^2 \cdot \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} \right\}$$

$$= \lim_{z \to -1} \frac{d}{dz} \left(\frac{z^2 - 2z}{z^2 + 4} \right)$$

$$= \lim_{z \to -1} \frac{(z^2+4)(2z-2) - (z^2-2z) \cdot 2z}{(z^2+4)^2}$$

$$= \lim_{z \to -1} \frac{2z^3 + 8z - 2z^2 - 8 - 2z^3 + 4z^2}{(z^2+4)^2}$$

$$= \lim_{z \to -1} \frac{2z^2 + 8z - 8}{(z^2+4)^2}$$

$$= \frac{2 - 8 - 8}{(1+4)^2} = \frac{-14}{25}$$

Residue at $z = 2i$ is [$z = 2i$ এ অবশেষ] $\lim\limits_{z \to 2i} \left\{ (z - 2i) \cdot \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} \right\}$

$$= \lim_{z \to 2i} \left\{ (z - 2i) \cdot \frac{z^2 - 2z}{(z+1)^2 (z+2i)(z-2i)} \right\}$$

$$= \lim_{z \to 2i} \frac{z^2 - 2z}{(z+1)^2 (z+2i)}$$

$$= \frac{4i^2 - 4i}{(2i+1)^2 (4i)} = \frac{-4 - 4i}{(4i^2 + 4i + 1) 4i}$$

$$= \frac{-4 - 4i}{(4i - 3) 4i} = \frac{-1 - i}{4i^2 - 3i} = \frac{-1 - i}{-4 - 3i}$$

$$= \frac{1 + i}{4 + 3i} \times \frac{4 - 3i}{4 - 3i} = \frac{4 - 3i + 4i - 3i^2}{16 - 9i^2}$$

$$= \frac{4 + i + 3}{16 + 9} = \frac{7 + i}{25}$$

Residue at $z = -2i$ is [$z = -2i$ এ অবশেষ]

$$\lim_{z \to -2i} \left\{ (z + 2i) \cdot \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} \right\}$$

$$= \lim_{z \to -2i} \left\{ (z + 2i) \cdot \frac{z^2 - 2z}{(z+1)^2 (z+2i)(z-2i)} \right\}$$

$$= \lim_{z \to -2i} \left\{ \frac{z^2 - 2z}{(z+1)^2 (z-2i)} \right\}$$

$$= \frac{4i^2 + 4i}{(-2i+1)^2 (-4i)} = \frac{-4 + 4i}{(4i^2 - 4i + 1)(-4i)}$$

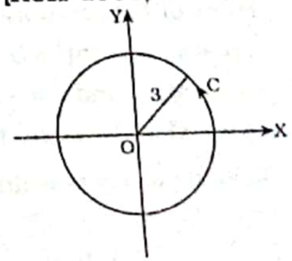
$$= \frac{1 - i}{(-4 - 4i + 1)i} = \frac{1 - i}{-4i + 4 + i} = \frac{1 - i}{4 - 3i}$$

$$= \frac{1 - i}{4 - 3i} \times \frac{4 + 3i}{4 + 3i} = \frac{4 - 4i + 3i - 3i^2}{16 - 9i^2}$$

$$= \frac{7 - i}{25}$$

Problem–12. Show that $\oint_C \frac{e^{tz}}{(z^2+1)^2} dz = \pi i(\sin t - t \cos t)$, where C is the circle $|z| = 3$ and $t > 0$ **[RUH–1980, 1982, 1996]**

Solution : Here the circle is $|z| = 3$.



The poles of $\frac{e^{tz}}{(z^2+1)^2}$ are obtained by solving the equation $(z^2+1)^2 = 0$

$$\Rightarrow \{(z+i)(z-i)\}^2 = 0$$

$\Rightarrow z = i, -i$ both of are double poles and lie inside C, since $|i| = |-i| = 1 < 3$.

Residue of $z = i$ is $\lim\limits_{z\to i} \frac{1}{1!}\frac{d}{dz}\left\{(z-i)^2 \cdot \frac{e^{tz}}{(z-i)^2(z+i)^2}\right\}$

$= \lim\limits_{z\to i} \frac{d}{dz}\left\{\frac{e^{tz}}{(z+i)^2}\right\}$

$= \lim\limits_{z\to i} \frac{(z+i)^2 \cdot te^{tz} - e^{tz} \cdot 2(z+i)}{(z+i)^4}$

$= \lim\limits_{z\to i} \frac{(z+i)\, te^{tz} - 2e^{tz}}{(z+i)^3}$

$= \frac{2i + e^{it} - 2e^{it}}{8i^3} = \frac{(it-1)e^{it}}{-4i}$

$= \frac{-(t+i)\, e^{it}}{4}$

Similarly residue at $z = -i$ is $\frac{-(t-i)\, e^{-it}}{4}$

[Replacing i by – i in the above result]

Therefore by Cauchy's residue theorem we have

$\oint_C \frac{e^{tz}}{(z^2+1)^2} dz = 2\pi i$ [sum of the residues]

$= -2\pi i\left[\frac{(t+i)\, e^{it}}{4} + \frac{(t-i)\, e^{-it}}{4}\right]$

$= -2\pi i\left[\frac{t(e^{it}+e^{-it})}{4} + \frac{i}{4}(e^{it} - e^{-it})\right]$

$= -2\pi i\left[\frac{2t\cos t}{4} + \frac{i}{4}\cdot 2i\sin t\right]$

$= -\pi i\, [t\cos t - \sin t]$

$= \pi i\, (\sin t - t\cos t)$ (Showed)

Problem–13. Show that $\oint_C \frac{e^z}{(z^2+\pi^2)^2} dz = \frac{1}{\pi}$, where C is the circle $|z| = 4$.

[CUH–1987]

Solution : Let $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$

Poles of f(z) are obtained from the equation

$(z+\pi i)^2(z-\pi i)^2 = 0$

$\Rightarrow z = \pi i, \pi i$ and $z = -\pi i, -\pi i$

$\Rightarrow z = \pi i$ and $-\pi i$ are two poles of f(z) each of double poles.

Residue at $z = \pi i$ is $\lim\limits_{z\to\pi i} \frac{1}{1!}\frac{d}{dz}\left\{(z-i\pi)^2 \cdot \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}\right\}$

$= \lim\limits_{z\to\pi i} \cdot \frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\}$

$= \lim\limits_{z\to\pi i} \frac{(z+\pi i)^2 \cdot e^z - e^z \cdot 2(z+\pi i)}{(z+\pi i)^4}$

$= \lim\limits_{z\to\pi i} \frac{e^z(z+\pi i - 2)}{(z+\pi i)^3} = \frac{e^{i\pi}(2i\pi - 2)}{(2i\pi)^3}$

$= \frac{2(i\pi - 1)\, e^{i\pi}}{-8i\pi^3} = \frac{(1-i\pi)\, e^{i\pi}}{4i\pi^3} = \frac{-(\pi+i)}{4\pi^3} e^{i\pi}$

Similarly, residue at $z = -\pi i$ is $\frac{-(\pi+i)}{4\pi^3} e^{-i\pi}$.

Therefore, by Cauchy's residue theorem we have

$\oint_C \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i$ [Sum of the residues]

$= 2\pi i\left[\frac{-(\pi+i)\, e^{i\pi}}{4\pi^3} - \frac{(\pi-i)}{4\pi^3} e^{-i\pi}\right]$

$= \frac{i}{2\pi^2}[-\pi(e^{i\pi} + e^{-i\pi}) - i\,(e^{i\pi} - e^{-i\pi})]$

$= \frac{i}{2\pi^2}[-2\pi\cos\pi - i \cdot 2i\sin\pi]$

$= \frac{i}{2\pi^2}[-2\pi \cdot (-1) - 2i^2 \times 0]$

$= \frac{i}{\pi}$ (Showed)

Problem–14. Show that

$$I = \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz = \frac{1}{2}(t-1) + \frac{1}{2}e^{-t}\cos t,$$

where C is the circle with equation $|z| = 3$.

[NU(Phy)–2004, CUH–1988, RUH–1984]

Solution : Poles of $\frac{e^{zt}}{z^2(z^2+2z+2)}$ are obtained by solving

$z^2(z^2+2z+2) = 0$

[$z^2(z^2+2z+2) = 0$ সমীকরণ সমাধান করে $\frac{e^{zt}}{z^2(z^2+2z+2)}$ এর পোল পাওয়া যাবে]

$\Rightarrow z^2 = 0$ and [এবং] $z^2 + 2z + 2 = 0$

$\Rightarrow z = 0, 0$ and [এবং] $(z+1)^2 = -1 = i^2$

$\Rightarrow z + 1 = \pm i$

$\Rightarrow z = -1 \pm i$

$|-1+i| = \sqrt{1+1} = \sqrt{2} < 3$

$|-1-i| = \sqrt{1+1} = \sqrt{2} < 3$

$\therefore$ $z = 0$ is a pole of order 2 inside C. [C এর ভিতর 2 ক্রমের পোল $z = 0$]

$z = -1 + i$ is a pole of order 1 inside C. [C এর ভিতর এক ক্রমের পোল $z = -1 + i$]

$z = -1 - i$ is a pole of order 1 inside C. [C এর ভিতর এক ক্রমের পোল $z = -1 - i$]

Now residue at $z = 0$ is [এখন $z = 0$ এ অবশেষ]

$$\lim_{z \to 0} \frac{1}{1!}\frac{d}{dz}\left\{z^2 \cdot \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\right\}$$

$$= \lim_{z \to 0} \frac{d}{dz}\left\{\frac{e^{zt}}{z^2 + 2z + 2}\right\}$$

$$= \lim_{z \to 0} \frac{(z^2 + 2z + 2) \cdot te^{zt} - e^{zt}(2z + 2)}{(z^2 + 2z + 2)^2}$$

$$= \frac{(0 + 0 + 2)\, te^0 - e^0(0 + 2)}{(0 + 0 + 2)^2} = \frac{2t - 2}{4} = \frac{t - 1}{2}$$

Residue at $z = -1 + i$ is [$z = -1 + i$ এ অবশেষ]

$$\lim_{z \to -1+i}\left\{(z + 1 - i) \cdot \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\right\}$$

$$= \lim_{z \to -1+i}\left\{(z + 1 - i) \cdot \frac{e^{zt}}{z^2(z + 1 - i)(z + 1 + i)}\right\}$$

$$= \lim_{z \to -1+i}\left\{\frac{e^{zt}}{z^2(z + 1 + i)}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1 + i)^2(-1 + i + 1 + i)}$$

$$= \frac{e^{(-1+i)t}}{(1 - 2i + i^2)(2i)}$$

$$= \frac{e^{(-1+i)t}}{(1 - 2i - 1)(2i)}$$

$$= \frac{e^{(-1+i)t}}{-4i^2} = \frac{e^{-t}}{4} e^{it}$$

Similarly, residue at $z = -1 - i$ is [অনুরূপে, $z = -1 - i$ এ অবশেষ]

$\frac{e^{-t}}{4} e^{-it}$ [Replacing i by $-i$ in the above result]

Therefore by Cauchy's residue theorem we have [অতএব কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$I = \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\, dz = \frac{1}{2}(t - 1) + \frac{e^{-t}}{4} e^{it} + \frac{e^{-t}}{4} e^{-it}$$

$$= \frac{1}{2}(t - 1) + \frac{e^{-t}}{4}(e^{it} + e^{-it})$$

$$= \frac{1}{2}(t - 1) + \frac{e^{-t}}{4} \cdot 2\cos t$$

$$= \frac{t - 1}{2} + \frac{1}{2} e^{-t} \cos t \qquad \text{(Showed)}$$

Problem-15. Show that $\oint_C \frac{zf'(z)}{f(z)}\, dz = 4\pi i$, where $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$ and C is the circle $|z| = 5$. **[JUH-1989]**

Solution : Given that $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$

$\Rightarrow f(z) = z^3(z - 2) + z(z - 2) - 10(z - 2)$

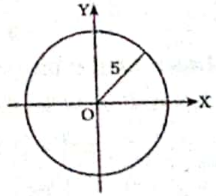
$= (z - 2)(z^3 + z - 10)$

$= (z - 2)\{z^2(z - 2) + 2z(z - 2) + 5(z - 2)\}$

$= (z - 2)^2 (z^2 + 2z + 5)$

$= (z - 2)^2 \{z - (-1 + 2i)\}\{z - (-1 - 2i)\}$

The zeros of f(z) are obtained from



$f(z) = 0$

$\Rightarrow (z - 2)^2 \{z - (-1 + 2i)\}\{z - (-1 - 2i)\} = 0$

$\Rightarrow z = 2, 2, -1 + 2i, -1 - 2i$

$|2| = 2 < 5$

$|-1 + 2i| = \sqrt{1 + 4} = \sqrt{5} < 5$

$|-1 - 2i| = \sqrt{1 + 4} = \sqrt{5} < 5$

There are 2 zeros of order 2 and two other simple zeros. All zeros are inside C.

Again, f(z) has no poles in C. Also here $g(z) = z$

$$\frac{1}{2\pi i}\oint_C g(z)\frac{f'(z)}{f(z)}\, dz = [2 \cdot g(2) + 1 \cdot g(-1 + 2i) + 1 \cdot g(-1 - 2i)] - 0$$

$$\Rightarrow \oint_C \frac{z\, f'(z)}{f(z)}\, dz = 2\pi i\, [2 \cdot 2 + (-1 + 2i) + (-1 - 2i)]$$

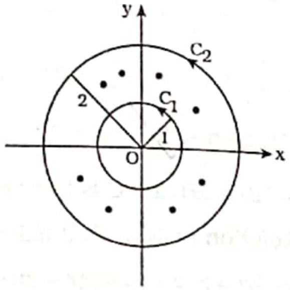
$$= 2\pi i\, [4 - 1 + 2i - 1 - 2i]$$

$$= 2\pi i\, (2) = 4\pi i \qquad \text{(Ans)}$$

Problem–16. Show that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

[DUH–2006, DUH–1989, RUH–1984, CUH–1987]

Solution : Let the circle $|z| = 1$ and $|z| = 2$ are C_1 and C_2 respectively.



Let $f(z) = 12$ and $g(z) = z^7 - 5z^3$.

Then on C_1 we have

$$|g(z)| = |z^7 - 5z^3| \le |z^7| + 5|z^3|$$

$$\Rightarrow |g(z)| \le 1 + 5$$

$$\Rightarrow |g(z)| \le 6 < 12 < f(z)$$

$$\Rightarrow |g(z)| < f(z).$$

Also, both $f(z)$ and $g(z)$ are analytic inside and on C_1.

Therefore, by Rouche's theorem

$$f(z) + g(z) = z^7 - 5z^3 + 12 \text{ and } f(z) = 12$$

have the same number of zeros inside C_1. Evidently, $f(z) = 12$ has no zero inside C_1 and so the given equation

$$f(z) + g(z) = z^7 - 5z^3 + 12 = 0 \;\ldots\ldots\; (1)$$

has no zeros inside C_1.

Again, For C_2, let $f(z) = z^7$ and $g(z) = -5z^3 + 12$.

$$\therefore |g(z)| = |-5z^3 + 12| \le |-5z^3| + |12|$$

$$\Rightarrow g(z) \le 40 + 12 = 52 < 2^7 = |f(z)|$$

$$\Rightarrow g(z) < |f(z)|$$

Both $f(z)$ and $g(z)$ are analytic inside and on C_2.

Therefore, by Rouche's theorem

$$f(z) + g(z) = z^7 - 5z^3 + 12 \text{ and } f(z) = z^7$$

have the same number of zeros inside C_2. Here $f(z) = z^7$ have seven zeros all lie inside C_2.

$$\therefore f(z) + g(z) = z^7 - 5z^3 + 12 = 0 \;\ldots\ldots\; (2)$$

has all zeros inside C_2

From (1) and (2) we have that the roots of the given equation lie between the circles C_1 and C_2, that is, $|z| = 1$ and $|z| = 2$.

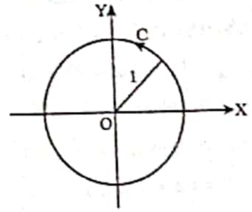
(Showed)

Problem–17. If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the circle $|z| = 1$.

[NUH–2004, 2007, 2012(Old), DUH–1986, RUMP–1988]

Solution : Let C be circle [মনে করি C বৃত্তটি] $|z| = 1$ and [এবং] $f(z) = az^n$ and [এবং] $g(z) = -e^z$.

On the circle C we have [C বৃত্তের উপর পাই]



$$|f(z)| = |az^n|$$

$$\Rightarrow |f(z)| = |a|\;|z^n| = a|z|^n$$

$$= a \cdot 1 = a > e \;\ldots\ldots\; (1)$$

$$|g(z)| = |-e^z| = |e^z|$$

$$= \left|1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right|$$

$$\Rightarrow |g(z)| \le 1 + |z| + \frac{1}{2!}|z|^2 + \frac{1}{3!}|z|^3 + \cdots$$

$$\Rightarrow |g(z)| \le 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$$

$$\Rightarrow |g(z)| \le e < a = |f(z)|$$

$$\Rightarrow |g(z)| < f(z) \quad \text{by (1)}$$

Therefore, by the Rouche's theorem, $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C. Here $f(z) = az^n$ gives n zeros inside C. Thus, the given equation has n zeros inside the circle $|z| = 1$. [অতএব, রচির উপপাদ্য দ্বারা C এর ভিতর $f(z) + g(z)$ ও $f(z)$ এর একই সংখ্যক শূন্য থাকবে। এখানে C এর ভিতর $f(z) = az^n$, n সংখ্যক শূন্য দেয়। অতএব প্রদত্ত সমীকরণের $|z| = 1$ বৃত্তের ভিতর n সংখ্যক মূল আছে।]

Problem–18. If $a > e$, then show that the equation $az^n = e^z$ has n roots inside the region $|z| < \frac{1}{2}$. **[NUH–2006, DUH–1989]**

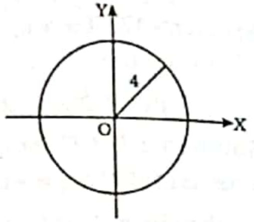
Solution : Do as 17.

Problem–19. Show that $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = -2$, where $f(z) = \frac{(z^2+1)^2}{(z^2+2z+2)^3}$ and C is the circle $|z| = 4$.

[RUH–1988, CUH–1983]

Solution : Here C is the circle $|z| = 4$ and $f(z) = \frac{(z^2+1)^2}{(z^2+2z+2)^3}$ which is analytic inside and on C.

Zeros of f(z) are obtained from



$$(z^2+1)^2=0$$
$$\Rightarrow (z+i)^2(z-i)^2=0$$
$$\Rightarrow z=-i,-i,i,i$$
$$|z|=|-i| \text{ or } |i|=1<4$$

There are two zeros each of order 2 lying in C.

Again pole of f(z) can be obtained from

$$(z^2+2z+2)^3=0$$
$$\Rightarrow z^2+2z+2=0$$
$$\Rightarrow (z+1)^2=-1=i^2$$
$$\Rightarrow z+1=\pm i$$
$$\Rightarrow z=-1\pm i$$
$$|z|=|-1+i|=\sqrt{1+1}=\sqrt{2}<4$$
$$\text{and } |z|=|-1-i|=\sqrt{1+1}=\sqrt{2}<4$$

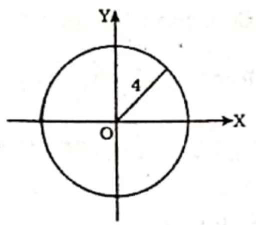
There are two poles each of order 3 lying in C. Thus by general argument theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz=N-P$$
$$=(2+2)-(3+3)=-2 \qquad \text{(Showed)}$$

Problem–20. For the function $f(z)=\dfrac{(z^2+1)^2(z+5)}{(z^2+2z+2)^3}$, find $\displaystyle\oint_C \frac{f'(z)}{f(z)}dz$ Where C is the circle $|z|=4$.

Solution : Here C is the circle $|z|=4$ and $f(z)=\dfrac{(z^2+1)^2(z+5)}{(z^2+2z+2)^3}$ which is analytic inside and on C.

Zeros of f(z) are obtained from the equation f(z) = 0



$$\Rightarrow (z^2+1)^2(z+5)=0$$
$$\Rightarrow (z+i)^2(z-i)^2(z+5)=0$$
$$\Rightarrow z=i,-i,-5$$

$z=i,-i$ lie in C but $z=-5$ lie outside C.

$z=i$ and $-i$ are zeros of order 2.

Poles of f(z) are obtained from the equation

$$(z^2+2z+2)^3=0$$
$$\Rightarrow z^2+2z+2=0$$
$$\Rightarrow (z+1)^2=-1=i^2$$
$$\Rightarrow z+1=\pm i$$
$$\Rightarrow z=-1\pm i$$
$$|z|=|-1+i|=\sqrt{1+1}=\sqrt{2}<4$$
$$\text{and } |z|=|-1-i|=\sqrt{1+1}=\sqrt{2}<4$$

$\therefore z=-1+i$ and $z=-1-i$ are poles each of order 3 and lie in C.

Thus by general argument theorem we have

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz=N-P$$
$$=(2+2)-(3+3)=-2$$
$$\Rightarrow \oint_C \frac{f'(z)}{f(z)}dz=-4\pi i$$

Problem–21. If C is the circle $|z|=\pi$, then show that

$$\oint_C \frac{f'(z)}{f(z)}dz=\begin{cases}14\pi i & \text{if } f(z)=\sin \pi z\\ 12\pi i & \text{if } f(z)=\cos \pi z\\ 2\pi i & \text{if } f(z)=\tan \pi z\end{cases}$$

[RUH–1985, 2000]

Solution : First Part : Here C is the circle $|z|=\pi=3{\cdot}14$. $f(z)=\sin \pi z$ is analytic inside and on C. It has no poles inside C. The zeros of f(z) can be obtained from

$$f(z)=\sin \pi z=0$$

$\Rightarrow \pi z=n\pi \Rightarrow z=n$, where $n=0,\pm 1,\pm 2,\pm 3,\pm 4,\cdots$ etc. Among these $0,1,-1,2,-2,3,-3$, lie with in C and each of them are simple zero. Hence by general argument theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz=N-P$$
$$=(1+1+1+1+1+1+1)-0=7-0=7$$
$$\Rightarrow \oint_C \frac{f'(z)}{f(z)}dz=14\pi i \qquad \text{(Showed)}$$

Second part : Here $f(z) = \cos \pi z$. There are no poles of $f(z)$ inside C. The zeros of $f(z)$ are obtained from

$$f(z) = \cos \pi z = 0$$
$$\Rightarrow \pi z = (2n+1)\frac{\pi}{2}$$
$$\Rightarrow z = n + \frac{1}{2}, \text{ where } n = 0, \pm 1, \pm 2, \cdots \text{ etc.}$$

Only the zeros $\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{-3}{2}, \frac{5}{2}, \frac{-5}{2}$ lie with in C and the order of each is one. Hence by general argument theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P$$
$$= (1+1+1+1+1+1) - 0 = 6 - 0 = 6$$
$$\Rightarrow \oint_C \frac{f'(z)}{f(z)}\,dz = 12\pi i \qquad \text{(Showed)}$$

Third Part : Here $f(z) = \tan \pi z = \frac{\sin \pi z}{\cos \pi z}$.

The zeros of $f(z)$ can be obtained from

$$f(z) = \tan \pi z = 0$$
$$\Rightarrow \sin \pi z = 0$$
$$\Rightarrow \pi z = n\pi$$
$$\Rightarrow z = n,\ n = 0, \pm 1, \pm 2, \pm 3, \cdots \text{ etc.}$$

The zeros with in C are $0, 1, -1, 2, -2, 3, -3$, each of order 1.

The poles are obtained from

$$\cos \pi z = 0 \Rightarrow \pi z = (2n+1)\frac{\pi}{2}$$
$$\Rightarrow z = n + \frac{1}{2},\ n = 0, \pm 1, \pm 2, \cdots \text{ etc.}$$

The poles inside C are $\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{-3}{2}, \frac{5}{2}, \frac{-5}{2}$ each of order 1. Hence by general argument theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P$$
$$= (1+1+1+1+1+1+1) - (1+1+1+1+1+1)$$
$$= 7 - 6 = 1$$
$$\Rightarrow \oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i \qquad \text{(Showed)}$$

Problem–22. Evaluate $\oint_C \frac{e^{3z}}{z+\pi i}\,dz$, where C is the circle $|z+1| = 4$. **[DUH–1994]**

Solution : The given circle is C : $|z+1| = 4$

$$\oint_C \frac{e^{3z}}{z+\pi i}\,dz = \oint_C f(z)\,dz$$

where $f(z) = \frac{e^{3z}}{z+\pi i}$

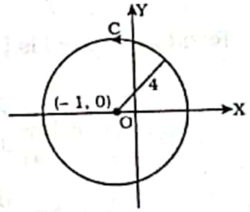
The poles of $f(z)$ is given by the equation $z + \pi i = 0 \Rightarrow z = -\pi i$

$$|z| = |-\pi i| = \pi < 4.$$

$\therefore$ The pole $z = -\pi i$ lie inside C.

Residue at $z = -\pi i$ is $\lim_{z \to -\pi i} (z + i\pi) f(z)$

$$= \lim_{z \to -\pi i} (z + i\pi)\frac{e^{3z}}{z+\pi i}$$
$$= \lim_{z \to -\pi i} e^{3z} = e^{-i3\pi}$$
$$= \cos 3\pi - i \sin 3\pi$$
$$= (-1) - 0 = -1$$



Therefore, by Cauchy's residue theorem

$$\oint_C \frac{e^{3z}}{z+\pi i}\,dz = 2\pi i(-1) = -2\pi i \qquad \text{Ans.}$$

Problem–23. Evaluate the integral $\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2}\,dz$, $C = \{z : z = 1 + 2e^{i\theta}, 0 \le \theta \le 2\pi\}$ using Cauchy's residue theorem. **[NUH–2005, 2008, 2010]**

Solution : Equation of the given curve is [প্রদত্ত বক্ররেখার সমীকরণ]

$$z = 1 + 2e^{i\theta}$$
$$\Rightarrow z - 1 = 2e^{i\theta}$$
$$\Rightarrow |z-1| = |2e^{i\theta}|$$
$$\Rightarrow |z-1| = 2 \qquad \because |e^{i\theta}| = 1$$

$\therefore$ The given cure is a circle whose centre is (1, 0) and radius is 2. [প্রদত্ত বক্ররেখা একটি বৃত্ত যার কেন্দ্র (1, 0) এবং ব্যাসার্ধ 2]

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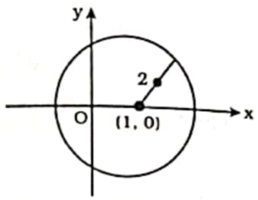
Now [এখন] $\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2} = \oint_C f(z)\,dz$, say

where [যেখানে] $f(z) = \frac{e^{-iz}}{(z+3)(z-i)^2}$

The poles of f(z) are given by $(z+3)(z-i)^2 = 0$

[$(z+3)(z-i)^2 = 0$ দ্বারা f(z) এর পোল পাওয়া যাবে]



$$\Rightarrow z = -3 \text{ and } z = i, i$$

$$|-3| = 3 > 2 \text{ and } |i| = 1 < 2$$

$\therefore$ The pole $z = i$ lies inside the circle which is a double pole (pole of order 2). [$z = i$ পোলটি বৃত্তের ভিতর অবস্থিত যাহা দ্বিপোল।]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim\limits_{z\to i} \frac{1}{\lfloor 2-1} \frac{d}{dz} [(z-i)^2 \cdot f(z)]$

$$= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{e^{-iz}}{(z+3)(z-i)^2}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{e^{-iz}}{z+3}\right]$$

$$= \lim_{z\to i} \frac{(z+3)(-ie^{-iz}) - e^{-iz} \cdot 1}{(z+3)^2}$$

$$= \frac{(i+3)(-ie) - e}{(i+3)^2}$$

$$= \frac{(1-3i-1)\,e}{i^2+6i+9}$$

$$= \frac{-3ie}{8+6i}$$

$$= \frac{-3ie}{2(4+3i)} \times \frac{4-3i}{4-3i}$$

$$= \frac{-12ie-9e}{2(16+9)} = \frac{-12ie-9e}{50}$$

$\therefore$ By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2} = 2\pi i \cdot (\text{Residue at } z = i)$$

$$= 2\pi i \cdot \frac{-12ie-9e}{50}$$

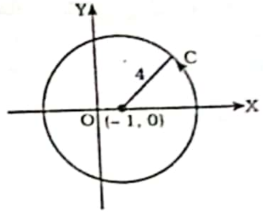
$$= \frac{(12-i9)\,\pi e}{25}. \quad \textbf{(Ans)}$$

Problem-24. Evaluate the value of $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$, where C is a curve (a) $|z-1| = 4$ and (b) $|z-2| + |z+2| = 6$ **[NUH–1999]**

Solution : (a) Given [দেওয়া আছে] $|z-1| = 4$

Let $I = \oint_C \frac{e^{3z}}{z-\pi i}\,dz = \oint_C f(z)\,dz$

where [যেখানে] $f(z) = \frac{e^{3z}}{z-\pi i}$



Pole of f(z) is $z = \pi i$ [f(z) এর পোল $z = \pi i$]

$$|z| = |\pi i| = \pi < 4$$

$\therefore$ The pole $z = \pi i$ lie inside C. [$z = \pi i$ পোলটি C এর ভিতর অবস্থিত]

Residue at $z = \pi i$ is [$z = \pi i$ এ অবশেষ]

$$\lim_{z\to\pi i} (z-\pi i)\, f(z)$$

$$= \lim_{z\to\pi i} (z-\pi i) \frac{e^{3z}}{z-\pi i} = e^{i3\pi}$$

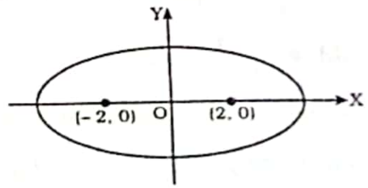
$$= \cos 3\pi + i \sin 3\pi$$

$$= -1 + i(0) = -1$$

Therefore, by Cauchy's residue theorem we have [অতএব কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{e^{3z}}{z-\pi i}\,dz = 2\pi i(-1) = -2\pi i \qquad \text{Ans.}$$

(b) In this case the curve is $|z-2| + |z+2| = 6$ which is the equation of an ellipse whose focii are (2, 0) and (– 2, 0) and length of the major axis is 6. Here the pole is $z = \pi i$.



$$\therefore |z| = |\pi i| = \pi = 3{\cdot}14 > 3$$

∴ The pole lies out side the ellipse.

Hence by Cauchy's integral theorem we have

$$\oint_C f(z)\,dz = 0$$

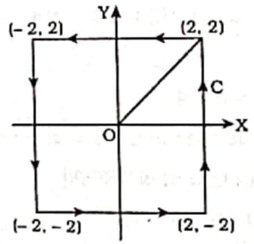
$$\Rightarrow \oint_C \frac{e^{3z}}{z-\pi i}\,dz = 0$$

Problem–25. Let C denotes the square whose sides lie along the lines $x = \pm 2$, $y = \pm 2$ described in the positive sense. Determine

(a) $\oint_C \frac{1}{z^2+9}\,dz$, (b) $\oint_C \frac{1}{z(z^2+9)}\,dz$, (c) $\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz$.

[NUH–1996]

Solution :



(a) $\oint_C \frac{1}{z^2+9}\,dz$

Poles are obtained from the equation $z^2+9=0$ [$z^2+9=0$ সমীকরণ হতে পোল পাওয়া যাবে]

$$\Rightarrow z^2 = -9 = (3i)^2$$

$$\Rightarrow z = \pm 3i$$

Each pole lie outside the square C. [প্রত্যেক পোল C বর্গের বাহিরে অবস্থিত]

$$\therefore \oint_C \frac{1}{z^2+9}\,dz = 0 \qquad \text{Ans.}$$

(b) $\oint_C \frac{1}{z(z^2+9)}\,dz$

Here the poles are [এখানে পোলগুলি হল] $z = 0$, $z = 3i$, $z = -3i$

Only the pole $z = 0$ lie inside C.

Residue at $z = 0$ is [$z = 0$ এ অবশেষ]

$$\lim_{z\to 0} z \cdot \frac{1}{z(z^2+9)} = \lim_{z\to 0} \frac{1}{z^2+9} = \frac{1}{9}$$

Therefore, by cauchy's residue theorem [অতএব, কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{1}{z(z^2+9)}\,dz = 2\pi i\left(\frac{1}{9}\right) = \frac{2\pi i}{9} \qquad \text{Ans.}$$

(c) $\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz$

Poles are obtained from the equation $(z^2+1)(z^2+9)=0$ [পোলগুলি $(z^2+1)(z^2+9)=0$ সমীকরণ হতে পাওয়া যায়।]

$$\Rightarrow (z+i)(z-i)(z+3i)(z-3i) = 0$$

$$\Rightarrow z = i, -i, 3i, -3i$$

Only the poles $i, -i$ lie inside C which are simple poles. [শুধুমাত্র $i, -i$ পোলগুলি C এর ভিতর অবস্থিত যারা সরল পোল]

Residue at $z = i$ is [$z = i$ এ অবশেষ] $\lim_{z\to i}\left\{(z-i)\cdot\frac{1}{(z^2+1)(z^2+9)}\right\}$

$$= \lim_{z\to i} \frac{1}{(z+i)(z^2+9)}$$

$$= \frac{1}{2i(i^2+9)} = \frac{1}{16i} = \frac{-i}{16}$$

Similarly, residue at $z = -i$ is [অনুরূপে, $z = -i$ এ অবশেষ] $\frac{i}{16}$

[Replacing i by $-i$ in the above result]

Therefore, by Cauchy's residue theorem we have [অতএব, কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz = 2\pi i\,(\text{sum of the residues})$$

$$= 2\pi i\left(\frac{-i}{16} + \frac{i}{16}\right) = 0 \text{ Ans.}$$

Problem–26. Expand $\log\left(\frac{1+z}{1-z}\right)$ in a Taylor series about $z=0$

[DUH–1988, 1990, DUMP–1991]

Solution : When $|z|<1$ then we have

$$\log(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}-\ldots\ldots \quad (1)$$

$$\text{and } \log(1-z)=-z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}-\frac{z^5}{5}\ldots\ldots \quad (2)$$

(1) – (2) gives, $\log(1+z)-\log(1-z)$

$$=2z+\frac{2z^3}{3}+\frac{2}{5}z^5+\cdots$$

$$\Rightarrow \log\left(\frac{1+z}{1-z}\right)=2\left(z+\frac{z^3}{3}+\frac{z^5}{5}+\cdots\right)$$

This series converses for $|z|<1$. This series also converges for $|z|=1$ except $z=-1$

Problem–26(a). Expand $f(z)=\sin z$ in a Taylor series about $z=\frac{\pi}{4}$ **[NUH–2007, 2010, 2012(Old)]**

Solution : Given that [দেওয়া আছে]

$$f(z)=\sin z \qquad \therefore f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$\therefore f'(z)=\cos z \qquad \therefore f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f''(z)=-\sin z \qquad \therefore f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f'''(z)=-\cos z \qquad \therefore f'''\left(\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f^{iv}(z)=\sin z \qquad \therefore f^{iv}\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

and so on.

Therefore, the Taylor series about $z=\frac{\pi}{4}$ is [অতএব $z=\frac{\pi}{4}$ এ টেইলর ধারা]

$$f(z)=f\left(\frac{\pi}{4}\right)+\left(z-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right)+\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)+\frac{\left(z-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right)+\cdots$$

$$=\frac{1}{\sqrt{2}}+\left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}-\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\frac{1}{\sqrt{2}}-\frac{\left(z-\frac{\pi}{4}\right)^3}{3!}\frac{1}{\sqrt{2}}+\cdots$$

$$=\frac{1}{\sqrt{2}}\left[1+\left(z-\frac{\pi}{4}\right)-\frac{\left(z-\frac{\pi}{4}\right)^2}{2!}-\frac{\left(z-\frac{\pi}{4}\right)^3}{3!}+\cdots\right]. \quad \textbf{(Ans)}$$

Problem–26(b). $z=0$ বিন্দুর চারিদিকে $f(z)=ln(1+z)$ ফাংশনটিকে Taylor এর ধারায় বিস্তার কর এবং ধারাটির জন্য convergence রিজিয়ন নির্ণয় কর। [State Taylor's theorem for the complex function $f(z)$. Expand $f(z)=ln\,(1+z)$ in Taylor's series about $z=0$ and determine the region of convergence for the series.]

Solution : দেওয়া আছে [Given] $f(z)=ln\,(1+z)$ $\therefore f(0)=ln\,1=0$

$$\therefore f'(z)=\frac{1}{1+z} \qquad f'(0)=\frac{1}{1+0}=1$$

$$f''(z)=\frac{-1}{(1+z)^2} \qquad f''(0)=-1$$

$$f'''(z)=(-1)(-2)\cdot\frac{1}{(1+z)^3} \qquad f'''(0)=(-1)(-2)=(-1)^2\lfloor\underline{2}$$

$$f''''(z)=(-1)(-2)(-3)\cdot\frac{1}{(1+z)^4}, \qquad f^{iv}(0)=(-1)(-2)(-3)=(-1)^3\lfloor\underline{3}$$

$$\cdots \quad \cdots \quad \cdots \qquad \cdots \quad \cdots \quad \cdots$$

$$f^n(z)=\frac{(-1)^{n-1}\lfloor\underline{n-1}}{(1+z)^n} \qquad f^n(0)=(-1)^{n-1}\lfloor\underline{n-1}$$

অতএব, $z=0$ বিন্দুর চারিদিকে টেইলরের উপপাদ্য [Therefore, the Taylor's series about the point $z=0$ is]

$$f(z)=f(0)+(z-0)f'(0)+\frac{(z-0)^2}{\lfloor\underline{2}}f''(0)+\frac{(z-0)^3}{\lfloor\underline{3}}f'''(0)$$

$$+\frac{(z-0)^4}{\lfloor\underline{4}}f^{iv}(0)+\cdots$$

$$=0+z\cdot 1+\frac{z^2}{\lfloor\underline{2}}\cdot(-1)+\frac{z^3}{\lfloor\underline{3}}(-1)^2\lfloor\underline{2}+\frac{z^4}{\lfloor\underline{4}}(-1)^3\lfloor\underline{3}+\cdots$$

$$=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\cdots$$

অভিসারী এলাকা (**Convergence region**) : আমরা জানি $ln(1 + z)$ অভিসারী হয় যখন $|z| < 1$ হয়। [We know that $ln(1 + z)$ is convergent when $|z| < 1$]

Problem-27. Expand the function $f(z) = \frac{1}{z-3}$ in a Laurent series for the region : (i) $|z| < 3$. (ii) $|z| > 3$ **[DUH-1983]**

Solution : (i) We express f(z) in a manner so that the binomial expansion is valid for $|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$.

$$\therefore f(z) = \frac{1}{z-3} = \frac{1}{-3\left(1-\frac{z}{3}\right)} = \frac{-1}{3}\left(1-\frac{z}{3}\right)^{-1}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right)$$

$$= -\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\frac{z^3}{81}-\cdots \quad \text{Ans.}$$

(ii) In this case $|z| > 3 \Rightarrow \frac{|z|}{3} > 1 \Rightarrow \left|\frac{3}{z}\right| < 1$

$$\therefore f(z) = \frac{1}{z-3} = \frac{1}{z\left(1-\frac{3}{z}\right)} = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\cdots\right)$$

$$= \frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\cdots \quad \text{Ans.}$$

Problem-28. Expand $f(z) = \frac{z}{(z+1)(z+2)}$ in a laurent series for the region $1 < |z| < 2$ **[DUH-1987]**

Solution : $1 < |z| < 2$

$$\Rightarrow 1 < |z| \text{ and } |z| < 2$$

$$\Rightarrow \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1$$

We expand f(z) in a way such that the bionomial expansion is valid for $1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$.

$$\therefore f(z) = \frac{z}{(z+1)(z+2)} = \frac{2}{z+2} - \frac{1}{z+1}$$

$$= \frac{2}{2\left(1+\frac{z}{2}\right)} - \frac{1}{z\left(1+\frac{1}{z}\right)}$$

$$= \left(1+\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right) - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)$$

$$= \left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\right) - \left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right) \quad \text{Ans.}$$

Problem-29. Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent series for the region (i) $0 < |z| < 2$; (ii) $|z| > 2$. **[DUMP-1991]**

Solution : (i) In this case $0 < |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$.

We expand f(z) in a manner so that bionomial expansion is valid in $0 < |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$.

$$\therefore f(z) = \frac{1}{z(z-2)} = \frac{1}{2}\left(\frac{1}{z-2}-\frac{1}{z}\right)$$

$$= \frac{-1}{4\left(1-\frac{z}{2}\right)} - \frac{1}{2z}$$

$$= -\frac{1}{4}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{2z}$$

$$= -\frac{1}{2z} - \frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right)$$

$$= -\frac{1}{2z}-\frac{1}{4}-\frac{z}{8}-\frac{z^2}{16}-\frac{z^3}{32}-\cdots \quad \text{Ans.}$$

(ii) In $|z| > 2$ we have $\left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$\therefore f(z) = \frac{1}{z(z-2)} = \frac{1}{2}\left(\frac{1}{z-2}-\frac{1}{z}\right)$$

$$= \frac{1}{2z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{2z}$$

$$= -\frac{1}{2z}+\frac{1}{2z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\frac{2^4}{z^4}+\cdots\right)$$

$$= -\frac{1}{2z}+\frac{1}{2z}+\frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+\frac{8}{z^5}+\cdots$$

$$=\frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+\frac{8}{z^5}+\cdots \qquad \text{Ans.}$$

Problem-30. Expand $f(z)=\frac{z^2}{(z-1)(z-2)}$ in a laurent series for the region $1<|z|<2$ **[NUH-04 (Old) 08, DUH-1971, 85, 88, 94]** and $0<|z|<1$. **[NUH-2004, 2008]**

Solution : Here [এখানে] $f(z)=\frac{z^2}{(z-1)(z-2)}=1+\frac{A}{z-1}+\frac{B}{z-2}$, say

Then [তখন] $z^2=(z-1)(z-2)+A(z-2)+B(z-1)$

Putting $z=1$ [$z=1$ বসাইয়া] $1=-A\Rightarrow A=-1$

Putting $z=2$ [$z=2$ বসাইয়া] $4=B$

$$\therefore f(z)=1-\frac{1}{z-1}+\frac{4}{z-2} \quad \cdots\cdots (1)$$

We expand f(z) in a manner so that the binomial expansion is valid in

$$1<|z|<2\Rightarrow\frac{1}{|z|}<1 \text{ and } \left|\frac{z}{2}\right|<1.$$

[আমরা f(z) কে এমন রীতিতে বিস্তৃত করব যেন দ্বিপদী বিস্তর $1<|z|<2\Rightarrow\frac{1}{|z|}<1$ এবং $\left|\frac{z}{2}\right|<1$ এ বৈধ হয়।]

$$\therefore f(z)=1-\frac{1}{z-1}+\frac{4}{z-2}$$

$$=1-\frac{1}{z\left(1-\frac{1}{z}\right)}-\frac{4}{2\left(1-\frac{z}{2}\right)}$$

$$=1-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-2\left(1-\frac{z}{2}\right)^{-1}$$

$$=1-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)-2\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right)$$

$$=\left(1-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\right)-2\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right)$$

$$=\cdots-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{z}-1-z-\frac{z^2}{2}-\frac{z^3}{2^2}-\cdots \qquad \text{Ans.}$$

For 2nd part we have [দ্বিতীয় অংশের জন্য পাই] $0<|z|<1$

$$\Rightarrow |z|<1 \text{ and [এবং] } |z|<2$$

$$\Rightarrow |z|<1 \text{ and [এবং] } \left|\frac{z}{2}\right|<1$$

$$\therefore f(z)=1-\frac{1}{z-1}+\frac{4}{z-2}$$

$$=1+\frac{1}{1-z}-\frac{4}{2\left(1-\frac{z}{2}\right)}$$

$$=1+(1-z)^{-1}-2\left(1-\frac{z}{2}\right)^{-1}$$

$$=1+(1+z+z^2+z^3+\cdots)-2\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$=\frac{z^2}{2}+\frac{3}{4}z^3+\cdots \quad \textbf{(Ans)}$$

Problem-31. Expand the function $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ for the following regions :

(i) $2<|z|<3$ (ii) $|z|<2$ (iii) $|z|>3$.

[NUH-06, 2010, DUH-90, DUMP-88, 89, RUH-80, 85]

Solution : Let [ধরি] $\frac{z^2-1}{(z+2)(z+3)}=1+\frac{A}{z+2}+\frac{B}{z+3}$

$$\Rightarrow z^2-1=(z+2)(z+3)+A(z+3)+B(z+2)$$

When [যখন] $z=-2$ then [তখন] $3=A\Rightarrow A=3$

When [যখন] $z=-3$ then [তখন] $8=-B\Rightarrow B=-8$

$$\therefore f(z)=\frac{z^2-1}{(z+2)(z+3)}=1+\frac{3}{z+2}-\frac{8}{z+3} \quad \cdots\cdots (1)$$

(i) In this case we expand f(z) in a manner so that the bionomial expansion valid in $2<|z|<3$ [এইক্ষেত্রে f(z) কে এমন রীতিতে বিস্তৃত করব যেন দ্বিপদী বিস্তার $2<|z|<3$ বৈধ হয়।]

$$\Rightarrow 2<|z| \text{ and [এবং] } |z|<3$$

$$\Rightarrow \frac{2}{|z|}<1 \text{ and [এবং] } \frac{|z|}{3}<1$$

$$\Rightarrow \left|\frac{2}{z}\right|<1 \text{ and [এবং] } \left|\frac{z}{3}\right|<1.$$

Thus from (1) we get [অতএব (1) হতে পাই]

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$= \cdots + \frac{12}{z^3} - \frac{6}{z^2} + \frac{3}{z} - \frac{5}{3} + \frac{8z}{3^2} - \frac{8z^2}{3^3} + \frac{8z^3}{3^4} - \cdots \quad \text{Ans.}$$

(ii) *In this case* [এইক্ষেত্রে] $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1.$

Also [আরো] $|z| < 2 < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$

Thus from (1) we get [অতএব (1) হতে পাই]

$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$= -\frac{1}{6} - \left(\frac{3}{2^2}-\frac{8}{3^2}\right)z + \left(\frac{3}{2^3}-\frac{8}{3^3}\right)z^2 - \left(\frac{3}{2^4}-\frac{8}{3^4}\right)z^3 + \cdots \text{ Ans.}$$

(iii) In this case [এইক্ষেত্রে] $|z| > 3 \Rightarrow \frac{|z|}{3} > 1 \Rightarrow \left|\frac{3}{z}\right| < 1$

Also [আরো] $|z| > 3 \Rightarrow |z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$. Thus from (1) [অতএব (1) হতে পাই]

$$\therefore f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right) - \frac{8}{z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}+\cdots\right)$$

$$= 1 + (3-8)\frac{1}{z} + (-6+24)\frac{1}{z^2} + (12-72)\frac{1}{z^3} + \cdots$$

$$= 1 - \frac{5}{z} + \frac{18}{z^2} - \frac{60}{z^3} + \cdots \quad \text{Ans.}$$

Problem-32. Expand the function $f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$ for the following regions : (i) $1 < |z| < 4$; (ii) $|z| < 1$; (iii) $|z| > 4$.

[DUMP-1988, RUMP-1986]

Solution : Let $\frac{(z-2)(z+2)}{(z+1)(z+4)} = 1 + \frac{A}{z+1} + \frac{B}{z+4}$

$\Rightarrow (z-2)(z+2) = (z+1)(z+4) + A(z+4) + B(z+1)$

When $z = -1$ then $(-3)(1) = 3A \Rightarrow A = -1$

When $z = -4$ then $(-6)(-2) = -3B \Rightarrow B = -4.$

$$\therefore f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)} = 1 - \frac{1}{z+1} - \frac{4}{z+4} \cdots\cdots (1)$$

(i) We expand f(z) in such a way that the bicnomial expansion valid in $1 < |z| < 4$

$\Rightarrow 1 < |z|$ and $|z| < 4$

$\Rightarrow \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{4}\right| < 1$

$$\therefore f(z) = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \left(1-\frac{z}{4}+\frac{z^2}{4^2}-\frac{z^3}{4^3}+\cdots\right)$$

$$= \cdots + \frac{1}{z^4} - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + \frac{z}{4} - \frac{z^2}{4^2} + \frac{z^3}{4^3} - \cdots \text{ Ans.}$$

(ii) In this case $|z| < 1$, so $|z| < 4 \Rightarrow \left|\frac{z}{4}\right| < 1.$

Thus from (1) we vet,

$$f(z) = 1 - \frac{1}{(1+z)} - \frac{4}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - (1+z)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - (1 - z + z^2 - z^3 + \cdots) - \left(1-\frac{z}{4}+\frac{z^2}{4^2}-\frac{z^3}{4^3}+\cdots\right)$$

$$= -1 + \left(1+\frac{1}{4}\right)z - \left(1+\frac{1}{4^2}\right)z^2 + \left(1+\frac{1}{4^3}\right)z^3 + \cdots \quad \text{Ans.}$$

(iii) In this case $|z| > 4 \Rightarrow \frac{|z|}{4} > 1 \Rightarrow \left|\frac{4}{z}\right| < 1$

Also $|z| > 4 \Rightarrow |z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$. Thus from (1),

$$f(z) = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{z\left(1+\frac{4}{z}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{4}{z}\left(1 - \frac{4}{z} + \frac{4^2}{z^2} - \frac{4^3}{z^3} + \cdots\right)$$

$$= 1 - (1+4)\frac{1}{z} + (1+4^2)\frac{1}{z^2} - (1+4^3)\frac{1}{z^3} + \cdots \quad \text{Ans.}$$

Problem–33. Expand $f(z) = \frac{z^3}{(z+1)(z-2)}$ in a laurent series in the powers of $(z+1)$ in the region $0 < |z+1| < 3$. [DUH-1978]

Solution : Given that $0 < |z+1| < 3 \Rightarrow \left|\frac{z+1}{3}\right| < 1$.

$$f(z) = \frac{z^3}{(z+1)(z-2)}$$

$$= \frac{z^3+1-1}{(z+1)(z-2)} = \frac{z^3+1}{(z+1)(z-2)} - \frac{1}{(z+1)(z-2)}$$

$$= \frac{z^2-z+1}{z-2} - \frac{1}{3}\left(\frac{1}{z-2} - \frac{1}{z+1}\right)$$

$$= \frac{z(z-2)+z+1}{z-2} - \frac{\frac{1}{3}}{z-2} + \frac{\frac{1}{3}}{z+1}$$

$$= z + \frac{z-2+3}{z-2} - \frac{\frac{1}{3}}{z-2} + \frac{\frac{1}{3}}{z+1}$$

$$= z + 1 + \frac{3}{z-2} - \frac{\frac{1}{3}}{z-2} + \frac{\frac{1}{3}}{z+1}$$

$$= (z+1) + \frac{\frac{1}{3}}{z+1} + \frac{\frac{8}{3}}{z-2}$$

$$= (z+1) + \frac{1}{3(z+1)} + \frac{8}{3}\cdot\frac{1}{z+1-3}$$

$$= (z+1) + \frac{1}{3(z+1)} - \frac{8}{9}\left(1 - \frac{z+1}{3}\right)^{-1}$$

$$= (z+1) + \frac{1}{3(z+1)} - \frac{8}{9}\left\{1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \cdots\right\}$$

$$= \frac{1}{3(z+1)} - \frac{8}{9} + \frac{19}{27}(z+1) - \frac{8}{9}\left\{\frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \cdots\right\} \quad \text{Ans.}$$

Problem–34. Expand $f(z) = \frac{1}{(z^2+1)(z+2)}$ in a laurent series for the region (i) $1 < |z| < 2$, (ii) $|z| > 1$.

[NUH-2005(Old), 2011, DUH-1978]

Solution : Given that [দেওয়া আছে] $1 < |z| < 2$

$$\Rightarrow 1 < |z| \text{ and } |z| < 2$$

$$\Rightarrow \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1$$

$$\Rightarrow \left|\frac{1}{z^2}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1.$$

Let [ধরি] $\frac{1}{(z^2+1)(z+2)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+1}$

$\Rightarrow 1 = A(z^2+1) + (Bz+C)(z+2)$

Putting $z = -2$ [$z = -2$ বসাইয়া] $1 = 5A \Rightarrow A = \frac{1}{5}$

Equating the coefficients of z^2 [z^2 এর সহগ সমীকৃত করে]

$$0 = A + B \Rightarrow B = -A = -\frac{1}{5}$$

Equating the coefficients of z [z এর সহগ সমীকৃত করে]

$$0 = 2B + C \Rightarrow C = -2B = \frac{2}{5}$$

$$\therefore f(z) = \frac{1}{(z^2+1)(z+2)}$$

$$\Rightarrow f(z) = \frac{\frac{1}{5}}{z+2} + \frac{-\frac{1}{5}z + \frac{2}{5}}{z^2+1}$$

$$= \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{z^2+1}\right]$$

We expand this so that the binomial expansions valid $\left|\frac{1}{z^2}\right| <$ and $\left|\frac{z}{2}\right| < 1$. [আমরা ইহাকে বিস্তৃত করব যেন দ্বিপদী বিস্তার $\left|\frac{1}{z^2}\right|$ এবং $\left|\frac{z}{2}\right|$ বৈধ হয়]

$$\therefore f(z) = \frac{1}{5}\left[\frac{1}{2\left(1+\frac{z}{2}\right)} - (z-2)\cdot\frac{1}{z^2\left(1+\frac{1}{z^2}\right)}\right]$$

$$= \frac{1}{5}\left[\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - \left(\frac{z-2}{z^2}\right)\left(1+\frac{1}{z^2}\right)^{-1}\right]$$

$$= \frac{1}{10}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right) - \frac{z-2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right).$$

(ii) $|z| > 2 \Rightarrow 2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1$.

$$\therefore f(z) = \frac{1}{5}\left[\frac{1}{z\left(1+\frac{2}{z}\right)} - (z-2)\cdot\frac{1}{z^2+1}\right]$$

$$= \frac{1}{5z}\left(1+\frac{2}{z}\right)^{-1} - \frac{z-2}{z^2+1}$$

$$= \frac{1}{5z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right) - \frac{z-2}{z^2+1}$$

Problem–35. Expand the function $f(z) = \frac{1}{(z+1)(z+3)}$ laurent series for the following region :

(i) $1 < |z| < 3$, (ii) $|z| > 3$, (iii) $0 < |z+1| < 2$, (iv) $|z| < 1$

[NUH–1998, 2003, 2005, DUH–2004]

Solution : Given [দেওয়া আছে] $f(z) = \frac{1}{(z+1)(z+3)}$

$$= \frac{1}{(z+1)(-1+3)} + \frac{1}{(-3+1)(z+3)}$$

$$= \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

(i) $1 < |z| < 3$

$\Rightarrow 1 < |z|$ and $|z| < 3$

$\Rightarrow |z| > 1$ and $|z| < 3$

$\Rightarrow \frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$.

We write f(z) in a manner so that the binomial expansion is valid for $1 < |z| < 3 \Rightarrow \frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$. [আমরা f(z) কে এমন রীতিতে লিখব যেন দ্বিপদী বিস্তার $1 < |z| < 3 \Rightarrow \frac{1}{|z|} < 1$ এবং $\frac{|z|}{3} < 1$ এ বৈধ হয়]

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

$$= \frac{1}{2z}\cdot\frac{1}{1+\frac{1}{z}} - \frac{1}{6}\cdot\frac{1}{1+\frac{z}{3}}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\cdots\right)$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots\right) - \left(\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\cdots\right) \quad \text{Ans.}$$

(ii) $|z| > 3$

$\Rightarrow |z| > 1$ and [এবং] $|z| > 3$

$\Rightarrow \frac{1}{|z|} < 1$ and [এবং] $\frac{3}{|z|} < 1$.

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

$$= \frac{1}{2z}\cdot\frac{1}{\left(1+\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{2z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}+\cdots\right)$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots\right) - \left(\frac{1}{2z}-\frac{3}{2z^2}+\frac{9}{2z^3}-\frac{27}{2z^4}+\cdots\right)$$

$$= \left(\frac{3}{2z^2}-\frac{1}{2z^2}\right) + \left(\frac{1}{2z^3}-\frac{9}{2z^3}\right) + \left(-\frac{1}{2z^4}+\frac{27}{2z^4}\right) + \cdots$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots \quad \text{Ans.}$$

(iii) $0 < |z+1| < 2$

$\Rightarrow |z_1| < 2$, where [যেখানে] $z_1 = z+1$

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$
$$= \frac{1}{2z_1} - \frac{1}{2(z_1+2)}$$
$$= \frac{1}{2z_1} - \frac{1}{4\left(1+\frac{z_1}{2}\right)}$$
$$= \frac{1}{2z_1} - \frac{1}{4}\left(1+\frac{z_1}{2}\right)^{-1}$$
$$= \frac{1}{2z_1} - \frac{1}{4}\left(1 - \frac{z_1}{2} + \frac{z_1^2}{2^2} - \frac{z_1^3}{2^3} + \cdots\right)$$
$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \frac{1}{32}(z+1)^3 + \cdots \quad \text{Ans.}$$

(iv) $|z| < 1 \Rightarrow |z| < 1$ and [এবং] $|z| < 3 \Rightarrow |z| < 1$ and [এবং] $\frac{|z|}{3} < 1$

$$\therefore f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$
$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$
$$= \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)$$
$$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{-z}{2} + \frac{z}{18}\right) + \left(\frac{z^2}{2} - \frac{z^2}{54}\right) + \left(-\frac{z^3}{2} + \frac{1}{162}z^3\right) + \cdots$$
$$= \frac{2}{6} - \frac{8}{18}z + \frac{26}{54}z^2 - \frac{80}{162}z^3 + \cdots$$
$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots \quad \text{Ans.}$$

Problem–36. Find the Laurent expansion of f(z) = \F(z – 1,(z + 2) (z + 3)) in each of the following region :

(i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$. **[NUH-2002]**

Solution : We have [দেওয়া আছে] $f(z) = \frac{z-1}{(z+2)(z+3)}$
$$= \frac{-2-1}{(z+2)(-2+3)} + \frac{-3-1}{(-3+2)(z+3)}$$
$$= \frac{-3}{z+2} + \frac{4}{z+3}$$

(i) In this case [এইক্ষেত্রে] $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

Also [আরো] $|z| < 2 < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$.

$$\therefore f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$$
$$= \frac{-3}{2\left(1+\frac{z}{2}\right)} + \frac{4}{3\left(1+\frac{z}{3}\right)}$$
$$= \frac{-3}{2}\left(1+\frac{z}{2}\right)^{-1} + \frac{4}{3}\left(1+\frac{z}{3}\right)^{-1}$$
$$= \frac{-3}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right) + \frac{4}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)$$
$$= \left(\frac{-3}{2} + \frac{4}{3}\right) + \left(\frac{3}{4} - \frac{4}{9}\right)z + \left(\frac{-3}{8} + \frac{4}{27}\right)z^2 + \left(\frac{3}{16} - \frac{4}{81}\right)z^3 + \cdots$$
$$= -\frac{1}{6} + \frac{11}{36}z - \frac{49}{216}z^2 + \frac{179}{1296}z^3 - \cdots$$

(ii) In this case [এইক্ষেত্রে] $2 < |z| < 3$

$\Rightarrow 2 < |z|$ and [এবং] $|z| < 3$

$\Rightarrow \left|\frac{2}{z}\right| < 1$ and [এবং] $\left|\frac{z}{3}\right| < 1$

$$\therefore f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$$
$$= \frac{-3}{z\left(1+\frac{2}{z}\right)} + \frac{4}{3\left(1+\frac{z}{3}\right)}$$
$$= \frac{-3}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{4}{3}\left(1+\frac{z}{3}\right)^{-1}$$
$$= \frac{-3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right) + \frac{4}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)$$
$$= \cdots - \frac{12}{z^3} + \frac{6}{z^2} - \frac{3}{z} + \frac{4}{3} - \frac{4z}{9} + \frac{4z^2}{27} - \frac{4z^3}{81} + \cdots$$

(iii) Here [এখানে] $|z| > 3 \Rightarrow \left|\frac{z}{3}\right| > 1 \Rightarrow \left|\frac{3}{z}\right| < 1$

Also [আরো] $|z| > 3 \Rightarrow |z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$\therefore f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$$
$$= \frac{-3}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{4}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{-3}{z}\left((1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right) + \frac{4}{z}\left(1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + \cdots\right)$$

$$= (-3+4)\frac{1}{z} + (6-12)\frac{1}{z^2} + (-12+36)\frac{1}{z^3} + \cdots$$

$$= \frac{1}{z} - \frac{6}{z^2} + \frac{24}{z^3} - \cdots \qquad \text{Ans.}$$

Problem–37. Find the Laurent expansion of the function $f(z) = \frac{z^2+1}{(z+1)(z-2)}$ in each of the regions

(i) $1 < |z| < 2$ **[NUH–01, 12]** (ii) $0 < |z| < 1$ **[NUH–01, 12]**

Solution : Given that [দেওয়া আছে] $f(z) = \frac{z^2+1}{(z+1)(z-2)}$

Let [এবং] $\frac{z^2+1}{(z+1)(z-2)} = 1 + \frac{A}{z+1} + \frac{B}{z-2}$

$\Rightarrow z^2 + 1 = (z+1)(z-2) + A(z-2) + B(z+1)$

When [যখন] $z = -1$, then [তখন] $2 = 0 + A(-3) + 0 \Rightarrow A = \frac{-2}{3}$

When [যখন] $z = 2$, then [তখন] $5 = 0 + 0 + 3B \Rightarrow B = \frac{5}{3}$

$\therefore f(z) = \frac{z^2+1}{(z+1)(z-2)} = 1 - \frac{2/3}{z+1} + \frac{5/3}{z-2} \cdots (1)$

(i) In this case [এইক্ষেত্রে] $1 < |z| < 2$

$\Rightarrow 1 < |z|$ and [এবং] $|z| < 2$

$\Rightarrow \left|\frac{1}{z}\right| < 1$ and [এবং] $\left|\frac{z}{2}\right| < 1$

Therefore, from (1) we have [অতএব (1) হতে পাই]

$$f(z) = 1 - \frac{2/3}{z\left(1 + \frac{1}{z}\right)} + \frac{5/3}{-2\left(1 - \frac{z}{2}\right)}$$

$$= 1 - \frac{2}{3z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{5}{6}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= 1 - \frac{2}{3z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{5}{6}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$

$$= \cdots - \frac{2}{3z^3} + \frac{2}{3z^2} - \frac{2}{3z} + \frac{1}{6} - \frac{5z}{12} - \frac{5z^2}{24} - \frac{5z^3}{48} - \cdots$$

(ii) Here [এখানে] $0 < |z| < 1 \Rightarrow |z| < 1$

Also [আরো] $|z| < 1 \Rightarrow |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$\therefore f(z) = 1 - \frac{2/3}{z+1} + \frac{5/3}{-2\left(1 - \frac{z}{2}\right)}$$

$$= 1 - \frac{2}{3}(1+z)^{-1} - \frac{5}{6}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= 1 - \frac{2}{3}(1 - z + z^2 - z^3 + \cdots) - \frac{5}{6}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right)$$

$$= \left(1 - \frac{2}{3} - \frac{5}{6}\right) + \left(\frac{2}{3} - \frac{5}{12}\right)z + \left(\frac{-2}{3} - \frac{5}{24}\right)z^2 + \left(\frac{2}{3} - \frac{5}{48}\right)z^3 + \cdots$$

$$= -\frac{1}{2} + \frac{1}{4}z - \frac{7}{8}z^2 + \frac{9}{16}z^3 + \cdots$$

Problem–38. Obtain Laurent expansion of $\frac{1}{z^2(z-3)^2}$ about the point $z = 3$. **[NUH–1995]**

Solution : Let [ধরি] $z - 3 = u \Rightarrow z = u + 3$.

$$\therefore \frac{1}{z^2(z-3)^2} = \frac{1}{u^2(u+3)^2}$$

$$= \frac{1}{9u^2\left(1 + \frac{u}{3}\right)^2}$$

$$= \frac{1}{9u^2}\left(1 + \frac{u}{3}\right)^{-2}$$

$$= \frac{1}{9u^2}\left[1 - (2)\frac{u}{3} + \frac{(-2)(-3)}{\lfloor 2}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{\lfloor 3}\left(\frac{u}{3}\right)^3 + \cdots\right]$$

$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots$$

$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4}{243}(z-3) + \cdots$$

$[\because u = z - 3]$ Ans.

Problem–39. Find the laurent series expansion of $f(z) = \frac{3}{z(2 - z - z^2)}$ in Powers of z, valid in the region (i) $0 < |z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$ **[NUH–1996]**

Solution : $f(z) = \dfrac{3}{z(2-z-z^2)}$

$$2 - z - z^2 = 2 - 2z + z - z^2 = 2(1-z) + z(1-z) = (1-z)(2+z)$$

$$= \frac{3}{z(1-z)(2+z)}$$

$$= 3\left[\frac{1}{z(1-0)(2+0)} + \frac{1}{1\cdot(1-z)(2+1)} + \frac{1}{-2(1+2)(2+z)}\right]$$

$$= 3\left[\frac{1}{2z} + \frac{1}{3(1-z)} - \frac{1}{6(2+z)}\right]$$

$$= \frac{3}{2z} - \frac{1}{z-1} - \frac{1}{2(z+2)}$$

(i) $0 < |z| < 1$

$\Rightarrow |z| < 1$ and [এবং] $|z| < 2$

$\Rightarrow |z| < 1$ and [এবং] $\dfrac{|z|}{2} < 1$

$$\therefore f(z) = \frac{3}{2z} - \frac{1}{z-1} - \frac{1}{2(z+2)}$$

$$= \frac{3}{2z} + \frac{1}{1-z} - \frac{1}{4\left(1+\frac{z}{2}\right)}$$

$$= \frac{3}{2z} + (1-z)^{-1} - \frac{1}{4}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{3}{2z} + \left(1 + z + z^2 + z^3 + \cdots\right) - \frac{1}{4}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right)$$

$$= \frac{3}{2z} + \left(1 - \frac{1}{4}\right) + \left(1 + \frac{1}{8}\right)z + \left(1 - \frac{1}{16}\right)z^2 + \cdots$$

$$= \frac{3}{2z} + \frac{3}{4} + \frac{9}{8}z + \frac{15}{16}z^2 + \cdots \quad \text{Ans.}$$

(ii) $1 < |z| < 2$

$\Rightarrow 1 < |z|$ and [এবং] $|z| < 2$

$\Rightarrow |z| > 1$ and [এবং] $|z| < 2$

$\Rightarrow \dfrac{1}{|z|} < 1$ and [এবং] $\dfrac{|z|}{2} < 1$

$$\therefore f(z) = \frac{3}{2z} - \frac{1}{z-1} - \frac{1}{2(z+2)}$$

$$= \frac{3}{2z} - \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{4\left(1+\frac{z}{2}\right)}$$

$$= \frac{3}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{4}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{3}{2z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) - \frac{1}{4}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right)$$

$$= \cdots - \frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{2z} - \left(\frac{1}{4} - \frac{z}{8} + \frac{z^2}{16} - \cdots\right) \quad \text{Ans.}$$

(iii) $|z| > 2 \Rightarrow \dfrac{|z|}{2} > 1 \Rightarrow \dfrac{2}{|z|} < 1$ and $|z| > 2 \Rightarrow |z| > 1 \Rightarrow \dfrac{1}{|z|} < 1$

$$\therefore f(z) = \frac{3}{2z} - \frac{1}{z-1} - \frac{1}{2(z+2)}$$

$$= \frac{3}{2z} - \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{2}{z}\right)}$$

$$= \frac{3}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{3}{2z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) - \frac{1}{2z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \frac{2^4}{z^4} + \cdots\right)$$

$$= \frac{-3}{z^3} + \frac{3}{z^4} - \frac{9}{z^5} + \cdots \quad \text{Ans.}$$

Problem–40. Expand $f(z) = \dfrac{3z-3}{(2z-1)(z-2)}$ in a laurent series for the region $|z| < 1$ and $|z| > 1$. **[NUH–2000, NU(Pre)–2006]**

Solution : Given [দেওয়া আছে] $f(z) = \dfrac{3z-3}{(2z-1)(z-2)}$

$$= \frac{\frac{3}{2} - 3}{(2z-1)\left(\frac{1}{2}-2\right)} + \frac{3\times 2 - 3}{(2\times 2 - 1)(z-2)}$$

$$= \frac{-3}{-3(2z-1)} + \frac{3}{3(z-2)}$$

$$= \frac{1}{2z-1} + \frac{1}{z-2}$$

Now for the region $|z| < 1$ we have [এখন $|z| < 1$ এলাকার জন্য পাই]

$$f(z) = \frac{1}{2z-1} + \frac{1}{z-2}$$

$$= \frac{1}{2}\cdot\frac{1}{z-\frac{1}{2}} + \frac{1}{z-2}$$

$$= \frac{1}{2} \cdot \frac{1}{-\frac{1}{2}(1-2z)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= -(1-2z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -[1 + (2z) + (2z)^2 + \cdots] - \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$= -(1 + 2z + 4z^2 + \cdots) - \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right)$$

$$= \frac{-3}{2} - \frac{9}{4}z - \frac{33}{8}z^2 - \cdots \qquad \text{Ans.}$$

For the region $|z| > 1$ [$|z| > 1$ এলাকার জন্য] $\Rightarrow \frac{1}{|z|} < 1$

$$f(z) = \frac{1}{2z-1} + \frac{1}{z-2}$$

$$= \frac{1}{2z\left(1 - \frac{1}{2z}\right)} + \frac{1}{z\left(1 - \frac{2}{z}\right)}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{2z}\right)^{-1} + \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 + \frac{1}{2z} + \frac{1}{2^2z^2} + \frac{1}{2^3z^3} + \cdots\right) + \frac{1}{z}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots\right)$$

$$= \left(\frac{1}{2} + 1\right)\frac{1}{z} + \left(\frac{1}{4} + 2\right)\frac{1}{z^2} + \left(\frac{1}{8} + 4\right)\frac{1}{z^3} + \cdots$$

$$= \frac{3}{2z} + \frac{9}{4z^2} + \frac{33}{8z^3} + \cdots \qquad \text{Ans.}$$

Problem-41. Show that $\cos h\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n})$,

Where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cos h(2\cos\theta)\, d\theta$.

[DUMP-1980, RUH-1973]

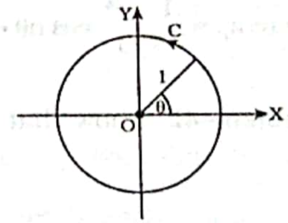
Solution : Let $f(z) = \cos h\left(z + \frac{1}{z}\right)$ which is analytic everywhere in the finite z-plane except at $z = 0$. Thus, $f(z)$ is analytic in the annular region $r \le |z| \le R$, where r is very small and R is very large. By Laurent's theorem we have

$$f(z) = \cos h\left(z + \frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} \;\ldots\ldots (1)$$

Where $a_n = \frac{1}{2\pi i}\oint_C \cos h\left(z + \frac{1}{z}\right)\frac{dz}{z^{n+1}} \;\ldots\ldots (2)$

$$b_n = \frac{1}{2\pi i}\oint_C \cos h\left(z + \frac{1}{z}\right) z^{n-1}\, dz \;\ldots\ldots (3)$$

and C is any circle with centre at the origin. Let us consider the circle C of unit radius. Then



$$|z| = 1 \Rightarrow z = e^{i\theta}$$

$$\Rightarrow dz = ie^{i\theta}\, d\theta.$$

and $z + \frac{1}{z} = e^{i\theta} + e^{-i\theta}$

$$= 2\cos\theta,\ 0 \le \theta \le 2\pi.$$

Putting these values in (2) we get,

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \cosh(2\cos\theta) \cdot \frac{ie^{i\theta}\, d\theta}{e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\,(\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta) \cdot \cos n\theta\, d\theta$$

$$- \frac{i}{2\pi}\int_0^{2\pi} \cos h(2\cos\theta) \cdot \sin n\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta) \cdot \cos n\theta\, d\theta - \frac{i}{2\pi} \times 0$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cdot \cos h(2\cos\theta)\, d\theta$$

$\because \int_0^{2\pi} F(\theta)\, d\theta = 0$ if $F(2\pi - \theta) = -F(\theta)$ which is true for $F(\theta) = \cosh(2\cos\theta)\sin n\theta$

From the values of a_n and b_n we see that if we replace n by $-n$ in the value of a_n we get the value of b_n.

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \cos(-n\theta) \cdot \cos h(2\cos\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cdot \cos h(2\cos\theta)\, d\theta = a_n$$

Thus from (1), $f(z) = \cosh\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n}$

$$\Rightarrow \cosh\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n})$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cdot \cosh(2\cos\theta)\, d\theta$ (Showed)

Problem–42. Show that $e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n$, where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta.$$

Solution : Let $f(z) = e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)}$. Then f(z) is analytic in the finite z-plane except the point z = 0. So f(z) is analytic in the annulas region $r \le |z| \le R$ where r is small and R is large. Hence by Laurent's theorem

$$f(z) = e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n \text{ ...... (1)}$$

Where $a_n = \frac{1}{2\pi i}\oint_C e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} \frac{dz}{z^{n+1}}$ ...... (2)

and C is any circle. Consider the circle C as $|z| = 1$.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta,\ 0 \le \theta \le 2\pi$

$$z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta.$$

$$\therefore a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{\frac{1}{2}c\cdot 2i\sin\theta} \cdot \frac{ie^{i\theta}\, d\theta}{e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ic\sin\theta} \cdot e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n\theta - c\sin\theta)}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} [\cos(n\theta - c\sin\theta) - i\sin(n\theta - c\sin\theta)]\, d\theta$$

$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta$, since the other term vanishes, as $\int_0^{2\pi} \sin(n\theta - c\sin\theta)\, d\theta$ denoted it by $\int_0^{2\pi} F(\theta)\, d\theta$ which satisfied $F(2\pi - \theta) = -F(\theta)$.

Thus, $e^{\frac{1}{2}c(z-1/z)} = \sum_{n=-\infty}^{\infty} a_n z^n$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta.$ (Showed)

Problem–43. Prove that $z = \pm i$ are branch points of the function $f(z) = (z^2 + 1)^{1/3}$.

Solution : We have $w = f(z) = (z^2 + 1)^{1/3}$

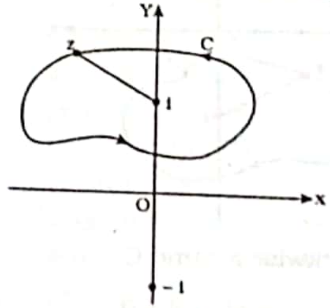
$$\Rightarrow w = [(z + i)(z - i)]^{1/3} = (z + i)^{1/3}(z - i)^{1/3}$$

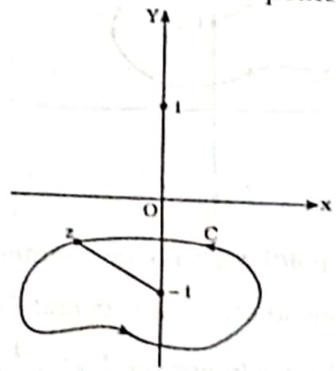
$\therefore \arg w = \frac{1}{3}\arg(z + i) + \frac{1}{3}\arg(z - i)$ $[\because \arg(z_1 z_2) = \arg z_1 + \arg z_2]$

$\Rightarrow$ Change in $\arg w = \frac{1}{3}$ [Change in arg (z + i)]

$+ \frac{1}{3}$ [Change in arg (z − i)]

Let C be a circuit enclosing the point i but not the point − i.





Let the point z goes once counterclockwise around C.

$\therefore$ Change in $\arg(z - i) = 2\pi$, Change in $\arg(z + i) = 0$

Thus, Change in $\arg w = \frac{1}{3} \times 0 + \frac{1}{3} \times 2\pi = \frac{2\pi}{3}$

Hence w does not return to its original value, so a change in branch has occured. Since $z = i$ altered the branch of the given function, so $z = i$ is a branch point of the given function.

In the same way considering as a circuit enclosing $z = -i$ but not $z = i$ we can show that $z = -i$ is another branch point.

Hence $z = \pm i$ are the branch points of the given function.

Problem–44. For the function $f(z) = (z^2 + 1)^{1/2}$ find branch points, branch line and show that a complete circuit around these points produces no change in the branches of the function.

Solution : Let $w = f(z) = (z^2 + 1)^{1/2}$

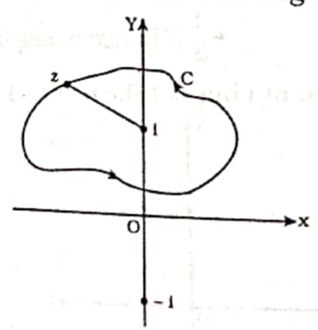
$\Rightarrow w = (z + i)^{1/2} (z - i)^{1/2}$

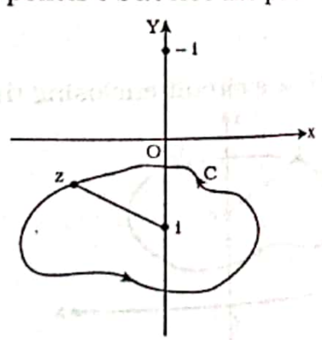
Then $\arg w = \frac{1}{2} \arg(z + i) + \frac{1}{2} \arg(z - i)$

$\Rightarrow$ Change in $\arg w = \frac{1}{2}$ [Change in $\arg(z + i)$]

$+ \frac{1}{2}$ [Change in $\arg(z - i)$]

Let C be the circuit enclosing the points i but not the point $-i$.





Let the point z goes once counter clockwise around C.

$\therefore$ Change in $\arg(z - i) = 2\pi$ and Change in $\arg(z + i) = 0$

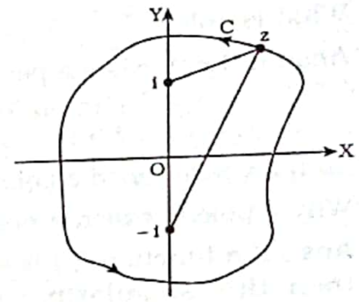
Thus Change in $\arg w = \frac{1}{2} \times 0 + \frac{1}{2} \times 2\pi = \pi$

Hence w does not return to its original values, so a change in branch has occurred. Since $z = i$ altered the branch of the given function, so $z = i$ is a branch point of the given function.

In the same way consider C as a circuit enclosing $z = -i$ but not $z = i$ we can show that $z = -i$ is another branch point. Hence $z = \pm i$ are the branch points of the given function.

2nd Part (Branch line) : The line segment $-1 \le y \le 1$ $(x = 0)$ is the branch line.

3rd Part : If C encloses both branch point $z = \pm i$ and z moves counterclockwise around C, then Change in $\arg(z - i) = 2\pi$ and Change in $\arg(z + i) = 2\pi$.



$\therefore$ Change in arg

$$w = \frac{1}{2} \times 2\pi + \frac{1}{2} \times 2\pi = 2\pi.$$

Hence a complete circuit around both the branch point does not change the branch.

Solved Brief/Quiz Questions
(সমাধানকৃত অতি সংক্ষিপ্ত প্রশ্ন)

1. What is a zero of an analytic function?

   Ans : A value of z for which the analytic function $f(z) = 0$ is called a zero of $f(z)$.

2. What is a zero of order n of an analytic function?

   Ans : If $f(z) = (z - z_0)^n g(z)$, where $g(z)$ is analytic, $g(z_0) \ne 0$ and n is a positive integer, then $z = z_0$ is called a zero of order n of the function $f(z)$.

3. What is a simple zero of an analytic function?

   Ans : If $f(z)$ has a zero of order one at $z = z_0$, then $f(z)$ is said to have a simple zero at $z = z_0$.

4. Define singular point of a complex function $f(z)$. **[NUH–2012]**

   Ans : A point at which an analytic function $f(z)$ fails to be analytic is called a singular point.

5. Define isolated singularity of a complex function $f(z)$.

   Ans : A singular point $z = z_0$ is called an isolated singularity of $f(z)$ if there is no other singularity with in a small circle surrounding the point $z = z_0$.

6. Define ordinary point of a complex function f(z).

Ans : If $z = z_0$ is not a singular point of f(z) and there can be found a small circle surrounding the point $z = z_0$ which encloses no singular point, then $z = z_0$ is called an ordinary point of f(z).

7. What is pole? **[NUH-2013]**

Ans : If there exists a positive integer n such that

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0.$$

then $z = z_0$ is called a pole of order n.

8. Why a pole is called a non-essential singularity?

Ans : If a function f(z) is single valued and has a singularity, then this singularity is either a pole or an essential singularity. For this reason a pole is sometimes called a non-essential singularity.

9. Define removable singularity.

Ans : A point z_0 is called a removable singularity of a complex function f(z) if $\lim_{z \to z_0} f(z)$ exists.

10. Define essential singularity.

Ans : A singular point which is not a pole, branch point or removable singularity is called an essential singularity.

11. Define meromorphic function.

Ans : A complex function f(z) which has poles as its only singularities in the finite part of the plane is called a meromorphic function.

12. Define entire function.

OR, What do you mean by entire function?

Ans : A complex function f(z) which has no singularities in the finite part of the plane is called an entire function.

13. Write a rule to determine a pole.

Ans : If $\lim_{z \to z_0} f(z) = \infty$ then $z = z_0$ is a pole of f(z).

14. Write a rule to determine a pole of order m.

Ans : If there are only m terms in the negative powers of $z - z_0$ of f(z) then $z = z_0$ is a pole of order m.

15. What is the pole of $f(z) = \frac{z^2 - 3z}{z - 2}$? [$f(z) = \frac{z^2 - 3z}{z - 2}$ এর পোল কত?] **[NUH-2012]**

Ans : The pole of $f(z) = \frac{z^2 - 3z}{z - 2}$ is 2, since

$$\lim_{z \to 2} (z - 2) f(z) = \lim_{z \to 2} (z - 2)\frac{z^2 - 3z}{z - 2} = 2^2 - 6 = -2 \neq 0.$$

16. What is branch point of complex number? **[NUH-2013]**

Ans : A multivalued function f(z) defined in some domain S is said to have a branch point at z_0 if, when z describes an arbitrary small circle about z_0, then for every branch F of f, F(z) does not return to its original value.

[একটি ডোমেন S এ বর্ণিত একটি বহুমানী ফাংশন f(z) এর z_0 বিন্দুতে ব্রাঞ্চ পয়েন্ট আছে যদি, f এর প্রত্যেক ব্রাঞ্চF এর জন্য z_0 বিন্দুর চারিদিকে z একটি ইচ্ছাধীন ক্ষুদ্র বৃত্ত বর্ণনা করে যেখানে F(z) তার আদি মান দিতে পারে না।]

17. What is meant by residue of complex number? **[NUH-2013]**

Ans : If the function f(z) is analytic within a circle C of radius r and centre a, except at z = a, then the coefficient a_{-1} of $\frac{1}{z - a}$ in the Laruent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} \frac{a_{-1}}{(z - a)^n}$$

$$= (a_0 + a_1 (z - a) + a_2(z - a)^2 + \cdots) + \left\{\frac{a_{-1}}{z - a} + \frac{a_{-2}}{(z - a)^2} + \cdots\right\}$$

around z = a is called the residue of f(z) at z = a. It is denoted by Res (a) or a_{-1}.

EXERCISE–4

Part–A : Brief Questions (অতি সংক্ষিপ্ত প্রশ্ন)

1. Define branch line.
2. Define branch point of a multivalued function f(z).
3. State maximum modulus theorem.
4. Write the statement of Rouche's theorem.
5. Why the function $f(z) = e^{-1/z^2}$ has no pole?