1. Solve each of the following sets of simultaneous congruences

(a) $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$

Soln:

Product of all moduli, $M = 3 \times 5 \times 7 = 105$

We can compute partial moduli, dividing M by each modulus:

$M_1 = \frac{105}{3} = 35$, $M_2 = \frac{105}{5} = 21$, $M_3 = \frac{105}{7} = 15$

Inverses of M_i mod m_i ; where m_i are 3, 5, 7.

1. 35 mod 3 = 2 inverse of 35 mod 3 = 2
2. 21 mod 5 = 1 inverse of 21 mod 5 = 1
3. 15 mod 7 = 1 inverse of 15 mod 7 = 1

Total weighted sum $= (1)(35)(2) + (2)\cdot(21)\cdot(1) + (3)\cdot(15)\cdot(1)$

$= 70 + 42 + 45$

$= 157$

$x \equiv 157 \bmod 105 \Rightarrow x = 52$ (or, 1 remainder 52)

$\therefore x \equiv 52 \pmod{105}$

(b) $x \equiv 5 \pmod{11}$, $x \equiv 14 \pmod{29}$, $x \equiv 15 \pmod{31}$

Products of all moduli, $M = 11 \times 29 \times 31$
$= 9889$

Partial moduli:

$M_1 = \frac{9889}{11} = 899$, $M_2 = \frac{9889}{29} = 341$, $M_3 = \frac{9889}{31} = 319$

Modular inverses of $M_i \bmod m_i$ or y_i:

we know, $M_i \cdot y_i = 1 \pmod{m_i}$

1. $M_i \bmod m_i = 899 \bmod 11 = 8$

$8 \cdot y_1 = 1 \pmod{11} \cdot 8 \times 7 = 56 \equiv 1 \Rightarrow y_1 = 7$

2. $341 \bmod 29 = 22$

$22 \cdot y_2 = 1 \bmod (29) \cdot 22 \times 4 = 88 \equiv 1 \Rightarrow y_2 = 4$

3. $319 \bmod 31 = 9 \quad 9y_3 = 1 \bmod (31) \times 9 \times 7 = 63 \equiv 1 \Rightarrow y_3 = 7$

Total sum $= (5) \cdot (899) \cdot (7) + (14)(341)(4) + (15)(319)(7)$

$= 84056$

$x \equiv 84056 \bmod 9889 \Rightarrow x = 4944$ or (8 remainder 4944)

$\therefore x \equiv 4944 \pmod{9889}$

(c) $x \equiv 5 \pmod 6$, $x \equiv 4 \pmod{11}$, $x \equiv 3 \pmod{17}$

Soln:

Products of the moduli, $M = m_1 \times m_2 \times m_3$
$= 6 \times 11 \times 17$
$= 1122$

Partial moduli: $M_1 = \frac{1122}{6} = 187$, $M_2 = \frac{1122}{11} = 102$, $M_3 = \frac{1122}{17} = 66$

Modular inverses:

$M_i \cdot y_i = 1 \pmod{m_i}$

1. $\cancel{M_1 \bmod y_1 = 1 \pmod 6}$ $M_i \bmod m_i = 187 \bmod 6 = 1$

$\cancel{187 \bmod y_1}$ $\cancel{1 \times y_1 = 1 \pmod 6}$ $1 \times 1 \equiv 1 \Rightarrow y_1 = 1$

2. $102 \bmod 11 = 3$ $3 \times 4 = 12 \equiv 1 \Rightarrow y_2 = 4$

3. $66 \bmod 17 = 15$ $15 \times 8 = 120 \equiv 1 \Rightarrow y_3 = 8$

total sum $= (5)(187)(1) + (4)(102)(4) + (3)(66)(8)$
$= 4151$

$x \equiv 4151 \pmod{1122} \Rightarrow x = 785$ or (3 remainder 785)

$x \equiv 785 \pmod{1122}$