Q 1. Set of elements of finite order is a subgroup?

Soln:

As stated false in general (nonabelian group). Counterexample: infinite dihedral group - reflections have order 2 but product of two reflections can be a rotation of infinite order.

True if G is abelian: if x, y have finite orders m, n then $(xy)^{lcm(m,n)} = e$, so the tortion element form a subgroup in abelian group.

2. Subgroup of index p where p is smallest prime divison of $|G|$ is normal:

Soln:

Action on cosets gives $\varphi: G \to S_p$. The image has order dividing $|G|$ and dividing $p!$, so any prime divison of $|im\varphi|$ is $\leq p$. By minimality of p the image is a p-group, so its order is 1 or p. It cannot be 1 (else $H = G$); so $|im\varphi| = p$ then $[G : \ker\varphi] = p$ and $\ker\varphi \subseteq H$ forces $\ker\varphi = H$.

Hence, H is normal.

3. If $a^4 = b^2$ and $ab = ba$ then $(ab)^6 = e$.

Soln: As written, this is not valid. From $a^4 = b^2$ and commutation you cannot deduce $(ab)^6 = e$ (take a of infinite order and $b = a^2$ as a counterexample).

Correct simple case: If $a^4 = b^2 = e$ and $ab = ba$, then $|a| \mid 4$, $|b| \mid 2$ so $|ab| \mid \text{lcm}(4,2) = 4$. Thus, $(ab)^4 = e$, hence in particular $(ab)^6 = e$. (So need the extra hypothesis that $a^4 = b^2 = e$.)

4. If $[G:H] = n$, then $x^n \in H$ for $x \in G$.

Soln:

The coset xH in G/H has finite order dividing $|G/H| = n$. Thus $(xH)^n = H$, i.e. $x^n \in H$.

5. Unique subgroup of order p^k for each $k \leq n$ implies normal sylow p-subgroup.

Soln: The subgroup of order p^n is unique by hypothesis, so it is invariant under conjugation and hence normal.

6. Union of all conjugates of a proper subgroup H (finite G).

Soln:

Claim: $\bigcup_{g\in G} gHg^{-1} \neq G$

Proof: Let the distinct conjugates be m in number, each has $|H|$ elements and they all contain the identity, so

$$\left|\bigcup_{g} gHg^{-1}\right| \leq m|H| - (m-1) = m(|H|-1)+1$$

But $m = [G : N_G(H)] \geq 2$ and $|H| < |G|$, ~~so m(H~~

so $m(|H|-1)+1 < |G|$. Thus the union is a proper subset of G. (if H were normal ~~the union is~~ $H \neq G$ anyway).