** Prove that, the set of rational numbers Q Q, equipped with the two binary operations of addition and multiplication, forms a field.

Sol n:

A set F with two binary operations + and · is a field if the following hold:

1. (F, +) is an abelian (commutative) group:
(a) clousure under +, (b) associativity of +, (c) identity element 0, (d) additive inverses, (e) commutativity of +;

2. $(F\setminus\{0\}, \cdot)$ is an abelian group:
(a) clousure under ·, (b) associativity of ·, (c) identity element 1, (d) ~~additive~~ multiplicative inverses for every nonzero element, (e) commutativity of ·.

3. Distributivity: $x\cdot(y+z) = x\cdot y + x\cdot z$ for all $x, y, z \in$

Finally, $0 \neq 1$ must hold (so the two ~~ind~~ identifies are distinct).

Verification for $\mathbb{Q}$:

Every rational number can be written as $\frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{Z} \setminus \{0\}$.

1. $(\mathbb{Q},+)$ is an abelian group

- Closure under addition:

if $x = \frac{a}{b}$ and $y = \frac{c}{d}$ then,

$$x+y = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd},$$

and $ad+bc$ and bd are integers with $bd \neq 0$.

Thus $x+y \in \mathbb{Q}$

→ Associativity: addion of rationals is associa-tive because it follows from associativity of integer addition.

For rationals x, y, z, $(x+y)+z = x+(y+z)$.

→ Additive Identity: 0 satisfies $x+0 = x$ for every rational x.

→ Additive inverses: For $x = \frac{a}{b}$, the additive inverse is $-x = \frac{-a}{b}$, which is rational and satisfies $x+(-x) = 0$.

→ Commutativity: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Hence, $(\mathbb{Q},+)$ is an abelian group.

~~2. (Q,+)~~

2. $(\mathbb{Q}\setminus\{0\}, \cdot)$ is an abelian group

• Clousure under multiplication:

with $x = \frac{a}{b}$, $y = \frac{c}{d}$

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and ac, bd are integers with $bd \neq 0$, so the product is in $\mathbb{Q}$. If neither x nor y is zero, then $ac \neq 0$ so the product is nonzero.

• Associativity: Multiplication of rationals is associative.

• Multiplicative identity: 1 satisfies $1 \cdot x = x$ for all $x \in \mathbb{Q}$

• Multiplicative inverses: For a non-zero rational $x = \frac{a}{b}$ with $a \neq 0$, the inverse is $\frac{b}{a}$ (an element of $\mathbb{Q}$), and $\frac{a}{b} \cdot \frac{b}{a} = 1$

• Commutativity: $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$ because integer multiplication is commutative.

Thus, $(\mathbb{Q}\setminus\{0\}, \cdot)$ is an abelian group.

3. Distributivity: For rationals $x=\frac{a}{b}, y=\frac{c}{d}, z=\frac{e}{f}$

$$x\cdot(y+z)=\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}\cdot\frac{(cf+de)}{df}=\frac{acf}{bdf}+\frac{ade}{bdf}$$

$$=\frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\cdot\frac{e}{f}$$

$$=x\cdot y+x\cdot z.$$

4. $0\neq 1$

In $\mathbb{Q}$, 0 is $\frac{0}{1}$ and 1 is $\frac{1}{1}$. These are different rationals, so $0\neq 1$. This prevents the degenerate one-element ring.

All field axioms hold for $\mathbb{Q}$: $(\mathbb{Q},+)$ is an abelian group, $(\mathbb{Q}\setminus\{0\},\cdot)$ is an abelian group, multiplication distributes over addition, and $0\neq 1$. Therefore $\mathbb{Q}$ with usual addition and multiplication is a field.